NON-BINARY BRANCHING PROCESS AND NON-MARKOVIAN EXPLORATION PROCESS

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Abstract. We study both a continuous time non-binary Galton−Watson random tree and its exploration (or height) process in the subcritical, critical and supercritical cases. We then renormalize our branching process and exploration process, and take the weak limit as the size of the population tends to infinity.

1. INTRODUCTION

We consider a general continuous time branching process, describing a population where multiple births are allowed, unlike in the paper [2]. We first describe the exploration process, or height process, of the corresponding genealogical tree. We next study the convergence as the population size tends to infinity, of a properly rescaled version of it, towards a reflecting Brownian motion with drift. The difficulty is that we have to deal with a non Markovian exploration process. It had not been described so far in the literature. Taking the large population limit requests new arguments, in comparison with the binary branching situation studied in [2].

We have carefully avoided to make any unnecessary assumption. In particular, we assume that the number of children born at a given birth event has a finite moment of order $2 + \delta$, for some $\delta > 0$ arbitrarily small, and no higher order moment. We hope to be able to treat in a near future the case without this assumption, and study the limit of the genealogical trees in case where the limiting branching process is a continuous state branching process with jumps.

In the supercritical case, as in [2,6], we need to reflect the exploration process below an arbitrary level Γ, in order for this process to accumulate an arbitrary amount of local time at zero. This means killing the population at time Γ. It turns out that for taking the large population limit, reflection is also needed in the critical case. On the other hand, reflection is not required in the subcritical case. In order to be as concise as possible, we study the limit of the exploration process reflected below Γ in the general case, and at the end show how the proof can be done without reflection in the subcritical case.

The paper is organised as follows. Section 2 is devoted to the description of the height curves. In Section 3 we describe the relation between the laws of height processes and non-binary continuous time Galton−Watson

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random trees. Finally, in Section 4 we present the results of convergence of the population process and the height process, in the limit of large populations. In this paper a unique letter C will denote a constant which may differ from line to line.

2. Description of the exploration process

In this section we will describe the exploration process of the non-binary tree associated to a continuous time branching process $(Z_t)_{t \geq 0}$. We fix $p > 0$ and consider a continuous piecewise linear function H from a subinterval of $\mathbb{R}_+$ into $\mathbb{R}_+$, which possesses the following properties: its slope is either p or $-p$; it starts at time $t = 0$ from 0 with the slope p ; whenever $H(t) = 0$, $H'_-(t) = -p$ and $H'_+(t) = p$; H is stopped at the time T_m of its mth return to zero, which is supposed to be finite. We will denote $\mathcal{H}_{p,m}$ the collection of all such functions. We will write $\mathcal{H}_p$ instead of $\mathcal{H}_{p,1}$. We now define a stochastic process whose trajectories belong to $\mathcal{H}_p$ as follows. We choose the slopes of the piecewise linear process H to be ± 2 (*i.e.* $p = 2$). We define the local time accumulated by H at level t up to time s:

$$L_s(t) = \lim_{\varepsilon \mapsto 0} \frac{1}{\varepsilon} \int_0^s \mathbf{1}_{\{t \leq H_r < t+\varepsilon\}} \mathrm{d}r.$$

$L_s(t)$ equals the number of pairs of branches of H which cross the level t between times 0 and s. In other words, $L_s(t)$ equals $1/2$ times the number of visits at the level t. Let $\{V_s,\ s \geq 0\}$ be the cádlág $\{-1, 1\}$-valued process which is such that, s−almost everywhere, $dH_s/ds = 2V_s$. Let $\{\Theta_k,\ k \geq 1\}$ be a sequence of independent and identically distributed (i.i.d) random variables with values in $\mathbb{N}$. Θ_k will be the number of newborn at the k–th birth event, where those events are numbered in the order in which they are explored, see below. Let $\{P_s^+,\ s \geq 0\}$ (resp. $\{P_s^-,\ s \geq 0\}$) be a Poisson process with intensity λ (resp. μ). We assume that the three processes $\{\Theta_k,\ k \geq 1\}$, $\{P_s^+,\ s \geq 0\}$ and $\{P_s^-,\ s \geq 0\}$ are independent.

The exploration process H is defined jointly with the process V by the following equation:

$$H_s = 2 \int_0^s V_r\ \mathrm{d}r,$$

$$V_s = 1 + 2 \int_0^s \mathbf{1}_{\{V_{r^-} = -1\}} \mathrm{d}P_r^+ - 2 \int_0^s \mathbf{1}_{\{V_{r^-} = +1\}} \mathrm{d}P_r^- + 2(L_s(0) - L_{0^+}(0))$$

$$+ 2 \sum_{k > 0, S_k^+ \leq s} \left(L_s\left(H_{S_k^+}\right) - L_{S_k^+}\left(H_{S_k^+}\right) \right) \wedge (\Theta_k - 1),$$

where the $(S_k^+, k \geq 1)$ are the successive jump times of the process

$$\widetilde{P}_s^+ = \int_0^s \mathbf{1}_{\{V_{r^-} = -1\}} \mathrm{d}P_r^+,$$

and where $L_s(t)$ denotes the number of visits to level t by the process H up to time s, $H_0 = 0$ and $V_0 = 1$. For any $k > 0$, $\Theta_k - 1$ denotes the number of reflections of H above the level $H_{S_k^+}$. Recall that Θ_k denotes the number of brothers and sisters born at the kth time of birth. In this work, we assume that Θ satisfies the following condition:

$$(\mathbf{H}): \quad \mathbb{E}\left[\Theta_1^{2+\delta}\right] < \infty, \quad \text{for some } \delta > 0 \text{ arbitrarily small.}$$

We define

$$a = \sum_{\ell \geq 1} \ell \mathbb{P}(\Theta_1 = \ell) \quad \text{and} \quad \zeta^2 = \sum_{\ell \geq 1} (\ell - a)^2 \mathbb{P}(\Theta_1 = \ell),$$

respectively the expectation and the variance of the number of births at each birth event. Let π denote the common law of the random variables $\{\Theta_k,\ k \geq 1\}$. We write Π for the triplet π, λ, μ (*i.e.* $\Pi = \{\pi, \lambda, \mu\}$)
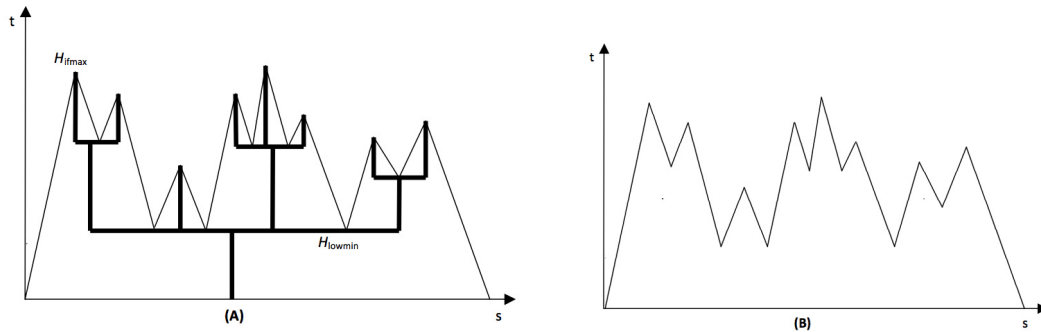
FIGURE 1. (A) The non-binary tree and its associated exploration process. (B) The exploration process. The t-axis is real time as well as exploration height, the s-axis is exploration time.

and we denote by $\mathbb{P}_\Pi$ the law of the random element $(H_s, \ s \geq 0)$ of $\mathcal{H}_2$. The random trajectory which we have constructed is an excursion above zero, provided the process returns to zero, which will always be the case below (see Fig. 1 (B)). We similarly define a law on $\mathcal{H}_{2,m}$ as the concatenation of m i.i.d such excursions. We denote by $\Im$ the set of finite rooted non binary trees which are defined as follows. An ancestor is born at time 0. Until she eventually dies, she produces a random number of offsprings. The same happens to each of her offspring, the offspring of her offspring, *etc.*, until eventually the population dies out (assuming for simplicity that we are in the subcritical case). We denote by $\Im_m$ the set of forests which are the union of m elements of $\Im$, *i.e.* the set of forests containing m trees. There is a well-known bijection between trees and exploration processes. Under the curve representing an element $H \in \mathcal{H}_p$, we can draw a tree as follows. The height H_{ifmax} of the leftmost local maximum of H is the lifetime of the ancestor and the height H_{lowmin} of the lowest nonzero local minimum is the birth time of the first offsprings of the ancestor. If there is no such local minimum, the ancestor dies childless. We draw a horizontal line at level H_{lowmin}. H makes $\Theta_1 + 1$ excursions above H_{lowmin}. The leftmost excursion is used to represent the fate of the ancestor and of the rest of her progeny, excluding the children born at the first birth event and their progeny. The Θ_1 others excursions describe the fate of the first offsprings and their progeny. If there is no other local minimum of H to the left or to the right of the first explored one, then there is no further birth: we draw a vertical line up to the unique local maximum, whose height is a death time. Continuing until there is no further local minimum/maximum to explore, this procedure defines a bijection Φ_p from $\mathcal{H}_p$ into $\Im$ (see Fig. 1). Repeating the same construction m times, we extend Φ_p to a bijection between $\mathcal{H}_{p,m}$ and $\Im_m$. Note that the horizontal distances between the vertical branches in the tree representation of the exploration process are arbitrary. See Figure 1(A).

To the exploration process H, we associate the continuous-time Galton−Watson tree (which is a random element of $\Im$) with the same law π and the same pair of parameters (μ, λ) as follows. The lifetime of each individual is exponential with parameter μ. The birth events come according to a Poisson process with rate λ and at each time of birth, a random number of children with law π are born. The behaviors of the various individuals are i.i.d. This defines a probability measure $\mathbb{Q}_\Pi$ on $\Im$. We use the same notation to denote the law on $\Im_m$ of m i.i.d random trees with $\mathbb{Q}_\Pi$ as their common law.

In the supercritical case, the exploration process defined above does not come back to 0 a.s. To overcome this difficulty, we use a trick which is due to Delmas [6], and reflect the process H^Γ below an arbitrary level $\Gamma > 0$ (which amounts to kill the whole population at time Γ). The height process $H^\Gamma = \{H_s^\Gamma, \ s \geq 0\}$ reflected below Γ is defined as above, with the addition of the rule that whenever the process reaches the level Γ, it stops and starts immediately going down with slope $-p$ for an exponential duration of time with expectation $1/\lambda$. Again, the process stops at its mth return to zero, when m trees have been explored. The reflected process H^Γ comes back to zero almost surely. Indeed, let A_n^Γ denote the event "H^Γ does not reach zero during its n first descents".

Since the levels of the local maxima are bounded by Γ, clearly $\mathbb{P}(A_n^\Gamma) \leq (1 - \exp(-\lambda\Gamma))^n$, which goes to zero as $n \to \infty$. Hence the result, see [2].

In this case, the exploration process H^Γ is defined jointly with the process V by the following equation:

$$H_s^\Gamma = 2 \int_0^s V_r \, \mathrm{d}r,$$

$$V_s = 1 + 2 \int_0^s \mathbf{1}_{\{V_{r^-} = -1\}} \mathrm{d}P_r^+ - 2 \int_0^s \mathbf{1}_{\{V_{r^-} = +1\}} \mathrm{d}P_r^- + 2(L_s^\Gamma(0) - L_{0^+}^\Gamma(0))$$

$$- 2L_s^\Gamma(\Gamma^-) + 2 \sum_{k>0, S_k^+ \leq s} \left(L_s^\Gamma(H_{S_k^+}^\Gamma) - L_{S_k^+}^\Gamma\left(H_{S_k^+}^\Gamma\right) \right) \wedge (\Theta_k - 1),$$

where $L_s^\Gamma(t)$ denotes the number of visits to level t by the process H^Γ up to time s.

For each $\Gamma \in (0, +\infty)$, and any sub-collection $\Pi = \{\pi, \lambda, \mu\}$, denotes by $\mathbb{P}_{\Pi,\Gamma}$ the law of the process H^Γ. Define $\mathbb{Q}_{\Pi,\Gamma}$ to be the law of the (π, λ, μ) Galton–Watson tree, killed at time $t = \Gamma$ (*i.e.* all individuals alive at time Γ^- are killed at time Γ).

This reflection below an arbitrary level Γ will be necessary both in the critical and in the supercritical cases. For our large population convergence result, we shall treat the subcritical case without this reflection in the last subsection of the paper.

3. CORRESPONDENCE OF LAWS

The aim of this section is to prove that for any sub-collection $\Pi = \{\pi, \lambda, \mu\}$ and $\Gamma \in (0, +\infty)$ [including possibly $\Gamma = +\infty$ in the subcritical case], $\mathbb{P}_{\Pi,\Gamma} \, \Phi_p^{-1} = \mathbb{Q}_{\Pi,\Gamma}$. Let us state some basic results for homogeneous Poisson processes, which will be useful in the sequel.

3.1. Preliminary results

Let $(T_k)_{k\geq 0}$ be a Poisson point process on $\mathbb{R}_+$ with intensity λ. This means that $T_0 = 0$ and $(T_{k+1} - T_k, k \geq 0)$ are i.i.d. exponential random variables with mean $1/\lambda$. Let $(N_t, t \geq 0)$ be the counting process associated with T, that is, for all $t \geq 0$,

$$N_t = \sup\{k \geq 0, T_k \leq t\}.$$

The following result is well-known and elementary.

Lemma 3.1. *Let M be a nonnegative random variable independent of T, and define*

$$R_M = \sup_{k\geq 0}\{T_k, T_k \leq M\}.$$

Then $M - R_M \stackrel{(d)}{=} V \wedge M$, where V and M are independent, and V has an exponential distribution with mean $1/\lambda$.

Moreover, on the event $\{R_M > s\}$, the conditional law of $N_{R_M^-} - N_s$ given R_M is Poisson with parameter $\lambda(R_M - s)$.

In addition, we have the following result, which is Lemma 3.2 in [2].

Lemma 3.2. *Let $T = (T_k)_{k\geq 0}$ be a Poisson point process on $\mathbb{R}_+$ with intensity λ, and let M be a positive random variable which is independent of T. Consider the integer-valued random variable K such that $T_K = R_M$ and a second Poisson point process $T' = (T_k')_{k\geq 0}$ with intensity λ, which is jointly independent of the first and of M. Then $\bar{T} = (\bar{T}_k)_{k\geq 0}$, defined by*

$$\bar{T}_k = \begin{cases} T_k & if \quad k < K, \\ \\ T_K + T'_{k-K+1} & if \quad k \geq K, \end{cases}$$

is a Poisson point process on $\mathbb{R}_+$ with intensity λ, which is independent of R_M.

3.2. Basic theorem

Let $\{U_\ell,\ \ell \geq 1\}$ and $\{V_\ell,\ \ell \geq 1\}$ be two sequences of independent and identically distributed (i.i.d) exponential random variables with means $1/\mu$ and $1/\lambda$, respectively. Let $(T_k^\ell, k \geq 1, \ell \geq 1)$ be a family of mutually independent Poisson processes with intensity λ. In the same way as Section 2, we introduce the sequence $\{\Theta_\ell,\ \ell \geq 1\}$ of i.i.d random variables with law π. We assume that the processes $(T_k^\ell, k \geq 1, \ell \geq 1)$, $\{U_\ell,\ \ell \geq 1\}$ and $\{\Theta_\ell,\ \ell \geq 1\}$ are mutually independent.

We are now in a position to prove the next theorem, which states a one-to-one correspondence between the tree associated with the exploration process H^Γ defined in Section 2, and a continuous-time non binary Galton−Watson tree with the same law π and the same pair of parameters (μ, λ), killed at time Γ.

Theorem 3.3. *For any Π and $\Gamma \in (0, +\infty)$ (including the case $\Gamma = +\infty$ when $\lambda < \mu$), the following representation holds*

$$\mathbb{P}_{\Pi,\Gamma}\ \Phi_p^{-1} = \mathbb{Q}_{\Pi,\Gamma}.$$

Proof. The individuals making up the population represented by the tree whose law is $\mathbb{Q}_\Pi$ are labeled $\ell = 1, 2, \dots$ 1 is the ancestor of the whole family. The subsequent individuals will be identified below. We will show that this tree is 'explored' by a process whose law is precisely $\mathbb{P}_\Pi$. U_ℓ will be the lifetime of individual ℓ. For any $\ell \geq 1$, the birth times of the offsprings of individual ℓ are $\{T_k^\ell,\ 1 \leq k \leq K_\ell\}$, where $K_\ell = \sup\{k,\ T_k^\ell \leq U_\ell\}$.

If ℓ is not the sister of an already explored individual born at the same time, *i.e.* if $m_{\ell-1} \notin \{m_1, m_2, \dots, m_{\ell-2}\}$, then we define $\Delta_\ell = \Theta_\ell - 1$ the number of sisters of individual ℓ born at time $m_{\ell-1}$. If ℓ is the sister of individual $j < \ell$, then we let $\Delta_\ell = \Delta_j - 1$.

Step 1. We start from the initial time $t = 0$ and climb up to level $M_1 = U_1 \wedge \Gamma$. We go down from M_1 until we find the most recent point of the Poisson process (T_k^1) (recall that this process gives the birth times of the offsprings of individual 1). By Lemma 3.1, we have descended a height $V_1 \wedge M_1$. We hence reach the level $m_1 = (M_1 - V_1) \vee 0$. If $m_1 = 0$, we stop, else we turn to the next step.

Step 2. We assign label 2 to the first offspring of the last birth event of offsprings of individual 1, born at time m_1 and we let $\Delta_2 = \Theta_2 - 1$ denote the number of unexplored sisters of individual 2 born at the same time her. Let us define $(\bar{T}_k^2)$ by

$$\bar{T}_k^2 = \begin{cases} T_k^1 & \text{if} \quad k < K_1, \\ T_{K_1}^1 + T_{k-K_1+1}^2 & \text{otherwise;} \end{cases}$$

where K_1 is such that $T_{K_1}^1 = m_1$.

Thanks to Lemma 3.2, $(\bar{T}_k^2)$ is a Poisson process with intensity λ on $\mathbb{R}_+$, which is independent of m_1 and in fact also of (U_1, V_1).

Starting from m_1, the exploration process climbs up to level $M_2 = (m_1 + U_2) \wedge \Gamma$. Starting from level M_2, if $\Delta_2 = 0$, we go down a height $M_2 \wedge V_2$, to find the most recent point of the Poisson process $(\bar{T}_k^2)$. At this time we are at level $m_2 = (M_2 - V_2) \vee 0$. If however $\Delta_2 \geq 1$, we go down a height $V_2 \wedge (M_2 - m_1)$ and in this case we are at level $m_2 = (M_2 - V_2) \vee m_1$. If $m_1 = m_2$, we change the value of Δ_2, and let it be equal to $\Delta_2 - 1$. If $m_2 = 0$, we stop. Otherwise we continue.

Suppose we have made $\ell - 1$ steps and $m_{\ell-1} \geq 0$, $\ell \geq 3$.

Step ℓ. We start from $m_{\ell-1}$ which is the birth time of individual ℓ. Note that by then for all $2 \leqslant j \leqslant \ell$, Δ_j is the number of sisters of individual j who still remain to be explored. We now define

$$\bar{T}_k^\ell = \begin{cases} \bar{T}_k^{\ell-1} & \text{if} \quad k < K_{\ell-1}, \\ \bar{T}_{K_{\ell-1}}^{\ell-1} + \bar{T}_{k-K_{\ell-1}+1}^\ell & \text{otherwise;} \end{cases}$$

Then $(\bar{T}_k^\ell)$ is a Poisson point process with intensity λ on $\mathbb{R}_+$ and is independent of $(m_1, M_1, \dots, m_{\ell-1}, M_{\ell-1})$.
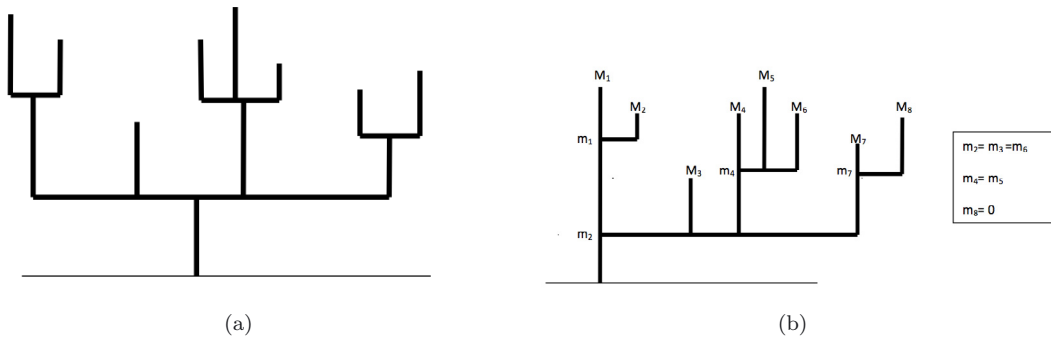
FIGURE 2. Two equivalent ways of representing a non binary tree.

Starting from $m_{\ell-1}$, the height process climbs up to level $M_\ell = (m_{\ell-1} + U_\ell) \wedge \Gamma$, which is the time of death of individual ℓ. We set

$$\ell^* = \begin{cases} \sup\{2 \leqslant j \leqslant \ell, \ \Delta_j > 0\}, & \text{if} \quad \inf_{2 \leqslant j \leqslant \ell} \ \Delta_j > 0, \\ 1, & \text{otherwise.} \end{cases}$$

Note that, if $\Delta_\ell > 0$, $\ell^* = \ell$. Coming down from level M_ℓ, if $\ell^* = 1$, we wait a time $V_\ell \wedge M_\ell$, to find the most recent point of the Poisson process $(\bar{T}_k^\ell)$. At this time we are at level $m_\ell = (M_\ell - V_\ell) \vee 0$. If however $\ell^* \geq 2$ we go down a height $V_\ell \wedge (M_\ell - m_{\ell^*-1})$ and in this case we are at level $m_\ell = (M_\ell - V_\ell) \vee m_{\ell^*-1}$. If $m_\ell = m_{\ell^*-1}$, we change the value of Δ_{ℓ^*}, and let it be equal to $\Delta_{\ell^*} - 1$. See Figure 2.

Since either we have a reflection at level Γ or we are in the subcritical case, zero is reached a.s. after a finite number of iterations. It is clear that the random variables M_i and m_i fully determine the law $\mathbb{Q}_{\Pi,\Gamma}$ of the non binary tree killed at time $t = \Gamma$ and they have both the same joint distribution as the levels of the successive local minima and maxima of the process H^Γ under $\mathbb{P}_{\Pi,\Gamma}$, see, *e.g.* [2]. □

4. WEAK CONVERGENCE

4.1. Renormalization

Let $x > 0$ be arbitrary, and let $N \geqslant 1$ be an integer which will eventually go to infinity. Let $(Z_t^{N,x})_{t \geq 0}$ denote the branching process which describes the number of offsprings at time t of $[Nx]$ ancestors in the population with birth rate $\lambda_N = N\sigma^2/2a + \alpha/a$ and death rate $\mu_N = N\sigma^2/2 + \beta$, where $\alpha \geq 0$, $\beta \geq 0$ and $\sigma > 0$. In this population, the number of children at each birth event is a random variable that has the law π, which does not depend upon N. Recall that a denotes the expectation of π and ζ^2 its variance. We now define the rescaled continuous time process

$$X_t^{N,x} := N^{-1} Z_t^{N,x}. \tag{4.1}$$

In particular, we have

$$X_0^{N,x} = [Nx]/N \longrightarrow x \quad as \quad N \to +\infty.$$

Let $H^{N,\Gamma}$ be the exploration process associated to $\{Z_t^{N,x}, \ 0 \leq t \leq \Gamma\}$ defined in the same way as previously, but with slopes $\pm 2N$, and where λ, μ are replaced by λ_N and μ_N to be specified below. We define also $L_s^{N,\Gamma}(t)$,

the local time accumulated by $H^{N,\Gamma}$ at level t up to time s, as

$$L_s^{N,\Gamma}(t) = \frac{4}{\kappa^2 b} \lim_{\varepsilon \mapsto 0} \frac{1}{\varepsilon} \int_0^s \mathbf{1}_{\{t \le H_r^{N,\Gamma} < t+\varepsilon\}} \mathrm{d}r, \tag{4.2}$$

where $b = \frac{1}{2a}(a + a^2 + \zeta^2)$ and $\kappa^2 = \sigma^2 b$. The motivation of the factor $4/\kappa^2 b$ will be clear after we have taken the limit as $N \to +\infty$. $L_s^{N,\Gamma}(t)$ equals $4/N\kappa^2 b$ times the number of pairs of t-crossings of $H^{N,\Gamma}$ between times 0 and s. In other words, $L_s^{N,\Gamma}(t)$ equals $(1/2) * (4/N\kappa^2 b)$ times the number of visits at the level t. Note that this process is neither right- nor left-continuous as a function of s.

4.2. Tightness criteria in function spaces

We shall start with the basic tightness criterion for random processes on the space of continuous functions $C([0, +\infty))$. For $T > 0$, we define $w_{x,T}(.)$ the modulus of continuity of $x \in C([0, +\infty))$ on the interval $[0, T]$ by

$$w_{x,T}(\rho) = w_T(x, \rho) = \sup_{|s-t| \le \rho, \ s,t \le T} |x(s) - x(t)|, \quad \rho > 0. \tag{4.3}$$

Consider now a sequence $\{X^n, n \ge 1\}$ of random processes with trajectories in $C([0, +\infty))$. The following proposition follows from Theorem 7.3 in [4].

Proposition 4.1. *The sequence $\{X^n, n \ge 1\}$ is tight in $C([0, \infty))$ iff the two following conditions hold:*

(i) *for each $\eta \ge 0$, there exist some d and some n_0 such that*

$$\mathbb{P}(|X^n(0)| \ge d) \le \eta, \quad n \ge n_0.$$

(ii) *for each $\epsilon > 0$, $T > 0$,*

$$\lim_{\rho \to 0} \limsup_{n \to \infty} \mathbb{P}(w_T(X^n, \rho) \ge \epsilon) = 0.$$

Let us present a sufficient condition for tightness which will be useful below. Consider a sequence $\{X_t^n, t \ge 0\}_{n \ge 1}$ of one-dimensional semi-martingales, which is such that for each $n \ge 1$,

$$X_t^n = X_0^n + \int_0^t \varphi_n(X_s^n)\mathrm{d}s + M_t^n, \quad t \ge 0;$$

where for each $n \ge 1$, M^n is a locally square-integrable martingale such that

$$\langle M^n \rangle_t = \int_0^t \psi_n(X_s^n)\mathrm{d}s, \quad t \ge 0;$$

φ_n and ψ_n are Borel measurable functions with values into $\mathbb{R}$ and $\mathbb{R}_+$ respectively. We define $V_t^n = X_0^n + \int_0^t \varphi_n(X_s^n)\mathrm{d}s$. Since our martingales $\{M_t^n, t \ge 0\}$ will be discontinuous, we need to consider their trajectories as elements of $D([0, \infty))$, the space of functions from $[0, \infty)$ into $\mathbb{R}$ which are right continuous and have left limits at any $t > 0$ (as usual such a function is called cádlág). We briefly write $\mathbb{D}$ for the space of adapted, cádlág stochastic processes. We shall always equip the space $D([0, \infty))$ with the Skorohod topology, for the definition of which we refer the reader to Billingsley [4] or Joffe, Métivier [8]. The following statement can be deduced from Theorem 13.4 and 16.10 of [4].

Proposition 4.2. *A sufficient condition for the above sequence $\{X_t^n, t \ge 0\}_{n \ge 1}$ of semi-martingales to be tight in $D([0, \infty))$ is that both*

the sequence of r.v.'s $\{X_0^n, n \ge 1\}$ is tight;

and for some $p > 1$,

$$\forall T > 0, \text{ the sequence of r.v.'s } \left\{ \int_0^T [|\varphi_n(X_s^n)| + \psi_n(X_t^n)]^p \mathrm{d}t, n \geq 1 \right\} \text{ is tight.}$$

Those conditions imply that both the bounded variation parts $\{V^n, n \geq 1\}$ and the martingale parts $\{M^n, n \geq 1\}$ are tight, and that the limit of any converging subsequence of $\{V^n\}$ is a.s. continuous.

If moreover, for any $T > 0$, as $n \longrightarrow \infty$,

$$\sup_{0 \leq t \leq T} |M_t^n - M_{t-}^n| \longrightarrow 0 \text{ in probability,}$$

then any limit X of a converging subsequence of the original sequence $\{X^n\}_{n \geq 1}$ is a.s. continuous.

In particular, the space $C([0, \infty))$ is closed in $D([0, \infty))$ equipped with the Skorohod topology. The next Lemma follows from considerations which can be found in [4].

Lemma 4.3. *Let $X_n, Y_n \in D([0, \infty))$, $n \geq 1$ and $X, Y \in C([0, \infty))$ be such that:*

(1) *for all $n \geq 1$, the function $t \to Y_n(t)$ is increasing;*
(2) *$X_n \to X$ and $Y_n \to Y$, both locally uniformly.*

Then Y is increasing and

$$\int_0^t X_n(s)\mathrm{d}Y_n(s) \to \int_0^t X(s)\mathrm{d}Y(s), \text{ locally, uniformly, in, } t \geq 0.$$

The following is a consequence of Theorem 13.5 of [4].

Proposition 4.4. *If $\{X_t^n, t \geq 0\}_{n \geq 1}$ and $\{Y_t^n, t \geq 0\}_{n \geq 1}$ are two tight sequences of random elements of $D([0, \infty))$ and $C([0, \infty))$ respectively, then $\{X_t^n + Y_t^n, t \geq 0\}_{n \geq 1}$ is tight in $D([0, \infty))$.*

To $x \in D([0, \infty); \mathbb{R})$, we associate for each $T > 0$ and $\rho > 0$ the quantity

$$\bar{w}_{x,T}(\rho) = \bar{w}_T(x, \rho) = \inf \max_{0 \leq i < n} \sup_{t_i \leq s < t \leq t_{i+1}} |x(t) - x(s)|,$$

where the infimum is taken over the set of all increasing sequences $0 = t_0 < t_1 < \ldots < t_n = T$ with the property that $\inf_{0 \leq i < n} |t_{i+1} - t_i| \geq \rho$. We state another tightness criterion, which is Theorem 13.2 from [4].

Proposition 4.5. *The sequence $\{X^n, n \geq 1\}$ is tight in $D([0, \infty); \mathbb{R})$ iff the two following conditions hold:*

(i) *for each $t \geq 0$, $\{X_t^n - X_{t-}^n, n \geq 1\}$ is tight in $\mathbb{R}$;*
(ii) *for each $\epsilon > 0$,*

$$\lim_{\rho \to 0} \limsup_{n \to \infty} \mathbb{P}(\bar{w}_T(X^n, \rho) \geq \epsilon) = 0.$$

One can compare $\bar{w}_{x,T}(\rho)$ with $w_{x,T}(\rho)$. Consider the maximum (absolute) jump in x:

$$j_T(x) = \sup_{0 < t \leq T} |x(t) - x(t-)|; \tag{4.4}$$

the supremum is achieved because only finitely many jumps can exceed a given positive number.

We have

$$w_{x,T}(\rho) \leq 2\bar{w}_{x,T}(\rho) + j_T(x) \tag{4.5}$$

and

$$\bar{w}_{x,T}(\rho) \leq w_{x,T}(2\rho)$$

(see Sect. 12, p. 123 in [4]).

As is well-know that, C-tightness implies D-tightness, in the sense that

Corollary 4.6. *A sufficient condition for the sequence* $\{X^n, n \geq 1\}$ *to be tight in* $D([0, \infty); \mathbb{R})$ *is that:*

(i) *for each* $\eta \geq 0$, *there exist some* d *and some* n_0 *such that*

$$\mathbb{P}(|X^n(0)| \geq d) \leqslant \eta, \quad n \geq n_0,$$

(ii) *for each* $T \geq 1$, $\epsilon > 0$,

$$\lim_{\rho \to 0} \limsup_{n \to \infty} \mathbb{P}(w_T(X^n, \rho) \geq \epsilon) = 0$$

and under that condition any limit of a converging subsequence is continuous (*Corollary of Thm.* 13.4 *in* [4]).

4.3. Tightness and Weak convergence of $X^{N,x}$

The following result describes the limit of the sequence of processes $\{X^{N,x}, N \geq 1\}$ defined in (4.1). The continuous time Galton−Watson process $\{X_t^{N,x}, t \geq 0\}$ is a Markov process with values in the set $E_N = \{k/N, k \geq 1\}$ with its infinitesimal generator given by:

$$Q^N f(y) = Ny\left(\frac{N\sigma^2}{2a} + \frac{\alpha}{a}\right)\left[\sum_{\ell \geq 1} p_\ell f(y + \frac{\ell}{N}) - f(y)\right] + Ny\left(\frac{N\sigma^2}{2} + \beta\right)\left[f\left(y - \frac{1}{N}\right) - f(y)\right],$$

for $f : E_N \longrightarrow \mathbb{R}$ such that

$$|f(y)| \leq C(1 + y^2), \ y \in E_N \tag{4.6}$$

and where p_ℓ is the probability that there are ℓ simultaneous births. Consequently for any $f \in \mathcal{C}(\mathbb{R})$ satisfying (4.6),

$$M_t^{f,N} := f(X_t^{N,x}) - f(X_0^{N,x}) - \int_0^t Q^N f(X_s^{N,x}) \mathrm{d}s$$

is a local martingale. Assuming that f is of class $\mathcal{C}^2(\mathbb{R})$ satisfying (4.6) and applying successively the above formula to the cases $f(y) = y$ and $f(y) = y^2$, we get that

$$X_t^{N,x} = X_0^{N,x} + (\alpha - \beta)\int_0^t X_s^{N,x}\mathrm{d}s + M_t^{(1),N} \tag{4.7}$$

and

$$\left(X_t^{N,x}\right)^2 = \left(X_0^{N,x}\right)^2 + 2(\alpha - \beta)\int_0^t \left(X_s^{N,x}\right)^2\mathrm{d}s + \left(\frac{\sigma^2 N + 2\alpha}{2aN}\sum_{\ell \geq 1}\ell^2 p_\ell + \frac{\sigma^2 N + 2\beta}{2N}\right)\int_0^t X_s^{N,x}\mathrm{d}s + M_t^{(2),N}, \tag{4.8}$$

where $\{M_t^{(1),N}, t \geq 0\}$ and $\{M_t^{(2),N}, t \geq 0\}$ are local martingales. Now combining (4.7), (4.8) and the Itô formula, we deduce that

$$\langle M^{(1),N}\rangle_t = \left(\kappa^2 + \frac{\frac{\alpha}{a}(\zeta^2 + a^2) + \beta}{N}\right)\int_0^t X_s^{N,x}\mathrm{d}s, \tag{4.9}$$

where $\kappa^2 = \frac{\sigma^2}{2a}(a + a^2 + \zeta^2)$.

We will establish some lemmas to prove the tightness of the process $X^{N,x}$.

Lemma 4.7. *For all* $T > 0$, *there exists a constant* $C > 0$ *such that for all* $N \geq 1$,

$$\sup_{0 \leq t \leq T} \mathbb{E}(X_t^{N,x}) \leq C.$$

Proof. Let $(\tau_n, n \geq 0)$ be a sequence of stopping times such that τ_n tends to infinity as n goes to infinity and for any n, $(M_{t \wedge \tau_n}^{(1),N})$ is a martingale and $X_{t \wedge \tau_n}^{N,x} \leq n$. Taking the expectation on both sides of equation (4.7) at times $t \wedge \tau_n$, we obtain

$$\mathbb{E}(X_{t \wedge \tau_n}^{N,x}) = \mathbb{E}(X_0^{N,x}) + (\alpha - \beta)\mathbb{E}\left(\int_0^{t \wedge \tau_n} X_s^{N,x} ds\right),$$

it follows that

$$\mathbb{E}(X_{t \wedge \tau_n}^{N,x}) \leq \mathbb{E}(X_0^{N,x}) + (\alpha - \beta)^+ \int_0^t \mathbb{E}(X_{s \wedge \tau_n}^{N,x}) ds.$$

From Gronwall and Fatou Lemmas, we deduce that for all $T > 0$ there exists a constant $C > 0$ such that

$$\sup_{N \geq 1} \sup_{0 \leq t \leq T} \mathbb{E}(X_t^{N,x}) \leq C. \qquad \square$$

We shall also need below the

Lemma 4.8. *For any $T > 0$,*

$$\sup_{N \geq 1} \mathbb{E}\left[\sup_{0 \leq t \leq T} (X_t^{N,x})^2\right] < \infty.$$

Proof. Since from (4.7), we have

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} (X_t^{N,x})^2\right] \leq 3\mathbb{E}\left(\left|X_0^{N,x}\right|^2\right) + 3(\alpha - \beta)^2 T \int_0^T \mathbb{E}\left[\sup_{0 \leq s \leq t} (X_s^{N,x})^2\right] ds + 3\mathbb{E}\left[\sup_{0 \leq t \leq T}\left(M_t^{(1),N}\right)^2\right].$$

Using Doob's inequality, we obtain

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} (X_t^{N,x})^2\right] \leq 3\mathbb{E}\left(\left|X_0^{N,x}\right|^2\right) + 3(\alpha - \beta)^2 T \int_0^T \mathbb{E}\left[\sup_{0 \leq s \leq t} (X_s^{N,x})^2\right] ds + 3C\mathbb{E}\left(\langle M^{(1),N}\rangle_T\right).$$

However, from (4.9) and Lemma 4.7, we have that

$$\mathbb{E}(\langle M^{(1),N}\rangle_T) \leq \left(\kappa^2 + \frac{\frac{\alpha}{a}(\sigma^2 + a^2) + \beta}{N}\right) CT = CT$$

for all $T > 0$. The above computations, combined with Gronwall's Lemma, lead to

$$\sup_{N \geq 1} \mathbb{E}\left[\sup_{0 \leq t \leq T} (X_t^{N,x})^2\right] < \infty. \qquad \square$$

Recall that, C denotes a constant which may differ from one line to the next.

Corollary 4.9. $\{M_t^{(1),N}, t \geq 0\}$ *and* $\{M_t^{(2),N}, t \geq 0\}$ *are in fact martingales.*

It now follows from Proposition 4.2, (4.7), (4.9), Lemma 4.8 and the fact $X_0^{N,x} \longrightarrow x$ that $\{X^{N,x}\}_{N \geq 1}$ is tight in $D([0, \infty))$.

Standard arguments exploiting (4.7) and (4.8) now allow us to deduce the convergence of the mass processes (for a detailed proof, see *e.g.* Thm. 18, p. 376 in [11]).

Proposition 4.10. *We have $X^{N,x} \Rightarrow X^x$ as $N \to \infty$ for the topology of locally uniform convergence, where X is the unique solution of the following Feller SDE:*

$$X_t^x = x + (\alpha - \beta)\int_0^t X_s^x ds + \kappa \int_0^t \sqrt{X_s^x} dW_s$$

where W is a standard Brownian motion.

4.4. Tightness and weak convergence of $H^{N,\Gamma}$

4.4.1. Some preliminary results on Galton−Watson branching processes

In this section, we state some results on Galton−Watson branching processes which will be useful in checking tightness of $H^{N,\Gamma}$ (the exploration process defined in Sect. 4.1). To do this, consider the branching process in continuous time $(Z_t^{N,\frac{1}{N}})_{t\geq 0}$ which describes the number of descendants alive at time t of a unique ancestor born at time zero, whose progeny is killed at time $t = \Gamma$, where Γ is a given constant. Every individual in this population, independently of the others, lives for an exponential time with parameter μ_N. The birth events happen according to a Poisson process with rate λ_N and at each time of birth, the random number of offsprings has the law π, which does not depend upon N. Recall that the parameters μ_N and λ_N were defined in Section 4.1. In what follows, in order to simplify our notations, we will write Z_t^N instead of $Z_t^{N,\frac{1}{N}}$. We define the length of the genealogical tree up to time Γ

$$S_N^\Gamma = \int_0^\Gamma Z_t^N \mathrm{d}t.$$

Since the births along this tree are occurring at rate λ_N, then the total number of individuals born before time Γ, denoted $\Lambda_{0,\Gamma}^N$, satisfies

$$\mathbb{E}\left(\Lambda_{0,\Gamma}^N \middle| S_N^\Gamma\right) = \lambda_N S_N^\Gamma a, \quad \text{where} \quad a = \sum_{\ell \geq 1} \ell \pi(\ell). \tag{4.10}$$

We first prove

Lemma 4.11. *For any $\Gamma > 0$, there exists a constant $C(\Gamma)$ such that*

$$\mathbb{E}(\Lambda_{0,\Gamma}^N) \leqslant C(\Gamma)N.$$

Proof. We have from (4.10)

$$\mathbb{E}(\Lambda_{0,\Gamma}^N) = a\mathbb{E}(\lambda_N S_N^\Gamma) = a\lambda_N \mathbb{E}\left(\int_0^\Gamma Z_t^N \mathrm{d}t\right) \tag{4.11}$$

However, from (4.1) and (4.7), we have that

$$\mathbb{E}(Z_t^N) = 1 + (\alpha - \beta)\int_0^t \mathbb{E}(Z_s^N)\mathrm{d}s,$$

it is easy to see that

$$\mathbb{E}(Z_t^N) = \exp\left[(\alpha - \beta)t\right].$$

Hence,

$$\mathbb{E}\left(\int_0^\Gamma Z_t^N \mathrm{d}t\right) = (\alpha - \beta)^{-1}\left(\exp\left[(\alpha - \beta)\Gamma\right] - 1\right) = C(\Gamma) \tag{4.12}$$

Now combining (4.11) and (4.12), we deduce that for any $N \geq 1$

$$\mathbb{E}(\Lambda_{0,\Gamma}^N) \leqslant C(\Gamma)N,$$

since $\lambda_N = N\sigma^2/2a + \alpha/a$. $\qquad\square$

In the just described population, we consider another branching process in continuous time $(\bar{Z}_t^N)_{t \geq t' > 0}$ which describes the number of individuals alive at time t who are the descendants of a unique ancestor born at time t', whose progeny is killed at time $t = \Gamma$. In this subpopulation, which starts with a unique ancestor born at time t', we denote by $\Lambda_{t',\Gamma}^N$, the total number of individuals born before time Γ. Thus, it is easy to see that $\Lambda_{t',\Gamma}^N$ is stochastically dominated by $\Lambda_{0,\Gamma}^N$.

In the same way as done in Section 2, we denote for any $k \geq 1$, Θ_k (resp. Υ_k) the number of newborn at the kth birth event (resp. the time of this birth event). We define U_j^N to be the lifetime of individual j. Note that $\{U_j^N, \ j \geq 1\}$ is a sequence of i.i.d random variables and is independent of $\Lambda_{\Upsilon_k,\Gamma}^N$. Recall that $U_1^N \sim \mathcal{E}(\mu_N)$.

Let $(T_k^N)_{k \geq 1}$ be the sequence of i.i.d random variables defined by

$$T_k^N = \sum_{j=1}^{\Theta_k} \Delta_{k,j}^N, \tag{4.13}$$

which is the time it takes to explore the total progeny of Θ_k individuals born at time Υ_k, where $\{\Delta_{k,j}^N, \ k \geq 1, j \geq 1\}$ is a sequence of i.i.d random variables, and $\Delta_{k,j}^N$ is the exploration time of the total progeny of the jth individual born at time Υ_k. Note first that

$$\Delta_{k,1}^N \overset{(d)}{=} \sum_{i=1}^{\Lambda_{\Upsilon_k,\Gamma}^N} \frac{U_i^N}{N}. \tag{4.14}$$

Note also that the two sequences $\{\Delta_{k,j}^N, \ k \geq 1, j \geq 1\}$ and $\{\Theta_k, \ k \geq 1\}$ are independent. Now, using Wald's identity, Lemma 4.11, (4.14) and the fact that $\Lambda_{\Upsilon_k,\Gamma}^N$ is stochastically bounded by $\Lambda_{0,\Gamma}^N$, we obtain

Corollary 4.12. *For any $\Gamma > 0$, there exists a constant $C'(\Gamma)$ such that for all $k \geq 1$, $1 \leq j \leq \Theta_k$*

$$\mathbb{E}(\Delta_{k,j}^N) \leqslant \frac{C'(\Gamma)}{N}$$

Thanks to these results, we are in position to study the asymptotic property of $H^{N,\Gamma}$.

4.4.2. Tightness and Weak convergence of $H^{N,\Gamma}$

In this section, we will need to write precisely the evolution of $\{H_s^{N,\Gamma}, \ s \geq 0\}$, the height process of the forest of trees representing the population $\{Z_t^{N,x}, \ 0 \leq t \leq \Gamma\}$. To this end, let $\{V_s^N, \ s \geq 0\}$ be the cádlág $\{-1,1\}$-valued process which is such that, $s-$almost everywhere, $dH_s^{N,\Gamma}/ds = 2NV_s^N$.

The $(\mathbb{R}_+ \times \{-1,1\})$-valued process $\{(H_s^{N,\Gamma}, V_s^N), \ s \geq 0\}$ solves the SDE

$$H_s^{N,\Gamma} = 2N \int_0^s V_r^N \mathrm{d}r,$$

$$V_s^N = 1 + 2 \int_0^s \mathbf{1}_{\{V_{r-}^N = -1\}} \mathrm{d}P_r^{N,+} - 2 \int_0^s \mathbf{1}_{\{V_{r-}^N = +1\}} \mathrm{d}P_r^{N,-} + \frac{\kappa^2 b N}{2} \left(L_s^{N,\Gamma}(0) - L_{0+}^{N,\Gamma}(0) \right)$$

$$- \frac{\kappa^2 b N}{2} L_s^{N,\Gamma}(\Gamma^-) + 2N \sum_{k > 0, S_k^{N,+} \leq s} \left(\frac{\kappa^2 b}{4} \left(L_s^{N,\Gamma}(H_{S_k^{N,+}}^{N,\Gamma}) - L_{S_k^{N,+}}^{N,\Gamma}(H_{S_k^{N,+}}^{N,\Gamma}) \right) \right) \wedge \frac{(\Theta_k - 1)}{N} \tag{4.15}$$

where $\{P_s^{N,+}, s \geq 0\}$ and $\{P_s^{N,-}, s \geq 0\}$ are two mutually independent Poisson processes, with respective intensities

$$a^{-1}b \left(N^2 \kappa^2 + 2N\alpha \right) \quad \text{and} \quad b \left(N^2 \kappa^2 + 2N\beta \right),$$

where the $S_k^{N,+}$ are the successive jump times of the process

$$\widetilde{P}_s^{N,+} = \int_0^s \mathbf{1}_{\{V_{r-}^N = -1\}} \mathrm{d}P_r^{N,+}, \tag{4.16}$$

and where $L_s^{N,\Gamma}(0)$ and $L_s^{N,\Gamma}(\Gamma^-)$ denote respectively the number of visits to 0 and Γ by the process $H^{N,\Gamma}$ up to time s, multiplied by $4/N\kappa^2 b$ (see (4.2)). Note that our definition of $L^{N,\Gamma}$ makes the mapping $t \to L_s^{N,\Gamma}(t)$ right continuous for each $s > 0$. Hence $L_s^{N,\Gamma}(t) = 0$ for $t \geq \Gamma$, while $L_s^{N,\Gamma}(\Gamma^-) = \lim_{n \to +\infty} L_s^{N,\Gamma}(\Gamma - \frac{1}{n}) > 0$ if $H^{N,\Gamma}$ has reached the level Γ by time s.

For the rest of this section we set

$$\nu = \kappa\sqrt{b}.$$

We deduce from (4.15)

$$\frac{V_s^N}{N\nu^2} = \frac{1}{N\nu^2} + Q_s^{N,+} - \frac{2}{N\nu^2}\int_0^s \mathbf{1}_{\{V_{r^-}^N = +1\}} \mathrm{d}P_r^{N,-} + \frac{1}{2}\Big(L_s^{N,\Gamma}(0) - L_{0^+}^{N,\Gamma}(0)\Big) - \frac{1}{2}L_s^{N,\Gamma}(\Gamma^-),$$

where

$$Q_s^{N,+} = \frac{2}{N\nu^2}\int_0^s \mathbf{1}_{\{V_{r^-}^N = -1\}} \mathrm{d}P_r^{N,+} + \frac{2}{\nu^2}\sum_{k>0, S_k^{N,+} \leq s}\left(\frac{\nu^2}{4}(L_s^{N,\Gamma}(H_{S_k^{N,+}}^{N,\Gamma}) - L_{S_k^{N,+}}^{N,\Gamma}(H_{S_k^{N,+}}^{N,\Gamma})) \right) \wedge \frac{(\Theta_k - 1)}{N}$$

$$= \frac{2}{N\nu^2}\int_0^s \left[1 + \frac{N\nu^2}{4}(L_s^{N,\Gamma}(H_r^{N,\Gamma}) - L_r^{N,\Gamma}(H_r^{N,\Gamma})) \wedge (\Theta_{\widetilde{P}_r^{N,+}} - 1)\right]\mathrm{d}\widetilde{P}_r^{N,+}$$

$$= Q_s^{N,+,1} - Q_s^{N,+,2}, \tag{4.17}$$

with

$$Q_s^{N,+,1} = \frac{2}{N\nu^2}\int_0^s \Theta_{\widetilde{P}_{r^-}^{N,+}+1}\mathrm{d}\widetilde{P}_r^{N,+} \quad \text{and}$$

$$Q_s^{N,+,2} = \frac{2}{N\nu^2}\int_0^s \left(\Theta_{\widetilde{P}_{r^-}^{N,+}+1} - 1 - \frac{N\nu^2}{4}(L_s^{N,\Gamma}(H_r^{N,\Gamma}) - L_r^{N,\Gamma}(H_r^{N,\Gamma}))\right)^+ \mathrm{d}\widetilde{P}_r^{N,+}. \tag{4.18}$$

Observe that $\widetilde{P}_s^{N,+} = \widetilde{P}_{s^-}^{N,+} + 1$, for $\mathrm{d}\widetilde{P}_s^{N,+}$ almost every s.

Writing the first line of (4.15) as

$$H_s^{N,\Gamma} = 2N\int_0^s \mathbf{1}_{\{V_r^N = +1\}}\mathrm{d}r - 2N\int_0^s \mathbf{1}_{\{V_r^N = -1\}}\mathrm{d}r,$$

denoting by $M_s^{1,N}$ and $M_s^{2,N}$ the two local martingales

$$M_s^{1,N} = Q_s^{N,+,1} - \frac{2}{N\nu^2}\int_0^s \mathbf{1}_{\{V_r^N = -1\}}\Theta_{\widetilde{P}_r^{N,+}+1}\left(\frac{N^2\nu^2 + 2N\alpha b}{a}\right)\mathrm{d}r \tag{4.19}$$

and

$$M_s^{2,N} = \frac{2}{N\nu^2}\int_0^s \mathbf{1}_{\{V_{r^-}^N = +1\}}\left(\mathrm{d}P_r^{N,-} - (\nu^2 N^2 + 2N\beta b)\mathrm{d}r\right) \tag{4.20}$$

and recalling (4.17), we deduce from (4.15),

$$H_s^{N,\Gamma} + \frac{V_s^N}{N\nu^2} = \frac{1}{N\nu^2} + M_s^{1,N} - M_s^{2,N} + \frac{2N}{a}\int_0^s \mathbf{1}_{\{V_r^N = -1\}}\big(\Theta_{\widetilde{P}_r^{N,+}+1} - a\big)\mathrm{d}r - Q_s^{N,+,2} - \frac{1}{2}L_s^{N,\Gamma}(\Gamma^-)$$

$$+ \frac{4\alpha b}{a\nu^2}\int_0^s \mathbf{1}_{\{V_r^N = -1\}}\Theta_{\widetilde{P}_r^{N,+}+1}\mathrm{d}r - \frac{4\beta b}{\nu^2}\int_0^s \mathbf{1}_{\{V_r^N = +1\}}\mathrm{d}r + \frac{1}{2}(L_s^{N,\Gamma}(0) - L_{0^+}^{N,\Gamma}(0)). \tag{4.21}$$

We shall need below the

Lemma 4.13. $\{M_s^{1,N}, s \geq 0\}$ and $\{M_s^{2,N}, t \geq 0\}$ are square integrable martingales.

Proof. Since $M^{1,N}$ is a purely discontinuous local martingale, its quadratic variation $[M^{1,N}]$ is given by the sum of the squares of its jumps, *i.e.*,

$$[M^{1,N}]_s = \frac{4}{N^2\nu^4} \int_0^s \Theta^2_{\widetilde{P}^{N,+}_{r-}+1} \mathrm{d}\widetilde{P}^{N,+}_r.$$

We deduce from this that the predictable quadratic variation $\langle M^{1,N}\rangle$ of $M^{1,N}$ is given by

$$\langle M^{1,N}\rangle_s = \int_0^s \Theta^2_{\widetilde{P}^{N,+}_r+1} \left(\frac{4\nu^2 N + 8\alpha b}{aN\nu^4}\right) \mathbf{1}_{\{V^N_r=-1\}} \mathrm{d}r.$$

It follows readily from condition (**H**) in Section 2 that there exists a constant $C > 0$ such that

$$\mathbb{E}(\langle M^{1,N}\rangle_T) \le CT, \quad \text{for all } T > 0, \ N \ge 1.$$

Hence $M^{1,N}$ is a square integrable martingale. We can prove similarly that $M^{2,N}$ is a square integrable martingale. $\qquad\square$

For $s > 0$, define

$$\widetilde{P}^{N,-}_s = \int_0^s \mathbf{1}_{\{V^N_{r-}=+1\}} \mathrm{d}P^{N,-}_r.$$

Let $(\lambda^N_+(s), \ s \ge 0)$ (resp. $(\lambda^N_-(s), \ s \ge 0)$) denote the intensity of the process $(\widetilde{P}^{N,+}_s, \ s \ge 0)$ (resp. $(\widetilde{P}^{N,-}_s, \ s \ge 0)$) where $\widetilde{P}^{N,+}_s$ was defined in (4.16).

Remark 4.14. Note that $\widetilde{P}^{N,+}$ (resp. $\widetilde{P}^{N,-}$) with intensity $\lambda^N_+(.)$ (resp. $\lambda^N_-(.)$) can be viewed as time-changed of P (resp. P'), a standard Poisson process *i.e.*

$$\widetilde{P}^{N,+}_s = P\left(\int_0^s \lambda^N_+(r)\mathrm{d}r\right) \quad and \quad \widetilde{P}^{N,-}_s = P'\left(\int_0^s \lambda^N_-(r)\mathrm{d}r\right).$$

The next Proposition will also be important in the proof of the tightness and weak convergence of $H^{N,\Gamma}$. Recall the definition (4.18) of the non negative stochastic process $Q^{N,+,2}$.

Proposition 4.15. *For any stopping time τ such that $\tau \le s$ a.s., $s > 0$ arbitrary,*

$$\mathbb{E}(Q^{N,+,2}_\tau) \longrightarrow 0, \quad as \quad N \to \infty.$$

Proof. For $r \in \mathbb{R}^+$ and $p \in \mathbb{N}^*$, the stopping time

$$\hat{\tau}^p_r = \inf\left\{s \ge 0, \ \frac{N\nu^2}{4}\left(L^{N,\Gamma}_s(H^{N,\Gamma}_r) - L^{N,\Gamma}_r(H^{N,\Gamma}_r)\right) \ge p\right\} - r \tag{4.22}$$

describes the time it takes to explore the total progeny of p individuals born at the real time $H^{N,\Gamma}_r$. Since, the r.v. $\Delta^N_{k,j}$ (defined in Sect. 4.4.1) describes the exploration time of the total progeny of the jth individual born at time Υ_k, one can see that $\hat{\tau}^p_r = \sum_{j=1}^p N^{-2}\Delta^N_{k,j}$. Hence we deduce from (4.13) and (4.22) that

$$\left\{\frac{N\nu^2}{4}(L^{N,\Gamma}_\tau(H^{N,\Gamma}_r) - L^{N,\Gamma}_r(H^{N,\Gamma}_r)) \le \Theta_{\widetilde{P}^{N,+}_r} - 1\right\} = \left\{r + N^{-2}T^N_{\widetilde{P}^{N,+}_r} \ge \tau\right\}.$$

Now we have

$$
Q_\tau^{N,+,2} = \frac{2}{N\nu^2} \int_0^\tau \left(\Theta_{\widetilde{P}_{r^-}^{N,+}+1} - 1 - \frac{N\nu^2}{4}(L_s^{N,\Gamma}(H_r^{N,\Gamma}) - L_r^{N,\Gamma}(H_r^{N,\Gamma})) \right)^+ \mathrm{d}\widetilde{P}_r^{N,+}
$$

$$
\leq \frac{2}{N\nu^2} \int_0^\tau \Theta_{\widetilde{P}_{r^-}^{N,+}+1} \mathbf{1}_{\left\{ \frac{N\nu^2}{4}\left(L_\tau^{N,\Gamma}(H_r^{N,\Gamma}) - L_r^{N,\Gamma}(H_r^{N,\Gamma})\right) \leq \Theta_{\widetilde{P}_{r^-}^{N,+}+1}-1 \right\}} \mathrm{d}\widetilde{P}_r^{N,+}
$$

$$
= \frac{2}{N\nu^2} \int_0^\tau \Theta_{\widetilde{P}_{r^-}^{N,+}+1} \mathbf{1}_{\left\{ r+N^{-2}T_{\widetilde{P}_{r^-}^{N,+}+1}^N \geq \tau \right\}} \mathrm{d}\widetilde{P}_r^{N,+}.
$$

Consider a partition $0 = s_0 < s_1 < s_2 < \ldots < s_n = s$ such that $h = s_{i+1} - s_i = \frac{s}{n}$, $\forall\, 0 \leq i \leq n$. It follows that

$$
Q_\tau^{N,+,2} \leq \frac{2}{N\nu^2} \sum_{i=1}^n \mathbf{1}_{\{s_i < \tau \leq s_{i+1}\}} \int_0^{s_{i+1}} \Theta_{\widetilde{P}_{r^-}^{N,+}+1} \mathbf{1}_{\left\{ r+N^{-2}T_{\widetilde{P}_{r^-}^{N,+}+1}^N \geq s_i \right\}} \mathrm{d}\widetilde{P}_r^{N,+}
$$

$$
\leq \frac{2}{N\nu^2} \sum_{i=1}^n \mathbf{1}_{\{s_i < \tau \leq s_{i+1}\}} \int_0^{s_i} \Theta_{\widetilde{P}_{r^-}^{N,+}+1} \mathbf{1}_{\left\{ r+N^{-2}T_{\widetilde{P}_{r^-}^{N,+}+1}^N \geq s_i \right\}} \mathrm{d}\widetilde{P}_r^{N,+}
$$

$$
+ \frac{2}{N\nu^2} \sum_{i=1}^n \mathbf{1}_{\{s_i < \tau \leq s_{i+1}\}} \int_{s_i}^{s_{i+1}} \Theta_{\widetilde{P}_{r^-}^{N,+}+1} \mathrm{d}\widetilde{P}_r^{N,+}
$$

$$
= J_\tau^N + K_\tau^N.
$$

In what follows, we set $q = 2 + \delta$, where $0 < \delta < 1$ appears in the assumption $(\mathbf{H})$ and we fix $\rho > 0$ arbitrarily small such that $\delta > \frac{\rho}{1-\rho}$. The Proposition is now a consequence of the two next lemmas. $\qquad\square$

Lemma 4.16. *For any stopping time τ such that $\tau \leq s$ a.s.,*

$$
\mathbb{E}(K_\tau^N) \longrightarrow 0, \quad as \quad N \to \infty.
$$

Proof. Let $q' > 1$ be such that $\frac{1}{q} + \frac{1}{q'} = 1$. Using Hölder's inequality, we obtain

$$
\mathbb{E}(K_\tau^N) \leq \frac{2}{N\nu^2} \left[\mathbb{E} \sum_{i=0}^{n-1} \mathbf{1}_{\{s_i < \tau \leq s_{i+1}\}} \right]^{\frac{1}{q'}} \left[\mathbb{E} \sum_{i=0}^{n-1} \left(\int_{s_i}^{s_{i+1}} \Theta_{\widetilde{P}_{r^-}^{N,+}+1} \mathrm{d}\widetilde{P}_r^{N,+} \right)^q \right]^{\frac{1}{q}}
$$

$$
\leq \frac{2}{N\nu^2} \left[\sum_{i=0}^{n-1} \mathbb{E} \left\{ \left(\int_{s_i}^{s_{i+1}} \Theta_{\widetilde{P}_{r^-}^{N,+}+1} \mathrm{d}\widetilde{P}_r^{N,+} \right)^q \right\} \right]^{\frac{1}{q}}
$$

$$
= \frac{2}{N\nu^2} \left[\sum_{i=0}^{n-1} \mathbb{E} \left\{ \left(\int_{s_i}^{s_{i+1}} \Theta_{\widetilde{P}_r^{N,+}+1} \lambda_+^N(r)\mathrm{d}r + \int_{s_i}^{s_{i+1}} \Theta_{\widetilde{P}_{r^-}^{N,+}+1}[\mathrm{d}\widetilde{P}_r^{N,+} - \lambda_+^N(r)\mathrm{d}r] \right)^q \right\} \right]^{\frac{1}{q}}.
$$

Since $q > 1$, for $x, y \geq 0$, $(x+y)^q \leq 2^{q-1}(x^q + y^q)$, hence

$$
\mathbb{E}(K_\tau^N) \leq \frac{2^{2-\frac{1}{q}}}{N\nu^2} \left[\sum_{i=0}^{n-1} \left(\mathbb{E} \left\{ \left(\int_{s_i}^{s_{i+1}} \Theta_{\widetilde{P}_r^{N,+}+1} \lambda_+^N(r)\mathrm{d}r \right)^q \right\} + \mathbb{E} \left\{ \left| \int_{s_i}^{s_{i+1}} \Theta_{\widetilde{P}_{r^-}^{N,+}+1}[\mathrm{d}\widetilde{P}_r^{N,+} - \lambda_+^N(r)\mathrm{d}r] \right|^q \right\} \right) \right]^{\frac{1}{q}}
$$

$$
\leq \frac{C}{N} \left[\sum_{i=0}^{n-1} \left(\mathbb{E} \left\{ \left(X_{s_{i+1}}^{1,N,s_i} \right)^q \right\} + \mathbb{E} \left\{ \left| X_{s_{i+1}}^{2,N,s_i} \right|^q \right\} \right) \right]^{\frac{1}{q}}, \tag{4.23}
$$

with for all $s_i \leq u \leq s_{i+1}$

$$X_u^{1,N,s_i} = \int_{s_i}^u \Theta_{\widetilde{P}_r^{N,+}+1} \lambda_+^N(r) \mathrm{d}r \quad \text{and} \quad X_u^{2,N,s_i} = \int_{s_i}^u \Theta_{\widetilde{P}_{r^-}^{N,+}+1} [\mathrm{d}\widetilde{P}_r^{N,+} - \lambda_+^N(r)\mathrm{d}r] \tag{4.24}$$

and where C will be a constant which may differ from line to line. Combining Hölder's inequality and the fact that $\lambda_+^N(s) \leq CN^2$, $\forall s > 0$ (recall that $\lambda_+^N(s) = a^{-1}(N^2\nu^2 + 2N\alpha b)\mathbf{1}_{\{V_s^N = -1\}}$), we have

$$\mathbb{E}\left[(X_{s_{i+1}}^{1,N,s_i})^q\right] \leq CN^{2q}h^{\frac{q}{q'}} \int_{s_i}^{s_{i+1}} \mathbb{E}[\Theta_1]^q \mathrm{d}r$$
$$\leq CN^{2q}h^q, \tag{4.25}$$

where we have used the assumption $\mathbb{E}[\Theta]^q < \infty$ and the fact that $h = s_{i+1} - s_i$. Consider now a function $g \in \mathcal{C}^2(\mathbb{R})$. It follows from Itô's formula

$$g(X_{s_{i+1}}^{2,N,s_i}) = g(X_{s_i}^{2,N,s_i}) + \int_{s_i}^{s_{i+1}} g'(X_{r^-}^{2,N,s_i}) \mathrm{d}X_r^{2,N,s_i}$$
$$+ \sum_{s_i \leq r \leq s_{i+1}} \left[g(X_{r^-}^{2,N,s_i} + \Delta X_r^{2,N,s_i}) - g(X_{r^-}^{2,N,s_i}) - g'(X_{r^-}^{2,N,s_i})\Delta X_r^{2,N,s_i}\right] \tag{4.26}$$

where $\Delta X_r^{2,N,s_i} = X_r^{2,N,s_i} - X_{r^-}^{2,N,s_i}$. It is easy to see that $\Delta X_r^{2,N,s_i} = \Theta_{\widetilde{P}_{r^-}^{N,+}+1}$. However, for any $x > 0$, we have from Taylor's formula

$$g(x + \Delta x) - g(x) - \Delta x g'(x) = \frac{1}{2}g''(x + \xi\Delta x)|\Delta x|^2, \quad \text{for some} \quad 0 \leq \xi \leq 1.$$

We set $\Psi(x, \Delta x) = g(x + \Delta x) - g(x) - \Delta x g'(x)$ in the case $g(x) = |x|^q$. Then $g'(x) = \mathrm{sgn}(x)q|x|^{q-1}$ and $g''(x) = q(q-1)|x|^{q-2}$. We deduce from the above formula and the fact that $0 < q - 2 < 1$,

$$\Psi(x, \Delta x) = \frac{q(q-1)}{2}|x + \xi\Delta x|^{q-2}|\Delta x|^2$$
$$\leq \frac{q(q-1)}{2}|\Delta x|^2 \left(|x|^{q-2} + |\Delta x|^{q-2}\right)$$
$$\leq \frac{q(q-1)}{2}\left(|\Delta x|^q + |\Delta x|^2(1 + |x|)\right)$$
$$\leq \frac{q(q-1)}{2}\left(|\Delta x|^q + |\Delta x|^2 + |x||\Delta x|^2\right). \tag{4.27}$$

From (4.26), we have

$$\left|X_{s_{i+1}}^{2,N,s_i}\right|^q = \int_{s_i}^{s_{i+1}} \mathrm{sgn}\left(X_{r^-}^{2,N,s_i}\right) q|X_{r^-}^{2,N,s_i}|^{q-1}\mathrm{d}X_r^{2,N,s_i} + \int_{s_i}^{s_{i+1}} \Psi\left(X_{r^-}^{2,N,s_i}, \Theta_{\widetilde{P}_{r^-}^{N,+}+1}\right) \mathrm{d}\widetilde{P}_r^{N,+}.$$

Note that the first term on the right is a martingale (see Lem. 4.13). Hence taking the expectation in both sides and using (4.27), we deduce that

$$\mathbb{E}\left[\left|X_{s_{i+1}}^{2,N,s_i}\right|^q\right] \leq \frac{q(q-1)}{2}\mathbb{E}\int_{s_i}^{s_{i+1}} \left(\left|\Theta_{\widetilde{P}_{r^-}^{N,+}+1}\right|^q + \left|\Theta_{\widetilde{P}_{r^-}^{N,+}+1}\right|^2 + \left|\Theta_{\widetilde{P}_{r^-}^{N,+}+1}\right|^2 \left|X_{r^-}^{2,N,s_i}\right|\right) \mathrm{d}\widetilde{P}_r^{N,+}. \tag{4.28}$$

However, it is easy to see that

$$\mathbb{E}\int_{s_i}^{s_{i+1}} |\Theta_{\widetilde{P}_{r^-}^{N,+}+1}|^q \mathrm{d}\widetilde{P}_r^{N,+} = \mathbb{E}\int_{s_i}^{s_{i+1}} \left|\Theta_{\widetilde{P}_r^{N,+}+1}\right|^q \lambda_+^N(r)\mathrm{d}r \leq CN^2h \tag{4.29}$$

and we obtain similarly,

$$\mathbb{E}\int_{s_i}^{s_{i+1}}\left|\Theta_{\widetilde{P}_{r^-}^{N,+}+1}\right|^2\mathrm{d}\widetilde{P}_r^{N,+}\le CN^2h. \tag{4.30}$$

Furthermore, recalling that the $S_k^{N,+}$ are the successive jump times of the process $\widetilde{P}^{N,+}$, we have

$$
\begin{aligned}
\mathbb{E}\int_{s_i}^{s_{i+1}}\left|X_{r^-}^{2,N,s_i}\right|\left|\Theta_{\widetilde{P}_{r^-}^{N,+}+1}\right|^2\mathrm{d}\widetilde{P}_r^{N,+} &= \mathbb{E}\sum_{k>0,\ s_i\le S_k^{N,+}\le s_{i+1}}\left|X_{(S_k^{N,+})^-}^{2,N,s_i}\right||\Theta_{k+1}|^2 \\
&= \sum_{k>0}\mathbb{E}\left(\mathbf{1}_{\left\{s_i\le S_k^{N,+}\le s_{i+1}\right\}}\left|X_{(S_k^{N,+})^-}^{2,N,s_i}\right||\Theta_{k+1}|^2\right) \\
&= \mathbb{E}|\Theta_1|^2\,\mathbb{E}\sum_{k>0}\mathbf{1}_{\left\{s_i\le S_k^{N,+}\le s_{i+1}\right\}}\left|X_{(S_k^{N,+})^-}^{2,N,s_i}\right| \\
&= \mathbb{E}|\Theta_1|^2\,\mathbb{E}\int_{s_i}^{s_{i+1}}\left|X_{r^-}^{2,N,s_i}\right|\mathrm{d}\widetilde{P}_r^{N,+} \\
&= \mathbb{E}|\Theta_1|^2\,\mathbb{E}\int_{s_i}^{s_{i+1}}\left|X_r^{2,N,s_i}\right|\lambda_+^N(r)\mathrm{d}r \\
&\le CN^2\mathbb{E}\int_{s_i}^{s_{i+1}}\left|X_r^{2,N,s_i}\right|\mathrm{d}r, \tag{4.31}
\end{aligned}
$$

However, from (4.24) we have that

$$\left|X_{s_{i+1}}^{2,N,s_i}\right|\le C\left(\int_{s_i}^{s_{i+1}}\left|\Theta_{\widetilde{P}_{r^-}^{N,+}+1}\right|\mathrm{d}\widetilde{P}_r^{N,+}+\int_{s_i}^{s_{i+1}}\left|\Theta_{\widetilde{P}_r^{N,+}+1}\right||\lambda_+^N(r)|\mathrm{d}r\right),$$

taking expectation in both side we deduce that

$$\mathbb{E}\left|X_{s_{i+1}}^{2,N,s_i}\right|\le CN^2h.$$

Combining this with (4.31), we deduce that

$$\mathbb{E}\int_{s_i}^{s_{i+1}}|\Theta_{\widetilde{P}_{r^-}^{N,+}+1}|^2|X_{r^-}^{2,N,s_i}|\mathrm{d}\widetilde{P}_r^{N,+}\le CN^4h^2.$$

Combining the last inequality with (4.28), (4.29), and (4.30), it follows that

$$\mathbb{E}\left[\left|X_{s_{i+1}}^{2,N,s_i}\right|^q\right]\le CN^2h+CN^4h^2.$$

Now, combining the last inequality with (4.23) and (4.25), we deduce that

$$\mathbb{E}(K_\tau^N)\le\frac{Cn^{\frac{1}{q}}}{N}\left(N^{2q}h^q+N^2h+N^4h^2\right)^{\frac{1}{q}}.$$

Using the inequality

$$\left(\sum_{i=1}^k a_i\right)^{\frac{1}{q}}\le\sum_{i=1}^k a_i^{\frac{1}{q}},\quad\text{for all }k\ge2,\ a_1,\dots,a_k\ge0, \tag{4.32}$$

we thus obtain

$$
\begin{aligned}
\mathbb{E}(K_\tau^N)&\le CNn^{\frac{1}{q}}h+Cn^{\frac{1}{q}}N^{\frac{2}{q}-1}h^{\frac{1}{q}}+Cn^{\frac{1}{q}}N^{\frac{4}{q}-1}h^{\frac{2}{q}} \\
&= C\left(\frac{N}{n^{1-\frac{1}{q}}}+N^{\frac{2}{q}-1}+\frac{N^{\frac{4}{q}-1}}{n^{\frac{1}{q}}}\right).
\end{aligned}
$$

We now choose $n = N^{2-\rho}$. It is easy to deduce from $\delta > \frac{\rho}{1-\rho}$ that

$$\mathbb{E}(K_\tau^N) \to 0, \ as \ N \to \infty.$$ $\square$

Lemma 4.17. *For any stopping time τ such that $\tau \leq s$ a.s.,*

$$\mathbb{E}(J_\tau^N) \longrightarrow 0, \quad as \quad N \to \infty.$$

Proof. We have

$$
J_\tau^N = \frac{2}{N\nu^2} \sum_{i=1}^n \mathbf{1}_{\{s_i < \tau \leq s_{i+1}\}} \int_0^{s_i} \Theta_{\widetilde{P}_{r^-}^{N,+}+1} \mathbf{1}_{\left\{ r + N^{-2} T^N_{\tilde{P}_{r^-}^{N,+}+1} \geq s_i \right\}} \, \mathrm{d}\widetilde{P}_r^{N,+}
$$

$$
\leq \frac{2}{N\nu^2} \sum_{i=1}^n \int_0^{s_i} \Theta_{\widetilde{P}_{r^-}^{N,+}+1} \mathbf{1}_{\left\{ r + N^{-2} T^N_{\tilde{P}_{r^-}^{N,+}+1} \geq s_i \right\}} \, \mathrm{d}\widetilde{P}_r^{N,+}
$$

$$
= \frac{2}{N\nu^2} \sum_{i=1}^n \sum_{k>0, S_k^{N,+} \leq s_i} \Theta_{k+1} \mathbf{1}_{\left\{ T^N_{k+1} \geq N^2 (s_i - S_k^{N,+}) \right\}}
$$

$$
= \frac{2}{N\nu^2} \sum_{i=1}^n \sum_{k=1}^\infty \mathbf{1}_{\left\{ S_k^{N,+} \leq s_i \right\}} \Theta_{k+1} \mathbf{1}_{\left\{ T^N_{k+1} \geq N^2 (s_i - S_k^{N,+}) \right\}}.
$$

Taking the expectation on both sides, we have

$$
\mathbb{E}(J_\tau^N) \leq \frac{2}{N\nu^2} \sum_{i=1}^n \sum_{k=1}^\infty \mathbb{E}\left[\mathbf{1}_{\left\{ S_k^{N,+} \leq s_i \right\}} \Theta_{k+1} \mathbf{1}_{\left\{ T^N_{k+1} \geq N^2 (s_i - S_k^{N,+}) \right\}} \right].
$$ (4.33)

We set $G_{k,s_i}^N = \mathbf{1}_{\left\{ S_k^{N,+} \leq s_i \right\}} \Theta_{k+1} \mathbf{1}_{\left\{ T^N_{k+1} \geq N^2 (s_i - S_k^{N,+}) \right\}}$. We have that

$$
\mathbb{E}\left(G_{k,s_i}^N \mid S_k^{N,+} = r \right) = \mathbf{1}_{\left\{ r \leq s_i \right\}} \mathbb{E}\left(\Theta_{k+1} \mathbf{1}_{\left\{ T^N_{k+1} \geq N^2 (s_i - r) \right\}} \right).
$$ (4.34)

However, using (4.13), we have

$$
\mathbb{E}\left(\Theta_1 \mathbf{1}_{\left\{ T_1^N \geq N^2 (s_i - r) \right\}} \right) = \mathbb{E}\left(\Theta_1 \mathbf{1}_{\left\{ \sum_{j=1}^{\Theta_1} \Delta_{1,j}^N \geq N^2 (s_i - r) \right\}} \right)
$$

$$
= \sum_{k=1}^\infty k \mathbb{P}\left(\Theta_1 = k, \sum_{j=1}^k \Delta_{1,j}^N \geq N^2 (s_i - r) \right)
$$

$$
= \sum_{k=1}^\infty k \mathbb{P}(\Theta_1 = k) \mathbb{P}\left(\sum_{j=1}^k \Delta_{1,j}^N \geq N^2 (s_i - r) \right).
$$

Since $\frac{3-\rho}{3} < 1$, there exists $0 < \eta < 1$ such that $\eta > \frac{3-\rho}{3}$. Using Markov's inequality, we obtain

$$
E\left(\Theta_1 \mathbf{1}_{\left\{ T_1^N \geq N^2 (s_i - r) \right\}} \right) \leq \frac{1}{N^{2\eta} (s_i - r)^\eta} \sum_{k=1}^\infty k \mathbb{P}(\Theta_1 = k) \mathbb{E}\left(\sum_{j=1}^k \Delta_{1,j}^N \right)^\eta.
$$

Using the inequality (4.32) with $\frac{1}{q}$ replaced by η, we obtain

$$
\mathbb{E}\left(\Theta_1 \mathbf{1}_{\left\{ T_1^N \geq N^2 (s_i - r) \right\}} \right) \leq \frac{1}{N^{2\eta} (s_i - r)^\eta} \mathbb{E}\left[\left(\Delta_{1,1}^N \right)^\eta \right] \mathbb{E}(\Theta_1^2).
$$

From Jensen's inequality, it follows that

$$\mathbb{E}\left(\Theta_1 \mathbf{1}_{\left\{T_1^N \geq N^2(s_i - r)\right\}}\right) \leq \frac{C}{N^{2\eta}(s_i - r)^\eta}\left[\mathbb{E}(\Delta_{1,1}^N)\right]^\eta.$$

From Corollary 4.12 , there exists a constant $C'(\Gamma)$ such that

$$\mathbb{E}(\Delta_{1,1}^N) \leqslant \frac{C'(\Gamma)}{N},$$

this implies

$$\mathbb{E}\left(\Theta_1 \mathbf{1}_{\left\{T_1^N \geq N^2(s_i - r)\right\}}\right) \leq \frac{C}{N^{3\eta}(s_i - r)^\eta}.$$

Now combining the last inequality with (4.33) and (4.34), we deduce that

$$\begin{aligned}
\mathbb{E}(J_\tau^N) &\leq \frac{2}{N\nu^2}\sum_{i=1}^{n}\sum_{k=1}^{\infty}\mathbb{E}\left[\frac{C}{N^{3\eta}(s_i - S_k^{N,+})^\eta}\mathbf{1}_{\left\{S_k^{N,+}\leq s_i\right\}}\right]\\
&\leq \frac{C}{N^{3\eta+1}}\sum_{i=1}^{n}\mathbb{E}\left[\sum_{k>0}\frac{1}{(s_i - S_k^{N,+})^\eta}\mathbf{1}_{\left\{S_k^{N,+}\leq s_i\right\}}\right]\\
&= \frac{C}{N^{3\eta+1}}\sum_{i=1}^{n}\mathbb{E}\int_0^{s_i}\frac{\mathrm{d}\widetilde{P}_r^{N,+}}{(s_i - r)^\eta}\\
&= \frac{C}{N^{3\eta+1}}\sum_{i=1}^{n}\mathbb{E}\int_0^{s_i}\lambda_+^N(r)\frac{\mathrm{d}r}{(s_i - r)^\eta}\\
&\leq \frac{CN}{N^{3\eta}}\sum_{i=1}^{n}\int_0^{s_i}\frac{\mathrm{d}r}{(s_i - r)^\eta}\\
&= \frac{CN}{N^{3\eta}}\sum_{i=1}^{n}\int_0^{s_i}\frac{\mathrm{d}r}{r^\eta}\\
&\leq \frac{CNn}{(1-\eta)N^{3\eta}}[s \vee 1].
\end{aligned}$$

Since $n = N^{2-\rho}$ and $\eta > \frac{3-\rho}{3}$, which implies that $\rho > 3(1-\eta)$, $\mathbb{E}(J_\tau^N) \to 0$, as $N \to \infty$.   □

We deduce from Proposition 4.15 the following corollary.

Corollary 4.18. *As $N \to \infty$,*
$$Q_\tau^{N,+,2} \to 0 \text{ in probability.}$$

It now follows from Aldous' tightness criterion (Thm. 16.10 p.178 in [4]) that

Lemma 4.19. *The sequence $\{Q^{N,+,2}, N \geq 1\}$ is tight in $\mathcal{D}([0,\infty))$.*

Corollary 4.20. $Q_s^{N,+,2} \longrightarrow 0$ *in probability, locally uniformly in s.*

Thanks to these results, we are in position to study the asymptotic behavior of $H^{N,\Gamma}$. We first recall

Remark 4.21. From the definition of $H^{N,\Gamma}$, we have, $H_s^{N,\Gamma} \leq \Gamma, \forall s > 0$.

For the proof of the weak convergence of $\{H_s^{N,\Gamma}, s \geq 0\}$, we will need the following lemma

Lemma 4.22. *For any $s > 0$,*

$$\int_0^s \mathbf{1}_{\{V_r^N=1\}} \mathrm{d}r \longrightarrow \frac{s}{2}; \quad \int_0^s \mathbf{1}_{\{V_r^N=-1\}} \mathrm{d}r \longrightarrow \frac{s}{2}$$

in probability, as $N \longrightarrow \infty$.

Proof. We have (the second line follows from (4.15))

$$\int_0^s \mathbf{1}_{\{V_r^N=1\}} \mathrm{d}r + \int_0^s \mathbf{1}_{\{V_r^N=-1\}} \mathrm{d}r = s,$$

$$\int_0^s \mathbf{1}_{\{V_r^N=1\}} \mathrm{d}r - \int_0^s \mathbf{1}_{\{V_r^N=-1\}} \mathrm{d}r = (2N)^{-1} H_s^{N,\Gamma}.$$

We conclude by adding and subtracting the two above identities and using Remark 4.21. □

For the rest of this section we set

$$A_s^N = \int_0^s \lambda_+^N(r) \mathrm{d}r, \quad \hat{A}_s^N = \int_0^s \lambda_-^N(r) \mathrm{d}r, \quad \bar{A}_s^N = (N^2\nu^2 + 2N\alpha b)s/2a,$$

$$\Delta_s^N = N^2\nu^2 s/2a, \quad \hat{\Delta}_s^N = N^2\nu^2 s/2 \quad \text{and} \quad \tilde{\Theta}_k = \Theta_k - a.$$

In the equation (4.21), we set

$$\Psi_s^N = \frac{2N}{a} \int_0^s \mathbf{1}_{\{V_r^N=-1\}} \left(\Theta_{\tilde{P}_r^{N,+}+1} - a \right) \mathrm{d}r.$$

From Remark 4.14, we deduce that

$$\Psi_s^N = \frac{2}{N\nu^2 + 2\alpha b} \int_0^s \tilde{\Theta}_{P(A_r^N)+1} dA_r^N = \frac{2}{N\nu^2 + 2\alpha b} \int_0^{A_s^N} \tilde{\Theta}_{P(u)+1} \mathrm{d}u$$

$$= \frac{2}{N\nu^2 + 2\alpha b} \left(\sum_{k=0}^{P(A_s^N)} \tilde{\Theta}_{k+1} \Xi_k - \tilde{\Theta}_{P(A_s^N)+1}(T^+(A_s^N) - A_s^N) \right), \tag{4.35}$$

where Ξ_k denotes the length of the time interval during which $P(u) = k$ and $T^+(A_s^N)$ is the first jump time of P after A_s^N. It is easily seen that Ξ_k has the standard exponential distribution and we notice that $(\Xi_1, \Theta_1, \Xi_2, \Theta_2, \ldots)$ is a sequence of independent random variables. By the same computations, we deduce from (4.19)

$$M_s^{1,N} = -\frac{2}{N\nu^2} \left(\sum_{k=0}^{P(A_s^N)} \Theta_{k+1} \tilde{\Xi}_k - \Theta_{P(A_s^N)+1}(T^+(A_s^N) - A_s^N) \right),$$

where $\tilde{\Xi}_k = \Xi_k - \mathbb{E}(\Xi_k)$. From (4.20), we have also

$$M_s^{2,N} = -\frac{2}{N\nu^2} \left(\sum_{k=0}^{P'(\hat{A}_s^N)} \tilde{\Xi}_k' - (T^-(\hat{A}_s^N) - \hat{A}_s^N) \right),$$

where $\tilde{\Xi}'_k = \Xi'_k - \mathbb{E}(\Xi'_k)$ and where Ξ'_k denotes the length of the time interval during which $P'(u) = k$ and $T^-(\hat{A}^N_s)$ is the first jump time of P' after $\hat{A}^N_s$. As previously Ξ'_k has the standard exponential distribution and $(\Xi'_1, \Xi_1, \Theta_1, \Xi'_2, \Xi_2, \Theta_2, \ldots)$ is a sequence of independent random variables.

If we define for $n \geq 1$

$$S^1_n = \sum_{k=0}^{n} \left(\tilde{\Theta}_{k+1} \Xi_k - \Theta_{k+1} \tilde{\Xi}_k \right), \quad \bar{S}^1_n = \sum_{k=0}^{n} \tilde{\Theta}_{k+1} \Xi_k \quad \text{and} \quad S^2_n = \sum_{k=0}^{n} \tilde{\Xi}'_k,$$

we obtain the following relations

$$\Psi^N_s + M^{1,N}_s = \frac{2}{N\nu^2} S^1_{P(A^N_s)} - \frac{2}{N\nu^2} \bar{S}^1_{P(A^N_s)} C_N - \Lambda^1_{A^N_s} + \bar{\Lambda}^1_{A^N_s}, \tag{4.36}$$

with

$$\Lambda^1_{A^N_s} = \frac{2}{N\nu^2 + 2\alpha b} \tilde{\Theta}_{P(A^N_s)+1}(T^+(A^N_s) - A^N_s), \tag{4.37}$$

$$\bar{\Lambda}^1_{A^N_s} = \frac{2}{N\nu^2} \Theta_{P(A^N_s)+1}(T^+(A^N_s) - A^N_s), \quad C_N = \frac{2\alpha b}{(N\nu^2 + 2\alpha b)}$$

and

$$M^{2,N}_s = -\frac{2}{N\nu^2} S^2_{P'(\hat{A}^N_s)} + \Lambda^2_{\hat{A}^N_s} \tag{4.38}$$

with

$$\Lambda^2_{\hat{A}^N_s} = \frac{2}{N\nu^2}(T^-(\hat{A}^N_s) - \hat{A}^N_s).$$

The following Proposition plays a key role in the asymptotic behavior of $H^{N,\Gamma}$

Proposition 4.23. *As $N \longrightarrow \infty$,*

$$\left(\Psi^N_s + M^{1,N}_s, M^{2,N}_s, \ s \geq 0 \right) \Longrightarrow \left(\frac{\sqrt{2}}{\nu} \sqrt{\frac{a^2 + \zeta^2}{a}} B^1_s, \frac{\sqrt{2}}{\nu} B^2_s, \ s \geq 0 \right) \ in \ (\mathcal{D}([0,\infty)))^2,$$

where B^1_s and B^2_s are two mutually independent standard Brownian motions.

We first prove the

Lemma 4.24. *As $N \longrightarrow \infty$, $\Lambda^1_{A^N_s} \longrightarrow 0$ in probability, locally uniformly in s.*

Proof. Let $T > 0$. From (4.37), we notice that

$$\left| \Lambda^1_{A^N_s} \right| \leq \frac{2}{N\nu^2 + 2\alpha b} \left| \tilde{\Theta}_{P(A^N_s)+1} \right| \Xi_{P(A^N_s)+1},$$

this implies

$$\sup_{0 \leqslant s \leqslant T} \left| \Lambda^1_{A^N_s} \right| \leq \sup_{0 \leq k \leq P(A^N_T)} \left(\frac{2}{N\nu^2 + 2\alpha b} \left| \tilde{\Theta}_{k+1} \right| \Xi_{k+1} \right). \tag{4.39}$$

However, for $\rho > 0$, we note that

$$\left\{ \sup_{0 \leq k \leq P(A^N_T)} \left| \tilde{\Theta}_{k+1} \right| \Xi_{k+1} > \frac{\rho(N\nu^2 + 2\alpha b)}{2} \right\} \subset \left\{ \sup_{0 \leq k \leq 2A^N_T} \left| \tilde{\Theta}_{k+1} \right| \Xi_{k+1} > \frac{\rho(N\nu^2 + 2\alpha b)}{2} \right\}$$

$$\cup \left\{ P(A^N_T) > 2A^N_T \right\}. \tag{4.40}$$

It follows from (4.39) and (4.40) that

$$\mathbb{P}\left(\sup_{0\leqslant s\leqslant T}|\Lambda^1_{A^N_s}|>\rho\right)\leq\mathbb{P}\left(\sup_{0\leq k\leq 2A^N_T}|\tilde{\Theta}_{k+1}|\Xi_{k+1}>\frac{\rho(N\nu^2+2\alpha b)}{2}\right)+\mathbb{P}\left(P(A^N_T)>2A^N_T\right)$$

$$\leq\mathbb{P}\left(\sup_{0\leq k\leq 2\bar{A}^N_T}|\tilde{\Theta}_{k+1}|\Xi_{k+1}>\frac{\rho(N\nu^2+2\alpha b)}{2}\right)+\mathbb{P}\left(P(A^N_T)>2A^N_T\right).$$

We deduce from the law of large numbers that the second term on the right converges to 0 a.e, as $N\longrightarrow\infty$. We will now show that the first term on the right converges to 0, as $N\longrightarrow\infty$. We have

$$\mathbb{P}\left(\sup_{0\leq k\leq 2\bar{A}^N_T}|\tilde{\Theta}_{k+1}|\Xi_{k+1}>\frac{\rho(N\nu^2+2\alpha b)}{2}\right)=1-\left[1-\mathbb{P}\left(|\tilde{\Theta}_1|\Xi_1>\frac{\rho(N\nu^2+2\alpha b)}{2}\right)\right]^{2\bar{A}^N_T}$$

$$=1-\left[1-\mathbb{E}\left(\exp\left(-\frac{\rho(N\nu^2+2\alpha b)}{2|\tilde{\Theta}_1|}\right)\right)\right]^{2\bar{A}^N_T}.$$

However, we notice that, as N tends to infinity,

$$\left[1-\mathbb{E}\left(\exp\left(-\frac{\rho(N\nu^2+2\alpha b)}{2|\tilde{\Theta}_1|}\right)\right)\right]^{2\bar{A}^N_T}\simeq\exp\left[-2\bar{A}^N_T\mathbb{E}\left(\exp\left(-\frac{\rho(N\nu^2+2\alpha b)}{2|\tilde{\Theta}_1|}\right)\right)\right].$$

Let

$$b_N=\bar{A}^N_T\mathbb{E}\left(\exp\left(-\frac{\rho(N\nu^2+2\alpha b)}{2|\tilde{\Theta}_1|}\right)\right).$$

We now show that $b_N\longrightarrow 0$, *as* $N\longrightarrow\infty$, which will imply the Lemma. We have

$$b_N=\frac{(N^2\nu^2+2N\alpha b)T}{2a}\int_0^1\mathbb{P}\left(\exp\left(-\frac{\rho(N\nu^2+2\alpha b)}{2|\tilde{\Theta}_1|}\right)>t\right)\mathrm{d}t$$

$$=\frac{\rho NT}{a}\left(\frac{N\nu^2+2\alpha b}{2}\right)^2\int_0^\infty y^{-2}\mathbb{P}(|\tilde{\Theta}_1|>y)\exp\left(-\frac{\rho(N\nu^2+2\alpha b)}{2y}\right)\mathrm{d}y$$

(recall that $\bar{A}^N_T=(N^2\nu^2+2N\alpha b)T/2a$). Define for $N\geq 1$,

$$f_N(y)=N\left(\frac{N\nu^2+2\alpha b}{2}\right)^2 y^{-2}\mathbb{P}(|\tilde{\Theta}_1|>y)\exp\left(-\frac{\rho(N\nu^2+2\alpha b)}{2y}\right).$$

It is easily seen that $f_N(y)\longrightarrow 0$, *as* $N\longrightarrow\infty$ and it is not very hard to show that

$$f_N(y)\leq N^3\nu^4 y^{-2}\mathbb{P}(|\tilde{\Theta}_1|>y)\exp\left(-\frac{\rho(N\nu^2+2\alpha b)}{2y}\right)$$

$$\leq\frac{2^3 3^3\nu^4}{\rho^3\nu^6}y^3 e^{-3}y^{-2}\mathbb{P}(|\tilde{\Theta}_1|>y)$$

$$=\frac{2^3 3^3}{\rho^3\nu^2}e^{-3}y\mathbb{P}(|\tilde{\Theta}_1|>y).$$

Hence, since $\tilde{\Theta}_1$ is square integrable, we deduce from the dominated convergence theorem that

$$b_N\longrightarrow 0,\ as\ N\longrightarrow\infty.\qquad\qquad\qquad\square$$

Following the same approach, we have the

Lemma 4.25. *As $N \longrightarrow \infty$, $\bar{\Lambda}^1_{A^N_s} \longrightarrow 0$ (resp. $\Lambda^2_{\hat{A}^N_s} \longrightarrow 0$) in probability, locally uniformly in s.*

Let us rewrite (4.36) in the form

$$\Psi^N_s + M^{1,N}_s = \frac{2}{N\nu^2} S^1_{[\Delta^N_s]} + \frac{2}{N\nu^2}\left(S^1_{P(A^N_s)} - S^1_{[\Delta^N_s]}\right) - \frac{2}{N\nu^2}\bar{S}^1_{P(A^N_s)}C_N - \Lambda^1_{A^N_s} + \bar{\Lambda}^1_{A^N_s},$$
$$= G^{1,N}_s + \bar{G}^{1,N}_s - \hat{G}^{1,N}_s - \Lambda^1_{A^N_s} + \bar{\Lambda}^1_{A^N_s}, \tag{4.41}$$

with

$$G^{1,N}_s = \frac{2}{N\nu^2}S^1_{[\Delta^N_s]}, \quad \bar{G}^{1,N}_s = \frac{2}{N\nu^2}\left(S^1_{P(A^N_s)} - S^1_{[\Delta^N_s]}\right) \quad \text{and} \quad \hat{G}^{1,N}_s = \frac{2}{N\nu^2}\bar{S}^1_{P(A^N_s)}C_N.$$

Lemma 4.24 combined with (4.4) leads to

Corollary 4.26. *For all $T > 0$, $j_T(G^{1,N}) \longrightarrow 0$, as $N \longrightarrow \infty$.*

For the proof of Proposition 4.23 we will need the following lemmas.

Lemma 4.27. *For any $s > 0$,*

$$(G^{1,N}_s, \ s \geq 0) \Longrightarrow \left(\frac{\sqrt{2}}{\nu}\sqrt{\frac{a^2 + \zeta^2}{a}}B^1_s, \ s \geq 0\right) \ in \ \mathcal{D}([0,\infty)),$$

where B^1 is a standard Brownian motion.

Proof. The result follows from Donsker's theorem (see, *e.g.* Thm. 14.1 p. 146 in [4]). $\square$

Corollary 4.28. *The sequence $\{G^{1,N}, \ N \geq 1\}$ is tight in $\mathcal{D}([0,\infty))$.*

Lemma 4.29. *As $N \longrightarrow \infty$, $\bar{G}^{1,N} \longrightarrow 0$ in probability, locally uniformly in s, where $\bar{G}^{1,N}$ was defined in (4.41).*

Proof. Let $\epsilon > 0$ be given and $T > 0$. We have

$$\mathbb{P}\left(\sup_{0\leqslant s\leqslant T}\left|\bar{G}^{1,N}_s\right| > \epsilon\right) = \mathbb{P}\left(\sup_{0\leqslant s\leqslant T}\frac{2}{N\nu^2}\left|S^1_{P(A^N_s)} - S^1_{[\Delta^N_s]}\right| > \epsilon\right)$$

$$\leq \mathbb{P}\left(\sup_{|s-r|\leqslant\rho,\ 0\leqslant s\leqslant T}\frac{2}{N\nu^2}\left|S^1_{[\Delta^N_r]} - S^1_{[\Delta^N_s]}\right| > \epsilon\right) + \mathbb{P}\left(\sup_{0\leqslant s\leqslant T}\left|P(A^N_s) - [\Delta^N_s]\right| > \left(\frac{N^2\nu^2}{2a}\right)\rho\right)$$

$$\leq \mathbb{P}\left(w_T\left(G^{1,N},\rho\right) > \epsilon\right) + \mathbb{P}\left(\sup_{0\leqslant s\leqslant T}\left|P(A^N_s) - [\Delta^N_s]\right| > \left(\frac{N^2\nu^2}{2a}\right)\rho\right). \tag{4.42}$$

Furthermore, we have

$$\frac{2a(P(A^N_s) - [\Delta^N_s])}{N^2\nu^2} \longrightarrow 0 \text{ a.s, } as \ N \longrightarrow \infty.$$

Indeed, it is readily seen $2a[\Delta^N_s]/N^2\nu^2 \longrightarrow s$ a.s, as $N \longrightarrow \infty$ (recall that $\Delta^N_s = N^2\nu^2 s/2a$) and we have

$$\frac{2aP(A^N_s)}{N^2\nu^2} = \left(\frac{P(A^N_s)}{A^N_s}\right)\left(\frac{2aA^N_s}{N^2\nu^2}\right),$$

we deduce from the law of large numbers that the first factor on the right converges to 1 a.e, as $N \longrightarrow \infty$ and from Lemma 4.22 that the second factor converges to s, as $N \longrightarrow \infty$.

Since, moreover, for each N the function $s \longrightarrow \{2aP(A_s^N)/N^2\nu^2\}$ is increasing, we deduce from the second Dini's theorem that the second term on the right in (4.42) converges to 0, as $N \longrightarrow \infty$. It follows that

$$\limsup_{N\to\infty} \mathbb{P}\left(\sup_{0\leqslant s\leqslant T} |\bar{G}_s^{1,N}| > \epsilon\right) \leq \limsup_{N\to\infty} \mathbb{P}\left(w_T(G^{1,N},\rho) > \epsilon\right).$$

Combining this inequality with (4.5) , we have

$$\limsup_{N\to\infty} \mathbb{P}\left(\sup_{0\leqslant s\leqslant T} |\bar{G}_s^{1,N}| > \epsilon\right) \leq \limsup_{N\to\infty} \mathbb{P}\left(\bar{w}_T(G^{1,N},\rho) > \frac{\epsilon}{4}\right) + \limsup_{N\to\infty} \mathbb{P}\left(j(G^{1,N}) > \frac{\epsilon}{2}\right)$$

$$\leq \lim_{\rho\to 0}\limsup_{N\to\infty} \mathbb{P}\left(\bar{w}_T(G^{1,N},\rho) > \frac{\epsilon}{4}\right) + \limsup_{N\to\infty} \mathbb{P}\left(j(G^{1,N}) > \frac{\epsilon}{2}\right).$$

Combining Proposition 4.5, Corollary 4.26 and Corollary 4.28, we deduce that

$$\limsup_{N\to\infty} \mathbb{P}\left(\sup_{0\leqslant s\leqslant T} |\bar{G}_s^{1,N}| > \epsilon\right) = 0.$$

The result follows $\square$

Lemma 4.30. *As $N \longrightarrow \infty$, $\hat{G}^{1,N} \longrightarrow 0$ in probability, locally uniformly in s, where $\hat{G}^{1,N}$ was defined in* (4.41).

Proof. We can rewrite $\hat{G}^{1,N}$ as

$$\hat{G}_s^{1,N} = \frac{2}{N\nu^2} \bar{S}_{[\Delta_s^N]}^1 C_N + \frac{2}{N\nu^2}\left(\bar{S}_{P(A_s^N)}^1 - \bar{S}_{[\Delta_s^N]}^1\right) C_N$$

Following the same approach as in the proofs of Lemmas 4.27 and 4.29, we have that for any $s > 0$,

$$\left(\frac{2}{N\nu^2}\bar{S}_{[\Delta_s^N]}^1, \ s \geq 0\right) \Longrightarrow \left(\frac{2}{\nu}\frac{\zeta}{\sqrt{a}}B_s^\circ, \ s \geq 0\right) \ in \ \mathcal{D}([0,\infty)),$$

where B° is a standard Brownian motion and that

$$\frac{2}{N\nu^2}\left(\bar{S}_{P(A_s^N)}^1 - \bar{S}_{[\Delta_s^N]}^1\right) \longrightarrow 0 \ \text{in probability, locally uniformly in s.}$$

Since, moreover, $C_N \longrightarrow 0$ (recall that $C_N = 2\alpha b/(N\nu^2 + 2\alpha b)$), the result follows readily by combining the above arguments. $\square$

We can rewrite (4.38) in the form

$$M_s^{2,N} = -\frac{2}{N\nu^2}S_{[\hat{\Delta}_s^N]}^2 - \frac{2}{N\nu^2}\left(S_{P'(\hat{A}_s^N)}^2 - S_{[\hat{\Delta}_s^N]}^2\right) + \Lambda_{\hat{A}_s^N}^2,$$

$$= G_s^{2,N} - \bar{G}_s^{2,N} + \Lambda_{\hat{A}_s^N}^2, \tag{4.43}$$

with

$$G_s^{2,N} = -\frac{2}{N\nu^2}S_{[\hat{\Delta}_s^N]}^2 \quad \text{and} \quad \bar{G}_s^{2,N} = \frac{2}{N\nu^2}\left(S_{P'(\hat{A}_s^N)}^2 - S_{[\hat{\Delta}_s^N]}^2\right).$$

Similarly as above, we deduce the following results.

Lemma 4.31. *For any $s > 0$,*

$$(G_s^{2,N}, \ s \geq 0) \Longrightarrow \left(\frac{\sqrt{2}}{\nu}B_s^2, \ s \geq 0\right) \ in \ \mathcal{D}([0,\infty)),$$

where B^2 is a standard Brownian motion.

Lemma 4.32. *As $N \longrightarrow \infty$, $\bar{G}^{2,N} \longrightarrow 0$ in probability, locally uniformly in s.*

Since, the sequences $(\tilde{\Theta}_k \Xi_k - \Theta_k \tilde{\Xi}_k)_{k \geq 1}$ and $(\tilde{\Xi}'_k)_{k \geq 1}$ are independent, the processes $\{G^{1,N}, \ N \geq 1\}$ and $\{G^{2,N}, \ N \geq 1\}$ are also independent. Consequently the assertion of Proposition 4.23 is now immediate by combining the above convergence results.

Let us state a basic result for counting processes, which will be useful in the sequel. For this, let Q be a counting process with stochastic intensity $(\lambda(t), t \geq 0)$. Let $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$ be the filtration generated by Q. Let $(S_k, k \geq 1)$ be the successive jump times of Q, and suppose that

$$Q_t - \int_0^t \lambda(r) \mathrm{d}r$$

is an $\mathcal{F}$-martingale.

Lemma 4.33. *The sequence*

$$\left(\int_{S_{k-1}}^{S_k} \lambda(r) \mathrm{d}r \right)_{k \geq 1}$$

is a sequence of i.i.d standard exponential random variables.

Proof. Let

$$\nu_t = \int_0^t \lambda(r) \mathrm{d}r$$

and let τ be the inverse of ν, that is,

$$\tau_u = \inf\{t > 0 : \nu_t > u\};$$

see Exercise 5.13 of Chapter I in [5]. Then, τ is right-continuous and strictly increasing, and $\nu_{\tau_u} = u$ by the continuity of ν. Clearly, (Q_{τ_u}) is adapted to the filtration $(\mathcal{F}_{\tau_u})$ and is again a counting process. Since $Q - \nu$ is assumed to be an $\mathcal{F}$-martingale and since τ_u is a stopping time, we deduce from Doob's optional stopping theorem that the process $(Q_{\tau_u} - u)$ is an $(\mathcal{F}_{\tau_u})$-martingale. By Proposition 6.13 in [5], the process $Q_u^\circ = Q_{\tau_u}$ is a standard Poisson process. It follows that

$$\int_{S_{k-1}}^{S_k} \lambda(r) \mathrm{d}r = \nu_{S_k} - \nu_{S_{k-1}} = T_k - T_{k-1},$$

where T_k is the kth jump time of Q°. Consequently the $T_k - T_{k-1}$ are independent and identically distributed standard exponential random variables. $\qquad \square$

To ease the reading, we rewrite (4.21) in the following form

$$H_s^{N,\Gamma} + \frac{V_s^N}{N\nu^2} = \frac{1}{N\nu^2} + M_s^{1,N} - M_s^{2,N} + \Psi_s^N - Q_s^{N,+,2} + F^N(s) + \frac{1}{2} \left(L_s^{N,\Gamma}(0) - L_{0+}^{N,\Gamma}(0) \right) - \frac{1}{2} L_s^{N,\Gamma}(\Gamma^-) \quad (4.44)$$

where

$$F^N(s) = \frac{4\alpha b}{a\nu^2} \int_0^s \mathbf{1}_{\{V_r^N = -1\}} \Theta_{\tilde{P}_r^{N,+}+1} \mathrm{d}r - \frac{4\beta b}{\nu^2} \int_0^s \mathbf{1}_{\{V_r^N = +1\}} \mathrm{d}r.$$

In the following, we give useful properties of the sequence of processes $(F^N, \ N \geq 1)$.

Lemma 4.34. *For any $s > 0$,*

$$F^N(s) \longrightarrow F(s) \ \text{in probability, as } N \longrightarrow \infty,$$

where $\quad F(s) = \frac{2(\alpha - \beta)}{\kappa^2} s.$

Proof. We have

$$F^N(s) = \frac{4\alpha b}{a\nu^2} \int_0^s \mathbf{1}_{\{V_r^N=-1\}} \Theta_{\widetilde{P}_r^{N,+}+1} \mathrm{d}r - \frac{4\beta b}{\nu^2} \int_0^s \mathbf{1}_{\{V_r^N=+1\}} \mathrm{d}r$$

$$= \frac{4\alpha b}{a\nu^2} \sum_{k=1}^{\widetilde{P}_s^{N,+}} \Theta_k \int_{S_{k-1}^{N,+}}^{S_k^{N,+}} \mathbf{1}_{\{V_r^N=-1\}} \mathrm{d}r + \varsigma_s^N - \frac{4\beta b}{\nu^2} \int_0^s \mathbf{1}_{\{V_r^N=+1\}} \mathrm{d}r$$

where ς_s^N describes the boundary effects at the points s, tending to 0 as N goes to ∞. Indeed, we have

$$\varsigma_s^N \le C\Theta_{\widetilde{P}_s^{N,+}} \int_{S_{\widetilde{P}_s^{N,+}}^{N,+}}^{S_{\widetilde{P}_s^{N,+}+1}^{N,+}} \mathbf{1}_{\{V_r^N=-1\}} \mathrm{d}r.$$

For $k \ge 1$ define,

$$\xi_k^N = \int_{S_{k-1}^{N,+}}^{S_k^{N,+}} \lambda_+^N(r)\mathrm{d}r, \tag{4.45}$$

recall that $\lambda_+^N(r) = a^{-1}(N^2\nu^2 + 2N\alpha b)\mathbf{1}_{\{V_{r-}^N=-1\}}$. Hence we have

$$\varsigma_s^N \le \left(\frac{aC}{N^2\nu^2 + 2N\alpha b}\right) \Theta_{\widetilde{P}_s^{N,+}} \xi_{\widetilde{P}_s^{N,+}+1}^N.$$

It follows readily from Lemma 4.33 that

$$\mathbb{E}(\varsigma_s^N)^2 \longrightarrow 0 \text{ as } N \longrightarrow \infty.$$

However, from Lemma 4.33, (4.45) and Remark 4.14, we deduce that

$$F^N(s) = \left(\frac{4\alpha b A_s^N}{\nu^2(N^2\nu^2 + 2N\alpha b)}\right) \left(\frac{P(A_s^N)}{A_s^N}\right) \left(\frac{1}{P(A_s^N)} \sum_{k=1}^{P(A_s^N)} \Theta_k \xi_k^N\right) - \frac{4\beta b}{\nu^2} \int_0^s \mathbf{1}_{\{V_r^N=+1\}} \mathrm{d}r + \varsigma_s^N.$$

From Lemma 4.22, we deduce that the first factor of the first term on the right converges to $2\alpha s/a\kappa^2$ as $N \longrightarrow \infty$. We have from the law of large numbers that the second factor converges to 1 a.e, as $N \longrightarrow \infty$. Moreover it follows from the strong law of large numbers that the third factor converges to a in probability, as $N \longrightarrow \infty$. Now combining the above arguments, we deduce that

$$F^N(s) \longrightarrow \frac{2(\alpha-\beta)}{\kappa^2} s \text{ in probability, as } N \longrightarrow \infty. \qquad \square$$

In addition

Lemma 4.35. *The sequence* $\{F^N, N \ge 1\}$ *is tight in* $\mathcal{C}([0,\infty))$.

Proof. We have

$$F^N(s) = \frac{4\alpha b}{a\nu^2} \int_0^s \mathbf{1}_{\{V_r^N=-1\}} \Theta_{\widetilde{P}_r^{N,+}+1} \mathrm{d}r - \frac{4\beta b}{\nu^2} \int_0^s \mathbf{1}_{\{V_r^N=+1\}} \mathrm{d}r$$

$$= b_s^N - d_s^N.$$

The sequence $\{b^N, N \geq 1\}$ is tight in $\mathcal{C}([0, \infty))$. Indeed, $b_0^N = 0$ and for all $0 < s < t$

$$
\begin{aligned}
\mathbb{E}\left(\sup_{s \leq t \leq s+\rho} |b_t^N - b_s^N|^2\right) &= \mathbb{E}\left(\sup_{s \leq t \leq s+\rho} \left|\frac{4\alpha b}{a\nu^2} \int_s^t \mathbf{1}_{\{V_r^N = -1\}} \Theta_{P_r^{N,+}+1} dr\right|^2\right) \\
&\leq \left(\frac{4\alpha b}{a\nu^2}\right)^2 \mathbb{E}\left(\int_s^{s+\rho} |\Theta_{P_r^{N,+}+1}| dr\right)^2 \\
&\leq \rho \left(\frac{4\alpha b}{a\nu^2}\right)^2 \mathbb{E}\left(\int_s^{s+\rho} \Theta_{P_r^{N,+}+1}^2 dr\right) \\
&\leq \left(\frac{4\alpha b}{a\nu^2}\right)^2 (\zeta^2 + a^2)\rho^2.
\end{aligned}
$$

We obtain similarly, the tightness of $\{d^N, N \geq 1\}$ in $\mathcal{C}([0, \infty))$. Consequently $\{F^N, N \geq 1\}$ is tight in $\mathcal{C}([0, \infty))$. $\qquad\square$

It follows readily from Lemmas 4.34 and 4.35.

Corollary 4.36. $F^N \longrightarrow F$ *in probability in* $\mathcal{C}([0, \infty), \mathbb{R}_+)$.

Let us rewrite (4.44) in the form

$$
H_s^{N,\Gamma} = R_s^N + \frac{1}{2} L_s^{N,\Gamma}(0) - \frac{1}{2} L_s^{N,\Gamma}(\Gamma^-), \tag{4.46}
$$

where

$$
R_s^N = \frac{1}{N\nu^2} - \frac{V_s^N}{N\nu^2} + M_s^{1,N} - M_s^{2,N} + \Psi_s^N + F^N(s) - \frac{1}{2} L_{0^+}^N(0) - Q_s^{N,+,2}, \quad s \geq 0. \tag{4.47}
$$

We have moreover

Lemma 4.37. *The sequence* $\{R^N, N \geq 1\}$ *is tight in* $\mathcal{D}([0, \infty))$.

Proof. We may rewrite (4.47) as

$$
R_s^N + \frac{V_s^N}{N\nu^2} + \frac{1}{2} L_{0^+}^N(0) + Q_s^{N,+,2} - \frac{1}{N\nu^2} = M_s^{1,N} - M_s^{2,N} + \Psi_s^N + F^N(s). \tag{4.48}
$$

Tightness of the right-hand side of (4.48) follows from Proposition 4.23, Lemma 4.35 and Proposition 4.4 . From (4.2), it is easily checked that $L_{0^+}^N(0) = 4/N\kappa^2 b$. Since, moreover $N^{-1} V_s^N \longrightarrow 0$ a.s uniformly with respect to s and $Q_s^{N,+,2} \longrightarrow 0$ in probability, locally uniformly in s, the sequence $\{R^N, N \geq 1\}$ is tight in $\mathcal{D}([0, \infty))$. $\qquad\square$

Recall that B^1 was defined in Lemma 4.27, B^2 in Lemma 4.31 and F in Lemma 4.34. An immediate consequence of the above results is

Proposition 4.38. *For any* $s > 0$,

$$
\left(R^N, \Psi^N + M^{1,N}, M^{2,N}, F^N\right) \Rightarrow \left(R, \frac{\sqrt{2}}{\nu}\sqrt{\frac{a^2 + \zeta^2}{a}} B^1, \frac{\sqrt{2}}{\nu} B^2, F\right) in (\mathcal{D}([0, \infty)))^3 \times (\mathcal{C}([0, \infty)))
$$

as $N \to \infty$, *where*

$$
R_s = \frac{2(\alpha - \beta)}{\kappa^2} s + \frac{2}{\kappa} B_s, \quad and \quad B_s = \sqrt{\frac{a^2 + \zeta^2}{a + a^2 + \zeta^2}} B_s^1 - \sqrt{\frac{a}{a + a^2 + \zeta^2}} B_s^2. \tag{4.49}
$$

We now deduce the tightness of $H^{N,\Gamma}$ from the above results concerning R^N, without having to worry about the local time terms.

Proposition 4.39. *For any $\Gamma > 0$, the sequence $\{H^{N,\Gamma}, \ N \geq 1\}$ is tight in $\mathcal{C}([0, \infty))$.*

Proof. We show that the sequence $\{H^{N,\Gamma}, \ N \geq 1\}$ satisfies the conditions of Proposition 4.1. Condition (i) follows easily from $H_0^{N,\Gamma} = 0$. In order to verify condition (ii), we will show that for each $\epsilon \geq 0$,

$$\lim_{\rho \to 0} \limsup_{N \to \infty} \mathbb{P}(w_T(H^{N,\Gamma}, \rho) \geq \epsilon) = 0.$$

Indeed, let $\epsilon > 0$ be given and $T > 0$. Since $L^{N,\Gamma}(0)$ (resp. $L^{N,\Gamma}(\Gamma^-)$) increases only when $H_s^{N,\Gamma} = 0$ (resp. when $H_s^{N,\Gamma} = \Gamma$), it follows from (4.46) that for any $0 < d < 1$,

$$\left\{ \sup_{|s-r| \leqslant \rho, \ 0 \leqslant r,s \leqslant T} \left| H_s^{N,\Gamma} - H_r^{N,\Gamma} \right| > \epsilon \right\} \subset \left\{ \sup_{|s-r| \leqslant \rho, \ 0 \leqslant r,s \leqslant T} \left| R_s^N - R_r^N \right| > d\epsilon \right\}.$$

Indeed, if

$$\sup_{|s-r| \leqslant \rho, \ 0 \leqslant r,s \leqslant T} \left| H_s^{N,\Gamma} - H_r^{N,\Gamma} \right| > \epsilon,$$

we notice that for any $0 < d < 1$, there exists (s', r') satisfying $0 \leq r' < s' \leq T$ and $s' - r' \leq \rho$ such that

$$\left| H_{s'}^{N,\Gamma} - H_{r'}^{N,\Gamma} \right| > d\epsilon, \quad \text{and} \quad 0 < H_u^{N,\Gamma} < \Gamma, \quad \forall \ u \in [r', s'],$$

which implies that

$$R_{s'}^N - R_{r'}^N = H_{s'}^{N,\Gamma} - H_{r'}^{N,\Gamma}.$$

However, from (4.5) we have

$$\mathbb{P}\left(\sup_{|s-r| \leqslant \rho, \ 0 \leqslant r,s \leqslant T} \left| R_s^N - R_r^N \right| > d\epsilon \right) \leq \mathbb{P}\left(\bar{w}_{R^N, T}(\rho) > \frac{d\epsilon}{4} \right) + \mathbb{P}\left(j_T(R^N) > \frac{d\epsilon}{2} \right).$$

Consequently

$$\mathbb{P}\left(\sup_{|s-r| \leqslant \rho, \ 0 \leqslant r,s \leqslant T} \left| H_s^{N,\Gamma} - H_r^{N,\Gamma} \right| > \epsilon \right) \leq \mathbb{P}\left(\bar{w}_{R^N, T}(\rho) > \frac{d\epsilon}{4} \right) + \mathbb{P}\left(j_T(R^N) > \frac{d\epsilon}{2} \right).$$

Combining this inequality with Lemma 4.37 and the fact that $j_T(R^N) \leq \frac{C}{N}$, we deduce that

$$\lim_{\rho \to 0} \limsup_{N \to \infty} \mathbb{P}(w_T(H^{N,\Gamma}, \rho) \geq \epsilon) = 0.$$

The result follows. $\square$

In the next Proposition, H^Γ is an arbitrary accumulation point of the sequence $H^{N,\Gamma}$, which will be uniquely identified below in Theorem 4.41.

Proposition 4.40. *The sequence $\{L^{N,\Gamma}(0), \ N \geq 1\}$ (resp. $\{L^{N,\Gamma}(\Gamma^-), \ N \geq 1\}$) is tight in $\mathcal{D}([0, \infty))$, the limit $L^\Gamma(0)$ (resp. $L^\Gamma(\Gamma^-)$) of any converging subsequence being continuous and increasing. Moreover, $L^\Gamma(0)$ (resp. $L^\Gamma(\Gamma^-)$) is the local time of H^Γ at level 0 (resp. at level Γ^-).*

Proof. Let us rewrite (4.46) in the form

$$H_s^{N,\Gamma} = R_s^N + \frac{1}{2}U_s^N$$

where $U_s^N = L_s^{N,\Gamma}(0) - L_s^{N,\Gamma}(\Gamma^-)$. It then follows from Proposition 4.4, Lemma 4.37 and Proposition 4.39 that $\{U^N, N \geq 1\}$ is tight in $\mathcal{D}([0,\infty))$. We now show that the sequence $\{L^{N,\Gamma}(0),\ N \geq 1\}$ satisfies the condition of Corollary 4.6. Indeed, let $\epsilon > 0$ be given and $T > 0$. Since $L^{N,\Gamma}(0)$ (resp. $L^{N,\Gamma}(\Gamma^-)$) increases only on the set of time when $H_s^{N,\Gamma} = 0$ (resp. $H_s^{N,\Gamma} = \Gamma$), we notice that

$$\sup_{|s-t|\leqslant\rho} \left|L_s^{N,\Gamma}(0) - L_r^{N,\Gamma}(0)\right| \leq \sup_{|s-t|\leqslant\rho} \left|U_s^N - U_r^N\right| \quad \text{unless} \quad \sup_{|s-t|\leqslant\rho} \left|H_s^{N,\Gamma} - H_r^{N,\Gamma}\right| > \frac{\Gamma}{2}.$$

It then follows from (4.3) and (4.5) that

$$\mathbb{P}\left(w_T\big(L^{N,\Gamma}(0),\rho\big) \geq \epsilon\right) \leq \mathbb{P}\left(w_T\big(U^N,\rho\big) \geq \epsilon\right) + \mathbb{P}\left(w_T\big(H^{N,\Gamma},\rho\big) > \frac{\Gamma}{2}\right)$$

$$\leq \mathbb{P}\left(\bar{w}_T\big(U^N,\rho\big) \geq \frac{\epsilon}{2}\right) + \mathbb{P}\left(j_T(U^N) \geq \frac{\epsilon}{2}\right) + \mathbb{P}\left(w_T\big(H^{N,\Gamma},\rho\big) > \frac{\Gamma}{2}\right).$$

The assertion of Corollary 4.6 is now immediate by combining Proposition 4.39, tightness in $\mathcal{D}([0,\infty))$ of $\{U^N, N \geq 1\}$ and the fact that $j_T(U^N) \leq \frac{C}{N}$. We deduce that the sequence $\{L^{N,\Gamma}(0),\ N \geq 1\}$ is tight in $\mathcal{D}([0,\infty))$. Now we show that the limit K of any converging subsequence is continuous and increasing. To this end, for each $l \geq 1$, we define the function $f_l : \mathbb{R}_+ \to [0,1]$ by $f_l(x) = (1 - lx)^+$. We have that for each $N, l \geq 1$, $s > 0$, since $L^{N,\Gamma}(0)$ increases only when $H^{N,\Gamma} = 0$,

$$\mathbb{E}\left(\int_0^s f_l(H_r^{N,\Gamma})dL_r^{N,\Gamma}(0) - L_s^{N,\Gamma}(0)\right) = 0.$$

Thanks to Lemma 4.3, we can take the limit in this last inequality as $N \to \infty$, yielding

$$\mathbb{E}\left(\int_0^s f_l(H_r^\Gamma)dK_r - K_s\right) = 0.$$

Then taking the limit as $l \to +\infty$ yields

$$\mathbb{E}\left(\int_0^s \mathbf{1}_{\{H_r^\Gamma=0\}}dK_r - K_s\right) = 0.$$

But the random variable under the expectation is clearly nonpositive, hence it is zero a.s., in other words

$$\int_0^s \mathbf{1}_{\{H_r^\Gamma=0\}}dK_r = K_s,\ \text{a.s},\ \ \forall s \geq 0.$$

which means that the process K increases only when $H_r^\Gamma = 0$. From the occupation times formula

$$\int_0^s g(H_r^\Gamma)dr = \int_0^s g(t)L_s^\Gamma(t)dt$$

applied to the function $g(h) = \mathbf{1}_{\{h=0\}}$, we deduce that the time spent by the process H^Γ at 0 has a.s. zero Lebesgue measure. Consequently

$$\int_0^s \mathbf{1}_{\{H_r^\Gamma=0\}}dB_r \equiv 0\ \ \text{a.s},$$

where B is the standard Brownian motion , which has been defined in (4.49). Hence a.s.

$$B_s = \int_0^s \mathbf{1}_{\{H_r^\Gamma > 0\}} \mathrm{d}B_r \quad \forall s \geq 0.$$

It then follows from Tanaka's formula applied to the process H^Γ and the function $h \to h^+$ that $K = L^\Gamma(0)$. The continuity of K follows from Corollary 4.6.

Following the same approach as for $L^{N,\Gamma}(0)$, we have $\{L^{N,\Gamma}(\Gamma^-), \ N \geq 1\})$ is tight in $\mathcal{D}([0,\infty))$, the limit $L^\Gamma(\Gamma^-)$ of any converging subsequence being continuous and increasing. In other words $L^\Gamma(\Gamma^-)$ is the local time of H^Γ at level Γ^-. $\qquad\square$

We are now ready to state the main result.

Theorem 4.41. *For each $\Gamma > 0$,*

$$\left(H^{N,\Gamma}, R^N, L^{N,\Gamma}(0), L^{N,\Gamma}(\Gamma^-) \right) \Rightarrow \left(H^\Gamma, R, L^\Gamma(0), L^\Gamma(\Gamma^-) \right) \ in \ (\mathcal{C}([0,\infty))) \times (\mathcal{D}([0,\infty)))^3$$

as $N \to \infty$, where R was defined in (4.49), $L^\Gamma(0)$ (resp. $L^\Gamma(\Gamma^-)$) is the local time of H^Γ at level 0 (resp. at level Γ^-) and H^Γ is the unique weak solution of the SDE

$$H_s^\Gamma = \frac{2(\alpha - \beta)}{\kappa^2}s + \frac{2}{\kappa}B_s + \frac{1}{2}L_s^\Gamma(0) - \frac{1}{2}L_s^\Gamma(\Gamma^-), \quad s \geq 0, \tag{4.50}$$

i.e. H^Γ equals $2/\kappa$ multiplied by Brownian motion with drift $(\alpha - \beta)s/\kappa$, reflected in the interval $[0, \Gamma]$.

Proof. Equation (4.50) follows by taking the limit in (4.46) combined with the above results. It is plain that H^Γ, being a limit (along a subsequence) of $H^{N,\Gamma}$, takes values in $[0, \Gamma]$. The fact that $L^\Gamma(0)$ (resp. $L^\Gamma(\Gamma^-)$) is the local time of H^Γ at level 0 (resp. at level Γ^-) proves that $\frac{\kappa}{2}H^\Gamma$ is a Brownian motion with drift $(\alpha - \beta)s/\kappa$, reflected in $[0, \Gamma]$, which characterizes its law. We can refer *e.g.* to the formulation of reflected SDEs in [13]. $\quad\square$

Corollary 4.42. *The following holds*

$$H^{N,\Gamma} \Longrightarrow H^\Gamma \ \ in \ \mathcal{C}([0,\infty)), \ as \ N \longrightarrow \infty.$$

4.5. The subcritical case

We now want to establish a similar statement for weak convergence of the height process in the subcritical case (*i.e.* $\alpha < \beta$) without reflecting the process $H^{N,\Gamma}$. In other words, in the subcritical case, we can choose $\Gamma = +\infty$, which simplifies the above construction. The two main difficulties are the need for a new bound for $\mathbb{E}(\Delta_{1,1}^N)$ (see Cor. 4.12), and a bound for $\mathbb{E}\left(\sup_{0 \leq s \leq T} H_s^N\right)$, since we cannot use Remark 4.21 anymore. Now we notice that in the subcritical case (*i.e.* $\alpha < \beta$), the constant $C(\Gamma)$ defined in (4.12) in the proof of Lemma 4.11, is bounded by $1/(\beta - \alpha)$ for all $\Gamma > 0$. In that case, we can choose $\Gamma = +\infty$. Consequently, an easy adaptation gives a result similar to Corollary 4.12. In the subcritical case, the equation (4.44) takes the following form

$$H_s^N + \frac{V_s^N}{N\nu^2} = \frac{1}{N\nu^2} + \tilde{M}_s^N + \Psi_s^N - Q_s^{N,+,2} + F^N(s) + \frac{1}{2}(L_s^N(0) - L_{0+}^N(0)), \tag{4.51}$$

where

$$\tilde{M}_s^N = M_s^{1,N} - M_s^{2,N}.$$

Since $M_s^{1,N}$ and $M_s^{2,N}$ are two orthogonal martingales, we deduce from (4.19) and (4.20) that

$$\langle \tilde{M}^N \rangle_s = \int_0^s \left(\Theta_{\tilde{P}_r^{N,+}+1}^2 \left(\frac{4\nu^2 N + 8\alpha b}{aN\nu^4} \right) \mathbf{1}_{\{V_r^N = -1\}} + \left(\frac{4\nu^2 N + 8\beta b}{N\nu^4} \right) \mathbf{1}_{\{V_r^N = 1\}} \right) \mathrm{d}r. \tag{4.52}$$

From (4.52) we deduce that $\{\tilde{M}_s^N, s \geq 0\}$ is in fact a martingale. Recall That Ψ^N was given by

$$\Psi_s^N = \frac{2}{N\nu^2 + 2\alpha b} \left(\sum_{k=0}^{P(A_s^N)} \tilde{\Theta}_{k+1} \Xi_k - \tilde{\Theta}_{P(A_s^N)+1}(T^+(A_s^N) - A_s^N) \right), \tag{4.53}$$

see (4.35). We now set for $\ell \geq 1$

$$\Phi_\ell^N = \frac{2}{N\nu^2 + 2\alpha b} \sum_{k=0}^{\ell} \tilde{\Theta}_{k+1} \Xi_k. \tag{4.54}$$

We will need the following lemmas.

Lemma 4.43. *There exists a constant $C > 0$ such that for all $T > 0$,*

$$\mathbb{E}\left(\Phi_{P(\bar{A}_T^N)}^N\right)^2 \leq CT,$$

(recall that $\bar{A}_s^N = (N^2\nu^2 + 2N\alpha b)s/2a$).

Proof. Since the random variables $P(\bar{A}_T^N)$, $\tilde{\Theta}_1$ and Ξ_1 are mutually independent, we have from (4.54) that

$$\left(\Phi_{P(\bar{A}_T^N)}^N\right)^2 = \frac{4}{(N\nu^2 + 2\alpha b)^2} \left[\sum_{k=0}^{P(\bar{A}_T^N)} \left(\tilde{\Theta}_{k+1}\Xi_k\right)^2 + \sum_{1 \leq i \neq k \leq P(\bar{A}_T^N)} \tilde{\Theta}_{k+1}\tilde{\Theta}_{i+1}\Xi_k\Xi_i \right].$$

Hence tacking expectation in both side, we deduce that

$$\mathbb{E}\left(\Phi_{P(\bar{A}_T^N)}\right)^2 = \frac{4}{(N\nu^2 + 2\alpha b)^2} \left[\mathbb{E}\left(P(\bar{A}_T^N)\right) \left(\mathbb{E}\left(\tilde{\Theta}_1\right)^2\right) \left(\mathbb{E}\left(\Xi_1\right)^2\right) \right.$$

$$\left. + \mathbb{E}\left(P(\bar{A}_T^N)(P(\bar{A}_T^N) - 1)\right) \left(\mathbb{E}\left(\tilde{\Theta}_1\right)\right)^2 \left(\mathbb{E}\left(\Xi_1\right)\right)^2 \right].$$

The second term on the right is zero because the random variable $\tilde{\Theta}_1$ is centered and since

$$\mathbb{E}\left(P(\bar{A}_T^N)\right) \leq CTN^2,$$

we deduce that

$$\mathbb{E}\left(\Phi_{P(\bar{A}_T^N)}^N\right)^2 \leq CT. \qquad \square$$

Lemma 4.44. *There exists a constant $C > 0$ such that for all $T > 0$,*

$$\mathbb{E}\left(\sup_{0 \leq s \leq T} |\Psi_s^N|\right) \leq C(1 + T).$$

Proof. From (4.53) and (4.54), it follows that

$$\Psi_s^N = \Phi_{P(A_s^N)}^N - \frac{2}{N\nu^2 + 2\alpha b}\tilde{\Theta}_{P(A_s^N)+1}(T^+(A_s^N) - A_s^N).$$

However, noting that $\tilde{\Theta}_k^- \leq a$, where $\tilde{\Theta}_k^- = \sup\{-\tilde{\Theta}_k, 0\}$,

$$\sup_{0 \leq s \leq T} |\Psi_s^N| \leq \sup_{0 \leq k \leq P(A_T^N)} |\Phi_k^N| + \frac{2a}{N\nu^2 + 2\alpha b} \sup_{0 \leq s \leq T} (T^+(A_s^N) - A_s^N).$$

It follows that

$$\mathbb{E}\left(\sup_{0\leq s\leq T}|\Psi_s^N|\right) \leq \left[\mathbb{E}\left(\sup_{0\leq k\leq P(A_T^N)}|\Phi_k^N|\right)^2\right]^{\frac{1}{2}} + \frac{2a}{N\nu^2+2\alpha b}\mathbb{E}\left(\sup_{0\leq s\leq T}(T^+(A_s^N)-A_s^N)\right)$$

$$\leq \left[\mathbb{E}\left(\sup_{0\leq k\leq P(\bar{A}_T^N)}|\Phi_k^N|\right)^2\right]^{\frac{1}{2}} + \frac{2a}{N\nu^2+2\alpha b}\mathbb{E}\left(\sup_{0\leq s\leq T}(T^+(A_s^N)-A_s^N)\right).$$

It is easy to check that $(\Phi_k^N,\ k\geq 0)$ is a discrete-time martingale. Moreover, note that $P(\bar{A}_T^N)$ is a stopping time. Hence, from Doob's inequality we have

$$\mathbb{E}\left(\sup_{0\leq s\leq T}|\Psi_s^N|\right) \leq 2\left[\mathbb{E}\left(\Phi_{P(\bar{A}_T^N)}^N\right)^2\right]^{\frac{1}{2}} + C.$$

The result now follows readily from Lemma 4.43. $\square$

Now we deduce the following basic estimate for H^N

Lemma 4.45. *There exists a constant $C > 0$ such that for all $T > 0$,*

$$\mathbb{E}\left(\sup_{0\leq s\leq T}H_s^N\right) \leq C(1+T).$$

Proof. Let us rewrite (4.51) in the form

$$H_s^N = W_s^N + \frac{1}{2}L_s^N(0),$$

where

$$W_s^N = \frac{1}{N\nu^2} - \frac{V_s^N}{N\nu^2} + \tilde{M}_s^N + \Psi_s^N + F^N(s) - \frac{1}{2}L_{0+}^N(0) - Q_s^{N,+,2}. \tag{4.55}$$

Set

$$s_N = \sup\left\{0 \leq r \leq s;\ L_r^N(0) - L_{r-}^N(0) > 0\right\},$$

then the fact that $L^N(0)$ is increasing, and increases only on the set of time when $H_s^N = 0$ proves that $H_{s_N}^N = 0$ and $L_s^N(0) = L_{s_N}^N(0)$. It follows that

$$H_s^N = W_s^N - W_{s_N}^N.$$

Hence

$$H_s^N \leq \sup_{0\leq r\leq s}\left[W_s^N - W_r^N\right],$$

this implies

$$\sup_{0\leq s\leq T}H_s^N \leq \sup_{0\leq r\leq s\leq T}\left[W_s^N - W_r^N\right] \leq 2\sup_{0\leq s\leq T}|W_s^N|. \tag{4.56}$$

However, from (4.2), it is easily checked that $L_{0+}^N(0) = 4/N\kappa^2 b$. Since, moreover $(Q_s^{N,+,2}, s \geq 0)$ is a process with values in $\mathbb{R}_+$ see (4.18), we have from (4.55) that

$$\sup_{0\leq s\leq T}|W_s^N| \leq \sup_{0\leq s\leq T}|\tilde{M}_s^N| + \sup_{0\leq s\leq T}|\Psi_s^N| + \frac{4\alpha b}{a\nu^2}\int_0^T \mathbf{1}_{\{V_r^N=-1\}}\Theta_{\tilde{P}_r^{N,+}+1}\mathrm{d}r$$

$$+ \frac{4\beta b}{\nu^2}\int_0^T \mathbf{1}_{\{V_r^N=+1\}}\mathrm{d}r + \sup_{0\leq s\leq T}Q_s^{N,+,2} + C.$$

Combining this inequality with (4.56), we deduce that

$$\sup_{0 \le s \le T} H_s^N \le 2 \sup_{0 \le s \le T} |\tilde{M}_s^N| + 2 \sup_{0 \le s \le T} |\Psi_s^N| + \frac{8\alpha b}{a\nu^2} \int_0^T \mathbf{1}_{\{V_r^N = -1\}} \Theta_{\tilde{P}_r^{N,+}+1} \mathrm{d}r$$
$$+ \frac{8\beta b}{\nu^2} \int_0^T \mathbf{1}_{\{V_r^N = +1\}} \mathrm{d}r + 2 \sup_{0 \le s \le T} Q_s^{N,+,2} + C.$$

Hence tacking expectation in both side, we deduce that

$$\mathbb{E}\left(\sup_{0 \le s \le T} H_s^N\right) \le 2\mathbb{E}\left(\sup_{0 \le s \le T} |\tilde{M}_s^N|\right) + 2\mathbb{E}\left(\sup_{0 \le s \le T} |\Psi_s^N|\right) + 2\mathbb{E}\left(\sup_{0 \le s \le T} Q_s^{N,+,2}\right) + C(1+T)$$
$$\le 2\left[\mathbb{E}\left(\sup_{0 \le s \le T} \tilde{M}_s^N\right)^2\right]^{\frac{1}{2}} + 2\mathbb{E}\left(\sup_{0 \le s \le T} |\Psi_s^N|\right) + 2\mathbb{E}\left(\sup_{0 \le s \le T} Q_s^{N,+,2}\right) + C(1+T).$$

This together with Lemma 4.19, Lemma 4.44, (4.52), Doob's L^2-inequality for martingales implies the result. $\quad\square$

This result is used to prove Lemma 4.22. The rest is entirely similar to the supercritical case. Therefore, we obtain a similar convergence result.

Theorem 4.46. $H^N \Longrightarrow H$ *in* $\mathcal{C}([0, \infty))$, *as* $N \longrightarrow \infty$, *where the process* H *equals* $2/\kappa$ *multiplied by Brownian motion with drift* $(\alpha - \beta)s/\kappa$, *reflected above* 0.

Remark 4.47. The critical case cannot be treated as the subcritical case. In other words, in the case $\alpha = \beta$, we cannot choose $\Gamma = +\infty$, since $\mathbb{E}(\Delta_{1,1}^N)$ would no longer be bounded (see Cor. 4.12).

Remark 4.48. From our convergence results, we can as in [2] (see also Thm. 3.1 in Delmas [6]) deduce the well-known second Ray–Knight theorem, in the subcritical, critical and supercritical cases.

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