Epidemic models with varying infectivity

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Abstract. We introduce an epidemic model with varying infectivity and general exposed and infectious periods, where the infectivity of each individual is a random function of the elapsed time since infection, those function being i.i.d. for the various individuals in the population. This approach models infection-age dependent infectivity, and extends the classical SIR and SEIR models. We focus on the infectivity process (total force of infection at each time), and prove a functional law of large number (FLLN). In the deterministic limit of this LLN, the infectivity process and the susceptible process are determined by a two-dimensional deterministic integral equation. From its solutions, we then derive the exposed, infectious and recovered processes, again using integral equations. We also use these equations to derive the basic reproduction number R_0 during the early stage of an epidemic, in terms of the average individual infectivity function and the exponential rate of growth of the epidemic, and note that a decreasing infectivity induces a decrease of R_0, computing its value in the case of the Covid–19 epidemic.

1. Introduction

Most of the literature on epidemics models is based upon ODE models which assume that the length of time during which a given individual is infectious follows an exponential distribution. More precisely, those deterministic models are law of large numbers limits, as the size of the population tends to infinity, of stochastic models where all transitions from one compartment to the next have exponential distributions, see [5] for a recent account. However, it is largely recognized that for most diseases, the durations of the exposed and infectious period are far from following an exponential distribution. In the case of influenza, a deterministic duration would probably be a better approximation. Recently in [15], the last two authors have obtained the functional law of large numbers (FLLN) limit for SIS, SIR, SEIR and SIRS models where in the stochastic model the duration of the stay in the I compartment (resp. both in the E and the I, resp. both in the I and the R compartments) follow a very arbitrary distribution. Of course, in this case the stochastic model is not a Markov model, which makes some of the proofs more delicate. Indeed, the fluctuating part of a Markov process is a martingale, and many tools exist to study tightness and limits of martingales, which are missing in the non–Markovian situation. Nevertheless, we were able in [15] to use *ad hoc* techniques in order to circumvent that difficulty, and we proved not only FLLNs, but also functional central limit theorems (FCLTs). While the classical "Markovian" deterministic

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models are ODEs, our more general and more realistic "non–Markovian" deterministic models are Volterra type integral equations of the same dimension as the classical ODE models. Recently in [7], the authors used the approach in [15] to describe the Covid-19 epidemic in France. The flexibility of the choice for the law of the infectious period was very helpful in order to write a realistic model with very few compartments, and our model follows better the data than Markov models.

The aim of the present paper is to go a step further in the direction of realistic models of epidemics. It has been established in [9] that in the case of the Covid-19 disease, the infectivity of infectious individuals decreases after symptom onset. In fact it is believed that in most infectious diseases, the infectivity of infectious individuals depends upon the time since infection. This was already argued in [12]. However, in this work the duration of the infectious period follows essentially an exponential distribution. See also a recent paper in the study of Covid-19 pandemic [8], which uses a transport PDE model (it is worth noting that following the original work in [12], PDEs have been commonly used to capture the effect of age of infection in the epidemic literature, see, e.g., [10, 18, 11, 14]). While our paper was already written, we discovered that the same deterministic model has already been described as an "age of infection epidemic model" in [2] and the recent book [3, Chapter 4.5]. But the fact that it is the law of large numbers limit of a well specified stochastic model, which is our main result, seems to be new. Section 4.5.1 of the same book also contains the same SEIR model as in [15], but again without any rigorous connection with a stochastic model.

The most realistic assumption is probably that this infectivity first increases continuously from 0, and then decreases back to 0. We shall however allow jumps in the random infectivity function, in order in particular to include the classical case of a constant infectivity during the infectious period. And we want to allow a very arbitrary law for the infectious (or exposed/infectious) period(s), as we did in [15]. In this work again, the FLLN limiting deterministic model is a Volterra type integral equation, which is of the same dimension as the corresponding classical ODE model, see Theorem 2.1. We treat only the case of SIR and SEIR models (see also Remark 2.4 on the SIS and SIRS models), but we intend to extend in later publications our approach to other types of models, including models with age classes and spatial distribution, see [16] for multi–patch models with general exposed and infectious durations. We will also establish in a further publication the FCLT associated to the FLLN established in the present paper.

Our approach in this paper is to assume that in the original stochastic finite population model, the infectivity of each individual is a random function of the time elapsed since his/her infection, those functions associated to various individuals being independent and identically distributed (i.i.d.). The total force of infection at each time is the aggregate infectivity of all the individuals that are currently infectious. We assume that the infectivity random functions can be piecewise continuous with a finite number of discontinuities, which includes all the commonly seen examples, in particular, constant infectivity over a given time interval as a special case. They are also allowed to start with a value zero for a period of time to generalize the SEIR model. These random functions then determine the durations of the exposed and infectious periods, and therefore, their corresponding probability distributions, which can be very general.

Under the i.i.d. assumptions of these infectivity random functions of the various individuals, we prove a functional law of large numbers for the infectivity process, together with the counting processes for the susceptible, exposed, infectious and recovered individuals. The infectivity and susceptible functions in the limit are uniquely determined by a two-dimensional Volterra integral equation. Given these two functions, the exposed, infectious and recovered functions in the limit are given by Volterra integral equations. They generalize the integral equations in the standard SIR/SEIR models with general exposed and infectious periods in [15]. Our proofs are based upon Poisson random measures associated with the infectivity process, which helps us to establish tightness and convergence, and further develops the techniques in [15].

Our limiting integral equations can be easily solved numerically. For the standard SIR/SEIR model with general exposed and infectious periods, the integral equations are implemented to estimate the state of the Covid-19 pandemic in France in [7]. We also refer the readers to another recent work by Fodor et al. [20] which argues that integral equations (in the case of deterministic infectious periods) should be used instead of ODEs since the latter may significantly underestimate the initial basic reproduction number R_0. We claim that our model may be used to better predict the trajectory of the epidemic, especially at the beginning of the epidemic and when certain control measures like lockdown and reopening are implemented.

We also study the linearized version of our model corresponding to the early phase of the epidemic, during which the proportion of susceptible individuals essentially does not change, whose solution is an exponential function (Theorem 2.2). We then deduce from that analysis a formula for the basic reproduction number R_0, as an explicit function of the average infectivity function and the exponential rate of growth ρ, which is obtained by observing the epidemic. This result extends the well known result in the exponential case (see for example equation (1) in [4]), and also the result recently established for the SEIR model with general exposed and infectious periods [7]. Note however that our formula for R_0 already appears on page 141 of the recent book [3]. We compute explicitly the value of R_0 for different values of two unknown parameters for the case of the early phase of the Covid–19 epidemic in France, assuming a decrease of the infectivity compatible with the results in [9]. We see that the decrease of the infectivity with age–infection induces a decrease of R_0.

The paper is organized as follows. In Section 2, we formulate our stochastic model, make precise all assumptions, state our two main results: the FLLN, which is stated as Theorem 2.1, and the result concerning the linearized model for the early phase of the epidemic, Theorem 2.2, and finally compute R_0 in the case of the early phase of the French Covid–19 epidemic. Section 3 is devoted to the proof of Theorem 2.2, and Section 4 to the proof of Theorem 2.1.

2. Model and Results

2.1. **Model description.** All random variables and processes are defined in a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider a generalized SEIR epidemic model where each infectious individual has an infectivity that is randomly varying with the time elapsed since infection. As usual, the population consists of four groups of individuals, susceptible, exposed, infectious and recovered.

Let N be the population size, and $S^N(t), E^N(t), I^N(t), R^N(t)$ denote the sizes of the four groups, respectively. We have the balance equation $N = S^N(t) + E^N(t) + I^N(t) + R^N(t)$ for $t \geq 0$. Assume that $R^N(0) = 0$, $S^N(0) > 0$ and $E^N(0) + I^N(0) > 0$ such that $S^N(0) + E^N(0) + I^N(0) = N$. Let $A^N(t)$ be the cumulative number of individuals that become infected in $(0, t]$ for $t \geq 0$ and denote the associated event times as τ_i^N, $i = 1, \ldots, A^N(t)$.

Note that an infected individual is either exposed or infectious. More precisely, he/she is first exposed, then infectious. Let us first consider those individuals who are infected after time 0 (i.e. they are in the S compartment at time 0). The i–th infected individual is infected at time τ_i^N. He/she is first Exposed during the time interval $[\tau_i^N, \tau_i^N + \zeta_i)$. Then he/she is infectious during the time interval $(\tau_i^N + \zeta_i, \tau_i^N + \zeta_i + \eta_i)$, and finally removed on the time interval $[\tau_i^N + \zeta_i + \eta_i, +\infty)$. In fact, to this individual is attached an infectivity process $\{\lambda_i(t) : t \geq 0\}$, which is a random right–continuous function such that

$$\lambda_i(t) \begin{cases} = 0, & \text{if } 0 \leq t < \zeta_i, \\ > 0, & \text{if } \zeta_i < t < \zeta_i + \eta_i, \\ = 0, & \text{if } t \geq \zeta_i + \eta_i. \end{cases} \tag{2.1}$$

We shall formulate some assumptions on the functions λ_i below. Let us just say for now that the collection of the functions $\{\lambda_i(\cdot)\}_{i \geq 1}$ are i.i.d. Since

$$\zeta_i = \inf\{t > 0,\ \lambda_i(t) > 0\}, \quad \text{and } \zeta_i + \eta_i = \inf\{t > 0,\ \lambda_i(r) = 0,\ \forall r \geq t\}, \tag{2.2}$$

the collection of random vectors $(\zeta_i, \eta_i)_{i \geq 1}$ is also i.i.d.

Each initially exposed individual is associated with an infectivity process $\lambda_j^0(t)$, $j = 1, \ldots, E^N(0)$, with càdlàg paths, which are assumed to be i.i.d. and be such that

$$\zeta_j^0 = \inf\{t > 0,\ \lambda_j^0(t) > 0\} > 0 \text{ a.s.} \quad \text{and } \zeta_j^0 + \eta_j^0 = \inf\{t > 0,\ \lambda_j^0(r) = 0,\ \forall r \geq t\}. \tag{2.3}$$

Each initially infectious individual is associated with an infectivity process $\lambda_k^{0,I}(t)$, $k = 1, \ldots, I^N(0)$, with càdlàg paths, which are also assumed to be i.i.d. and such that

$$\inf\{t > 0,\ \lambda_k^{0,I}(t) > 0\} = 0 \text{ a.s.} \quad \text{and } \eta_k^{0,I} = \inf\{t > 0,\ \lambda_k^{0,I}(r) = 0,\ \forall r \geq t\}. \tag{2.4}$$

We will write (ζ, η) (resp. (ζ^0, η^0), resp. $\eta^{0,I}$) for a vector which has the same law as (ζ_i, η_i) (resp. (ζ_j^0, η_j^0), resp. $\eta_k^{0,I}$). Let $H(du, dv)$ denote the law of (ζ, η), $H_0(du, dv)$ that of (ζ^0, η^0) and $F_{0,I}$ the c.d.f. of $\eta^{0,I}$. We define moreover

$$\Phi(t) := \int_0^t \int_0^{t-u} H(du, dv) = \mathbb{P}(\zeta + \eta \leq t),\ \Psi(t) := \int_0^t \int_{t-u}^{\infty} H(du, dv) = \mathbb{P}(\zeta \leq t < \zeta + \eta),$$

$$\Phi_0(t) := \int_0^t \int_0^{t-u} H_0(du, dv) = \mathbb{P}(\zeta^0 + \eta^0 \leq t),\ \Psi_0(t) := \int_0^t \int_{t-u}^{\infty} H_0(du, dv) = \mathbb{P}(\zeta^0 \leq t < \zeta^0 + \eta^0),$$

$$F_{0,I}(t) := \mathbb{P}(\eta^{0,I} \leq t)\,.$$

We shall also write

$$H(du, dv) = G(du)F(dv|u), \quad H_0(du, dv) = G_0(du)F_0(dv|u),$$

i.e., G is the c.d.f. of ζ and $F(\cdot|u)$ is the conditional law of η, given that $\zeta = u$, G_0 is the c.d.f. of ζ^0 and $F_0(\cdot|u)$ is the conditional law of η^0, given that $\zeta^0 = u$. In the case of independent

exposed and infectious periods, it is reasonable that the infectious periods of the initially exposed individuals have the same distribution as the newly exposed ones, that is, $F_0 = F$. Note that in the independent case, $\Psi(t) = G(t) - \Phi(t)$ and $\Psi_0(t) = G_0(t) - \Phi_0(t)$. Also, let $G_0^c = 1 - G_0$, $G^c = 1 - G$, $F_{0,I}^c = 1 - F_{0,I}$, and $F^c = 1 - F$.

The total force of infection which is exerted on the susceptibles at time t can be written as

$$\mathfrak{I}^N(t) = \sum_{j=1}^{E^N(0)} \lambda_j^0(t) + \sum_{k=1}^{I^N(0)} \lambda_k^{0,I}(t) + \sum_{i=1}^{A^N(t)} \lambda_i(t - \tau_i^N)\,, \quad t \geq 0. \tag{2.5}$$

Thus, the instantaneous infectivity rate function at time t is

$$\Upsilon^N(t) = \frac{S^N(t)}{N}\mathfrak{I}^N(t), \quad t \geq 0. \tag{2.6}$$

The infection process $A^N(t)$ can be expressed by

$$A^N(t) = \int_0^t \int_0^\infty \mathbf{1}_{u \leq \Upsilon^N(s)} Q(ds, du), \quad t \geq 0, \tag{2.7}$$

where Q is a standard Poisson random measure (PRM) on $\mathbb{R}_+^2$, and we use $\mathbf{1}_{\{\cdot\}}$ for the indicator function. One may observe that besides the PRM Q, the randomness in the epidemic dynamics comes only from the infectivity processes $\{\lambda_j^0(t)\}$, $\{\lambda_k^{0,I}(t)\}$ and $\{\lambda_i(t)\}$ (the infectious periods $\{\eta_j^0\}$, $(\eta_k^{0,I})$ and $\{\eta_i\}$ are induced from them).

The epidemic dynamics of the model can be described by

$$\begin{aligned}
S^N(t) &= S^N(0) - A^N(t)\,,\\
E^N(t) &= \sum_{j=1}^{E^N(0)} \mathbf{1}_{\zeta_j^0 > t} + \sum_{i=1}^{A^N(t)} \mathbf{1}_{\tau_i^N + \zeta_i > t}\,,\\
I^N(t) &= \sum_{j=1}^{E^N(0)} \mathbf{1}_{\zeta_j^0 \leq t < \zeta_j^0 + \eta_j^0} + \sum_{k=1}^{I^N(0)} \mathbf{1}_{\eta_k^{0,I} > t} + \sum_{i=1}^{A^N(t)} \mathbf{1}_{\tau_i^N + \zeta_i \leq t < \tau_i^N + \zeta_i + \eta_i}\,,\\
R^N(t) &= \sum_{j=1}^{E^N(0)} \mathbf{1}_{\zeta_j^0 + \eta_j^0 \leq t} + \sum_{k=1}^{I^N(0)} \mathbf{1}_{\eta_k^{0,I} \leq t} + \sum_{i=1}^{A^N(t)} \mathbf{1}_{\tau_i^N + \zeta_i + \eta_i \leq t}\,.
\end{aligned} \tag{2.8}$$

In the case where $\zeta_j^0 = 0$ and $\zeta_i = 0$, the model is a generalized SIR model, and $E^N(t) \equiv 0$.

2.2. **FLLN.** We first make the following assumptions on the distribution functions, infectivity functions, and the initial quantities.

Assumption 2.1. *The c.d.f. G satisfies the following assumption: G can be written as $G = G_1 + G_2$, where $G_1(t) = \sum_i a_i \mathbf{1}(t \geq t_i)$ for a finite or countable number of positive numbers a_i and the corresponding t_i such that $\sum_i a_i \leq 1$ and $t_0 < t_1 < \ldots t_k < \ldots$, and G_2 is Hölder continuous with exponent $\frac{1}{2} + \theta$ for some $\theta > 0$, that is, $G_2(t + \delta) - G_2(t) \leq c\delta^{1/2+\theta}$ for some $c > 0$. Moreover, the conditional c.d.f. $F(\cdot|u)$ satisfies the same assumption, uniformly in u.*

We now state our assumptions on λ^0, $\lambda^{0,I}$ and λ.

Assumption 2.2. *The random functions $\lambda(t)$ (resp. $\lambda^0(t)$ and resp. $\lambda^{0,I}(t)$), of which $\lambda_1(t), \lambda_2(t), \dots$ (resp. $\lambda_1^0(t), \lambda_2^0(t), \dots$ and resp. $\lambda_1^{0,I}(t), \lambda_2^{0,I}(t), \dots$) are i.i.d. copies, satisfying the following properties. There exists a constant $\lambda^* < \infty$ such that $\sup_{t\in[0,T]} \max\{\lambda^0(t), \lambda^{0,I}(t), \lambda(t)\} \le \lambda^*$ almost surely. In addition, there exist a given number $k \ge 1$, a random sequence $0 = \xi^0 \le \xi^1 \le \cdots \le \xi^k = \eta$ and random functions $\lambda^j \in C(\mathbb{R}_+; \mathbb{R}_+)$, $1 \le j \le k$ such that*

$$\lambda(t) = \sum_{j=1}^{k} \lambda^j(t) \mathbf{1}_{[\xi^{j-1}, \xi^j)}(t)\,. \tag{2.9}$$

Moreover, denoting by F_j the c.d.f. of ξ^j, we assume that each F_j satisfies assumption 2.1, and that there exists a nondecreasing function $\varphi \in C(\mathbb{R}_+; \mathbb{R}_+)$ with $\varphi(0) = 0$, such that $|\lambda^j(t) - \lambda^j(s)| \le \varphi(|t-s|)$ almost surely, for all $t, s \ge 0$, $1 \le j \le k$.

Let $\bar\lambda^0(t) = \mathbb{E}[\lambda^0(t)]$, $\bar\lambda^{0,I}(t) = \mathbb{E}[\lambda^{0,I}(t)]$ and $\bar\lambda(t) = \mathbb{E}[\lambda(t)]$ for $t \ge 0$. It is clear that $\bar\lambda^0(t), \bar\lambda^{0,I}(t)$ and $\bar\lambda(t)$ are all càdàg.

Remark 2.1. *We think that $\lambda(t)$ being continuous is a good model of reality. However, the early phase of the function $\lambda(t)$ is not well known, since patients are tested only after symptom onset, and usually (this is the case in particular for the Covid–19) they may have been infectious (i.e., with $\lambda(t) > 0$) prior to that. Consequently we should not exclude the possibility that $\lambda(t)$ jumps to its maximum at time ζ, and the decreases continuously to 0.*

Moreover, in order to include the "classical" models where $\lambda(t)$ is first 0 during the exposed period, and then equal to a positive constant during the infectious period, as well as possible models of infectivity that would be piecewise constant, we allow $\lambda(t)$ to have a given number of jumps.

Let $\bar X^N := N^{-1} X^N$ for any process X^N. Let $D = D([0, +\infty), \mathbb{R})$ denote the space of $\mathbb{R}$–valued càdlàg functions defined on $[0, +\infty)$. Throughout the paper, convergence in D means convergence in the Skorohod J_1 topology, see Chapter 3 of [1]. Also, D^k stands for the k-fold product equipped with the product topology.

Assumption 2.3. *Assume that there exist deterministic constants $\bar E(0), \bar I(0) \in [0, 1]$ such that $0 < \bar E(0) + \bar I(0) < 1$, and $(\bar E^N(0), \bar I^N(0)) \to (\bar E(0), \bar I(0)) \in \mathbb{R}^2_+$ in probability as $N \to \infty$.*

Theorem 2.1. *Under Assumptions 2.1, 2.2 and 2.3,*

$$\big(\bar S^N, \bar{\mathfrak{I}}^N, \bar E^N, \bar I^N, \bar R^N\big) \to \big(\bar S, \bar{\mathfrak{I}}, \bar E, \bar I, \bar R\big) \quad \text{in} \quad D^5 \quad \text{as} \quad N \to \infty, \tag{2.10}$$

in probability, locally uniformly in t. The limits $\bar S$ and $\bar{\mathfrak{I}}(t)$ are the unique solution of the following system of Volterra integral equations

$$\bar S(t) = 1 - \bar I(0) - \int_0^t \bar S(s) \bar{\mathfrak{I}}(s) ds\,, \tag{2.11}$$

$$\bar{\mathfrak{I}}(t) = \bar E(0) \bar\lambda^0(t) + \bar I(0) \bar\lambda^{0,I}(t) + \int_0^t \bar\lambda(t-s) \bar S(s) \bar{\mathfrak{I}}(s) ds\,, \tag{2.12}$$

and the limit $(\bar E, \bar I, \bar R)$ is given by the following integral equations:

$$\bar E(t) = \bar E(0) G_0^c(t) + \int_0^t G^c(t-s) \bar S(s) \bar{\mathfrak{I}}(s) ds\,, \tag{2.13}$$

$$\bar{I}(t) = \bar{I}(0)F^c_{0,I}(t) + \bar{E}(0)\Psi_0(t) + \int_0^t \Psi(t-s)\bar{S}(s)\bar{\mathfrak{I}}(s)ds\,, \tag{2.14}$$

$$\bar{R}(t) = \ \bar{I}(0)F_{0,I}(t) + \bar{E}(0)\Phi_0(t) + \int_0^t \Phi(t-s)\bar{S}(s)\bar{\mathfrak{I}}(s)ds\,. \tag{2.15}$$

The limit $\bar{S}$ is in C, and the limits $\bar{\mathfrak{I}}, \bar{E}, \bar{I}, \bar{R}$ are in D. If $\bar{\lambda}^0$ and $\bar{\lambda}^{0,I}$ are continuous, then $\bar{\mathfrak{I}}$ is in C, and if G_0 and $F_{0,I}$ are continuous, then $\bar{E}, \bar{I}, \bar{R}$ are in C.

Remark 2.2. *If we suppose only that Assumption 2.3 is valid, and* $\sup_{t\in[0,T]} \max\{\lambda^0(t), \lambda^{0,I}(t), \lambda(t)\} \leq \lambda^*$ *almost surely, then Theorem 2.1 remains valid, but with the convergence in probability in D^5 being replaced by the convergence in probability in $L^p_{loc}(0,+\infty;\mathbb{R}^5)$, for any $p \geq 1$.*

Note that in the case where $\zeta = \zeta^0 = 0$ a.s. (i.e., an infected individual is immediately infectious) and $E^N(0) = 0$, there is no exposed period, then we find the generalized SIR model, which reads

$$\begin{aligned}
\bar{S}(t) &= 1 - \bar{I}(0) - \int_0^t \bar{S}(s)\bar{\mathfrak{I}}(s)ds\,,\\
\bar{\mathfrak{I}}(t) &= \bar{I}(0)\bar{\lambda}^0(t) + \int_0^t \bar{\lambda}(t-s)\bar{S}(s)\bar{\mathfrak{I}}(s)ds\,,\\
\bar{I}(t) &= \bar{I}(0)F^c_{0,I}(t) + \int_0^t F^c(t-s)\bar{S}(s)\bar{\mathfrak{I}}(s)ds\,,\\
\bar{R}(t) &= \ \bar{I}(0)F_{0,I}(t) + \int_0^t F(t-s)\bar{S}(s)\bar{\mathfrak{I}}(s)ds\,,
\end{aligned}$$

where F is the c.d.f. of the infectious duration η of newly infected individuals and $F_{0,I}$ is the c.d.f. of the infectious duration $\eta^{0,I}$ of initially infectious individuals.

Remark 2.3. *The above result generalizes both our SIR and our SEIR FLLN results in* [15].

The SIR model in [15] *is the particular case of the present result, where $\lambda(t) = \lambda \mathbf{1}_{t<\eta}$, η being the random duration of the infectious period. In this case, $\bar{\lambda}(t) = \lambda F^c(t)$, if F is the c.d.f. of η, and $F^c = 1 - F$. Note that in this case $\bar{\mathfrak{I}}(t) = \lambda \bar{I}(t)$. Therefore, if we divide the $\bar{\mathfrak{I}}$ equation by λ, we find equation (2.14), which is also equation (2.4) in* [15]. *If we assume that the law of η is exponential, then we are in the case of the classical SIR model.*

The SEIR model in [15] *corresponds to the situation where $\lambda(t) = \lambda \mathbf{1}_{\zeta\leq t<\zeta+\eta}$, where ζ is the duration of the exposed period (the time when the individual is infected, but not yet infectious), and η is as above, while $\lambda^0(t) = \lambda \mathbf{1}_{\zeta^0\leq t<\zeta^0+\eta^0}$. Then $\bar{\lambda}(t) = \lambda[\mathbb{P}(\zeta \leq t) - \mathbb{P}(\zeta + \eta \leq t)] = \lambda\Psi(t)$. If we divide the $\bar{\mathfrak{I}}$ equation by λ, we find equation (2.14), which is also (3.15) in* [15]. *If moreover ζ and η are independent exponential random variables, then we are reduced to the classical SEIR model.*

Remark 2.4. *For the generalized SIS model, since $\bar{S}(t) = 1 - \bar{I}(t)$, it is clear that the epidemic dynamics in the FLLN is determined by the two–dimensional functions $\left(\bar{\mathfrak{I}}, \bar{I}\right)$ via the following integral equations:*

$$\begin{aligned}
\bar{\mathfrak{I}}(t) &= \bar{I}(0)\bar{\lambda}^{0,I}(t) + \int_0^t \bar{\lambda}(t-s)(1-\bar{I}(s))\bar{\mathfrak{I}}(s)ds\,,\\
\bar{I}(t) &= \bar{I}(0)F^c_{0,I}(t) + \int_0^t F^c(t-s)(1-\bar{I}(s))\bar{\mathfrak{I}}(s)ds\,.
\end{aligned}$$

Recall that as shown in Theorem 2.3 of [15]*, in the SIS with general infectious periods,* $\bar{\mathfrak{I}}(s) = \lambda \bar{I}(s)$*, and the epidemic dynamics is determined by the one–dimensional integral equation for* $\bar{I}$*.*

For the generalized SIRS model, the variables (ζ_i, η_i) *in our setup represent the infectious and recovered/immune periods of newly infected individuals, and similarly the variables* (ζ_j^0, η_j^0) *represent the infectious and immune periods of initially infectious individuals. We assume that there is no initially immune individuals. Let* I^N, R^N *be the processes counting infectious and recovered/immune individuals (corresponding to the notation* E^N *and* I^N *in the SEIR model). Of course, instead of* (2.1)*, the infectivity function* $\lambda(t)$ *should be positive only in the infectious periods* $[0, \zeta_i)$*. Similarly,* $\lambda_j^0(t)$ *should be positive only over* $[0, \zeta_j^0)$*. The definitions of the variables* (ζ_i, η_i)*,* (ζ_j^0, η_j^0) *in* (2.2) *and* (2.3) *also need to be modified accordingly in the natural way. The distribution functions* $G_0, F_{0,R}$ *are for initially infectious and immune periods, and* G, F *for newly infectious and immune periods, similarly for the notation* $\Psi, \Psi_0, \Phi, \Phi_0$*. Then the epidemic dynamics of the generalized SIRS model in the FLLN is determined by the three–dimensional functions* $\big(\bar{\mathfrak{I}}, \bar{I}, \bar{R}\big)$ *via the following integral equations:*

$$\bar{\mathfrak{I}}(t) = \bar{I}(0)\bar{\lambda}^0(t) + \int_0^t \bar{\lambda}(t-s)\big(1 - \bar{I}(s) - \bar{R}(s)\big)\bar{\mathfrak{I}}(s)ds\,,$$
$$\bar{I}(t) = \bar{I}(0)G_0^c(t) + \int_0^t G^c(t-s)\big(1 - \bar{I}(s) - \bar{R}(s)\big)\bar{\mathfrak{I}}(s)ds\,,$$
$$\bar{R}(t) = \bar{I}(0)\Psi_0(t) + \int_0^t \Psi(t-s)\big(1 - \bar{I}(s) - \bar{R}(s)\big)\bar{\mathfrak{I}}(s)ds\,.$$

Also recall that as shown in Theorem 3.3 of [15]*, in the SIRS model with general infectious and recovered periods,* $\bar{\mathfrak{I}}(s) = \lambda \bar{I}(s)$*, and the epidemic dynamics is determined by the two–dimensional integral equation for* $\big(\bar{I}, \bar{R}\big)$*.*

2.3. **Estimation of the parameters during the early phase of an epidemic.** In this subsection, we rewrite the infectivity function $\lambda_i(t)$ in the following form : $\mu \times \lambda_i(t)$. Here $\lambda_i(t)$ depends on each individual. We assume that its mean is known. The better the specific disease is known, the better this function $\bar{\lambda}(t)$ is known. It depends upon how the virus load evolves after infection on average. Now the factor μ is of a different nature. It is related to social aspects: it is different in densely population cities and in rural areas. It is affected by measures like lockdown, which have been used during the spring of 2020 by many countries, in order to control the Covid–19 epidemic. The factor μ is unknown. It is important to estimate μ, in order to be able to use the model for predictions. We assume that μ and (λ, λ_0) are independent, so that the mean of $\lambda(t)$ equals $\bar{\mu} \times \bar{\lambda}(t)$, where $\bar{\mu} = \mathbb{E}[\mu]$, and as above $\bar{\lambda}(t) = \mathbb{E}[\lambda(t)]$. Similarly for λ^0.

Now we simplify the formulation of our model. We do not change the model, but merge the two classes E and I into a unique class I of infected individuals, infected meaning either exposed or infectious. An individual who gets infected at time τ_i^N is exposed on the time interval $[\tau_i^N, \tau_i^N + \zeta_i)$, then infectious on the time interval $[\tau_i^N + \zeta_i, \tau_i^N + \zeta_i + \eta_i)$, then he/she moves into the compartment R at time $\tau_i^N + \zeta_i + \eta_i$. The model is the same as in the previous subsections, but we follow globally the infected individuals, and do not distinguish in the dynamic of the epidemic between exposed and infectious individuals. We denote by F (resp. F_0) the c.d.f. of the r.v. $\zeta + \eta$ (resp. of the r.v.

$\zeta^0 + \eta^0$). Hence our LLN limiting equations, rewritten with the new factor $\bar{\mu}$, become

$$\begin{aligned}
\bar{S}(t) &= 1 - \bar{I}(0) - \int_0^t \bar{S}(s)\bar{\mathfrak{I}}(s)ds\,,\\
\bar{\mathfrak{I}}(t) &= \bar{I}(0)\bar{\mu}\bar{\lambda}^0(t) + \bar{\mu}\int_0^t \bar{\lambda}(t-s)\bar{S}(s)\bar{\mathfrak{I}}(s)ds\,,\\
\bar{I}(t) &= \bar{I}(0)F_0^c(t) + \int_0^t F^c(t-s)\bar{S}(s)\bar{\mathfrak{I}}(s)ds\,,\\
\bar{R}(t) &= \bar{R}(0) + \bar{I}(0)F_0(t) + \int_0^t F(t-s)\bar{S}(s)\bar{\mathfrak{I}}(s)ds\,.
\end{aligned}$$

(Note that Theorem 2.1 can be easily extended to allow $\bar{R}(0) > 0$.) In order to avoid any ambiguity, note that $\bar{\mathfrak{I}}(t)$ is the limit of $\bar{\mathfrak{I}}^N(t) = N^{-1}\mathfrak{I}^N(t)$, where

$$\mathfrak{I}^N(t) = \sum_{j=1}^{I^N(0)} \mu\lambda_j^0(t) + \sum_{i=1}^{A^N(t)} \mu\lambda_i(t-\tau_i^N)\,.$$

We will now linearize the above system. This means that we shall consider a phase of the epidemic during which the proportion of susceptible individuals does not change significantly (since in particular we consider a large population). For simplicity, we assume that $S(t) \simeq 1$, but if we were considering a second wave of an epidemic where part of the population is immunized, or consider a situation where part of the population has been vaccinated, that constant (or frozen) proportion could be smaller than one. Next we multiply the remaining quantities by N, the total population size, so we suppress the bars (which in particular introduces a different notation for the linearized system). In other words, we consider the system

$$\begin{aligned}
\mathfrak{I}(t) &= I(0)\bar{\mu}\bar{\lambda}^0(t) + \bar{\mu}\int_0^t \bar{\lambda}(t-s)\mathfrak{I}(s)ds\,,\\
I(t) &= I(0)F_0^c(t) + \int_0^t F^c(t-s)\mathfrak{I}(s)ds\,,\\
R(t) &= R(0) + I(0)F_0(t) + \int_0^t F(t-s)\mathfrak{I}(s)ds\,.
\end{aligned} \tag{2.16}$$

The initial time and condition for the epidemic is typically unknown, so we assume that it has started some time in the past, and we look for a solution of the linear system

$$\begin{aligned}
\mathfrak{I}(t) &= \bar{\mu}\int_{-\infty}^t \bar{\lambda}(t-s)\mathfrak{I}(s)ds\,,\\
I(t) &= \int_{-\infty}^t F^c(t-s)\mathfrak{I}(s)ds\,,\\
R(t) &= \int_{-\infty}^t F(t-s)\mathfrak{I}(s)ds\,.
\end{aligned} \tag{2.17}$$

As is well known, the early phase of an epidemic is characterized by the fact that the above quantities increase at exponential speed (which is to be expected in case of a linear system; at least if they increase). Therefore we look for a solution of the form

$$\mathfrak{I}(t) = \boldsymbol{\iota}e^{\rho t}, \quad I(t) = \boldsymbol{i}e^{\rho t}, \quad R(t) = \boldsymbol{r}e^{\rho t}\,, \tag{2.18}$$

with $\boldsymbol{i} + \boldsymbol{r} = 1$. Note that this initial condition is very arbitrary, it means intuitively that we start with all individuals susceptible, except one.

In the next theorem, we will not need that Assumptions 2.1 and 2.2 are valid, but only that $\sup_{t\in[0,T]} \max\{\lambda^0(t), \lambda(t)\} \le \lambda^*$ almost surely for some finite constant λ^*, and in addition that $\int_0^\infty \bar\lambda(t)dt < \infty$.

Theorem 2.2. *If $\bar\mu > 0$ and $\rho > 0$ satisfy*

$$\bar\mu = \left(\int_0^\infty \bar\lambda(t)e^{-\rho t}dt\right)^{-1}, \tag{2.19}$$

and if we choose

$$\boldsymbol{\iota} = \rho, \quad \boldsymbol{i} = \mathbb{E}[1 - e^{-\rho(\zeta+\eta)}], \quad \boldsymbol{r} = \mathbb{E}[e^{-\rho(\zeta+\eta)}], \tag{2.20}$$

then $(\mathfrak{I}(t), I(t), R(t)) = (\boldsymbol{\iota}e^{\rho t}, \boldsymbol{i}e^{\rho t}, \boldsymbol{r}e^{\rho t})$ solves (2.17) for all $t \in \mathbb{R}$.

In addition, in case $\rho > 0$, if Θ denotes an independent exponential random variable with parameter ρ, then $(\mathfrak{I}(t), I(t), R(t)) = (\boldsymbol{\iota}e^{\rho t}, \boldsymbol{i}e^{\rho t}, \boldsymbol{r}e^{\rho t})$ solves (2.16) with

$$\begin{aligned} \bar\lambda^0(t) &= \mathbb{E}[\lambda(t+\Theta)|\Theta \le \zeta + \eta]\,, \\ F_0(t) &= \mathbb{P}(\Theta + t > \zeta + \eta|\Theta \le \zeta + \eta)\,. \end{aligned} \tag{2.21}$$

Moreover, we have

$$I(0) = \boldsymbol{i} = \mathbb{P}(\Theta \le \zeta + \eta), \quad R(0) = \boldsymbol{r} = \mathbb{P}(\Theta > \zeta + \eta)\,.$$

In the case of $\rho < 0$, provided $\mathbb{E}[e^{-\rho(\zeta+\eta)}] < \infty$, $(\mathfrak{I}(t), I(t)) = (\boldsymbol{\iota}e^{\rho t}, \boldsymbol{i}e^{\rho t})$ solves the first two lines of (2.17) for all $t \in \mathbb{R}$ if

$$\boldsymbol{\iota} = -\rho, \quad \boldsymbol{i} = \mathbb{E}[e^{-\rho(\zeta+\eta)} - 1].$$

Note that the relation (2.19) between $\bar\mu$ and ρ is valid for $\rho < 0$, which might happen, for example, during a lockdown period. On the other hand, since $t \to R(t)$ is increasing, the formula $R(t) = \boldsymbol{r}e^{\rho t}$ can be true only when $\rho > 0$.

Corollary 2.1. *The basic reproduction number is given by the formula*

$$R_0 = \frac{\int_0^\infty \bar\lambda(t)dt}{\int_0^\infty \bar\lambda(t)e^{-\rho t}dt} = \frac{\rho\int_0^\infty \bar\lambda(t)dt}{\mathbb{E}[\lambda(\Theta)]}\,, \tag{2.22}$$

where the second formula is valid only in the case $\rho > 0$, and in the case $\rho < 0$, $\mathbb{E}[e^{-\rho(\zeta+\eta)}] < \infty$ implies that $R_0 > 0$.

Remark 2.5. *Note that the exponent ρ is a quantity which is deduced from the observation of the epidemic (it is closely related to the "doubling time" of the number of cases). The above results give us $\bar\mu$ and R_0 in terms of ρ and the function $\bar\lambda(t)$. If $\lambda(t)$ is deterministic, then η is also deterministic and thus*

$$R_0 = \frac{\int_\zeta^{\zeta+\eta} \lambda(s)ds}{\int_\zeta^{\zeta+\eta} \lambda(s)e^{-\rho s}ds}.$$

If in addition $\bar\lambda(t) \equiv \lambda > 0$, then this simplifies to the well known result

$$R_0 = \frac{\rho\eta}{e^{-\rho\zeta}(1 - e^{-\rho\eta})}.$$

Remark 2.6. *Theorem 2.2 and its Corollary generalize Proposition 2 and Corollary 3 in* [7]*, in the case* $\lambda(t) = \lambda \mathbf{1}_{\zeta \le t < \zeta+\eta}$ *for some constant* $\lambda > 0$*, and the pair* (ζ, η) *is an arbitrary* $\mathbb{R}_+^2$*–valued random vector. In that case, our formulas for* R_0 *reduces to*

$$R_0 = \frac{\rho \mathbb{E}[\eta]}{\mathbb{E}[e^{-\rho\zeta}(1 - e^{-\rho\eta})]}\,.$$

In the particular case where ζ *and* η *are independent exponential random variables, with parameter* ν *and* γ*, the above formula becomes*

$$R_0 = \left(1 + \frac{\rho}{\nu}\right)\left(1 + \frac{\rho}{\gamma}\right)\,.$$

From this we deduce the formula in the classical SIR case by choosing $\nu = +\infty$*, i.e.,*

$$R_0 = 1 + \frac{\rho}{\gamma}.$$

2.4. **Application to the Covid–19 epidemic.** We now want to explain how the type of model described in this paper can be used to model the Covid–19 epidemic. As we shall see, the disadvantage, with respect to the classical "Markovian" models (where the infectivity is constant and fixed across the population, and the Exposed and Infectious periods follow an exponential distribution), of replacing a system of ODEs by a system of Volterra integral equations, is compensated by the benefits that the flexibility induced by the fact that the law of λ is arbitrary allows us to reduce the number of compartments in the model, so that we can replace a system of ODEs by a system of Volterra type equations of smaller dimension.

To be more specific, let us describe the SEIRU model of [13]. An individual who is infected is first "Exposed" E, then "Infectious" I. Soon after, the infectious individual either develops significant symptoms, and then will be soon "Reported" R, and isolated so that he/she does not infect any more; while the alternative is that this infectious individual is asymptomatic: he/she develops no or very mild symptoms, so remains "Unreported" U, and continues to infect susceptible individuals for a longer period. At any rate, any infectious individual will sooner or later become "Removed" Rem. In this model, there are 6 compartments: S like susceptible, E like exposed, I like infectious, R like reported, U like unreported, and Rem like removed.

Our approach allows us to have a more realistic version of this model with only 3 compartments (see Figure 1): S like susceptible, I like infected (first exposed, then infectious), R like removed (which includes the Reported individuals, since they do not infect any more, and will recover soon or later). We do not need to distinguish between the exposed and infectious, since the function λ is allowed to remain equal to zero during a certain time interval starting from the time of infection. More importantly, since the law of λ is allowed to be bimodal, we can accommodate in the same compartment I individuals who remain infectious for a short duration of time, and others who will remain infectious much longer (but probably with a lower infectivity). Moreover, since we know, see [9], that the infectivity decreases after a maximum which in the case of symptomatic individuals, seems to take place shortly before symptom onset, our varying infectivity model allows us to use a model corresponding to what the medical science tells us about this illness. Note that our version of the SEIRU model from [13] is the same as the one which we have already used in [7] (except

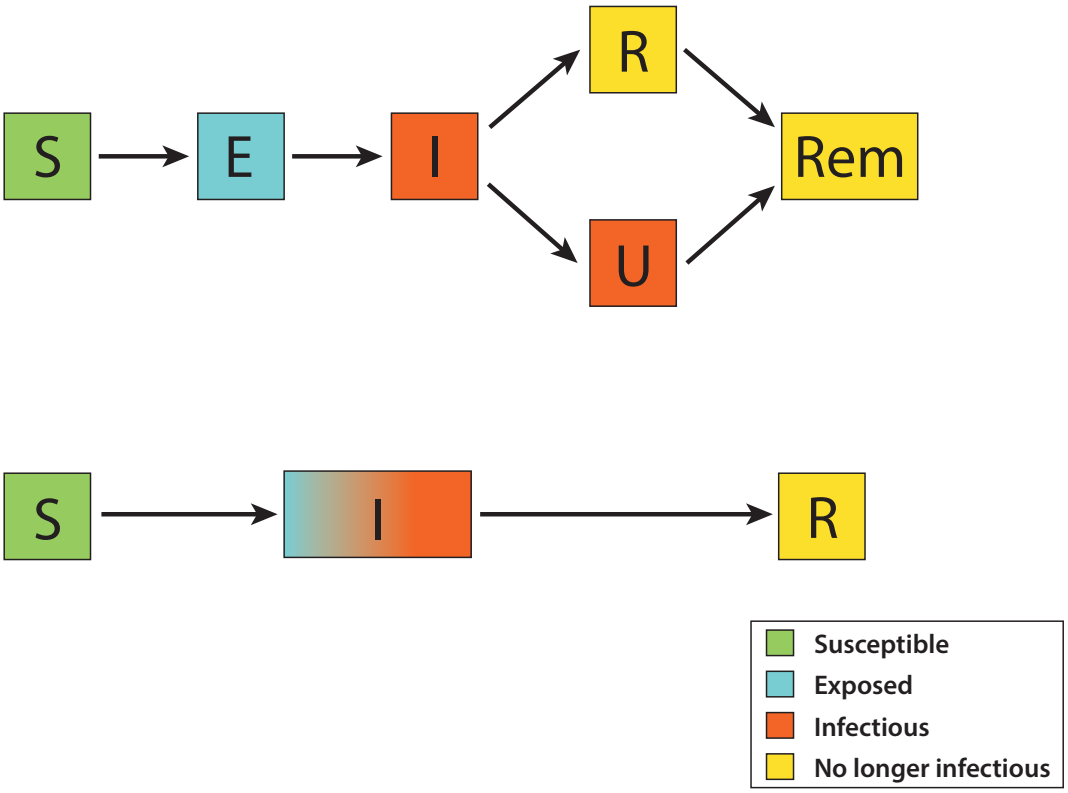

FIGURE 1. Flow chart of the SEIRU model of [13] and of our SIR model. We are able to replace the six compartments of the SEIRU model with only three compartments by using the equations described in Theorem 2.1.

that there we had to distinguish the E and the I compartments). However, the main novelty here is that the infectivity decreases after a maximum near the beginning of the infectious period.

Let us now compute R_0 in terms of the exponential growth rate in the early phase of the French Covid–19 epidemic. We adopt values for the various quantities which are compatible with the results described in [9].

We choose $\bar{\lambda}(t) = ae^{-0.08t}$, t being expressed in number of days. Since a simplifies in the computation of R_0, we choose w.l.o.g. $a = 1$.

We assume that the length of the exposed period is 3 days. For the reported individuals, the length of the infectious period is 3.5 days, while it equals 10 days for the unreported individuals. An infected individual is reported with probability p_R, and unreported with probability p_U. So $p_R + p_U = 1$. We expect that the unreported individuals are less infectious by an unknown factor $0 < \alpha \leq 1$. Consequently, we deduce from Corollary 2.1 that

$$R_0 = \frac{p_R \int_3^{6.5} e^{-0.08t} dt + \alpha p_U \int_3^{13} e^{-0.08t} dt}{p_R \int_3^{6.5} e^{-(0.08+\rho)t} dt + \alpha p_U \int_3^{13} e^{-(0.08+\rho)t} dt}\,.$$

We compute this quantity with $\rho = 0.27$, which is our estimate of the exponential growth rate of the epidemic in France prior to lockdown, see [7], which yields

$$R_0 = 4.375 \times \frac{0.192 p_R + 0.433 \alpha p_U}{0.247 p_R + 0.339 \alpha p_U}\,.$$

We obtain the following values for R_0, for different values of the unknown parameters p_R and α.

	$p_R = 0.8$	$p_R = 0.6$	$p_R = 0.4$	$p_R = 0.2$
$\alpha = 1$	$R_0 = 3.96$	$R_0 = 4.45$	$R_0 = 4.87$	$R_0 = 5.25$
$\alpha = 0.75$	$R_0 = 3.85$	$R_0 = 4.29$	$R_0 = 4.73$	$R_0 = 5.16$
$\alpha = 0.5$	$R_0 = 3.72$	$R_0 = 4.09$	$R_0 = 4.51$	$R_0 = 5.00$

We note that R_0 is increasing with p_U and with α. We also note that with the same durations of the exposed and infectious periods, but with λ constant, R_0 would be larger (see [7]), which is not surprising, since in the present model the decrease of $\bar\lambda(t)$ reduces the effect of the factor $e^{-\rho t}$ in the integrals in the denominator, which makes $R_0 > 1$ for $\rho > 0$.

3. Proof of Theorem 2.2 and Corollary 2.1

The Corollary follows readily from the Theorem and the fact that, from its definition,

$$R_0 = \bar\mu \int_0^\infty \bar\lambda(t)dt\,.$$

Let us now prove the Theorem. We deduce from the first equation in (2.17) that necessarily

$$\bar\mu = \left(\int_0^\infty \bar\lambda(t)e^{-\rho t}dt\right)^{-1}\,.$$

We first consider the case $\rho > 0$. We note that

$$I(t) = \boldsymbol{i}e^{\rho t} = \boldsymbol{\iota}\int_{-\infty}^t F^c(t-s)e^{\rho s}ds\,,$$

which implies

$$\boldsymbol{i} = \boldsymbol{\iota}\int_0^\infty F^c(t)e^{-\rho t}dt = \boldsymbol{\iota}\mathbb{E}\int_0^{\zeta+\eta} e^{-\rho t}dt = \frac{\boldsymbol{\iota}}{\rho}\mathbb{E}[1-e^{-\rho(\zeta+\eta)}]\,. \tag{3.1}$$

In the same way, we find

$$\boldsymbol{r} = \boldsymbol{\iota}\int_0^{+\infty} F(t)e^{-\rho t}dt = \frac{\boldsymbol{\iota}}{\rho}\mathbb{E}\left[e^{-\rho\,(\zeta+\eta)}\right]\,.$$

Together with the condition $\boldsymbol{i}+\boldsymbol{r} = 1$, this yields (2.20).

Now plugging the expressions in (2.17) into the left hand sides of the equations in (2.16), we obtain that

$$\begin{aligned} I(0)\bar\lambda^0(t) &= \int_{-\infty}^0 \bar\lambda(t-s)\mathfrak{I}(s)ds\,,\\ I(0)F_0^c(t) &= \int_{-\infty}^0 F^c(t-s)\mathfrak{I}(s)ds\,,\\ R(0)+I(0)F_0(t) &= \int_{-\infty}^0 F(t-s)\mathfrak{I}(s)ds\,. \end{aligned} \tag{3.2}$$

Substituting $I(0) = \boldsymbol{i}$ and $\mathfrak{I}(s) = \rho e^{\rho s}$ in the first equation, we obtain

$$\boldsymbol{i}\bar\lambda^0(t) = \int_{-\infty}^0 \bar\lambda(t-s)\rho e^{\rho s}ds = \int_0^\infty \bar\lambda(t+s)\rho e^{-\rho s}ds = \mathbb{E}[\bar\lambda(t+\Theta)]\,,$$

that is

$$\bar\lambda^0(t) = \frac{\mathbb{E}[\bar\lambda(t+\Theta);\Theta+t\le\zeta+\eta]}{\mathbb{P}(\Theta\le\zeta+\eta)}\,,$$

which gives the first line of (2.21). Substituting $I(0) = \boldsymbol{i}$ and $\mathfrak{I}(s) = \rho e^{\rho s}$ in the second equation, we obtain

$$\boldsymbol{i}F_0^c(t) = \rho \int_0^\infty F^c(t+s)e^{-\rho s}ds = \mathbb{P}(\Theta + t \le \zeta + \eta)\,,$$

that is

$$F_0^c(t) = \mathbb{P}(\Theta + t \le \zeta + \eta | \Theta \le \zeta + \eta)\,,$$

which implies the second line of (2.21).

When $\rho < 0$, we can check that (3.1) holds with $\boldsymbol{\iota} = -\rho$. This completes the proof.

4. Proof of the FLLN

4.1. **Convergence of $(\bar{S}^N, \bar{\mathfrak{I}}^N)$.** For the process $A^N(t)$, we have the decomposition

$$A^N(t) = M_A^N(t) + \int_0^t \Upsilon^N(s)ds, \tag{4.1}$$

where

$$M_A^N(t) = \int_0^t \int_0^\infty \mathbf{1}_{u \le \Upsilon^N(s)} \overline{Q}(ds, du),$$

with $\overline{Q}(ds, du) = Q(ds, du) - dsdu$ being the compensated PRM. It is clear that the process $\{M_A^N(t) : t \ge 0\}$ is a square-integrable martingale (see, e.g., [6, Chapter VI]) with respect to the filtration $\{\mathcal{F}_t^N : t \ge 0\}$ defined by

$$\mathcal{F}_t^N := \sigma\Big\{E^N(0), I^N(0), \{\lambda_j^0(\cdot)\}_{j\ge1}, \{\lambda_k^{0,I}(\cdot)\}_{k\ge1}, \{\lambda_i(\cdot)\}_{i\ge1}, \int_0^{t'}\int_0^\infty \mathbf{1}_{u\le\Upsilon^N(s)}Q(ds,du) : t' \le t\Big\}.$$

It has a finite quadratic variation

$$\langle M_A^N\rangle(t) = \int_0^t \Upsilon^N(s)ds, \quad t \ge 0.$$

Under Assumption 2.2, we have

$$0 \le N^{-1}\int_s^t \Upsilon^N(u)du \le \lambda^*(t-s), \quad \text{w.p. } 1 \quad \text{for} \quad 0 \le s \le t. \tag{4.2}$$

Thus, this implies that, in probability as $N \to \infty$,

$$\langle \bar{M}_A^N\rangle(t) = N^{-2}\int_0^t \Upsilon^N(s)ds \to 0 \quad \text{in} \quad D,$$

and by the FCLT for martingales (see, e.g., [19]),

$$\bar{M}_A^N \to 0 \quad \text{in} \quad D. \tag{4.3}$$

As a consequence, we obtain the following lemma.

Lemma 4.1. *Under Assumptions 2.2 and 2.3, the processes $\{(\bar{A}^N, \bar{S}^N) : N \in \mathbb{N}\}$ are tight in D^2. The limit of each convergence subsequence of $\{\bar{A}^N\}$, denoted by $\bar{A}$, satisfies*

$$\bar{A} = \lim_{N\to\infty} \bar{A}^N = \lim_{N\to\infty} N^{-1}\int_0^\cdot \Upsilon^N(u)du, \tag{4.4}$$

and

$$0 \le \bar{A}(t) - \bar{A}(s) \le \lambda^*(t-s), \quad \textit{w.p. 1} \quad \textit{for} \quad 0 \le s \le t. \tag{4.5}$$

Given the limit $\bar{A}$ of a consequent subsequence of $\{\bar{A}^N\}$, $\bar{S}^N \to \bar{S} = \bar{S}(0) - \bar{A} = 1 - \bar{I}(0) - \bar{A}$ in D in probability as $N \to \infty$.

Let

$$\bar{\mathfrak{I}}^N_{0,1}(t) := N^{-1} \sum_{k=1}^{I^N(0)} \lambda^{0,I}_k(t), \quad \bar{\mathfrak{I}}^N_{0,2}(t) := N^{-1} \sum_{j=1}^{E^N(0)} \lambda^0_j(t), \quad t \geq 0.$$

Lemma 4.2. *Under Assumptions 2.2 and 2.3,*

$$\left(\bar{\mathfrak{I}}^N_{0,1}, \bar{\mathfrak{I}}^N_{0,2}\right) \to \left(\bar{\mathfrak{I}}_{0,1}, \bar{\mathfrak{I}}_{0,2}\right) \quad \text{in} \quad D^2 \quad \text{as} \quad N \to \infty, \tag{4.6}$$

in probability, where

$$\bar{\mathfrak{I}}_{0,1}(t) := \bar{I}(0)\bar{\lambda}^{0,I}(t), \quad \bar{\mathfrak{I}}_{0,2}(t) := \bar{E}(0)\bar{\lambda}^0(t), \quad t \geq 0.$$

Proof. Define the processes

$$\tilde{\mathfrak{I}}^N_{0,1}(t) := N^{-1} \sum_{k=1}^{N\bar{I}(0)} \lambda^{0,I}_k(t), \quad \tilde{\mathfrak{I}}^N_{0,2}(t) := N^{-1} \sum_{j=1}^{N\bar{E}(0)} \lambda^0_j(t), \quad t \geq 0. \tag{4.7}$$

By the i.i.d. assumptions for the sequences $\{\lambda^0_j(t)\}$ and $\{\lambda^{0,I}_k(t)\}$, and their independence, and by the LLN for random elements in D (see Theorem 1 in [17]), we directly obtain the joint convergence in probability

$$\left(\tilde{\mathfrak{I}}^N_{0,1}, \tilde{\mathfrak{I}}^N_{0,2}\right) \to \left(\bar{\mathfrak{I}}_{0,1}, \bar{\mathfrak{I}}_{0,2}\right) \quad \text{in} \quad D^2 \quad \text{as} \quad N \to \infty.$$

It then suffices to show that in probability

$$\left(\tilde{\mathfrak{I}}^N_{0,1} - \bar{\mathfrak{I}}^N_{0,1}, \tilde{\mathfrak{I}}^N_{0,2} - \bar{\mathfrak{I}}^N_{0,2}\right) \to 0 \quad \text{in} \quad D^2 \quad \text{as} \quad N \to \infty. \tag{4.8}$$

We have

$$\tilde{\mathfrak{I}}^N_{0,1} - \bar{\mathfrak{I}}^N_{0,1} = \operatorname{sign}(\bar{I}(0) - \bar{I}^N(0)) N^{-1} \sum_{k=N(\bar{I}^N(0)\wedge\bar{I}(0))}^{N(\bar{I}^N(0)\vee\bar{I}(0))} \lambda^{0,I}_k(t),$$

and thus,

$$\mathbb{E}\left[N^{-1} \sum_{k=N(\bar{I}^N(0)\wedge\bar{I}(0))}^{k=N(\bar{I}^N(0)\vee\bar{I}(0))} \lambda^{0,I}_k(t) \middle| \mathcal{F}^N_0 \right] \leq \lambda^{0,I}_k(t) \left| \bar{I}^N(0) - \bar{I}(0) \right|.$$

By the almost sure boundedness of $\lambda^{0,I}_k(t)$ in Assumption 2.2, and by the convergence $\bar{I}^N(0) - \bar{I}(0) \to 0$ in probability under Assumption 2.3, we obtain that $\lambda^{0,I}_k(t)\left|\bar{I}^N(0) - \bar{I}(0)\right| \to 0$ in probability. Thus we have shown that $\tilde{\mathfrak{I}}^N_{0,1} - \bar{\mathfrak{I}}^N_{0,1} \to 0$ in probability. Similarly for the convergence $\tilde{\mathfrak{I}}^N_{0,2} - \bar{\mathfrak{I}}^N_{0,2} \to 0$ in probability. This completes the proof. □

Let

$$\bar{\mathfrak{I}}^N_1(t) := N^{-1} \sum_{i=1}^{A^N(t)} \lambda_i(t - \tau^N_i), \quad t \geq 0.$$

Before we prove the convergence of $\bar{\mathfrak{I}}^N_1 \to \bar{\mathfrak{I}}_1$ in D, let us first establish a technical result which will be useful in the next proof.

Lemma 4.3. *Let $\{X^N\}_{N\geq 1}$ be a sequence of random elements in D. If the two conditions*

(i) *for all $\epsilon>0$, $\sup_{0\leq t\leq T}\mathbb{P}\big(|X^N(t)|>\epsilon\big)\to 0$, as $N\to\infty$, and*

(ii) *for all $\epsilon>0$, $\limsup_N \sup_{0\leq t\leq T}\frac{1}{\delta}\mathbb{P}\big(\sup_{0\leq u\leq\delta}|X^N(t+u)-X^N(t)|>\epsilon\big)\to 0$, as $\delta\to 0$*

are satisfied, then $X^N(t)\to 0$ in probability uniformly in t.

Proof. The Lemma is a direct consequence of the inequality (4.21) in [15], which we repeat here for the reader's convenience:

$$\mathbb{P}\left(\sup_{0\leq t\leq T}|X^N(t)|>\varepsilon\right)\leq\frac{T}{\delta}\sup_{0\leq t\leq T}\mathbb{P}(|X^N(t)|>\varepsilon/2)$$
$$+\frac{T}{\delta}\sup_{0\leq t\leq T}\mathbb{P}\left(\sup_{0\leq u\leq\delta}|X^N(t+u)-X^N(t)|>\epsilon/2\right).$$

□

Lemma 4.4. *Under Assumptions 2.1 and 2.2, if $\bar{A}$ is the limit of the converging subsequence of $\{\bar{A}^N\}$,*

$$\bar{\mathfrak{I}}_1^N\to\bar{\mathfrak{I}}_1\quad in\quad D\quad as\quad N\to\infty, \tag{4.9}$$

in probability, where

$$\bar{\mathfrak{I}}_1(t):=\int_0^t\bar{\lambda}(t-s)d\bar{A}(s),\quad t\geq 0.$$

Proof. Let

$$\breve{\mathfrak{I}}_1^N(t):=N^{-1}\sum_{i=1}^{A^N(t)}\bar{\lambda}(t-\tau_i^N)=\int_0^t\bar{\lambda}(t-s)d\bar{A}^N(s),\quad t\geq 0. \tag{4.10}$$

Under Assumption 2.2, applying the continuous mapping theorem, we obtain that in probability,

$$\breve{\mathfrak{I}}_1^N\to\bar{\mathfrak{I}}_1\quad\text{in}\quad D\quad\text{as}\quad N\to\infty. \tag{4.11}$$

Then it suffices to show that in probability

$$V^N:=\bar{\mathfrak{I}}_1^N-\breve{\mathfrak{I}}_1^N\to 0\quad\text{in}\quad D\quad\text{as}\quad N\to\infty. \tag{4.12}$$

We have

$$V^N(t)=N^{-1}\sum_{i=1}^{A^N(t)}\chi_i^N(t),\quad \chi_i^N(t):=\lambda_i(t-\tau_i^N)-\bar{\lambda}(t-\tau_i^N).$$

$\chi_i^N(t)$ clearly satisfies $\mathbb{E}\big[\chi_i^N(t)\big]=0$ and $\mathbb{E}\big[\chi_i^N(t)\chi_j^N(t)\big]=0$. Thus,

$$\mathbb{E}\big[V^N(t)^2\big]=N^{-2}\mathbb{E}\bigg[\sum_{i=1}^{A^N(t)}\nu(t-\tau_i^N)\bigg]=N^{-1}\mathbb{E}\bigg[\int_0^t\nu(t-s)d\bar{A}^N(s)\bigg],$$

where $\nu(t):=E[(\lambda_i(t)-\bar{\lambda}(t))^2]$ and $\nu(t)<\infty$ under Assumption 2.2. We easily obtain that for each $t\geq 0$,

$$V^N(t)\to 0\ \text{ in probability,}\quad\text{as}\quad N\to\infty\,.$$

It remains to establish condition (ii) of Lemma 4.3, i.e., that for any $\epsilon > 0$,

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\left[\frac{T}{\delta}\right]\sup_{t\in[0,T]}\mathbb{P}\left(\sup_{u\in[0,\delta]}\left|V^N(t+u)-V^N(t)\right|>\epsilon\right)=0. \tag{4.13}$$

We have for $t, u \geq 0$,

$$\begin{aligned}\left|V^N(t+u)-V^N(t)\right| &\leq \left|N^{-1}\sum_{i=1}^{A^N(t)}\left(\lambda_i(t+u-\tau_i^N)-\lambda_i(t-\tau_i^N)\right)\right| \\ &\quad+\left|N^{-1}\sum_{i=1}^{A^N(t)}\left(\bar\lambda(t+u-\tau_i^N)-\bar\lambda(t-\tau_i^N)\right)\right| \\ &\quad+\left|N^{-1}\sum_{i=A^N(t)}^{A^N(t+u)}\left(\lambda_i(t+u-\tau_i^N)-\bar\lambda(t+u-\tau_i^N)\right)\right| \\ &= \Delta_{t,u}^{1,N}+\Delta_{t,u}^{2,N}+\Delta_{t,u}^{3,N}.\end{aligned}$$

We first note that by (4.2),

$$\begin{aligned}\sup_{0\leq u\leq\delta}\Delta_{t,u}^{N,3} &\leq \lambda^*\left(\bar A^N(t+\delta)-\bar A^N(t)\right) \\ &\leq (\lambda^*)^2\delta+\lambda^*\left(\bar M_A^N(t+\delta)-\bar M_A^N(t)\right),\end{aligned}$$

so that by (4.3), for any $T>0$, $\epsilon>0$, provided $\delta<\varepsilon/(6(\lambda^*)^2)$,

$$\begin{aligned}\mathbb{P}\left(\sup_{0\leq u\leq\delta}\Delta_{t,u}^{N,3}>\epsilon/3\right) &\leq \mathbb{P}\left(\left|\bar M_A^N(t+\delta)-\bar M_A^N(t)\right|>\varepsilon/\lambda^*\right) \\ &\to 0, \quad \text{as } N\to\infty,\end{aligned}$$

and consequently,

$$\limsup_{N\to\infty}\left[\frac{T}{\delta}\right]\sup_{t\in[0,T]}\mathbb{P}\left(\sup_{u\in[0,\delta]}\left|\Delta_{t,u}^{N,3}\right|>\epsilon\right)=0. \tag{4.14}$$

We now consider the first term

$$\begin{aligned}\Delta_{t,u}^{N,1} &\leq N^{-1}\sum_{i=1}^{A^N(t)}\sum_{j=1}^{k}|\lambda_i^j(t+u-\tau_i^N)-\lambda_i^j(t-\tau_i^N)|\mathbf{1}_{\xi_i^{j-1}\leq t-\tau_i^N<t+u-\tau_i^N<\xi_i^j} \\ &\quad+\lambda^*N^{-1}\sum_{i=1}^{A^N(t)}\sum_{j=1}^{k}\mathbf{1}_{t-\tau_i^N\leq\xi_i^j<t+u-\tau_i^N} \\ &\leq \varphi(u)\bar A^N(t)+\lambda^*\sum_{j=1}^{k}N^{-1}\sum_{i=1}^{A^N(t)}\mathbf{1}_{t-\tau_i^N\leq\xi_i^j<t+u-\tau_i^N}.\end{aligned}$$

The right hand side being nondecreasing in u, we deduce that

$$\sup_{0\leq u\leq\delta}\Delta_{t,u}^{N,1}\leq\varphi(\delta)\lambda^*T+\varphi(\delta)\bar M_A^N(t)+\lambda^*\sum_{j=1}^{k-1}N^{-1}\sum_{i=1}^{A^N(t)}\mathbf{1}_{t-\tau_i^N\leq\xi_i^j<t+\delta-\tau_i^N}.$$

So, provided $\delta > 0$ is small enough such that $\varphi(\delta) < \varepsilon/(9\lambda^* T)$,

$$\mathbb{P}\left(\sup_{0\le u\le\delta}\Delta^{N,1}_{t,u} > \epsilon/3\right) \le \mathbb{P}\left(\varphi(\delta)\bar{M}^N_A(t) > \epsilon/9\right) + \mathbb{P}\left(\lambda^*\sum_{j=1}^{k-1} N^{-1}\sum_{i=1}^{A^N(t)} \mathbf{1}_{t-\tau^N_i\le\xi^j_i<t+\delta-\tau^N_i} > \epsilon/9\right).$$

The first term tends to 0 as $N\to\infty$ thanks to (4.3), and the second term is bounded by

$$\begin{aligned}
&\frac{9^2}{\epsilon^2}\mathbb{E}\left[\left(\lambda^*\sum_{j=1}^{k-1} N^{-1}\sum_{i=1}^{A^N(t)} \mathbf{1}_{t-\tau^N_i\le\xi^j_i<t+\delta-\tau^N_i}\right)^2\right]\\
&\le \frac{2\times 9^2}{\epsilon^2}\mathbb{E}\left[\left(\lambda^*\sum_{j=1}^{k-1} N^{-1}\int_0^t\int_0^\infty\int_{t-s}^{t+\delta-s} \mathbf{1}_{u\le\Upsilon^N(s)}\overline{Q}_j(ds,du,d\xi)\right)^2\right]\\
&\quad+\frac{2\times 9^2}{\epsilon^2}\mathbb{E}\left[\left(\lambda^*\sum_{j=1}^{k-1} N^{-1}\int_0^t \big(F_j(t+\delta-s)-F_j(t-s)\big)\Upsilon^N(s)ds\right)^2\right],
\end{aligned}$$

where $Q_j(ds,du,d\xi)$ is a PRM on $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+$ with mean measure $dsduF_j(d\xi)$, and $\overline{Q}_j(ds,du,d\xi)$ is the corresponding compensated PRM. Observe that

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t\int_0^\infty\int_{t-s}^{t+\delta-s} \mathbf{1}_{u\le\Upsilon^N(s)}\overline{Q}_j(ds,du,d\xi)\right)^2\right]\\
&= N^{-2}\mathbb{E}\left[\int_0^t \big(F_j(t+\delta-s)-F_j(t-s)\big)\Upsilon^N(s)ds\right]\\
&\le N^{-1}\lambda^*\int_0^t \big(F_j(t+\delta-s)-F_j(t-s)\big)ds
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\left(N^{-1}\int_0^t \big(F_j(t+\delta-s)-F_j(t-s)\big)\Upsilon^N(s)ds\right)^2\right]\\
&\le\left(\lambda^*\int_0^t \big(F_j(t+\delta-s)-F_j(t-s)\big)ds\right)^2. \qquad (4.15)
\end{aligned}$$

The first term tends to 0 as $N\to\infty$. The second term is treated as in the proof of Lemma 4.2 in [15], since F_j satisfies the conditions in Assumption 2.1. We deduce that

$$\limsup_{N\to\infty}\left[\frac{T}{\delta}\right]\sup_{t\in[0,T]}\mathbb{P}\left(\sup_{u\in[0,\delta]}\left|\Delta^{N,1}_{t,u}\right|>\epsilon\right)\to 0,\quad\text{as }\delta\to 0. \qquad (4.16)$$

We next consider $\Delta^{N,2}_{t,u}$. We have

$$\begin{aligned}
&\left|N^{-1}\sum_{i=1}^{A^N(t)}\big(\bar\lambda(t+u-\tau^N_i)-\bar\lambda(t-\tau^N_i)\big)\right|\\
&=\Bigg|\sum_{j=1}^{k}\int_0^t\mathbb{E}\Big\{\lambda^j(t+u-s)(\mathbf{1}_{\xi^{j-1}\le t+u-s}-\mathbf{1}_{\xi^j\le t+u-s})\\
&\qquad\qquad-\lambda^j(t-s)(\mathbf{1}_{\xi^{j-1}\le t-s}-\mathbf{1}_{\xi^j\le t-s})\Big\}d\bar A^N(s)\Bigg|
\end{aligned}$$

$$= \Big| \sum_{j=1}^{k} \int_0^t \mathbb{E}\Big\{ [\lambda^j(t+u-s) - \lambda^j(t-s)](\mathbf{1}_{\xi^{j-1} \le t+u-s} - \mathbf{1}_{\xi^j \le t+u-s}) \\ + \lambda^j(t-s)(\mathbf{1}_{\xi^{j-1} \le t+u-s} - \mathbf{1}_{\xi^{j-1} \le t-s} - \mathbf{1}_{\xi^j \le t+u-s} + \mathbf{1}_{\xi^j \le t-s}) \Big\} d\bar{A}^N(s) \Big|$$

$$\le \varphi(u)\bar{A}^N(t) + \lambda^* \sum_{j=1}^{k} \int_0^t [F_j(t+u-s) - F_j(t-s)] d\bar{A}^N(s)\,.$$

The right hand side being nondecreasing in u, we deduce that

$$\sup_{0 \le u \le \delta} \Delta^{2,N}_{t,u} \le \varphi(\delta)\bar{A}^N(t) + \lambda^* \sum_{j=1}^{k} \int_0^t [F_j(t+\delta-s) - F_j(t-s)] d\bar{A}^N(s).$$

The first term on the right is the same as the one which appeared in the upper bound of $\Delta^{N,1}_{t,u}$. We need only consider the second term. We note that

$$\int_0^t [F_j(t+\delta-s) - F_j(t-s)] d\bar{A}^N(s) = \int_0^t [F_j(t+\delta-s) - F_j(t-s)] d\bar{M}^N_A(s) \\ + \int_0^t [F_j(t+\delta-s) - F_j(t-s)] \bar{\Upsilon}^N(s) ds\,.$$

Now,

$$\mathbb{E}\left[\left(\int_0^t [F_j(t+\delta-s) - F_j(t-s)] d\bar{A}^N(s) \right)^2 \right] \\ \le 2\mathbb{E}\left[\left(\bar{M}^N_A(t) \right)^2 \right] + 2\mathbb{E}\left[\left(\int_0^t [F_j(t+\delta-s) - F_j(t-s)] \bar{\Upsilon}^N(s) ds \right)^2 \right].$$

The first term on the right tends to 0 as $N \to \infty$. The second term can be bounded by $2(\lambda^*)^2$ times

$$\left(\int_0^t [F_j(t+\delta-s) - F_j(t-s)] ds \right)^2.$$

which is again the bound in (4.15). The same argument again applies. Hence (4.16) holds with $\Delta^{N,1}_{t,u}$ replaced by $\Delta^{N,2}_{t,u}$. Therefore, we have proved (4.13). This completes the proof of the lemma. □

By Lemmas 4.1 and 4.4, we obtain in probability,

$$\int_0^{\cdot} \bar{\Upsilon}^N(s) ds = \int_0^{\cdot} \bar{S}^N(s)\bar{\mathfrak{I}}^N(s) ds \to \int_0^{\cdot} \bar{S}(s)\bar{\mathfrak{I}}(s) ds \quad \text{in} \quad D \quad \text{as} \quad N \to \infty. \tag{4.17}$$

This implies that in probability,

$$\bar{A}^N \to \bar{A} = \int_0^{\cdot} \bar{S}(s)\bar{\mathfrak{I}}(s) ds \quad \text{in} \quad D \quad \text{as} \quad N \to \infty. \tag{4.18}$$

Therefore, the limits $(\bar{S}, \bar{\mathfrak{I}})$ satisfy the integral equations (2.11) and (2.12) in Theorem 2.1. Finally, the existence and uniqueness of solution to the integral equations follow from applying Gronwall's inequality in a straightforward way, and the whole sequence converges. This completes the proof of the joint convergence of $(\bar{S}^N, \bar{\mathfrak{I}}^N) \to (\bar{S}, \bar{\mathfrak{I}})$ in D^2 in probability.

4.2. **Convergence of $(\bar{E}^N, \bar{I}^N, \bar{R}^N)$.** The proof for the convergence of $(\bar{E}^N, \bar{I}^N, \bar{R}^N)$ follows essentially the same argument as in Section 6 of [15]. We highlight the key steps and differences below.

For the initially exposed and infectious individuals, let

$$\bar{E}^N_0(t) := N^{-1} \sum_{j=1}^{E^N(0)} \mathbf{1}_{\zeta^0_j > t}, \quad \bar{I}^N_{0,1}(t) := N^{-1} \sum_{k=1}^{I^N(0)} \mathbf{1}_{\eta^{0,I}_k > t}, \quad \bar{I}^N_{0,2}(t) := N^{-1} \sum_{j=1}^{E^N(0)} \mathbf{1}_{\zeta^0_j + \eta^0_j > t},$$

$$\bar{R}^N_{0,1}(t) := N^{-1} \sum_{k=1}^{I^N(0)} \mathbf{1}_{\eta^{0,I}_k \le t}, \quad \bar{R}^N_{0,2}(t) := N^{-1} \sum_{j=1}^{E^N(0)} \mathbf{1}_{\zeta^0_j + \eta^0_j \le t}.$$

By the FLLN for empirical processes, we obtain the following lemma.

Lemma 4.5. *Under Assumptions 2.2 and 2.3,*

$$\big(\bar{E}^N_0, \bar{I}^N_{0,1}, \bar{I}^N_{0,2}, \bar{R}^N_{0,1}, \bar{R}^N_{0,2}\big) \to \big(\bar{E}_0, \bar{I}_{0,1}, \bar{I}_{0,2}, \bar{R}_{0,1}, \bar{R}_{0,2}\big) \quad \text{in} \quad D^5 \quad \text{as} \quad N \to \infty, \tag{4.19}$$

in probability, jointly with the convergence $\big(\bar{\mathfrak{I}}^N_{0,1}, \bar{\mathfrak{I}}^N_{0,2}\big) \to \big(\bar{\mathfrak{I}}_{0,1}, \bar{\mathfrak{I}}_{0,2}\big)$ *in (4.6), where*

$$\bar{E}_0(t) = \bar{E}(0) G^c_0(t), \quad \bar{I}_{0,1}(t) = \bar{I}(0) F^c_{0,I}(t), \quad \bar{I}_{0,2}(t) = \bar{E}(0) \Psi_0(t),$$
$$\bar{R}_{0,1}(t) = I(0) F_{0,I}(t), \quad \bar{R}_{0,2}(t) = \bar{E}(0) \Phi_0(t).$$

Proof. Recall the definition of $\big(\tilde{\mathfrak{I}}^N_{0,1}, \tilde{\mathfrak{I}}^N_{0,2}\big)$ in (4.7). Similarly, define $\big(\tilde{E}^N_0, \tilde{I}^N_{0,1}, \tilde{I}^N_{0,2}, \tilde{R}^N_{0,1}, \tilde{R}^N_{0,2}\big)$ by replacing $E^N(0)$ and $I^N(0)$ with$N\bar{E}(0)$ and $N\bar{I}(0)$, respectively, in the definitions of $\big(\bar{E}^N_0, \bar{I}^N_{0,1}, \bar{I}^N_{0,2}, \bar{R}^N_{0,1}, \bar{R}^N_{0,2}\big)$. To prove the joint convergence of these newly defined processes, because of the independence of the sequences $\{\lambda^0_j\}_{j\ge1}$ and $\{\lambda^{0,I}_k\}_{k\ge1}$, it suffices to show the joint convergence of $\big(\tilde{\mathfrak{I}}^N_{0,1}, \tilde{I}^N_{0,1}, \tilde{R}^N_{0,1}\big)$ and $\big(\tilde{\mathfrak{I}}^N_{0,2}, \tilde{E}^N_0, \tilde{I}^N_{0,2}, \tilde{R}^N_{0,2}\big)$, separately. By the FLLN for empirical processes, and by the i.i.d. assumption of $\{\lambda^{0,I}_k\}_{k\ge1}$ and the definition of $\eta^{0,I}_k$ from $\lambda^{0,I}_k$ in (2.4), we obtain the joint convergence in probability

$$\big(\tilde{\mathfrak{I}}^N_{0,1}, \tilde{I}^N_{0,1}, \tilde{R}^N_{0,1}\big) \to \big(\bar{\mathfrak{I}}_{0,1}, \bar{I}_{0,1}, \bar{R}_{0,1}\big) \quad \text{in} \quad D^3 \quad \text{as} \quad N \to \infty.$$

Similarly, by he i.i.d. assumption of $\{\lambda^0_j\}_{j\ge1}$ and the definition of (ζ^0_j, η^0_j) from λ^0_j in (2.3), we obtain the joint convergence in probability

$$\big(\tilde{\mathfrak{I}}^N_{0,2}, \tilde{E}^N_0, \tilde{I}^N_{0,2}, \tilde{R}^N_{0,2}\big) \to \big(\bar{\mathfrak{I}}_{0,2}, \bar{E}_0, \bar{I}_{0,2}, \bar{R}_{0,2}\big) \quad \text{in} \quad D^4 \quad \text{as} \quad N \to \infty.$$

Then it suffices to show that in probability, as $N \to \infty$,

$$\big(\tilde{\mathfrak{I}}^N_{0,1} - \bar{\mathfrak{I}}^N_{0,1}, \tilde{\mathfrak{I}}^N_{0,2} - \bar{\mathfrak{I}}^N_{0,2}, \tilde{E}^N_0 - \bar{E}^N_0, \tilde{I}^N_{0,1} - \bar{I}^N_{0,1}, \tilde{I}^N_{0,2} - \bar{I}^N_{0,2}, \tilde{R}^N_{0,1} - \bar{R}^N_{0,1}, \tilde{R}^N_{0,2} - \bar{R}^N_{0,2}\big) \to 0 \quad \text{in} \quad D^7.$$

The convergence of $\big(\tilde{\mathfrak{I}}^N_{0,1} - \bar{\mathfrak{I}}^N_{0,1}, \tilde{\mathfrak{I}}^N_{0,2} - \bar{\mathfrak{I}}^N_{0,2}\big) \to 0$ is proved in (4.8). For the other processes, see a similar argument in (4.10) and (4.11) in Section 4 of [15]. This completes the proof. □

For the newly infected individuals, let

$$\bar{E}^N_1(t) := N^{-1} \sum_{j=1}^{A^N(t)} \mathbf{1}_{\tau^N_i + \zeta_i > t}, \quad \bar{I}^N_1(t) := N^{-1} \sum_{i=1}^{A^N(t)} \mathbf{1}_{\tau^N_i + \zeta_i \le t < \tau^N_i + \zeta_i + \eta_i},$$

$$\bar{R}^N_1(t) := N^{-1} \sum_{i=1}^{A^N(t)} \mathbf{1}_{\tau^N_i + \zeta_i + \eta_i \le t}.$$

Lemma 4.6. *Under Assumptions 2.1, 2.2 and 2.3,*

$$\left(\bar{E}_1^N, \bar{I}_1^N, \bar{R}_1^N\right) \to \left(\bar{E}_1, \bar{I}_1, \bar{R}_1\right) \quad \text{in} \quad D^3 \quad \text{as} \quad N \to \infty, \tag{4.20}$$

in probability, jointly with the convergence of $\bar{\mathfrak{I}}_1^N \to \bar{\mathfrak{I}}_1$ *in* (4.9)*, where*

$$\bar{E}_1(t) := \int_0^t G^c(t-s)\bar{S}(s)\bar{\mathfrak{I}}(s)ds\,, \quad \bar{I}_1(t) := \int_0^t \Psi(t-s)\bar{S}(s)\bar{\mathfrak{I}}(s)ds\,,$$
$$\bar{R}_1(t) := \int_0^t \Phi(t-s)\bar{S}(s)\bar{\mathfrak{I}}(s)ds\,.$$

Proof. Let

$$\breve{E}_1^N(t) := N^{-1}\sum_{j=1}^{A^N(t)} G^c(t-\tau_i^N) = \int_0^t G^c(t-s)d\bar{A}^N(s)\,,$$
$$\breve{I}_1^N(t) := N^{-1}\sum_{i=1}^{A^N(t)} \Psi(t-\tau_i^N) = \int_0^t \Psi(t-s)d\bar{A}^N(s)\,,$$
$$\breve{R}_1^N(t) := N^{-1}\sum_{i=1}^{A^N(t)} \Phi(t-\tau_i^N) = \int_0^t \Phi(t-s)d\bar{A}^N(s)\,.$$

Recall the definition of $\breve{\mathfrak{I}}_1^N(t)$ in (4.10) and its convergence in (4.11).

Define the map Γ from a nondecreasing function $x \in D$ to the following integral functions:

$$\left(\int_0^\cdot \bar{\lambda}(\cdot - s)dx(s), \int_0^\cdot G^c(\cdot - s)dx(s), \int_0^\cdot \Psi(\cdot - s)dx(s), \int_0^\cdot \Phi(\cdot - s)dx(s)\right) \in D^4.$$

Then given the convergence of A^N in (4.18), and applying the continuous mapping theorem, we obtain the joint convergence in probability:

$$\left(\breve{\mathfrak{I}}_1^N, \breve{E}_1^N, \breve{I}_1^N, \breve{R}_1^N\right) \to \left(\bar{\mathfrak{I}}_1, \bar{E}_1, \bar{I}_1, \bar{R}_1\right) \quad \text{in} \quad D^4 \quad \text{as} \quad N \to \infty.$$

Then it suffices to show that in probability,

$$\left(\bar{\mathfrak{I}}_1^N - \breve{\mathfrak{I}}_1^N, \bar{E}_1^N - \breve{E}_1^N, \bar{I}_1^N - \breve{I}_1^N, \bar{R}_1^N - \breve{R}_1^N\right) \to 0 \quad \text{in} \quad D^4 \quad \text{as} \quad N \to \infty.$$

The convergence $\bar{\mathfrak{I}}_1^N - \breve{\mathfrak{I}}_1^N \to 0$ is shown in Lemma 4.4 (see equation (4.12)). The convergence $\bar{E}_1^N - \breve{E}_1^N \to 0$ follows essentially the same argument as Lemma 4.2 in [15]. The convergence $\bar{I}_1^N - \breve{I}_1^N \to 0$ follows essentially the same argument as Lemma 6.1 in [15]. Similarly for $\bar{R}_1^N - \breve{R}_1^N \to 0$. This completes the proof. □

Finally, observing that the processes $(\bar{E}^N, \bar{I}^N, \bar{R}^N)$ can be written as sums of the processes in (4.19) and (4.20), and thus applying the continuous mapping theorem, we directly obtain the joint convergence $(\bar{E}^N, \bar{I}^N, \bar{R}^N) \to (\bar{E}, \bar{I}, \bar{R})$ in D^3 in probability.

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