

Malliavin Calculus for the Stochastic 2D Navier–Stokes Equation

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Abstract

We consider the incompressible, two-dimensional Navier-Stokes equation with periodic boundary conditions under the effect of an additive, white-in-time, stochastic forcing. Under mild restrictions on the geometry of the scales forced, we show that any finite-dimensional projection of the solution possesses a smooth, strictly positive density with respect to Lebesgue measure. In particular, our conditions are viscosity independent. We are mainly interested in forcing that excites a very small number of modes. All of the results rely on proving the nondegeneracy of the infinite-dimensional Malliavin matrix. © 2006 Wiley Periodicals, Inc.

1 Introduction

We consider the movement of a two-dimensional, incompressible fluid with mean flow zero under periodic boundary conditions. We analyze the problem using the vorticity formulation of the form

$$(1.1) \quad \begin{cases} \frac{\partial w}{\partial t}(t, x) + B(w, w)(t, x) = \nu \Delta w(t, x) + \frac{\partial W}{\partial t}(t, x) \\ w(0, x) = w_0(x), \end{cases}$$

where $x = (x_1, x_2) \in \mathbb{T}^2$, the two-dimensional torus $[0, 2\pi] \times [0, 2\pi]$, $\nu > 0$ is the viscosity constant, $\frac{\partial W}{\partial t}$ is a white-in-time stochastic forcing to be specified below, and

$$B(w, \tilde{w}) = \sum_{i=1}^2 u_i(x) \frac{\partial \tilde{w}}{\partial x_i}(x)$$

where $u = \mathcal{K}(w)$. Here \mathcal{K} is the Biot-Savart integral operator, which will be defined below.

First we define a convenient basis in which we will perform all explicit calculations. Setting $\mathbb{Z}_+^2 = \{(j_1, j_2) \in \mathbb{Z}^2 : j_2 > 0\} \cup \{(j_1, j_2) \in \mathbb{Z}^2 : j_1 > 0, j_2 = 0\}$,

$\mathbb{Z}_-^2 = -\mathbb{Z}_+^2$, and $\mathbb{Z}_0^2 = \mathbb{Z}_+^2 \cup \mathbb{Z}_-^2$, we define a real Fourier basis for functions on \mathbb{T}^2 with zero spatial mean by

$$e_k(x) = \begin{cases} \sin(k \cdot x), & k \in \mathbb{Z}_+^2, \\ \cos(k \cdot x), & k \in \mathbb{Z}_-^2. \end{cases}$$

We write $w(t, x) = \sum_{k \in \mathbb{Z}_0^2} \alpha_k(t) e_k(x)$ for the expansion of the solution in this basis. With this notation in the two-dimensional periodic setting we have the expression

$$(1.2) \quad \mathcal{K}(w) = \sum_{k \in \mathbb{Z}_0^2} \frac{k^\perp}{\|k\|^2} \alpha_k e_{-k},$$

where $k^\perp = (-k_2, k_1)$ and $\|k\|^2 = k_1^2 + k_2^2$. See, for example, [17] for more details on the deterministic vorticity formulation in a periodic domain. We use the vorticity formulation for simplicity. All of our results can be translated into statements about the velocity formulation of the problem.

We take the forcing to be of the form

$$(1.3) \quad W(t, x) = \sum_{k \in \mathcal{Z}_*} W_k(t) e_k(x)$$

where \mathcal{Z}_* is a finite subset of \mathbb{Z}_0^2 and $\{W_k : k \in \mathcal{Z}_*\}$ is a collection of mutually independent, standard scalar Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The fact that we force a finite collection of Fourier modes becomes important starting in Section 3. Up until then the analysis applies to a force acting on any linearly independent collection of functions from \mathbb{T}^2 into \mathbb{R} that have spatial mean zero. The collection could even be infinite with a mild summability assumption.

We assume that $w_0 \in \mathbb{L}^2 = \{w \in L^2(\mathbb{T}^2, \mathbb{R}) : \int w \, dx = 0\}$. We will use $\|\cdot\|$ to denote the norm on \mathbb{L}^2 and $\langle \cdot, \cdot \rangle$ to denote the inner product. We also define $\mathbb{H}^s = \{w \in H^s(\mathbb{T}^2, \mathbb{R}) : \int w \, dx = 0\}$. Under these assumptions, it is standard that $w \in C([0, +\infty); \mathbb{L}^2) \cap L^2_{loc}((0, +\infty); \mathbb{H}^1)$ [7, 12, 23]. We will denote by $\|\cdot\|_s$ the natural norm on \mathbb{H}^s given by $\|f\|_s = \|\Lambda^s f\|$ where $\Lambda^2 = (-\Delta)$.

Our first goal is to prove the following theorem, which will be the consequence of the more general results given later in the text. In particular, it follows from Theorem 3.1, Theorem 6.1, and Corollary 8.2 when combined with Proposition 3.2.

THEOREM 1.1 *Consider the forcing*

$$\begin{aligned} W(t, x) = & W_1(t) \sin(x_1) + W_2(t) \cos(x_1) + W_3(t) \sin(x_1 + x_2) \\ & + W_4(t) \cos(x_1 + x_2). \end{aligned}$$

Then for any $t > 0$ and any finite-dimensional subspace S of \mathbb{L}^2 , the law of the orthogonal projection $\Pi w(t, \cdot)$ of $w(t, \cdot)$ onto S is absolutely continuous with respect to the Lebesgue measure on S . Furthermore, the density is C^∞ and everywhere strictly positive.

A version of Theorem 1.1 for Galerkin approximations of (1.1) was one of the main ingredients of the ergodic and exponential mixing results proven in [9]. There the algebraic structure of the nonlinearity was exploited to show that the associated diffusion was hypoelliptic. Here we use similar observations on the algebraic structure generated by the vector fields. However, new tools are required because there exists little theory applying Malliavin calculus in an infinite-dimensional setting. Relevant exceptions are [11, 15, 28].

In [11], Malliavin calculus is used to establish the existence of a density when all but a finite number of degrees of freedom were forced. In contrast to the present paper, the technique developed there fundamentally requires that only a finite number of directions are unforced. The ideas developed in the present paper could also likely be applied to the setting of [11].

An essential tool in our approach is a representation of the Malliavin covariance matrix through the solution of a backward (stochastic) partial differential equation, which was first invented by Ocone (see [28]) and which is particularly useful when dealing with certain classes of SPDEs, since in this case (as opposed to that of finite-dimensional SDEs), the fundamental solution of the linearized equation cannot be easily inverted. Ocone used that representation in the case where the original equation is a so-called bilinear SPDE (that is, both the coefficients of dt and $dW(t)$ are linear in the solution). In contrast, we use it in the case of a nonlinear PDE with additive noise. It seems that these are the only two cases where Ocone's representation of the Malliavin matrix through a backward (S)PDE can be used, without being exposed to the trouble of handling a stochastic PDE involving anticipative stochastic integrals. In Ocone's case, the backward PDE is a stochastic one, while in our case it is a PDE with random coefficients.

There has been a lot of activity in recent years exploring the ergodic properties of the stochastic Navier-Stokes equations and other dissipative stochastic partial differential equations. The central new idea was to make use of the pathwise contractive properties of the dynamics on the small scales and the mixing/smoothing due to the stochastic forcing on the larger scales. In [18] a determining-modes-type theorem (see [13]) was developed in the stochastic setting. This showed how controlling the behavior of a finite number of low modes on a time interval of infinite length was sufficient to control the entire system.

An important advance was made concurrently in [5, 10, 16], where it was shown that if all of the low modes are directly forced, the system is ergodic. The first two covered the case of white-in-time forcing while the last considered impulsive forcing. The assumptions of these papers can be restated as: the diffusion is elliptic on the unstable subspace of the pathwise dynamics (see [21] on this point of view).

The present paper establishes the needed control on the low modes when a "partial hypoelliptic" assumption is satisfied. We show that the forcing need not excite directly all the unstable modes because the nonlinearity transmits the randomness to the nondirectly excited unstable directions. Already the results of this paper

have been used in an essential way in [14] to prove the ergodicity of the stochastic Navier-Stokes equations under mild, viscosity-independent assumptions on the geometry of the forcing.

This article is organized as follows: In Section 2 we discuss the elements of Malliavin calculus needed in the paper. In particular, we give an alternative representation of the quadratic form associated to the Malliavin matrix. This representation is critical to the rest of the article. In Section 3 we explore the structure of the nonlinearity as it relates to nondegeneracy of the Malliavin matrix, which in turn implies the existence of a density. In Section 4 we prove an abstract lemma on the quadratic variation of nonadapted processes of a particular form that is the key to the results of the preceding section. In Section 5 we discuss the relationship to brackets of vector fields and the usual proof of nondegeneracy of the Malliavin matrix. In doing so we sketch an alternative proof of the existence of a density. In Section 6 we prove that the density, whose existence is given in Section 3, is in fact C^∞ . This requires the abstract results of Section 7, which amount to quantitative versions of the results in Section 4. Finally, in Section 8 we prove that the density of the finite-dimensional projections of $w(t)$ are everywhere positive under the same conditions that guarantee smoothness. We then give a number of concluding remarks and finish with five appendices containing technical estimates on the stochastic Navier-Stokes equation. In particular, Appendix C proves that the solution is smooth in the Malliavin sense, and Appendix E gives control of the Lipschitz constants in terms of various quantities associated to the solution.

2 Representation of the Malliavin Covariance Matrix

One way to solve the vorticity equation is by letting $w'(t, x) = w(t, x) - W(t, x)$ and solving the resulting PDE with random coefficients for w' . It easily follows from that approach that for each $t > 0$, there exists a continuous map

$$\Phi_t : C([0, t]; \mathbb{R}^{\mathcal{Z}_*}) \rightarrow \mathbb{L}^2$$

such that

$$w(t) = \Phi_t(W_{[0,t]}).$$

In other words, the solution of equation (1.1) can be constructed pathwise. We shall exploit this in Section 8.

For $k \in \mathcal{Z}_*$, $h \in L^2_{\text{loc}}(\mathbb{R}_+)$, $t > 0$, we define, if it exists, the Malliavin derivative of $w(t)$ in the direction (k, h) as

$$D^{k,h}w(t) = L^2(\Omega, \mathbb{L}^2)\text{-}\lim_{\varepsilon \rightarrow 0} \frac{\Phi_t(W + \varepsilon H e_k) - \Phi_t(W)}{\varepsilon},$$

where $H(t) = \int_0^t h(s) ds$. In fact, this convergence holds pathwise, and it is a Fréchet derivative.

We will show that the above derivative exists for each $h \in L^2_{\text{loc}}(\mathbb{R}_+)$, and moreover, that for each $s \in [0, t]$ and $k \in \mathcal{Z}_*$ there exists a random element $V_{k,s}(t)$ in \mathbb{L}^2 such that

$$D^{k,h}w(t) = \int_0^t V_{k,s}(t)h(s)ds.$$

$V_{k,s}(t)$ is then identified with $D^k_s w(t) \stackrel{\text{def}}{=} D^{k,\delta_s}w(t)$ and solves equation (2.1) below.

PROPOSITION 2.1 *For each $s > 0$ and $k \in \mathcal{Z}_*$, the linear parabolic PDE*

$$(2.1) \quad \begin{cases} \frac{\partial}{\partial t} V_{k,s}(t) = v \Delta V_{k,s}(t) - B(w(t), V_{k,s}(t)) - B(V_{k,s}(t), w(t)), & t \geq s, \\ V_{k,s}(s) = e_k \end{cases}$$

has a unique solution

$$V_{k,s} \in C([s, +\infty); \mathbb{L}^2) \cap L^2_{\text{loc}}([s, +\infty); \mathbb{H}^1).$$

PROOF: See, for example, Constantin and Foias [6]. □

At times we will consider the linearized equation (2.1) with arbitrary initial conditions. We write $J_{s,t}\phi$ for the solution to (2.1) at time t with initial condition ϕ at time s less than t . In this notation $V_{k,s}(t) = J_{s,t}e_k$.

Furthermore, Lemma B.1 from the appendix implies that for all deterministic initial conditions $w(0)$, $p \geq 1$, $\eta > 0$, and $T < \infty$,

$$\mathbb{E} \sup_{0 \leq s \leq t \leq T} \|V_{k,s}(t)\|^{2p} < c \exp(\eta \|w(0)\|^2)$$

for some $c = c(v, p, T, \eta)$.

Clearly, if $h \in L^2_{\text{loc}}(\mathbb{R}_+)$,

$$V_{k,h}(t) \stackrel{\text{def}}{=} \int_0^t V_{k,s}(t)h(s)ds$$

is the unique solution in $C([0, +\infty); \mathbb{L}^2) \cap L^2_{\text{loc}}([0, +\infty); \mathbb{H}^1)$ of the parabolic PDE

$$(2.2) \quad \begin{cases} \frac{\partial V_{k,h}(t)}{\partial t} = v \Delta V_{k,h}(t) - B(w(t), V_{k,h}(t)) \\ \quad - B(V_{k,h}(t), w(t)) + h(t)e_k, & t \geq 0, \\ V_{k,h}(0) = 0. \end{cases}$$

It is not hard to see that, in the sense of convergence in $L^2(\Omega; \mathbb{L}^2)$,

$$V_{k,h}(t) = \lim_{\varepsilon \rightarrow 0} \frac{\Phi_t(W + \varepsilon H e_k) - \Phi_t(W)}{\varepsilon}.$$

It then follows that

$$w(t) \in H^1(\Omega, \mathbb{L}^2) \stackrel{\text{def}}{=} \left\{ X : \Omega \rightarrow \mathbb{L}^2 : \mathbb{E} \|X\|^2, \mathbb{E} \int_0^t \|D^k_s X\|^2 ds < \infty, \right. \\ \left. \text{for all } k \in \mathcal{Z}_* \text{ and finite } t > 0 \right\}$$

(see Nualart [25, pp. 27, 62] for more details). Furthermore, its associated infinite-dimensional Malliavin covariance matrix is given by

$$(2.3) \quad \mathcal{M}(t) = \sum_{k \in \mathcal{Z}_*} \int_0^t V_{k,s}(t) \otimes V_{k,s}(t) ds,$$

that is to say, it is the operator mapping $\phi \in \mathbb{L}^2$ to $\mathcal{M}(t)(\phi) \in \mathbb{L}^2$ given by

$$(2.4) \quad \mathcal{M}(t)(\phi) = \sum_{k \in \mathcal{Z}_*} \int_0^t \langle V_{k,s}(t), \phi \rangle V_{k,s}(t) ds.$$

It follows from theorem 2.1.2 in Nualart [25] that Theorem 1.1 is a consequence of the fact that for each $\phi \in \mathbb{L}^2$ with $\phi \neq 0$,

$$\langle \mathcal{M}(t)\phi, \phi \rangle = \sum_{k \in \mathcal{Z}_*} \int_0^t \langle V_{k,s}(t), \phi \rangle^2 ds > 0 \text{ a.s.}$$

We now want to give an alternative representation of this quantity, using a backward PDE that is the adjoint of equation (2.2).

PROPOSITION 2.2 *For each $t > 0$, $\phi \in \mathbb{L}^2$, the linear backward parabolic PDE*

$$(2.5) \quad \begin{cases} \frac{\partial}{\partial s} U^{t,\phi}(s) + v \Delta U^{t,\phi}(s) \\ + B(w(s), U^{t,\phi}(s)) - C(U^{t,\phi}(s), w(s)) = 0, & 0 \leq s \leq t, \\ U^{t,\phi}(t) = \phi \end{cases}$$

has a unique solution

$$U^{t,\phi} \in C([0, t]; \mathbb{L}^2) \cap L^2([0, t]; \mathbb{H}^1).$$

Here $C(\cdot, w(s))$ is the \mathbb{L}^2 adjoint of the time-dependent, linear operator $B(\cdot, w(s))$ and thus is defined by the relation $\langle B(u, w(s)), v \rangle = \langle C(v, w(s)), u \rangle$.

PROOF: We use the same argument as in Proposition 2.1. □

As before, Lemma B.1 from the appendix implies that there exist a positive constant η so that for all deterministic initial conditions $w(0)$, $\phi \in \mathbb{L}^2$, $p \geq 1$, and $T < \infty$

$$\mathbb{E} \sup_{0 \leq s \leq t \leq T} \|U^{t,\phi}(s)\|^{2p} < c \|\phi\|^{2p} \exp(\eta \|w(0)\|^2)$$

for some $c = c(v, p, T, \eta)$.

PROPOSITION 2.3 *For each $k \in \mathcal{Z}_*$ and $\phi \in \mathbb{L}^2$, the function*

$$r \rightarrow \langle V_{k,s}(r), U^{t,\phi}(r) \rangle$$

from $[s, t]$ into \mathbb{R} is constant.

PROOF: We first show that this mapping belongs to $W^{1,1}(s, t; \mathbb{R})$. It is clearly a continuous function and moreover

$$V, U \in C([s, t]; \mathbb{L}^2) \cap L^2((s, t); \mathbb{H}^1);$$

hence by interpolation

$$V, U \in L^4((s, t), \mathbb{H}^{1/2}) \cap L^2((s, t); \mathbb{H}^1).$$

Since we also know that

$$w \in L^4((s, t); \mathbb{H}^{1/2}) \cap L^2((s, t); \mathbb{H}^1),$$

we know that the products

$$\|U\|_{1/2}\|w\|_1, \|V\|_{1/2}\|w\|_1, \|w\|_{1/2}\|U\|_1, \text{ and } \|w\|_{1/2}\|V\|_1$$

all belong to $L^{4/3}((s, t); \mathbb{R})$. On the other hand, it follows from the fact that $\|\mathcal{K}U\|_1 = \|U\|$ and estimate (6.10) in Constantin and Foias [6] that

$$\begin{aligned} |\langle C(U, w), \psi \rangle| &\leq c\|U\|_{1/2}\|w\|_1\|\psi\|_{1/2}, \\ |\langle B(V, w), \psi \rangle| &\leq c\|V\|_{1/2}\|w\|_1\|\psi\|_{1/2}, \\ |\langle B(w, U), \psi \rangle| &\leq c\|w\|_{1/2}\|U\|_1\|\psi\|_{1/2}, \\ |\langle B(w, V), \psi \rangle| &\leq c\|w\|_{1/2}\|V\|_1\|\psi\|_{1/2}. \end{aligned}$$

Hence we conclude that $C(U, w)$, $B(V, w)$, $B(w, U)$, and $B(w, V)$ all belong to $L^{4/3}((s, t); \mathbb{H}^{-1/2})$. From (2.1) and (2.5), we see that both $\frac{d}{dt}V$ and $\frac{d}{dt}U$ consist of three terms. The first belongs to $L^2((s, t); \mathbb{H}^{-1})$, and the last two belong to $L^{4/3}((s, t); \mathbb{H}^{-1/2})$. Hence $\langle V_{k,s}(r), \frac{d}{dr}U^{t,\phi}(r) \rangle$ and $\langle \frac{d}{dr}V_{k,s}(r), U^{t,\phi}(r) \rangle$ are in $L^1(s, t; \mathbb{R})$, and the statement that

$$r \rightarrow \langle V_{k,s}(r), U^{t,\phi}(r) \rangle$$

is a.e. differentiable then follows a variant of theorem 2 in Dautray and Lions [8, chap. 18, sec. 1]. Moreover, for almost every r

$$\begin{aligned} \frac{d}{dr}\langle V(r), U(r) \rangle &= \langle A(w(r))V(r), U(r) \rangle - \langle V(r), A^*(w(r))U(r) \rangle \\ &= 0 \end{aligned}$$

where $A(w(t))$ is the linear operator on the right-hand side of (2.1), and $A^*(w(t))$ is its \mathbb{L}^2 adjoint. The result follows. □

We can now rewrite the Malliavin covariance matrix using U in place of V . For a fixed ϕ , this is an improvement as $U^{t,\phi}(r)$ is a single solution to a PDE while $V_{k,t}(r)$ is a continuum of solutions indexed by the parameter s .

COROLLARY 2.4 For any $\phi \in \mathbb{L}^2$,

$$\langle \mathcal{M}(t)\phi, \phi \rangle = \sum_{k \in \mathbb{Z}_*} \int_0^t \langle e_k, U^{t,\phi}(s) \rangle^2 ds.$$

PROOF: This follows from the fact that for any $0 \leq s \leq t$ and $k \in \mathcal{Z}_*$,

$$\langle V_{k,s}(s), U^{t,\phi}(s) \rangle = \langle V_{k,s}(t), U^{t,\phi}(t) \rangle,$$

i.e.,

$$\langle e_k, U^{t,\phi}(s) \rangle = \langle V_{k,s}(t), \phi \rangle.$$

□

From this corollary, one immediately deduces the following result:

COROLLARY 2.5 *Assume that for some fixed $\phi \in \mathbb{L}^2$,*

$$\langle \mathcal{M}(t)\phi, \phi \rangle = 0$$

on a subset Ω_1 of Ω . Then for all $k \in \mathcal{Z}_$ and $s \leq t$, $\langle e_k, U^{t,\phi}(s) \rangle = 0$ on Ω_1 . In particular, $\langle e_k, \phi \rangle = 0$.*

3 Hypocoellipticity

3.1 Final Assumptions and the Main Existence Result

We define \mathcal{Z}_0 to be the symmetric part of the forcing set \mathcal{Z}_* given by $\mathcal{Z}_0 = \mathcal{Z}_* \cap (-\mathcal{Z}_*)$ and then the collection

$$\mathcal{Z}_n = \{\ell + j \in \mathbb{Z}_0^2 : j \in \mathcal{Z}_0, \ell \in \mathcal{Z}_{n-1} \text{ with } \ell^\perp \cdot j \neq 0, \|j\| \neq \|\ell\|\},$$

and lastly

$$\mathcal{Z}_\infty = \bigcup_{n=1}^\infty \mathcal{Z}_n.$$

Notice that the above union starts at 1 and that the \mathcal{Z}_n are symmetric in that $\mathcal{Z}_n = -\mathcal{Z}_n$. This follows by recurrence, starting with $\mathcal{Z}_0 = -\mathcal{Z}_0$. We are mainly concerned with the case where $\mathcal{Z}_0 = \mathcal{Z}_*$, since this corresponds to noise that is stationary in x .

We can now state the main theorem. Defining

$$(3.1) \quad S_0 = \text{Span}(e_k : k \in \mathcal{Z}_*), \quad S_n = \text{Span}\left(e_k : k \in \bigcup_{j=1}^n \mathcal{Z}_j \cup \mathcal{Z}_*\right),$$

$$n \in \{1, \dots, \infty\},$$

we have the following result, which implies the first part of Theorem 1.1 in the case when $S_\infty = \mathbb{L}^2$.

THEOREM 3.1 *For any $t > 0$ and any finite-dimensional subspace S of S_∞ , the law of the orthogonal projection $\Pi w(t, \cdot)$ of $w(t, \cdot)$ onto S is absolutely continuous with respect to the Lebesgue measure on S .*

The above result guarantees an absolutely continuous density on finite-dimensional subsets of S_∞ . However, it does not imply the lack of density for other subsets because in constructing S_∞ we have used only part of the available information. In the proof below, it will become clear that we make use only of the directions generated by frequencies where both the sin and cos are stochastically forced. We do this in the name of simplicity and utility. Verifying any more complicated condition is difficult. However, as translation invariance implies that both the sin and cos mode of a given frequency are forced, it seems a reasonable compromise. In the end, we are primarily interested in producing conditions that give insight into how the nonlinearity spreads the randomness. In particular, we now give an easy proposition that, in conjunction with Theorem 3.1, proves the first part of Theorem 1.1 given in the introduction. After that we will quote a more general result from [14] that is proven using similar ideas to those below.

PROPOSITION 3.2 *If $\{(0, 1), (1, 1)\} \subset \mathcal{Z}_0$, then $S_\infty = \mathbb{L}^2$.*

PROOF: Clearly by adding and subtracting the vectors $(0, 1)$ and $(1, 1)$, one can generate all of \mathbb{Z}_0^2 . The only question is whether the conditions $\ell^\perp \cdot j \neq 0$ and $\|j\| \neq \|\ell\|$ from the definition of \mathcal{Z}_n ever create a situation that blocks continuing generating the lattice.

First, notice that $(1, 0) = (1, 1) - (0, 1)$, $(-1, 0) = (0, 1) - (1, 1)$, and $(-1, 1) = (-1, 0) - (1, 1) + (0, 1) + (1, 1)$, and if these moves are made from left to right, none of the restrictions are violated. Hence all of the vectors of the same length as the vectors in \mathcal{Z}_0 can be reached and henceforth the restriction $\|j\| \neq \|\ell\|$ will not be binding. The requirement that $\ell^\perp \cdot j \neq 0$ does not cause a problem. The line $\ell^\perp \cdot (0, 1) = 0$ can be approached from above and does not obstruct generating the rest of the lattice. The line $\ell^\perp \cdot (1, 1) = 0$ does separate the lattice. However, we know we can reach $(1, 0)$ and have the point $(0, 1)$ to start with. Hence we can reach all of the points on either side of the line. Those on the line can be reached from points on either side. \square

It is clear from the preceding lemma that many other choices of forcing will also lead to $S_\infty = \mathbb{L}^2$. For instance, if $\{(1, 0), (1, 1)\} \subset \mathcal{Z}_0$, then $S_\infty = \mathbb{L}^2$. It is also interesting to force a collection of modes distant from the origin and allow the noise to propagate both up to the large scales and down to still smaller scales. We now give a simple proposition giving sufficient conditions in such a setting.

PROPOSITION 3.3 *Let $M, K \in \mathbb{N}$ with $M, K > 2$ and $\|M - K\| > 2$. Then if $\{(M + 1, 0), (M, 0), (0, K + 1), (0, K)\} \subset \mathcal{Z}_0$, $S_\infty = \mathbb{L}^2$.*

PROOF: The idea is to use $(M + 1, 0) - (M, 0) = (1, 0)$, $(M, 0) - (M + 1, 0) = (-1, 0)$, $(0, K + 1) - (0, K) = (0, 1)$, and $(0, K) - (0, K + 1) = (0, -1)$ in order to generate the whole lattice. The only difficulty could be the above restrictions. The restrictions of the form $\ell^\perp \cdot j \neq 0$ only prevent applying $(M + 1, 0) - (M, 0)$ or $(M, 0) - (M + 1, 0)$ to points on the x -axis and $(0, K) - (0, K + 1)$ and $(0, K + 1) - (0, K)$.

1) – (0, K) to points on the y-axis. However, this is not a serious restriction as all of the points on the x-axis can be reached by moving down from above, and all of the points on the y-axis can be reached by moving horizontally. Furthermore, the y-axis can be crossed by using strictly horizontal moves.

The only remaining restriction consists of the points $k \in \mathbb{Z}_0^2$ with $\|k\| \in \{K, K + 1, M, M + 1\}$. For example, assume that $\|k\| = K$ and one wanted to move to the left by applying $(0, K) - (0, K + 1)$. While this direct move is illegal, one can accomplish the same effect by moving up, then left, and finally down. The requirement that $\|M - K\| > 2$ ensures that we will not be blocked from moving up using $(M + 1, 0) - (M, 0)$ given that $k = \|K\|$. Once we have moved up, we will be free to move to the left and then back down. The other cases are analogous. \square

Guided by these results, in [14] the following is proven:

PROPOSITION 3.4 *One has $S_\infty = \mathbb{L}^2$ if and only if the following hold:*

- (1) *Integer linear combinations of elements of \mathcal{Z}_0 generate \mathbb{Z}_0^2 .*
- (2) *There exist at least two elements in \mathcal{Z}_0 with unequal Euclidean norm.*

This gives a very satisfactory characterization of the setting when $S_\infty = \mathbb{L}^2$, which is the case of primary interest.

3.2 Proof of Theorem 3.1

Since we already know that $w(t) \in H^1(\Omega, \mathbb{L}^2)$, Theorem 3.1 follows from theorem 2.1.2 in Nualart [25] and the fact that for any $\phi \in S_\infty, \phi \neq 0$,

$$\langle \mathcal{M}(t)\phi, \phi \rangle > 0 \quad \text{a.s.}$$

Hence to prove Theorem 3.1, it suffices to show the following:

PROPOSITION 3.5 *There exists a subset $\Omega_1 \subset \Omega$ of full measure so that on Ω_1 if $\langle \mathcal{M}(t)\phi, \phi \rangle = 0$ for some $\phi \in \mathbb{L}^2$, then $\Pi_\infty\phi = 0$ where Π_∞ is the \mathbb{L}^2 orthogonal projection onto S_∞ .*

Notice that Proposition 3.5 is equivalent to

$$(3.2) \quad \mathbb{P}\left(\bigcap_{\substack{\phi \in \mathbb{L}^2 \\ \Pi_\infty\phi \neq 0}} \{\langle \mathcal{M}(t)\phi, \phi \rangle > 0\}\right) = 1.$$

To prove the proposition, we need to better understand the structure of the equations. To this end, we now write the equations for the spatial Fourier coefficients of U and w to better expose the interactions between the systems' various degrees of freedom. In this and the general structures of the nonlinearity exploited, we follow E and Mattingly [9]; however, the tack of the analysis is different. (We also take the chance to correct a small error in [9]. There the summation was restricted to modes

in the first quadrant, when it should have ranged over the entire upper half-plane.) Again setting $w(t, x) = \sum_{k \in \mathbb{Z}_0^2} \alpha_k(t) e_k$, we have for $\ell \in \mathbb{Z}_0^2$

$$(3.3) \quad \begin{aligned} \frac{d}{dt} \alpha_\ell(t) + \nu \|\ell\|^2 \alpha_\ell(t) + \frac{1}{2} \sum_{(j,k,\ell) \in \mathcal{I}_+} c(j, k) \alpha_j(t) \alpha_k(t) \\ - \frac{1}{2} \sum_{(j,k,\ell) \in \mathcal{I}_-} c(j, k) \alpha_j(t) \alpha_k(t) = \mathbf{1}_{\mathcal{Z}_*}(\ell) \frac{d}{dt} W_\ell(t) \end{aligned}$$

where $\mathbf{1}_{\mathcal{Z}_*}$ is the indicator function of \mathcal{Z}_* , $c(j, k) = \frac{1}{2}(j^\perp \cdot k)(\|j\|^{-2} - \|k\|^{-2})$ and

$$\begin{aligned} \mathcal{I}_+ = \{ & (j, k, \ell) \in (\mathbb{Z}_+^2, \mathbb{Z}_-^2, \mathbb{Z}_+^2) \cup (\mathbb{Z}_-^2, \mathbb{Z}_+^2, \mathbb{Z}_+^2) \cup (\mathbb{Z}_+^2, \mathbb{Z}_+^2, \mathbb{Z}_-^2) \mid k + j + \ell = 0 \} \\ & \cup \{ (j, k, \ell) \in (\mathbb{Z}_+^2, \mathbb{Z}_-^2, \mathbb{Z}_+^2) \cup (\mathbb{Z}_-^2, \mathbb{Z}_-^2, \mathbb{Z}_-^2) \cup (\mathbb{Z}_+^2, \mathbb{Z}_+^2, \mathbb{Z}_-^2) \mid \ell = j - k \} \\ & \cup \{ (j, k, \ell) \in (\mathbb{Z}_-^2, \mathbb{Z}_+^2, \mathbb{Z}_+^2) \cup (\mathbb{Z}_-^2, \mathbb{Z}_-^2, \mathbb{Z}_-^2) \cup (\mathbb{Z}_+^2, \mathbb{Z}_+^2, \mathbb{Z}_-^2) \mid \ell = k - j \}, \\ \mathcal{I}_- = \{ & (j, k, \ell) \in (\mathbb{Z}_-^2, \mathbb{Z}_+^2, \mathbb{Z}_+^2) \cup (\mathbb{Z}_+^2, \mathbb{Z}_-^2, \mathbb{Z}_+^2) \cup (\mathbb{Z}_-^2, \mathbb{Z}_-^2, \mathbb{Z}_-^2) \mid \ell = j + k \}. \end{aligned}$$

Setting

$$U^{t,\phi}(s, x) = \sum_{k \in \mathbb{Z}_0^2} \beta_k^\phi(s) e_k(x) \quad \text{and} \quad \phi(x) = \sum_{k \in \mathbb{Z}_0^2} \phi_k e_k(x),$$

we also have the backward equations

$$(3.4) \quad \begin{aligned} \frac{d}{ds} \beta_\ell^\phi(s) = \nu \|\ell\|^2 \beta_\ell^\phi(s) + \sum_{(j,k,\ell) \in \mathcal{I}_+^*} c(j, \ell) \alpha_j(t) \beta_k^\phi(t) \\ - \sum_{(j,k,\ell) \in \mathcal{I}_-^*} c(j, \ell) \alpha_j(t) \beta_k^\phi(t), \quad s < t, \\ \beta_\ell^\phi(t) = \phi_\ell, \end{aligned}$$

where

$$\begin{aligned} \mathcal{I}_+^* = \{ & (j, k, \ell) \in (\mathbb{Z}_+^2, \mathbb{Z}_+^2, \mathbb{Z}_-^2) \cup (\mathbb{Z}_-^2, \mathbb{Z}_+^2, \mathbb{Z}_+^2) \cup (\mathbb{Z}_+^2, \mathbb{Z}_-^2, \mathbb{Z}_+^2) \mid j + k + \ell = 0 \} \\ & \cup \{ (j, k, \ell) \in (\mathbb{Z}_+^2, \mathbb{Z}_+^2, \mathbb{Z}_-^2) \cup (\mathbb{Z}_-^2, \mathbb{Z}_-^2, \mathbb{Z}_-^2) \cup (\mathbb{Z}_+^2, \mathbb{Z}_-^2, \mathbb{Z}_+^2) \mid \ell = j - k \} \\ & \cup \{ (j, k, \ell) \in (\mathbb{Z}_-^2, \mathbb{Z}_+^2, \mathbb{Z}_+^2) \cup (\mathbb{Z}_-^2, \mathbb{Z}_-^2, \mathbb{Z}_-^2) \cup (\mathbb{Z}_+^2, \mathbb{Z}_-^2, \mathbb{Z}_+^2) \mid \ell = j + k \}, \\ \mathcal{I}_-^* = \{ & (j, k, \ell) \in (\mathbb{Z}_-^2, \mathbb{Z}_+^2, \mathbb{Z}_+^2) \cup (\mathbb{Z}_+^2, \mathbb{Z}_+^2, \mathbb{Z}_-^2) \cup (\mathbb{Z}_-^2, \mathbb{Z}_-^2, \mathbb{Z}_-^2) \mid \ell = k - j \}. \end{aligned}$$

We now continue the proof of Proposition 3.5. Notice that the β_ℓ^ϕ are continuous in time for every $\ell \in \mathcal{Z}_0$, every $\phi \in \mathbb{L}^2$, and every realization of the stochastic forcing. Hence if $\beta_\ell^\phi \equiv 0$ for some realization of noise, then $\phi_\ell = 0$. (The notation $x \equiv 0$ means $x(s) = 0, s \in [0, t)$.) Thus to prove the lemma, it would be sufficient to show that there exists a fixed set Ω_1 with positive probability so that for any $\phi \in S_\infty$, if $\langle \mathcal{M}(t)\phi, \phi \rangle = 0$ on Ω_1 , then $\beta_\ell^\phi \equiv 0$ for all $\ell \in \mathcal{Z}_* \cup \mathcal{Z}_\infty$. This will be proven inductively.

The base case of the induction is given by Corollary 2.5. In the present notation, it simply says that for any $\omega \in \Omega$ if $\langle \mathcal{M}(t)\phi, \phi \rangle = 0$ for some $\phi \in \mathbb{L}^2$, then $\beta_\ell^\phi \equiv 0$

for all $\ell \in \mathcal{Z}_*$. In particular, for any $\omega \in \Omega$ if $\langle \mathcal{M}(t)\phi, \phi \rangle = 0$, then $\beta_\ell^\phi \equiv \beta_{-\ell}^\phi \equiv 0$ for all $\ell \in \mathcal{Z}_0$. The proof of Theorem 3.1 would then be complete if we show that there exists a single subset $\Omega_1 \subset \Omega$ of full measure so that if $\beta_\ell^\phi \equiv \beta_{-\ell}^\phi \equiv 0$ on Ω_1 for all $\ell \in \mathcal{Z}_n$, then $\beta_\ell^\phi \equiv \beta_{-\ell}^\phi \equiv 0$ on Ω_1 for all $\ell \in \mathcal{Z}_{n+1}$. This inductive step is given by the next lemma, which once proved completes the proof of Theorem 3.1.

LEMMA 3.6 *There exists a fixed subset Ω_1 of full measure so that for any $\phi \in \mathbb{L}^2$ and $\ell \in \mathbb{Z}_0^2$, if $\beta_\ell^\phi \equiv 0$ and $\beta_{-\ell}^\phi \equiv 0$ on Ω_1 , then for all $j \in \mathcal{Z}_0$ such that $j^\perp \cdot \ell \neq 0$ and $\|j\| \neq \|\ell\|$*

$$\beta_{\ell+j}^\phi \equiv \beta_{-(\ell+j)}^\phi \equiv \beta_{\ell-j}^\phi \equiv \beta_{j-\ell}^\phi \equiv 0$$

on Ω_1 .

PROOF: We begin with some simple observations that will be critical shortly. Notice that from (3.3)–(3.4), one sees that for $\ell \in \mathcal{Z}_*$, $\alpha_\ell(s)$ has the form

$$(3.5) \quad \alpha_\ell(s) = \alpha_\ell(0) + \int_0^s \gamma_\ell(r) dr + W_\ell(s)$$

where the γ_ℓ are some stochastic processes depending on the initial conditions and noise realizations. Hence these coordinates are the sum of a Brownian motion and a part that has finite first variation and is continuous in time for all $\omega \in \Omega$.

Similarly, for $\ell \notin \mathcal{Z}_*$,

$$(3.6) \quad \alpha_\ell(s) = \alpha_\ell(0) + \int_0^s \gamma_\ell(r) dr,$$

and hence these coordinates are continuous and have finite first variation in time for every $\omega \in \Omega$ since they are not directly forced. Similarly, notice that β_ℓ^ϕ is continuous and of finite first variation. In particular, we emphasize that these properties of α_ℓ and β_ℓ^ϕ hold on all of Ω for all $\ell \in \mathbb{Z}_0^2$ and $\phi \in \mathbb{L}^2$.

Now, if $\beta_\ell^\phi \equiv 0$ or $\beta_{-\ell}^\phi \equiv 0$, then $\frac{d}{ds}\beta_\ell^\phi(s) \equiv 0$ or $\frac{d}{ds}\beta_{-\ell}^\phi(s) \equiv 0$, respectively, as the coordinates are constant. Notice that from (3.4)–(3.6) these derivatives have the form

$$X(s) + \sum_{k \in \mathcal{Z}_*} Y_k(s) W_k(s)$$

where the X and Y_k are continuous and bounded-variation processes. Also, notice that they are *not adapted* to the past of the W_k 's! Nonetheless, it follows from Lemma 4.1 in the next section that if $\{X(\cdot), Y_k(\cdot) : k \in \mathcal{Z}_*\}$ are continuous and of bounded variation, then

$$(3.7) \quad X(s) + \sum_{k \in \mathcal{Z}_*} Y_k(s) W_k(s) = 0, \quad 0 \leq s \leq t,$$

implies that

$$(3.8) \quad Y_k(s) = 0 \quad \text{for all } 0 \leq s \leq t \text{ and } k \in \mathcal{Z}_*$$

on a set $\Omega_1 \subset \Omega$ of full measure, which does not depend on ℓ, k , or ϕ , and hence we can use a single exceptional set for all of the steps in the induction. To summarize,

we have shown that there is a single fixed set $\Omega_1 \subset \Omega$, of full measure, so that for any $\phi \in \mathbb{L}^2$, if $\beta_\ell^\phi \equiv \beta_{-\ell}^\phi \equiv 0$ on Ω_1 , then $Y_k \equiv 0$ for all $k \in \mathcal{Z}_*$. We now identify the Y_k to discover what (3.8) implies.

Define $|\ell| = \pm\ell$ depending on whether $\ell \in \mathbb{Z}_\pm^2$ and $\text{sgn}(\ell) = \pm 1$ depending on whether $\ell \in \mathbb{Z}_\pm^2$. (Care should be taken not to confuse $\|\ell\|$, which is in \mathbb{R}_+ , with $|\ell|$, which is in \mathbb{Z}_\pm^2 .) Then from (3.3)–(3.4), we see that for each $\ell \in \mathbb{Z}_0^2$,

$$\begin{aligned} \frac{d}{ds}\beta_\ell^\phi(s) &= X_\ell(s) + \sum_{\substack{j \in \mathcal{Z}_* \\ (j,\ell) \in I_S}} c(j, \ell) [\beta_{-|\ell-j|}^\phi(s) + \text{sgn}(\ell)\beta_{-|\ell+j|}^\phi(s)]W_j(s) \\ &+ \sum_{\substack{j \in \mathcal{Z}_* \\ (j,\ell) \in I_A}} c(j, \ell) [\beta_{|\ell-j|}^\phi(s) - \text{sgn}(\ell+j)\beta_{|\ell+j|}^\phi(s)]W_j(s), \end{aligned}$$

where $I_A = (\mathbb{Z}_+^2, \mathbb{Z}_-^2) \cup (\mathbb{Z}_-^2, \mathbb{Z}_+^2)$, $I_S = (\mathbb{Z}_+^2, \mathbb{Z}_+^2) \cup (\mathbb{Z}_-^2, \mathbb{Z}_-^2)$, and $X_\ell(s)$ is a continuous stochastic process with bounded variation. Hence by Lemma 4.1 we obtain that terms in brackets in the above equation are identically zero.

Recall that by assumption $\frac{d}{ds}\beta_\ell^\phi(s) = 0$, $\frac{d}{ds}\beta_{-\ell}^\phi(s) = 0$, and $\{j, -j\} \subset \mathcal{Z}_*$. Without loss of generality, we assume that $\ell, j \in \mathbb{Z}_+$ since this can always be achieved by renaming ℓ and j . The preceding reasoning using Lemma 4.1 applied to (j, ℓ) , $(-j, -\ell)$, $(-j, \ell)$, and $(j, -\ell)$ implies that

$$\begin{aligned} c(j, \ell) [\beta_{-|\ell-j|}^\phi(s) + \beta_{-|\ell+j|}^\phi(s)] &= 0, \\ c(j, \ell) [\beta_{-|\ell-j|}^\phi(s) - \beta_{-|\ell+j|}^\phi(s)] &= 0, \\ c(j, \ell) [\text{sgn}(\ell-j)\beta_{|\ell-j|}^\phi(s) - \beta_{|\ell+j|}^\phi(s)] &= 0, \\ c(j, \ell) [\text{sgn}(\ell-j)\beta_{|\ell-j|}^\phi(s) + \beta_{|\ell+j|}^\phi(s)] &= 0, \end{aligned}$$

for all $s < t$ on a subset of Ω_1 of full measure. Provided that $j^\perp \cdot \ell \neq 0$ and $\|j\| \neq \|\ell\|$, one has that $c(\ell, j) \neq 0$. Hence the left-hand sides are linearly independent, and one concludes that $\beta_{\ell-j}^\phi \equiv \beta_{\ell+j}^\phi \equiv \beta_{j-\ell}^\phi \equiv \beta_{-(\ell+j)}^\phi \equiv 0$ on a subset of Ω_1 of full measure. \square

We now collect some of the information from the preceding proof for later use.

PROPOSITION 3.7 *Let $U^{\phi,t}$ be the solution of (2.5) for any choice of terminal condition ϕ and terminal time t . Recall the definition of S_n from (3.1). Let Π_0 be the projection onto S_0 and Π_0^\perp its orthogonal complement. Then for $s < t$*

$$\frac{\partial}{\partial s}U^{\phi,t}(s) = X^\phi(s) + \sum_{j \in \mathcal{Z}_*} Y_j^\phi(s)W_j(s)$$

where

$$\begin{aligned} X^\phi(s) &= -\nu\Delta U^{\phi,t}(s) - B(\Pi_0^\perp w(s), U^{\phi,t}(s)) + C(U^{\phi,t}(s), \Pi_0^\perp w(s)) \\ &- B(R(s), U^{\phi,t}(s)) + C(U^{\phi,t}(s), R(s)), \end{aligned}$$

$$R(s) = \Pi_0 w(0) + \int_0^s \nu \Delta \Pi_0 w(r) + \Pi_0 B(w(r), w(r)) dr,$$

$$Y_j^\phi(s) = -B(e_j, U^{\phi,t}(s)) + C(U^{\phi,t}(s), e_j).$$

For all $\ell, j \in \mathbb{Z}_+^2$, we have

$$\begin{aligned} \langle Y_j^\phi(s), e_\ell \rangle &= \pi^2 c(j, \ell) [\beta_{-|\ell-j|}^\phi(s) + \beta_{-(\ell+j)}^\phi(s)], \\ \langle Y_{-j}^\phi(s), e_{-\ell} \rangle &= \pi^2 c(j, \ell) [\beta_{-|\ell-j|}^\phi(s) - \beta_{-(\ell+j)}^\phi(s)], \\ \langle Y_{-j}^\phi(s), e_\ell \rangle &= \pi^2 c(j, \ell) [\operatorname{sgn}(\ell - j) \beta_{|\ell-j|}^\phi(s) - \beta_{\ell+j}^\phi(s)], \\ \langle Y_j^\phi(s), e_{-\ell} \rangle &= -\pi^2 c(j, \ell) [\operatorname{sgn}(\ell - j) \beta_{|\ell-j|}^\phi(s) + \beta_{\ell+j}^\phi(s)]. \end{aligned}$$

4 A Quadratic Variation Lemma

The following lemma is the main technical result used to prove the existence of a density.

LEMMA 4.1 *Let \mathcal{A} be a collection of real-valued stochastic processes such that there exists a fixed subset $\Omega_{\mathcal{A}} \subset \Omega$ of full measure such that on $\Omega_{\mathcal{A}}$ any element of \mathcal{A} is continuous and has finite first variation.*

Fix a finite collection $\{W_1, \dots, W_N\}$ of independent Wiener processes and a sequence of partitions $\{s_j^n\}_{j=0}^{m(n)}$ with $s_{j+1}^n - s_j^n \rightarrow 0$ as $n \rightarrow \infty$ and

$$0 = s_1^n \leq \dots \leq s_{m(n)}^n = t.$$

Then there exists a fix subset $\Omega' \subset \Omega_{\mathcal{A}}$ of full measure and a fixed subsequence of partitions $\{t_j^n\}_{j=0}^{k(n)}$ of $\{s_j^n\}_{j=0}^{m(n)}$ so that if

$$Z(s) = X(s) + \sum_{i=1}^N Y_i(s) W_i(s)$$

with $X, Y_1, \dots, Y_N \in \mathcal{A}$, then on the set Ω'

$$\sum_{j=1}^{k(n)} |Z(t_j^n) - Z(t_{j-1}^n)|^2 \longrightarrow \sum_{i=1}^N \int_0^t Y_i^2(s) ds \quad \text{as } n \rightarrow \infty.$$

To prove this lemma we will invoke the following auxiliary results, whose proofs will be given after the proof of Lemma 4.1.

LEMMA 4.2 *Let $\{W(s) : 0 \leq s \leq t\}$ be a standard Brownian motion and $\{s_j^n\}_{j=0}^{m(n)}$ a sequence of partitions as in Lemma 4.1. The sequence of measures*

$$\left\{ \sum_{j=1}^{m(n)} (W(s_j^n) - W(s_{j-1}^n))^2 \delta_{s_{j-1}^n}, n = 1, 2, \dots \right\}$$

converges weakly as $n \rightarrow \infty$ to the Lebesgue measure on $[0, t]$ in probability.

LEMMA 4.3 *Let $\{W(s) : 0 \leq s \leq t\}$ and $\{\bar{W}(s) : 0 \leq s \leq t\}$ be two mutually independent Brownian motions and $\{s_j^n\}_{j=0}^{m(n)}$ a sequence of partitions as in Lemma 4.1. The sequence of signed measures*

$$\left\{ \sum_{j=1}^{m(n)} (W(s_j^n) - W(s_{j-1}^n))(\bar{W}(s_j^n) - \bar{W}(s_{j-1}^n))\delta_{s_{j-1}^n}, n = 1, 2, \dots \right\}$$

converges weakly to zero in probability as $n \rightarrow \infty$.

PROOF OF LEMMA 4.1: In light of Lemmas 4.2 and 4.3, since the collection of Brownian motions is finite, we can select a single set of full measure $\Omega' \subset \Omega_{\mathcal{A}}$ and a single subsequence of partitions $\{t_j^n\}_{j=0}^{k(n)}$ such that the weak convergences given in Lemmas 4.2 and 4.3 hold on Ω' . Furthermore, we can assume that on Ω' the $W_i(s)$ are continuous and have quadratic variation s . For notational brevity, we will write t_j instead of t_j^n .

Consider the quantity $\sum_j |Z(t_j) - Z(t_{j-1})|^2$. If we express it in terms of the X , Y , and W 's, we note that it contains four types of terms,

$$\begin{aligned} & \sum_j |X(t_j) - X(t_{j-1})|^2, \\ & \sum_j |Y(t_j)W(t_j) - Y(t_{j-1})W(t_{j-1})|^2, \\ & \sum_j (X(t_j) - X(t_{j-1}))(Y(t_j)W(t_j) - Y(t_{j-1})W(t_{j-1})), \\ & \sum_j (Y(t_j)W(t_j) - Y(t_{j-1})W(t_{j-1}))(\bar{Y}(t_j)\bar{W}(t_j) - \bar{Y}(t_{j-1})\bar{W}(t_{j-1})), \end{aligned}$$

where W and \bar{W} are mutually independent scalar Brownian motions. The first and third terms are easily shown to tend to zero on Ω' as $n \rightarrow \infty$, since X is of bounded variation, and X , Y , and W are continuous.

Consider the second term

$$\begin{aligned} & \sum_j |Y(t_j)W(t_j) - Y(t_{j-1})W(t_{j-1})|^2 \\ & = \sum_j Y(t_{j-1})^2(W(t_j) - W(t_{j-1}))^2 + \sum_j W^2(t_j)(Y(t_j) - Y(t_{j-1}))^2 \\ & \quad + 2 \sum_j W(t_j)Y(t_{j-1})(W(t_j) - W(t_{j-1})) \times (Y(t_j) - Y(t_{j-1})). \end{aligned}$$

Again on Ω' , the second and last terms above tend to 0, and

$$\sum_j Y(t_{j-1})^2(W(t_j) - W(t_{j-1}))^2 \rightarrow \int_0^t Y(s)^2 ds$$

on Ω' by the convergence given in Lemma 4.2.

Finally,

$$\begin{aligned} & \sum_j (Y(t_j)W(t_j) - Y(t_{j-1})W(t_{j-1}))(\bar{Y}(t_j)\bar{W}(t_j) - \bar{Y}(t_{j-1})\bar{W}(t_{j-1})) \\ &= \sum_j [Y(t_{j-1})(W(t_j) - W(t_{j-1})) + (Y(t_j) - Y(t_{j-1}))W(t_j)] \\ & \quad \times [\bar{Y}(t_{j-1})(\bar{W}(t_j) - \bar{W}(t_{j-1})) + (\bar{Y}(t_j) - \bar{Y}(t_{j-1}))\bar{W}(t_j)] \\ &= \sum_j Y(t_{j-1})\bar{Y}(t_{j-1})(W(t_j) - W(t_{j-1}))(\bar{W}(t_j) - \bar{W}(t_{j-1})) + \varepsilon_n, \end{aligned}$$

where $\varepsilon_n \rightarrow 0$ a.s. as $n \rightarrow \infty$. Again by Lemma 4.3, the sum tends to 0 on Ω' . \square

PROOF OF LEMMA 4.2: For any measure, the fact that $\mu_n \Rightarrow \mu$ follows from $\mu_n([0, s]) \rightarrow \mu([0, s])$ for all $s \in [0, t]$, s rational. From any subsequence of the given sequence, one can extract a further subsequence such that $\mu_n([0, s]) \rightarrow \mu([0, s])$ for all s rational, $0 \leq s \leq t$ a.s. Hence along that subsequence $\mu_n \Rightarrow \mu$ a.s.; hence the whole sequence converges weakly in probability. \square

PROOF OF LEMMA 4.3: Write $\Delta_j W$ for $W(s_j^n) - W(s_{j-1}^n)$, and write $\Delta_j \bar{W}$ for $\bar{W}(s_j^n) - \bar{W}(s_{j-1}^n)$. Note that for all $0 \leq r < s \leq t$ as $n \rightarrow \infty$,

$$\sum_{r < s_j^n \leq s} \Delta_j W \Delta_j \bar{W} \rightarrow 0 \quad \text{in probability as } n \rightarrow \infty.$$

Consequently, if f is a step function,

$$\sum_{j=1}^n \Delta_j W \Delta_j \bar{W} f(s_{j-1}^n) \rightarrow 0 \quad \text{in probability as } n \rightarrow \infty.$$

Moreover, for any two functions f and g

$$\begin{aligned} & \left| \sum_{j=1}^n \Delta_j W \Delta_j \bar{W} (f(s_{j-1}^n) - g(s_{j-1}^n)) \right| \\ & \leq \sup_{0 \leq s \leq t} |f(s) - g(s)| \left(\sum_{j=1}^n (\Delta_j W)^2 \right)^{1/2} \left(\sum_{j=1}^n (\Delta_j \bar{W})^2 \right)^{1/2}, \end{aligned}$$

and the right-hand side tends to

$$t \times \sup_{0 \leq s \leq t} |f(s) - g(s)|$$

in probability as $n \rightarrow \infty$.

Let now f be a continuous function, and g be a step function. Choose

$$\delta = 2t \sup_{0 \leq s \leq t} |f(s) - g(s)|$$

We have

$$\begin{aligned} & \mathbb{P} \left\{ \left| \sum f(s_{j-1}^n) \Delta_j W \Delta_j \bar{W} \right| > \delta \right\} \\ & \leq \mathbb{P} \left\{ \left| \sum_{j=1}^n g(s_{j-1}^n) \Delta_j W \Delta_j \bar{W} \right| > \frac{\delta}{3} \right\} \\ & \quad + \mathbb{P} \left\{ \left| \sum_{j=1}^n (f(s_{j-1}^n) - g(s_{j-1}^n)) \Delta_j W \Delta_j \bar{W} \right| > \frac{2\delta}{3} \right\}, \end{aligned}$$

and it follows from the above arguments that the latter tends to zero as $n \rightarrow \infty$. Since δ can be made arbitrarily small by an appropriate choice of the step function g , the lemma is proved. \square

5 Relation to Brackets of Vector Fields

We now sketch another possible proof of our Theorem 3.1, which brings in explicitly the brackets of certain vector fields. A vector field over the space \mathbb{L}^2 is a mapping from a dense subset of \mathbb{L}^2 into itself. We begin by rewriting (1.1) as

$$\frac{\partial w}{\partial t}(t) = F_0(w(t)) + \sum_{i=1}^N F_i \frac{\partial W_i}{\partial t}(t).$$

The diffusion vector fields in our case are constant vector fields defined by

$$F_i = e_{k_i}, \quad 1 \leq i \leq N,$$

where N is the cardinality of \mathcal{Z}_* and $\{k_1, \dots, k_N\}$ is any ordering of the set \mathcal{Z}_* . Similarly, the drift vector field is denoted by $F_0(w) = v \Delta w - B(w, w)$. In this notation, (2.2) becomes

$$\begin{cases} \frac{\partial U^{t,\phi}}{\partial s}(s) + (\nabla_w F_0)^*(w(s))U^{t,\phi}(s) = 0, & 0 \leq s \leq t, \\ U^{t,\phi}(t) = \phi, \end{cases}$$

where $\nabla_w F_0$ is the Fréchet derivative of F_0 in the \mathbb{L}^2 topology and $(\nabla_x F_0)^*$ is its \mathbb{L}^2 adjoint. If it is well-defined, we define the bracket $[F, G]$ between two \mathbb{L}^2 vector fields F and G as $[F, G] = (\nabla_w F)G - (\nabla_w G)F$. (Part of being well-defined is that the range of G and F are contained in the domain of $\nabla_w F$ and $\nabla_w G$, respectively.)

The argument in this alternate proof is based on the two next results.

LEMMA 5.1 *Let G be a vector field on \mathbb{L}^2 that is twice Fréchet differentiable in the \mathbb{L}^2 topology and such that $[F_i, G], i = 0, \dots, N$, are vector fields on \mathbb{L}^2 . Then we have*

$$\begin{aligned} & \langle U^{t,\phi}(s), G(w(s)) \rangle - \langle U^{t,\phi}(0), G(w(0)) \rangle \\ & = \int_0^s \langle U^{t,\phi}(r), [F_0, G](w(r)) \rangle dr + \sum_{i=1}^N \int_0^s \langle U^{t,\phi}(r), [F_i, G](w(r)) \rangle \circ dW_r^i \end{aligned}$$

where \circ means that it is a Stratonovich (anticipating) integral; in Itô-Skorohod language, it takes the form

$$\int_0^s \langle U^{t,\phi}(r), [F_0, G](w(r)) \rangle dr + \sum_{i=1}^N \int_0^s \langle U^{t,\phi}(r), [F_i, G](w(r)) \rangle dW_r^i + \sum_{i=1}^N \int_0^s \left[\frac{1}{2} \langle \nabla_w^2 G(w(r))(F_i, F_i) U^{t,\phi}(r) \rangle + \langle [F_i, G](w(r)) D_r^i U^{t,\phi}(r) \rangle \right] dr$$

PROOF: The formula in Skorohod language follows from theorem 6.1 in [27] via an easy finite-dimensional approximation. Its translation in the Stratonovich form follows from theorem 7.3 in the same paper (see also theorem 3.1.1 in [25]). □

We can now prove the following:

PROPOSITION 5.2 *Let $\Omega_0 \subset \Omega$. Under the same assumptions on G as in the above lemma,*

$$\langle U^{t,\phi}(s), G(w(s)) \rangle \equiv 0 \quad \text{on the set } \Omega_0$$

implies that

$$\langle U^{t,\phi}(s), [F_i, G](w(s)) \rangle \equiv 0 \quad \text{a.s. on the set } \Omega_0, \quad i = 0, \dots, N.$$

PROOF: The assumption implies that the quadratic variation on $[0, t]$ of the process $\{\langle U^{t,\phi}(s), G(w(s)) \rangle\}$ vanishes almost surely on Ω_0 . Then from [25, theorem 3.2.1], for $i = 1, \dots, N$,

$$\int_0^t \langle U^{t,\phi}(s), [F_i, G](w(s)) \rangle^2 ds = 0$$

a.s. on the set Ω_0 , i.e.,

$$\langle U^{t,\phi}(s), [F_i, G](w(s)) \rangle \equiv 0 \quad \text{a.s. on the set } \Omega_0, \quad i = 1, \dots, N.$$

This implies that for $1 \leq i \leq N$,

$$\int_0^s \langle U^{t,\phi}(r), [F_i, G](w(r)) \rangle \circ dW_r^i \equiv 0 \quad \text{a.s. on the set } \Omega_0$$

(see definition 3.1.1 in [25]), from which it follows (see the previous lemma) that

$$\langle U^{t,\phi}(s), [F_0, G](w(s)) \rangle \equiv 0 \quad \text{a.s. on the set } \Omega_0.$$

□

Now call \mathcal{L} all well-defined \mathbb{L}^2 vector fields in the ideal generated by the vector fields F_1, \dots, F_N in the Lie algebra generated by F_0, \dots, F_N . In other words, at each $u \in \mathbb{L}^2$, $\mathcal{L}(u)$ consists of F_1, \dots, F_N , and all brackets

$$(5.1) \quad [F_{i_n}, [F_{i_{n-1}}, \dots [F_{i_2}, F_{i_1}] \dots]](u),$$

which are well-defined vector fields on \mathbb{L}^2 where $1 \leq i_1 \leq N$, and for $j > 1$, $0 \leq i_j \leq N$. Iterating the argument in the proposition, we deduce the following result:

COROLLARY 5.3 *Given $\phi \in \mathbb{L}^2$, let*

$$\Omega_0 := \{\langle \mathcal{M}(t)\phi, \phi \rangle = 0\}.$$

If $\mathbb{P}(\Omega_0) > 0$, then

$$\langle \phi, G(w(s)) \rangle \equiv 0 \quad \forall G \in \mathcal{L} \text{ a.s. on } \Omega_0.$$

In particular, ϕ is orthogonal to all constant vector fields in \mathcal{L} .

In the case of the stochastic Navier-Stokes equations given in (1.1), all of the brackets given in (5.1) are well-defined if the e_k used to define the forcing have exponentially decaying Fourier components. This follows from the fact that all of the brackets of the form (5.1) contain differential operators with polynomial symbols and the fact that, with this type of forcing, on any finite time interval $[0, T]$ there exists a positive random variable γ so that $\sup_{[0, T]} \|e^{\gamma|\nabla|} w(s)\|^2 < \infty$ almost surely. Here $\|e^{\gamma|\nabla|} w\|^2 = \sum_k e^{2\gamma|k|} |w_k|^2$ where $w(t, x) = \sum_k w_k(t) e^{ik \cdot x}$. See, for example, [19] for a stronger version of this result or [18, 22] for simpler versions.

Furthermore, in [9] it is implicitly shown by the construction used that the span of the constant vector fields contains S_∞ . Thus, under the same conditions as before we see that the law of arbitrary finite-dimensional projections of $w(t)$ have a density with respect to Lebesgue measure.

6 Smoothness

In the preceding sections, we proved the existence of a density. We now address the smoothness of the density. While the former simply required that the projected Malliavin matrix be invertible, the proof of smoothness requires control on the norm of the inverse of the projected Malliavin matrix together with “smoothness in the Malliavin sense.” The following is the main result of this section; however, it rests heavily on the general results proven in Section 7 as well as some technical results from the appendices.

THEOREM 6.1 *Let S be any finite-dimensional subspace of S_∞ and Π the orthogonal projection in \mathbb{L}^2 onto S . For any $t > 0$, the law of Πw_t has a C^∞ density with respect to the Lebesgue measure on S .*

PROOF: We use corollary 2.1.2 of [25]. Lemma C.1 from the appendix establishes condition (i) from that corollary, while condition (ii) of the same corollary follows from the next theorem. \square

The following is a quantitative version of Proposition 3.5. It gives a quantitative control of the smallest eigenvalue of a finite-dimensional projection of the Malliavin matrix.

THEOREM 6.2 *Let Π be the orthogonal projection of \mathbb{L}^2 onto a finite-dimensional subspace of S_∞ . For any $T > 0$, $\eta > 0$, $p \geq 1$, and $K > 0$, there exists a constant $c = c(\nu, \eta, p, |\mathcal{Z}_*|, T, K, \Pi)$ and $\epsilon_0 = \epsilon_0(\nu, K, |\mathcal{Z}_*|, T, \Pi)$ so that for all $\epsilon \in (0, \epsilon_0]$,*

$$\mathbb{P}\left(\inf_{\phi \in S(K, \Pi)} \langle \mathcal{M}(T)\phi, \phi \rangle < \epsilon\right) \leq c \exp(\eta \|w(0)\|^2) \epsilon^p$$

where $S(K, \Pi) = \{\phi \in S_\infty : \|\phi\|_1 \leq 1, \|\Pi\phi\| \geq K\}$.

Remark 6.3. Notice that this lemma implies that the eigenvectors with “small” eigenvalues have small projections in the “lower” modes. The definition of lower modes depends on the definition of small eigenvalues. This separation between the eigenvectors with small eigenvalues and the low modes is one of the keys to the ergodic results proved in [14].

Remark 6.4. Also notice that there is a mismatch in the topology in Theorem 6.2 in that the test functions are bounded in \mathbb{H}^1 but the inner product is in \mathbb{L}^2 . This can likely be rectified; since the backward adjoint linearized flow $\bar{J}_{s,T}^*$ maps \mathbb{L}^2 into \mathbb{H}^1 for any $s \leq T$, it is possible to obtain estimates on $\mathbb{P}(\langle \phi, \mathcal{M}(T)\phi \rangle < \epsilon)$ for $\phi \in \mathbb{L}^2$. Just as in the proof of Theorem 6.2, where we exclude a small neighborhood of time 0 to allow w_t to regularize, we could exclude the s in a small neighborhood of the terminal T to allow $U^{T,\phi}(s) = J_{s,T}^* \phi$ to regularize.

PROOF: Recall the definition of S_n and \mathcal{Z}_n from (3.1), and let Π_n be the orthonormal projection onto S_n . Since $S(K, \Pi) \subset S_\infty$ and Π projects onto a finite-dimensional subspace of S_∞ , for n sufficiently large, $\|\Pi_n \phi\| > \frac{1}{2}K$ for all $\phi \in S(K, \Pi)$. Fix such an n .

We now construct a basis of S_n compatible with the structure of \mathcal{Z}_k , $k \leq n$. Fixing any ordering of \mathcal{Z}_* , set $\{f_i : i = 1, \dots, N = |\mathcal{Z}_*|\} = \mathcal{Z}_*$. Clearly $\{f_i\}_{i=1}^N$ is a basis for S_0 . Set $J_0 = N$. By the construction of S_n it is clear that $S_n \setminus S_{n-1}$ is equal to the $\text{Span}(e_k : k \in \mathcal{Z}'_n)$ where $\mathcal{Z}'_n \stackrel{\text{def}}{=} \mathcal{Z}_n \cap_{j=0}^{n-1} \mathcal{Z}_j^c \cap \mathcal{Z}_*^c$. For $n \geq 1$, set $J_n = J_{n-1} + |\mathcal{Z}'_n|$ and $\{f_i : i = J_{n-1} + 1, \dots, J_n\} = \{e_k : k \in \mathcal{Z}'_n\}$, again fixing an arbitrary ordering of the right-hand side. Clearly $\{f_i\}_{i=1}^{J_n}$ chosen in this way is an orthogonal basis for S_n .

Fix some $t_0 \in (0, T)$. Recall that by Corollary 2.4

$$\mathbb{P}\left(\inf_{\phi \in S(K, \Pi)} \langle \mathcal{M}(T)\phi, \phi \rangle < \epsilon\right) = \mathbb{P}\left(\inf_{\phi \in S(K, \Pi)} \sum_{k \in \mathcal{Z}_*} \int_0^T \langle U^{T,\phi}(s), e_k \rangle^2 ds < \epsilon\right).$$

Let X^ϕ and Y_j^ϕ be as in Proposition 3.7. For $j = 1, \dots, J_n$, define

$$\chi_j^\phi(t) = -\left\langle X^\phi(T-t) + \sum_{k \in \mathcal{Z}_*} Y_k^\phi(T-t) W_k(T) f_j \right\rangle,$$

$\Upsilon_{j,k}^\phi(t) = \langle Y_k^\phi(T-t), f_j \rangle$, $G_j^\phi(t) = \langle U^{T,\phi}(T-t), f_j \rangle$, and $\bar{W}_k(t) = W_k(T) - W_k(T-t)$. Notice that for $t \in [0, T]$ and $j = 1, \dots, J_n$,

$$G_j^\phi(t) = \langle \phi, f_j \rangle + \int_0^t \left[\chi_j^\phi(s) + \sum_{k \in \mathcal{Z}_*} \Upsilon_{j,k}^\phi(s) \bar{W}_k(s) \right] ds.$$

Furthermore, in light of the observations in the proof of Lemma 3.6, we see that this sequence of equations satisfies the assumptions of the next subsection with J_0 as defined above and $J = J_n$. Next recall that $U^{t,\phi}(s, x) = \sum_{k \in \mathbb{Z}_0^2} \beta_k^\phi(s) e_k(x)$. Combining this, the last equalities in Proposition 3.7, and the argument already used at the end of the proof of Lemma 3.6, we see that each G_j^ϕ is a linear combination of the $\{\Upsilon_{i,k}^\phi : i < j, k \in \mathcal{Z}_*\}$ with coefficients that are constant in time.

Below $\|\cdot\|_{1,[a,b]}$ and $\|\cdot\|_{\infty,[a,b]}$, respectively, denote the Lipschitz and L^∞ norm on $[a, b]$; see the beginning of Section 7 for the precise definitions.

Fix a $t \in (0, T)$ and set $T_1 = T - t$. Bounds, uniform for $\phi \in S(K, \Pi)$, on the p^{th} moments of the L^∞ norm and Lipschitz constants of χ_j^ϕ and $\Upsilon_{j,k}^\phi$ over the interval $[0, T_1]$ are given by Lemma E.2. (Recall that these processes have been time reversed.) Hence given any $p \geq 1$, $q > 0$, and $\eta > 0$, there exists a $c = c(\eta, q, p, t, \nu, \mathcal{E}_1, T) > 0$ so that for any $\epsilon > 0$ if one defines

$$(6.1) \quad \Omega_b(\epsilon, q) = \bigcap_{\phi \in S(K, \Pi)} \left\{ \sup_{k,j} (\|\chi_j^\phi\|_{1,[0,T_1]}, \|\Upsilon_{j,k}^\phi\|_{1,[0,T_1]}) \leq \epsilon^{-q} \right\},$$

then the estimate $\mathbb{P}(\Omega_b(\epsilon, q)^c) \leq c \exp(\eta \|w(0)\|^2) \epsilon^p$ holds.

Next Corollary 7.3 and Proposition 7.1 state that there exist $q = q(|S|, N)$ and $\epsilon_0 = \epsilon_0(T, \mathcal{Z}_*, |S|)$ so that for all $\epsilon \in (0, \epsilon_0]$ there is a $\Omega_\sharp(\epsilon)$ so that for all $\phi \in S(K, \Pi)$ one has

$$(6.2) \quad \left\{ \sup_{i=1,\dots,N} \int_0^{T_1} |G_i^\phi(s)|^2 ds < \epsilon; \sup_{i=1,\dots,J_n} \sup_{s \in [0, T_1]} |G_i^\phi(s)| > \epsilon^q \right\} \cap \Omega_b(\epsilon, q) \subset \Omega_\sharp(\epsilon)$$

and $\mathbb{P}(\Omega_\sharp(\epsilon)) \leq c\epsilon^p$ for all $p \geq 1$ and $\eta > 0$ with a $c = c(T, |\mathcal{Z}_*|, |S|, p, \eta, \nu)$. Notice that because of the uniformity in (6.1), $\Omega_\sharp(\epsilon)$ does not depend on the sequence of G 's. Since $\sup_{k \in \mathcal{Z}_*} \int_t^T \langle U^{T,\phi}(s), e_k \rangle^2 ds = \sup_{i=1,\dots,N} \int_0^{T_1} |G_i^\phi(s)|^2 ds$ and $\sup_{s \in [0, T_1]} |G_i^\phi(s)| = \|\langle U^{T,\phi}, f_i \rangle\|_{\infty,[t,T]}$, the inclusion given in (6.2) becomes

$$(6.3) \quad \left\{ \sup_{k \in \mathcal{Z}_*} \int_t^T \langle U^{T,\phi}(s), e_k \rangle^2 ds < \epsilon; \sup_{i=1,\dots,J_n} \|\langle U^{T,\phi}, f_i \rangle\|_{\infty,[t,T]} > \epsilon^q \right\} \subset \Omega_\sharp(\epsilon) \cup \Omega_b(\epsilon, q)^c$$

for all $\epsilon \in (0, \epsilon_0]$. Since $\langle U^{T,\phi}(T), f_i \rangle = \langle \phi, f_i \rangle$, by the choice of the subspace S_n one has $\sup_i \|\langle U^{T,\phi}, f_i \rangle\|_{\infty,[t,T]} \geq K/(2\sqrt{J_n})$. Thus for $\epsilon \in (0, \epsilon_0 \wedge$

$(K/[2\sqrt{J_n}]^{1/q})$ one has $\epsilon^q \leq K/(2\sqrt{J_n})$, which transforms (6.3) into

$$\left\{ \sum_{k \in \mathcal{Z}_*} \int_t^T \langle U^{T,\phi}(s), e_k \rangle^2 ds < \epsilon \right\} \subset \Omega_{\#}(\epsilon) \cup \Omega_b(\epsilon, q)^c.$$

Since ϕ is an arbitrary direction in $S(K, \Pi)$, and $\Omega_{\#}(\epsilon)$ and $\Omega_b(\epsilon, q)$ are independent of ϕ , we have

$$\begin{aligned} & \left\{ \inf_{\phi \in S(K, \Pi)} \sum_{k \in \mathcal{Z}_*} \int_t^T \langle U^{T,\phi}(s), e_k \rangle^2 ds < \epsilon \right\} = \\ & \bigcup_{\phi \in S(K, \Pi)} \left\{ \sum_{k \in \mathcal{Z}_*} \int_t^T \langle U^{T,\phi}(s), e_k \rangle^2 ds < \epsilon \right\} \subset \Omega_{\#}(\epsilon) \cup \Omega_b(\epsilon, q)^c. \end{aligned}$$

In summary we have shown that for any $p \geq 1$ and $\eta > 0$, there exists $q > 0$ so that the above inclusion holds and a $c > 0$ so that $\mathbb{P}(\Omega_{\#}(\epsilon)) + \mathbb{P}(\Omega_b(\epsilon, q)^c) \leq c \exp(\eta \|w(0)\|^2) \epsilon^p$. □

7 Controlling the Chance of Being Small

This section contains the main estimate used to control the chance of certain processes being small when their quadratic variation is large. The estimates of this section are simply quantitative versions of the results of Section 4. There are also the analogues of results used in the standard Malliavin calculus as applied to finite-dimensional SDEs. There the estimates were developed by Stroock [31] and Norris [24]. Here we do not have adapted processes. Instead, we exploit the smoothness in time to obtain estimates.

For the entirety of this section, we fix a time T and consider only the interval of time $[0, T]$. For any real-valued function of time f , define the α -Hölder constant over the time interval $[0, T]$ by

$$(7.1) \quad \mathcal{H}_\alpha(f) = \sup_{\substack{s, r \in [0, T] \\ 0 < |s-r| \leq 1}} \frac{|f(s) - f(r)|}{|s - r|^\alpha}$$

and the L^∞ norm by

$$(7.2) \quad \|f\|_\infty = \sup_{s \in [0, T]} |f(s)|.$$

We also define $\|f\|_\alpha = \max(\|f\|_\infty, \mathcal{H}_\alpha(f))$. At times we will also need versions of the above norms over shorter intervals of time. For $[a, b] \subset [0, T]$ we will write $\mathcal{H}_{\alpha, [a, b]}(f)$, $\|f\|_{\infty, [a, b]}$, and $\|f\|_{\alpha, [a, b]}$ for the norms with the same definitions as above except that the supremum over $[0, T]$ is replaced with a supremum over $[a, b]$. We also extend the definitions of the Lipschitz constant in time $\mathcal{H}_\alpha(f)$ to functions of time taking values in \mathbb{L}^2 by replacing the absolute value in the definitions given in (7.1) and (7.2) by the norm on \mathbb{L}^2 . Similarly, we extend the

definitions $\mathcal{H}_{\alpha,[a,b]}(f)$, $\|f\|_\infty$, and $\|f\|_{\infty,[a,b]}$ to functions of time taking values in \mathbb{L}^2 .

7.1 A Ladder of Estimates

For each $1 \leq j \leq J$, let $\{W_i^{(j)} : i = 1, \dots, N\}$ be a collection of mutually independent standard Wiener processes with $W_i(0) = 0$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We say that the collection of processes $\mathcal{G} = \{G^{(j)}(t, \omega) : 1 \leq j \leq J\}$ forms a ladder of order J with base size J_0 if first $1 \leq J_0 < J$ and

$$G^{(j)}(t, \omega) = G_0^{(j)} + \int_0^t H^{(j)}(s, \omega) ds$$

$$H^{(j)}(s, \omega) = X^{(j)}(s, \omega) + \sum_{i=1}^N Y_i^{(j)}(s) W_i^{(j)}(s, \omega)$$

where $j = 1, \dots, J < \infty$ and $\omega \in \Omega$.

Second, we require that for j greater than J_0 , the $G^{(j)}$ are determined by the functions at the previous levels. More precisely, for each j with $j > J_0$ there exists an integer $K = K(j)$, a collection $\{g_k(t) : k = 1, \dots, K\}$ of bounded, deterministic functions of time, and a collection $\{f_k(t, \omega) : k = 1, \dots, K\}$ of stochastic processes with

$$(7.3) \quad f_k \in \{Y_i^{(l)}, X^{(l)}, G^{(n)} : 1 \leq i \leq N, 1 \leq l \leq j - 1, 1 \leq n \leq j - 1\}$$

so that

$$G^{(j)}(t, \omega) = \sum_{k=1}^K g_k(t) f_k(t, \omega) \quad \text{a.s.}$$

This assumption can be restated by saying that, for $j > J_0$, $G^{(j)}$ must be at each moment of time in the span of the preceding X , Y , and G . And furthermore, the coefficients in the linear combination producing $G^{(j)}$ must be uniformly bounded on $[0, T]$.

It is important to remark that we do *not* assume that the $Y_i^{(j)}$ or $X^{(j)}$ are adapted to the Wiener processes. Typical assumptions regarding adaptedness will be replaced with assumptions on the regularity of the processes in time.

The goal of this section is to prove that under certain assumptions, if the first J_0 of the $G^{(j)}$ are small in some sense, then all of the X , Y , and remaining G are also small with high probability. The ladder structure connects the j^{th} level with the other levels.

Fix a time $T > 0$. For any choice of the positive parameter Δ define $\delta = \Delta^{5/3}$. For $k = 0, 1, \dots$, define $t_k = k\Delta \wedge T$. For each fixed k , define $s_\ell(k) = (t_k + \ell\delta) \wedge t_{k+1}$ for $\ell = 0, 1, \dots$. Set $\delta_\ell^k = s_\ell(k) - s_{\ell-1}(k)$ and $\delta_\ell^k f = f(s_\ell(k)) - f(s_{\ell-1}(k))$. Lastly, define $m = \inf\{k : t_k = T\}$ and $M(k) = \inf\{\ell : s_\ell(k) = t_{k+1}\}$. Notice that $\Delta^{-2/3} = \Delta/\delta \leq M(k) \leq \Delta/\delta + 1 = \Delta^{-2/3} + 1$ for all k and $m \leq T\Delta^{-1} + 1$.

Define the following subsets of the probability space Ω :

$$\Omega_a(\Delta) = \left\{ \inf_{0 \leq k \leq m} \inf_{1 \leq i \leq N} \frac{1}{M(k)} \sum_{\ell=1}^{M(k)} \frac{(\delta_\ell^k W_i)^2}{\delta_\ell^k} \leq \frac{1}{2} \right\},$$

$$\Omega_b(\Delta) = \left\{ \sup_{0 \leq k \leq m} \sup_{\substack{(i,j) \in \{1, \dots, N\}^2 \\ i \neq j}} \frac{1}{M(k)} \left| \sum_{\ell=1}^{M(k)} \frac{(\delta_\ell^k W_i)(\delta_\ell^k W_j)}{\delta_\ell^k} \right| \geq \frac{\Delta^{3/14}}{3N^2} \right\},$$

$$\Omega_c(\Delta) = \left\{ \sup_{1 \leq i \leq N} \|W_i\|_{1/4} > \Delta^{-\frac{1}{28}} \right\},$$

$\Omega'_\#(\epsilon) = \Omega_a(\epsilon^{14/75}) \cup \Omega_b(\epsilon^{14/75}) \cup \Omega_c(\epsilon^{14/75})$, and finally

$$\Omega_\#(\epsilon) = \bigcup_{j=1}^J \Omega'_\#(\epsilon^{(\frac{1}{152})^j}).$$

The following bound follows readily from Corollary 7.11 and Lemma 7.12 in Section 7.4.

PROPOSITION 7.1 *For any $p \geq 1$, there exists a constant $c = c(T, J, N, p)$, in particular independent of ϵ , such that*

$$\mathbb{P}(\Omega_\#(\epsilon)) \leq c\epsilon^p.$$

The next key result in this section is the following proposition, which shows why the previous estimate is important:

PROPOSITION 7.2 *Fix a positive integer J and for $q > 0$ define*

$$\Omega_{q*}(\mathcal{G}, \epsilon) = \left\{ \sup_{\substack{1 \leq j \leq J \\ 1 \leq i \leq N}} (\|X^{(j)}\|_1, \|Y_i^{(j)}\|_1) \leq \epsilon^{-q} \right\},$$

$$\Omega_q(\mathcal{G}, \epsilon) = \left\{ \sup_{1 \leq j \leq J_0} \|G^{(j)}\|_\infty < \epsilon ; \sup_{\substack{1 \leq j \leq J \\ 1 \leq i \leq N}} (\|X^{(j)}\|_\infty, \|Y_i^{(j)}\|_\infty, \|G^{(j)}\|_\infty) > \epsilon^q \right\}.$$

Then there exists positive constants $q_0 = q_0(J)$ and $\epsilon_0 = \epsilon_0(T, J, N)$ so that for any $\epsilon \in (0, \epsilon_0]$, $q \in (0, q_0]$, and ladder $\mathcal{G} = \{G^{(j)} : 1 \leq j \leq J\}$ of order J with base size J_0 less than J ,

$$\Omega_{q*}(\mathcal{G}, \epsilon) \cap \Omega_q(\mathcal{G}, \epsilon) \subset \Omega_\#(\epsilon).$$

In words, this proposition states that if the first J_0 of the $G^{(j)}$ are small and all of the quantities $\|X^{(j)}\|_1$ and $\|Y_i^{(j)}\|_1$ are not too big, then it is unlikely that the remaining $G^{(j)}$ are big.

PROOF OF PROPOSITION 7.2: The idea of the proof is to iterate Lemma 7.5 below. Begin by setting $g_* = \sup_j \sum_{k=1}^{K(j)} |g_k|_\infty$. Now define $\epsilon_0 = \tilde{\epsilon}_0 = \epsilon$ and for $j > 1$ define $\epsilon_j = \tilde{\epsilon}_{j-1}^{1/151}$, $\tilde{\epsilon}_j = \epsilon_j^{151/152}$. With these choices, $\epsilon_j < \tilde{\epsilon}_j < \epsilon_{j+1} <$

$\tilde{\epsilon}_{j+1}$. We also choose q_0 sufficiently small so that $\tilde{\epsilon}_{J+1} < \epsilon^q$ for $\epsilon \in [0, 1)$ and $q \in (0, q_0]$. Define the following subsets of Ω :

$$\begin{aligned}
 A(j) &= \Omega_{q^*}(\mathcal{G}, \epsilon) \cap \Omega_q(\mathcal{G}, \epsilon) \cap \left\{ \sup_{1 \leq i \leq N} (\|X^{(l)}\|_\infty, \|Y_i^{(l)}\|_\infty) \leq \epsilon_{l+1} \text{ for } l \leq j \right\}, \\
 A^+(j) &= A(j) \cap \{\|G^{(l)}\|_\infty \leq \tilde{\epsilon}_l \leq \epsilon_{l+1} \text{ for } l \leq j + 1\}, \\
 B(j) &= \left\{ \sup_{1 \leq i \leq N} (\|X^{(j)}\|_\infty, \|Y_i^{(j)}\|_\infty) > \tilde{\epsilon}_j^{1/151} = \epsilon_{j+1} \right\}, \\
 C(j) &= \Omega_{q^*}(\mathcal{G}, \epsilon) \cap \left\{ \|G^{(j)}\|_\infty \leq \tilde{\epsilon}_j; \right. \\
 &\quad \left. \sup_{1 \leq i \leq N} (\|X^{(j)}\|_\infty, \|Y_i^{(j)}\|_\infty) > \tilde{\epsilon}_j^{1/151} = \epsilon_{j+1} \right\}.
 \end{aligned}$$

First notice that since $\epsilon < \epsilon^q$ for $\epsilon \in (0, 1]$ and $q \in (0, q_0]$, the event

$$\left\{ \sup_{1 \leq j \leq J_0} \|G^{(j)}\|_\infty < \epsilon; \sup_{1 \leq j \leq J_0} \|G^{(j)}\|_\infty > \epsilon^q \right\}$$

is empty. Hence

$$\Omega_q(\mathcal{G}, \epsilon) = \left\{ \sup_{1 \leq j \leq J_0} \|G^{(j)}\|_\infty < \epsilon; \sup_{\substack{1 \leq j \leq J \\ 1 \leq i \leq N \\ J_0 < \ell \leq J}} (\|X^{(j)}\|_\infty, \|Y_i^{(j)}\|_\infty, \|G^{(\ell)}\|_\infty) > \epsilon^q \right\}.$$

Next note that for any $j > J_0$, because $G^{(j)}(t) = \sum_{k=1}^K g_k(t) f_k(t)$, we have $\|G^{(j)}\|_\infty \leq g_* \sup_k \|f_k\|_\infty$. Here the f_k are from earlier in the ladder in the sense of (7.3). Hence for any j , we have that if

$$\sup_{1 \leq i \leq N} (\|X^{(l)}\|_\infty, \|Y_i^{(l)}\|_\infty, \|G^{(l)}\|_\infty) \leq \epsilon_{l+1} \quad \text{for } l \leq j - 1,$$

then $\|G^{(j)}\|_\infty \leq g_* \epsilon_j < \epsilon_j^{151/152} = \tilde{\epsilon}_j$ for ϵ sufficiently small. Restricting to ϵ small enough so that $g_* \epsilon_j < \epsilon_j^{151/152}$ for all $1 \leq j \leq J$ implies that $A^+(j) = A(j)$ for all $1 \leq j \leq J$.

Next observe that $A(j) \cap B(j+1)^c = A(j+1)$ and hence $A^+(j) \cap B(j+1)^c = A^+(j+1)$ for ϵ sufficiently small. Iterating this observation, with the convention $A^+(0) = \Omega_{q^*}(\mathcal{G}, \epsilon) \cap \Omega_q(\mathcal{G}, \epsilon)$, we obtain

$$\begin{aligned}
 A^+(0) &= [A^+(0) \cap B(1)^c] \cup [A^+(0) \cap B(1)] \\
 &= A^+(1) \cup [A^+(0) \cap B(1)] \\
 &= [A^+(1) \cap B(2)^c] \cup [A^+(1) \cap B(2)] \cup [A^+(0) \cap B(1)]
 \end{aligned}$$

$$\begin{aligned}
 &= A^+(2) \cup [A^+(1) \cap B(2)] \cup [A^+(0) \cap B(1)] \\
 &= A^+(J) \cup \bigcup_{j=0}^{J-1} [A^+(j) \cap B(j+1)].
 \end{aligned}$$

Since q was picked sufficiently small so that $\epsilon_{J+1} < \epsilon^q$, we observe that $A^+(J)$ is empty since on $A^+(J)$

$$\sup_{\substack{1 \leq j \leq J \\ 1 \leq i \leq N \\ J_0 < l \leq J}} (\|X^{(j)}\|_\infty, \|Y_i^{(j)}\|_\infty, \|G^{(l)}\|_\infty) \leq \epsilon_{J+1} < \epsilon^q < \sup_{\substack{1 \leq j \leq J \\ 1 \leq i \leq N \\ J_0 < l \leq J}} (\|X^{(j)}\|_\infty, \|Y_i^{(j)}\|_\infty, \|G^{(l)}\|_\infty),$$

which cannot be satisfied.

Recall that $\Omega_{\#}(\epsilon) = \bigcup_{j=1}^J \Omega'_{\#}(\tilde{\epsilon}_j)$. Let $\Omega'_*(H^{(j)}, \tilde{\epsilon}_j)$ be the set defined below in Lemma 7.5. For all q sufficiently small, $\Omega_{q*}(\mathcal{G}, \epsilon) \subset \Omega'_*(H^{(j)}, \tilde{\epsilon}_j)$ for $j = 1, \dots, J + 1$. Decrease q_0 so this holds. With this choice Lemma 7.5 implies that $C(j) \subset \Omega'_{\#}(\tilde{\epsilon}_j)$. Since clearly $A^+(j) \cap B(j+1) \subset C(j+1)$, combining all of these observations produces

$$\Omega_{q*}(\mathcal{G}, \epsilon) \cap \Omega_q(\mathcal{G}, \epsilon) = A^+(0) \subset \bigcup_{j=1}^J C(j) \subset \bigcup_{j=1}^J \Omega'_{\#}(\tilde{\epsilon}_j) = \Omega_{\#}(\epsilon).$$

□

Lastly, we give a version of the preceding proposition that begins with L^p estimates in time on the $\{G^{(j)} : 1 \leq j \leq J_0\}$ rather than L^∞ estimates.

COROLLARY 7.3 Fix $T > 0$. For any $\ell > 0$, define

$$\begin{aligned}
 &\Omega_{q,\ell}(\mathcal{G}, \epsilon) \\
 &= \left\{ \sup_{1 \leq j \leq J_0} \int_0^T |G^{(j)}(s)|^\ell ds < \epsilon; \sup_{\substack{1 \leq j \leq J \\ 1 \leq i \leq N}} (\|X^{(j)}\|_\infty, \|Y_i^{(j)}\|_\infty, \|G^{(j)}\|_\infty) > \epsilon^q \right\}.
 \end{aligned}$$

There exist positive constants $q = q(J, \ell)$ and $\epsilon_0 = \epsilon_0(J, T, N, \ell)$ so that for all $\epsilon \in (0, \epsilon_0]$

$$\Omega_{q*}(\mathcal{G}, \epsilon) \cap \Omega_{q,\ell}(\mathcal{G}, \epsilon) \subset \Omega_{\#}(\epsilon).$$

PROOF: We begin by translating the bound

$$\sup_{1 \leq j \leq J_0} \int_0^T |G^{(j)}(s)|^\ell ds < \epsilon$$

into a bound of the form $\sup_{1 \leq j \leq J_0} \|G^{(j)}\|_\infty \leq \epsilon^\beta$ for some $\beta \in (0, 1)$.

Notice that

$$|G^{(j)}(s) - G^{(j)}(r)| = \left| \int_r^s H^{(j)}(t) dt \right| \leq |s - r| \|H^{(j)}\|_\infty.$$

Without loss of generality we assume $q < \frac{1}{150}$. Hence on

$$\Omega_{q*}(\mathcal{G}, \epsilon) \cap \{\sup_i \|W_i\|_\infty > \epsilon^{-1/150}\}^c$$

we have

$$\begin{aligned} \|H^{(j)}\|_\infty &\leq \|X^{(j)}\|_\infty + \sum_{i=1}^N \|Y_i^{(j)}\|_\infty \|W_i^{(j)}\|_\infty \leq \epsilon^{-1/150} + N\epsilon^{-1/75} \\ &\leq (N + 1)\epsilon^{-1/75}. \end{aligned}$$

In other words, $\mathcal{H}_1(G) \leq (N + 1)\epsilon^{-1/75}$ on $\Omega_{q*}(\mathcal{G}, \epsilon) \cap \{\sup_i \|W_i\|_\infty > \epsilon^{-1/150}\}^c$. Then Lemma 7.6 below implies $\sup_{j \leq J_0} \|G^{(j)}\|_\infty \leq (2 + N)\epsilon^{\beta_0} < \epsilon^{\beta_1}$ for ϵ sufficiently small where

$$\beta_0 = \frac{74}{75} \frac{1}{1 + \ell} \quad \text{and} \quad \beta_1 = \frac{1}{2} \frac{1}{1 + \ell}.$$

Now notice that $\Omega_*(\mathcal{G}, \epsilon^{\beta_1}) \subset \Omega_*(\mathcal{G}, \epsilon)$ because $\beta_1 < 1$. Hence the result follows from Proposition 7.2 and the fact that $\{\sup_i \|W_i\|_\infty > \epsilon^{-1/150}\} \subset \Omega_\#(\epsilon)$. \square

7.2 The Basic Estimates

Let

$$G(t) = G_0 + \int_0^t H(s) ds.$$

Now let $H(s)$ be any stochastic process of the form

$$(7.4) \quad H(s) = X(s) + \sum_{i=1}^N Y_i(s) W_i(s) \stackrel{\text{def}}{=} X(s) - Z(s)$$

where $X(s)$ and $Y_1(s), \dots, Y_N(s)$ are Lipschitz-continuous stochastic processes and $\{W_1(s), \dots, W_N(s)\}$ are mutually independent, standard Wiener processes with $W_i(0) = 0, 1 \leq i \leq N$.

Next, given $\epsilon > 0$, define the following subsets of the probability space:

$$(7.5) \quad \begin{aligned} \hat{\Omega}_*(H, \epsilon) &= \left\{ \sup_{1 \leq i \leq N} (\|Y_i\|_\infty) \leq \epsilon^{-1/28}; \sup_{1 \leq i \leq N} (\mathcal{H}_1(Y_i), \mathcal{H}_1(X)) \leq \epsilon^{-1/28} \right\}, \\ \Omega'_*(H, \epsilon) &= \left\{ \sup_{1 \leq i \leq N} (\|X\|_1, \|Y_i\|_1) \leq \epsilon^{-1/150} \right\}. \end{aligned}$$

LEMMA 7.4 *Let $\epsilon > 0$. Assume $\mathcal{H}_\alpha(H) \leq c\epsilon^{-\gamma}$ for some fixed $\alpha > \gamma > 0$. Then $\|G\|_\infty \leq \epsilon$ implies $\|H\|_\infty \leq (2 + c)\epsilon^{(\alpha-\gamma)/(1+\alpha)}$.*

PROOF: For any $s \in [0, t]$, let $r_1 \leq s \leq r_2$ such that $r_2 - r_1 = \epsilon^{(1+\gamma)/(1+\alpha)}$. Notice that, by the assumption, $H(r) \geq H(s) - c\epsilon^{-\gamma}|r_2 - r_1|^\alpha$ for any $r \in [r_1, r_2]$. Hence we have

$$2\epsilon \geq G(r_2) - G(r_1) = \int_{r_1}^{r_2} H(r) dr \geq |r_2 - r_1| (H(s) - c\epsilon^{-\gamma}|r_2 - r_1|^\alpha).$$

Rearranging this gives

$$H(s) \leq \frac{2\epsilon}{|r_2 - r_1|} + c\epsilon^{-\gamma}|r_2 - r_1|^\alpha = (2 + c)\epsilon^{\frac{\alpha-\gamma}{1+\alpha}}.$$

The same argument from above gives a complementary lower bound and completes the result. \square

Next, we have the following result:

LEMMA 7.5 *There exists a $\epsilon_0 = \epsilon_0(N)$ so that for every $\epsilon \in (0, \epsilon_0]$ and stochastic process $G(t)$ of the form given above, one has*

$$\Omega'_*(H, \epsilon) \cap \left\{ \|G\|_\infty < \epsilon; \sup_{1 \leq i \leq N} (\|X\|_\infty, \|Y_i\|_\infty) > \epsilon^{1/151} \right\} \subset \Omega'_\#(\epsilon).$$

PROOF: The result will follow from Lemma 7.9 of the next subsection after some groundwork is laid. As before, set $\Delta = \epsilon^{14/75}$ and recall that by definition $\Omega'_\#(\epsilon) = \Omega_a(\Delta) \cup \Omega_b(\Delta) \cup \Omega_c(\Delta)$. Then notice that $\mathcal{H}_{1/4}(f) \leq \mathcal{H}_1(f)$, $\mathcal{H}_{1/4}(f + g) \leq \mathcal{H}_{1/4}(f) + \mathcal{H}_{1/4}(g)$, and $\mathcal{H}_{1/4}(fg) \leq \mathcal{H}_{1/4}(g)\|f\|_\infty + \mathcal{H}_{1/4}(f)\|g\|_\infty$ for all functions f and g . Hence on $\Omega_c(\Delta)^c \cap \Omega'_*(H, \epsilon)$

$$\begin{aligned} \mathcal{H}_{1/4}(H) &\leq \mathcal{H}_1(X) + \sum_i \mathcal{H}_1(Y_i)\|W_i\|_\infty + \mathcal{H}_{1/4}(W_i)\|Y_i\|_\infty \\ &\leq \epsilon^{-1/150} + 2N\epsilon^{-1/75} \leq (1 + 2N)\epsilon^{-1/75}. \end{aligned}$$

Hence by Lemma 7.4 one has that on $\Omega_c(\Delta)^c \cap \Omega'_*(H, \epsilon)$, $\|G\|_\infty < \epsilon$ implies $\|H\|_\infty \leq (1 + 2N)\epsilon^{71/375} < \Delta = \epsilon^{14/75}$ for all ϵ sufficiently small. Next observe that because $\Delta = \epsilon^{14/75}$, $\Omega'_*(H, \epsilon) \subset \hat{\Omega}_*(H, \Delta)$ where $\hat{\Omega}_*(H, \Delta)$ is the set that was defined in equation (7.5). In light of this, Lemma 7.9 implies that

$$(7.6) \quad \Omega'_*(H, \epsilon) \cap \left\{ \|G\|_\infty < \epsilon; \sup_i \|Y_i\|_\infty > \epsilon^{1/75} \right\} \subset \Omega'_\#(\epsilon).$$

Now on

$$\Omega_c(\Delta)^c \cap \Omega'_*(H, \epsilon) \cap \left\{ \|G\|_\infty < \epsilon; \sup_i \|Y_i\|_\infty \leq \epsilon^{1/75} \right\},$$

one has that $\|X\|_\infty \leq \|H\|_\infty + \sum \|Y_i\|_\infty \|W_i\|_\infty \leq \epsilon^{14/75} + N\epsilon^{1/75}\epsilon^{-1/150}$, which is less than $\epsilon^{1/151}$ for ϵ sufficiently small. Hence

$$\Omega_c(\Delta)^c \cap \Omega'_*(H, \epsilon) \cap \left\{ \|G\|_\infty < \epsilon; \sup_i \|Y_i\|_\infty \leq \epsilon^{1/75} \right\} \cap \left\{ \|X\|_\infty > \epsilon^{1/151} \right\}$$

is empty. Combining this observation with (7.6) implies

$$\Omega'_*(H, \epsilon) \cap \left\{ \|G\|_\infty < \epsilon \right\} \cap \left[\left\{ \sup_i \|Y_i\|_\infty > \epsilon^{1/75} \right\} \cup \left\{ \|X\|_\infty > \epsilon^{1/151} \right\} \right] \subset \Omega'_\#(\epsilon).$$

Since $\left\{ \sup_i \|Y_i\|_\infty > \epsilon^{1/151} \right\} \subset \left\{ \sup_i \|Y_i\|_\infty > \epsilon^{1/75} \right\}$, the proof is complete. \square

LEMMA 7.6 *For any $\epsilon > 0$ and $\ell > 0$, $\int_0^T |G(s)|^\ell ds < \epsilon$ and $\mathcal{H}_\alpha(G) < c\epsilon^{-\gamma}$ imply $\|G\|_\infty < (1 + c)\epsilon^{(\alpha-\gamma)/(1+\ell\alpha)}$.*

PROOF: By Chebyshev’s inequality for any $\beta > 0$, we have $\lambda\{x : |G(x)| > \epsilon^\beta\} \leq \epsilon^{1-\ell\beta}$ where λ is Lebesgue measure. Hence $\|G\|_\infty \leq \epsilon^\beta + c\epsilon^{(1-\ell\beta)\alpha}\epsilon^{-\gamma}$. Setting $\beta = \frac{\alpha-\gamma}{1+\ell\alpha}$ proves the result. \square

7.3 The Main Technical Estimate

Let $H(s)$ be as in (7.4) from the preceding section. Define the following piecewise constant approximation of the Z from the definition of H :

$$(7.7) \quad Z^*(s) = - \sum_{i=1}^N Y_i^*(s) W_i(s)$$

where $Y_i^*(s) = \sum_{k=1}^m \mathbf{1}_{I_k}(s) \sup_{s \in I_k} Y_i(s)$ and $\mathbf{1}_{I_k}$ is the indicator function of the set $I_k = [t_{k-1}, t_k)$.

For any k and process $\zeta(s)$ defined on $[t_{k-1}, t_k]$, we define the δ -scale quadratic variation on $[t_{k-1}, t_k]$ by

$$Q_k(\zeta) = \sum_{\ell=1}^{M(k)} [\delta_\ell^k \zeta]^2.$$

LEMMA 7.7 *On $\hat{\Omega}_*(H, \Delta) \cap \Omega_c(\Delta)^c$ for any $k = 1, \dots, m$, we have the following estimates: $Q_k(X) \leq 2\Delta^{109/42}$ and $Q_k(Z^*) \leq 2Q_k(Z) + 4N^2\Delta^{25/21}$. Moreover, on $\hat{\Omega}_*(H, \Delta) \cap \Omega_c(\Delta)^c \cap \{\sup_{s \in I_k} |H(s)| < \Delta\}$, $Q_k(Z) \leq 2Q_k(X) + 8\Delta^{4/3} + 8\Delta^2$ and $Q_k(Z^*) \leq (40 + 4N^2)\Delta^{25/21}$.*

PROOF: For brevity, we suppress the k -dependence of $M(k)$ and $s_\ell(k)$. The first inequality follows from

$$\begin{aligned} Q_k(X) &= \sum_{\ell=1}^M [X(s_\ell) - X(s_{\ell-1})]^2 \\ &\leq \sum_{\ell=1}^M [\delta \Delta^{-1/28}]^2 \leq M\delta^2 \Delta^{-1/14} = (\Delta^{-23} + 1)\Delta^{10/3} \Delta^{-1/14} \\ &[\delta \Delta^{-1/28}]^2 \leq M\delta^2 \Delta^{-1/14} = \Delta^{109/42} + \Delta^{139/42} < 2\Delta^{109/42}. \end{aligned}$$

To see the second implication, first notice that

$$\begin{aligned} Q_k(Z^*) &\leq \sum_{\ell=1}^M (2[|Z^*(s_{\ell-1}) - Z(s_{\ell-1})| + |Z^*(s_\ell) - Z(s_\ell)|])^2 \\ &\quad + 2[Z(s_{\ell-1}) - Z(s_\ell)]^2 \\ &\leq 2Q_k(Z) + 4 \sum_{\ell=0}^{M-1} [Z^*(s_\ell) - Z(s_\ell)]^2 + 4 \sum_{\ell=1}^M [Z^*(s_\ell) - Z(s_\ell)]^2. \end{aligned}$$

Next note that

$$\begin{aligned} \sum_{\ell=1}^M [Z^*(s_\ell) - Z(s_\ell)]^2 &\leq \sum_{\ell=1}^M \left[\sum_{i=1}^N [Y_i(s_\ell) - Y_i^*] W_i(s_\ell) \right]^2 \\ &\leq \Delta^{-1/7} \frac{N^2}{\delta} \sum_{\ell=1}^M (\ell\delta)^2 \delta \leq \frac{N^2}{3\delta} \Delta^{20/7} = \frac{N^2}{3} \Delta^{25/21} \end{aligned}$$

and similarly

$$\sum_{\ell=0}^{M-1} [Z^*(s_\ell) - Z(s_\ell)]^2 \leq \frac{N^2}{3} \Delta^{25/21}.$$

Combining this estimate with the previous gives the second result.

The third result follows from $H(s) = X(s) - Z(s)$ and

$$\begin{aligned} Q_k(Z) &= \sum_{\ell=1}^M [Z(s_\ell) - Z(s_{\ell-1})]^2 \\ &\leq \sum_{\ell=1}^M 2[|Z(s_\ell) - X(s_\ell)| + |Z(s_{\ell-1}) - X(s_{\ell-1})|]^2 + 2[X(s_\ell) - X(s_{\ell-1})]^2 \\ &\leq 8\Delta^2 M + 2Q_k(X) \leq 8\Delta^2 \left(\frac{\Delta}{\delta} + 1 \right) + 2Q_k(X). \end{aligned}$$

Lastly, combining the three previous estimates produces the final estimate. □

To aid in the analysis of Y^* , consider a general process of the form

$$\zeta(s) = \sum_{i=1}^N a_i(s) W_i(s)$$

where the W_i are independent standard Wiener processes and the $a_i(s)$ are constant on the intervals $I_k = [(k - 1)\Delta, k\Delta)$ for each $k = 1, \dots, m$. As before, for $k = 1, \dots, m$, we define

$$Q_k(\zeta) = \sum_{\ell=1}^{M(k)} \left(\sum_{i=1}^N a_i(s_\ell(k)) \delta_\ell^k W_i \right)^2,$$

where $s_\ell(k)$ and δ_ℓ^k are as defined at the start of Section 7.1. Notice that if we define

$$U = \sum_{i=1}^N a_i^2 \sum_{\ell=1}^{M(k)} (\delta_\ell^k W_i)^2, \quad V = \sum_{\substack{(i,j) \in \{1, \dots, N\}^2 \\ i \neq j}} a_i a_j \sum_{\ell=1}^{M(k)} (\delta_\ell^k W_i) (\delta_\ell^k W_j),$$

where $a_i = a_i((k + 1)\Delta)$ and $\delta_\ell^k f = (\delta_\ell^k f) / \sqrt{\delta_\ell^k}$, then $Q_k(\zeta) = \frac{\Delta}{M(k)} (U + V)$.

LEMMA 7.8 For $\sigma > \frac{8}{7}$, $\Delta \in (0, 6^{-7/(7\sigma-8)})$, and $k = 1, \dots, m$,

$$\left\{ Q_k(\zeta) < \Delta^\sigma; \Delta^{1/14} < \sup_i |a_i| \leq \Delta^{-1/28} \right\} \subset \Omega_a(\Delta) \cup \Omega_b(\Delta)$$

PROOF: First notice that because $\frac{8}{7} < \sigma$ and $\Delta < 6^{-7/(7\sigma-8)}$, $\Delta^\sigma - \frac{1}{2}\Delta^{8/7} < -\frac{1}{3}\Delta^{8/7}$, so

$$\begin{aligned} & \left\{ Q_k(\zeta) < \Delta^\sigma; \Delta^{1/14} < \sup_i |a_i| \leq \Delta^{-1/28} \right\} \\ & \subset \left\{ \frac{\Delta}{M(k)}U < \frac{1}{2}\Delta^{8/7}; \sup_i |a_i| > \Delta^{1/14} \right\} \\ & \cup \left\{ \frac{\Delta}{M(k)}V < \Delta^\sigma - \frac{1}{2}\Delta^{8/7}; \sup_i |a_i| < \Delta^{-1/28} \right\} \\ & \subset \left\{ \frac{\Delta}{M(k)}U < \frac{1}{2}\Delta^{8/7}; \sup_i |a_i| > \Delta^{1/14} \right\} \\ & \cup \left\{ \frac{\Delta}{M(k)}|V| > \frac{1}{3}\Delta^{8/7}; \sup_i |a_i| < \Delta^{-1/28} \right\} \end{aligned}$$

Now

$$\left\{ U \leq \frac{1}{2}\Delta^{1/7}M(k); \sup_i |a_i| > \Delta^{1/14} \right\} \subset \left\{ \inf_i \sum_{\ell=1}^{M(k)} (\hat{\delta}_\ell^k W_i)^2 \leq \frac{1}{2}M(k) \right\} \subset \Omega_a(\Delta)$$

and

$$\begin{aligned} & \left\{ |V| > \frac{1}{3}\Delta^{1/7}M(k); \sup_i |a_i| < \Delta^{1/28} \right\} \\ & \subset \left\{ \sup_{\substack{(i,j) \\ i \neq j}} |a_i| |a_j| \left| \sum_{\ell=1}^{M(k)} (\hat{\delta}_\ell^k W_i)(\hat{\delta}_\ell^k W_j) \right| > \frac{\Delta^{1/7}}{3N^2}M(k); \sup_i |a_i| < \Delta^{-1/28} \right\} \\ & \subset \left\{ \sup_{\substack{(i,j) \\ i \neq j}} \left| \sum_{\ell=1}^{M(k)} (\hat{\delta}_\ell^k W_i)(\hat{\delta}_\ell^k W_j) \right| > \frac{\Delta^{3/14}}{3N^2}M(k) \right\} \subset \Omega_b(\Delta). \end{aligned}$$

□

The following result is the main result of this section:

LEMMA 7.9 For all $\Delta \in (0, (40 + 4N^2)^{-42}]$ and all stochastic processes $H(s)$ of the form (7.4)

$$\hat{\Omega}_*(H, \Delta) \cap \left\{ \|H\|_\infty < \Delta; \sup_i \|Y_i\|_\infty > \Delta^{1/14} \right\} \subset \Omega_a(\Delta) \cup \Omega_b(\Delta) \cup \Omega_c(\Delta).$$

We will prove this result by showing that on $\hat{\Omega}_*(H, \Delta)$ as $\Delta \rightarrow 0$, if

$$\sup_{s \in [0, T]} |H(s)| < \Delta \quad \text{and} \quad \sup_i \sup_{s \in [0, T]} |Y_i(s)| > \Delta^{1/14},$$

then the approximate quadratic variation of the Wiener processes at the scale δ is abnormally small or $\sup_i \|W_i\|_\infty > \Delta^{-1/28}$.

PROOF OF LEMMA 7.9: From the last estimate in Lemma 7.7, we have that on $\hat{\Omega}_*(H, \Delta) \cap \Omega_c(\Delta)^c \cap \{\sup_{s \in I_k} |H(s)| < \Delta\}$, $Q_k(Z^*) \leq (40 + 4N^2)\Delta^{25/21} < \Delta^{49/42}$ for $\Delta \in (0, (40 + 4N^2)^{-42}]$. Here Z^* is the approximation defined in (7.7). Now Lemma 7.8 with $\sigma = \frac{49}{42} (> \frac{8}{7})$ implies that

$$(7.8) \quad \Omega_c(\Delta)^c \cap \hat{\Omega}_*(H, \Delta) \cap \left\{ \sup_{s \in I_k} |H(s)| < \Delta; \sup_i \sup_{s \in I_k} |Y_i(s)| > \Delta^{1/14} \right\} \subset \Omega_a(\Delta) \cup \Omega_b(\Delta)$$

for all $k = 1, 2, \dots$, and $\Delta \in (0, (40 + 4N^2)^{-42}]$.

Continuing, we have that

$$\begin{aligned} &\Omega_c(\Delta)^c \cap \hat{\Omega}_*(H, \Delta) \cap \left\{ \|H\|_\infty < \Delta; \sup_i \|Y_i\|_\infty > \Delta^{1/14} \right\} \\ &\subset \bigcup_{k=1}^m \left\{ \sup_{s \in I_k} |H(s)| < \Delta; \sup_i \sup_{s \in I_k} |Y_i(s)| > \Delta^{1/14} \right\} \cap \hat{\Omega}_*(H, \Delta) \cap \Omega_c(\Delta)^c \\ &\subset \Omega_a(\Delta) \cup \Omega_b(\Delta) \end{aligned}$$

□

7.4 Estimates on the Size of Ω_a , Ω_b , and Ω_c

Since the events described by Ω_a and Ω_b are simply statements about collections of independent standard normal random variables, the following two estimates will give us the needed control:

LEMMA 7.10 For $c \in (0, 1)$ and $M > \frac{2}{1-c}$, setting $\gamma = c - 1 - \ln(c) > 0$

$$\begin{aligned} &\mathbb{P}\left(\sum_{\ell=1}^M \eta_\ell^2 \leq cM\right) \leq \frac{1}{\sqrt{\pi M}} \exp\left(-\frac{1}{2}\gamma M\right) \\ &\mathbb{P}\left(\left|\sum_{\ell=1}^M \eta_\ell \tilde{\eta}_\ell\right| \geq cM\right) \leq 2\mathbb{P}\left(\sum_{\ell=1}^M \eta_\ell \tilde{\eta}_\ell \geq cM\right) \leq 2 \exp\left(-\frac{c^2}{4}M\right) \end{aligned}$$

where $\{\eta_\ell, \tilde{\eta}_\ell\}$ are a collection of $2M$ mutually independent standard $N(0, 1)$ random variables.

PROOF: Notice that $\sum_{\ell=1}^M \eta_\ell^2$ is distributed as a χ^2 random variable with M degrees of freedom. Hence we have

$$\mathbb{P}\left(\sum_{\ell=1}^M \eta_\ell^2 \leq cM\right) = \frac{2^{-M/2}}{\Gamma(M/2)} \int_0^{cM} x^{M/2-1} e^{-x/2} dx$$

Since $c < 1$ and $M > \frac{2}{1-c}$, the integrand is bounded by $(cM)^{M/2-1} \exp(-c \frac{M}{2})$. Combining this with $\Gamma(\frac{M}{2}) \geq \sqrt{\pi M} (\frac{M}{2e})^{M/2}$ implies that

$$\begin{aligned} \mathbb{P}\left(\sum_{\ell=1}^M \eta_\ell^2 \leq cM\right) &\leq \frac{M^{-M/2} e^{M/2}}{\sqrt{\pi M}} (cM)^{M/2} \exp\left(-c \frac{M}{2}\right) \\ &\leq \frac{1}{\sqrt{\pi M}} \exp\left(\frac{M}{2}[-c + 1 + \ln(c)]\right). \end{aligned}$$

Noticing that $-c + 1 + \ln(c) < 0$ for $c \in (0, 1)$ finishes the proof of the first statement.

For the second, note that for $\lambda \in (-1, 1)$, $\mathbb{E} \exp(\lambda \eta_\ell \tilde{\eta}_\ell) = (1 - \lambda^2)^{-1/2}$. Hence for $\lambda \in [0, \frac{1}{2}]$,

$$\begin{aligned} \mathbb{P}\left(\sum_{\ell=1}^M \eta_\ell \tilde{\eta}_\ell \geq cM\right) &\leq \exp\left(-\lambda cM + \frac{1}{2} |\ln(1 - \lambda^2)| M\right) \\ &\leq \exp(-\lambda cM + \lambda^2 M). \end{aligned}$$

Taking $\lambda = \frac{c}{2}$ gives the result. □

COROLLARY 7.11 For $\Delta \leq \frac{1}{2} \wedge T$,

$$\begin{aligned} \mathbb{P}(\Omega_a(\Delta)) &\leq \frac{2TN}{\sqrt{\pi}} \Delta^{-2/3} \exp\left(-\frac{1}{20} \Delta^{-2/3}\right) \\ \mathbb{P}(\Omega_b(\Delta)) &\leq 6N^2 T \Delta^{-1} \exp\left(-\frac{1}{3N^2} \Delta^{-19/42}\right). \end{aligned}$$

In particular, if $\gamma = \min(\frac{1}{3N^2}, \frac{1}{20})$ then

$$\mathbb{P}(\Omega_a(\Delta) \cup \Omega_b(\Delta)) \leq 8TN^2 \frac{\exp(-\gamma \Delta^{-2/5})}{\Delta}$$

PROOF: First observe that the $\{\hat{\delta}_\ell^k W_i\}$ are independent $N(0, 1)$ random variables. Since

$$\Omega_a \subset \bigcup_{k=0}^m \bigcup_{i=1}^N \left\{ \sum_{\ell=1}^{M(k)} (\hat{\delta}_\ell^k W_i)^2 \leq \frac{M(k)}{2} \right\},$$

$m \leq \frac{T}{\Delta} + 1$, and $\Delta^{-2/3} - 1 \leq M \leq \Delta^{-2/3}$, the first result follows from Lemma 7.10 and bounding the previous expression by the sum of the probability of the sets on the right-hand side.

Proceeding in a fashion similar to the first estimate, the second bound follows from

$$\Omega_b \subset \bigcup_{k=0}^m \bigcup_{\substack{(i,j) \in \{1, \dots, N\}^2 \\ i \neq j}} \left\{ \left| \sum_{\ell=1}^{M(k)} (\hat{\delta}_\ell^k W_i)(\hat{\delta}_\ell^k W_j) \right| \geq \frac{\Delta^{3/14} M(k)}{3N^2} \right\}.$$

Combining the first two estimates gives the last quoted result. □

LEMMA 7.12 *For any $p \geq 1$ there exists a $c = c(p, N, T)$ so that $\mathbb{P}(\Omega_c(\epsilon)) \leq c\epsilon^p$.*

PROOF: It is enough to show that $\mathbb{E}\|W_i\|_\infty^\gamma < c(T, \gamma)$ and $\mathbb{E}[\mathcal{H}_{1/4}(W_i)^\gamma] < c(T, \gamma)$ for any $\gamma \geq 1$. The first follows from Doob's inequalities for the continuous martingale $W_i(s)$. The finiteness of the moments of the modules of continuity of the Wiener processes is given by theorem 2.1 [29, p. 26] and the observation at the top of page 28 in the same reference. Together these imply that $\mathbb{E}[\mathcal{H}_\alpha(W_i)^\gamma] < \infty$ for all $\gamma > 0$ as long as $\alpha \in (0, \frac{1}{2})$. Since $\alpha = \frac{1}{4}$ in our setting, the proof is complete. □

8 Strict Positivity of the Density

We will now give conditions under which for any $t > 0$ and some orthogonal projection Π of \mathbb{L}^2 onto a finite-dimensional subspace S , the density $p(t, x)$ of the law of $\Pi w(t)$ with respect to Lebesgue measure on S satisfies

$$(8.1) \quad p(t, x) > 0 \quad \text{for all } x \in S.$$

Our proof will make use of a criterion for strict positivity of the density of a random variable, which was first established in the case of finite-dimensional diffusions by Ben Arous and Léandre [4]. It was then extended to general random variables defined on Wiener space by Aida, Kusuoka, and Stroock [2]. We follow the presentation in Nualart [26]. However, there is one major difference between our case and the classical situation treated in those references. As noted at the start of Section 2, our SPDE can be solved pathwise. This means that the Wiener process $W(t) = (W_k(t))_{k \in \mathcal{Z}_*}$ can be replaced by a fixed trajectory in $\Omega_{[0,t]} \stackrel{\text{def}}{=} C([0, t]; \mathbb{R}^{\mathcal{Z}_*})$. Hence, we do not really need the notion of a skeleton. Because of this we can prove a result that is slightly more general than usual in that our controls need not belong to the Cameron-Martin space.

Let $Q \in \mathcal{L}(\mathbb{R}^{\mathcal{Z}_*}; \mathbb{L}^2)$ be such that if $\{q_k, k \in \mathcal{Z}_*\}$ is a standard basis for $\mathbb{R}^{\mathcal{Z}_*}$, then $Qq_k = e_k$.

The main result of this section is the following:

THEOREM 8.1 *Assume that $S_\infty = \mathbb{L}^2$. Let $t > 0$, and let Π be an orthogonal projection of \mathbb{L}^2 onto a finite-dimensional subspace $S \subset S_\infty = \mathbb{L}^2$. Let $x \in S$ be such that for some $0 < s < t$ and all $w \in \mathbb{L}^2$ there exists $H \in C([s, t]; \mathbb{R}^{\mathcal{Z}_*})$ such that the solution of*

$$(8.2) \quad \begin{cases} \frac{\partial w^H}{\partial r}(r) + B(w^H(r), w^H(r)) = v\Delta w^H(r) + Q\frac{\partial H}{\partial r}(r), & r > s, \\ w^H(s) = w, \end{cases}$$

satisfies $\Pi w^H(t) = x$. Then the density $p(t, \cdot)$ of the law of the random variable $\Pi w(t)$ satisfies $p(t, x) > 0$.

Note that equation (8.2) makes perfect sense even when H is not differentiable in time. To see this, define $\bar{w}^H(s) \stackrel{\text{def}}{=} w^H(s) - QH(s)$ and note that the equation for $\bar{w}^H(s)$ is a well-known type of equation to which existence and uniqueness results apply (cf. [6, 13]).

COROLLARY 8.2 *Let Π be any orthogonal projection of \mathbb{L}^2 onto a finite-dimensional space. If $S_\infty = \mathbb{L}^2$, then the density $p(t, \cdot)$ of the random variable $\Pi w(t)$ satisfies $p(t, x) > 0$ for all $x \in \Pi\mathbb{L}^2$.*

PROOF OF COROLLARY 8.2: The results almost follow from the published version of theorem 9 in Agrachev and Sarychev [1], which states that under some assumptions for any $s \in (0, t)$, the controllability assumption of Theorem 8.1 is satisfied with a control $H \in W^{1,\infty}(s, t; \mathbb{R}^{\mathcal{Z}_*})$. The increasing family of sets that describes the way the randomness spreads is slightly different from the S_n . Furthermore, they do not state the result for arbitrary projections but only for the span of a finite number of Fourier modes. However, in private communications, the authors have verified that the sets S_n may be used and that an arbitrary finite-dimensional projection may be taken. As an aside, Romito [30] has proven this formulation of controllability of the Galerkin approximations under our assumptions. \square

Remark 8.3. It is worth pointing out that the exact controllability of the projections is far from the exact controllability in all of \mathbb{L}^2 . In fact, the latter does not hold with smooth-in-space and L^2 -in-time controls. This would imply that the density was supported on \mathbb{L}^2 , which is not true as its support is contained in functions that are analytic in space [18, 19].

The rest of the section is devoted to the proof of Theorem 8.1. Our proof is based on the following result, which is a variant of proposition 4.2.2 in Nualart [26].

PROPOSITION 8.4 *Let $F \in C(\Omega_{[0,t]}; S)$ where $S = \Pi\mathbb{L}^2$ is a finite-dimensional vector space such that $H \mapsto F(H)$ is twice differentiable in the directions of $H^1(0, t; \mathbb{R}^{\mathcal{Z}_*})$, and there exist $DF(\cdot) \in C(\Omega_{[0,t]}; [L^2(0, t; S)]^{\mathcal{Z}_*})$ and $D^2F(\cdot) \in C(\Omega_{[0,t]}; [L^2((0, t)^2; S)]^{\mathcal{Z}_* \times \mathcal{Z}_*})$ such that for all $j, \ell \in \mathcal{Z}_*$ and $h, g \in L^2(0, t; \mathbb{R})$,*

$$\begin{aligned} \frac{d}{d\epsilon} F\left(H + \epsilon \int_0^\cdot q_j h(s) ds\right) \Big|_{\epsilon=0} &= \int_0^t D_{j,s} F(H) h(s) ds, \\ \frac{d^2}{d\epsilon d\delta} F\left(H + \epsilon \int_0^\cdot q_j h(s) ds + \delta \int_0^\cdot q_\ell g(s) ds\right) \Big|_{\substack{\epsilon=0 \\ \delta=0}} &= \int_0^t \int_0^t D_{j,s;\ell,r}^2 F(H) h(s) g(r) ds dr. \end{aligned}$$

We assume, moreover, that

$$(8.3) \quad H \mapsto (F(H), DF(H), D^2F(H))$$

is continuous and locally bounded from $\Omega_{[0,t]}$ into

$$S \times [L^2(0, t; S)]^{\mathbb{Z}_*} \times [L^2([0, t]^2; S)]^{\mathbb{Z}_* \times \mathbb{Z}_*},$$

and that there exists $H \in \Omega_{[0,t]}$ such that $F(H) = x$ and

$$(8.4) \quad \det \left(\sum_{k \in \mathbb{Z}_*} \int_0^t D_{k,s} F^i(H) D_{k,s} F^j(H) ds \right) > 0.$$

Then, if the law of $F(W)$ has a density $p(t, \cdot)$, $p(t, x) > 0$.

PROOF: Since the proof is almost identical to that of proposition 4.2.2 in [26], we only indicate the differences with the latter proof.

In this proof, H is one specific element of $\Omega_{[0,t]}$ satisfying $F(H) = x$ and (8.4). Let $h_j(s) = D_{\cdot,s} F^j(H)$, $j \in \mathbb{Z}_*$. Clearly, $h_j \in L^2(0, t; \mathbb{R}^{\mathbb{Z}_*})$. For each $z \in \mathbb{R}^{\mathbb{Z}_*}$, let

$$(T_z W)(t) \stackrel{\text{def}}{=} W(t) + \sum_{j \in \mathbb{Z}_*} z_j \int_0^t h_j(s) ds$$

and

$$g(z, W) = F(T_z W) - F(W).$$

From our assumptions, for any $W \in \Omega_{[0,t]}$, $g(\cdot, W) \in C^2(B_1(0); \mathbb{R}^{\mathbb{Z}_*})$, and for any $\beta > 1$, there exists $C(\beta)$ such that

$$\|W\|_{\infty,t} \leq C(\beta) \Rightarrow \|g(\cdot, W)\|_{C^2(B_1(0))} \leq \beta,$$

where $\|W\|_{\infty,t} \stackrel{\text{def}}{=} \sup_{0 \leq s \leq t} |W(s)|$, and the notation $B_\alpha(0)$ stands for the open ball in $\mathbb{R}^{\mathbb{Z}_*}$ centered at 0 with radius α .

Assume for a moment that in addition

$$|\det g'(0)| \geq \frac{1}{\beta}.$$

It then follows from lemma 4.2.1 in [26] that there exist $c_\beta \in (0, \frac{1}{\beta})$ and $\delta_\beta > 0$ such that $g(\cdot, W)$ is diffeomorphic from $B_{c_\beta}(0)$ onto a neighborhood of $B_{\delta_\beta}(0)$.

We now define the random variable \mathcal{H}_β , which plays exactly the same role in the rest of our proof as H_β in [26] but is defined slightly differently. We let

$$\mathcal{H}_\beta = k_\beta(\|W\|_{\infty,t}) \alpha_\beta(|\det \langle DF^i(W), DF^j(H) \rangle|),$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(0, t; \mathbb{R}^{\mathbb{Z}_*})$; $k_\beta, \alpha_\beta \in C(\mathbb{R}; [0, 1])$, $k_\beta(x) = 0$ whenever $|x| \geq \beta$, and $k_\beta(x) = 1$ whenever $|x| \leq \beta - 1$; $\alpha_\beta(x) = 0$ whenever $|x| \leq \frac{1}{\beta}$, $\alpha_\beta(x) > 0$ whenever $|x| > \frac{1}{\beta}$, and $\alpha_\beta(x) = 1$ if $|x| \geq \frac{2}{\beta}$.

The rest of the proof follows exactly the argument in [26, pp. 181–182]. We only have to make explicit the sequence T_N^H , $N = 1, 2, \dots$, of absolutely continuous transformations of $\Omega_{[0,t]}$ equipped with Wiener measure, which is used at the

end of the proof. For $N = 1, 2, \dots$, let $t_i^N = \frac{it}{N}$, $0 \leq i \leq N$. We define

$$(T_N^H W)(t) = W(t) + \int_0^t (\dot{H}_N(s) - \dot{W}_N(s)) ds,$$

where

$$\begin{aligned} \dot{H}_N(s) &= \sum_{i=1}^{N-1} \frac{H(t_i^N) - H(t_{i-1}^N)}{t_i^N - t_{i-1}^N} \mathbf{1}_{[t_i^N, t_{i+1}^N)}(s), \\ \dot{W}_N(s) &= \sum_{i=1}^{N-1} \frac{W(t_i^N) - W(t_{i-1}^N)}{t_i^N - t_{i-1}^N} \mathbf{1}_{[t_i^N, t_{i+1}^N)}(s). \end{aligned}$$

For any $W \in \Omega_{[0,t]}$, as $N \rightarrow \infty$,

$$\sup_{0 \leq s \leq t} |(T_N^H W)(s) - H(s)| \rightarrow 0.$$

Hence, by the continuity of F and DF , we also have

$$\begin{aligned} F(T_N^H W) &\rightarrow F(H), \\ DF(T_N^H W) &\rightarrow DF(H), \end{aligned}$$

and moreover

$$\lim_{M \rightarrow \infty} \sup_N \mathbb{P}(\|T_N^H W\|_{\infty,t} > M) = 0.$$

This provides exactly the version of (H2) from Nualart, which is needed here to complete the proof. □

All that remains is to prove the following lemma:

PROPOSITION 8.5 *Under the assumptions of Theorem 8.1, if $F = \Pi w(t)$, there exists $H \in \Omega_{[0,t]}$ such that $F(H) \stackrel{\text{def}}{=} \Pi w^H(t) = x$, and (8.4) holds.*

PROOF: Let $s \in (0, t)$ be the time that appears in the assumption of Theorem 8.1. Since $S_\infty = \mathbb{L}^2$, it follows from (3.2) that

$$\mathbb{P}\left(\bigcap_{\substack{\phi \in \mathbb{H}^1 \\ \phi \neq 0}} \{ \langle \mathcal{M}(s)\phi, \phi \rangle > 0 \} \right) = 1.$$

We choose an arbitrary Brownian trajectory W in this set of measure 1. We then choose, according to the assumption of Theorem 8.1, an $H_x \in \Omega_{[0,t]}$ satisfying $H_x(s) = 0$ such that the solution $\{w^{H_x}(r) : s \leq r \leq t\}$ of (8.2) with initial condition $w^{H_x}(s) = w(s; w_0, W)$, the value at time s of the solution of (1.1) corresponding to the trajectory W that we chose above, satisfies $\Pi w^{H_x}(t) = x$.

It remains to show that condition (8.4) is satisfied, with $H \in \Omega_{[0,t]}$ defined by

$$H(r) = \begin{cases} W(r), & \text{if } 0 \leq r \leq s, \\ W(s) + H_x(r), & \text{if } s < r \leq t. \end{cases}$$

We shall write

$$w^H(r) = \begin{cases} w(r, W), & \text{if } 0 \leq r \leq s, \\ w^{H_x}(r), & \text{if } s < r \leq t. \end{cases}$$

All we need to show is that for any fixed $\phi \in S, \phi \neq 0$,

$$\sum_{k \in \mathcal{Z}_*} \int_0^t \langle D_{k,r} w^H(t), \phi \rangle^2 dr > 0.$$

Note that

$$\begin{aligned} \sum_{k \in \mathcal{Z}_*} \int_0^t \langle D_{k,r} w^H(t), \phi \rangle^2 dr &\geq \sum_{k \in \mathcal{Z}_*} \int_0^s \langle D_{k,r} w^H(t), \phi \rangle^2 dr \\ &= \sum_{k \in \mathcal{Z}_*} \int_0^s \langle D_{k,r} w^H(s), U^{t,\phi}(s) \rangle^2 ds, \end{aligned}$$

where $\{U^{t,\phi}(s), 0 \leq s \leq t\}$ solves the backward equation (2.5) associated to the corresponding w trajectory, and the last identity follows from Proposition 2.3. In view of our specific choice of W , since from Lemma B.1 $U^{t,\phi}(s) \in \mathbb{H}^1$, it suffices to check that $U^{t,\phi}(s) \neq 0$. This follows from “backward uniqueness” (maybe we should say “forward uniqueness” since the $U^{t,\phi}$ equation is a backward equation!); see theorem I.1 in Bardos and Tartar [3], whose assumptions are clearly verified by the $U^{t,\phi}(\cdot)$ equation (2.5). Specifically, since if $\phi = 0$, then $U^{t,\phi}(s) = 0$ for all $s \leq t$, one knows that no other terminal condition ϕ at time t can lead to $U^{t,\phi}(s) = 0$ for $s \leq t$. □

9 Conclusion

We have proven under reasonable nondegeneracy conditions that the law of any finite-dimensional projection of the solution of the stochastic Navier-Stokes equation with additive noise possesses a smooth, strictly positive density with respect to Lebesgue measure. In particular, it was shown that four degrees of freedom are sufficient to guarantee nondegeneracy.

It is reasonable to ask if four is the minimal size set that produces finite-dimensional projections with a smooth density. The nondegeneracy condition concentrates on the wave numbers where both the sin and cos are forced. Since this represents the translation-invariant scales in the forcing, it is a mild restriction to require that whenever either of the sin or cos of a given wave number is forced, then both are forced. Under this assumption, forcing only two degrees of freedom corresponds to forcing both degrees of freedom associated to a single wave number k . It is easy to see that the subspace $\{u \in \mathbb{L}^2 : \langle u, \sin(j \cdot x) \rangle = \langle u, \cos(j \cdot x) \rangle = 0 \text{ for } j \neq k\}$ is invariant under the dynamics with such a forcing. Hence if the initial condition lies in this two-dimensional subspace, the conclusions of Theorem 1.1 fail to hold. See [14] for a more complete discussion of this and other cases where the nondegeneracy condition fails.

We have concentrated on the two-dimensional Navier-Stokes equations forced by a finite number of Wiener processes. However, there are a number of ways one could extend these results. The choice of a forcing with finitely many modes was made for simplicity in a number of technical lemmas, in particular in Section 7.1. There appears to be no fundamental obstruction to extending the method to the cases with infinitely many forcing terms if the covariances satisfy an appropriate summability condition. In addition, the methods of this paper should apply equally to other polynomial nonlinearities, such as stochastic reaction diffusion equations with additive noise. In contrast, handling nonadditive forcing in a nonlinear equation would require nontrivial extensions of the present work. Since in the linearization, stochastic integrals of nonadapted processes would appear, it is not certain that the line of argument in this paper would succeed.

Appendix A: Estimates on the Enstrophy

Define $\mathcal{E}_0 = \sum_{k \in \mathcal{Z}_*} \|e_k\|^2$ where the e_k were the functions used to define the forcing in (1.3). In general, we define the α^{th} spatial moment of the forcing to be $\mathcal{E}_\alpha = \sum_{k \in \mathcal{Z}_*} |k|^{2\alpha} \|e_k\|^2$. With this notation, we have the following estimate:

LEMMA A.1 *Given any $\epsilon \in (0, 1)$, there exists a $\gamma = \gamma(\epsilon)$ such that*

$$\mathbb{P} \left\{ \sup_{s \in [0, t]} \|w(s)\|^2 + 2\epsilon v \int_0^s \|w(r)\|_1^2 dr - \mathcal{E}_0 s > \|w(0)\|^2 + K \right\} < e^{-\gamma K}$$

for all $K \geq 0$.

PROOF: The lemma follows from the exponential martingale estimate after one notices that the quadratic variation of the martingale in the equation for the enstrophy is controlled by $\int_0^t \|w(r)\|_1^2 dr$. See lemma A.2 or lemma A.1 in [20] for the details and related lemmas in exactly this setting or Lemma A.3 below for a similar argument. \square

From the previous result, we obtain the following:

COROLLARY A.2 *There exists a constant $\eta_0 = \eta_0(T, \nu) > 0$ so that for any $\eta \in (0, \eta_0]$ there exists a constant $c = c(T, \eta, \nu)$ so that*

$$\mathbb{E} \exp \left(\eta \sup_{0 \leq s \leq T} \|w(s)\|^2 \right) \leq c \mathbb{E} \exp(\eta \|w(0)\|^2)$$

and

$$\mathbb{E} \exp \left(\nu \eta \int_0^T \|w(s)\|_1^2 ds \right) \leq c \mathbb{E} \exp(\eta \|w(0)\|^2).$$

We will also need the following result, which gives quantitative estimates on the regularization of the \mathbb{H}^1 norm:

LEMMA A.3 *Given a time $T > 0$ and $p \geq 0$, there exists a positive constant $c = c(\nu, T, p, \mathcal{E}_1)$ so that*

$$\mathbb{E} \sup_{0 \leq s \leq T} [\|w(s)\|^2 + s\nu\|w(s)\|_1^2]^p \leq c[1 + \|w(0)\|^{4p}].$$

PROOF: Defining $\zeta(s) = \|w(s)\|^2 + s\nu\|w(s)\|_1^2$, we have for all $s \in [0, T]$

$$\begin{aligned} \zeta_s - \mathcal{E}_0 s - \frac{1}{2} \mathcal{E}_1 s^2 &= \|w(0)\|^2 + \int_0^s 2r\nu[-\nu\|w(r)\|_2^2 + \langle B(w(r), w(r)), \Delta w(r) \rangle] dr \\ &\quad - \nu \int_0^s \|w(r)\|_1^2 dr + \sum_{k \in \mathcal{Z}_*} \int_0^s 2(1 + r\nu|k|^2) \langle w(r), e_k \rangle dW_k(r). \end{aligned}$$

Since

$$|\langle B(w, w), \Delta w \rangle| \leq c\|w\|_{1/2} \|w\|_1 \|w\|_2 \leq \nu\|w\|_2^2 + c(\nu)\|w\|^4,$$

one has

$$(A.1) \quad \sup_{0 \leq s \leq T} \zeta_s \leq c(1 + T^2) + cT^2 \sup_{0 \leq s \leq T} \|w(s)\|^4 + \sup_{0 \leq s \leq T} N_s$$

where

$$N_s = -\nu \int_0^s [\|w(r)\|^2 + r\|w(r)\|_2^2] dr + M_s$$

and

$$M_s = \sum_{k \in \mathcal{Z}_*} \int_0^s 2(1 + r\nu|k|^2) \langle w(r), e_k \rangle dW_k(r).$$

Notice that for all $s \in [0, T]$ and $\alpha > 0$ sufficiently small, $N_s \leq M_s - \frac{\alpha}{2}[M, M]_s$, where $[M, M]_s$ is the quadratic variation of the martingale M_s . Hence the exponential martingale estimate implies $\mathbb{P}(\sup_{s \leq T} N_s > \beta) \leq \mathbb{P}(\sup_{s \leq T} M_s - \frac{\alpha}{2}[M, M]_s > \beta) \leq \exp(-\alpha\beta)$. This implies that the last term in (A.1) can be bounded by a constant depending only on α, T, ν , and the power p . By Corollary A.2, the third term in (A.1) can be bounded by a constant that depends on the initial condition as stated as well as α, T, ν , and the power p . \square

Appendix B: Estimates on the Linearization and Its Adjoint

Define the action of linearized operator $J_{s,t}$ on a $\phi \in \mathbb{L}^2$ by

$$(B.1) \quad \begin{cases} \frac{\partial}{\partial t} J_{s,t} \phi = \nu \Delta J_{s,t} \phi + B(w(t), J_{s,t} \phi) + B(J_{s,t} \phi, w(t)), & 0 \leq s \leq t, \\ J_{s,s} \phi = \phi, \end{cases}$$

and its time-reversed, \mathbb{L}^2 -adjoint $\bar{J}_{s,t}^*$ acting on $\phi \in \mathbb{L}^2$ by

$$(B.2) \quad \begin{cases} \frac{\partial}{\partial s} \bar{J}_{s,t}^* \phi + \nu \Delta \bar{J}_{s,t}^* \phi + B(w(s), \bar{J}_{s,t}^* \phi) \\ \quad - C(\bar{J}_{s,t}^* \phi, w(s)) = 0, & 0 \leq s \leq t, \\ \bar{J}_{t,t}^* \phi = \phi. \end{cases}$$

If we define the operator $Q : \mathbb{R}^{\mathcal{Z}^*} \rightarrow \mathbb{L}^2$ by $(x_k)_{k \in \mathcal{Z}^*} \mapsto \sum x_k e_k$, then $V_{k,s}(t) = J_{s,t} Q q_k$ where $\{q_k : k \in \mathcal{Z}^*\}$ is the standard basis for $\mathbb{R}^{\mathcal{Z}^*}$ and $U^{t,\phi}(s) = \bar{J}_{s,t}^* \phi$. Similarly, for $h \in L^2_{\text{loc}}(\mathbb{R}^{\mathcal{Z}^*}_+)$, $D_h w(t) = \int_0^t J_{s,t} Q h(s) ds$.

LEMMA B.1 *For any $T_0 > 0$, $\eta > 0$, and $\alpha \in \{0, 1\}$, there exist constants $\gamma = \gamma(\nu, \eta, T_0^\alpha)$ and $c = c(\nu, T_0^\alpha, \eta)$ so that for all $\phi \in \mathbb{L}^2$ and $T \leq T_0$*

$$\begin{aligned} & \sup_{0 \leq s \leq t \leq T} \|J_{s,t} \phi\|^2 + \gamma(t-s)^\alpha \|J_{s,t} \phi\|_1^2 \\ & \leq \exp\left(\eta \int_0^T \|w(r)\|_1^2 dr + cT\right) (\|\phi\|^2 + (1-\alpha)\|\phi\|_1^2), \\ & \sup_{0 \leq s \leq t \leq T} \|\bar{J}_{s,t}^* \phi\|^2 + \gamma(t-s)^\alpha \|\bar{J}_{s,t}^* \phi\|_1^2 \\ & \leq \exp\left(\eta \int_0^T \|w(r)\|_1^2 dr + cT\right) (\|\phi\|^2 + (1-\alpha)\|\phi\|_1^2), \end{aligned}$$

where on the right-hand side $(1-\alpha)\|\phi\|_1^2 = 0$ when $\alpha = 1$ by convention for all ϕ even if $\|\phi\|_1 = \infty$.

PROOF: We start by deriving a number of bounds on the nonlinear terms. Using Lemma D.1 and standard interpolation inequalities produces for any $\delta > 0$ and $\eta > 0$ and some c ,

$$\begin{aligned} 2|\langle B(\phi, w), \phi \rangle| & \leq c\|\phi\|^{3/2} \|\phi\|_1^{1/2} \|w\|_1 \leq \delta\|\phi\|_1^2 + \|\phi\|^2 \left(\eta\|w\|_1^2 + \frac{c}{\delta\eta^2} \right), \\ 2|\langle B(\phi, w), \Delta\phi \rangle| & \leq c\|w\|_1 \|\phi\|_1^{1/2} \|\phi\|^{1/2} \|\phi\|_2 \\ & \leq \delta\|\phi\|_2^2 + \eta\|w\|_1^2 \|\phi\|_1^2 + \frac{c}{\eta\delta^2} \|w\|_1^2 \|\phi\|^2, \\ 2|\langle B(w, \phi), \Delta\phi \rangle| & \leq c\|w\|_1 \|\phi\|_1 \|\phi\|^{1/4} \|\phi\|_2^{3/4} \\ & \leq \frac{c}{(\eta\delta)^{3/2}} \|w\|_1 \|\phi\|_1 \|\phi\| + (\eta\delta)^{1/2} \|w\|_1 \|\phi\|_1 \|\phi\|_2 \\ & \leq \delta\|\phi\|_1^2 + \frac{c}{\eta^3\delta^4} \|w\|_1^2 \|\phi\|^2 + \frac{\delta}{2} \|\phi\|_2^2 + \eta\|w\|_1^2 \|\phi\|_1^2, \\ 2|\langle B(\Delta\phi, w), \phi \rangle| & \leq c\|\phi\|_{1+\epsilon} \|\phi\|_{1-\epsilon} \|w\|_1 \\ & \leq c\|w\|_1 \|\phi\| \|\phi\|_2 \leq \delta\|\phi\|_2^2 + \frac{c}{\delta} \|w\|_1^2 \|\phi\|^2. \end{aligned}$$

We begin by bounding the J expression with $\alpha = 0$. Setting $\zeta_r = \|J_{s,s+r}\phi\|^2 + \gamma \|J_{s,s+r}\phi\|_1^2$, $w_r = w(s+r)$, and $J_r = J_{s,s+r}\phi$ and using the above estimates with appropriately chosen constants produces

$$\begin{aligned} \frac{\partial \zeta_r}{\partial r} &= \frac{\partial}{\partial r} \|J_r\|^2 + \gamma \frac{\partial}{\partial r} \|J_r\|_1^2 \\ &= -2\nu \|J_r\phi\|_1^2 + 2\langle B(J_r, w), J_r \rangle \\ &\quad + 2\gamma [-\nu \|J_r\|_2^2 + \langle B(J_r, w), \Delta J_r \rangle + \langle B(w, J_r), \Delta J_r \rangle] \\ &\leq \left(\frac{c}{\eta^2\nu} + \frac{\eta}{2} \|w_r\|_1^2 + \gamma c \left(\frac{1}{\eta^2\nu^2} + \frac{1}{\eta^3\nu^4} \right) \|w_r\|_1^2 \right) \|J_r\|^2 + \gamma \eta \|w_r\|_1^2 \|J_r\|_1^2 \end{aligned}$$

Hence for γ sufficiently small there exists a constant c so for all $r > 0$

$$\frac{\partial \zeta_r}{\partial r} \leq \left[\frac{c}{\eta^2\nu} + \eta \|w_r\|_1^2 \right] \zeta_r,$$

which proves the first result for $J_{s,t}$. The result for $\alpha = 1$ is identical except that we take $\zeta_r = \|J_{s,s+r}\phi\|^2 + \gamma r \|J_{s,s+r}\phi\|_1^2$ and hence there is an extra term from differentiating the coefficient of $\|J_{s,s+r}\phi\|_1^2$ and the fact that the resulting constants depend on the time interval $[0, T]$ over which r ranges. See the proof of Lemma A.3.

Turning to $\bar{J}_{s,t}^*$, we set $\bar{\zeta}_r = \|\bar{J}_{t-r,t}^*\phi\|^2 + \gamma \|\bar{J}_{t-r,t}^*\phi\|_1^2$, $\bar{w}_r = w(t-r)$, and $\bar{J}_r^* = \bar{J}_{t-r,t}^*\phi$ for all $r \in [0, t]$. The bar is to remind us that the process is time reversed. The argument proceeds as in the previous case. Again using the estimates above, we obtain

$$\frac{\partial \bar{\zeta}_r}{\partial r} \leq \left(\frac{c}{\nu(1+\eta^2)} + \left(\frac{\eta}{2} + \frac{\gamma c}{\nu} \right) \|w_r\|_1^2 \right) \|\bar{J}_r^*\|^2 + \gamma \eta \|w_r\|_1^2 \|\bar{J}_r^*\|_1^2.$$

Hence for γ small enough and $r \in [0, t]$,

$$\frac{\partial \bar{\zeta}_r}{\partial r} \leq \left[\frac{c}{\nu(1+\eta^2)} + \eta \|w_r\|_1^2 \right] \bar{\zeta}_r,$$

which proves the result. □

Appendix C: Higher Malliavin Derivatives of $w(t)$

For notational brevity define $\tilde{B}(f, g) = B(f, g) + B(g, f)$ for $f, g \in \mathbb{L}^2$. For $k \in \mathcal{Z}_*$, we define $J_{s,t}^{(1)} e_k = J_{s,t} e_k$. For $n > 1$ define $J_{s,t}^{(n)}$ acting on $\phi = (e_{k_1}, \dots, e_{k_n})$ with $k_1, \dots, k_n \in \mathcal{Z}_*$ and with time parameters $s = (s_1, \dots, s_n) \in \mathbb{R}_+^n$ by the equations

$$\begin{aligned} \frac{\partial}{\partial t} J_{s;t}^{(n)} \phi &= \nu \Delta J_{s;t}^{(n)} \phi + \tilde{B}(w(t), J_{s;t}^{(n)} \phi) + F_{s;t}^{(n)} \phi, & t > \bigvee s \\ J_{s;t}^{(n)} &= 0, & t \leq \bigvee s, \end{aligned}$$

where $\vee s = s_1 \vee \dots \vee s_n$. The operator $F_{s,t}^{(n)}$ applied to ϕ is defined by

$$F_{s;t}^{(n)} \phi = \sum_{(\alpha, \beta) \in \text{part}(n)} \tilde{B}(J_{s_\alpha;t}^{(|\alpha|)} \phi_\alpha, J_{s_\beta;t}^{(|\beta|)} \phi_\beta).$$

Here $\text{part}(n)$ is the set of partitions of $\{1, \dots, n\}$ into two sets, neither of them empty. $|\alpha|$ is the number of elements in α , $\phi_\alpha = (\phi_{\alpha_1}, \dots, \phi_{\alpha_{|\alpha|}})$, and $s_\alpha = (s_{\alpha_1}, \dots, s_{\alpha_{|\alpha|}})$. The partition (α, β) and (β, α) are viewed as the same partition.

First observe that when $n = 1$, Lemma B.1 says that for any $\eta > 0$ there is a $c = c(T, \eta)$ so for all $\phi \in \mathbb{R}^{\mathcal{Z}_*}$

$$\sup_{0 \leq s \leq t \leq T} \|J_{s;t}^{(1)} e_k\|_1 \leq c \exp\left(\eta \int_0^T \|w(r)\|_1^2 dr\right) \leq c \exp\left(\eta \int_0^T \|w(r)\|_1^2 dr\right).$$

For $n > 1$ with again $\phi = (e_{k_1}, \dots, e_{k_n})$ and $s \in \mathbb{R}_+^n$, we have the following estimate on $F^{(n)}$:

$$\begin{aligned} \|F_{s;t}^{(n)} \phi\| &\leq \sum_{(\alpha, \beta) \in \text{part}(n)} \|\tilde{B}(J_{s_\alpha;t}^{(|\alpha|)} \phi_\alpha, J_{s_\beta;t}^{(|\beta|)} \phi_\beta)\| \\ &\leq c \sum_{(\alpha, \beta) \in \text{part}(n)} \|J_{s_\alpha;t}^{(|\alpha|)} \phi_\alpha\|_1 \|J_{s_\beta;t}^{(|\beta|)} \phi_\beta\|_1. \end{aligned}$$

Then the variation-of-constants formula implies that $J_{s;t}^{(n)} \phi = \int_{\vee s}^t J_{r,t} F_{s;r}^{(n)} \phi dr$ and hence

$$\begin{aligned} \|J_{s;t}^{(n)} \phi\|_1 &\leq \int_{\vee s}^t \|J_{r,t} F_{s;r}^{(n)} \phi\|_1 dr \\ &\leq c \exp\left(\eta \int_0^T \|w(r)\|_1^2 dr\right) \left(\int_{\vee s}^t \frac{1}{(r - \vee s)^{1/2}} dr\right) \sup_{\substack{\tau=(\tau_1, \dots, \tau_n) \\ 0 \leq \tau_i \leq t \\ \vee \tau \leq r \leq t}} \|F_{\tau;r}^{(n)} \phi\| \\ &\leq c \exp\left(\eta \int_0^T \|w(r)\|_1^2 dr\right) \sum_{(\alpha, \beta) \in \text{part}(n)} \sup_{\tau,r} \|J_{\tau_\alpha;r}^{(|\alpha|)} \phi_\alpha\|_1 \sup_{\tau,r} \|J_{\tau_\beta;r}^{(|\beta|)} \phi_\beta\|_1. \end{aligned}$$

Proceeding inductively, one obtains for any $\eta > 0$ the existence for a $c = c(T, \eta, n)$ and $\gamma = \gamma(n)$ so that

$$\|J_{s;t}^{(n)} \phi\|_1 \leq \int_s^t \|J_{r,t} F_{s,r} \phi\|_1 dr \leq c \exp\left(\gamma \eta \int_0^T \|w(r)\|_1^2 dr\right).$$

Now with ϕ and s as above,

$$D_{s_1, k_1; \dots; s_n, k_n}^n w(t) = J_{s,t}^{(n)} \phi,$$

and since η is an arbitrary positive constant, by redefining it, one obtains that for any $\eta > 0$, there exists a $c = c(T, \eta, n)$ so that

$$\sup_{t \in (0, T]} \|D_{s_1, k_1; \dots; s_n, k_n}^n w(t)\|_1 \leq c \exp\left(\eta \int_0^T \|w(r)\|_1^2 dr\right).$$

Combining this estimate with Lemma A.2, we obtain the following result: Letting $\mathbb{D}^\infty(\mathbb{H}_1)$ be the space of random variables taking values in \mathbb{H}_1 that are infinitely differentiable in the Malliavin sense and such that those derivatives have all moments finite, we have the following (see [25, p. 62] for the definition of \mathbb{D}^∞):

LEMMA C.1 *For any $\eta > 0$, $t > 0$, $p \geq 1$, and $n \geq 1$ there exists a constant $c = c(t, \nu, \eta, p, n)$ so that*

$$\mathbb{E} \left[\left(\sum_{k_1, \dots, k_n \in \mathbb{Z}_*} \int_0^t \cdots \int_0^t \|D_{s_1, k_1; \dots; s_n, k_n}^n w(t)\|_1^2 ds_1 \cdots ds_n \right)^{p/2} \right] \leq c \exp(\eta \|w(0)\|^2).$$

Hence $w(t) \in \mathbb{D}^\infty(\mathbb{H}_1)$ for all $t > 0$.

Appendix D: Estimates on the Nonlinearity

In the following lemma, we collect a few standard estimates on the nonlinearity and derive a few consequences from them.

LEMMA D.1 *Let $\alpha_i \geq 0$ and either $\alpha_1 + \alpha_2 + \alpha_3 > 1$ or both $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and $\alpha_i \neq 1$ for any i . Then the following estimates hold for all $f, g, h \in \mathbb{L}^2$ if the right-hand side is well-defined:*

$$|\langle B(f, g), h \rangle| \leq c \|f\|_{\alpha_1-1} \|g\|_{\alpha_2+1} \|h\|_{\alpha_3}.$$

In addition, we have the following estimate: For any $\epsilon > 0$ there exists a $c = c(\epsilon)$ with

$$|\langle \nabla B(f, g), \nabla g \rangle| \leq c \|f\|_1 \|g\|_1 \|g\|_{1+\epsilon}.$$

PROOF: For the first result, see proposition 6.1 of [6] and recall that our B is slightly different than theirs and that $\|\mathcal{K}f\|_1 = \|f\|_0$. After translation, the result follows. For the second result we need to rearrange things. Setting $u = (u_1, u_2) = \mathcal{K}f$, we have

$$|\langle \nabla B(f, g), \nabla g \rangle| = \left| \int_{\mathbb{T}^2} \nabla[(u \cdot \nabla)g] \cdot \nabla g \, dx \right|.$$

Observe that $\nabla[(u \cdot \nabla)g] \cdot \nabla g = [(u \cdot \nabla)\nabla g] \cdot \nabla g + (\nabla u \nabla g) \cdot \nabla g$. Because $\nabla \cdot u = 0$, $[(u \cdot \nabla)\nabla g] \cdot \nabla g = \frac{1}{2} \sum_i \nabla \cdot (u \frac{\partial g}{\partial x_i})^2$. Hence, the integral of the first term is zero by Stokes' theorem and the fact that we are on the torus.

The integral of the second term over the domain is made of a finite number of terms of the form

$$\int \frac{\partial u_j}{\partial x_i} \frac{\partial g}{\partial x_j} \frac{\partial g}{\partial x_i} \, dx.$$

This term is dominated by

$$\left| \frac{\partial u_j}{\partial x_i} \right|_{L^r} \left| \frac{\partial g}{\partial x_j} \right|_{L^p} \left| \frac{\partial g}{\partial x_i} \right|_{L^q}$$

for any $r, p, q > 1$ with $\frac{1}{r} + \frac{1}{q} + \frac{1}{p} = 1$. Recall that in two dimensions, the Sobolev space $W^{1,2}$ is embedded in L^r for any $r < \infty$. Hence by taking $p = \frac{1}{2}$, $q > 2$ sufficiently close to 2, and r correspondingly large, we obtain the bound

$$c \left\| \frac{\partial u}{\partial x_i} \right\|_1 \left\| \frac{\partial g}{\partial x_j} \right\| \left\| \frac{\partial g}{\partial x_i} \right\|_\epsilon$$

for any $\epsilon > 0$ and some $c = c(\epsilon)$, and thus $c \|f\|_1 \|g\|_1 \|g\|_{1+\epsilon}$ bounds the estimate. □

Appendix E: Lipschitz and Supremum Estimates

Let S be a subspace of \mathbb{L}^2 spanned by a finite number of $\cos(x \cdot k)$ and $\sin(x \cdot k)$. Let Π be the orthogonal projection onto S . Also, let Π_0 be the orthogonal projection onto the directions directly forced by Wiener processes as defined in Proposition 3.7.

Recall from Section 7 the definitions of $\mathcal{H}_{\alpha,[a,b]}(f)$, $\|f\|_\infty$, and $\|f\|_{\infty,[a,b]}$ applied to functions of time taking values in \mathbb{L}^2 .

For $0 \leq s < t \leq T$ one has

$$\Pi_0^\perp w(t) = \Pi_0^\perp e^{v\Delta(t-s)} w(s) - \Pi_0^\perp \int_s^t e^{v\Delta(t-r)} B(w(r), w(r)) dr.$$

Since $\int f(x) dx = \int g(x) dx = 0$, $\|B(f, g)\| \leq c \|\nabla f\| \|\nabla g\|$, and

$$\|(e^{v\Delta(t-s)} - 1)f\| \leq (1 - e^{-\nu\lambda(t-s)}) \|f\|$$

for some fixed $\lambda > 0$, one has

$$\begin{aligned} \|\Pi_0^\perp [w(t) - w(s)]\| &\leq 2[1 - e^{-\lambda\nu(t-s)}] \|w(s)\| + 2 \sup_{0 \leq r \leq T} \|\nabla w(r)\|^2 |t - s| \\ &\leq c|t - s| [1 + \|\nabla w\|_\infty^2]. \end{aligned}$$

Similarly,

$$\begin{aligned} \|U^{T,\phi}(t) - U^{T,\phi}(s)\| &\leq c|t - s| \|U^{T,\phi}(t)\| + c \|\nabla w\|_\infty \|\nabla U^{T,\phi}\|_\infty |t - s| \\ &\leq c[1 + \|\nabla w\|_\infty^2 + \|\nabla U^{T,\phi}\|_\infty^2] |t - s|. \end{aligned}$$

Also, if

$$R(t) = \Pi_0 w(0) + \int_0^t v \Delta \Pi_0 w(r) - \Pi_0 B(w(r), w(r)) dr,$$

then

$$\|R(t) - R(s)\| \leq c \int_s^t \|w(r)\| dr + c \int_s^t \|w(r)\|^2 dr \leq c[1 + \|w\|_\infty^2] |t - s|$$

Next observe that $\|\Pi B(f, g)\| \leq c\|f\|\|g\|$ and $\|\Pi C(f, g)\| \leq c[\|f\| \|\nabla g\| \wedge \|g\| \|\nabla f\|]$. Thus

$$\begin{aligned} & \|\Pi C(f(t), g(t)) - \Pi C(f(s), g(s))\| \\ & \leq \|\Pi C(f(t) - f(s), g(t))\| + \|\Pi C(f(s), g(t) - g(s))\| \\ & \leq c[\|\nabla g\|_\infty \|f(t) - f(s)\| + \|\nabla f\|_\infty \|g(t) - g(s)\|] \end{aligned}$$

and

$$\begin{aligned} & \|\Pi B(f(t), g(t)) - \Pi B(f(s), g(s))\| \\ & \leq c[\|g\|_\infty \|f(t) - f(s)\| + \|f\|_\infty \|g(t) - g(s)\|]. \end{aligned}$$

We combine these observations in the following lemma:

LEMMA E.1 *Let Π and Π_0 be as above. In the notation of (7.1) and (7.2),*

$$\begin{aligned} \mathcal{H}_1(\Pi_0^\perp w) & \leq c[1 + \|\nabla w\|_\infty^2] \\ \mathcal{H}_1(U^{T,\phi}) & \leq c[1 + \|\nabla w\|_\infty^2 + \|\nabla U^{T,\phi}\|_\infty^2] \\ \mathcal{H}_1(R) & \leq c[1 + \|w\|_\infty^2] \\ \mathcal{H}_1(\Pi B(f, g)) & \leq c[\|f\|_\infty \mathcal{H}_1(g) + \|g\|_\infty \mathcal{H}_1(f)] \\ \mathcal{H}_1(\Pi C(f, g)) & \leq c[\|\nabla f\|_\infty \mathcal{H}_1(g) + \|\nabla g\|_\infty \mathcal{H}_1(f)] \end{aligned}$$

for all $f, g \in \mathbb{L}^2$ smooth enough so that each term on the right-hand side is finite.

Lastly we specialize these estimates to the setting of Proposition 3.7. Let X^ϕ , R , and Y_k^ϕ be as defined in Proposition 3.7 for $w(t)$ and $U^{T,\phi}$ on the interval $[0, T]$. Define $\chi^\phi = \Pi X^\phi$ and $\Upsilon_k^\phi = \Pi Y_k^\phi$. We wish to obtain control of the Lipschitz constants over an interval $[t, T]$ with $t \in (0, T)$.

Using the estimates from Lemma E.1 and the fact that

$$\|\nabla R\|_{\infty,[t,T]} \leq c\|R\|_{\infty,[t,T]} \leq c(1 + \|w\|_{\infty,[t,T]}^2),$$

we obtain

$$\begin{aligned} \mathcal{H}_{1,[t,T]}(\chi^\phi) & \leq c[\mathcal{H}_{1,[t,T]}(U^{T,\phi}) + \|\nabla w\|_{\infty,[t,T]} \mathcal{H}_{1,[t,T]}(U^{T,\phi}) \\ & \quad + \|\nabla U^{T,\phi}\|_{\infty,[t,T]} \mathcal{H}_{1,[t,T]}(\Pi_0^\perp w) + \|\nabla R\|_{\infty,[t,T]} \mathcal{H}_{1,[t,T]}(U^{T,\phi}) \\ & \quad + \|\nabla U^{T,\phi}\|_{\infty,[t,T]} \mathcal{H}_{1,[t,T]}(R)] \\ & \leq c[1 + \|\nabla w\|_{\infty,[t,T]}^4 + \|\nabla U^{T,\phi}\|_{\infty,[t,T]}^4] \end{aligned}$$

and

$$\begin{aligned} \mathcal{H}_{1,[t,T]}(\Upsilon_k^\phi) & \leq c[\mathcal{H}_{1,[t,T]}(U^{T,\phi}) + \|U^{T,\phi}\|_{\infty,[t,T]}^2] \\ & \leq c[1 + \|\nabla w\|_{\infty,[t,T]}^2 + \|\nabla U^{T,\phi}\|_{\infty,[t,T]}^2]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|\chi^\phi\|_{\infty,[t,T]} &\leq c[\|U^{T,\phi}\|_{\infty,[t,T]} + \|\nabla U^{T,\phi}\|_{\infty,[t,T]}\|\nabla w\|_{\infty,[t,T]} \\ &\quad + \|\nabla U^{T,\phi}\|_{\infty,[t,T]}\|R\|_{\infty,[t,T]}] \\ &\leq c[1 + \|\nabla U^{T,\phi}\|_{\infty,[t,T]}^2 + \|\nabla w\|_{\infty,[t,T]}^2] \end{aligned}$$

and $\|\Upsilon_k^\phi\|_{\infty,[t,T]} \leq c\|U^{T,\phi}\|_{\infty,[t,T]}$.

Recall that $\|f\|_{\alpha,[t,T]} = \max(\|f\|_{\infty,[t,T]}, \mathcal{H}_{\alpha,[t,T]}(f))$. Combining the above estimates with Lemma B.1 produces

$$\begin{aligned} \|\chi^\phi\|_{1,[t,T]}, \|\Upsilon_k^\phi\|_{1,[t,T]} &\leq c[1 + \|\nabla U^{T,\phi}\|_{\infty,[t,T]}^4 + \|\nabla w\|_{\infty,[t,T]}^4] \\ &\leq c\left[1 + \|\nabla w\|_{\infty,[t,T]}^4 + \exp\left(\eta \int_0^T \|w(r)\|_1^2 dr\right)\right] \end{aligned}$$

for all indices i , and ϕ with $\|\nabla\phi\| \leq M$. Here $\eta > 0$ is arbitrary but c depends on the choice of η and M . In light of Corollary A.2 and Lemma A.3, which control the right-hand side, we obtain the following result:

LEMMA E.2 *Given an $M > 0$, define $S(M) = \{\phi : \|\nabla\phi\| \leq M\}$. Then for any $T > 0$, $t \in (0, T)$, $p \geq 1$, and $\eta > 0$ there exists a positive constant $c = c(\eta, p, t, \nu, \mathcal{E}_1, M, T)$ such that*

$$\mathbb{E}\left(\sup_{\phi \in S(M)} [\|\chi^\phi\|_{1,[t,T]}^p + \sup_{k \in \mathcal{Z}_*} \|\Upsilon_k^\phi\|_{1,[t,T]}^p]\right) \leq c \exp(\eta \|w(0)\|^2).$$

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