Multi-patch epidemic models with general exposed and infectious periods

GUODONG PANG AND ÉTIENNE PARDOUX

Abstract. We study multi-patch epidemic models where individuals may migrate from one patch to another in either of the susceptible, exposed/latent, infectious and recovered states. We assume that infections occur both locally with a rate that depends on the patch as well as "from distance" from all the other patches. The exposed and infectious periods have general distributions, and are not affected by the possible migrations of the individuals. The migration processes in either of the three states are assumed to be Markovian, and independent of the exposed and infectious periods. We establish a functional law of large number (FLLN) and a function central limit theorem (FCLT) for the susceptible, exposed/latent, infectious and recovered processes. In the FLLN, the limit is determined by a set of Volterra integral equations. In the special case of deterministic exposed and infectious periods, the limit becomes a system of ODEs with delays. In the FCLT, the limit is given by a set of stochastic Volterra integral equations driven by a sum of independent Brownian motions and continuous Gaussian processes with an explicit covariance structure.

1. Introduction

Multi-patch epidemic models have been used to study various infectious diseases, for example, nosocomial infection [22], vector-borne diseases [21], HIV/AIDS transmission [20], SARS epidemic [24], and so on. They are often used to capture the heterogeneity between different geographic locations, for example, a densely populated city and a less populated rural area. It also helps to study the effect of migrations or lock-down measures among different population groups or locations. In the Covid-19 pandemic, it has been observed that the infectivity in different regions may vary and is impacted by various social-distance and lock-down measures [29, 31].

ODE models are often used to study the dynamics of such multi-patch epidemic models. It is well known that the ODE dynamics arises from the Markovian assumptions in the stochastic multi-patch epidemic model, that is, the infection process is Poisson, the infectious (and/or exposed/latent) periods are exponentially distributed and the migration processes are also Markovian [1, 8, 23, 24, 4, 21]. Some ODE/PDE models are also used to study their dynamics when the infection rates are age-dependent (depending on how long the population has been infected), see, for example, [22, 12]. These models also assume exponentially distributed infectious periods and Markovian migration processes.

In this paper, we study multi-patch SEIR models, in which the exposed/latent and infectious periods have a general joint distribution (possibly correlated), while the migration processes are Markovian. The infection is assumed to be both local, and from distance. That is, the infection rate in a given patch depends on the susceptible population in that patch, and on the infectious population in all the patches. Individuals may migrate from one patch to another in each of the Susceptible, Exposed (Latent), Infected and Recovered stages. The reason for infection at distance is twofold. First, if we set some of the migration rates to 0, we could consider some of the patches as substructures of the population, like age classes, which infect each other. Second, some of the movements of the population should not be considered as migrations, but visits from one patch to another, during a week–end or holiday, with a return at home at the end of a short period. Such movements may produce infection of a susceptible individual in patch i by an infectious individual

Key words and phrases. multi-patch SIR/SEIR model, general infectious (and/or exposing, latent) periods, migration, FLLN, FCLT, multi-dimensional (stochastic) Volterra integral equations, Poisson random measure.

from patch $i' \neq i$. Those displacements would be very complicated to model as such. We think that infection at distance is a reasonable way to model infections due to such movements.

If the exposed and infectious periods were independent and exponentially distributed, one could derive the deterministic approximations described by ODEs for the epidemic dynamics (see, for example, those used in [31]). As for the one-patch SEIR models with general exposed and infectious periods (possibly correlated), we have to use Volterra integral equations (involving the distribution function of the exposed and infectious periods) as deterministic approximations for the epidemic dynamics [28]. In particular, the LLN limiting models are models with memory.

We describe the evolution dynamics by tracking the time epochs of becoming exposed and/or infectious and the location of an individual at these event times. Specifically, in the multi-patch SEIR model, each individual tracks the time epochs of becoming exposed, infected and recovered, and is associated with two Markov chains that are used to track their movement starting when the individual becomes exposed and infectious, respectively. For the initially exposed and/or infected individuals, we also assume that their remaining exposing and/or infectious periods have general distributions, which may be different from those of the newly exposed/infected individuals. For these initially exposed/infectious individuals, we also track their movement among the patches using Markov chains while being exposed/infectious.

Given the representations with these time epochs and location processes, we show a functional law of large numbers (FLLN) and a functional central limit theorem (FCLT) for the associated counting processes in the multi-patch SEIR model. The FLLN limits are determined by a set of Volterra integral equations. When the infectious (and exposed/latent) periods are deterministic, we can write the fluid integral equations as a set of ODEs with delay (Remark 2.1). The limit processes in the FCLT are determined by a set of stochastic Volterra integral equations, driven by a sum of independent Brownian motions and continuous Gaussian processes with a certain covariance structure. When the infectious (and exposed/latent) periods are deterministic, the limits become stochastic differential equations with linear drifts and delay (Remark 2.3). We discuss how the results simplify in the SIR model as a special case, and also how the approach and results can be extended to study multi-patch SIS and SIRS models (Section 2.4).

In the proofs of these results, we employ Poisson random measures (PRMs) that are constructed as the sums of the Dirac masses at the time epochs of becoming exposed and infectious, the infectious and exposing periods and the Markov chain starting from the location of each individual at those epochs. While to the Markov migration process are naturally associated martingales, which are easily proved to be tight in the appropriate path space, the non–Markovian nature of the epidemic process does not produce obvious martingales. However, as in our previous work [28], we are able to find various martingales attached to various and non–standard filtrations, which are constructed from our representation of the epidemic by integrals with respect to Poisson Random Measures. We use the martingale properties and convergence theorems as critical tools in the proofs. For the single-patch SIR and SEIR models with general infectious and exposing periods, an approach using PRMs that are constructed at the time epochs of becoming infectious (and/or exposed), was developed in [28]. The approach is further developed in this paper for multi-patch SIR and SEIR models, to track the locations of each individual at each event epoch. Incorporating infection from distance in addition to local infections in the model also brings in new technical challenges in the proofs of both the FLLN and FCLT.

This paper contributes to the limited literature on stochastic epidemic models with general infectious periods. We refer the readers to the overview in Chapter 3.4 of [8] on the common approaches to study non-Markovian epidemic models and LLN and CLT for the final sizes of the epidemic; see also the recent method using piecewise Markov deterministic processes in [10] and [17] for the SIR model. FLLNs and FCLTs are proved for some age and density dependent population models in [33, 34, 35], which includes the SIR model with the infection rate depending on the number of infectious individuals as a special case. Reinert [30] proves an FLLN for the empirical

measure of the SIR epidemic dynamics using Stein's method, while no FCLT has been proved with that approach. In [28], both FLLN and FCLT were established for the epidemic dynamics in the classical models (SIR, SIS, SEIR, SIRS) where the PRM representations of the dynamics plays a fundamental role in the proofs. The FCLT limit for the SIR model in [28] is similar to that in [33, 34, 35], however, the proof approaches are completely different; in addition, the distribution function of the infectious periods is assumed to be continuously differentiable in [33, 34, 35] while no condition is imposed in [28]. We highlight that the distribution functions of the exposed and infectious periods in this paper are general without requiring any conditions. The integral equations for the SEIR model in [28] are also used to estimate the state of the Covid-19 epidemic in [16]. For SIR and SEIR models with varying infectivity, where each individual is associated with an i.i.d. random infectivity, which is a function of the time elapsed since infection, FLLN and FCLT have recently been established in [15, 27]. Although Volterra integral equations were used to describe the proportion of infectious population in the SIS, SIR or SEIR model without proving an FLLN (see [6, 7, 11, 13, 19, 32]), as far as we know no Volterra integral equations have been proposed so far for multi-patch epidemic models with general infectious (and/or exposed) periods. Our work shows both FLLN and FCLT for non-Markovian multi-patch models, and identify (stochastic) multidimensional Volterra integral equations as their limits.

It is also worth mentioning the multi-type epidemic models where the population splits up into multiple groups of individuals and each group may infect any other group in addition to itself (no migration), see Chapters 6.1 and 6.2 in [1] and [2, 3]. The special case of our model with zero migration rates covers that situation. In those models, proportionate mixing taking into account control measures like social distance or lockdowns can also be incorporated.

1.1. **Notation.** Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers, and $\mathbb{R}^k(\mathbb{R}_+^k)$ denotes the space of k-dimensional vectors with real (nonnegative) coordinates, with $\mathbb{R}(\mathbb{R}_+)$ for $k=1$. For $x,y\in\mathbb{R}$, denote $x\wedge y=\min\{x,y\}$ and $x\vee y=\max\{x,y\}$. For $x\in\mathbb{R}$, let $x^+=x\vee 0$ and $x^-=-(x\wedge 0)$. Let $D=D([0,T],\mathbb{R})$ denote the space of $\mathbb{R}$–valued càdlàg functions defined on $[0,T]$. Throughout the paper, convergence in D means convergence in the Skorohod J_1 topology, see chapter 3 of [5]. Also, D^k stands for the k-fold product equipped with the product topology. Let C be the subset of D consisting of continuous functions. Let C^1 consist of differentiable functions whose derivative is continuous. For any function $x\in D$, we use the notation $\|x\|_T=\sup_{t\in[0,T]}|x(t)|$. For two functions $x,y\in D$, we use $x\circ y(t)=x(y(t))$ denote their composition. All random variables and processes are defined on a common complete probability space $(\Omega,\mathcal{F},\mathbb{P})$. The notation $\Rightarrow$ means convergence in distribution. We use $\mathbf{1}_{\{.\}}$ for indicator function, and occasionally we shall write $\mathbf{1}\{.\}$ in case the first notation is not readable enough. We use small-o notation for real-valued functions f and non-zero g: $f(x)=o(g(x))$ if $\limsup_{x\to\infty}|f(x)/g(x)|=0$. We use $\hat{\imath}$ to denote the unit imaginary number.

2. Model and Results

2.1. **Model description.** We consider a multi-patch epidemic model, where individuals in each patch experience the Susceptible–Exposed (Latent)–Infectious–Recovered (SEIR) process. The patches may refer to populations in different locations, for example, a densely populated city and a less populated rural area. As explained in the introduction, susceptible individuals in each patch are infected both locally, by infectious individuals located in the same patch, and at distance, by infectious individuals from other patches. The rate of infection is different in each patch (because of the differences in the density of population or in the type of available public transportations), while the law of the infectious period is the same (due to the same illness).

Let N be the total population size and L be the number of patches. The set of patches will be denoted $\mathcal{L}=\{1,\dots,L\}$. (We use indices i,i',ℓ,ℓ' for elements in $\mathcal{L}$, and occasionally i'',ℓ''.) For each patch $i\in\mathcal{L}$, let $S_i^N(t),E_i^N(t),I_i^N(t),R_i^N(t)$ count the numbers of individuals that are

susceptible, exposed (latent), infectious and recovered in patch i at time t, respectively. We have the balance equation:

$$N \;=\; \sum_{i=1}^{L}(S_i^N(t)+E_i^N(t)+I_i^N(t)+R_i^N(t))\,,\quad t\geq 0\,. \tag{2.1}$$

Assume that $S_i^N(0)>0, \sum_{i=1}^L (E_i^N(0)+I_i^N(0))>0$ and $R_i^N(0)=0$ for each $i\in\mathcal{L}$.

Let λ_i be the infection rate of patch $i\in\mathcal{L}$. Define the following processes, for some $0\leq\gamma\leq 1$,

$$\Upsilon_i^N(t) \;=\; \frac{S_i^N(t)\sum_{\ell=1}^L \kappa_{i\ell} I_\ell^N(t)}{N^{1-\gamma}(S_i^N(t)+E_i^N(t)+I_i^N(t)+R_i^N(t))^\gamma}\,,\quad i\in\mathcal{L}\,, \tag{2.2}$$

where $\kappa_{ii}=1$ and $\kappa_{i\ell}\geq 0$ for $i\neq\ell$ represent the infectivity from distance. The rate of new infections in patch i at time t is $\lambda_i\Upsilon_i^N(t)$. Let us explain the role of the parameter γ. Let $\bar\kappa_i:=\sum_{\ell=1}^L\kappa_{i\ell}$ and $\bar\kappa:=\max_{i\in\mathcal{L}}\bar\kappa_i$.

In the homogeneous population, where $L=1$, (2.1) tells us that in the unique patch, $S^N(t)+E^N(t)+I^N(t)+R^N(t)=N$, hence $\Upsilon^N(t)$ is the same, irrespective of the value of γ. The rationale of this form of the infection rate is as follows. Each infectious individual meets others at rate β. If the individual who is met is susceptible, which happens with probability equal to the proportion of susceptible individuals, i.e., $S^N(t)/N$, then the encounter results in a new infection with probability p. If we let $\lambda=\beta\times p$, we find the above formula for the rate of infection in case $L=1$. Now, consider the case $L>1$. We do not factorize λ into $\beta\times p$ anymore, or equivalently do as if $p=1$. In the case $\gamma=1$, the rate of encounters of individuals in patch i by a given infectious is fixed, equal to λ_i for an infectious of the same patch, and equal to $\lambda_i\kappa_{ii'}$ for an infectious from patch i', whatever the total population in patch i at time t may be. In case $\gamma=0$, the same rate is proportional to the total population of patch i at time t. In the intermediate cases, the rate lies between those two extremes. The case $\gamma=1$ seems to be used in most spatial epidemics models. The values of λ_i's can correct for the different densities of population of the various patches, resulting in more or less encounters. However, especially in the stochastic model, the population size in each patch may fluctuate significantly, which is a motivation for using a model with $\gamma<1$.

We shall prove the FLLN for any value of $\gamma\in[0,1]$, and the FCLT only for $\gamma\in[0,1)$ in the general case, and for all $\gamma\in[0,1]$ in the case that infections are only local, i.e., $\kappa_{i\ell}=0$ for $i\neq\ell$. The reason for this restriction is that in the case $\gamma=1$ and $\sum_{\ell\neq i}\kappa_{i\ell}>0$, we are not able to establish the estimate (4.18) in Lemma 4.3 below.

Note that $\Upsilon_i^N(t)\leq\left(\frac{S_i^N(t)}{N}\right)^{1-\gamma}\sum_{\ell=1}^L\kappa_{i\ell}I_\ell^N(t)$, so that in the case $\gamma<1$, $\Upsilon_i^N(t)=0$ whenever $S_i^N(t)+I_i^N(t)+R_i^N(t)=0$. By convention, we shall assume that the same holds in case $\gamma=1$, i.e., $\frac{0}{0}=0$. $\lambda_i\Upsilon_i^N(t)$ is the rate of new infections in patch i at time t. It is of course 0 if patch i is empty.

Let $A_i^N(t)$ be the cumulative counting process of individuals in patch i that get infected on the time interval $(0,t]$. Then we can give a representation of the process $A_i^N(t)$ via the standard Poisson random measure (PRM) Q_i on $\mathbb{R}_+^2$ (with mean measure $dsd\mathfrak{a}$), the various $\{Q_i,\ i\in\mathcal{L}\}$ being mutually independent,

$$A_i^N(t) \;=\; \int_0^t\int_0^\infty \mathbf{1}_{\mathfrak{a}\leq\lambda_i\Upsilon_i^N(s)}Q_i(ds,d\mathfrak{a})\,,\quad t\geq 0\,. \tag{2.3}$$

Equivalently, we could write

$$A_i^N(t) \;=\; P_{A,i}\left(\lambda_i\int_0^t\Upsilon_i^N(s)ds\right)\,,\quad t\geq 0\,, \tag{2.4}$$

where $P_{A,i}$ is a unit-rate Poisson process, and independent from each other for $i \in \mathcal{L}$. But the first description will be more useful for us. We let $\{\tau^N_{j,i},\ j \geq 1\}$ denote the successive jump times of the process A^N_i, for $i \in \mathcal{L}$.

The $E^N_i(0)$ initially exposed individuals experience the exposed and infectious periods before recovery. Let $\{\eta^0_{k,i} : k = 1, \ldots, E^N_i(0)\}$ be the remaining exposed periods of the initially exposed individuals in patch i. After the exposed period, let $\{\zeta_{-k,i} : k = 1, \ldots, E^N_i(0)\}$ be the durations of their infectious periods. The $I^N_i(0)$ initially infected individuals experience a remaining infectious period before recovery, and let $\zeta^0_{k,i}$, $k = 1, \ldots, I^N_i(0)$, denote their remaining infectious periods. The $A^N_i(t)$ newly infected individuals in patch i experience the exposed and infectious periods. Let $\{\eta_{j,i} : j \in \mathbb{N}\}$ and $\{\zeta_{j,i} : j \in \mathbb{N}\}$ be the associated exposing and infectious periods.

Assume that $\{\zeta^0_{k,i}\}$, $\{(\eta^0_{k,i}, \zeta_{-k,i})\}$ and $\{(\eta_{j,i}, \zeta_{j,i})\}$ are all i.i.d. sequences of random variables having distribution functions F_0, $H_0(du, dv)$ and $H(du, dv)$, respectively, and they are also mutually independent. Note that $\zeta_{j,i}$ is defined for $j \in \mathbb{Z}$ and $i \in \mathcal{L}$ (those with $j < 0$ code the infectious periods of the initially exposed individuals, while those with $j > 0$ code the infectious periods of the newly exposed individuals). Let G_0 and F be the marginals of H_0 for $\eta^0_{k,i}$ and $\zeta_{-k,i}$, and G and F be the marginals of H for $\eta_{k,i}$ and $\zeta_{k,i}$, respectively. (It is reasonable to assume that the marginal distributions of $\zeta_{-k,i}$ and $\zeta_{k,i}$ are the same.) Also let $F_0(\cdot|u)$ and $F(\cdot|u)$ be the conditional c.d.f.'s of $\zeta_{-k,i}$ and $\zeta_{k,i}$, given that $\eta^0_{k,i} = u$ and $\eta_{k,i} = u$, respectively. Let $G^c_0 = 1 - G_0$, $G^c = 1 - G$, $F^c_0 = 1 - F_0$ and $F^c = 1 - F$.

Individuals may migrate from patch ℓ to ℓ' in any of the four epidemic stages, with rates $\nu_{S,\ell,\ell'}$, $\nu_{E,\ell,\ell'}$, $\nu_{I,\ell,\ell'}$ and $\nu_{R,\ell,\ell'}$ for the susceptible, exposed, infectious and recovered ones, respectively. For each individual, the times between migrations in each of the stages are exponentially distributed.

In order to track the location/patch of the individual j who gets exposed at time $\tau^N_{j,\ell}$ in patch ℓ, we first use the Markov process X^j_ℓ, taking values in $\mathcal{L}$, associated with the rates $\nu_{E,\ell,\ell'}$ for $\ell' \in \mathcal{L}$. It takes effect from the time $\tau^N_{j,\ell}$ of becoming exposed, until the time $\tau^N_{j,\ell} + \eta_{j,\ell}$ when this individual becomes infectious. Given that this individual has migrated to patch ℓ' at the end of the exposed period (she/he may have done several migrations to other patches during the exposed period), that is, $X^j_\ell(\eta_{j,\ell}) = \ell'$, we then use another Markov process $Y^j_{\ell'}$ to track the location/patch of the individual during the infectious period $\zeta_{j,\ell'}$, associated with rates $\nu_{I,\ell',\ell''}$ for $\ell'' \in \mathcal{L}$. This process $Y^j_{\ell'}$ only takes effect from the time of becoming infectious $\tau^N_{j,\ell} + \eta_{j,\ell}$, until the time of recovery $\tau^N_{j,\ell} + \eta_{j,\ell} + \zeta_{j,\ell'}$. Suppose that the individual has migrated to patch i at the end of the infectious period, that is, $Y^j_{\ell'}(\zeta_{j,\ell'}) = i$. The individual will then belong to the compartment of recovered individuals, and will migrate among patches according to the rates $\nu_{R,i,i'}$ for $i' \in \mathcal{L}$. Similarly we use $X^{0,k}_\ell$ and $Y^{-k}_{\ell'}$ for the initially exposed individuals $k = 1, \ldots, E^N_\ell(0)$ that have been exposed at time 0 in patch ℓ. They are again associated with the rates $\nu_{E,\ell,\ell'}$ for $\ell' \in \mathcal{L}$, and $\nu_{I,\ell',\ell''}$ for $\ell'' \in \mathcal{L}$, respectively. In addition, we also use $Y^{0,k}_\ell$ for the initially infectious individuals $k = 1, \ldots, I^N_\ell(0)$ that have been infectious at time 0 in patch ℓ. It is again associated with the rates $\nu_{I,\ell,\ell'}$ for $\ell' \in \mathcal{L}$. We assume that for each j, X^j_ℓ and $Y^j_{\ell'}$ are independent for $\ell, \ell' \in \mathcal{L}$, and for each k, $X^{0,k}_\ell$ and $Y^{-k}_{\ell'}$ are independent for $\ell, \ell' \in \mathcal{L}$. We also assume that all these Markov processes $\{X^j_\ell, Y^j_\ell\}_{j,\ell}$, $\{X^{0,k}_\ell, Y^{-k}_\ell\}_{k,\ell}$ and $\{Y^{0,k}_\ell\}_{k,\ell}$ are mutually independent.

Let $p_{\ell,\ell'}(t) = \mathbb{P}(X^j_\ell(t) = \ell' | X^j_\ell(0) = \ell)$ and $q_{\ell,\ell'}(t) = \mathbb{P}(Y^j_\ell(t) = \ell' | Y^j_\ell(0) = \ell)$ for $\ell, \ell' \in \mathcal{L}$, $j \geq 1$ and $t \geq 0$. For each ℓ, the processes $\{X^{0,k}_\ell\}_k$ have the same transition function $(p_{\ell,\ell'}(\cdot))_{\ell' \in \mathcal{L}}$ as $\{X^j_\ell\}_j$, and the processes $\{Y^{-k}_\ell\}_k$ and $\{Y^{0,k}_\ell\}_k$ have the same transition function $(q_{\ell,\ell'}(\cdot))_{\ell' \in \mathcal{L}}$ as $\{Y^j_\ell\}_j$.

The multi-patch SEIR epidemic evolution dynamics can be described as follows:

$$S_i^N(t) \;=\; S_i^N(0) - A_i^N(t) + \sum_{\ell=1,\ell\neq i}^{L}\left(P_{S,\ell,i}\left(\nu_{S,\ell,i}\int_0^t S_\ell^N(s)ds\right) - P_{S,i,\ell}\left(\nu_{S,i,\ell}\int_0^t S_i^N(s)ds\right)\right), \tag{2.5}$$

$$E_i^N(t) \;=\; \sum_{\ell=1}^{L}\sum_{k=1}^{E_\ell^N(0)} \mathbf{1}_{t<\eta_{k,\ell}^0}\mathbf{1}_{X_\ell^{0,k}(t)=i} + \sum_{\ell=1}^{L}\sum_{j=1}^{A_\ell^N(t)} \mathbf{1}_{\tau_{j,\ell}^N+\eta_{j,\ell}>t}\mathbf{1}_{X_\ell^j(t-\tau_{j,\ell}^N)=i}\;, \tag{2.6}$$

$$\begin{aligned} I_i^N(t) \;=\;& \sum_{\ell=1}^{L}\sum_{k=1}^{I_\ell^N(0)} \mathbf{1}_{t<\zeta_{k,\ell}^0}\mathbf{1}_{Y_\ell^{0,k}(t)=i} + \sum_{\ell=1}^{L}\sum_{k=1}^{E_\ell^N(0)} \mathbf{1}_{\eta_{k,\ell}^0\le t}\left(\sum_{\ell'=1}^{L}\mathbf{1}_{X_\ell^{0,k}(\eta_{k,\ell}^0)=\ell'}\mathbf{1}_{\eta_{k,\ell}^0+\zeta_{-k,\ell'}>t}\mathbf{1}_{Y_{\ell'}^{-k}(t-\eta_{k,\ell}^0)=i}\right) \\ &+ \sum_{\ell=1}^{L}\sum_{j=1}^{A_\ell^N(t)} \mathbf{1}_{\tau_{j,\ell}^N+\eta_{j,\ell}\le t}\left(\sum_{\ell'=1}^{L}\mathbf{1}_{X_\ell^j(\eta_{j,\ell})=\ell'}\mathbf{1}_{\tau_{j,\ell}^N+\eta_{j,\ell}+\zeta_{j,\ell'}>t}\mathbf{1}_{Y_{\ell'}^j(t-\tau_{j,\ell}^N-\eta_{j,\ell})=i}\right), \end{aligned} \tag{2.7}$$

$$\begin{aligned} R_i^N(t) \;=\;& \sum_{\ell=1}^{L}\sum_{k=1}^{I_\ell^N(0)} \mathbf{1}_{\zeta_{k,\ell}^0\le t}\mathbf{1}_{Y_\ell^{0,k}(\zeta_{k,\ell}^0)=i} + \sum_{\ell=1}^{L}\sum_{k=1}^{E_\ell^N(0)} \left(\sum_{\ell'=1}^{L}\mathbf{1}_{X_\ell^{0,k}(\eta_{k,\ell}^0)=\ell'}\mathbf{1}_{\eta_{k,\ell}^0+\zeta_{-k,\ell'}\le t}\mathbf{1}_{Y_{\ell'}^{-k}(\zeta_{-k,\ell'})=i}\right) \\ &+ \sum_{\ell=1}^{L}\sum_{j=1}^{A_\ell^N(t)} \left(\sum_{\ell'=1}^{L}\mathbf{1}_{X_\ell^j(\eta_{j,\ell})=\ell'}\mathbf{1}_{\tau_{j,\ell}^N+\eta_{j,\ell}+\zeta_{j,\ell'}\le t}\mathbf{1}_{Y_{\ell'}^j(\zeta_{j,\ell'})=i}\right) \\ &+ \sum_{\ell=1,\,\ell\neq i}^{L}\left(P_{R,\ell,i}\left(\nu_{R,\ell,i}\int_0^t R_\ell^N(s)ds\right) - P_{R,i,\ell}\left(\nu_{R,i,\ell}\int_0^t R_i^N(s)ds\right)\right), \end{aligned} \tag{2.8}$$

where $P_{S,i,\ell}, P_{R,i,\ell}$, $i,\ell \in \mathcal{L}$, are mutually independent unit-rate Poisson processes, which are globally independent of the Q_i's. For the dynamics of $E_i^N(t)$, the first term represents the number of initially exposed individuals from patch ℓ that remain exposed and are in patch i at time t, and the second term represents the number of newly exposed individuals from patch ℓ that remain exposed and are in patch i at time t. In the expression for $I_i^N(t)$, the first term counts the number of initially infectious individuals from all the patches that remain infectious and are in patch i at time t, and the second term counts the numbers of initially exposed individuals from all the patches that have become infectious and are in patch i at time t (for tracking purposes, the location at the epochs of becoming infectious is recorded). Also note that we use the Markov process $Y_\ell^{0,k}$ to indicate that these are for the initially exposed individuals. The third term counts the number of newly exposed individuals at all patches that have become infectious and are in patch i at time t, and we also track the patch in which each individual has become infectious. In the expression for $R_i^N(t)$, the first term represents the number of initially infectious individuals from patch ℓ that have recovered by time t and were in patch i at the time of recovery, the second term represents the number of initially exposed individuals from patch ℓ that have recovered by time t, and were in patch i at the time of recovery, while becoming infectious in patch ℓ', the third term represents the number of newly exposed individuals from patch ℓ that have recovered by time t, and were in patch i at the time of recovery while becoming infectious in patch ℓ'.

It is not easy to take the limit as $N\to\infty$ in the formulas of E_i^N and I_i^N above. We now derive the following representations, which will be very helpful in the proofs of our results.

Lemma 2.1. *We have*

$$E_i^N(t) \;=\; E_i^N(0) - \sum_{\ell=1}^{L}\sum_{k=1}^{E_\ell^N(0)} \mathbf{1}_{\eta_{k,\ell}^0\le t}\mathbf{1}_{X_\ell^{0,k}(\eta_{k,\ell}^0)=i} + A_i^N(t) - \sum_{\ell=1}^{L}\sum_{j=1}^{A_\ell^N(t)} \mathbf{1}_{\tau_{j,\ell}^N+\eta_{j,\ell}\le t}\mathbf{1}_{X_\ell^j(\eta_{j,\ell})=i}$$

$$+\sum_{\ell\neq i} P_{E,\ell,i}\left(\nu_{E,\ell,i}\int_0^t E_\ell^N(s)ds\right)-\sum_{\ell\neq i} P_{E,i,\ell}\left(\nu_{E,i,\ell}\int_0^t E_i^N(s)ds\right), \tag{2.9}$$

$$\begin{aligned}
I_i^N(t) \;=\; & I_i^N(0)-\sum_{\ell=1}^{L}\sum_{k=1}^{I_\ell^N(0)}\mathbf{1}_{\zeta_{k,\ell}^0\le t}\mathbf{1}_{Y_\ell^{0,k}(\zeta_{k,\ell}^0)=i}\\
&+\sum_{\ell=1}^{L}\sum_{k=1}^{E_\ell^N(0)}\mathbf{1}_{\eta_{k,\ell}^0\le t}\mathbf{1}_{X_\ell^{0,k}(\eta_{k,\ell}^0)=i}-\sum_{\ell=1}^{L}\sum_{k=1}^{E_\ell^N(0)}\left(\sum_{\ell'=1}^{L}\mathbf{1}_{X_\ell^{0,k}(\eta_{k,\ell}^0)=\ell'}\mathbf{1}_{\eta_{k,\ell}^0+\zeta_{-k,\ell'}\le t}\mathbf{1}_{Y_{\ell'}^{-k}(\zeta_{-k,\ell'})=i}\right)\\
&+\sum_{\ell=1}^{L}\sum_{j=1}^{A_\ell^N(t)}\mathbf{1}_{\tau_{j,\ell}^N+\eta_{j,\ell}\le t}\mathbf{1}_{X_\ell^j(\eta_{j,\ell})=i}-\sum_{\ell=1}^{L}\sum_{j=1}^{A_\ell^N(t)}\left(\sum_{\ell'=1}^{L}\mathbf{1}_{X_\ell^j(\eta_{j,\ell})=\ell'}\mathbf{1}_{\tau_{j,\ell}^N+\eta_{j,\ell}+\zeta_{j,\ell'}\le t}\mathbf{1}_{Y_{\ell'}^j(\zeta_{j,\ell'})=i}\right)\\
&+\sum_{\ell\neq i} P_{I,\ell,i}\left(\nu_{I,\ell,i}\int_0^t I_\ell^N(s)ds\right)-\sum_{\ell\neq i} P_{I,i,\ell}\left(\nu_{I,i,\ell}\int_0^t I_i^N(s)ds\right).
\end{aligned} \tag{2.10}$$

where $P_{E,i,\ell}$ and $P_{I,i,\ell}$, $i,\ell\in\mathcal{L}$, are all unit-rate Poisson processes, mutually independent, and also independent of $P_{A,i}$, $P_{S,i,\ell}$ and $P_{R,i,\ell}$.

Before turning to the proof, let us comment on these formulas. In the expression of $E_i^N(t)$, the first and third term count the number of initially exposed individuals in patch i, and the number of those whose became exposed in patch i on the time interval $[0,t]$. The second and fourth terms subtract the numbers of initially and newly exposed individuals in any patch who have become infectious before time t in patch i. Finally the last two term count the numbers of migrations of exposed individuals to and from patch i. The expression of $I_i^N(t)$ is similar, except that the second term subtracts the number of initially infectious individuals in any patch who have recovered before time t in patch i, and the fourth and sixth terms subtract the numbers of initially and newly exposed individuals in any patch who have recovered before time t in patch i, where the patch in which they became infectious is also tracked.

Proof. In the representation of $E_i^N(t)$, we observe that

$$\sum_{k=1}^{E_i^N(0)}\mathbf{1}_{t<\eta_{k,i}^0}\mathbf{1}_{X_i^{0,k}(t)=i}=E_i^n(0)-\sum_{k=1}^{E_i^N(0)}\mathbf{1}_{\eta_{k,i}^0\le t}\mathbf{1}_{X_\ell^{0,k}(\eta_{k,i}^0)=i}-\sum_{\ell\neq i}V_{i,\ell}^{N,0}(t)$$

and for $\ell\neq i$,

$$\sum_{k=1}^{E_\ell^N(0)}\mathbf{1}_{t<\eta_{k,\ell}^0}\mathbf{1}_{X_\ell^{0,k}(t)=i}=V_{\ell,i}^{N,0}(t)-\sum_{k=1}^{E_\ell^N(0)}\mathbf{1}_{\eta_{k,\ell}^0\le t}\mathbf{1}_{X_\ell^{0,k}(\eta_{k,\ell}^0)=i}$$

where $V_{i,\ell}^{N,0}(t)$ is the number of initially exposed individuals from patch i that are in patch ℓ at the time $t\wedge\eta_{k,\ell}^0$ for $k=1,\dots,E_\ell^N(0)$. We also observe that

$$\sum_{j=1}^{A_i^N(t)}\mathbf{1}_{\tau_{j,i}^N+\eta_{j,i}>t}\mathbf{1}_{X_i^j(t-\tau_{j,\ell}^N)=i}=A_i^N(t)-\sum_{j=1}^{A_i^N(t)}\mathbf{1}_{\tau_{j,i}^N+\eta_{j,i}\le t}\mathbf{1}_{X_i^j(\eta_{j,i})=i}-\sum_{\ell\neq i}V_{i,\ell}^N(t)$$

and for $\ell\neq i$,

$$\sum_{j=1}^{A_\ell^N(t)}\mathbf{1}_{\tau_{j,\ell}^N+\eta_{j,\ell}>t}\mathbf{1}_{X_\ell^j(t-\tau_{j,\ell}^N)=i}=V_{\ell,i}^N(t)-\sum_{j=1}^{A_\ell^N(t)}\mathbf{1}_{\tau_{j,\ell}^N+\eta_{j,\ell}\le t}\mathbf{1}_{X_\ell^j(\eta_{j,\ell})=i}$$

where $V^N_{i,\ell}(t)$ denotes the number of individuals who became exposed at time $\tau^N_{j,\ell} \in (0,t)$ in patch i, and are in patch ℓ at time $t \wedge (\tau^N_{j,\ell} + \eta_{j,\ell})$ for $j = 1, \dots, A^N_\ell(t)$.

It is clear that

$$\begin{aligned}&\sum_{\ell \neq i} V^N_{\ell,i}(t) - \sum_{\ell \neq i} V^N_{i,\ell}(t) + \sum_{\ell \neq i} V^{N,0}_{\ell,i}(t) - \sum_{\ell \neq i} V^{N,0}_{i,\ell}(t)\\ &= \sum_{j \neq i} P_{E,\ell,i}\left(\nu_{E,\ell,i} \int_0^t E^N_\ell(s)ds\right) - \sum_{\ell \neq i} P_{E,i,\ell}\left(\nu_{E,i,\ell} \int_0^t E^N_i(s)ds\right).\end{aligned}$$

Thus, using the above identities, we obtain the expression in (2.9). A similar argument gives the expression in (2.10). □

Note that some of the key components in the dynamics above can be represented via PRMs. The infection process A^N_ℓ has the same representation in (2.3) using the PRM Q_ℓ. Define a PRM $\check{Q}_\ell(ds, d\mathfrak{a}, du, d\theta)$ on $\mathbb{R}^3_+ \times \mathcal{L}$, which is the sum of the Dirac masses at the points $(\tau^N_{j,\ell}, \mathfrak{A}^N_{j,\ell}, \eta_{j,\ell}, X^j_\ell(\eta_{j,\ell}))$ with mean measure $ds \times d\mathfrak{a} \times G(du) \times \mu^X_\ell(u, d\theta)$, where for each $u > 0$, $\mu^X_\ell(u, \{\ell'\}) = p_{\ell,\ell'}(u)$, and an infection occurs at time $\tau^N_{j,\ell}$ if and only if $\mathfrak{A}^N_{j,\ell} \leq \lambda_\ell \Upsilon^N(\tau^N_{j,\ell})$. We can then write for $\ell, \ell' \in \mathcal{L}$,

$$\sum_{j=1}^{A^N_\ell(t)} \mathbf{1}_{\tau^N_{j,\ell} + \eta_{j,\ell} \leq t} \mathbf{1}_{X^j_\ell(\eta_{j,\ell}) = \ell'} = \int_0^t \int_0^\infty \int_0^{t-s} \int_{\{\ell'\}} \mathbf{1}_{\mathfrak{a} \leq \lambda_\ell \Upsilon^N_\ell(s^-)} \check{Q}_\ell(ds, d\mathfrak{a}, du, d\theta). \tag{2.11}$$

We denote the corresponding compensated PRM $\overline{Q}_\ell(ds, d\mathfrak{a}, du, d\theta) = \check{Q}_\ell(ds, d\mathfrak{a}, du, d\theta) - ds \times d\mathfrak{a} \times G(du) \times \mu^X_\ell(u, d\theta)$ for $\ell, \ell' \in \mathcal{L}$.

Define another PRM $\breve{Q}_\ell(ds, d\mathfrak{a}, du, d\theta, dv, d\vartheta)$ on $\mathbb{R}^4_+ \times \mathcal{L}^2$, which is the sum of the Dirac masses at the points $(\tau^N_{j,\ell}, \mathfrak{A}^N_{j,\ell}, \eta_{j,\ell}, \zeta_{j,\ell'}, X^j_\ell(\eta_{j,\ell}), Y^j_{\ell'}(\zeta_{j,\ell'}))$ with mean measure $ds \times d\mathfrak{a} \times H(du, dv) \times \mu^X_\ell(u, d\theta) \times \mu^Y_\theta(v, d\vartheta)$, where for each $u > 0$, $\mu^X_\ell(u, \{\ell'\}) = p_{\ell,\ell'}(u)$, and for each $v > 0$, $\mu^Y_\ell(v, \{\ell'\}) = q_{\ell,\ell'}(v)$, and again an infection occurs at time $\tau^N_{j,\ell}$ if and only if $\mathfrak{A}^N_{j,\ell} \leq \lambda_\ell \Upsilon^N((\tau^N_{j,\ell})^-)$. We can then write for $\ell, i \in \mathcal{L}$,

$$\begin{aligned}&\sum_{j=1}^{A^N_\ell(t)} \mathbf{1}_{\tau^N_{j,\ell} + \eta_{j,\ell} \leq t} \left(\sum_{\ell'=1}^{L} \mathbf{1}_{X^j_\ell(\eta_{j,\ell}) = \ell'} \mathbf{1}_{\tau^N_{j,\ell} + \eta_{j,\ell} + \zeta_{j,\ell'} \leq t} \mathbf{1}_{Y^j_{\ell'}(\zeta_{j,\ell'}) = i}\right)\\ &= \int_0^t \int_0^\infty \int_0^{t-s} \int_0^{t-s-u} \int_{\mathcal{L}} \int_{\{i\}} \mathbf{1}_{\mathfrak{a} \leq \lambda_\ell \Upsilon^N_\ell(s^-)} \breve{Q}_i(ds, d\mathfrak{a}, du, dv, d\theta, d\vartheta).\end{aligned} \tag{2.12}$$

We denote the corresponding compensated PRM $\widetilde{Q}_\ell(ds, d\mathfrak{a}, du, dv, d\theta, d\vartheta) = \breve{Q}_\ell(ds, d\mathfrak{a}, du, dv, d\theta, d\vartheta) - ds \times d\mathfrak{a} \times H(du, dv) \times \mu^X_\ell(u, d\theta) \times \mu^Y_\theta(v, d\vartheta)$ for $\ell, i \in \mathcal{L}$.

2.2. **FLLN.** For any process Z^N, let $\bar{Z}^N := N^{-1} Z^N$.

Assumption 2.1. *There exist constants* $0 < \bar{S}_i(0) \leq 1, 0 \leq \bar{E}_i(0) < 1, 0 \leq \bar{I}_i(0) < 1$ *with* $\sum_{i=1}^L [\bar{E}_i(0) + \bar{I}_i(0)] > 0$ *such that* $\sum_{i=1}^L (\bar{S}_i(0) + \bar{E}_i(0) + \bar{I}_i(0)) = 1$ *and* $(\bar{S}^N_i(0), \bar{E}^N_i(0), \bar{I}^N_i(0),\ i \in \mathcal{L}) \to (\bar{S}_i(0), \bar{E}^N_i(0), \bar{I}_i(0),\ i \in \mathcal{L})$ *in probability in* $\mathbb{R}^{3L}$ *as* $N \to \infty$.

The following FLLN is proved in Section 3.

Theorem 2.1. *Under Assumption 2.1,*

$$(\bar{S}^N_i, \bar{E}^N_i, \bar{I}^N_i, \bar{R}^N_i,\ i \in \mathcal{L}) \to (\bar{S}_i,\ \bar{E}_i, I_i, \bar{R}_i,\ i \in \mathcal{L}) \quad \text{in} \quad D^{4L} \quad \text{as} \quad N \to \infty, \tag{2.13}$$

in probability, locally uniformly on $[0,T]$*, where* $(\bar{S}_i(t), \bar{E}_i(t), \bar{I}_i(t), \bar{R}_i(t),\ i \in \mathcal{L}) \in C^{4L}$ *is the unique solution of the following system of deterministic integral equations:*

$$\bar{S}_i(t) \;=\; \bar{S}_i(0) - \lambda_i \int_0^t \bar{\Upsilon}_i(s)ds + \sum_{\ell=1,\ell\neq i}^{L} \int_0^t (\nu_{S,\ell,i}\bar{S}_\ell(s) - \nu_{S,i,\ell}\bar{S}_i(s))ds\,, \tag{2.14}$$

$$\begin{aligned}\bar{E}_i(t) \;=\; & \bar{E}_i(0) - \sum_{\ell=1}^{L} \bar{E}_\ell(0) \int_0^t p_{\ell,i}(u)dG_0(u) + \lambda_i \int_0^t \bar{\Upsilon}_i(s)ds \\ & - \sum_{\ell=1}^{L} \lambda_\ell \int_0^t \int_0^{t-s} p_{\ell,i}(u)dG(u)\bar{\Upsilon}_\ell(s)ds + \sum_{\ell=1,\ell\neq i}^{L} \int_0^t (\nu_{E,\ell,i}\bar{E}_\ell(s) - \nu_{E,i,\ell}\bar{E}_i(s))ds\,,\end{aligned} \tag{2.15}$$

$$\begin{aligned}\bar{I}_i(t) \;=\; & \bar{I}_i(0) - \sum_{\ell=1}^{L} \bar{I}_\ell(0) \int_0^t q_{\ell,i}(s)dF_0(s) + \sum_{\ell=1}^{L} \bar{E}_\ell(0)\Big(\int_0^t p_{\ell,i}(u)dG_0(u) - \Phi^0_{\ell,i}(t)\Big) \\ & + \sum_{\ell=1}^{L} \lambda_\ell \Big(\int_0^t \int_0^{t-s} p_{\ell,i}(u)dG(u)\bar{\Upsilon}_\ell(s)ds - \int_0^t \Phi_{\ell,i}(t-s)\bar{\Upsilon}_\ell(s)ds\Big) \\ & + \sum_{\ell=1,\ell\neq i}^{L} \int_0^t \big(\nu_{I,\ell,i}\bar{I}_\ell(s) - \nu_{I,i,\ell}\bar{I}_i(s)\big)ds\,,\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}\bar{R}_i(t) \;=\; & \sum_{\ell=1}^{L} \bar{I}_\ell(0) \int_0^t q_{\ell,i}(s)dF_0(s) + \sum_{\ell=1}^{L} \bar{E}_\ell(0)\Phi^0_{\ell,i}(t) + \sum_{\ell=1}^{L} \lambda_\ell \int_0^t \Phi_{\ell,i}(t-s)\bar{\Upsilon}_\ell(s)ds \\ & + \sum_{\ell=1,\ell\neq i}^{L} \int_0^t (\nu_{R,\ell,i}\bar{R}_\ell(s) - \nu_{R,i,\ell}\bar{R}_i(s))ds\,,\end{aligned} \tag{2.17}$$

with

$$\bar{\Upsilon}_i(t) \;:=\; \frac{\bar{S}_i(t)\sum_{j=1}^{L}\kappa_{ij}\bar{I}_j(t)}{(\bar{S}_i(t)+\bar{E}_i(t)+\bar{I}_i(t)+\bar{R}_i(t))^\gamma}\,, \tag{2.18}$$

$$\Phi^0_{\ell,i}(t) \;:=\; \int_0^t \sum_{\ell'=1}^{L} p_{\ell,\ell'}(u) \int_0^{t-u} q_{\ell' i}(v)H_0(du,dv)\,, \tag{2.19}$$

and

$$\Phi_{\ell,i}(t) \;:=\; \int_0^t \sum_{\ell'=1}^{L} p_{\ell,\ell'}(u) \int_0^{t-u} q_{\ell' i}(v)H(du,dv)\,. \tag{2.20}$$

Note that if the exposed and infectious periods are independent for each individual, we have

$$\Phi^0_{\ell,i}(t) \;:=\; \int_0^t \left(\sum_{\ell'=1}^{L} p_{\ell,\ell'}(u) \int_0^{t-u} q_{\ell' i}(v)F(dv) \right) G_0(du)\,, \tag{2.21}$$

and

$$\Phi_{\ell,i}(t) \;:=\; \int_0^t \left(\sum_{\ell'=1}^{L} p_{\ell,\ell'}(u) \int_0^{t-u} q_{\ell' i}(v)F(dv) \right) G(du)\,. \tag{2.22}$$

Remark 2.1. *Suppose the exposed and infectious periods are deterministic, taking values of $t_e > 0$ and $t_o > 0$. Also, assume that the remaining exposed and infectious periods of the initially exposed and infectious are uniformly distributed over $(0, t_e)$ and $(0, t_o)$, respectively. This means that they are the corresponding stationary excess (or equilibrium) distributions of the deterministic ones. Recall that for any c.d.f. F on $\mathbb{R}_+$, the stationary excess (equilibrium) distribution $F_e(x) := \int_0^x (1 - F(t))dt / \int_0^\infty (1 - F(t))dt$ for $x \geq 0$. Then the FLLN equations of $\bar{E}_i, \bar{I}_i, \bar{R}_i$ become*

$$\begin{aligned}\bar{E}_i(t) &= \bar{E}_i(0) - \sum_{\ell=1}^L \bar{E}_\ell(0)\frac{1}{t_e}\int_0^t p_{\ell,i}(u)du + \lambda_i \int_0^t \bar{\Upsilon}_s(s)ds \\ &\quad - \sum_{\ell=1}^L \lambda_\ell p_{\ell,i}(t_e)\int_0^{t-t_e} \bar{\Upsilon}_\ell(s)ds + \sum_{\ell=1,\ell\neq i}^L \int_0^t (\nu_{E,\ell,i}\bar{E}_\ell(s) - \nu_{E,i,\ell}\bar{E}_i(s))ds\,,\end{aligned}$$

$$\begin{aligned}\bar{I}_i(t) &= \bar{I}_i(0) - \sum_{\ell=1}^L \bar{I}_\ell(0)\frac{1}{t_o}\int_0^t q_{\ell,i}(s)ds + \sum_{\ell=1}^L \bar{E}_\ell(0)\frac{1}{t_e}\Big(\int_0^t p_{\ell,i}(u)du - \int_0^{t-t_o}\sum_{\ell'=1}^L p_{\ell,\ell'}(s)q_{\ell',i}(t_o)ds\Big) \\ &\quad + \sum_{\ell=1}^L \lambda_\ell\Big(p_{\ell,i}(t_e)\int_0^{t-t_e}\bar{\Upsilon}_\ell(s)ds - \sum_{\ell'=1}^L p_{\ell,\ell'}(t_e)q_{\ell' i}(t_o)\int_0^{t-t_e-t_o}\bar{\Upsilon}_\ell(s)ds\Big) \\ &\quad + \sum_{\ell=1,\ell\neq i}^L \int_0^t (\nu_{I,\ell,i}\bar{I}_\ell(s) - \nu_{I,i,\ell}\bar{I}_i(s))ds\,,\end{aligned}$$

and

$$\begin{aligned}\bar{R}_i(t) &= \sum_{\ell=1}^L \bar{I}_\ell(0)\frac{1}{t_o}\int_0^t q_{\ell,i}(s)ds + \sum_{\ell=1}^L \bar{E}_\ell(0)\frac{1}{t_e}\int_0^{t-t_o}\sum_{\ell'=1}^L p_{\ell,\ell'}(s)q_{\ell',i}(t_o)ds \\ &\quad + \sum_{\ell=1}^L \lambda_\ell \sum_{\ell'=1}^L p_{\ell,\ell'}(t_e)q_{\ell' i}(t_o)\int_0^{t-t_e-t_o}\bar{\Upsilon}_\ell(s)ds + \sum_{\ell=1,\ell\neq i}^L \int_0^t (\nu_{R,\ell,i}\bar{R}_\ell(s) - \nu_{R,i,\ell}\bar{R}_i(s))ds\,.\end{aligned}$$

It is easy to see that we obtain a set of ODEs with delay after taking derivative.

2.3. **FCLT.** For any process Z^N, let $\hat{Z}^N := \sqrt{N}(\bar{Z}^N - \bar{Z})$ be the diffusion-scaled process where $\bar{Z}^N := N^{-1}Z^N$ and $\bar{Z}$ is its limit.

Assumption 2.2. *There exist constants $0 < \bar{S}_i(0) \leq 1, 0 \leq \bar{E}_i(0) < 1, 0 \leq \bar{I}_i(0) < 1$ with $\sum_{i=1}^L [\bar{E}_i(0) + \bar{I}_i(0)] > 0$ such that $\sum_{i=1}^L (\bar{S}_i(0) + \bar{E}_i(0) + \bar{I}_i(0)) = 1$, and random variables $\hat{S}_i(0)$, $\hat{E}_i(0)$ and $\hat{I}_i(0)$, $i \in \mathcal{L}$, such that $(\hat{S}_i^N(0), \hat{E}_i^N(0), \hat{I}_i^N(0), i \in \mathcal{L}) \Rightarrow (\hat{S}_i(0), \hat{E}_i(0), \hat{I}_i(0), i \in \mathcal{L})$ in $\mathbb{R}^{3L}$ as $N \to \infty$. In addition, $\sup_N \mathbb{E}\big[(\hat{Z}^N(0))^2\big] < \infty$ for $\hat{Z}^N(0) = \hat{S}_i^N(0), \hat{E}_i^N(0), \hat{I}_i^N(0), \hat{R}_i^N(0)$, $i \in \mathcal{L}$.*

Theorem 2.2. *Under Assumption 2.2, if F_0 and G_0 are continuous, in the two cases (i) $\gamma \in [0, 1)$ or (ii) $\gamma \in [0, 1]$ and $\sum_{j\neq i} \kappa_{ij} = 0$,*

$$(\hat{S}_i^N, \hat{E}_i^N, \hat{I}_i^N, \hat{R}_i^N, i \in \mathcal{L}) \to (\hat{S}_i(t), \hat{E}_i, \hat{I}_i(t), \hat{R}_i(t), i \in \mathcal{L}) \quad \text{in} \quad D^{4L} \quad \text{as} \quad N \to \infty, \tag{2.23}$$

where the limit is the unique solution of the following system of stochastic Volterra integral equations driven by continuous Gaussian processes:

$$\begin{aligned}\hat{S}_i(t) &= \hat{S}_i(0) - \lambda_i \int_0^t \hat{\Upsilon}_i(s)ds + \sum_{\ell=1,\ell\neq i}^L \int_0^t (\nu_{S,\ell,i}\hat{S}_\ell(s) - \nu_{S,i,\ell}\hat{S}_i(s))ds \\ &\quad - \hat{M}_{A,i}(t) + \sum_{\ell=1,\ell\neq i}^L \big(\hat{M}_{S,\ell,i}(t) - \hat{M}_{S,i,\ell}(t)\big)\,,\end{aligned} \tag{2.24}$$

$$\begin{aligned}\hat{E}_i(t) \;=\; & \hat{E}_i(0)\left(1-\sum_{\ell=1}^{L}\int_0^t p_{\ell,i}(s)G_0(ds)\right)+\lambda_i\int_0^t\hat{\Upsilon}_i(s)ds-\sum_{\ell=1}^{L}\lambda_\ell\int_0^t\int_0^{t-s}p_{\ell,i}(u)G(du)\hat{\Upsilon}_\ell(s)ds\\ & +\sum_{\ell=1,\ell\neq i}^{L}\int_0^t(\nu_{I,\ell,i}\hat{E}_\ell(s)-\nu_{I,i,\ell}\hat{E}_i(s))ds-\sum_{\ell=1}^{L}\left(\hat{E}^0_{\ell,i}(t)+\hat{E}_{\ell,i}(t)\right)\\ & +\hat{M}_{A,i}(t)+\sum_{\ell=1,\ell\neq i}^{L}\left(\hat{M}_{E,\ell,i}(t)-\hat{M}_{E,i,\ell}(t)\right), \end{aligned}\tag{2.25}$$

$$\begin{aligned}\hat{I}_i(t) \;=\; & \hat{I}_i(0)\left(1-\sum_{\ell=1}^{L}\int_0^t q_{\ell,i}(s)F_0(ds)\right)+\hat{E}_i(0)\sum_{\ell=1}^{L}\left(\int_0^t p_{\ell,i}(s)G_0(ds)-\Phi^0_{\ell,i}(t)\right)\\ & -\sum_{\ell=1}^{L}\lambda_\ell\int_0^t\left(\int_0^{t-s}p_{\ell,i}(u)G(du)+\Phi_{\ell,i}(t-s)\right)\hat{\Upsilon}_\ell(s)ds\\ & +\sum_{\ell\neq i}\int_0^t\left(\nu_{I,\ell,i}\hat{I}_\ell(s)-\nu_{I,i,\ell}\hat{I}_i(s)\right)ds+\sum_{\ell=1}^{L}\left(\hat{M}_{I,\ell,i}(t)-\hat{M}_{I,i,\ell}(t)\right)\\ & +\sum_{\ell=1}^{L}\left(\hat{E}^0_{\ell,i}(t)+\hat{E}_{\ell,i}(t)\right)-\sum_{\ell=1}^{L}\left(\hat{I}^{0,1}_{\ell,i}(t)+\hat{I}^{0,2}_{\ell,i}(t)+\hat{I}_{\ell,i}(t)\right), \end{aligned}\tag{2.26}$$

$$\begin{aligned}\hat{R}_i(t) \;=\; & \hat{I}^N_i(0)\sum_{\ell=1}^{L}\int_0^t q_{\ell,i}(s)F_0(ds)+\hat{E}^N_i(0)\sum_{\ell=1}^{L}\Phi^0_{\ell,i}(t)\\ & +\sum_{\ell=1}^{L}\lambda_\ell\int_0^t\left(\int_0^{t-s}p_{\ell,i}(u)G(du)+\Phi_{\ell,i}(t-s)\right)\hat{\Upsilon}_\ell(s)ds+\sum_{\ell\neq i}\int_0^t\left(\nu_{R,\ell,i}\hat{R}_\ell(s)-\nu_{R,i,\ell}\hat{R}_i(s)\right)ds\\ & +\sum_{\ell=1}^{L}\left(\hat{M}_{R,\ell,i}(t)-\hat{M}_{R,i,\ell}(t)\right)+\sum_{\ell=1}^{L}\left(\hat{I}^{0,1}_{\ell,i}(t)+\hat{I}^{0,2}_{\ell,i}(t)+\hat{I}^{N}_{\ell,i}(t)\right). \end{aligned}\tag{2.27}$$

Here, with the notation $\bar{I}_{(i)}(t)=\sum_{\ell=1}^{L}\kappa_{i\ell}\bar{I}_\ell(t)$,

$$\begin{aligned}\hat{\Upsilon}_i(t) \;=\; & \frac{1}{(\bar{S}_i(t)+\bar{E}_i(t)+\bar{I}_i(t)+\bar{R}_i(t))^{(1+\gamma)}}\Big([(1-\gamma)\bar{S}_i(t)+\bar{E}_i(t)+\bar{I}_i(t)+\bar{R}_i(t)]\bar{I}_{(i)}(t)\hat{S}_i(t)\\ & +[\bar{S}_i(t)(\bar{S}_i(t)+\bar{E}_i(t)+\bar{I}_i(t)+\bar{R}_i(t))-\gamma\bar{S}_i(t)\bar{I}_{(i)}(t)]\hat{I}_i(t)-\gamma\bar{S}_i(t)\bar{I}_{(i)}(t)[\hat{E}_i(t)+\hat{R}_i(t)]\Big)\\ & +\frac{\bar{S}_i(t)\sum_{j\neq i}\hat{I}_j(t)}{(\bar{S}_i(t)+\bar{E}_i(t)+\bar{I}_i(t)+\bar{R}_i(t))^{\gamma}}, \end{aligned}\tag{2.28}$$

$$\begin{aligned}\hat{M}_{A,i}(t) &= B_{A,i}\left(\int_0^t\lambda_i\bar{\Upsilon}_i(s)ds\right), \quad \hat{M}_{S,i,\ell}(t) = B_{S,i,\ell}\left(\nu_{S,i,\ell}\int_0^t\bar{S}_i(s)ds\right),\\ \hat{M}_{E,i,\ell}(t) &= B_{E,i,\ell}\left(\nu_{I,i,\ell}\int_0^t\bar{E}_i(s)ds\right), \quad \hat{M}_{I,i,\ell}(t) = B_{I,i,\ell}\left(\nu_{I,i,\ell}\int_0^t\bar{I}_i(s)ds\right),\\ \hat{M}_{R,i,\ell}(t) &= B_{R,i,\ell}\left(\nu_{R,i,\ell}\int_0^t\bar{R}_i(s)ds\right), \quad i\neq\ell, \end{aligned}$$

with $B_{A,i}$, $B_{S,i,\ell}$, $B_{E,i,\ell}$, $B_{I,i,\ell}$, $B_{R,i,\ell}$ *being mutually independent standard Brownian motions, and with the deterministic functions* $\bar{S}_i,\bar{E}_i,\bar{I}_i,\bar{R}_i$ *being the limits in Theorem 2.1. The processes*

$\hat{E}^0_{i,\ell}(t), \hat{E}_{i,\ell}(t), \hat{I}^{0,1}_{i,\ell}(t), \hat{I}^{0,2}_{i,\ell}(t), \hat{I}_{i,\ell}(t)$ are continuous Gaussian processes, independent of the Brownian motions above, with mean zero and covariance functions:

$$\begin{aligned}
\mathrm{Cov}(\hat{E}^0_{i,\ell}(t), \hat{E}^0_{i,\ell}(t')) &= \bar{E}_i(0)\Big(\int_0^{t\wedge t'} p_{i,\ell}(s)G_0(ds) - \int_0^t p_{i,\ell}(s)G_0(ds) \int_0^{t'} p_{i,\ell}(s)G_0(ds)\Big),\\
\mathrm{Cov}(\hat{E}^0_{i,\ell}(t), \hat{E}^0_{i',\ell'}(t')) &= 0, \quad \textit{for} \quad i \neq i', \textit{ and for } \quad i = i',\ \ell \neq \ell',
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cov}(\hat{I}^{0,1}_{i,\ell}(t), \hat{I}^{0,1}_{i,\ell}(t')) &= \bar{I}_i(0)\Big(\int_0^{t\wedge t'} q_{i,\ell}(s)F_0(ds) - \int_0^t q_{i,\ell}(s)F_0(ds) \int_0^{t'} q_{i,\ell}(s)F_0(ds)\Big),\\
Cov(\hat{I}^{0,1}_{i,\ell}(t), \hat{I}^{0,1}_{i',\ell'}(t')) &= 0, \quad \textit{for} \quad i \neq i', \textit{ and for } \quad i = i',\ \ell \neq \ell',
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cov}(\hat{I}^{0,2}_{i,\ell}(t), \hat{I}^{0,2}_{i,\ell}(t')) &= \bar{E}_i(0)\Phi^0_{i,\ell}(t\wedge t') - \bar{E}_i(0)\Phi^0_{i,\ell}(t)\Phi^0_{i,\ell}(t'),\\
\mathrm{Cov}(\hat{I}^{0,2}_{i,\ell}(t), \hat{I}^{0,2}_{i',\ell'}(t')) &= 0, \quad \textit{for} \quad i \neq i', \textit{ and for } \quad i = i',\ \ell \neq \ell',
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cov}(\hat{E}_{i,\ell}(t), \hat{E}_{i,\ell}(t')) &= \lambda_i \int_0^{t\wedge t'} \int_0^{t\wedge t'-s} p_{i,\ell}(u)G(du)\bar{\Upsilon}_i(s)ds,\\
\mathrm{Cov}(\hat{E}_{i,\ell}(t), \hat{E}_{i',\ell'}(t')) &= 0, \quad \textit{for} \quad i \neq i', \textit{ and for } \quad i = i',\ \ell \neq \ell',
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cov}(\hat{I}_{i,\ell}(t), \hat{I}_{i,\ell}(t')) &= \lambda_i \int_0^{t\wedge t'} \Phi_{i,\ell}(t\wedge t' - s)\bar{\Upsilon}_i(s)ds,\\
\mathrm{Cov}(\hat{I}_{i,\ell}(t), \hat{I}_{i',\ell'}(t')) &= 0, \quad \textit{for} \quad i \neq i', \textit{ and for } \quad i = i',\ \ell \neq \ell'.
\end{aligned}$$

The processes $\big(\hat{E}^0_{i,\ell}(t), \hat{I}^{0,2}_{i,\ell}(t)\big)$, $\big(\hat{I}^{0,1}_{i,\ell}(t)\big)$, and $\big(\hat{E}_{i,\ell}(t), \hat{I}_{i,\ell}(t)\big)$ are independent from each other, and

$$\begin{aligned}
\mathrm{Cov}\big(\hat{E}^0_{i,\ell}(t), \hat{I}^{0,2}_{i,\ell'}(t')\big) &= \bar{E}_i(0)\Big(\int_0^t p_{i,\ell}(u) \int_0^{t'-u} q_{\ell,\ell'}(v)H_0(du,dv) - \int_0^t p_{i,\ell}(s)G_0(ds)\Phi^0_{i,\ell'}(t')\Big),\\
\mathrm{Cov}\big(\hat{E}^0_{i,\ell}(t), \hat{I}^{0,2}_{i',\ell'}(t')\big) &= 0, \quad \textit{for} \quad i \neq i',
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Cov}\big(\hat{E}_{i,\ell}(t), \hat{I}_{i,\ell'}(t')\big) &= \lambda_i \int_0^{t\wedge t'} \int_0^{t\wedge t'-s} p_{i,\ell}(u) \int_0^{t'-s-u} q_{\ell,\ell'}(v)H(du,dv)\bar{\Upsilon}_i(s)ds\\
&\quad - \lambda_i \int_0^{t\wedge t'} \int_0^{t\wedge t'-s} p_{i,\ell}(u)G(du)\Phi_{i,\ell}(t'-s)\bar{\Upsilon}_i(s)ds,\\
\mathrm{Cov}\big(\hat{E}_{i,\ell}(t), \hat{I}_{i',\ell'}(t')\big) &= 0, \quad \textit{for} \quad i \neq i'.
\end{aligned}$$

Remark 2.2. *The continuity of F_0 and G_0 will be important in our proofs of the FCLT, in particular for the convergence of the processes associated with the initially exposed and infectious individuals in Lemma 4.2. It is not needed for the FLLN. Note that this assumption is not really restrictive, in the sense that even when F or G is a Dirac measure (deterministic duration), the time in the past when the initially exposed (or infectious) individuals have been infected (or have become infectious) would most naturally be assumed to follow a uniform distribution on some interval dictated by F or G as discussed in Remark 2.1.*

Remark 2.3. *Suppose that the c.d.f.'s F_0, G_0, F, G have the same conditions in Remark 2.1. Then the limits in Theorem 2.2 become stochastic differential equations with linear drifts and delay. In particular,*

$$\sum_{\ell=1}^L \lambda_\ell \int_0^t \int_0^{t-s} p_{\ell,i}(u)G(du)\hat{\Upsilon}_\ell(s)ds = \sum_{\ell=1}^L \lambda_\ell p_{\ell,i}(t_e) \int_0^{t-t_e} \hat{\Upsilon}_\ell(s)ds\,,$$

$$\sum_{\ell=1}^{L}\lambda_\ell\int_0^t\Phi_{\ell,i}(t-s)\hat{\Upsilon}_\ell(s)ds \;=\; \sum_{\ell=1}^{L}\lambda_\ell\sum_{\ell'=1}^{L}p_{\ell,\ell'}(t_e)q_{\ell'i}(t_o)\int_0^{t-t_e-t_o}\hat{\Upsilon}_\ell(s)ds\,.$$

2.4. **On the multi–patch SIR, SIS and SIRS models.**

2.4.1. *Multi–patch SIR model.* The multi-patch SEIR model includes the mulit-patch SIR model as a special case, without the exposed periods and the associated Markov chain X. The infectious process $I_i^N(t)$ becomes

$$I_i^N(t) \;=\; \sum_{\ell=1}^{L}\sum_{k=1}^{I_\ell^N(0)}\mathbf{1}_{\zeta_{k,\ell}^0>t}\mathbf{1}_{Y_\ell^{0,k}(t)=i}+\sum_{\ell=1}^{L}\sum_{j=1}^{A_\ell^N(t)}\mathbf{1}_{\tau_{j,\ell}^N+\zeta_{j,\ell}>t}\mathbf{1}_{Y_\ell^j(t-\tau_{j,\ell}^N)=i}\;, \tag{2.29}$$

which can be also expressed as

$$\begin{aligned}I_i^N(t) \;=\; & I_i^N(0)+A_i^N(t)-\sum_{\ell=1}^{L}\sum_{k=1}^{I_\ell^N(0)}\mathbf{1}_{\zeta_{k,\ell}^0\le t}\mathbf{1}_{Y_\ell^{0,k}(\zeta_{k,\ell}^0)=i}-\sum_{\ell=1}^{L}\sum_{j=1}^{A_\ell^N(t)}\mathbf{1}_{\tau_{j,\ell}^N+\zeta_{j,\ell}\le t}\mathbf{1}_{Y_\ell^j(\zeta_{j,\ell})=i}\\ & -\sum_{\ell\neq i}P_{I,i,\ell}\left(\nu_{I,i,\ell}\int_0^t I_i^N(s)ds\right)+\sum_{\ell\neq i}P_{I,\ell,i}\left(\nu_{I,\ell,i}\int_0^t I_\ell^N(s)ds\right)\,,\end{aligned} \tag{2.30}$$

The process $\Upsilon_i^N(t)$ in (2.2) becomes

$$\Upsilon_i^N(t) \;=\; \frac{S_i^N(t)\sum_{\ell=1}^{L}\kappa_{i\ell}I_\ell^N(t)}{N^{1-\gamma}(S_i^N(t)+I_i^N(t)+R_i^N(t))^\gamma}\,,\quad i\in\mathcal{L}\,,$$

In the FLLN, we obtain

$$\bar{S}_i(t) \;=\; \bar{S}_i(0)-\lambda_i\int_0^t\bar{\Upsilon}_i(s)ds+\sum_{\ell=1,\ell\neq i}^{L}\int_0^t\left(\nu_{S,\ell,i}\bar{S}_\ell(s)-\nu_{S,i,\ell}\bar{S}_i(s)\right)ds\,, \tag{2.31}$$

$$\begin{aligned}\bar{I}_i(t) \;=\; & \bar{I}_i(0)-\int_0^t\sum_{\ell=1}^{L}\bar{I}_\ell(0)q_{\ell,i}(s)F_0(ds)+\lambda_i\int_0^t\bar{\Upsilon}_i(s)ds\\ & -\int_0^t\sum_{\ell=1}^{L}\left(\int_0^{t-s}q_{\ell,i}(u)F(du)\right)\lambda_\ell\bar{\Upsilon}_\ell(s)ds+\sum_{\ell\neq i}\int_0^t\left(\nu_{I,\ell,i}\bar{I}_\ell(s)-\nu_{I,i,\ell}\bar{I}_i(s)\right)ds\,,\end{aligned} \tag{2.32}$$

$$\begin{aligned}\bar{R}_i(t) \;=\; & \int_0^t\sum_{\ell=1}^{L}\bar{I}_\ell(0)q_{\ell,i}(s)F_0(ds)+\int_0^t\sum_{\ell}\left(\int_0^{t-s}q_{\ell,i}(u)F(du)\right)\lambda_\ell\bar{\Upsilon}_\ell(s)ds\\ & +\sum_{\ell=1,\ell\neq i}^{L}\int_0^t\left(\nu_{R,\ell,i}\bar{R}_\ell(s)-\nu_{R,i,\ell}\bar{R}_i(s)\right)ds\,,\end{aligned} \tag{2.33}$$

with $\bar{\Upsilon}_i$ defined by

$$\bar{\Upsilon}_i(t) \;=\; \frac{\bar{S}_i(t)\sum_{\ell=1}^{L}\kappa_{i\ell}\bar{I}_\ell(t)}{(\bar{S}_i(t)+\bar{I}_i(t)+\bar{R}_i(t))^\gamma}\,. \tag{2.34}$$

In the FCLT, we obtain

$$\hat{S}_i(t) \;=\; \hat{S}_i(0)-\lambda_i\int_0^t\hat{\Upsilon}_i(s)ds+\sum_{\ell=1,\ell\neq i}^{L}\int_0^t(\nu_{S,\ell,i}\hat{S}_\ell(s)-\nu_{S,i,\ell}\hat{S}_i(s))ds$$

$$- \hat{M}_{A,i}(t) + \sum_{\ell=1,\ell\neq i}^{L} \left(\hat{M}_{S,\ell,i}(t) - \hat{M}_{S,i,\ell}(t)\right), \tag{2.35}$$

$$\begin{aligned}\hat{I}_i(t) \;=\; & \hat{I}_i(0)\Big(1 - \sum_{\ell=1}^{L}\int_0^t q_{\ell,i}(s)F_0(ds)\Big) + \lambda_i\int_0^t \hat{\Upsilon}_i(s)ds - \sum_{\ell=1}^{L}\lambda_\ell\int_0^t\int_0^{t-s} q_{\ell,i}(u)F(du)\hat{\Upsilon}_\ell(s)ds\\ & + \sum_{\ell=1,\ell\neq i}^{L}\int_0^t(\nu_{I,\ell,i}\hat{I}_\ell(s) - \nu_{I,i,\ell}\hat{I}_i(s))ds - \sum_{\ell=1}^{L}\left(\hat{I}^0_{\ell,i}(t) + \hat{I}_{\ell,i}(t)\right)\\ & + \hat{M}_{A,i}(t) + \sum_{\ell=1,\ell\neq i}^{L}\left(\hat{M}_{I,\ell,i}(t) - \hat{M}_{I,i,\ell}(t)\right),\end{aligned} \tag{2.36}$$

$$\begin{aligned}\hat{R}_i(t) \;=\; & \hat{I}_i(0)\sum_{\ell=1}^{L}\int_0^t q_{\ell,i}(s)F_0(ds) + \sum_{\ell=1}^{L}\lambda_\ell\int_0^t\int_0^{t-s} q_{\ell,i}(u)F(du)\hat{\Upsilon}_\ell(s)ds\\ & + \sum_{\ell=1,\ell\neq i}^{L}\int_0^t(\nu_{R,\ell,i}\hat{R}_\ell(s) - \nu_{R,i,\ell}\hat{R}_i(s))ds\\ & + \sum_{\ell=1}^{L}\left(\hat{I}^0_{\ell,i}(t) + \hat{I}_{\ell,i}(t)\right) + \sum_{\ell=1,\ell\neq i}^{L}\left(\hat{M}_{R,\ell,i}(t) - \hat{M}_{R,i,\ell}(t)\right).\end{aligned} \tag{2.37}$$

Here, with the notation $\bar{I}_{(i)}(t) = \sum_{\ell=1}^{L}\kappa_{i\ell}\bar{I}_\ell(t)$,

$$\begin{aligned}\hat{\Upsilon}_i(t) = & \frac{1}{(\bar{S}_i(t)+\bar{I}_i(t)+\bar{R}_i(t))^{(1+\gamma)}}\Big([(1-\gamma)\bar{S}_i(t)+\bar{I}_i(t)+\bar{R}_i(t)]\bar{I}_{(i)}(t)\hat{S}_i(t)\\ & + \left[\bar{S}_i(t)(\bar{S}_i(t)+\bar{I}_i(t)+\bar{R}_i(t)) - \gamma\bar{S}_i(t)\bar{I}_{(i)}(t)\right]\hat{I}_i(t) - \gamma\bar{S}_i(t)\bar{I}_{(i)}(t)\hat{R}_i(t)\Big)\\ & + \frac{\bar{S}_i(t)\sum_{\ell\neq i}\kappa_{i\ell}\hat{I}_\ell(t)}{(\bar{S}_i(t)+\bar{I}_i(t)+\bar{R}_i(t))^{\gamma}},\end{aligned} \tag{2.38}$$

$$\begin{aligned}\hat{M}_{A,i}(t) \;&=\; B_{A,i}\left(\int_0^t \lambda_i\bar{\Upsilon}_i(s)ds\right), \quad \hat{M}_{S,i,\ell}(t) \;=\; B_{S,i,\ell}\left(\nu_{S,i,\ell}\int_0^t \bar{S}_i(s)ds\right),\\ \hat{M}_{I,i,\ell}(t) \;&=\; B_{I,i,\ell}\left(\nu_{I,i,\ell}\int_0^t \bar{I}_i(s)ds\right), \quad \hat{M}_{R,i,\ell}(t) \;=\; B_{R,i,\ell}\left(\nu_{R,i,\ell}\int_0^t \bar{R}_i(s)ds\right), \quad i\neq j,\end{aligned}$$

with $B_{A,i}$, $B_{S,i,\ell}$, $B_{I,i,\ell}$, $B_{R,i,\ell}$ being mutually independent standard Brownian motions, and with the deterministic functions $\bar{S}_i, \bar{I}_i, \bar{R}_i$ given above. The processes $\hat{I}^0_{i,\ell}$ and $\hat{I}_{i,\ell}$ are continuous Gaussian processes with mean zero and covariance functions:

$$Cov(\hat{I}^0_{i,\ell}(t), \hat{I}^0_{i',\ell'}(t')) \;=\; \begin{cases}\bar{I}_i(0)\Big(\int_0^{t\wedge t'} q_{i,\ell}(s)F_0(ds) - \int_0^t q_{i,\ell}(s)F_0(ds)\int_0^{t'} q_{i,\ell}(s)F_0(ds)\Big), & \text{if} \quad i=i', \ell=\ell',\\ 0, & \textit{otherwise},\end{cases}$$

$$Cov(\hat{I}_{i,\ell}(t), \hat{I}_{i',\ell'}(t')) \;=\; \begin{cases}\lambda_i\int_0^{t\wedge t'}\int_0^{t\wedge t'-s} q_{i,j}(u)F(du)\bar{\Upsilon}_i(s)ds, & \text{if} \quad i=i',\, \ell=\ell',\\ 0, & \textit{otherwise}.\end{cases}$$

In addition, $\hat{I}^0_{i,\ell}$ and $\hat{I}_{i,\ell}$ are independent, and also independent of the Brownian terms.

2.4.2. *Multi–patch SIS model.* The analysis of the multi-patch SIR model can be easily extended to the multi-patch SIS model, where the population in each patch has susceptible and infectious groups, and when infectious individuals recover, they become susceptible immediately. The epidemic evolution dynamics is described as

$$\begin{aligned} S_i^N(t) &= S_i^N(0) - A_i^N(t) + \sum_{\ell=1}^{L}\sum_{k=1}^{I_\ell^N(0)} \mathbf{1}_{\zeta^0_{k,\ell}\le t}\mathbf{1}_{Y_\ell^{0,k}(\zeta^0_{k,\ell})=i} + \sum_{\ell=1}^{L}\sum_{j=1}^{A_\ell^N(t)} \mathbf{1}_{\tau^N_{j,\ell}+\zeta_{j,\ell}\le t}\mathbf{1}_{Y_\ell^{j}(\zeta_{j,\ell})=i} \\ &\quad - \sum_{\ell=1,\ell\neq i}^{L} P_{S,i,\ell}\left(\nu_{S,i,\ell}\int_0^t S_i^N(s)ds\right) + \sum_{\ell=1,\ell\neq i}^{L} P_{S,\ell,i}\left(\nu_{S,\ell,i}\int_0^t S_i^N(s)ds\right), \end{aligned} \tag{2.39}$$

$$I_i^N(t) = \sum_{\ell=1}^{L}\sum_{k=1}^{I_\ell^N(0)} \mathbf{1}_{t<\zeta^0_{k,\ell}}\mathbf{1}_{Y_\ell^{0,k}(t)=i} + \sum_{\ell=1}^{L}\sum_{j=1}^{A_\ell^N(t)} \mathbf{1}_{\tau^N_{j,\ell}+\zeta_{j,\ell}>t}\mathbf{1}_{Y_\ell^{j}(t-\tau^N_{j,\ell})=i}, \tag{2.40}$$

where A_i^n is given as in (2.3) with $\Upsilon_i^N(t) = \frac{S_i^N(t)\sum_{\ell=1}^L \kappa_{i\ell} I_\ell^N(t)}{(S_i^N(t)+I_i^N(t))^\gamma}$, for $i \in \mathcal{L}$. Thus, in the FLLN, we obtain the same limit $\bar{I}_i$ in (2.32) as in the multi-patch SIR model, and the limit $\bar{S}_i(t)$:

$$\begin{aligned} \bar{S}_i(t) &= \bar{S}_i(0) - \lambda_i \int_0^t \bar{\Upsilon}_i(s)ds \int_0^t \sum_\ell q_{\ell,i}(s)F_0(ds) + \int_0^t \sum_\ell \left(\int_0^{t-s} q_{\ell,i}(u)F(du)\right)\lambda_\ell \bar{\Upsilon}_\ell(s)ds \\ &\quad + \sum_{\ell=1,\ell\neq i}^{L}\int_0^t \left(\nu_{S,\ell,i}\bar{S}_j(s) - \nu_{S,i,\ell}\bar{S}_i(s)\right)ds, \end{aligned}$$

where $\bar{\Upsilon}_i(t) := \frac{\bar{S}_i(t)\sum_{\ell=1}\kappa_{i\ell}\bar{I}_\ell(t)}{(\bar{S}_i(t)+\bar{I}_i(t))^\gamma}$. Similarly in the FCLT, we obtain the same limit $\hat{I}_i$ as in (2.36) for the multi-patch SIR model, and the limit $\hat{S}_i(t)$:

$$\begin{aligned} \hat{S}_i(t) &= \hat{S}_i(0) - \lambda_i\int_0^t \hat{\Upsilon}_i(s)ds + \sum_{\ell=1}^{L}\lambda_\ell \int_0^t\int_0^{t-s} q_{\ell,i}(u)F(du)\hat{\Upsilon}_\ell(s)ds + \sum_{\ell=1}^{L}\left(\hat{I}^0_{\ell,i}(t) + \hat{I}_{\ell,i}(t)\right) \\ &\quad + \sum_{\ell=1,\ell\neq i}^{L}\int_0^t (\nu_{S,\ell,i}\hat{S}_\ell(s) - \nu_{S,i,\ell}\hat{S}_i(s))ds - \hat{M}_{A,i}(t) + \sum_{\ell=1,\ell\neq i}^{L}\left(\hat{M}_{S,\ell,i}(t) - \hat{M}_{S,i,\ell}(t)\right), \end{aligned}$$

where

$$\begin{aligned} \hat{\Upsilon}_i(t) = \frac{1}{(\bar{S}_i(t)+\bar{I}_i(t))^{(1+\gamma)}}&\left\{[(1-\gamma)\bar{S}_i(t)+\bar{I}_i(t)]\bar{I}_{(i)}(t)\hat{S}_i(t) + [\bar{S}_i(t)(\bar{S}_i(t)+\bar{I}_i(t)) - \bar{S}_i(t)\hat{I}_{(i)}(t)]\hat{I}_i(t)\right\} \\ &+ \frac{\bar{S}_i(t)\sum_{\ell\neq i}\hat{I}_\ell(t)}{(\bar{S}_i(t)+\bar{I}_i(t))^\gamma}. \end{aligned}$$

2.4.3. *Multi-patch SIRS model.* The analysis for the multi-patch SEIR model can be easily extended to multi-patch SIRS model, where in each patch, the population is grouped into susceptible, infectious, and recovered individuals and individuals become susceptible after experiencing a recovery period. In this model, the infectious and recovered processes I_i^N, R_i^N correspond to the exposed and infectious processes E_i^N, I_i^N in the SEIR model. In the description of the epidemic dynamics, we need to change the dynamics of S_i^N in (2.5) by adding the individuals that have become susceptible after recovery, i.e., the first three terms in R_i^N in (2.8). This is similar to the susceptible process S_i^N in (2.39) for the SIS model. Then it is straightforward to write down the limit processes in the FLLN and FCLT for the processes $(S_i^N, I_i^N, R_i^N, i \in \mathcal{L})$ (corresponding to $(S_i^N, E_i^N, I_i^N, i \in \mathcal{L})$ in the SEIR model).

3. Proof of the FLLN

In this section we prove Theorem 2.1. We first provide a representation of the processes $A_i^N(t)$ using the PRM $Q_i(ds, d\mathfrak{a})$. Let $\bar{Q}_i(ds, d\mathfrak{a}) = Q_i(ds, d\mathfrak{a}) - dsd\mathfrak{a}$ be the compensated PRM. Then for each $i \in \mathcal{L}$,

$$A_i^N(t) \;=\; \lambda_i \int_0^t \Upsilon_i^N(s)ds + M_{A,i}^N(t)\,, \quad t \geq 0\,, \tag{3.1}$$

where

$$M_{A,i}^N(t) \;:=\; \int_0^t \int_0^\infty \mathbf{1}_{\mathfrak{a} \leq \lambda_i \Upsilon_i^N(s^-)} \bar{Q}_i(ds, d\mathfrak{a})\,. \tag{3.2}$$

The process $\{M_{A,i}^N(t) : t \geq 0\}$ is a square-integrable martingale with respect to the filtration $\{\mathcal{F}_{A,i}^N(t) : t \geq 0\}$, defined by

$$\mathcal{F}_{A,i}^N(t) \;:=\; \sigma\big\{S_i^N(0), I_i^N(0), i \in \mathcal{L}\big\} \vee \sigma\big\{A_i^N(s) : 0 \leq s \leq t\big\}\,, \quad t \geq 0\,.$$

It has the predictable quadratic variation:

$$\langle M_{A,i}^N \rangle(t) \;=\; \lambda_i \int_0^t \Upsilon_i^N(s)ds\,, \quad t \geq 0\,.$$

Lemma 3.1. *The sequence $\big\{\big(\bar{A}_1^N, \ldots, \bar{A}_L^N\big) : N \geq 1\big\}$ is tight in D^L. The limit of each convergent subsequence of $\big\{\big(\bar{A}_1^N, \ldots, \bar{A}_L^N\big)\big\}$, denoted as $\big(\bar{A}_1, \ldots, \bar{A}_L\big)$, satisfies*

$$\bar{A}_i \;=\; \lim_{N\to\infty} \bar{A}_i^N \;=\; \lim_{N\to\infty} N^{-1}\lambda_i \int_0^\cdot \Upsilon_i^N(s)ds\,,$$

and

$$0 \;\leq\; \bar{A}_i(t) - \bar{A}_i(s) \;\leq\; \lambda_i \bar{\kappa}_i (t-s)\,, \quad \textit{for} \quad 0 < s \leq t\,, \quad \textit{w.p.}\,1\,.$$

Proof. First, since

$$0 \;\leq\; \frac{\lambda_i}{N} \int_s^t \Upsilon_i^N(u)du \;\leq\; \lambda_i \bar{\kappa}_i (t-s), \quad \text{w.p.1}\,. \quad t \geq s \geq 0\,, \tag{3.3}$$

and

$$\langle \bar{M}_{A,i}^N \rangle(t) \leq \frac{\lambda_i}{N^2} \int_s^t \Upsilon_i^N(u)du,$$

it follows readily from Doob's inequality that, as $N \to \infty$,

$$\bar{M}_{A,i}^N \;\to\; 0 \quad \text{in probability, locally uniformly in } t\,. \tag{3.4}$$

Tightness of $\{(\bar{A}_1^N, \ldots, \bar{A}_L^N)\}$ in D^L then follows from the representation in (3.1) and the two properties in (3.3) and (3.4). □

In the following we consider a convergent subsequence of $\{(\bar{A}_1^N, \ldots, \bar{A}_L^N)\}$.

Lemma 3.2. *With the limit $\big(\bar{A}_1, \ldots, \bar{A}_L\big)$ of the convergent subsequence of $\big\{\big(\bar{A}_1^N, \ldots, \bar{A}_L^N\big)\big\}$, under Assumption 2.1,*

$$(\bar{S}_1^N, \ldots, \bar{S}_L^N) \;\Rightarrow\; (\bar{S}_1, \ldots, \bar{S}_L) \quad \textit{in} \quad D^L \quad \textit{as} \quad N \to \infty$$

where the limit $(\bar{S}_1, \ldots, \bar{S}_L)$ is the unique solution to the ODEs: for each $i \in \mathcal{L}$,

$$\bar{S}_i(t) \;=\; \bar{S}_i(0) - \bar{A}_i(t) + \sum_{\ell \neq i} \int_0^t \big(\nu_{S,\ell,i} \bar{S}_\ell(s) - \nu_{S,i,\ell} \bar{S}_i(s)\big)\, ds\,, \quad t \geq 0\,. \tag{3.5}$$

Proof. We can rewrite the processes $\bar{S}_i^N$ as

$$\bar{S}_i^N(t) \;=\; \bar{S}_i^N(0) + \sum_{\ell \neq i} \int_0^t \left(\nu_{S,\ell,i} \bar{S}_\ell^N(s) - \nu_{S,i,\ell} \bar{S}_i^N(s) \right) ds - \bar{A}_i^N(t)) \,,$$

where

$$\bar{M}_{S,\ell,i}^N(t) \;:=\; \frac{1}{N} \left(P_{S,\ell,i} \left(\nu_{S,\ell,i} \int_0^t S_\ell^N(s) ds \right) - \nu_{S,\ell,i} \int_0^t S_\ell^N(s) ds \right) .$$

The processes $\bar{M}_{S,\ell,i}^N$ are square integrable martingales with respect to the filtration $\{\mathcal{F}_S^N(t) : t \geq 0\}$, defined by

$$\mathcal{F}_S^N(t) \;:=\; \bigvee_{i=1}^L \mathcal{F}_{A,i}^N(t) \vee \sigma \left\{ P_{S,\ell,i} \left(\nu_{S,\ell,i} \int_0^s S_\ell^N(u) du \right) : 0 \leq s \leq t, \ell, i \in \mathcal{L},\, \ell \neq i \right\} , \quad t \geq 0 \,.$$

They have the predictable quadratic variation:

$$\langle \bar{M}_{S,\ell,i}^N \rangle(t) \;=\; \frac{1}{N} \nu_{S,\ell,i} \int_0^t \bar{S}_i^N(s) ds \to 0 \quad \text{as} \quad n \to \infty \,.$$

Thus, as $N \to \infty$,

$$\left(\bar{M}_{S,\ell,i}^N(t),\, \ell, i \in \mathcal{L}, \ell \neq i \right) \to 0 \;\text{ locally uniformly in } t \,,$$

and under Assumption 2.1, for $1 \leq i \leq L$, the processes $\bar{S}^N(0) - \bar{A}_i^N$ jointly converge weakly to $\bar{S}(0) - \bar{A}_i$ in D. Hence it follows from the continuous mapping theorem that

$$(\bar{S}_1^N, \ldots, \bar{S}_L^N) \Rightarrow (\bar{S}, \ldots, \bar{S}_L) \;\text{ in } D^L,$$

where $(\bar{S}(t), \ldots, \bar{S}_L(t))$ is the unique solution of the system of ODEs (3.5). □

For $\ell, i \in \mathcal{L}$, let

$$\left.\begin{aligned}
E_{\ell,i}^{N,0}(t) \;&:=\; \sum_{k=1}^{E_\ell^N(0)} \mathbf{1}_{\eta_{k,\ell}^0 \leq t} \mathbf{1}_{X_\ell^{0,k}(\eta_{k,\ell}^0)=i} \,, \\
E_{\ell,i}^{N}(t) \;&:=\; \sum_{j=1}^{A_\ell^N(t)} \mathbf{1}_{\tau_{j,\ell}^N + \eta_{j,\ell} \leq t} \mathbf{1}_{X_\ell^j(\eta_{j,\ell})=i} \,, \\
I_{\ell,i}^{N,0,1}(t) \;&:=\; \sum_{k=1}^{I_\ell^N(0)} \mathbf{1}_{\zeta_{k,\ell}^0 \leq t} \mathbf{1}_{Y_\ell^{0,k}(\zeta_{k,\ell}^0)=i} \,, \\
I_{\ell,i}^{N,0,2}(t) \;&:=\; \sum_{k=1}^{E_\ell^N(0)} \mathbf{1}_{\eta_{k,\ell}^0 \leq t} \left(\sum_{\ell'=1}^L \mathbf{1}_{X_\ell^{0,k}(\eta_{k,\ell}^0)=\ell'} \mathbf{1}_{\eta_{k,\ell}^0 + \zeta_{-k,\ell'} \leq t} \mathbf{1}_{Y_{\ell'}^{-k}(\zeta_{-k,\ell'})=i} \right) , \\
I_{\ell,i}^{N}(t) \;&:=\; \sum_{j=1}^{A_\ell^N(t)} \mathbf{1}_{\tau_{j,\ell}^N + \eta_{j,\ell} \leq t} \left(\sum_{\ell'=1}^L \mathbf{1}_{X_\ell^j(\eta_{j,\ell})=\ell'} \mathbf{1}_{\tau_{j,\ell}^N + \eta_{j,\ell} + \zeta_{j,\ell'} \leq t} \mathbf{1}_{Y_{\ell'}^j(\zeta_{j,\ell'})=i} \right) .
\end{aligned}\right\} \tag{3.6}$$

We first treat the components associated with the initial quantities.

Lemma 3.3. *Under Assumption 2.1,*

$$\left(\bar{E}_{\ell,i}^{N,0},\, \bar{I}_{\ell,i}^{N,0,1},\, \bar{I}_{\ell,i}^{N,0,2},\, \ell, i \in \mathcal{L} \right) \to \left(\bar{E}_{\ell,i}^0,\, \bar{I}_{\ell,i}^{0,1},\, \bar{I}_{\ell,i}^{0,2},\, \ell, i \in \mathcal{L} \right) \quad \textit{in} \quad D^{3L^2} \quad \textit{as} \quad N \to \infty,$$

in probability, uniformly in t, where for $\ell, i \in \mathcal{L}$ and $t \geq 0$,

$$\bar{E}_{\ell,i}^0(t) \;:=\; \bar{E}_\ell(0) \int_0^t p_{\ell,i}(s) G_0(ds) \,, \quad \bar{I}_{\ell,i}^{0,1}(t) \;:=\; \bar{I}_\ell(0) \int_0^t q_{\ell,i}(s) F_0(ds) \,, \tag{3.7}$$

and

$$\bar{I}^{0,2}_{\ell,i}(t) \;:=\; \bar{E}_\ell(0)\Phi^0_{\ell,i}(t)\,, \tag{3.8}$$

with $\Phi^0_{\ell,i}(t)$ *defined in* (2.19).

Proof. We define $\tilde{E}^{N,0}_{\ell,i}$, $\tilde{I}^{N,0,1}_{\ell,i}$ and $\tilde{I}^{N,0,2}_{\ell,i}$ similarly as $\bar{E}^{N,0}_{\ell,i}$, $\bar{I}^{N,0,1}_{\ell,i}$ and $\bar{I}^{N,0,2}_{\ell,i}$, but with $E^N_\ell(0)$ and $I^N_\ell(0)$ replaced by $[N\bar{E}_\ell(0)]$ and $[N\bar{I}_\ell(0)]$ respectively. As a consequence of Assumption 2.1, the differences $\bar{E}^{N,0}_{\ell,i}(t)-\tilde{E}^{N,0}_{\ell,i}(t)$, $\bar{I}^{N,0,1}_{\ell,i}(t)-\tilde{I}^{N,0,1}_{\ell,i}(t)$ and $\bar{I}^{N,0,2}_{\ell,i}(t)-\tilde{I}^{N,0,2}_{\ell,i}(t)$ are easily shown to tend to 0 in probability, locally uniformly in t, as $N\to\infty$.

The convergence of $\{\tilde{E}^{N,0}_{\ell,i}\}$ follows the same argument as that of $\{\tilde{I}^{N,0,1}_{\ell,i}\}$, so we establish the convergence of $\{\tilde{I}^{N,0,1}_{\ell,i}\}$.

We first focus on $\tilde{I}^{N,0,1}_{1,1}(t)$. Note that, since $\zeta^0_{k,1}$ and $Y^{0,k}_1$ are independent,

$$\mathbb{E}\left[\mathbf{1}_{\zeta^0_{k,1}\le t}\mathbf{1}_{Y^{0,k}_1(\zeta^0_{k,1})=1}\right]=\mathbb{E}\left[\int_0^t \mathbf{1}_{Y^{0,k}_1(s)=1}dF_0(s)\right]=\int_0^t q_{1,1}(s)F_0(ds),$$

where the expectation is taken under the condition that $Y^{0,k}_1(0)=1$. Note that the pairs $(\zeta^0_{k,1},Y^{0,k}_1(\cdot))$ are independent over k, and have the same distributions. Thus, by the LLN of i.i.d. random variables, we obtain that for each $t\ge 0$, as $N\to\infty$,

$$\tilde{I}^{N,0}_{1,1}(t)\to\bar{I}^0_{1,1}(t)\ \text{ in probability.}$$

In order to establish locally uniform convergence in t, is suffices to establish tightness in D, which (see the Corollary of Theorem 7.4 in [5]) will follow from the fact that

$$\limsup_{N\to\infty}\sup_{0\le t\le T}\frac{1}{\delta}\mathbb{P}\left(\sup_{0\le u\le\delta}|\tilde{I}^{N,0,1}_{1,1}(t+u)-\tilde{I}^{N,0,1}_{1,1}(t)|>\varepsilon\right)\to 0,\quad\text{as }\delta\to 0. \tag{3.9}$$

By the independence of the pairs $\{(\zeta^0_{k,1},Y^{0,k}_1(\zeta^0_{k,1})),\ k\ge 1\}$,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\le s\le t+\delta}\left|\tilde{I}^{N,0,1}_{1,1}(s)-\tilde{I}^{N,0,1}_{1,1}(t)\right|>\epsilon\right)=\mathbb{P}\left(N^{-1}\sum_{k=1}^{N\bar{I}_1(0)}\mathbf{1}_{t<\zeta^0_{k,1}\le t+\delta}\mathbf{1}_{Y^{0,k}_1(\zeta^0_{k,1})=1}>\epsilon\right)\\
&\le\mathbb{P}\left(N^{-1}\sum_{k=1}^{N\bar{I}_1(0)}\left[\mathbf{1}_{t<\zeta^0_{k,1}\le t+\delta}\mathbf{1}_{Y^{0,k}_1(\zeta^0_{k,1})=1}-\int_t^{t+\delta}q_{1,1}(u)F_0(du)\right]>\epsilon/2\right)\\
&\qquad+\mathbf{1}\left\{\int_t^{t+\delta}q_{1,1}(u)F_0(du)>\epsilon/2\bar{I}_1(0)\right\}\\
&\le\frac{4}{\epsilon^2}\mathbb{E}\left[\left(N^{-1}\sum_{k=1}^{N\bar{I}_1(0)}\left[\mathbf{1}_{t<\zeta^0_{k,1}\le t+\delta}\mathbf{1}_{Y^{0,k}_1(\zeta^0_{k,1})=1}-\int_t^{t+\delta}q_{1,1}(u)F_0(du)\right]\right)^2\right]\\
&\qquad+\mathbf{1}\left\{\int_t^{t+\delta}q_{1,1}(u)F_0(du)>\epsilon/2\bar{I}_1(0)\right\}\\
&=\frac{4\bar{I}_1(0)}{\epsilon^2N}\int_t^{t+\delta}q_{1,1}(s)F_0(ds)\left[1-\int_t^{t+\delta}q_{1,1}(s)F_0(ds)\right]\\
&\qquad+\mathbf{1}\left\{\int_t^{t+\delta}q_{1,1}(u)F_0(du)>\epsilon/2\bar{I}_1(0)\right\}\\
&\to\mathbf{1}\left\{\int_t^{t+\delta}q_{1,1}(u)F_0(du)>\epsilon/2\bar{I}_1(0)\right\},\quad\text{as}\quad N\to\infty\,.
\end{aligned}$$

Finally the last term vanishes for $\delta > 0$ small enough, uniformly w.r.t. $t \in [0,T]$. Hence, (3.9) follows. The convergence of the other processes $\tilde{I}^{N,0,1}_{\ell,i}$ follows similarly.

We next sketch the proof for the convergence of $\tilde{I}^{N,0,2}_{\ell,i}$ since it follows similar steps as that of $\{\tilde{I}^{N,0,1}_{\ell,i}\}$. We have

$$\mathbb{E}\left[\mathbf{1}_{\eta^0_{k,\ell}\le t}\left(\sum_{\ell'=1}^{L}\mathbf{1}_{X^{0,k}_\ell(\eta^0_{k,\ell})=\ell'}\mathbf{1}_{\eta^0_{k,\ell}+\zeta_{-k,\ell'}\le t}\mathbf{1}_{Y^{-k}_{\ell'}(\zeta_{-k,\ell'})=i}\right)\right] \;=\; \Phi^0_{\ell,i}(t)\,,$$

which implies by the LLN of i.i.d. variables that for each $t \ge 0$, $\bar{I}^{N,0,2}_{\ell,i}(t) \Rightarrow \bar{I}^{0,2}_{\ell,i}(t)$ as $N \to \infty$. The convergence of finite dimensional distribution is a straightforward extension. For tightness we use the same approach as above. We start with

$$\begin{aligned}\left|\tilde{I}^{N,0,2}_{\ell,i}(t+s) - \tilde{I}^{N,0,2}_{\ell,i}(t)\right| \le{}& \frac{1}{N}\sum_{k=1}^{E^N_\ell(0)}\mathbf{1}_{t<\eta^0_{k,\ell}\le t+s}\left(\sum_{\ell'=1}^{L}\mathbf{1}_{X^{0,k}_\ell(\eta^0_{k,\ell})=\ell'}\mathbf{1}_{\eta^0_{k,\ell}+\zeta_{-k,\ell'}\le t+s}\mathbf{1}_{Y^{-k}_{\ell'}(\zeta_{-k,\ell'})=i}\right)\\ &+\frac{1}{N}\sum_{k=1}^{E^N_\ell(0)}\mathbf{1}_{\eta^0_{k,\ell}\le t}\left(\sum_{\ell'=1}^{L}\mathbf{1}_{X^{0,k}_\ell(\eta^0_{k,\ell})=\ell'}\mathbf{1}_{t<\eta^0_{k,\ell}+\zeta_{-k,\ell'}\le t+s}\mathbf{1}_{Y^{-k}_{\ell'}(\zeta_{-k,\ell'})=i}\right).\end{aligned}$$

We next note that each of the two terms on the right hand side is increasing in s, so that

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\le s\le\delta}\left|\tilde{I}^{N,0,2}_{\ell,i}(t+s) - \tilde{I}^{N,0,2}_{\ell,i}(t)\right| > \epsilon\right)\\
&\le \mathbb{P}\left(\frac{1}{N}\sum_{k=1}^{E^N_\ell(0)}\mathbf{1}_{t<\eta^0_{k,\ell}\le t+\delta}\left(\sum_{\ell'=1}^{L}\mathbf{1}_{X^{0,k}_\ell(\eta^0_{k,\ell})=\ell'}\mathbf{1}_{\eta^0_{k,\ell}+\zeta_{-k,\ell'}\le t+\delta}\mathbf{1}_{Y^{-k}_{\ell'}(\zeta_{-k,\ell'})=i}\right) > \epsilon/2\right)\\
&\quad+\mathbb{P}\left(\frac{1}{N}\sum_{k=1}^{E^N_\ell(0)}\mathbf{1}_{\eta^0_{k,\ell}\le t}\left(\sum_{\ell'=1}^{L}\mathbf{1}_{X^{0,k}_\ell(\eta^0_{k,\ell})=\ell'}\mathbf{1}_{t<\eta^0_{k,\ell}+\zeta_{-k,\ell'}\le t+\delta}\mathbf{1}_{Y^{-k}_{\ell'}(\zeta_{-k,\ell'})=i}\right) > \epsilon/2\right)\\
&\le \mathbb{P}\left(\frac{1}{N}\sum_{k=1}^{E^N_\ell(0)}\left[\mathbf{1}_{t<\eta^0_{k,\ell}\le t+\delta}\left(\sum_{\ell'=1}^{L}\mathbf{1}_{X^{0,k}_\ell(\eta^0_{k,\ell})=\ell'}\mathbf{1}_{\eta^0_{k,\ell}+\zeta_{-k,\ell'}\le t+\delta}\mathbf{1}_{Y^{-k}_{\ell'}(\zeta_{-k,\ell'})=i}\right)\right.\right.\\
&\qquad\left.\left.-\int_t^{t+\delta}\sum_{\ell'=1}^{L}p_{\ell,\ell'}(u)\int_0^{t+\delta-u}q_{\ell',i}(v)H_0(du,dv)\right] > \epsilon/4\right)\\
&\quad+\mathbf{1}\left\{\int_t^{t+\delta}\sum_{\ell'=1}^{L}p_{\ell,\ell'}(u)\int_0^{t+\delta-u}q_{\ell',i}(v)H_0(du,dv) > \epsilon/4\bar{E}_\ell(0)\right\}\\
&\quad+\mathbb{P}\left(\frac{1}{N}\sum_{k=1}^{E^N_\ell(0)}\left[\mathbf{1}_{\eta^0_{k,\ell}\le t}\left(\sum_{\ell'=1}^{L}\mathbf{1}_{X^{0,k}_\ell(\eta^0_{k,\ell})=\ell'}\mathbf{1}_{t<\eta^0_{k,\ell}+\zeta_{-k,\ell'}\le t+\delta}\mathbf{1}_{Y^{-k}_{\ell'}(\zeta_{-k,\ell'})=i}\right)\right.\right.\\
&\qquad\left.\left.-\int_0^{t}\sum_{\ell'=1}^{L}p_{\ell,\ell'}(u)\int_{t-u}^{t+\delta-u}q_{\ell',i}(v)H_0(du,dv)\right] > \epsilon/4\right)\\
&\quad+\mathbf{1}\left\{\int_0^{t}\sum_{\ell'=1}^{L}p_{\ell,\ell'}(u)\int_{t-u}^{t+\delta-u}q_{\ell',i}(v)H_0(du,dv) > \epsilon/4\bar{E}_\ell(0)\right\}\\
&\le \frac{16\bar{E}_\ell(0)}{\epsilon^2 N}\int_t^{t+\delta}\sum_{\ell'=1}^{L}p_{\ell,\ell'}(u)\int_0^{t+\delta-u}q_{\ell',i}(v)H_0(du,dv)\end{aligned}$$

$$\times \left(1 - \int_t^{t+\delta} \sum_{\ell'=1}^{L} p_{\ell,\ell'}(u) \int_0^{t+\delta-u} q_{\ell',i}(v) H_0(du,dv)\right)$$
$$+ \mathbf{1}\Big\{ \int_t^{t+\delta} \sum_{\ell'=1}^{L} p_{\ell,\ell'}(u) \int_0^{t+\delta-u} q_{\ell',i}(v) H_0(du,dv) > \epsilon/4\bar{E}_\ell(0)\Big\}$$
$$+ \frac{16\bar{E}_\ell(0)}{\epsilon^2 N} \int_0^t \sum_{\ell'=1}^{L} p_{\ell,\ell'}(u) \int_{t-u}^{t+\delta-u} q_{\ell',i}(v) H_0(du,dv)$$
$$\times \left(1 - \int_0^{t} \sum_{\ell'=1}^{L} p_{\ell,\ell'}(u) \int_{t-u}^{t+\delta-u} q_{\ell',i}(v) H_0(du,dv)\right)$$
$$+ \mathbf{1}\Big\{ \int_0^{t} \sum_{\ell'=1}^{L} p_{\ell,\ell'}(u) \int_{t-u}^{t+\delta-u} q_{\ell',i}(v) H_0(du,dv) > \epsilon/4\bar{E}_\ell(0)\Big\}$$
$$\to \mathbf{1}\Big\{ \int_t^{t+\delta} \sum_{\ell'=1}^{L} p_{\ell,\ell'}(u) \int_0^{t+\delta-u} q_{\ell',i}(v) H_0(du,dv) > \epsilon/4\bar{E}_\ell(0)\Big\}$$
$$+ \mathbf{1}\Big\{ \int_0^{t} \sum_{\ell'=1}^{L} p_{\ell,\ell'}(u) \int_{t-u}^{t+\delta-u} q_{\ell',i}(v) H_0(du,dv) > \epsilon/4\bar{E}_\ell(0)\Big\},$$

as $N \to \infty$. And the two terms in the limit both vanish for small enough δ. Thus, for any $\epsilon > 0$,

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \frac{T}{\delta} \sup_{t\in[0,T]} \mathbb{P}\left(\sup_{t\le s\le t+\delta} \left|\tilde{I}^{N,0,2}_{\ell,i}(s) - \tilde{I}^{N,0,2}_{\ell,i}(t)\right| > \epsilon \right) \to 0\,.$$

Therefore, we can conclude that $\tilde{I}^{N,0,2}_{\ell,i} \to \bar{I}^{0,2}_{\ell,i}$ in probability, uniformly in t, as $N \to \infty$. Then, since G_0 is continuous, we can verify the continuity of the covariance function, and thus the continuity of the limit processes $\bar{I}^{0,2}_{\ell,i}$. This completes the proof. □

In the next proof, we will make use of the following result, which is Lemma 4.4 in [15], which we repeat here of the convenience of the reader. In the next statement, $D_\uparrow(\mathbb{R}_+)$ (resp. $C_\uparrow(\mathbb{R}_+)$) denotes the set of real-valued nondecreasing function on $\mathbb{R}_+$, which belong to $D(\mathbb{R}_+)$ (resp. $C(\mathbb{R}_+)$).

Lemma 3.4. *Let $f \in D(\mathbb{R}_+)$ and $\{g_N\}_{N\ge 1}$ be a sequence of elements of $D_\uparrow(\mathbb{R}_+)$ which is such that $g_N \to g$ locally uniformly as $N \to \infty$, where $g \in C_\uparrow(\mathbb{R}_+)$. Then, for any $t > 0$, as $N \to \infty$,*

$$\int_{[0,t]} f(s) g_N(ds) \to \int_{[0,t]} f(s) g(ds)\,.$$

Proof. The assumption implies that the sequence of measures $g_N(ds)$ converges weakly, as $N \to \infty$, towards the measure $g(ds)$. Since, moreover, f is bounded and the set of discontinuities of f is of $g(ds)$ measure 0, the convergence is essentially a minor improvement of the Portmanteau theorem, see Theorem 2.1 in [5]. □

We next prove the convergence of $\bar{E}^N_{\ell,i}$ and $\bar{I}^N_{\ell,i}$ for $\ell, i \in \mathcal{L}$. Define the auxiliary processes: for $\ell, i \in \mathcal{L}$,

$$\breve{E}^N_{\ell,i}(t) = \mathbb{E}\big[\bar{E}^N_{\ell,i}(t)|\mathcal{F}^N_{A,\ell}(t)\big], \quad \breve{I}^N_{\ell,i}(t) = \mathbb{E}\big[\bar{I}^N_{\ell,i}(t)|\mathcal{F}^N_{A,\ell}(t)\big], \quad t \ge 0.$$

We first prove these processes converge to the desired limits in the following lemma, and then show that these processes are asymptotically equivalent to $\bar{E}^N_{\ell,i}$ and $\bar{I}^N_{\ell,i}$, $\ell, i \in \mathcal{L}$.

Lemma 3.5. *With the limit* $(\bar{A}_1, \ldots, \bar{A}_L)$ *of the convergent subsequence of* $\{(\bar{A}_1^N, \ldots, \bar{A}_L^N)\}$*, under Assumption 2.1,*

$$(\breve{E}_{\ell,i}^N, \breve{I}_{\ell,i}^N, \ell, i \in \mathcal{L}) \Rightarrow (\bar{E}_{\ell,i}(t), \bar{I}_{\ell,i}, \ell, i \in \mathcal{L}) \quad in \quad D^{2L^2} \quad as \quad N \to \infty, \tag{3.10}$$

in probability, where for $\ell, i \in \mathcal{L}$ *and* $t \geq 0$,

$$\bar{E}_{\ell,i}(t) := \int_0^t \left(\int_0^{t-s} p_{\ell,i}(u) G(du) \right) d\bar{A}_\ell(s), \tag{3.11}$$

and

$$\bar{I}_{\ell,i}(t) := \int_0^t \Phi_{\ell,i}(t-s) d\bar{A}_\ell(s), \tag{3.12}$$

with $\Phi_{\ell,i}$ *defined in* (2.20).

Proof. Observe that for $\ell, i \in \mathcal{L}$,

$$\begin{aligned}
\breve{E}_{\ell,i}^N(t) &= \mathbb{E}\left[\bar{E}_{\ell,i}^N(t) | \mathcal{F}_{A,\ell}^N(t)\right] = \frac{1}{N} \sum_{j=1}^{A_\ell^N(t)} \mathbb{E}\left[\mathbf{1}_{\tau_{j,\ell}^N + \eta_{j,\ell} \leq t} \mathbf{1}_{X_\ell^j(\eta_{j,\ell}) = i} | \tau_{j,\ell}^N\right] \\
&= \frac{1}{N} \sum_{j=1}^{A_\ell^N(t)} \mathbb{E}\left[\int_0^{t-\tau_{j,\ell}^N} \mathbf{1}_{X_1^j(u)=i} F(du) \Big| \tau_{j,\ell}^N\right] = \frac{1}{N} \sum_{j=1}^{A_\ell^N(t)} \int_0^{t-\tau_{j,\ell}^N} p_{\ell,i}(u) F(du) \\
&= \int_0^t \left(\int_0^{t-s} p_{\ell,i}(u) F(du) \right) d\bar{A}_\ell^N(s),
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
\breve{I}_{\ell,i}^N(t) &= N^{-1} \sum_{j=1}^{A_\ell^N(t)} \mathbb{E}\left[\mathbf{1}_{\tau_{j,\ell}^N + \eta_{j,\ell} \leq t} \left(\sum_{\ell'=1}^{L} \mathbf{1}_{X_\ell^j(\eta_{j,\ell}) = \ell'} \mathbf{1}_{\tau_{j,\ell}^N + \eta_{j,\ell} + \zeta_{j,\ell'} \leq t} \mathbf{1}_{Y_{\ell'}^j(\zeta_{j,\ell'}) = i} \right) \Big| \tau_{j,\ell}^N \right] \\
&= N^{-1} \sum_{j=1}^{A_\ell^N(t)} |Phi_{\ell,i}(t - \tau_{j,\ell}^N) = \int_0^t \Phi_{\ell,i}(t-s) d\bar{A}_\ell^N(s)\,.
\end{aligned} \tag{3.14}$$

Then the convergence follows by applying Lemma 3.4. □

Lemma 3.6. *Under the assumptions of Lemma 3.5, for* $\ell, i \in \mathcal{L}$*, and for any* $\epsilon > 0$,

$$P\left(\sup_{t \in [0,T]} \left| \bar{E}_{\ell,i}^N(t) - \breve{E}_{\ell,i}^N(t) \right| > \epsilon \right) \to 0 \quad as \quad N \to \infty, \tag{3.15}$$

and

$$P\left(\sup_{t \in [0,T]} \left| \bar{I}_{\ell,i}^N(t) - \breve{I}_{\ell,i}^N(t) \right| > \epsilon \right) \to 0 \quad as \quad N \to \infty. \tag{3.16}$$

Proof. We first consider $\bar{E}_{\ell,i}^N$. We have

$$\bar{E}_{\ell,i}^N(t) - \breve{E}_{\ell,i}^N(t) = \frac{1}{N} \sum_{j=1}^{A_\ell^N(t)} \chi_{j,\ell}^N(t),$$

where

$$\chi_{j,\ell}^N(t) := \mathbf{1}_{\tau_{j,\ell}^N + \eta_{j,\ell} \leq t} \mathbf{1}_{X_\ell^j(\eta_{j,\ell}) = i} - \int_0^{t-\tau_{j,\ell}^N} q_{\ell,i}(u) F(du).$$

Let

$$G_{\ell,i}(t) := \int_0^t q_{\ell,i}(u)F(du), \quad \text{for} \quad \ell, i \in \mathcal{L}. \tag{3.17}$$

Then it is clear that for each j, $\mathbb{E}\big[\chi^N_{j,\ell}(t)|\tau^N_{j,\ell}\big] = 0$, and $\mathbb{E}\big[\chi^N_{j,\ell}(t)^2|\tau^N_{j,\ell}\big] = G_{\ell,i}(t-\tau^N_{j,\ell})(1-G_{\ell,i}(t-\tau^N_{j,\ell}))$. And by the independence of the pairs $\big(\zeta_{j,\ell}, Y^j_\ell(\cdot)\big)$ and $\big(\zeta_{j',\ell}, Y^{j'}_\ell(\cdot)\big)$, we have $\mathbb{E}\big[\chi^N_{j,\ell}(t)\chi^N_{j',\ell}(t)|\mathcal{F}^N_{A,\ell}(t)\big] = 0$, for $j \neq j'$. Thus, we obtain

$$\begin{aligned}
\mathbb{E}\big[\big(\bar{E}^N_{\ell,i}(t) - \breve{E}^N_{\ell,i}(t)\big)^2\big|\mathcal{F}^N_{A,\ell}(t)\big] &= \frac{1}{N^2}\sum_{j=1}^{A^N_\ell(t)} \mathbb{E}\big[\chi^N_{j,\ell}(t)^2|\tau^N_{j,\ell}\big] \\
&= \frac{1}{N}\int_0^t G_{\ell,i}(t-u)(1-G_{\ell,i}(t-u))d\bar{A}^N_\ell(u) \le \frac{\bar{A}^N_\ell(t)}{N}, \\
\mathbb{E}\big[\big(\bar{E}^N_{\ell,i}(t) - \breve{E}^N_{\ell,i}(t)\big)^2\big] &\le \frac{1}{N}\lambda_\ell\bar{\kappa}_\ell\, t,
\end{aligned}$$

which implies that for any $t > 0$ and $\epsilon > 0$,

$$\mathbb{P}\left(\left|\bar{E}^N_{\ell,i}(t) - \breve{E}^N_{\ell,i}(t)\right| > \epsilon\right) \le \frac{1}{N\epsilon^2}\lambda_\ell\bar{\kappa}_\ell\, t \to 0, \quad \text{as} \quad N \to \infty.$$

Next, for $t, u > 0$,

$$\begin{aligned}
&\big|(\bar{E}^N_{\ell,i}(t+u) - \breve{E}^N_{\ell,i}(t+u)) - (\bar{E}^N_{\ell,i}(t) - \breve{E}^N_{\ell,i}(t))\big| \\
&= \left|\frac{1}{N}\sum_{j=1}^{A^N_\ell(t+u)} \chi^N_{j,\ell}(t+u) - \frac{1}{N}\sum_{j=1}^{A^N_\ell(t)} \chi^N_{j,\ell}(t)\right| \\
&= \left|\frac{1}{N}\sum_{j=1}^{A^N_\ell(t)} (\chi^N_{j,\ell}(t+u) - \chi^N_{j,\ell}(t)) + \frac{1}{N}\sum_{j=A^N_\ell(t)+1}^{A^N_\ell(t+u)} \chi^N_{j,\ell}(t+u)\right| \\
&\le \frac{1}{N}\sum_{j=1}^{A^N_\ell(t)} \mathbf{1}_{t<\tau^N_{j,\ell}+\eta_{j,\ell}\le t+u}\mathbf{1}_{X^j_\ell(\eta_{j,\ell})=1} + \int_0^t\int_{t-s}^{t+u-s} q_{\ell,i}(v)F(dv)d\bar{A}^N_\ell(s) \\
&\quad + \big|\bar{A}^N_\ell(t+u) - \bar{A}^N_\ell(t)\big|.
\end{aligned} \tag{3.18}$$

Observe that the three terms on the right hand side are nondecreasing in u. Thus we obtain

$$\begin{aligned}
&\mathbb{P}\left(\sup_{u\in[0,\delta]}\big|(\bar{E}^N_{\ell,i}(t+u) - \breve{E}^N_{\ell,i}(t+u)) - (\bar{E}^N_{\ell,i}(t) - \breve{E}^N_{\ell,i}(t))\big| > \epsilon\right) \\
&\le \mathbb{P}\left(\frac{1}{N}\sum_{j=1}^{A^N_\ell(t)} \mathbf{1}_{t<\tau^N_{j,\ell}+\eta_{j,\ell}\le t+\delta}\mathbf{1}_{X^j_\ell(\eta_{j,\ell})=i} > \epsilon/3\right) \\
&\quad + \mathbb{P}\left(\int_0^t\int_{t-s}^{t+\delta-s} q_{\ell,i}(v)F(dv)d\bar{A}^N_\ell(s) > \epsilon/3\right) + \mathbb{P}\left(\big|\bar{A}^N_\ell(t+\delta) - \bar{A}^N_\ell(t)\big| > \epsilon/3\right).
\end{aligned} \tag{3.19}$$

Using the PRM $\breve{Q}_\ell(ds, d\mathfrak{a}, du, d\theta)$ and its compensated PRM $\overline{Q}_\ell(ds, d\mathfrak{a}, du, d\theta)$, we have

$$\mathbb{E}\left[\left(\frac{1}{N}\sum_{j=1}^{A^N_\ell(t)} \mathbf{1}_{t<\tau^N_{j,\ell}+\eta_{j,\ell}\le t+\delta}\mathbf{1}_{X^j_\ell(\eta_{j,\ell})=i}\right)^2\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\left(\frac{1}{N}\int_0^t\int_0^\infty\int_{t-s}^{t+\delta-s}\int_{\{i\}}\mathbf{1}_{\mathfrak{a}\le\lambda_\ell\Upsilon_\ell^N(s)}\check{Q}_\ell(ds,d\mathfrak{a},du,d\theta)\right)^2\right]\\
&\le 2\mathbb{E}\left[\left(\frac{1}{N}\int_0^t\int_0^\infty\int_{t-s}^{t+\delta-s}\int_{\{i\}}\mathbf{1}_{\mathfrak{a}\le\lambda_\ell\Upsilon_\ell^N(s)}\overline{Q}_\ell(ds,d\mathfrak{a},du,d\theta)\right)^2\right]\\
&\quad + 2\mathbb{E}\left[\left(\int_0^t\int_{t-s}^{t+\delta-s}q_{\ell,i}(u)F(du)\lambda_\ell\bar{\Upsilon}_\ell^N(s)ds\right)^2\right]\\
&= \frac{2}{N}\mathbb{E}\left[\int_0^t\int_{t-s}^{t+\delta-s}q_{\ell,i}(u)F(du)\lambda_\ell\bar{\Upsilon}_\ell^N(s)ds\right] + 2\mathbb{E}\left[\left(\int_0^t\int_{t-s}^{t+\delta-s}q_{\ell,i}(u)F(du)\lambda_\ell\bar{\Upsilon}_\ell^N(s)ds\right)^2\right]\\
&\le \frac{2}{N}\lambda_\ell\bar{\kappa}_\ell\int_0^t\int_{t-s}^{t+\delta-s}q_{\ell,i}(u)F(du)ds + 2\left(\lambda_\ell\bar{\kappa}_\ell\int_0^t\int_{t-s}^{t+\delta-s}q_{\ell,i}(u)F(du)ds\right)^2.
\end{aligned}
$$

The first term converges to zero as $N\to\infty$, and the second term satisfies

$$
\begin{aligned}
\frac{1}{\delta}\left(\lambda_\ell\bar{\kappa}_\ell\int_0^t\int_{t-s}^{t+\delta-s}q_{\ell,i}(u)F(du)ds\right)^2 &\le \frac{1}{\delta}\left(\lambda_\ell\bar{\kappa}_\ell\int_0^t(F(t+\delta-s)-F(t-s))ds\right)^2\\
&\le \frac{1}{\delta}\big(\lambda_\ell\bar{\kappa}_\ell\big)^2\left(\int_t^{t+\delta}F(u)du-\int_0^\delta F(u)du\right)^2\\
&\le \delta\big(\lambda_\ell\bar{\kappa}_\ell\big)^2\to 0 \quad \text{as} \quad \delta\to 0. \qquad (3.20)
\end{aligned}
$$

The second term on the right hand side of (3.19) can be treated similarly as the second term right above. Now for the third term, using (3.1),

$$
\mathbb{E}\left[\left|\bar{A}_\ell^N(t+\delta)-\bar{A}_\ell^N(t)\right|^2\right] \le 2\mathbb{E}\left[\left|\bar{M}_{A,\ell}^N(t+\delta)-\bar{M}_{A,\ell}^N(t)\right|^2\right] + 2\mathbb{E}\left[\left|\lambda_\ell N^{-1}\int_t^{t+\delta}\Upsilon_\ell^N(s)ds\right|^2\right]. \qquad (3.21)
$$

By (3.4), the first term converges to zero as $N\to\infty$. The second term is bounded by $2\big(\lambda_\ell\bar{\kappa}_\ell\delta\big)^2$ by (3.3).

Thus, combining the above, we obtain

$$
\lim_{\delta\to 0}\limsup_{N\to\infty}\left[\frac{T}{\delta}\right]\sup_{t\in[0,T]}\mathbb{P}\left(\sup_{u\in[0,\delta]}\left|(\bar{E}_{\ell,i}^N(t+u)-\check{E}_{\ell,i}^N(t+u))-(\bar{E}_{\ell,i}^N(t)-\check{E}_{\ell,i}^N(t))\right|>\epsilon\right)=0.
$$

Therefore, we have proved (3.15)

We next show (3.16), which follows from similar steps as above. We highlight the main differences below.

For each $t\ge 0$, we have

$$
\bar{I}_{\ell,i}^N(t)-\check{I}_{\ell,i}^N(t)=\frac{1}{N}\sum_{j=1}^{A_\ell^N(t)}\chi_{j,\ell,i}^N(t),
$$

where

$$
\chi_{j,\ell,i}^N(t):=\mathbf{1}_{\tau_{j,\ell}^N+\eta_{j,\ell}\le t}\left(\sum_{\ell'=1}^L\mathbf{1}_{X_\ell^j(\eta_{j,\ell})=\ell'}\mathbf{1}_{\tau_{j,\ell}^N+\eta_{j,\ell}+\zeta_{j,\ell'}\le t}\mathbf{1}_{Y_{\ell'}^j(\zeta_{j,\ell'})=i}\right)-\Phi_{\ell,i}(t-\tau_{j,\ell}^N).
$$

It is clear that $\mathbb{E}\big[\chi_{j,\ell,i}^N(t)|\tau_{j,\ell}^N\big]=0$ and $\mathbb{E}\big[\chi_{j,\ell,i}^N(t)^2|\tau_{j,\ell}^N\big]=\Phi_{\ell,i}(t-\tau_{j,\ell}^N)(1-\Phi_{\ell,i}(t-\tau_{j,\ell}^N))$ where $\Phi_{\ell,i}(t)$ is defined in (2.20). Moreover $\mathbb{E}\big[\chi_{j,\ell,i}^N(t)\chi_{j',\ell,i}^N(t)|\mathcal{F}_{A,\ell}^N(t)\big]=0$ due to the independence of the pairs

$\big(\eta_{j,\ell}, \zeta_{j,\ell'}, X^j_\ell(\cdot), Y^j_{\ell'}(\cdot)\big)$ and $\big(\eta_{j',\ell}, \zeta_{j',\ell'}, X^{j'}_\ell(\cdot), Y^{j'}_{\ell'}(\cdot)\big)$. Thus, we obtain

$$\begin{aligned}\mathbb{E}\big[\big(\bar I^N_{\ell,i}(t) - \breve I^N_{\ell,i}(t)\big)^2 \big| \mathcal{F}^N_{A,\ell}(t)\big] &= \frac{1}{N^2}\sum_{j=1}^{A^N_\ell(t)} \mathbb{E}\big[\chi^N_{j,\ell,i}(t)^2 | \tau^N_{j,\ell}\big] \\ &= \frac{1}{N}\int_0^t \Phi_{\ell,i}(t-u)(1-\Phi_{\ell,i}(t-u))d\bar A^N_\ell(u) \le \frac{\lambda_\ell \bar\kappa_\ell t}{N},\end{aligned}$$

which implies that for any $\epsilon > 0$,

$$\mathbb{P}\left(\big|\bar I^N_{\ell,i}(t) - \breve I^N_{\ell,i}(t)\big| > \epsilon\right) \le \frac{1}{N\epsilon^2}\left(\lambda_\ell \sum_{\ell'=1}^L \kappa_{\ell,\ell'}\right) t \to 0, \quad \text{as} \quad N \to \infty.$$

Next, for $t, s > 0$, we have

$$\begin{aligned}&\big|\big(\bar I^N_{\ell,i}(t+s) - \breve I^N_{\ell,i}(t+s)\big) - \big(\bar I^N_{\ell,i}(t) - \breve I^N_{\ell,i}(t)\big)\big| \\ &= \left|\frac{1}{N}\sum_{j=1}^{A^N_\ell(t)} (\chi^N_{j,\ell,i}(t+s) - \chi^N_{j,\ell,i}(t)) + \frac{1}{N}\sum_{j=A^N_\ell(t)+1}^{A^N_\ell(t+s)} \chi^N_{j,\ell,i}(t+s)\right| \\ &\le \frac{1}{N}\sum_{j=1}^{A^N_\ell(t)} \mathbf{1}_{t<\tau^N_{j,\ell}+\eta_{j,\ell}\le t+s}\left(\sum_{\ell'=1}^L \mathbf{1}_{X^j_\ell(\eta_{j,\ell})=\ell'}\mathbf{1}_{\tau^N_{j,\ell}+\eta_{j,\ell}+\zeta_{j,\ell'}\le t+s}\mathbf{1}_{Y^j_{\ell'}(\zeta_{j,\ell'})=i}\right) \\ &\quad + \frac{1}{N}\sum_{j=1}^{A^N_\ell(t)} \mathbf{1}_{\tau^N_{j,\ell}+\eta_{j,\ell}\le t}\left(\sum_{\ell'=1}^L \mathbf{1}_{X^j_\ell(\eta_{j,\ell})=\ell'}\mathbf{1}_{t<\tau^N_{j,\ell}+\eta_{j,\ell}+\zeta_{j,\ell'}\le t+s}\mathbf{1}_{Y^j_{\ell'}(\zeta_{j,\ell'})=i}\right) \\ &\quad + \frac{1}{N}\sum_{j=1}^{A^N_\ell(t)} \int_{t-\tau^N_{j,\ell}}^{t+s-\tau^N_{j,\ell}} \sum_{\ell'=1}^L p_{\ell,\ell'}(u)\int_0^{t+s-\tau^N_{j,\ell}-u} q_{\ell',i}(v)H(du,dv) \\ &\quad + \frac{1}{N}\sum_{j=1}^{A^N_\ell(t)} \int_0^{t-\tau^N_{j,\ell}} \sum_{\ell'=1}^L p_{\ell,\ell'}(u)\int_{t-\tau^N_{j,\ell}-u}^{t+s-\tau^N_{j,\ell}-u} q_{\ell',i}(v)H(du,dv) \\ &\quad + \big|\bar A^N_\ell(t+s) - \bar A^N_\ell(t)\big|.\end{aligned}$$

Observe that each of the five terms on the right hand side is increasing in s. Thus, we have

$$\begin{aligned}&\mathbb{P}\left(\sup_{s\in[0,\delta]}\big|\big(\bar I^N_{\ell,i}(t+s) - \breve I^N_{\ell,i}(t+s)\big) - \big(\bar I^N_{\ell,i}(t) - \breve I^N_{\ell,i}(t)\big)\big| > \epsilon\right) \\ &\le \mathbb{P}\left(\frac{1}{N}\sum_{j=1}^{A^N_\ell(t)} \mathbf{1}_{t<\tau^N_{j,\ell}+\eta_{j,\ell}\le t+\delta}\left(\sum_{\ell'=1}^L \mathbf{1}_{X^j_\ell(\eta_{j,\ell})=\ell'}\mathbf{1}_{\tau^N_{j,\ell}+\eta_{j,\ell}+\zeta_{j,\ell'}\le t+\delta}\mathbf{1}_{Y^j_{\ell'}(\zeta_{j,\ell'})=i}\right) > \epsilon/5\right) \\ &\quad + \mathbb{P}\left(\frac{1}{N}\sum_{j=1}^{A^N_\ell(t)} \mathbf{1}_{\tau^N_{j,\ell}+\eta_{j,\ell}\le t}\left(\sum_{\ell'=1}^L \mathbf{1}_{X^j_\ell(\eta_{j,\ell})=\ell'}\mathbf{1}_{t<\tau^N_{j,\ell}+\eta_{j,\ell}+\zeta_{j,\ell'}\le t+\delta}\mathbf{1}_{Y^j_{\ell'}(\zeta_{j,\ell'})=i}\right) > \epsilon/5\right) \\ &\quad + \mathbb{P}\left(\int_0^t\int_{t-s}^{t+\delta-s}\sum_{\ell'=1}^L p_{\ell,\ell'}(u)\int_0^{t+\delta-s-u} q_{\ell',i}(v)H(du,dv)d\bar A^N_\ell(s) > \epsilon/5\right) \\ &\quad + \mathbb{P}\left(\int_0^t\int_0^{t-s}\sum_{\ell'=1}^L p_{\ell,\ell'}(u)\int_{t-s-u}^{t+\delta-s-u} q_{\ell',i}(v)H(du,dv)d\bar A^N_\ell(s) > \epsilon/5\right)\end{aligned}$$

$$+\mathbb{P}\left(\left|\bar{A}_\ell^N(t+\delta)-\bar{A}_\ell^N(t)\right|>\epsilon/5\right). \tag{3.22}$$

The last term is treated in the same way as the last term in (3.18) using the bound in (3.21). For the first two terms, we use the PRM representation in (2.12). We have

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{N}\sum_{j=1}^{A_\ell^N(t)}\mathbf{1}_{t<\tau_{j,\ell}^N+\eta_{j,\ell}\le t+\delta}\sum_{\ell'=1}^{L}\mathbf{1}_{X_\ell^j(\eta_{j,\ell})=\ell'}\mathbf{1}_{\tau_{j,\ell}^N+\eta_{j,\ell}+\zeta_{j,\ell'}\le t+\delta}\mathbf{1}_{Y_{\ell'}^j(\zeta_{j,\ell'})=i}\right)^2\right]\\
&\le 2\mathbb{E}\left[\left(\frac{1}{N}\int_0^t\int_0^\infty\int_{t-s}^{t+\delta-s}\int_0^{t+\delta-s-u}\int_{\mathcal{L}}\int_{\{i\}}\mathbf{1}_{\mathfrak{a}\le\lambda_\ell\Upsilon_\ell^N(s)}\widetilde{Q}_i(ds,d\mathfrak{a},du,dv,d\theta,d\vartheta)\right)^2\right]\\
&\quad+2\mathbb{E}\left[\left(\int_0^t\left(\int_{t-s}^{t+\delta-s}\int_0^{t+\delta-s-u}\sum_{\ell'=1}^{L}p_{\ell,\ell'}(u)q_{\ell',i}(v)H(du,dv)\right)\lambda_\ell\bar{\Upsilon}_\ell^N(s)ds\right)^2\right]\\
&\le\frac{2}{N}\mathbb{E}\left[\int_0^t\left(\int_{t-s}^{t+\delta-s}\int_0^{t+\delta-s-u}\sum_{\ell'=1}^{L}p_{\ell,\ell'}(u)q_{\ell',i}(v)H(du,dv)\right)\lambda_\ell\bar{\Upsilon}_\ell^N(s)ds\right]\\
&\quad+2\mathbb{E}\left[\left(\int_0^t\left(\int_{t-s}^{t+\delta-s}\int_0^{t+\delta-s-u}\sum_{\ell'=1}^{L}p_{\ell,\ell'}(u)q_{\ell',i}(v)H(du,dv)\right)\lambda_\ell\bar{\Upsilon}_\ell^N(s)ds\right)^2\right]\\
&\le\frac{2\lambda_\ell\bar{\kappa}_\ell}{N}\int_0^t\left(\int_{t-s}^{t+\delta-s}\int_0^{t+\delta-s-u}\sum_{\ell'=1}^{L}p_{\ell,\ell'}(u)q_{\ell',i}(v)H(du,dv)\right)ds\\
&\quad+2\left(\lambda_\ell\bar{\kappa}_\ell\int_0^t\left(\int_{t-s}^{t+\delta-s}\int_0^{t+\delta-s-u}\sum_{\ell'=1}^{L}p_{\ell,\ell'}(u)q_{\ell',i}(v)H(du,dv)\right)ds\right)^2.
\end{aligned}$$

It is clear that the first term converges to zero as $N\to\infty$, and the second term without the constant $2(\lambda_\ell\bar{\kappa}_\ell)^2$ satisfies

$$\begin{aligned}
&\frac{1}{\delta}\left(\int_0^t\left(\int_{t-s}^{t+\delta-s}\int_0^{t+\delta-s-u}\sum_{\ell'=1}^{L}p_{\ell,\ell'}(u)q_{\ell',i}(v)H(du,dv)\right)ds\right)^2\\
&\le\frac{1}{\delta}\left(L\int_0^t\int_{t-s}^{t+\delta-s}F(t+\delta-s-u|u)G(du)ds\right)^2\\
&\le\frac{1}{\delta}\left(L\int_0^t\left(G(t+\delta-s)-G(t-s)\right)ds\right)^2\to 0\quad\text{as}\quad\delta\to 0,
\end{aligned}$$

where the last step follows from the same argument as in (3.20).

Similarly, for the second term on the right hand side of (3.22), we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{N}\sum_{j=1}^{A_\ell^N(t)}\mathbf{1}_{\tau_{j,\ell}^N+\eta_{j,\ell}\le t}\sum_{\ell'=1}^{L}\mathbf{1}_{X_\ell^j(\eta_{j,\ell})=\ell'}\mathbf{1}_{t<\tau_{j,\ell}^N+\eta_{j,\ell}+\zeta_{j,\ell'}\le t+\delta}\mathbf{1}_{Y_{\ell'}^j(\zeta_{j,\ell'})=i}\right)^2\right]\\
&\le\frac{2\lambda_\ell\bar{\kappa}_\ell}{N}\int_0^t\left(\int_0^{t-s}\int_{t-s-u}^{t+\delta-s-u}\sum_{\ell'=1}^{L}p_{\ell,\ell'}(u)q_{\ell',i}(v)H(du,dv)\right)ds\\
&\quad+2\left(\lambda_\ell\bar{\kappa}_\ell\int_0^t\left(\int_0^{t-s}\int_{t-s-u}^{t+\delta-s-u}\sum_{\ell'=1}^{L}p_{\ell,\ell'}(u)q_{\ell',i}(v)H(du,dv)\right)ds\right)^2.
\end{aligned}$$

Here it is clear that the first term converges to zero as $N \to \infty$, and the second term without the constant $2(\lambda_\ell \bar{\kappa}_\ell)^2$ satisfies

$$\begin{aligned}
&\frac{1}{\delta}\left(\int_0^t \left(\int_0^{t-s}\int_{t-s-u}^{t+\delta-s-u} \sum_{\ell'=1}^L p_{\ell,\ell'}(u) q_{\ell',i}(v) H(du,dv)\right) ds\right)^2 \\
&\le \frac{1}{\delta}\left(L\int_0^t \int_0^{t-s} (F(t+\delta-s-u|u) - F(t-s-u|u)) G(du) ds\right)^2 \\
&= \frac{1}{\delta}\left(L\int_0^t \int_0^{t-u} (F(t-u+\delta-s|u) - F(t-u-s|u)) ds G(du)\right)^2 \\
&= \frac{1}{\delta}\left(L\int_0^t \left(\int_\delta^{t+\delta-u} F(s|u) ds - \int_0^{t-u} F(s|u) ds\right) G(du)\right)^2 \\
&\le \frac{1}{\delta}\left(L\int_0^t \int_{t-u}^{t+\delta-u} F(s|u) ds G(du)\right)^2 \\
&\le L^2 \delta \to 0, \quad \text{as} \quad \delta \to 0.
\end{aligned} \tag{3.23}$$

Now for the third and fourth terms on the right hand side of (3.22), we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t \left(\int_{t-s}^{t+\delta-s}\int_0^{t+\delta-s-u} \sum_{\ell'=1}^L p_{\ell,\ell'}(u) q_{\ell',i}(v) H(du,dv)\right) d\bar{A}_\ell^N(s)\right)^2\right] \\
&\le \left(\lambda_\ell \bar{\kappa}_\ell \int_0^t \left(\int_{t-s}^{t+\delta-s}\int_0^{t+\delta-s-u} \sum_{\ell'=1}^L p_{\ell,\ell'}(u) q_{\ell',i}(v) H(du,dv)\right) ds\right)^2,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t \left(\int_0^{t-s}\int_{t-s-u}^{t+\delta-s-u} \sum_{\ell'=1}^L p_{\ell,\ell'}(u) q_{\ell',i}(v) H(du,dv)\right) d\bar{A}_\ell^N(s)\right)^2\right] \\
&\le \left(\lambda_\ell \bar{\kappa}_\ell \int_0^t \left(\int_0^{t-s}\int_{t-s-u}^{t+\delta-s-u} \sum_{\ell'=1}^L p_{\ell,\ell'}(u) q_{\ell',i}(v) H(du,dv)\right) ds\right)^2.
\end{aligned}$$

These two terms are bounded by a constant times δ^2. Thus we have shown that

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\left[\frac{T}{\delta}\right] \sup_{t\in[0,T]} \mathbb{P}\left(\sup_{s\in[0,\delta]} \left|(\bar{I}_{\ell,i}^N(t+s) - \check{I}_{\ell,i}^N(t+s)) - (\bar{I}_{\ell,i}^N(t) - \check{I}_{\ell,i}^N(t))\right| > \epsilon\right) = 0.$$

Therefore, we have shown that (3.16) holds. □

By the above two lemmas we have shown the following result.

Lemma 3.7. *With the limit $(\bar{A}_1, \ldots, \bar{A}_L)$ of the convergent subsequence of $\{(\bar{A}_1^N, \ldots, \bar{A}_L^N)\}$, under Assumption 2.1,*

$$(\bar{E}_{\ell,i}^N, \bar{I}_{\ell,i}^N,\ \ell, i \in \mathcal{L}) \Rightarrow (\bar{E}_{\ell,i}(t), \bar{I}_{\ell,i}, \ell, i \in \mathcal{L}) \quad \text{in} \quad D^{2L^2} \quad \text{as} \quad N \to \infty, \tag{3.24}$$

in probability, where $\bar{E}_{\ell,i}(t)$ and $\bar{I}_{\ell,i}(t)$ are given in (3.11) and (3.12), respectively.

We are now ready to prove the convergence of $(\bar{E}_i^N, \bar{I}_i^N, \bar{R}_i^N)$.

Lemma 3.8. *With the limit $(\bar{A}_1, \ldots, \bar{A}_L)$ of the convergent subsequence of $\{(\bar{A}_1^N, \ldots, \bar{A}_L^N)\}$, under Assumption 2.1,*

$$(\bar{E}_i^N, \bar{I}_i^N, \bar{R}_i^N,\ i \in \mathcal{L}) \Rightarrow (\bar{E}_i, \bar{I}_i, \bar{R}_i,\ i \in \mathcal{L}) \quad \text{in} \quad D^{3L} \quad \text{as} \quad N \to \infty,$$

where the limits are the unique solution to the systems of ODEs: for $i \in \mathcal{L}$,

$$\bar{E}_i(t) \;=\; \bar{E}_i(0) + \bar{A}_i(t) - \sum_{\ell=1}^{L} \left(\bar{E}^0_{\ell,i}(t) + \bar{E}_{\ell,i}(t)\right) + \sum_{\ell=1,\ell\neq i}^{L} \int_0^t (\nu_{E,\ell,i}\bar{E}_\ell(s) - \nu_{E,i,\ell}\bar{E}_i(s))ds\,, \tag{3.25}$$

$$\begin{aligned}\bar{I}_i(t) = \bar{I}_i(0) + \sum_{\ell=1}^{L} \left(\bar{E}^0_{\ell,i}(t) + \bar{E}_{\ell,i}(t)\right) - \sum_{\ell=1}^{L} \left(\bar{I}^{0,1}_{\ell,i}(t) + \bar{I}^{0,2}_{\ell,i}(t) + \bar{I}_{\ell,i}(t)\right) \\ + \sum_{\ell=1,\ell\neq i}^{L} \int_0^t (\nu_{I,\ell,i}\bar{I}_\ell(s) - \nu_{I,i,\ell}\bar{I}_i(s))ds,\end{aligned} \tag{3.26}$$

$$\bar{R}_i(t) = \sum_{\ell=1}^{L} \left(\bar{I}^{0,1}_{\ell,i}(t) + \bar{I}^{0,2}_{\ell,i}(t) + \bar{I}_{\ell,i}(t)\right) + \sum_{\ell\neq i} \int_0^t \left(\nu_{R,\ell,i}\bar{R}_\ell(s) - \nu_{R,i,\ell}\bar{R}_i(s)\right) ds, \tag{3.27}$$

with $\bar{E}^0_{\ell,i}$, $\bar{I}^{0,1}_{\ell,i}$ and $\bar{I}^{0,2}_{\ell,i}$ being given in (3.7) and (3.8), and with $\bar{E}_{\ell,i}$ and $\bar{I}_{\ell,i}$ being defined in (3.11) and (3.12), respectively.

Proof. The proof for the process $E^N_i(t)$ is similar to that of $I^N_i(t)$, so we focus on $I^N_i(t)$. By the representations of $I^N_i(t)$ in (2.10), we have

$$\begin{aligned}\bar{I}^N_i(t) = \bar{I}^N_i(0) + \sum_{\ell=1}^{L} \left(\bar{E}^{N,0}_{\ell,i}(t) + \bar{E}^N_{\ell,i}(t)\right) - \sum_{\ell=1}^{L} \left(\bar{I}^{N,0,1}_{\ell,i}(t) + \bar{I}^{N,0,2}_{\ell,i}(t) + \check{I}^N_{\ell,i}(t)\right) + \sum_{\ell=1}^{L} \Delta^N_{I,\ell,i}(t) \\ + \sum_{\ell=1}^{L} \left(\bar{M}^N_{I,\ell,i}(t) - \bar{M}^N_{I,i,\ell}(t)\right) + \sum_{\ell\neq i} \int_0^t \left(\nu_{I,\ell,i}\bar{I}^N_\ell(s) - \nu_{I,i,\ell}\bar{I}^N_i(s)\right) ds,\end{aligned} \tag{3.28}$$

where

$$\Delta^N_{I,\ell,i}(t) = \check{I}^N_{\ell,i}(t) - \bar{I}^N_{\ell,i}(t), \tag{3.29}$$

and for $\ell \neq i$,

$$\bar{M}^N_{I,\ell,i}(t) = \frac{1}{N}\left(P_{I,\ell,i}\left(\nu_{I,\ell,i}\int_0^t I^N_\ell(s)ds\right) - \nu_{I,\ell,i}\int_0^t I^N_\ell(s)ds\right). \tag{3.30}$$

As in the proof of Lemma 3.2 for the convergence of $\left(\bar{M}^N_{S,\ell,i},\, \ell, i \in \mathcal{L}, \ell \neq i\right)$, we obtain that for any $\ell, i \in \mathcal{L}, \ell \neq i$, as $N \to \infty$,

$$\bar{M}^N_{I,\ell,i}(t) \to 0 \;\text{ in probability, locally uniformly in } t. \tag{3.31}$$

From Assumption 2.1 and Lemma 3.3,

$$\bar{I}^N_i(0) - \sum_{\ell=1}^{L} \bar{I}^{N,0}_{\ell,i}(t) \to \bar{I}_i(0) - \int_0^t \sum_{\ell=1}^{L} \bar{I}_\ell(0) q_{\ell,i}(s) F_0(ds)$$

in probability, locally uniformly in t. Moreover, from Lemma 3.6 and the last arguments,

$$\sum_{\ell=1}^{L} \Delta^N_{I,\ell,i}(t) + \sum_{\ell=1}^{L} \left(\bar{M}^N_{I,\ell,i}(t) - \bar{M}^N_{I,i,\ell}(t)\right) \to 0$$

in probability, locally uniformly in t. Hence, it follows from (3.28) and the continuous mapping theorem that, along any subsequence along which $(\bar{A}^N_1, \ldots, \bar{A}^N_L) \Rightarrow (\bar{A}_1, \ldots, \bar{A}_L)$ in D^L, $\left(\bar{I}^N_1, \ldots, \bar{I}^N_L\right) \Rightarrow \left(\bar{I}_1, \ldots, \bar{I}_L\right)$, where the limit is the unique solution of the system of ODEs.

We next prove the convergence of $(\bar{R}_1^N, \dots, \bar{R}_L^N)$. Similar to (2.10), we obtain the following representations for the process $R_i^N(t)$:

$$R_i^N(t) = \sum_{\ell=1}^{L} \sum_{k=1}^{I_\ell^N(0)} \mathbf{1}_{\zeta_{k,\ell}^0 \le t} \mathbf{1}_{Y_\ell^{0,k}(\zeta_{k,\ell}^0)=i} + \sum_{\ell=1}^{L} \sum_{k=1}^{E_\ell^N(0)} \mathbf{1}_{\eta_{k,\ell}^0 \le t} \left(\sum_{\ell'=1}^{L} \mathbf{1}_{X_\ell^{0,k}(\eta_{k,\ell}^0)=\ell'} \mathbf{1}_{\eta_{k,\ell}^0+\zeta_{-k,\ell'} \le t} \mathbf{1}_{Y_{\ell'}^{-k}(\zeta_{-k,\ell'})=i} \right)$$
$$+ \sum_{\ell=1}^{L} \sum_{j=1}^{A_\ell^N(t)} \mathbf{1}_{\tau_{j,\ell}^N+\eta_{j,\ell} \le t} \left(\sum_{\ell'=1}^{L} \mathbf{1}_{X_\ell^j(\eta_{j,\ell})=\ell'} \mathbf{1}_{\tau_{j,\ell}^N+\eta_{j,\ell}+\zeta_{j,\ell'} \le t} \mathbf{1}_{Y_{\ell'}^j(\zeta_{j,\ell'})=i} \right)$$
$$+ \sum_{\ell \ne i} P_{I,\ell,i} \left(\nu_{I,\ell,i} \int_0^t I_\ell^N(s) ds \right) - \sum_{\ell \ne i} P_{I,i,\ell} \left(\nu_{I,i,\ell} \int_0^t I_i^N(s) ds \right). \tag{3.32}$$

Thus, we can represent the processes $\bar{R}_i^N(t)$ by

$$\bar{R}_i^N(t) = \sum_{\ell=1}^{L} \left(\bar{I}_{\ell,i}^{N,0,1}(t) + \bar{I}_{\ell,i}^{N,0,2}(t) + \breve{I}_{\ell,i}^N(t) \right) + \sum_{\ell=1}^{L} \Delta_{I,\ell,i}^N(t) + \sum_{\ell=1}^{L} \left(\bar{M}_{R,\ell,i}^N(t) - \bar{M}_{R,i,\ell}^N(t) \right)$$
$$+ \sum_{\ell \ne i} \int_0^t \left(\nu_{R,\ell,i} \bar{R}_\ell(s) - \nu_{R,i,\ell} \bar{R}_i(s) \right) ds, \tag{3.33}$$

where $\Delta_{I,\ell,i}^N(t)$ is given (3.29), and for $\ell \ne i$,

$$\bar{M}_{R,\ell,i}^N(t) = \frac{1}{N} \left(P_{R,\ell,i} \left(\nu_{R,\ell,i} \int_0^t R_\ell^N(s) ds \right) - \nu_{R,\ell,i} \int_0^t R_\ell^N(s) ds \right). \tag{3.34}$$

Arguments very similar to the proof above allow us to conclude. □

From the above arguments, since we have the joint convergence in Lemmas 3.3 and 3.7, we can conclude the joint convergence of $(\bar{S}_i^N, \bar{E}_i^N, \bar{I}_i^N, \bar{R}_i^N,\ i \in \mathcal{L})$. However, we have not yet quite explicited the limiting equations, since we have not expressed $\bar{A}_i(t)$ in terms of $(\bar{S}_i(t), \bar{E}_i(t), \bar{I}_i(t), \bar{R}_i(t), \bar{I}_\ell(t), \ell \ne i)$. It seems easy to do that since we know that

$$\bar{\Upsilon}_i^N(t) = \frac{\bar{S}_i^N(t) \sum_{\ell=1}^{L} \kappa_{i\ell} \bar{I}_\ell^N(t)}{(\bar{S}_i^N(t) + \bar{E}_i^N(t) + \bar{I}_i^N(t) + \bar{R}_i^N(t))^\gamma}.$$

However, there is a difficulty in the case where both $\gamma = 1$ and $\sum_{\ell \ne i} \kappa_{i\ell} \ne 0$. Define the function $\psi(s,e,i,r,u) = \frac{s(i+u)}{(s+e+i+r)^\gamma}$ on $[0,1]^4 \times [0, \bar{\kappa}]$. If either $0 \le \gamma < 1$ or $\sup_i \sum_{\ell \ne i} \kappa_{i\ell} = 0$, ψ is continuous. However, if both $\gamma = 1$ and $\sup_i \sum_{j \ne i} \kappa_{ij} > 0$, then ψ is not continuous at any point of the form $(0,0,0,0,u)$, with $u > 0$. Hence, if we want to include that case in our model, we need to prove that for any $i \in \mathcal{L}$ and $T > 0$, $\inf_{0 \le t \le T} (\bar{S}_i(t) + \bar{E}_i(t) + \bar{I}_i(t) + \bar{R}_i(t)) > 0$. Fortunately, we can prove such an estimate, although we do not have yet established the exact system of equations of the $(\bar{S}_i(t), \bar{E}_i(t), \bar{I}_i(t), \bar{R}_i(t)),\ i \in \mathcal{L}$.

Lemma 3.9. *Let $(\bar{S}_i(t), \bar{E}_i(t), \bar{I}_i(t), \bar{R}_i(t),\ i \in \mathcal{L})$, $0 \le t \le T$, be any weak limit as $N \to \infty$ of $(\bar{S}_i^N(t), \bar{E}_i^N(t), \bar{I}_i^N(t), \bar{R}_i^N(t),\ i \in \mathcal{L})$. For any $i \in \mathcal{L}$ and $T > 0$, there exists a constant $C_{i,T} > 0$ which is such that for $0 \le t \le T$,*

$$\bar{S}_i(t) + \bar{E}_i(t) + \bar{I}_i(t) + \bar{R}_i(t) \ge C_{i,T}. \tag{3.35}$$

Proof. Let $\bar{U}_i(t) := \bar{S}_i(t) + \bar{E}_i(t) + \bar{I}_i(t) + \bar{R}_i(t)$ for $i \in \mathcal{L}$ and $0 \le t \le T$. For any $i, \ell \in \mathcal{L}$, let

$$\bar{\nu}_{i,\ell} = \begin{cases} \nu_{S,i,\ell} \vee \nu_{E,i,\ell} \vee \nu_{I,i,\ell} \vee \nu_{R,i,\ell}, & \text{if } i \ne \ell, \\ 0, & \text{if } i = \ell. \end{cases}$$

We know that $(\bar{S}_i(t), \bar{E}_i(t), \bar{I}_i(t), \bar{R}_i(t))$ is a solution of (3.5), (3.25), (3.26) and (3.27). Then we have

$$\bar{U}_i(t) = \bar{U}_i(0) - \int_0^t \Big(\sum_{\ell=1}^{L} \bar{\nu}_{i,\ell} \Big) \bar{U}_i(s) ds + \int_0^t \bar{V}_i(s) ds, \tag{3.36}$$

where

$$\begin{aligned} \bar{V}_i(t) = & \sum_{\ell \neq i} \left(\nu_{S,\ell,i} \bar{S}_\ell(t) + \nu_{E,\ell,i} \bar{E}_\ell(t) + \nu_{I,\ell,i} \bar{I}_\ell(t) + \nu_{R,\ell,i} \bar{R}_\ell(t) \right) \\ & + \sum_{\ell \neq i} \left([\bar{\nu}_{i,\ell} - \nu_{S,i,\ell}] \bar{S}_i(t) + [\bar{\nu}_{i,\ell} - \nu_{E,i,\ell}] \bar{E}_i(t) + [\bar{\nu}_{i,\ell} - \nu_{I,i,\ell}] \bar{I}_i(t) + [\bar{\nu}_{i,\ell} - \nu_{R,i,\ell}] \bar{R}_i(t) \right) . \end{aligned}$$

Differentiating equation (3.36) and exploiting the inequality $W_i(t) \geq 0$ for all $t \geq 0$, we deduce that

$$\bar{U}_i(t) \geq \bar{U}_i(0) e^{-(\sum_{\ell=1}^{L} \bar{\nu}_{i,\ell}) t} \geq \bar{U}_i(0) e^{-(\sum_{\ell=1}^{L} \bar{\nu}_{i,\ell}) T} := C_{i,T},$$

for any $0 \leq t \leq T$. □

We can now explicit the processes $\bar{A}_i(t)$.

Lemma 3.10. *Let $(\bar{A}_i(t), \bar{S}_i(t), \bar{E}_i(t), \bar{I}_i(t), \bar{R}_i(t),\ i \in \mathcal{L})$, $0 \leq t \leq T$ be any weak limit as $N \to \infty$ of $(\bar{A}_i^N(t), \bar{S}_i^N(t), \bar{E}_i^N(t), \bar{I}_i^N(t), \bar{R}_i^N(t),\ i \in \mathcal{L})$. Then for any $i \in \mathcal{L}$ and $0 \leq t \leq T$,*

$$\bar{A}_i(t) = \int_0^t \bar{\Upsilon}_i(t),$$

where

$$\bar{\Upsilon}_i(t) := \frac{\bar{S}_i(t) \sum_{\ell=1}^{L} \kappa_{i\ell} \bar{I}_\ell(t)}{(\bar{S}_i(t) + \bar{E}(t) + \bar{I}_i(t) + \bar{R}_i(t))^\gamma} .$$

Proof. Let $\mathcal{D}^4$ denote the set of càdlàg functions from $\mathbb{R}_+$ into $[0,1]^4 \times [0, \bar{\kappa}]$. For any $\gamma \in [0,1]$, the function $\psi(s,e,i,r,u) = \frac{s(i+u)}{(s+e+i+r)^\gamma}$ is continuous for the Skorohod topology, on the subset of $\mathcal{D}^4$ which is such that for any $T > 0$, $\inf_{0 \leq t \leq T} \{s(t) + e(t) + i(t) + r(t)\} > 0$. Thus, we deduce from the joint convergence

$$(\bar{A}_i^N, \bar{S}_i^N, \bar{E}_i^N, \bar{I}_i^N, \bar{R}_i^N,\ i \in \mathcal{L}) \Rightarrow (\bar{A}_i, \bar{S}_i, \bar{E}_i, \bar{I}_i, \bar{R}_i,\ i \in \mathcal{L})$$

and Lemma 3.9 that

$$\bar{\Upsilon}_i^N \Rightarrow \bar{\Upsilon}_i := \frac{\bar{S}_i \sum_{\ell=1}^{L} \kappa_{i\ell} \bar{I}_\ell}{(\bar{S}_i + \bar{E}_i + \bar{I}_i + \bar{R}_i)^\gamma} \quad \text{in} \quad D \quad \text{as} \quad N \to \infty.$$

Consequently,

$$(\bar{A}_1^N, \ldots, \bar{A}_L^N) \Rightarrow (\bar{A}_1, \ldots, \bar{A}_L) = \left(\lambda_1 \int_0^{\cdot} \bar{\Upsilon}_1(s) ds, \ldots, \lambda_L \int_0^{\cdot} \bar{\Upsilon}_L(s) ds \right) \quad \text{in} \quad D^L \quad \text{as} \quad N \to \infty.$$

Therefore, any limit satisfies the system of integral equations given in Theorem 2.1. □

The uniqueness of solutions to the set of integral equations in Theorem 2.1 follows from the next Lemma, from which we deduce that the whole sequence converges, and since the limit is deterministic, the convergence is in probability. This completes the proof of Theorem 2.1.

Lemma 3.11. *The system of equations (2.31), (2.32), (2.33), together with (2.34), has at most one solution.*

Proof. In view of Lemma 3.9, if we take the difference between two solutions, any convex combination of those two solutions satisfies the lower bound (3.35). Since

$$\Upsilon_i(t) = \psi\left(\bar{S}_i(t), \bar{E}_i(t), \bar{I}_i(t), \bar{R}_i(t), \sum_{\ell \neq i} \kappa_{i\ell} \bar{I}_\ell(t)\right),$$

and at each time $t \in [0,T]$, the derivatives of ψ with respect to each of its variables is bounded in absolute value by the supremum of 1 and $\bar{\kappa}_i \bar{U}_i^{-\gamma}(t) \leq \bar{\kappa}_i C_{i,T}^{-\gamma}$, we can now apply a standard argument based upon Gronwall's Lemma in order to deduce uniqueness. □

4. Proof of the FCLT

In this section, we prove Theorem 2.2. We first give the following representations of the diffusion-scaled processes. The process $\hat{A}_i^N(t)$ can be decomposed as:

$$\hat{A}_i^N(t) = \lambda_i \int_0^t \hat{\Upsilon}_i^N(s)ds + \hat{M}_{A,i}^N(t), \quad t \geq 0, \tag{4.1}$$

where

$$\hat{\Upsilon}_i^N(t) = \sqrt{N}(\bar{\Upsilon}_i^N(t) - \Upsilon_i(t)), \tag{4.2}$$

and

$$\hat{M}_{A,i}^N(t) := \frac{1}{\sqrt{N}} \int_0^t \int_0^\infty \mathbf{1}_{\mathfrak{a} \leq \lambda_i \Upsilon_i^N(s)} \bar{Q}_i(ds, d\mathfrak{a}). \tag{4.3}$$

For the process $\hat{S}_i^N(t)$, we have

$$\begin{aligned}\hat{S}_i^N(t) = \hat{S}_i^N(0) - \lambda_i \int_0^t \hat{\Upsilon}_i^N(s)ds + \sum_{\ell=1,\ell\neq i}^L \int_0^t (\nu_{S,\ell,i}\hat{S}_\ell^N(s) - \nu_{S,i,\ell}\hat{S}_i^N(s))ds \\ - \hat{M}_{A,i}(t) + \sum_{\ell=1,\ell\neq i}^L \left(\hat{M}_{S,\ell,i}(t) - \hat{M}_{S,i,\ell}(t)\right),\end{aligned} \tag{4.4}$$

where

$$\hat{M}_{S,\ell,i}^N(t) := \frac{1}{\sqrt{N}} \left(P_{S,\ell,i}\left(\nu_{S,\ell,i} \int_0^t S_\ell^N(s)ds\right) - \nu_{S,\ell,i} \int_0^t S_\ell^N(s)ds \right).$$

For the process $\hat{E}_i^N(t)$, by the representation in (2.9), using the definitions of $E_{\ell,i}^{N,0}(t)$ and $E_{\ell,i}^N(t)$ in (3.6), we obtain

$$\begin{aligned}\hat{E}_i^N(t) = \hat{E}_i^N(0)\left(1 - \sum_{\ell=1}^L \int_0^t p_{\ell,i}(s)G_0(ds)\right) + \lambda_i \int_0^t \hat{\Upsilon}_i^N(s)ds - \sum_{\ell=1}^L \lambda_\ell \int_0^t \int_0^{t-s} p_{\ell,i}(u)G(du)\hat{\Upsilon}_\ell^N(s)ds \\ + \sum_{\ell=1,\ell\neq i}^L \int_0^t (\nu_{E,\ell,i}\hat{E}_\ell^N(s) - \nu_{E,i,\ell}\hat{E}_i^N(s))ds - \sum_{\ell=1}^L \left(\hat{E}_{\ell,i}^{N,0}(t) + \hat{E}_{\ell,i}^N(t)\right) \\ + \hat{M}_{A,i}^N(t) + \sum_{\ell=1,\ell\neq i}^L \left(\hat{M}_{E,\ell,i}^N(t) - \hat{M}_{E,i,\ell}^N(t)\right),\end{aligned} \tag{4.5}$$

where for $\ell, i \in \mathcal{L}$,

$$\hat{E}_{\ell,i}^{N,0}(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{E_\ell^N(0)} \left(\mathbf{1}_{\eta_{k,\ell}^0 \leq t} \mathbf{1}_{X_\ell^{0,k}(\eta_{k,\ell}^0)=i} - \int_0^t p_{\ell,i}(s)G_0(ds)\right), \tag{4.6}$$

$$\hat{E}^N_{\ell,i}(t) = \frac{1}{\sqrt{N}} \left(\sum_{j=1}^{A^N_\ell(t)} \mathbf{1}_{\tau^N_{j,\ell}+\eta_{j,\ell}\le t} \mathbf{1}_{X^j_\ell(\eta_{j,\ell})=i} - N\lambda_\ell \int_0^t \left(\int_0^{t-s} p_{\ell,i}(u)G(du) \right) \bar{\Upsilon}^N_\ell(s)ds \right), \tag{4.7}$$

and for $\ell \neq i$,

$$\hat{M}^N_{E,\ell,i}(t) = \frac{1}{\sqrt{N}} \left(P_{E,\ell,i} \left(\nu_{E,\ell,i} \int_0^t E^N_\ell(s)ds \right) - \nu_{E,\ell,i} \int_0^t E^N_\ell(s)ds \right).$$

For the process $\hat{I}^N_i(t)$, we obtain

$$\begin{aligned}
\hat{I}^N_i(t) = {} & \hat{I}^N_i(0) \left(1 - \sum_{\ell=1}^L \int_0^t q_{\ell,i}(s)F_0(ds) \right) + \hat{E}^N_i(0) \sum_{\ell=1}^L \left(\int_0^t p_{\ell,i}(s)G_0(ds) - \Phi^0_{\ell,i}(t) \right) \\
& + \sum_{\ell=1}^L \lambda_\ell \int_0^t \int_0^{t-s} p_{\ell,i}(u)G(du)\hat{\Upsilon}^N_\ell(s)ds - \sum_{\ell=1}^L \lambda_\ell \int_0^t \Phi_{\ell,i}(t-s)\hat{\Upsilon}^N_\ell(s)ds \\
& + \sum_{\ell\neq i} \int_0^t \left(\nu_{I,\ell,i}\hat{I}^N_\ell(s) - \nu_{I,i,\ell}\hat{I}^N_i(s) \right) ds + \sum_{\ell=1}^L \left(\hat{M}^N_{I,\ell,i}(t) - \hat{M}^N_{I,i,\ell}(t) \right) \\
& + \sum_{\ell=1}^L \left(\hat{E}^{N,0}_{\ell,i}(t) + \hat{E}^N_{\ell,i}(t) \right) - \sum_{\ell=1}^L \left(\hat{I}^{N,0,1}_{\ell,i}(t) + \hat{I}^{N,0,2}_{\ell,i}(t) + \hat{I}^N_{\ell,i}(t) \right),
\end{aligned} \tag{4.8}$$

where $\hat{E}^{N,0}_{\ell,i}(t)$ and $\hat{E}^N_{\ell,i}(t)$ are defined in (4.6) and (4.7), respectively, and

$$\hat{I}^{N,0,1}_{\ell,i}(t) := \frac{1}{\sqrt{N}} \sum_{k=1}^{I^N_\ell(0)} \left(\mathbf{1}_{\zeta^0_{k,\ell}\le t} \mathbf{1}_{Y^{0,k}_\ell(\zeta^0_{k,\ell})=i} - \int_0^t q_{\ell,i}(s)F_0(ds) \right), \tag{4.9}$$

$$\hat{I}^{N,0,2}_{\ell,i}(t) := \frac{1}{\sqrt{N}} \sum_{k=1}^{E^N_\ell(0)} \left(\sum_{\ell'=1}^L \mathbf{1}_{X^{0,k}_\ell(\eta^0_{k,\ell})=\ell'} \mathbf{1}_{\eta^0_{k,\ell}+\zeta_{-k,\ell'}\le t} \mathbf{1}_{Y^{-k}_{\ell'}(\zeta_{-k,\ell'})=i} - \Phi^0_{\ell,i}(t) \right), \tag{4.10}$$

$$\hat{I}^N_{\ell,i}(t) := \frac{1}{\sqrt{N}} \left(\sum_{j=1}^{A^N_\ell(t)} \sum_{\ell'=1}^L \mathbf{1}_{X^j_\ell(\eta_{j,\ell})=\ell'} \mathbf{1}_{\tau^N_{j,\ell}+\eta_{j,\ell}+\zeta_{j,\ell'}\le t} \mathbf{1}_{Y^j_{\ell'}(\zeta_{j,\ell'})=i} - N\lambda_\ell \int_0^t \Phi_{\ell,i}(t-s)\bar{\Upsilon}^N_\ell(s)ds \right), \tag{4.11}$$

and for $\ell \neq i$,

$$\hat{M}^N_{I,\ell,i}(t) = \frac{1}{\sqrt{N}} \left(P_{I,\ell,i} \left(\nu_{I,\ell,i} \int_0^t I^N_\ell(s)ds \right) - \nu_{I,\ell,i} \int_0^t I^N_\ell(s)ds \right). \tag{4.12}$$

For the process $\hat{R}^N_i(t)$, we have

$$\begin{aligned}
\hat{R}^N_i(t) = {} & \hat{I}^N_i(0) \sum_{\ell=1}^L \int_0^t q_{\ell,i}(s)F_0(ds) + \hat{E}^N_i(0) \sum_{\ell=1}^L \Phi^0_{\ell,i}(t) \\
& + \sum_{\ell=1}^L \lambda_\ell \int_0^t \Phi_{\ell,i}(t-s)\hat{\Upsilon}^N_\ell(s)ds + \sum_{\ell\neq i} \int_0^t \left(\nu_{R,\ell,i}\hat{R}^N_\ell(s) - \nu_{R,i,\ell}\hat{R}^N_i(s) \right) ds \\
& + \sum_{\ell=1}^L \left(\hat{M}^N_{R,\ell,i}(t) - \hat{M}^N_{R,i,\ell}(t) \right) + \sum_{\ell=1}^L \left(\hat{I}^{N,0,1}_{\ell,i}(t) + \hat{I}^{N,0,2}_{\ell,i}(t) + \hat{I}^N_{\ell,i}(t) \right),
\end{aligned} \tag{4.13}$$

where for $\ell, i \in \mathcal{L}$, and $\ell \neq i$,

$$\hat{M}^N_{R,\ell,i}(t) = \frac{1}{\sqrt{N}} \left(P_{R,\ell,i} \left(\nu_{R,\ell,i} \int_0^t R^N_\ell(s) ds \right) - \nu_{R,\ell,i} \int_0^t R^N_\ell(s) ds \right). \tag{4.14}$$

We establish the convergence of some key components in these representations in Lemmas 4.1, 4.2 and 4.6.

Lemma 4.1. *Under Assumption 2.2,*

$$\begin{aligned} &\big(\hat{M}^N_{A,i}, \hat{M}^N_{E,\ell,i}, \hat{M}^N_{S,\ell,i}, \hat{M}^N_{I,\ell,i}, \hat{M}^N_{R,\ell,i},\ \ell, i \in \mathcal{L}, \ell \neq i\big) \\ &\Rightarrow \big(\hat{M}_{A,i}, \hat{M}_{E,\ell,i}, \hat{M}_{S,\ell,i}, \hat{M}_{I,\ell,i}, \hat{M}_{R,\ell,i},\ \ell, i \in \mathcal{L}, \ell \neq i\big) \quad \text{in} \quad D^{L+4L(L-1)} \quad \text{as} \quad N \to \infty, \end{aligned}$$

where the limits are as given in Theorem 2.2.

Proof. This follows from a standard martingale convergence argument, see, e.g., Theorem 1.4 in Chapter 7 of [14]. The main step consists in proving that the quadratic variations converge (involving the convergence of fluid-scaled processes). We omit the details for brevity. □

Lemma 4.2. *Under Assumption 2.2,*

$$\big(\hat{E}^{N,0}_{\ell,i},\ \hat{I}^{N,0,1}_{\ell,i},\ \hat{I}^{N,0,2}_{\ell,i},\ \ell, i \in \mathcal{L}\big) \Rightarrow \big(\hat{E}^{0}_{\ell,i},\ \hat{I}^{0,1}_{\ell,i},\ \hat{I}^{0,2}_{\ell,i},\ \ell, i \in \mathcal{L}\big) \quad \text{in} \quad D^{3L^2} \quad \text{as} \quad N \to \infty,$$

where the limits are as given in Theorem 2.2.

Proof. Since the proofs for the convergence of $\big(\hat{E}^{N,0}_{\ell,i}, \ell, i \in \mathcal{L}\big)$ and $\big(\hat{I}^{N,0,1}_{\ell,i}, \ell, i \in \mathcal{L}\big)$ follow from the same argument, we only prove the convergence of $\big(\hat{I}^{N,0,1}_{\ell,i}, \ell, i \in \mathcal{L}\big) \Rightarrow \big(\hat{I}^{0,1}_{\ell,i}, \ell, i \in \mathcal{L}\big)$ in D^{L^2}. Define $\tilde{I}^{N,0,1}_{\ell,i}(t)$ by replacing $\bar{I}^N_\ell(0)$ with $N\bar{I}_\ell(0)$ in (4.9) for $\ell, i \in \mathcal{L}$.

Let us consider the convergence of $\tilde{I}^{N,0,1}_{1,1}$. Observe that the pairs $\big(\zeta_{k,1}, Y^{0,k}_1(\cdot)\big)$ and $\big(\zeta_{k',1}, Y^{0,k'}_1(\cdot)\big)$ are independent and have the same law. Thus, its proof follows in a similar approach for empirical processes, see, e.g., Theorem 14.3 in [5]. There are some differences due to the Markov process $Y^{0,k}_1$, which we highlight below. So, we apply Theorem 13.5 in [5].

For each $t > 0$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} &\mathbb{E}\left[\exp\left(\hat{\imath}\alpha \tilde{I}^{N,0,1}_{1,1}(t)\right)\right] = \mathbb{E}\left[\prod_{k=1}^{N\bar{I}_1(0)} \exp\left(\hat{\imath}\alpha \frac{1}{\sqrt{N}} \left(\mathbf{1}_{\zeta^0_{k,1} \le t} \mathbf{1}_{Y^{0,k}_1(\zeta^0_{k,1})=1} - \int_0^t q_{1,1}(s) F_0(ds)\right)\right)\right] \\ &= \prod_{k=1}^{N\bar{I}_1(0)} \mathbb{E}\left[\exp\left(\hat{\imath}\alpha \frac{1}{\sqrt{N}} \left(\mathbf{1}_{\zeta^0_{k,1} \le t} \mathbf{1}_{Y^{0,k}_1(\zeta^0_{k,1})=1} - \int_0^t q_{1,1}(s) F_0(ds)\right)\right)\right] \\ &= \left(1 - \frac{\alpha^2}{2N} \mathbb{E}\left[\left(\mathbf{1}_{\zeta^0_{k,1} \le t} \mathbf{1}_{Y^{0,k}_1(\zeta^0_{k,1})=1} - \int_0^t q_{1,1}(s) F_0(ds)\right)^2\right] + o(N^{-1})\right)^{N\bar{I}_1(0)} \\ &= \left(1 - \frac{\alpha^2}{2N} \int_0^t q_{1,1}(s) F_0(ds) \left(1 - \int_0^t q_{1,1}(s) F_0(ds)\right) + o(N^{-1})\right)^{N\bar{I}_1(0)} \\ &\xrightarrow{N\to\infty} \exp\left(-\frac{\alpha^2}{2} \bar{I}_1(0) \int_0^t q_{1,1}(s) F_0(ds) \left(1 - \int_0^t q_{1,1}(s) F_0(ds)\right)\right) = \mathbb{E}\left[\exp\left(\hat{\imath}\alpha \hat{I}^{0,1}_{1,1}(t)\right)\right]. \end{aligned}$$

Similarly, it can be also shown that for any $0 < s < t$,

$$\mathbb{E}\left[\exp\left(\hat{\imath}\alpha \left(\tilde{I}^{N,0,1}_{1,1}(t) - \tilde{I}^{N,0,1}_{1,1}(s)\right)\right)\right] \to \mathbb{E}\left[\exp\left(\hat{\imath}\alpha \left(\hat{I}^{0,1}_{1,1}(t) - \hat{I}^{0,1}_{1,1}(s)\right)\right)\right] \quad \text{as} \quad N \to \infty.$$

Thus, for the convergence of finite dimensional distributions, with $t_1 < t_2 < \cdots < t_k$ and $\alpha_\ell, \ell = 1, \ldots, k$, we can write $\sum_{\ell=1}^k \hat{\imath}\alpha_\ell \tilde{I}^{N,0,1}_{1,1}(t_\ell)$ using the increments $\tilde{I}^{N,0,1}_{1,1}(t_\ell) - \tilde{I}^{N,0,1}_{1,1}(t_{\ell-1})$, which have covariances equal to zero over disjoint intervals.

Next, to prove tightness, we employ Theorem 13.5 and verify condition (13.14) in [5]. We show that for $r \le s \le t$ and for $N \ge 1$,

$$\mathbb{E}\left[\left|\tilde{I}^{N,0,1}_{1,1}(s) - \tilde{I}^{N,0,1}_{1,1}(r)\right|^2 \left|\tilde{I}^{N,0,1}_{1,1}(t) - \tilde{I}^{N,0,1}_{1,1}(s)\right|^2\right] \le C(\phi(s)-\phi(r))(\phi(t)-\phi(s)) \le C(\phi(t)-\phi(r))^2$$

for some constant C and $\phi(t) = \int_0^t q_{1,1}(u)F_0(du)$ which is a nonnegative, nondecreasing and continuous function. Recall that F_0 is assumed to be continuous. This will enforce condition (13.14) in [5], which according to Theorem 13.5 implies tightness in D. Let

$$\Delta I^k_{r,s} = \mathbf{1}_{r<\zeta^0_{k,1}\le s}\mathbf{1}_{Y_1^{0,k}(\zeta^0_{k,1})=1} - \int_r^s q_{1,1}(u)F_0(du),$$

and

$$\Delta I^k_{s,t} = \mathbf{1}_{s<\zeta^0_{k,1}\le t}\mathbf{1}_{Y_1^{0,k}(\zeta^0_{k,1})=1} - \int_s^t q_{1,1}(u)F_0(du).$$

Note that $\mathbb{E}[\Delta I^k_{r,s}] = 0$, $\mathbb{E}[\Delta I^k_{s,t}] = 0$,

$$\mathbb{E}[(\Delta I^k_{r,s})^2] = \int_r^s q_{1,1}(u)F_0(du)\left(1 - \int_r^s q_{1,1}(u)F_0(du)\right),$$

and

$$\mathbb{E}[(\Delta I^k_{s,t})^2] = \int_s^t q_{1,1}(u)F_0(du)\left(1 - \int_s^t q_{1,1}(u)F_0(du)\right).$$

By direct calculations, following similar steps in the proof of (14.9) in [5], we obtain

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{k=1}^{N\bar{I}_1(0)}\Delta I^k_{r,s}\right|^2\left|\sum_{k=1}^{N\bar{I}_1(0)}\Delta I^k_{s,t}\right|^2\right] &= N\bar{I}_1(0)\mathbb{E}[\Delta I^2_{r,s}\Delta I^2_{s,t}] + N\bar{I}_1(0)(N\bar{I}_1(0)-1)\mathbb{E}[\Delta I^2_{r,s}]\mathbb{E}[\Delta I^2_{s,t}] \\ &\quad + 2N\bar{I}_1(0)(N\bar{I}_1(0)-1)(\mathbb{E}[\Delta I_{r,s}\Delta I_{s,t}])^2,\end{aligned}$$

$$N^{-2}\mathbb{E}\left[\left|\sum_{k=1}^{N\bar{I}_1(0)}\Delta I^k_{r,s}\right|^2\left|\sum_{k=1}^{N\bar{I}_1(0)}\Delta I^k_{s,t}\right|^2\right] \le C\int_r^s q_{1,1}(u)F_0(du)\int_s^t q_{1,1}(u)F_0(du).$$

Thus we have shown the convergence $\tilde{I}^{N,0,1}_{1,1} \Rightarrow \hat{I}^{0,1}_{1,1}$ in D.

For the joint convergence $\left(\tilde{I}^{N,0,1}_{\ell,i},\, \ell, i = 1,\dots L\right)$, since the variables and processes associated with patch ℓ and patch ℓ' are independent, it suffices to show the joint convergences $\left(\tilde{I}^{N,0,1}_{\ell,i},\, i \in \mathcal{L}\right)$ for different ℓ's separately. For the joint convergence $\left(\tilde{I}^{N,0,1}_{\ell,i},\, i \in \mathcal{L}\right)$, we obtain tightness from that of each process as established above, so it suffices to show the joint convergence of their finite dimensional distributions. Take $\ell = 1, i = 1, 2$ as an example. For $0 < t_1 < t_2$ and $\alpha_1, \alpha_2 \in \mathbb{R}$,

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\hat{\imath}\alpha_1\tilde{I}^{N,0,1}_{1,1}(t_1) + \hat{\imath}\alpha_2\tilde{I}^{N,0,1}_{1,2}(t_2)\right)\right] \\ &= \mathbb{E}\Bigg[\prod_{k=1}^{N\bar{I}_1(0)}\exp\Bigg(\hat{\imath}\alpha_1\frac{1}{\sqrt{N}}\left(\mathbf{1}_{\zeta^0_{k,1}\le t_1}\mathbf{1}_{Y_1^{0,k}(\zeta^0_{k,1})=1} - \int_0^{t_1} q_{1,1}(s)F_0(ds)\right) \\ &\qquad + \hat{\imath}\alpha_2\frac{1}{\sqrt{N}}\left(\mathbf{1}_{\zeta^0_{k,1}\le t_2}\mathbf{1}_{Y_1^{0,k}(\zeta^0_{k,1})=2} - \int_0^{t_2} q_{1,2}(s)F_0(ds)\right)\Bigg)\Bigg] \\ &= \Bigg(1 - \frac{1}{2N}\mathbb{E}\Bigg[\Bigg(\alpha_1\left(\mathbf{1}_{\zeta^0_{k,1}\le t_1}\mathbf{1}_{Y_1^{0,k}(\zeta^0_{k,1})=1} - \int_0^{t_1} q_{1,1}(s)F_0(ds)\right) \\ &\qquad + \alpha_2\left(\mathbf{1}_{\zeta^0_{k,1}\le t_2}\mathbf{1}_{Y_1^{0,k}(\zeta^0_{k,1})=2} - \int_0^{t_2} q_{1,2}(s)F_0(ds)\right)\Bigg)^2\Bigg] + o(N^{-1})\Bigg)^{N\bar{I}_1(0)}\end{aligned}$$

$$
\begin{aligned}
&= \Big(1 - \frac{\alpha_1^2}{2N}\int_0^{t_1} q_{1,1}(s)F_0(ds)\Big(1 - \int_0^t q_{1,1}(s)F_0(ds)\Big) \\
&\quad - \frac{\alpha_2^2}{2N}\int_0^{t_2} q_{1,2}(s)F_0(ds)\Big(1 - \int_0^t q_{1,2}(s)F_0(ds)\Big)\Big)^{N\bar{I}_1(0)} \\
&\xrightarrow{N\to\infty} \exp\Big(-\frac{\alpha_1^2}{2}\bar{I}_1(0)\int_0^{t_1} q_{1,1}(s)F_0(ds)\Big(1 - \int_0^t q_{1,1}(s)F_0(ds)\Big) \\
&\quad - \frac{\alpha_2^2}{2}\bar{I}_1(0)\int_0^{t_2} q_{1,2}(s)F_0(ds)\Big(1 - \int_0^t q_{1,2}(s)F_0(ds)\Big)\Big) \\
&= \mathbb{E}\left[\exp\Big(\hat{\imath}\alpha_1\hat{I}_{1,1}^{0,1}(t_1) + \hat{\imath}\alpha_2\hat{I}_{1,2}^{0,1}(t_2)\Big)\right].
\end{aligned}
$$

This calculation can be extended to the computation of finite dimensional distributions of $\big(\tilde{I}_{1,1}^{N,0,1}, \tilde{I}_{1,2}^{N,0,1}\big)$, and then that of $\big(\tilde{I}_{\ell,i}^{N,0,1},\, i \in \mathcal{L}\big)$. Therefore, we can conclude the joint convergence of $\big(\tilde{I}_{\ell,i}^{N,0,1},\, i \in \mathcal{L}\big)$.

Next, to prove the convergence of $\hat{I}_{\ell,i}^{N,0,1}$, it suffices to show that $\hat{I}_{\ell,i}^{N,0,1} - \tilde{I}_{\ell,i}^{N,0,1} \Rightarrow 0$ in D for each $\ell, i \in \mathcal{L}$. We consider the convergence $\hat{I}_{1,1}^{N,0,1} - \tilde{I}_{1,1}^{N,0,1} \Rightarrow 0$ in D. We have

$$
\begin{aligned}
&\mathrm{sign}(I_1^N(0) - N\bar{I}_1(0))\Big(\hat{I}_{1,1}^{N,0,1}(t) - \tilde{I}_{1,1}^{N,0,1}(t)\Big) \\
&= \frac{1}{\sqrt{N}}\sum_{k=I_1^N(0)\wedge N\bar{I}_1(0)}^{I_1^N(0)\vee N\bar{I}_1(0)}\Big(\mathbf{1}_{\zeta_{k,1}^0\le t}\mathbf{1}_{Y_1^{0,k}(\zeta_{k,\ell}^0)=1} - \int_0^t q_{1,1}(s)F_0(ds)\Big) \\
&= \frac{1}{\sqrt{N}}\sum_{k=I_1^N(0)\wedge N\bar{I}_1(0)}^{I_1^N(0)\vee N\bar{I}_1(0)}\mathbf{1}_{\zeta_{k,1}^0\le t}\mathbf{1}_{Y_1^{0,k}(\zeta_{k,\ell}^0)=1} - |\hat{I}_1^N(0)|\int_0^t q_{1,1}(s)F_0(ds)\,. \qquad (4.15)
\end{aligned}
$$

It is clear that by Assumption 2.2,

$$
\mathbb{E}\Big[\big(\hat{I}_{1,1}^{N,0}(t) - \tilde{I}_{1,1}^{N,0}(t)\big)^2\Big] = \int_0^t q_{1,1}(s)F_0(ds)\Big(1 - \int_0^t q_{1,1}(s)F_0(ds)\Big)\,\mathbb{E}\big[|\bar{I}_1^N(0) - \bar{I}_1(0)|\big] \to 0
$$

as $N \to \infty$. To show that $\big\{\hat{I}_{1,1}^{N,0,1} - \hat{I}_{1,1}^{N,0,1}\big\}_N$ is tight, by Assumption 2.2 and (4.15), it suffices to prove the tightness of the first term on the right hand side of (4.15), which we denote as $\Delta_{1,1}^{N,0,1}(t)$. By the Corollary of Theorem 7.4 in [5], it suffices to show that for all $\epsilon > 0$,

$$
\limsup_N \sup_{0\le t\le T}\frac{1}{\delta}\mathbb{P}\Big(\sup_{0\le u\le\delta}\big|\Delta_{1,1}^{N,0,1}(t+u) - \Delta_{1,1}^{N,0,1}(t)\big| > \epsilon\Big) \to 0, \quad \text{as} \quad \delta \to 0. \qquad (4.16)
$$

Since $\Delta_{1,1}^{N,0,1}(t+u)$ is increasing in u, we only need to consider

$$
\mathbb{E}\big[\big|\Delta_{1,1}^{N,0,1}(t+\delta) - \Delta_{1,1}^{N,0,1}(t)\big|^2\big] = \mathbb{E}\big[|\bar{I}_1^N(0) - \bar{I}_1(0)|\big]\int_t^{t+\delta} q_{1,1}(s)F_0(ds) \le \mathbb{E}\big[|\bar{I}_1^N(0) - \bar{I}_1(0)|\big]\delta
$$

whose $\limsup_N$ is equal to zero under Assumption 2.2, and thus we conclude that (4.16) holds. Therefore we have shown $\hat{I}_{1,1}^{N,0,1} - \tilde{I}_{1,1}^{N,0,1} \to 0$ in D in probability, and conclude the joint convergence $\big(\hat{I}_{\ell,i}^{N,0,1}, \ell, i \in \mathcal{L}\big) \Rightarrow \big(\hat{I}_{\ell,i}^{0,1}, \ell, i \in \mathcal{L}\big)$ in D^{L^2}. To prove that the limit processes $\big(\hat{I}_{\ell,i}^{0,1}, \ell, i \in \mathcal{L}\big)$ are continuous when F_0 is continuous, since they are Gaussian, it suffices to show continuity in the quadratic mean [18], that is, for all $t > 0$, $\lim_{s\to t}\mathbb{E}\big[\big|\hat{I}_{\ell,i}^{0,1}(t) - \hat{I}_{\ell,i}^{0,1}(s)\big|^2\big] = 0$. This is easily checked from the continuity of the covariance functions.

We next focus on the processes $\big(\hat{I}_{\ell,i}^{N,0,2}, \ell, i \in \mathcal{L}\big)$. Define $\tilde{I}_{\ell,i}^{N,0,2}(t)$ by replacing $E_\ell^N(0)$ with $N\bar{E}_\ell(0)$ in the expression of $\tilde{I}_{\ell,i}^{N,0,2}(t)$ in (4.10). We first prove the joint convergence $\big(\tilde{I}_{\ell,i}^{N,0,2}, \ell, i \in \mathcal{L}\big) \Rightarrow$

$\big(\hat{I}^{0,2}_{\ell,i}, \ell, i \in \mathcal{L}\big)$ in D^{L^2} as $N \to \infty$. We again apply Theorem 13.5 in [5]. By direct calculations, we obtain for $t \geq 0$,

$$\mathbb{E}\left[\exp\left(\hat{\imath}\alpha \tilde{I}^{N,0,2}_{\ell,i}(t)\right)\right] \xrightarrow{N\to\infty} \mathbb{E}\left[\exp\left(\hat{\imath}\alpha \hat{I}^{0,2}_{\ell,i}(t)\right)\right] = \exp\left(-\frac{\alpha^2}{2}\bar{E}_\ell(0)\Phi^0_{\ell,i}(t)(1-\Phi^0_{\ell,i}(t))\right)$$

and for $t > t' \geq 0$,

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\hat{\imath}\alpha\left(\tilde{I}^{N,0,2}_{\ell,i}(t) - \tilde{I}^{N,0,2}_{\ell,i}(t')\right)\right)\right]\\ &\xrightarrow{N\to\infty} \mathbb{E}\left[\exp\left(\hat{\imath}\alpha\left(\hat{I}^{0,2}_{\ell,i}(t) - \hat{I}^{0,2}_{\ell,i}(t')\right)\right)\right] = \exp\left(-\frac{\alpha^2}{2}\bar{E}_\ell(0)(\Phi^0_{\ell,i}(t) - \Phi^0_{\ell,i}(t'))\right).\end{aligned}$$

Hence, we can establish the convergence of finite dimensional distributions of $\hat{I}^{N,0,2}_{\ell,i}$ similarly as that of $\tilde{I}^{N,0,1}_{1,1}(t)$ above. For tightness, we obtain for $t' \leq t \leq t''$ and for $N \geq 1$,

$$\mathbb{E}\left[\left|\hat{I}^{N,0,2}_{\ell,i}(t') - \hat{I}^{N,0,2}_{\ell,i}(t)\right|^2\left|\hat{I}^{N,0}_{\ell,i}(t'') - \hat{I}^{N,0}_{\ell,i}(t)\right|^2\right] \leq C(\phi(t)-\phi(t'))(\phi(t'')-\phi(t)) \leq C(\phi(t'')-\phi(t'))^2$$

where $\phi(t) = \int_0^t \sum_{\ell'=1}^L p_{\ell,\ell'}(u)\int_0^{t-u} q_{\ell',i}(v)F(dv|u)G_0(du)$. Note that since G_0 is continuous, this function $\phi(t)$ is a nonnegative, nondecreasing and continuous function. This proves the convergence of $\tilde{I}^{N,0,2}_{\ell,i} \Rightarrow \hat{I}^{0,2}_{\ell,i}$ in D as $N \to \infty$.

For the joint convergence of $\tilde{I}^{N,0,2}_{\ell,i}$ and $\tilde{I}^{N,0,2}_{\ell',i'}$, we can follow a similar argument as the joint convergence $\big(\tilde{I}^{N,0,1}_{\ell,i},\ \ell, i = 1,\dots L\big)$ above. Thus we have shown the joint convergence $\big(\tilde{I}^{N,0,2}_{\ell,i}, \ell, i \in \mathcal{L}\big) \Rightarrow \big(\hat{I}^{0,2}_{\ell,i}, \ell, i \in \mathcal{L}\big)$ in D^{L^2}. To conclude $\big(\hat{I}^{N,0,2}_{\ell,i}, \ell, i \in \mathcal{L}\big) \Rightarrow \big(\hat{I}^{0,2}_{\ell,i}, \ell, i \in \mathcal{L}\big)$ in D^{L^2}, it remains to show that $\hat{I}^{N,0,2}_{\ell,i} - \tilde{I}^{N,0,2}_{\ell,i} \to 0$ in D in probability for each $\ell, i \in \mathcal{L}$. We have

$$\mathbb{E}\Big[\big(\hat{I}^{N,0,2}_{\ell,i}(t) - \tilde{I}^{N,0,2}_{\ell,i}(t)\big)^2\Big] = \Phi^0_{\ell,i}(t)(1-\Phi^0_{\ell,i}(t))\mathbb{E}\big[\big|\bar{E}^N_\ell(0) - \bar{E}_\ell(0)\big|\big] \to 0 \quad \text{as} \quad N \to \infty,$$

under Assumption 2.2. To prove tightness of $\big\{\hat{I}^{N,0,2}_{\ell,i} - \tilde{I}^{N,0,2}_{\ell,i}\big\}_N$, observing that

$$\begin{aligned}&\operatorname{sign}(E^N_\ell(0) - N\bar{E}_\ell(0))\big(\hat{I}^{N,0,2}_{\ell,i}(t) - \tilde{I}^{N,0,2}_{\ell,i}(t)\big)\\ &= \frac{1}{\sqrt{N}}\sum_{k=E^N_\ell(0)\wedge N\bar{E}_\ell(0)}^{E^N_\ell(0)\vee N\bar{E}_\ell(0)}\sum_{\ell'=1}^L \mathbf{1}_{X^{0,k}_\ell(\eta^0_{k,\ell})=\ell'}\mathbf{1}_{\eta^0_{k,\ell}+\zeta_{-k,\ell'}\leq t}\mathbf{1}_{Y^{-k}_{\ell'}(\zeta_{-k,\ell'})=i} - \big|\hat{E}^N_\ell(0)\big|\Phi^0_{\ell,i}(t)\,,\end{aligned}$$

it suffices, applying the Corollary of Theorem 7.4 in [5] to the first term, denoted by $\Delta^{N,0,2}_{\ell,i}(t)$, to show that

$$\limsup_N \sup_{0\leq t\leq T}\frac{1}{\delta}\mathbb{P}\left(\sup_{0\leq u\leq\delta}\left|\Delta^{N,0,2}_{\ell,i}(t+u) - \Delta^{N,0,2}_{\ell,i}(t)\right| > \epsilon\right) \to 0, \quad \text{as} \quad \delta \to 0. \tag{4.17}$$

Since it is increasing in t, we consider for $\delta > 0$,

$$\mathbb{E}\big[\big|\Delta^{N,0,2}_{\ell,i}(t+\delta) - \Delta^{N,0,2}_{\ell,i}(t)\big|^2\big] = \mathbb{E}\big[\big|\bar{E}^N_\ell(0) - \bar{E}_\ell(0)\big|\big]\big(\Phi^0_{\ell,i}(t+\delta) - \Phi^0_{\ell,i}(t)\big).$$

The $\limsup_N$ of the above is equal to zero by Assumption 2.2, so it is clear that (4.17) holds. Thus we have shown $\big(\hat{I}^{N,0,2}_{\ell,i}, \ell, i \in \mathcal{L}\big) \Rightarrow \big(\hat{I}^{0,2}_{\ell,i}, \ell, i \in \mathcal{L}\big)$ in D^{L^2}.

For the joint convergence of $\big(\hat{E}^{N,0}_{\ell,i},\ \hat{I}^{N,0,1}_{\ell,i},\ \hat{I}^{N,0,2}_{\ell,i},\ \ell, i \in \mathcal{L}\big)$, by the independence of the variables associated with $I^N_\ell(0)$ and $E^N_\ell(0)$, it suffices to show the joint convergence of $\big(\hat{E}^{N,0}_{\ell,i}, \hat{I}^{N,0,2}_{\ell,i},\ \ell, i \in \mathcal{L}\big)$. We define $\tilde{E}^{N,0}_{\ell,i}(t)$ by replacing $E^N(0)$ with $N\bar{E}(0)$ in the expression of $\tilde{E}^{N,0}_{\ell,i}(t)$ in (4.6). Similar to the proof above, we have $\hat{E}^{N,0}_{\ell,i} - \tilde{E}^{N,0}_{\ell,i} \to 0$ in D in probability. It then suffices to show the joint convergence of $\big(\tilde{E}^{N,0}_{\ell,i}, \tilde{I}^{N,0,2}_{\ell,i},\ \ell, i \in \mathcal{L}\big)$ and moreover, since they are tight individually, it suffices to

show the convergence of their joint finite dimensional distributions. Note that $\tilde{E}^{N,0}_{\ell,i}(t)$ and $\tilde{I}^{N,0,2}_{\ell',i}(t')$ are independent for $\ell \neq \ell'$. We calculate that for $\alpha, \alpha' \in \mathbb{R}$ and $t, t' > 0$,

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(\hat{\imath}\alpha\tilde{E}^{N,0}_{\ell,i}(t) + \hat{\imath}\alpha'\tilde{I}^{N,0,2}_{\ell,i'}(t')\Big)\Big] \\
&= \exp\Big(-\frac{\alpha^2}{2}\bar{E}_\ell(0)\int_0^t p_{\ell,i}(s)dG_0(s)\Big) + \exp\Big(-\frac{(\alpha')^2}{2}\bar{E}_\ell(0)\Phi^0_{\ell,i'}(t')(1-\Phi^0_{\ell,i'}(t'))\Big) \\
&\quad + \exp\Big(-\alpha\alpha'\bar{E}_\ell(0)\Big(\int_0^t p_{\ell,i}(u)\int_0^{t'-u} q_{i,i'}(v)H_0(du,dv) - \int_0^t p_{\ell,i}(s)G_0(ds)\Phi^0_{\ell,i'}(t')\Big)\Big).
\end{aligned}$$

This can be extended easily to finite dimensional distributions of $\big(\tilde{E}^{N,0}_{\ell,i}(t_1), \dots, \tilde{E}^{N,0}_{\ell,i}(t_k), \tilde{I}^{N,0,2}_{\ell,i'}(t_1), \dots, \tilde{I}^{N,0,2}_{\ell,i'}(t_k),\ \ell, i \in \mathcal{L}\big)$ for $t_1 < t_2 < \cdots < t_k$, $k \geq 1$. Therefore we have shown the joint convergence of $\big(\hat{E}^{N,0}_{\ell,i}, \hat{I}^{N,0,2}_{\ell,i},\ \ell, i \in \mathcal{L}\big)$. Finally for the continuity of the limit processes, it suffices to show the continuity in the quadratic mean [18], which follows from the continuity of the covariance functions. This completes the proof of the lemma. □

For the next lemma on the moment estimates, we shall need the following technical result.

Lemma 4.3. *In the two cases $\gamma \in [0,1)$ and $\sum_{\ell \neq u}\kappa_{i\ell} = 0$, there exists a constant C such that for any $0 \leq t \leq T$, $1 \leq i \leq L$,*

$$|\hat{\Upsilon}^N_i(t)| \leq C\left(|\hat{S}^N_i(t)| + |\hat{E}^N_i(t)| + |\hat{I}^N_i(t)| + |\hat{R}^N_i(t)|\right) + \sum_{\ell \neq i}\kappa_{i\ell}|\hat{I}^N_\ell(t)|\,. \tag{4.18}$$

Proof. We consider again the map $\psi : [0,1]^4 \times [0,\bar{\kappa}] \mapsto \mathbb{R}_+$:

$$\psi(s,e,i,r,u) = \frac{s(i+u)}{(s+e+i+r)^\gamma}\,.$$

We have

$$\begin{aligned}
0 \leq \psi'_s(s,e,i,r,u) &= \frac{((1-\gamma)s+e+i+r)(i+u)}{(s+e+i+r)^{1+\gamma}} \leq \frac{\bar{\kappa}}{(s+e+i+r)^\gamma}, \\
0 \geq \psi'_e(s,e,i,r,u) &= -\gamma\frac{s(i+u)}{(s+e+i+r)^{1+\gamma}} \geq -\frac{\bar{\kappa}}{(s+e+i+r)^\gamma}, \\
\psi'_i(s,e,i,r,u) &= \frac{s(s+e+i+r)-\gamma s(i+u)}{(s+e+i+r)^{1+\gamma}}, \quad |\psi'_i(s,e,i,r,u)| \leq \frac{\bar{\kappa}}{(s++e+i+r)^\gamma}, \\
0 \geq \psi'_r(s,e,i,r,u) &= -\gamma\frac{s(i+u)}{(s+e+i+r)^{1+\gamma}} \geq -\frac{\bar{\kappa}}{(s+e+i+r)^\gamma}, \\
0 \leq \psi'_u(s,e,i,r,u) &= \frac{s}{(s+e+i+r)^\gamma} \leq 1\,.
\end{aligned}$$

Moreover, if we define for $0 \leq a \leq 1$, $g(a) = \psi(s+a(s'-s), e+a(e'-e), i+a(i'-i), r+a(r'-r), u+a(u'-u))$, we have

$$\begin{aligned}
&\psi(s',e',i',r',u') - \psi(s,e,i,r,u) = \int_0^1 g'(a)da \\
&= \left(\int_0^1 \psi'_s(s+a(s'-s), e+a(e'-e), i+a(i'-i), r+a(r'-r), u+a(u'-u))da\right)[s'-s] \\
&\quad + \left(\int_0^1 \psi'_e(s+a(s'-s), e+a(e'-e), i+a(i'-i), r+a(r'-r), u+a(u'-u))da\right)[e'-e] \\
&\quad + \left(\int_0^1 \psi'_i(s+a(s'-s), e+a(e'-e), i+a(i'-i), r+a(r'-r), u+a(u'-u))da\right)[i'-i]
\end{aligned}$$

$$+\left(\int_0^1 \psi'_r(s+a(s'-s),e+a(e'-e),i+a(i'-i),r+a(r'-r),u+a(u'-u))da\right)[r'-r]$$
$$+\left(\int_0^1 \psi'_u(s+a(s'-s),e+a(e'-e),i+a(i'-i),r+a(r'-r),u+a(u'-u))da\right)[u'-u]\,.$$

We have

$$\bar\Upsilon_i^N(t)=\psi\Big(\bar S_i^N(t),\bar E_i^N(t),\bar I_i^N(t),\bar R_i^N(t),\sum_{\ell\neq i}\bar I_\ell^N(t)\Big),\quad \bar\Upsilon_i(t)=\psi\Big(\bar S_i(t),\bar E_i(t),\bar I_i(t),\bar R_i(t),\sum_{\ell\neq i}\bar I_\ell(t)\Big)\,.$$

Suppose first that $\gamma<1$. Clearly, the result will follow from the last formulas, if we prove that there exists $C>0$ such that for all $t\in[0,T]$, $N\geq 1$,

$$\int_0^1((1-a)\bar S_i(t)+a\bar S_i^N(t)+(1-a)\bar E_i(t)+a\bar E_i^N(t)+(1-a)\bar I_i(t)+a\bar I_i^N(t)+(1-a)\bar R_i(t)+a\bar R_i^N(t))^{-\gamma}da\leq C\,.$$

We have

$$\begin{aligned}
&\int_0^1\Big((1-a)\bar S_i(t)+a\bar S_i^N(t)+(1-a)\bar E_i(t)+a\bar E_i^N(t)+(1-a)\bar I_i(t)+a\bar I_i^N(t)\\
&\qquad\qquad+(1-a)\bar R_i(t)+a\bar R_i^N(t)\Big)^{-\gamma}da\\
&\leq(\bar S_i(t)+\bar E_i(t)+\bar I_i(t)+\bar R_i(t))^{-\gamma}\int_0^1\frac{da}{a^\gamma}\\
&=\frac{(\bar S_i(t)+\bar E_i(t)+\bar I_i(t)+\bar R_i(t))^{-\gamma}}{1-\gamma}\leq\frac{C_{i,T}^{-\gamma}}{1-\gamma}\,,
\end{aligned}$$

for any $0\leq t\leq T$, where we have used Lemma 3.9 for the last inequality.

It is easy to check that in the case where the variable u disappear from the above formulas, the derivatives of f are bounded on $[0,1]^3$, and the result holds in the case $\gamma=1$ as well. □

We will now prove the following estimate.

Lemma 4.4. *For each $i\in\mathcal{L}$,*

$$\sup_N\mathbb{E}\Big[\sup_{t\in[0,T]}\big|\hat\Upsilon_i^N(t)\big|^2\Big]<\infty.\tag{4.19}$$

Proof. We first show that for each $i\in\mathcal{L}$,

$$\sup_N\sup_{t\in[0,T]}\mathbb{E}\Big[\big|\hat\Upsilon_i^N(t)\big|^2\Big]<\infty.\tag{4.20}$$

We shall use (4.18). In the representations of $\hat S_i^N(t)$, $\hat E_i^N(t)$, $\hat I_i^N(t)$ and $\hat R_i^N(t)$ in (4.4), (4.5), (4.8) and (4.13), respectively, the following hold: there exists a constant $C>0$ such that for each $i,\ell\in\mathcal{L}$,

$$\begin{aligned}
&\sup_N\mathbb{E}\big[\big|\hat S_i^N(0)\big|^2\big]\leq C,\quad \sup_N\mathbb{E}\big[\big|\hat E_i^N(0)\big|^2\big]\leq C,\quad \sup_N\mathbb{E}\big[\big|\hat I_i^N(0)\big|^2\big]\leq C,\quad \sup_N\mathbb{E}\big[\big|\hat R_i^N(0)\big|^2\big]\leq C,\\
&\sup_N\sup_{t\in[0,T]}\mathbb{E}\big[(\hat M_{A,i}^N(t))^2\big]\leq\lambda_i\int_0^T\sup_N\bar\Upsilon_i^N(s)ds\leq\lambda_i\bar\kappa_iT,\quad \sup_N\sup_{t\in[0,T]}\mathbb{E}\big[(\hat M_{S,i,\ell}^N(t))^2\big]\leq C\nu_{S,i,\ell}T,\\
&\sup_N\sup_{t\in[0,T]}\mathbb{E}\big[(\hat M_{E,i,\ell}^N(t))^2\big]\leq C\nu_{E,i,\ell}T,\quad \sup_N\sup_{t\in[0,T]}\mathbb{E}\big[(\hat M_{I,i,\ell}^N(t))^2\big]\leq C\nu_{E,i,\ell}T,\\
&\sup_N\sup_{t\in[0,T]}\mathbb{E}\big[(\hat M_{R,i,\ell}^N(t))^2\big]\leq C\nu_{R,i,\ell}T.
\end{aligned}\tag{4.21}$$

Thus, by taking squares of the processes $\hat{S}_i^N(t)$, $\hat{E}_i^N(t)$, $\hat{I}_i^N(t)$ and $\hat{R}_i^N(t)$ in (4.4), (4.5), (4.8) and (4.13), we can apply Cauchy-Schwartz inequality and Gronwall's inequality to conclude that

$$\sup_N \sup_{t\in[0,T]} \mathbb{E}\left[\left|\hat{S}_i^N(t)\right|^2\right] < \infty, \quad \sup_N \sup_{t\in[0,T]} \mathbb{E}\left[\left|\hat{E}_i^N(t)\right|^2\right] < \infty,$$
$$\sup_N \sup_{t\in[0,T]} \mathbb{E}\left[\left|\hat{I}_i^N(t)\right|^2\right] < \infty, \quad \sup_N \sup_{t\in[0,T]} \mathbb{E}\left[\left|\hat{R}_i^N(t)\right|^2\right] < \infty, \tag{4.22}$$

and thus (4.20) holds.

We next prove (4.19). By (4.21) and Doob's inequality, we obtain the martingale terms satisfy

$$\sup_N \mathbb{E}\left[\sup_{t\in[0,T]} (\hat{M}_{A,i}^N(t))^2\right] < \infty, \quad \sup_N \mathbb{E}\left[\sup_{t\in[0,T]} (\hat{M}_{S,i,\ell}^N(t))^2\right] < \infty, \quad \sup_N \mathbb{E}\left[\sup_{t\in[0,T]} (\hat{M}_{E,i,\ell}^N(t))^2\right] < \infty,$$
$$\sup_N \mathbb{E}\left[\sup_{t\in[0,T]} (\hat{M}_{I,i,\ell}^N(t))^2\right] < \infty, \quad \sup_N \mathbb{E}\left[\sup_{t\in[0,T]} (\hat{M}_{R,i,\ell}^N(t))^2\right] < \infty. \tag{4.23}$$

Then, by the expression of $\hat{S}_i^N(t)$ in (4.4), the bounds in (4.18) and (4.22), we easily obtain that the property in (4.19) holds for $\hat{S}_i^N(t)$.

For $\hat{E}_i^N(t)$, given the bounds in (4.18), (4.22) and (4.23), it suffices to show that the property in (4.19) holds for $\hat{E}_{\ell,i}^{N,0}(t)$ and $\hat{E}_{\ell,i}^N(t)$. Similarly, for $\hat{I}_i^N(t)$, it suffices to show that the property in (4.19) holds for $\hat{I}_{\ell,i}^{N,0,1}(t)$, $\hat{I}_{\ell,i}^{N,0,2}(t)$ and $\hat{I}_{\ell,i}^N(t)$. We will first treat the processes associated with the initial quantities $\hat{E}_{\ell,i}^{N,0}(t)$, $\hat{I}_{\ell,i}^{N,0,1}(t)$ and $\hat{I}_{\ell,i}^{N,0,2}(t)$. The processes $\hat{E}_{\ell,i}^{N,0}(t)$, $\hat{I}_{\ell,i}^{N,0,1}(t)$ can be treated in the same way, so we will only prove $\hat{I}_{\ell,i}^{N,0,1}(t)$.

Recall the expression of $\hat{I}_{\ell,i}^{N,0,1}(t)$ in (4.9). Also recall the process $\tilde{I}_{\ell,i}^{N,0,1}(t)$ by defined in the proof of Lemma 4.2, that is, replacing $I_\ell^N(0)$ in (4.9) with $N\bar{I}_\ell(0)$, so that $\hat{I}_{\ell,i}^{N,0,1}(t) = \tilde{I}_{\ell,i}^{N,0,1}(t) + (\hat{I}_{\ell,i}^{N,0,1}(t) - \tilde{I}_{\ell,i}^{N,0,1}(t))$. The process $\hat{I}_{\ell,i}^{N,0,1}(t)$ is driven by the sequence of two dimensional r.v.'s $(\zeta_{k,\ell}^0, Y_\ell^{0,k}(\zeta_{k,\ell}^0))$, $k \geq 1$. We add a sequence of i.i.d. r.v.'s, globally independent of the above sequence, U_k, $k \geq 1$, which all have the uniform distribution on the interval $[T, T+1]$. We define a sequence of r.v.'s $\tilde{\zeta}_{k,\ell}^0$, $k \geq 1$, as follows

$$\tilde{\zeta}_{k,\ell}^0 = \begin{cases} \zeta_{k,\ell}^0, & \text{if} \quad Y_\ell^{0,k}(\zeta_{k,\ell}^0) = i, \\ \zeta_{k,\ell}^0 + U_k, & \text{if} \quad Y_\ell^{0,k}(\zeta_{k,\ell}^0) \neq i\,. \end{cases}$$

We have that for all $t \in [0,T]$,

$$\mathbf{1}_{\zeta_{k,\ell}^0 \leq t} \mathbf{1}_{Y_\ell^{0,k}(\zeta_{k,\ell}^0)=i} = \mathbf{1}_{\tilde{\zeta}_{k,\ell}^0 \leq t}\,,$$

and

$$\mathbb{E}\big[\mathbf{1}_{\tilde{\zeta}_{k,\ell}^0 \leq t}\big] = \int_0^t q_{\ell,i}(s) F_0(ds).$$

By writing

$$\frac{1}{\sqrt{\bar{I}_\ell(0)}} \tilde{I}_{\ell,i}^{N,0,1}(t) = \frac{1}{\sqrt{N\bar{I}_\ell(0)}} \sum_{k=1}^{N\bar{I}_\ell(0)} \big(\mathbf{1}_{\tilde{\zeta}_{k,\ell}^0 \leq t} - \mathbb{E}\big[\mathbf{1}_{\tilde{\zeta}_{k,\ell}^0 \leq t}\big]\big),$$

we apply the Dvoretsky–Kiefer–Wolfowitz inequality (with Massart's optimal constant [25]) and obtain

$$\frac{1}{\bar{I}_\ell(0)} \mathbb{E}\left[\sup_{0\leq t\leq T} \big(\tilde{I}_{\ell,i}^{N,0,1}(t)\big)^2\right] < 2\int_0^\infty e^{-2x} dx = 1.$$

On the other hand,

$$\hat{I}^{N,0,1}_{\ell,i}(t) - \tilde{I}^{N,0,1}_{\ell,i}(t) = \frac{1}{\sqrt{N}} \sum_{k=N\bar{I}_\ell(0)\wedge I^N_\ell(0)+1}^{N\bar{I}_\ell(0)\vee I^N_\ell(0)} \big(\mathbf{1}_{\tilde{\zeta}^0_{k,\ell}\le t} - \mathbb{E}\big[\mathbf{1}_{\tilde{\zeta}^0_{k,\ell}\le t}\big]\big),$$

$$\sup_{t\ge 0} |\hat{I}^{N,0,1}_{\ell,i}(t) - \tilde{I}^{N,0,1}_{\ell,i}(t)| \le \sqrt{N}|\bar{I}_\ell(0) - \bar{I}^N_\ell(0)| = |\hat{I}^N_\ell(0)|,$$

hence from Assumption 2.2,

$$\sup_N \mathbb{E}\Big[\sup_{t\ge 0} \big(\hat{I}^{N,0,1}_{\ell,i}(t) - \tilde{I}^{N,0,1}_{\ell,i}(t)\big)^2\Big] < \infty.$$

Combining the above, we have shown that the property in (4.19) holds for $\hat{I}^{N,0,1}_{\ell,i}(t)$.

For the process $\hat{I}^{N,0,2}_{\ell,i}(t)$, we can extend the approach above as follows. Define $\tilde{I}^{N,0,2}_{\ell,i}(t)$ by replacing $E^N_\ell(0)$ with $N\bar{E}_\ell(0)$ in the definition of $\hat{I}^{N,0,2}_{\ell,i}(t)$ in (4.10). Write $\hat{I}^{N,0,2}_{\ell,i}(t) = \tilde{I}^{N,0,2}_{\ell,i}(t) + \hat{I}^{N,0,2}_{\ell,i}(t) - \tilde{I}^{N,0,2}_{\ell,i}(t)$. For fixed ℓ, ℓ', we have a sequence of i.i.d. random vectors $\big(\eta^0_{k,\ell}, X^{0,k}_\ell(\eta^0_{k,\ell}), \zeta_{-k,\ell'}, Y^{-k}_{\ell'}(\zeta_{-k,\ell'})\big)_{k\ge 1}$. We also add a sequence of i.i.d. r.v.'s globally independent of the previous sequence, U_k, $k \ge 1$, uniformly distributed on $[T, T+1]$. Define

$$\varsigma^0_{k,\ell,\ell'} = \begin{cases} \eta^0_{k,\ell} + \zeta_{-k,\ell'}, & \text{if} \quad X^{0,k}_\ell(\eta^0_{k,\ell}) = \ell', \quad \text{and} \quad Y^{-k}_{\ell'}(\zeta_{-k,\ell'}) = i\,, \\ \eta^0_{k,\ell} + \zeta_{-k,\ell'} + U_k, & \text{if} \quad X^{0,k}_\ell(\eta^0_{k,\ell}) \ne \ell', \quad \text{or} \quad Y^{-k}_{\ell'}(\zeta_{-k,\ell'}) \ne i\,. \end{cases}$$

Then we have

$$\mathbf{1}_{X^{0,k}_\ell(\eta^0_{k,\ell})=\ell'} \mathbf{1}_{\eta^0_{k,\ell}+\zeta_{-k,\ell'}\le t} \mathbf{1}_{Y^{-k}_{\ell'}(\zeta_{-k,\ell'})=i} = \mathbf{1}_{\varsigma^0_{k,\ell,\ell'}\le t}\,,$$

and

$$\mathbb{E}\Big[\mathbf{1}_{\varsigma^0_{k,\ell,\ell'}\le t}\Big] = \int_0^t p_{\ell,\ell'}(u) \int_0^{t-u} q_{\ell' i}(v) H_0(du, dv)\,.$$

Note that $\Phi^0_{\ell,i}(t)$ is the sum of the right hand side of the above equation over $\ell' \in \mathcal{L}$, as given in (2.19). Then we can write

$$\tilde{I}^{N,0,2}_{\ell,i}(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N\bar{E}_\ell(0)} \sum_{\ell'=1}^{L} \Big(\mathbf{1}_{\varsigma^0_{k,\ell,\ell'}\le t} - \mathbb{E}\Big[\mathbf{1}_{\varsigma^0_{k,\ell,\ell'}\le t}\Big]\Big) = \sum_{\ell'=1}^{L} \tilde{I}^{N,0,2}_{\ell,\ell',i}(t)\,,$$

where

$$\tilde{I}^{N,0,2}_{\ell,\ell',i}(t) := \frac{1}{\sqrt{N}} \sum_{k=1}^{N\bar{E}_\ell(0)} \Big(\mathbf{1}_{\varsigma^0_{k,\ell,\ell'}\le t} - \mathbb{E}\Big[\mathbf{1}_{\varsigma^0_{k,\ell,\ell'}\le t}\Big]\Big)\,.$$

We can apply the Dvoretsky–Kiefer–Wolfowitz inequality to obtain the desired estimate for $\tilde{I}^{N,0,2}_{\ell,\ell',i}(t)$ for each fixed ℓ, ℓ', that is,

$$\mathbb{E}\Big[\sup_{0\le t\le T} \big(\tilde{I}^{N,0,2}_{\ell,\ell',i}(t)\big)^2\Big] < \bar{E}_\ell(0).$$

Hence,

$$\mathbb{E}\Big[\sup_{0\le t\le T} \big(\tilde{I}^{N,0,2}_{\ell,i}(t)\big)^2\Big] < 2L\bar{E}_\ell(0).$$

The difference $\hat{I}^{N,0,2}_{\ell,i}(t) - \tilde{I}^{N,0,2}_{\ell,i}(t)$ is again easy to treat. So we obtain the estimate for $\hat{I}^{N,0,2}_{\ell,i}(t)$.

We next consider the processes associated with the newly infected individuals $\hat{E}^N_{\ell,i}(t)$ and $\hat{I}^N_{\ell,i}(t)$. Recall the expression of $\hat{E}^N_{\ell,i}(t)$ and $\hat{I}^N_{\ell,i}(t)$ in (4.7) and (4.11), respectively. Recall the expressions in

(2.11) using the PRM $\check{Q}_\ell(ds,du,dy,d\theta)$ and (2.12) using the PRM $\breve{Q}_\ell(ds,du,dy,dz,d\vartheta,d\theta)$. Also recall that $\overline{Q}_\ell$ and $\widetilde{Q}_\ell$ are the corresponding compensated PRMs. Thus we can write

$$\hat{E}^N_{\ell,i}(t)=\frac{1}{\sqrt{N}}\int_0^t\int_0^\infty\int_0^{t-s}\int_{\{i\}}\mathbf{1}_{\mathfrak{a}\le\lambda_\ell\Upsilon^N_\ell(s^-)}\overline{Q}_\ell(ds,d\mathfrak{a},du,d\theta), \tag{4.24}$$

and

$$\hat{I}^N_{\ell,i}(t)=\frac{1}{\sqrt{N}}\int_0^t\int_0^\infty\int_0^{t-s}\int_0^{t-s-u}\int_{\mathcal{L}}\int_{\{i\}}\mathbf{1}_{\mathfrak{a}\le\lambda_\ell\Upsilon^N_\ell(s^-)}\widetilde{Q}_\ell(ds,d\mathfrak{a},du,dv,d\theta,d\vartheta). \tag{4.25}$$

Define

$$\tilde{E}^N_{\ell,i}(t)=\frac{1}{\sqrt{N}}\int_0^t\int_0^\infty\int_0^{t-s}\int_{\{i\}}\mathbf{1}_{\mathfrak{a}\le\lambda_\ell N\Upsilon_\ell(s)}\overline{Q}_\ell(ds,d\mathfrak{a},du,d\theta), \tag{4.26}$$

and

$$\tilde{I}^N_{\ell,i}(t)=\frac{1}{\sqrt{N}}\int_0^t\int_0^\infty\int_0^{t-s}\int_0^{t-s-u}\int_{\mathcal{L}}\int_{\{i\}}\mathbf{1}_{\mathfrak{a}\le\lambda_\ell N\Upsilon_\ell(s)}\widetilde{Q}_\ell(ds,d\mathfrak{a},du,dv,d\theta,d\vartheta), \tag{4.27}$$

where $\bar{\Upsilon}_\ell(t)$ is given in (2.34) and is deterministic. It is not hard to check that $\tilde{E}^N_{\ell,i}(t)$ (resp. $\tilde{I}^N_{\ell,i}(t)$) is a martingale w.r.t. the filtration $\mathcal{F}^E_t$ (resp. $\mathcal{F}^I_t$), where $\mathcal{F}^E_t$ is generated by the restriction of $\check{Q}$ to the set of $(s,\mathfrak{a},u,\theta)$ which are such that $s+u\le t$, and $\mathcal{F}^I_t$ is generated by the restriction of $\breve{Q}$ to the set of $(s,\mathfrak{a},u,v,\theta,\vartheta)$ which are such that $s+u+v\le t$. Those martingales have the quadratic variations

$$\langle\tilde{E}^N_{\ell,i}\rangle(t)=\lambda_\ell\int_0^t\int_0^{t-s}q_{\ell,i}(u)F(du)\bar{\Upsilon}_\ell(s)ds,\quad \langle\tilde{I}^N_{\ell,i}\rangle(t)=\lambda_\ell\int_0^t\Phi_{\ell,i}(t-s)\bar{\Upsilon}_\ell(s)ds. \tag{4.28}$$

Then by Doob's inequality, we obtain

$$\sup_N\mathbb{E}\Big[\sup_{0\le t\le T}\big(\tilde{E}^N_{\ell,i}(t)\big)^2\Big]\le\sup_N 4\,\mathbb{E}\big[\big(\tilde{E}^N_{\ell,i}(T)\big)^2\big]=4\lambda_\ell\int_0^T\int_0^{T-s}q_{\ell,i}(u)F(du)\bar{\Upsilon}_\ell(s)ds<\infty,$$
$$\sup_N\mathbb{E}\Big[\sup_{0\le t\le T}\big(\tilde{I}^N_{\ell,i}(t)\big)^2\Big]\le\sup_N 4\,\mathbb{E}\big[\big(\tilde{I}^N_{\ell,i}(T)\big)^2\big]=4\lambda_\ell\int_0^T\Phi_{\ell,i}(T-s)\bar{\Upsilon}_\ell(s)ds<\infty.$$

We next show that

$$\sup_N\mathbb{E}\left[\sup_{0\le t\le T}\left|\hat{E}^N_{\ell,i}(t)-\tilde{E}^N_{\ell,i}(t)\right|^2\right]<\infty,\quad \sup_N\mathbb{E}\left[\sup_{0\le t\le T}\left|\hat{I}^N_{\ell,i}(t)-\tilde{I}^N_{\ell,i}(t)\right|^2\right]<\infty\,. \tag{4.29}$$

Let us establish the first estimate in (4.29). The second can be obtained by the exact same argument. We will use below the identity

$$\frac{1}{\sqrt{N}}\int_0^\infty\left|\mathbf{1}_{\mathfrak{a}\le\lambda_\ell\Upsilon^N_\ell(s^-)}-\mathbf{1}_{\mathfrak{a}\le\lambda_\ell N\Upsilon_\ell(s)}\right|d\mathfrak{a}=\lambda_\ell\left|\hat{\Upsilon}^N_\ell(s^-)\right|.$$

We have

$$\hat{E}^N_{\ell,i}(t)-\tilde{E}^N_{\ell,i}(t)=\frac{1}{\sqrt{N}}\int_0^t\int_0^\infty\int_0^{t-s}\int_{\{i\}}\left(\mathbf{1}_{\mathfrak{a}\le\lambda_\ell\Upsilon^N_\ell(s^-)}-\mathbf{1}_{\mathfrak{a}\le\lambda_\ell N\Upsilon_\ell(s)}\right)\overline{Q}_\ell(ds,d\mathfrak{a},du,d\theta),$$

and

$$\begin{aligned}&\sup_{0\le t\le T}\left|\hat{E}^N_{\ell,i}(t)-\tilde{E}^N_{\ell,i}(t)\right|\\&\le\frac{1}{\sqrt{N}}\int_0^T\int_0^\infty\int_0^{T-s}\int_{\{i\}}\left|\mathbf{1}_{\mathfrak{a}\le\lambda_\ell\Upsilon^N_\ell(s^-)}-\mathbf{1}_{\mathfrak{a}\le\lambda_\ell N\Upsilon_\ell(s)}\right|\check{Q}_\ell(ds,d\mathfrak{a},du,d\theta)+\int_0^T\lambda_\ell\left|\hat{\Upsilon}^N_\ell(t)\right|dt\end{aligned}$$

$$= \frac{1}{\sqrt{N}} \int_0^T \int_0^\infty \int_0^{T-s} \int_{\{i\}} \left| \mathbf{1}_{\mathfrak{a} \le \lambda_\ell \Upsilon_\ell^N(s^-)} - \mathbf{1}_{\mathfrak{a} \le \lambda_\ell N \Upsilon_\ell(s)} \right| \overline{Q}_\ell(ds, d\mathfrak{a}, du, d\theta) + 2 \int_0^T \lambda_\ell \left| \hat{\Upsilon}_\ell^N(t) \right| dt \,.$$

Thus we obtain

$$\mathbb{E}\left[\sup_{0 \le t \le T} \left| \hat{E}_{\ell,i}^N(t) - \tilde{E}_{\ell,i}^N(t) \right|^2 \right] \le 2\lambda_\ell \int_0^T \mathbb{E}\left| \bar{\Upsilon}_\ell^N(t) - \Upsilon_\ell(t) \right| dt + 4\lambda_\ell^2 T \int_0^T \mathbb{E}\left[\left| \hat{\Upsilon}_\ell^N(t) \right|^2 \right] dt,$$

hence the first part of (4.29), thanks to (4.20). Plugging the above estimates in (4.4), (4.5), (4.8) and (4.13), using (4.18) and Gronwall's Lemma we finally establish (4.19). □

In the next Lemma, we shall need the following well–known result on integrals with respect to a PRM, which follows rather easily from Theorem VI.2.9 in [9].

Lemma 4.5. *Let Q be a PRM on some measurable space $(E, \mathcal{E})$, with mean measure ν, and $\widetilde{Q}$ the associated compensated measure. Let $f : E \mapsto \mathbb{C}$ be measurable and such that $e^f - 1 - f$ is ν integrable. Then*

$$\mathbb{E}\left[\exp\left(\int_E f(x) \widetilde{Q}(dx) \right) \right] = \exp\left(\int_E \left[e^{f(x)} - 1 - f(x) \right] \nu(dx) \right) .$$

We are now ready to prove the convergence of the components associated with the newly exposed individuals.

Lemma 4.6. *Under Assumption 2.2,*

$$\left(\hat{E}_{\ell,i}^N, \hat{I}_{\ell,i}^N,\ \ell, i \in \mathcal{L} \right) \Rightarrow \left(\hat{E}_{\ell,i}, \hat{I}_{\ell,i},\ \ell, i \in \mathcal{L} \right) \quad \text{in} \quad D^{2L^2} \quad \text{as} \quad N \to \infty, \tag{4.30}$$

where the limits are as given in Theorem 2.2.

Proof. Recall the processes $\tilde{E}_{\ell,i}^N$ and $\tilde{I}_{\ell,i}^N$ defined in (4.26) and (4.27) using the compensated PRMs $\overline{Q}_\ell$ and $\widetilde{Q}_\ell$, respectively. Each of these processes being a martingale, they are easily shown to be tight. We now establish their joint final dimensional convergence. By their definitions of the two PRMs, we can regard $\breve{Q}_\ell$ (resp. $\overline{Q}_\ell$) as the image of $\breve{Q}_\ell$ (resp. $\widetilde{Q}_\ell$) by the projection Π from $\mathbb{R}_+^4 \times \mathcal{L}^2$ onto $\mathbb{R}_+^3 \times \mathcal{L}$, defined by $\Pi(s, \mathfrak{a}, u, v, \theta, \vartheta) = (s, \mathfrak{a}, u, \theta)$. In other words, we can write, together with (4.27),

$$\tilde{E}_{\ell,i}^N(t) = \frac{1}{\sqrt{N}} \int_0^t \int_0^\infty \int_0^{t-s} \int_0^\infty \int_{\{i\}} \int_{\mathcal{L}} \mathbf{1}_{\mathfrak{a} \le \lambda_\ell N \Upsilon_\ell(s)} \overline{Q}_\ell(ds, d\mathfrak{a}, du, dv, d\theta, d\vartheta).$$

Consequently, for any $\alpha_E, \alpha_I, \alpha'_E, \alpha'_I \in \mathbb{R}$, and for each $\ell, i, i' \in \mathcal{L}$ and $t, t' > 0$,

$$\begin{aligned} &\alpha_E \tilde{E}_{\ell,i}^N(t) + \alpha_I \tilde{I}_{\ell,i'}^N(t) + \alpha'_E \tilde{E}_{\ell,i}^N(t') + \alpha'_I \tilde{I}_{\ell,i'}^N(t') \\ &= \int_0^t \int_0^\infty \int_0^{t-s} \int_0^\infty \int_{\mathcal{L}} \int_{\mathcal{L}} \left[f_N(s, \mathfrak{a}, u, v, \theta, \vartheta) + f'_N(s, \mathfrak{a}, u, v, \theta, \vartheta) \right] \overline{Q}_\ell(ds, d\mathfrak{a}, du, dv, d\theta, d\vartheta), \end{aligned}$$

where

$$\begin{aligned} f_N(s, \mathfrak{a}, u, v, \theta, \vartheta) &= \frac{1}{\sqrt{N}} \mathbf{1}_{\mathfrak{a} \le \lambda_\ell N \Upsilon_\ell(s)} \mathbf{1}_{[0,t]}(s) \mathbf{1}_{[0,t-s]}(u) \left(\alpha_E \delta_{\{i\}}(d\theta) + \alpha_I \mathbf{1}_{[0,t-s-u]}(v) \delta_{\{i'\}}(d\vartheta) \right), \\ f'_N(s, \mathfrak{a}, u, v, \theta, \vartheta) &= \frac{1}{\sqrt{N}} \mathbf{1}_{\mathfrak{a} \le \lambda_\ell N \Upsilon_\ell(s)} \mathbf{1}_{[0,t']}(s) \mathbf{1}_{[0,t'-s]}(u) \left(\alpha'_E \delta_{\{i\}}(d\theta) + \alpha'_I \mathbf{1}_{[0,t'-s-u]}(v) \delta_{\{i'\}}(d\vartheta) \right). \end{aligned}$$

By Lemma 4.5, we have

$$\begin{aligned} &\mathbb{E}\left[\exp\left\{ \hat{\imath}\alpha_E \tilde{E}_{\ell,i}^N(t) + \hat{\imath}\alpha_I \tilde{I}_{\ell,i'}^N(t) + \hat{\imath}\alpha'_E \tilde{E}_{\ell,i}^N(t') + \hat{\imath}\alpha'_I \tilde{I}_{\ell,i'}^N(t') \right\} \right] \\ &= \exp\left(\int_0^t \int_0^\infty \int_0^{t-s} \int_0^\infty \int_{\mathcal{L}} \int_{\mathcal{L}} \left(e^{\hat{\imath}[f_N + f'_N]} - 1 - \hat{\imath}[f_N + f'_N] \right) (s, \mathfrak{a}, u, v, \theta, \vartheta) \right. \end{aligned}$$

$$
\begin{aligned}
&\qquad ds d\mathfrak{a} H(du,dv)\mu_\ell^X(u,d\theta)\mu_\theta^Y(v,d\vartheta)\Big)\\
&\longrightarrow \exp\Bigg(-\frac{1}{2}\int_0^t \lambda_\ell \Upsilon_\ell(s)\Big\{\alpha_E^2\int_0^{t-s} p_{\ell,i}(u)G(du)+\alpha_I^2\Phi_{\ell,i'}(t-s)\\
&\qquad\qquad +2\alpha_E\alpha_I\int_0^{t-s}\int_0^{t-s-u} p_{\ell,i}(u)q_{i,i'}(v)H(du,dv)\Big\}ds\\
&\quad -\frac{1}{2}\int_0^{t'} \lambda_\ell \Upsilon_\ell(s)\Big\{(\alpha'_E)^2\int_0^{t'-s} p_{\ell,i}(u)G(du)+(\alpha'_I)^2\Phi_{\ell,i'}(t'-s)\\
&\qquad\qquad +2\alpha'_E\alpha'_I\int_0^{t'-s}\int_0^{t'-s-u} p_{\ell,i}(u)q_{i,i'}(v)H(du,dv)\Big\}ds\\
&\quad -\int_0^{t\wedge t'} \lambda_\ell \Upsilon_\ell(s)\Big\{\alpha_E\alpha'_E\int_0^{t\wedge t'-s} p_{\ell,i}(u)G(du)+\alpha_I\alpha'_I\Phi_{\ell,i'}(t\wedge t'-s)\\
&\qquad\qquad +\alpha_E\alpha'_I\int_0^{t\wedge t'-s}\int_0^{t'-s-u} p_{\ell,i}(u)q_{i,i'}(v)H(du,dv)\\
&\qquad\qquad +\alpha_I\alpha'_E\int_0^{t\wedge t'-s}\int_0^{t-s-u} p_{\ell,i}(u)q_{i,i'}(v)H(du,dv)\Big\}ds\Bigg)\\
&=\mathbb{E}\left[\exp\left\{\hat{\imath}\alpha_E\hat{E}_{\ell,i}(t)+\hat{\imath}\alpha_I\hat{I}_{\ell,i'}(t)+\hat{\imath}\alpha'_E\hat{E}_{\ell,i}(t')+\hat{\imath}\alpha'_I\hat{I}_{\ell,i'}(t')\right\}\right].
\end{aligned}
$$

Would we consider more distinct times, we would clearly deduce that the whole vector converges to a Gaussian random vector. The only point which requests a detailed computation is the determination of the covariances, which can de deduced from the above formula, and obvious similar formulas. In particular, it is easily seen for $\ell \neq \ell'$, that the covariance of $\hat{E}_{\ell,i}(t)$ and $(\hat{E}_{\ell',i'}(t'),\hat{I}_{\ell',i''}(t''))$ is zero. Similarly, for $i' \neq i$, the covariances of $\hat{E}_{\ell,i}(t)$ and $\hat{E}_{\ell,i'}(t')$, of $\hat{I}_{\ell,i}(t)$ and $\hat{I}_{\ell,i'}(t')$ are zero. The formulas for the covariances of the pair $(\hat{E},\hat{I})$ in the statement of Theorem 2.2 are easy to deduce from the above computation.

It then remains to show that, for each $\ell, i \in \mathcal{L}$, as $N\to\infty$, $\hat{E}^N_{\ell,i}(t)-\tilde{E}^N_{\ell,i}(t)\to 0$ and $\hat{I}^N_{\ell,i}(t)-\tilde{I}^N_{\ell,i}(t)\to 0$ in probability, locally uniformly in t.

We focus on the process $\hat{E}^N_{\ell,i}-\tilde{E}^N_{\ell,i}$. It is clear that

$$
\begin{aligned}
&\mathbb{E}\big[\hat{E}^N_{\ell,i}(t)-\tilde{E}^N_{\ell,i}(t)\big]=0,\\
&\mathbb{E}\big[\big(\hat{E}^N_{\ell,i}(t)-\tilde{E}^N_{\ell,i}(t)\big)^2\big]=\lambda_\ell\mathbb{E}\int_0^t\int_0^{t-s} q_{\ell,i}(u)F(du)\big|\bar{\Upsilon}^N_\ell(s)-\bar{\Upsilon}_\ell(s)\big|ds\to 0 \quad \text{as} \quad N\to\infty,
\end{aligned}
$$

where the convergence holds by Theorem 2.1 and the dominated convergence theorem. We next show that the sequence $\{\hat{E}^N_{\ell,i}-\tilde{E}^N_{\ell,i}\}_N$ is tight. Observe that

$$
\begin{aligned}
\hat{E}^N_{\ell,i}(t)-\tilde{E}^N_{\ell,i}(t)=\frac{1}{\sqrt{N}}\int_0^t\int_{\lambda_\ell N(\bar{\Upsilon}^N_\ell(s)\wedge\bar{\Upsilon}_\ell(s))}^{\lambda_\ell N(\bar{\Upsilon}^N_\ell(s)\vee\bar{\Upsilon}_\ell(s))}\int_0^{t-s}\int_{\{i\}}\mathrm{sign}(\bar{\Upsilon}^N_\ell(s)-\bar{\Upsilon}_\ell(s))\check{Q}_\ell(ds,d\mathfrak{a},du,d\theta)\\
-\lambda_\ell\int_0^t\int_0^{t-s} q_{\ell,i}(u)F(du)\hat{\Upsilon}^N_\ell(s)ds.
\end{aligned}
$$

We can decompose $\mathrm{sign}(\bar{\Upsilon}^N_\ell(s)-\bar{\Upsilon}_\ell(s))=\mathbf{1}_{\bar{\Upsilon}^N_\ell(s)-\bar{\Upsilon}_\ell(s)>0}-\mathbf{1}_{\bar{\Upsilon}^N_\ell(s)-\bar{\Upsilon}_\ell(s)<0}$, and write $\hat{\Upsilon}^N_\ell(s)=\hat{\Upsilon}^N_\ell(s)^+-\hat{\Upsilon}^N_\ell(s)^-$, such that each of these will induce a process that is nondecreasing in t. It is

also clear that tightness of these processes will be implied by the tightness of the following processes:

$$\Xi_1^N(t) = \frac{1}{\sqrt{N}} \int_0^t \int_{\lambda_\ell N(\bar{\Upsilon}_\ell^N(s)\wedge\bar{\Upsilon}_\ell(s))}^{\lambda_\ell N(\bar{\Upsilon}_\ell^N(s)\vee\bar{\Upsilon}_\ell(s))} \int_0^{t-s} \int_{\{i\}} \check{Q}_\ell(ds, d\mathfrak{a}, du, d\theta),$$
$$\Xi_2^N(t) = \lambda_\ell \int_0^t \int_0^{t-s} q_{\ell,i}(u)F(du)|\hat{\Upsilon}_\ell^N(s)|ds.$$

Since these two processes are nondecreasing in t, it suffices to show that for any $\epsilon > 0$, and $\iota = 1, 2$,

$$\limsup_{N\to\infty} \frac{1}{\delta}\mathbb{P}\left(\left|\Xi_\iota^N(t+\delta) - \Xi_\iota^N(t)\right| > \epsilon\right) \to 0 \quad \text{as} \quad \delta \to 0. \tag{4.31}$$

For the process $\Xi_1^N(t)$, we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\Xi_1^N(t+\delta) - \Xi_1^N(t)\right|^2\right] \\
&= \mathbb{E}\Bigg[\Bigg(\frac{1}{\sqrt{N}} \int_t^{t+\delta} \int_{\lambda_\ell N(\bar{\Upsilon}_\ell^N(s)\wedge\bar{\Upsilon}_\ell(s))}^{\lambda_\ell N(\bar{\Upsilon}_\ell^N(s)\vee\bar{\Upsilon}_\ell(s))} \int_0^{t+\delta-s} \int_{\{i\}} \check{Q}_\ell(ds, d\mathfrak{a}, du, d\theta) \\
&\quad + \frac{1}{\sqrt{N}} \int_0^t \int_{\lambda_\ell N(\bar{\Upsilon}_\ell^N(s)\wedge\bar{\Upsilon}_\ell(s))}^{\lambda_\ell N(\bar{\Upsilon}_\ell^N(s)\vee\bar{\Upsilon}_\ell(s))} \int_{t-s}^{t+\delta-s} \int_{\{i\}} \check{Q}_\ell(ds, d\mathfrak{a}, du, d\theta)\Bigg)^2\Bigg] \\
&\le 2\mathbb{E}\Bigg[\Bigg(\frac{1}{\sqrt{N}} \int_t^{t+\delta} \int_{\lambda_\ell N(\bar{\Upsilon}_\ell^N(s)\wedge\bar{\Upsilon}_\ell(s))}^{\lambda_\ell N(\bar{\Upsilon}_\ell^N(s)\vee\bar{\Upsilon}_\ell(s))} \int_0^{t+\delta-s} \int_{\{i\}} \check{Q}_\ell(ds, d\mathfrak{a}, du, d\theta)\Bigg)^2\Bigg] \\
&\quad + 2\mathbb{E}\Bigg[\Bigg(\frac{1}{\sqrt{N}} \int_0^t \int_{\lambda_\ell N(\bar{\Upsilon}_\ell^N(s)\wedge\bar{\Upsilon}_\ell(s))}^{\lambda_\ell N(\bar{\Upsilon}_\ell^N(s)\vee\bar{\Upsilon}_\ell(s))} \int_{t-s}^{t+\delta-s} \int_{\{i\}} \check{Q}_\ell(ds, d\mathfrak{a}, du, d\theta)\Bigg)^2\Bigg] \\
&\le 4\mathbb{E}\Bigg[\Bigg(\frac{1}{\sqrt{N}} \int_t^{t+\delta} \int_{\lambda_\ell N(\bar{\Upsilon}_\ell^N(s)\wedge\bar{\Upsilon}_\ell(s))}^{\lambda_\ell N(\bar{\Upsilon}_\ell^N(s)\vee\bar{\Upsilon}_\ell(s))} \int_0^{t+\delta-s} \int_{\{i\}} \overline{Q}_\ell(ds, d\mathfrak{a}, du, d\theta)\Bigg)^2\Bigg] \\
&\quad + 4\mathbb{E}\left[\left(\lambda_\ell \int_t^{t+\delta} \int_0^{t+\delta-s} q_{\ell,i}(u)F(du)\left|\hat{\Upsilon}_\ell^N(s)\right|ds\right)^2\right] \\
&\quad + 4\mathbb{E}\Bigg[\Bigg(\frac{1}{\sqrt{N}} \int_0^t \int_{\lambda_\ell N(\bar{\Upsilon}_\ell^N(s)\wedge\bar{\Upsilon}_\ell(s))}^{\lambda_\ell N(\bar{\Upsilon}_\ell^N(s)\vee\bar{\Upsilon}_\ell(s))} \int_{t-s}^{t+\delta-s} \int_{\{i\}} \overline{Q}_\ell(ds, d\mathfrak{a}, du, d\theta)\Bigg)^2\Bigg] \\
&\quad + 4\mathbb{E}\left[\left(\lambda_\ell \int_0^t \int_{t-s}^{t+\delta-s} q_{\ell,i}(u)F(du)\left|\hat{\Upsilon}_\ell^N(s)\right|ds\right)^2\right] \\
&\le 4\lambda_\ell \int_t^{t+\delta} \int_0^{t+\delta-s} q_{\ell,i}(u)F(du)\mathbb{E}\left[\left|\bar{\Upsilon}_\ell^N(s) - \bar{\Upsilon}_\ell(s)\right|\right] ds + 4\lambda_\ell^2\delta^2 \sup_{s\in[0,T]} \mathbb{E}\left[\left|\hat{\Upsilon}_\ell^N(s)\right|^2\right] \\
&\quad + 4\lambda_\ell \int_0^t \int_{t-s}^{t+\delta-s} q_{\ell,i}(u)F(du)\mathbb{E}\left[\left|\bar{\Upsilon}_\ell^N(s) - \bar{\Upsilon}_\ell(s)\right|\right] ds \\
&\quad + 4\mathbb{E}\left[\left(\lambda_\ell \int_0^t \int_{t-s}^{t+\delta-s} q_{\ell,i}(u)F(du)\left|\hat{\Upsilon}_\ell^N(s)\right|ds\right)^2\right].
\end{aligned} \tag{4.32}$$

It is clear that $\mathbb{E}\left[\left|\bar{\Upsilon}_\ell^N(s) - \bar{\Upsilon}_\ell(s)\right|\right] \to 0$ as $N \to \infty$ by the convergence $\bar{\Upsilon}_\ell^N \Rightarrow \bar{\Upsilon}_\ell$ and the dominated convergence theorem. Thus, the first and third terms converge to zero as $N \to \infty$. Thanks to (4.20), δ^{-1} times the second term converges to zero as $\delta \to 0$, which is exactly what we

wish. Thus, in order to prove (4.31) for $\Xi_1^N(t)$, it suffices to show that

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\frac{1}{\delta}\mathbb{E}\left[\left(\lambda_\ell\int_0^t\int_{t-s}^{t+\delta-s}q_{\ell,i}(u)F(du)\big|\hat\Upsilon_\ell^N(s)\big|ds\right)^2\right]=0. \tag{4.33}$$

The expectation is bounded by

$$\mathbb{E}\Big[\sup_{0\le s\le T}\big|\hat\Upsilon_\ell^N(s)\big|^2\Big]\left(\lambda_\ell\int_0^t\int_{t-s}^{t+\delta-s}q_{\ell,i}(u)F(du)ds\right)^2.$$

Hence we obtain (4.33) by the same argument as in (3.20), and the bound in Lemma 4.4.

For the process $\Xi_2^N(t)$, we have

$$\begin{aligned}\mathbb{E}\left[\big|\Xi_2^N(t+\delta)-\Xi_2^N(t)\big|^2\right]&\le 2\mathbb{E}\left[\left(\lambda_\ell\int_t^{t+\delta}\int_0^{t+\delta-s}q_{\ell,i}(u)F(du)\big|\hat\Upsilon_\ell^N(s)\big|ds\right)^2\right]\\&\quad+2\mathbb{E}\left[\left(\lambda_\ell\int_0^t\int_{t-s}^{t+\delta-s}q_{\ell,i}(u)F(du)\big|\hat\Upsilon_\ell^N(s)\big|ds\right)^2\right]\\&\le 4\lambda_1^2\delta^2\sup_{s\in[0,T]}\mathbb{E}\left[\big|\hat\Upsilon_\ell^N(s)\big|^2\right]+4\mathbb{E}\left[\left(\lambda_\ell\int_0^t\int_{t-s}^{t+\delta-s}q_{\ell,i}(u)F(du)\big|\hat\Upsilon_\ell^N(s)\big|ds\right)^2\right].\end{aligned}$$

The argument for these two terms follow from that for the second and fourth terms above for $\Xi_1^N(t)$.

We next prove that as $N\to\infty$, $\hat I_{\ell,i}^N(t)-\tilde I_{\ell,i}^N(t)\to 0$ in probability, locally uniformly in t for each $\ell,i\in\mathcal{L}$. It follows a similar argument so we only highlight differences below. It is clear that

$$\begin{aligned}&\mathbb{E}\big[\hat I_{\ell,i}^N(t)-\tilde I_{\ell,i}^N(t)\big]=0,\\&\mathbb{E}\big[\big(\hat I_{\ell,i}^N(t)-\tilde I_{\ell,i}^N(t)\big)^2\big]=\lambda_\ell\int_0^t\Phi_{\ell,i}(t-s)\big|\bar\Upsilon_\ell^N(s)-\bar\Upsilon_\ell(s)\big|ds\to 0\quad\text{as}\quad N\to\infty,\end{aligned}$$

where the convergence holds by Theorem 2.1 and the dominated convergence theorem. To show that the sequence $\{\hat I_{\ell,i}^N-\tilde I_{\ell,i}^N\}$ is tight, we write

$$\begin{aligned}&\hat I_{\ell,i}^N(t)-\tilde I_{\ell,i}^N(t)\\&=\frac{1}{\sqrt{N}}\int_0^t\int_0^\infty\int_0^{t-s}\int_0^{t-s-u}\int_{\mathcal{L}}\int_{\{\ell\}}\Big(\mathbf{1}_{\mathfrak{a}\le\lambda_\ell\Upsilon_\ell^N(s)}-\mathbf{1}_{\mathfrak{a}\le\lambda_\ell N\bar\Upsilon_\ell(s)}\Big)\breve Q_i(ds,d\mathfrak{a},du,dv,d\theta,d\vartheta),\\&\quad-\lambda_\ell\int_0^t\Phi_{\ell,i}(t-s)\hat\Upsilon_\ell^N(s)ds,\end{aligned}$$

and observe that it suffices to prove tightness of the following processes

$$\begin{aligned}\mathcal{I}_1^N(t)&=\frac{1}{\sqrt{N}}\int_0^t\int_{\lambda_\ell N(\bar\Upsilon_\ell^N(s)\wedge\bar\Upsilon_\ell(s))}^{\lambda_\ell N(\bar\Upsilon_\ell^N(s)\vee\bar\Upsilon_\ell(s))}\int_0^{t-s}\int_0^{t-s-u}\int_{\mathcal{L}}\int_{\{\ell\}}\breve Q_i(ds,d\mathfrak{a},du,dv,d\theta,d\vartheta)\\\mathcal{I}_2^N(t)&=\lambda_\ell\int_0^t\Phi_{\ell,i}(t-s)\big|\hat\Upsilon_\ell^N(s)\big|ds.\end{aligned}$$

By the monotone property of these two processes in t, we then show that for any $\epsilon>0$, and $\iota=1,2$,

$$\limsup_{N\to\infty}\frac{1}{\delta}\mathbb{P}\left(\big|\mathcal{I}_\iota^N(t+\delta)-\mathcal{I}_\iota^N(t)\big|>\epsilon\right)\to 0\quad\text{as}\quad\delta\to 0. \tag{4.34}$$

Similar to the derivation in (4.32), we obtain

$$\mathbb{E}\left[\big|\mathcal{I}_1^N(t+\delta)-\mathcal{I}_1^N(t)\big|^2\right]\le 4\lambda_\ell\int_t^{t+\delta}\Phi_{\ell,i}(t+\delta-s)\mathbb{E}\left[\big|\bar\Upsilon_\ell^N(s)-\bar\Upsilon_\ell(s)\big|\right]ds$$

$$+ 4\lambda_\ell^2\delta^2 \sup_{s\in[0,T]} \mathbb{E}\left[\left|\hat{\Upsilon}_\ell^N(s)\right|^2\right] + 4\lambda_\ell \int_0^t \left(\Phi_{\ell,i}(t+\delta-s) - \Phi_{\ell,i}(t-s)\right)\mathbb{E}\left[\left|\bar{\Upsilon}_\ell^N(s) - \bar{\Upsilon}_\ell(s)\right|\right] ds$$

$$+ 4\mathbb{E}\left[\left(\lambda_\ell \int_0^t \left(\Phi_{\ell,i}(t+\delta-s) - \Phi_{\ell,i}(t-s)\right)\left|\hat{\Upsilon}_\ell^N(s)\right| ds\right)^2\right]. \tag{4.35}$$

The first and third terms converge to zero as $N \to \infty$ by the convergence $\mathbb{E}\left[\left|\bar{\Upsilon}_\ell^N(s) - \bar{\Upsilon}_\ell(s)\right|\right] \to 0$ and applying the dominated convergence theorem. For the last term in (4.35), we have

$$\mathbb{E}\left[\left(\lambda_\ell \int_0^t \left(\Phi_{\ell,i}(t+\delta-s) - \Phi_{\ell,i}(t-s)\right)\left|\hat{\Upsilon}_\ell^N(s)\right| ds\right)^2\right]$$

$$\le 2\mathbb{E}\left[\left(\lambda_\ell \int_0^t \left(\int_0^{t+\delta-s} \sum_{\ell'=1}^L p_{\ell,\ell'}(u) \int_{t-s-u}^{t+\delta-s-u} q_{\ell' i}(v) H(du,dv)\right)\left|\hat{\Upsilon}_\ell^N(s)\right| ds\right)^2\right]$$

$$+ 2\mathbb{E}\left[\left(\lambda_\ell \int_0^t \left(\int_{t-s}^{t+\delta-s} \int_0^{t+\delta-s-u} \sum_{\ell'=1}^L p_{\ell,\ell'}(u) q_{\ell' i}(v) H(du,dv)\right)\left|\hat{\Upsilon}_\ell^N(s)\right| ds\right)^2\right]$$

$$\le 2\mathbb{E}\left[\sup_{s\in[0,T]} \left|\hat{\Upsilon}_i^N(s)\right|^2\right]\left(\lambda_\ell \int_0^t \left(\int_0^{t+\delta-s} \sum_{\ell'=1}^L p_{\ell,\ell'}(u) \int_{t-s-u}^{t+\delta-s-u} q_{\ell' i}(v) H(du,dv)\right) ds\right)^2$$

$$+ 2\mathbb{E}\left[\sup_{s\in[0,T]} \left|\hat{\Upsilon}_i^N(s)\right|^2\right]\left(\lambda_\ell \left(\int_{t-s}^{t+\delta-s} \int_0^{t+\delta-s-u} \sum_{\ell'=1}^L p_{\ell,\ell'}(u) q_{\ell' i}(v) H(du,dv)\right) ds\right)^2$$

$$\le 2\mathbb{E}\left[\sup_{s\in[0,T]} \left|\hat{\Upsilon}_i^N(s)\right|^2\right]\left(\lambda_\ell L \int_0^t \int_0^{t+\delta-s} (F(t+\delta-s-u|u) - F(t-s-u|u)) G(du) ds\right)^2$$

$$+ 2\mathbb{E}\left[\sup_{s\in[0,T]} \left|\hat{\Upsilon}_i^N(s)\right|^2\right]\left(\lambda_\ell L \int_0^t (G(t+\delta-s) - G(t-s)) ds\right)^2. \tag{4.36}$$

Then the first term is treated with the same argument as in (3.23) while the second as in (3.20).

We then consider

$$\mathbb{E}\left[\left|\mathcal{I}_2^N(t+\delta) - \mathcal{I}_2^N(t)\right|^2\right] \le 2\mathbb{E}\left[\left(\lambda_\ell \int_t^{t+\delta} \Phi_{\ell,i}(t+\delta-s)\left|\hat{\Upsilon}_\ell^N(s)\right| ds\right)^2\right]$$

$$+ 2\mathbb{E}\left[\left(\lambda_\ell \int_0^t (\Phi_{\ell,i}(t+\delta-s) - \Phi_{\ell,i}(t-s))\left|\hat{\Upsilon}_\ell^N(s)\right| ds\right)^2\right].$$

Here the first term is bounded by

$$2\mathbb{E}\left[\sup_{s\in[0,T]} \left|\hat{\Upsilon}_\ell^N(s)\right|^2\right]\left(\lambda_\ell \int_t^{t+\delta} \Phi_{\ell,i}(t+\delta-s) ds\right)^2 \le 2\lambda_\ell^2\delta^2\mathbb{E}\left[\sup_{s\in[0,T]} \left|\hat{\Upsilon}_\ell^N(s)\right|^2\right]$$

and the second term is treated as above in (4.36). This completes the proof of (4.34), and thus $\hat{I}_{\ell,i}^N(t) - \tilde{I}_{\ell,i}^N(t) \to 0$ in probability, locally uniformly in t. This completes the proof. □

Completing the proof of Theorem 2.2. Let $\widetilde{S}_i^N(t), \widetilde{E}_i^N(t), \widetilde{I}_i^N(t), \widetilde{R}_i^N(t)$ be defined as in (4.4), (4.5), (4.8) and (4.13) correspondingly with $\widetilde{S}_i^N(0) = \hat{S}_i^N(0), \widetilde{E}_i^N(0) = \hat{E}_i^N(0), \widetilde{I}_i^N(0) = \hat{I}_i^N(0)$ and $\hat{\Upsilon}_i^N(t)$ being replaced by $\widetilde{\Upsilon}_i^N(t)$ defined by

$$\widetilde{\Upsilon}_i^N(t) = \psi_s(\bar{S}_i(t), \bar{I}_i(t), \bar{R}_i(t), \sum_{\ell\neq i} \kappa_{i\ell}\bar{I}_\ell(t))\widetilde{S}_i^N(t) + \psi_e(\bar{S}_i(t), \bar{I}_i(t), \bar{R}_i(t), \sum_{\ell\neq i} \kappa_{i\ell}\bar{I}_\ell(t))\widetilde{E}_i^N(t)$$

$$+\psi_i(\bar S_i(t),\bar I_i(t),\bar R_i(t),\sum_{\ell\neq i}\kappa_{i\ell}\bar I_\ell(t))\widetilde I_i^N(t)+\psi_r(\bar S_i(t),\bar I_i(t),\bar R_i(t),\sum_{\ell\neq i}\kappa_{i\ell}\bar I_\ell(t))\widetilde R_i^N(t)$$
$$+\psi_u(\bar S_i(t),\bar I_i(t),\bar R_i(t),\sum_{\ell\neq i}\kappa_{i\ell}\bar I_\ell(t))\sum_{\ell\neq i}\kappa_{i\ell}\widetilde I_\ell^N(t).$$

and the other components remain unchanged. Then by Lemma 5.1 below (with $m=L$), and by the convergence results in Lemmas 4.1, 4.2 and 4.6, we obtain that $(\widetilde S_i^N,\widetilde E_i^N,\widetilde I_i^N,\widetilde R_i^N,i\in\mathcal{L})\Rightarrow(\hat S_i,\hat E_i^N,\hat I_i,\hat R_i,i\in\mathcal{L})$ in D^{3L}. It remains to show that $(\widetilde S_i^N-\hat S_i^N,\widetilde E_i^N-\hat E_i^N,\widetilde I_i^N-\hat I_i^N,\widetilde R_i^N-\hat R_i^N,i\in\mathcal{L})\Rightarrow 0$. We have

$$\hat S_i^N(t)-\widetilde S_i^N=-\lambda_i\int_0^t(\hat\Upsilon_i^N(s)-\widetilde\Upsilon_i^N(s))ds$$
$$+\sum_{\ell=1,\ell\neq i}^L\int_0^t(\nu_{S,\ell,i}(\hat S_\ell^N(s)-\widetilde S_\ell^N(s))-\nu_{S,i,\ell}(\hat S_i^N(s)-\widetilde S_i^N(s)))ds\,,$$

$$\hat E_i^N(t)-\widetilde E_i^N(t)=\lambda_i\int_0^t(\hat\Upsilon_i^N(s)-\widetilde\Upsilon_i^N(s))ds-\sum_{\ell=1}^L\lambda_\ell\int_0^t\int_0^{t-s}p_{\ell,i}(u)G(du)(\hat\Upsilon_\ell^N(s)-\widetilde\Upsilon_i^N(s))ds$$
$$+\sum_{\ell=1,\ell\neq i}^L\int_0^t(\nu_{E,\ell,i}(\hat E_\ell^N(s)-\widetilde E_\ell^N(s))-\nu_{E,i,\ell}(\hat E_i^N(s)-\widetilde E_\ell^N(s)))ds\,,$$

$$\hat I_i^N(t)-\widetilde I_i^N(t)=\sum_{\ell=1}^L\lambda_\ell\int_0^t\int_0^{t-s}p_{\ell,i}(u)G(du)(\hat\Upsilon_\ell^N(s)-\widetilde\Upsilon_\ell^N(s))ds$$
$$-\sum_{\ell=1}^L\lambda_\ell\int_0^t\Phi_{\ell,i}(t-s)(\hat\Upsilon_\ell^N(s)-\widetilde\Upsilon_\ell^N(s))ds$$
$$+\sum_{\ell\neq i}\int_0^t\Big(\nu_{I,\ell,i}(\hat I_\ell^N(s)-\widetilde I_\ell^N(s))-\nu_{I,i,\ell}(\hat I_i^N(s)-\widetilde I_\ell^N(s))\Big)ds,$$

$$\hat R_i^N(t)-\widetilde R_i^N(t)=\sum_{\ell=1}^L\lambda_\ell\int_0^t\Phi_{\ell,i}(t-s)(\hat\Upsilon_\ell^N(s)-\widetilde\Upsilon_\ell^N(s))ds$$
$$+\sum_{\ell=1,\ell\neq i}^L\int_0^t(\nu_{R,\ell,i}(\hat R_\ell^N(s)-\widetilde R_\ell^N(s))-\nu_{R,i,\ell}(\hat R_i^N(s)-\widetilde R_i^N(s)))ds.$$

Let

$$\psi_{s,a}^N(t):=\psi_s'\Big(\bar S_i(t)+a(\bar S_i^N(t)-\bar S_i(t)),\bar E_i(t)+a(\bar E_i^N(t)-\bar E_i(t)),\bar I_i(t)+a(\bar I_i^N(t)-\bar I_i(t)),$$
$$\bar R_i(t)+a(\bar R_i^N(t)-\bar R_i(t)),\sum_{\ell\neq i}\bar I_\ell(t)+a(\sum_{\ell\neq i}\bar I_\ell^N(t)-\sum_{\ell\neq i}\bar I_\ell(t))\Big).$$

Similarly for $\psi_{e,a}^N(t)$, $\psi_{i,a}^N(t)$, $\psi_{r,a}^N(t)$ and $\psi_{u,a}^N(t)$. Also write $\psi_{e,0}(t),\psi_{i,0}(t),\psi_{r,0}(t),\psi_{u,0}(t),\psi_{u,0}(t)$ when $a=0$ (noting that they no longer depend on N in this case). We then have

$$\hat\Upsilon_i^N(t)-\widetilde\Upsilon_i^N(t)$$
$$=\sqrt N\Big(\psi(\bar S_i^N(t),\bar E_i^N(t),\bar I_i^N(t),\bar R_i^N(t),\sum_{\ell\neq i}\bar I_\ell^N(t))-f(\bar S_i(t),\bar E_i(t),\bar I_i(t),\bar R_i(t),\sum_{\ell\neq i}\bar I_\ell(t))\Big)-\widetilde\Upsilon_i^N(t)$$

$$
\begin{aligned}
&= \int_0^1 \psi^N_{s,a}(t)da(\hat{S}^N_i(t) - \widetilde{S}^N_i(t)) + \int_0^1 \psi^N_{e,a}(t)da(\hat{E}^N_i(t) - \widetilde{E}^N_i(t)) + \int_0^1 \psi^N_{i,a}(t)da(\hat{I}^N_i(t) - \widetilde{I}^N_i(t)) \\
&\quad + \int_0^1 \psi^N_{r,a}(t)da(\hat{R}^N_i(t) - \widetilde{R}^N_i(t)) + \int_0^1 \psi^N_{u,a}(t)da \sum_{\ell\neq i}(\hat{I}^N_\ell(t) - \widetilde{I}^N_j(t)) \\
&\quad + \Big(\int_0^1 \psi^N_{s,a}(t)da - \psi_{s,0}(t)\Big)\widetilde{S}^N_i(t) + \Big(\int_0^1 \psi^N_{e,a}(t)da - \psi_{e,0}(t)\Big)\widetilde{E}^N_i(t) + \Big(\int_0^1 \psi^N_{i,a}(t)da - \psi_{i,0}(t)\Big)\widetilde{I}^N_i(t) \\
&\quad + \Big(\int_0^1 \psi^N_{r,a}(t)da - \psi_{r,0}(t)\Big)\widetilde{R}^N_i(t) + \Big(\int_0^1 \psi^N_{u,a}(t)da - \psi_{u,0}(t)\Big)\sum_{\ell\neq i}\widetilde{I}^N_\ell(t).
\end{aligned}
$$

We can use the bounds in the proof of Lemma 4.3 to show that $\limsup_N \sup_{t\in[0,T]} \int_0^1 \psi^N_{s,a}(t)da < \infty$ and the same holds for $\psi^N_{e,a}(t)$, $\psi^N_{i,a}(t)$, $\psi^N_{r,a}(t)$ and $\psi^N_{u,a}(t)$. It is also clear that $\int_0^1 \psi^N_{s,a}(t)da - \psi_{s,0}(t) \to 0$ as $N \to \infty$, uniformly in t, and similarly for the others. In addition, similarly as Lemma 4.3, we can also show that $\sup_N \sup_{t\in[0,T]} \mathbb{E}\left[\left|\widetilde{S}^N_i(t)\right|^2\right] < \infty$ and the same for $\widetilde{E}^N_i(t)$, $\widetilde{I}^N_i(t)$ and $\widetilde{R}^N_i(t)$. Then by Gronwall's inequality, we conclude that $(\widetilde{S}^N_i - \hat{S}^N_i, \widetilde{I}^N_i - \hat{I}^N_i, \widetilde{R}^N_i - \hat{R}^N_i, i \in \mathcal{L}) \Rightarrow 0$. This completes the proof of the theorem. □

5. Appendix

We define a $4m$-dimensional integral mapping F: given $a_i, b_i, c_i, d_i, \phi_i, \psi_i, \varphi_i, \chi_i \in D$, some constants $\alpha_i, \beta_i, \gamma_i, \kappa_i > 0$ and functions $F_{\ell,i}$, $G_{\ell,i}$ for $\ell, i = 1, \dots, m$, let x_i, y_i, z_i, w_i be the solutions to the following integral mapping:

$$
\begin{aligned}
x_i(t) &= x_i(0) + \phi_i(t) - \int_0^t (a_i(s)x_i(s) + b_i(s)y_i(s) + c_i(s)z_i(s) + d_i(s)w_i(s))ds \\
&\quad + \sum_{\ell=1,\ell\neq i}^m \int_0^t (\alpha_{\ell,i}x_\ell(s) - \alpha_{i,\ell}x_i(s))ds\,, \\
y_i(t) &= y_i(0) + \psi_i(t) + \int_0^t (a_i(s)x_i(s) + b_i(s)y_i(s) + c_i(s)z_i(s) + d_i(s)w_i(s))ds \\
&\quad - \sum_{\ell=1}^m \int_0^t F_{\ell,i}(t-s)(a_\ell(s)x_\ell(s) + b_\ell(s)y_\ell(s) + c_\ell(s)z_\ell(s) + d_\ell(s)w_\ell(s))ds \\
&\quad + \sum_{\ell=1,\ell\neq i}^m \int_0^t (\beta_{\ell,i}y_\ell(s) - \beta_{i,\ell}y_i(s))ds\,, \\
z_i(t) &= z_i(0) + \varphi_i(t) - \sum_{\ell=1}^m \int_0^t F_{\ell,i}(t-s)(a_\ell(s)x_\ell(s) + b_\ell(s)y_\ell(s) + c_\ell(s)z_\ell(s) + d_\ell(s)w_\ell(s))ds \\
&\quad - \sum_{\ell=1}^m \int_0^t G_{\ell,i}(t-s)(a_\ell(s)x_\ell(s) + b_\ell(s)y_\ell(s) + c_\ell(s)z_\ell(s) + d_\ell(s)w_\ell(s))ds \\
&\quad + \sum_{\ell=1,\ell\neq i}^m \int_0^t (\gamma_{\ell,i}z_\ell(s) - \gamma_{i,\ell}z_i(s))ds, \\
w_i(t) &= w_i(0) + \chi_i(t) + \sum_{\ell=1}^m \int_0^t F_{\ell,i}(t-s)(a_\ell(s)x_\ell(s) + b_\ell(s)y_\ell(s) + c_\ell(s)z_\ell(s) + d_\ell(s)w_\ell(s))ds
\end{aligned}
$$

$$+\sum_{\ell=1}^{m}\int_0^t G_{\ell,i}(t-s)(a_\ell(s)x_\ell(s)+b_\ell(s)y_\ell(s)+c_\ell(s)z_\ell(s)+d_\ell(s)w_\ell(s))ds$$
$$+\sum_{\ell=1,\ell\neq i}^{m}\int_0^t(\kappa_{\ell,i}w_\ell(s)-\kappa_{i,\ell}w_i(s))ds.$$

The existence and uniqueness of its solution and the continuity property are stated in the following lemma.

Lemma 5.1. *Assume that $F_{\ell,i}$ and $G_{\ell,i}$, $\ell, i = 1,\dots,m$ are measurable, bounded and continuous functions satisfying $F_{\ell,i}(0) = 0$ and $G_{\ell,i}(0) = 0$, and let the constants $\alpha_i, \beta_i, \gamma_i, \kappa_i > 0$ and the functions $\phi_i, \psi_i, \varphi_i, \chi_i$ be given. There exists a unique solution $(x_i, y_i, z_i, w_i,\ i = 1,\dots,m) \in D^{4m}$ to the set of integrable equations defining the mapping $\tilde{F}$. The mapping is continuous in the Skorohod J_1 topology, that is, if $(a_i^n, b_i^n, c_i^n, d_i^n, \phi_i^N, \psi_i^n, \varphi_i^n, \chi_i^n\ i = 1,\dots,m) \to (a_i, b_i, c_i, d_i, \phi_i, \psi_i, \varphi_i, \chi_i,\ i = 1,\dots,m)$ in $D([0,T],\mathbb{R}^{8m})$ as $n\to\infty$ and $(x_i^n(0), y_i^n(0), z_i^n(0), w_i^n(0),\ i = 1,\dots,m) \to (x_i(0), y_i(0), z_i(0), w_i(0),\ i = 1,\dots,m)$, then $(x_i^n, y_i^n, z_i^n, w_i^n,\ i = 1,\dots,m) \to (x_i, y_i, z_i, w_i,\ i = 1,\dots,m)$ in $D([0,T],\mathbb{R}^{4m})$ as $n\to\infty$. In addition, if $\phi_i, \psi_i, \varphi_i, \chi_i$ are continuous, then $(x_i, y_i, z_i, w_i\, i = 1,\dots,m) \in C^{4m}$ and the mapping F is continuous uniformly on compact sets in $[0,T]$.*

Proof. For the existence and uniqueness of solutions, we can apply the Schauder-Tychonoff fixed point theorem, and modify the proofs in Theorems 1.2 and 2.3 in Chapter II of [26] (where these results are shown for Volterra integral equations with continuous functions). The continuity can be proved similarly as Lemma 9.1 in [28]. □

References

[1] Hakan Andersson and Tom Britton. *Stochastic epidemic models and their statistical analysis*, volume 151. Springer Science & Business Media, 2012.

[2] Frank Ball. A unified approach to the distribution of total size and total area under the trajectory of infectives in epidemic models. *Advances in Applied Probability*, 18(2):289–310, 1986.

[3] Frank Ball and Damian Clancy. The final size and severity of a generalised stochastic multitype epidemic model. *Advances in Applied Probability*, 25(4):721–736, 1993.

[4] Derdei Bichara and Abderrahman Iggidr. Multi-patch and multi-group epidemic models: a new framework. *Journal of Mathematical Biology*, 77(1):107–134, 2018.

[5] Patrick Billingsley. *Convergence of probability measures*. John Wiley & Sons, 1999.

[6] Fred Brauer. On a nonlinear integral equation for population growth problems. *SIAM Journal on Mathematical Analysis*, 6(2):312–317, 1975.

[7] Fred Brauer, Carlos Castillo-Chavez, and Zhilan Feng. *Mathematical Models in Epidemiology*. Springer, 2019.

[8] Tom Britton and Etienne Pardoux. Stochastic epidemics in a homogeneous community. *Stochastic Epidemic Models with Inference (T. Britton and E. Pardoux eds). Part I. Lecture Notes in Math. 2255*, pages 1–120, 2019.

[9] Erhan Çınlar. *Probability and Stochastics*, volume 261. Springer Science & Business Media, 2011.

[10] Damian Clancy. SIR epidemic models with general infectious period distribution. *Statistics & Probability Letters*, 85:1–5, 2014.

[11] Kenneth L Cooke. An epidemic equation with immigration. *Mathematical Biosciences*, 29(1-2):135–158, 1976.

[12] Mamadou L Diagne, Ousmane Seydi, and Aissata AB Sy. A two-group age of infection epidemic model with periodic behavioral changes. *Discrete & Continuous Dynamical Systems-B*, 22(11):0, 2019.

[13] Odo Diekmann. Limiting behaviour in an epidemic model. *Nonlinear Analysis: Theory, Methods & Applications*, 1(5):459–470, 1977.

[14] Stewart N Ethier and Thomas G Kurtz. *Markov processes: characterization and convergence*, volume 282. John Wiley & Sons, 2009.

[15] Raphaël Forien, Guodong Pang, and Étienne Pardoux. Epidemic models with varying infectivity. *arXiv:2006.15377*, 2020.

[16] Raphaël Forien, Guodong Pang, and Étienne Pardoux. Estimating the state of the Covid–19 epidemic in France using a model with memory. *Royal Society Open Science*, 8(3):202327, 2021.

[17] A Gómez-Corral and M López-García. On SIR epidemic models with generally distributed infectious periods: Number of secondary cases and probability of infection. *International Journal of Biomathematics*, 10(02):1750024, 2017.

[18] Marjorie G Hahn. Central limit theorems in $D[0,1]$. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 44(2):89–101, 1978.
[19] Herbert W Hethcote and Pauline van den Driessche. An SIS epidemic model with variable population size and a delay. *Journal of Mathematical Biology*, 34(2):177–194, 1995.
[20] Wenzhang Huang, Kenneth L Cooke, and Carlos Castillo-Chavez. Stability and bifurcation for a multiple-group model for the dynamics of hiv/aids transmission. *SIAM Journal on Applied Mathematics*, 52(3):835–854, 1992.
[21] Aberrahman Iggidr, Gauthier Sallet, and Max O Souza. On the dynamics of a class of multi-group models for vector-borne diseases. *Journal of Mathematical Analysis and Applications*, 441(2):723–743, 2016.
[22] Pierre Magal and Connell McCluskey. Two-group infection age model including an application to nosocomial infection. *SIAM Journal on Applied Mathematics*, 73(2):1058–1095, 2013.
[23] Pierre Magal, Ousmane Seydi, and Glenn Webb. Final size of an epidemic for a two-group SIR model. *SIAM Journal on Applied Mathematics*, 76(5):2042–2059, 2016.
[24] Pierre Magal, Ousmane Seydi, and Glenn Webb. Final size of a multi-group SIR epidemic model: Irreducible and non-irreducible modes of transmission. *Mathematical Biosciences*, 301:59–67, 2018.
[25] Pascal Massart. The tight constant in the Dvoretzky-Kiefer-Wolfowitz inequality. *The Annals of Probability*, 18:1269–1283, 1990.
[26] R. K. Miller. *Nolinear Volterra Integral Equations*. Benjiamin Press, Menlo Park, Cal., 1971.
[27] Guodong Pang and Étienne Pardoux. Functional central limit theorems for epidemic models with varying infectivity. *arXiv:2009.11861*, 2020.
[28] Guodong Pang and Étienne Pardoux. Functional limit theorems for non-Markovian epidemic models. *Working paper*, arXiv:2003.03249, 2020.
[29] Melanie Prague, Linda Wittkop, Quentin Clairon, Dan Dutartre, Rodolphe Thiebaut, and Boris Pierre Hejblum. Population modeling of early COVID-19 epidemic dynamics in French regions and estimation of the lockdown impact on infection rate. *https://doi.org/10.1101/2020.04.21.20073536*, 2020.
[30] Gesine Reinert. The asymptotic evolution of the general stochastic epidemic. *The Annals of Applied Probability*, 5(4):1061–1086, 1995.
[31] Nita H Shah, Nisha Sheoran, Ekta Jayswal, Dhairya Shukla, Nehal Shukla, Jagdish Shukla, and Yash Shah. Modelling COVID-19 transmission in the United States through interstate and foreign travels and evaluating impact of governmental public health interventions. *Journal of Mathematical Analysis and Applications*, page 124896, 2020.
[32] Pauline van den Driessche and James Watmough. A simple SIS epidemic model with a backward bifurcation. *Journal of Mathematical Biology*, 40(6):525–540, 2000.
[33] Frank JS Wang. Limit theorems for age and density dependent stochastic population models. *Journal of Mathematical Biology*, 2(4):373–400, 1975.
[34] Frank JS Wang. A central limit theorem for age-and density-dependent population processes. *Stochastic Processes and their Applications*, 5(2):173–193, 1977.
[35] Frank JS Wang. Gaussian approximation of some closed stochastic epidemic models. *Journal of Applied Probability*, pages 221–231, 1977.

The Harold and Inge Marcus Department of Industrial and Manufacturing Engineering, College of Engineering, Pennsylvania State University, University Park, PA 16802 USA
Email address: gup3@psu.edu

Aix–Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373 13453 Marseille, France
Email address: etienne.pardoux@univ.amu.fr