

PROBABILISTIC INTERPRETATION OF A SYSTEM OF QUASILINEAR PARABOLIC PDES

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Using a forward–backward stochastic differential equations (FBSDE) associated to a transmutation process driven by a finite sequence of Poisson processes, we obtain a probabilistic interpretation for a non-degenerate system of quasilinear parabolic partial differential equations (PDEs). The novelty is that the linear second order differential operator is different on each line of the system.

Keywords: System of parabolic quasilinear PDEs; Forward–backward stochastic differential equations; Poisson process; Malliavin Calculus

INTRODUCTION

In this paper, we are interested in the problem of providing a stochastic representation of the solutions of the following system of quasilinear parabolic partial differential equations (PDEs)

$$(\mathcal{E}) \begin{cases} \mathbf{V}(t, x) \in [0, T] \times \mathbf{R}^d, \quad \forall i \in \mathcal{K} \\ \frac{\partial u_i}{\partial t}(t, x) + L^i u_i(t, x) + \bar{f}_i(t, x, u(t, x)), \quad \nabla_x u_i(t, x) \sigma_i(t, x, u_i(t, x)) = 0 \\ u_i(T, x) = h_i(x) \end{cases}$$

where the coefficients \bar{b} , \bar{f} , σ and h are given from above, the second order differential operator L is defined by

$$\mathbf{V}(t, x, y) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}, \quad L^i u_i(t, x) = \frac{1}{2} \sum_{j,m=1}^d a_i(t, x, u_i(t, x))_{jm} \frac{\partial^2 u_i(t, x)}{\partial x_j \partial x_m} \\ + \sum_{j=1}^d (\bar{b}_i)^j(t, x, u(t, x)) \nabla_x u_i(t, x) \\ \times \sigma_i(t, x, u_i(t, x)) \frac{\partial u_i(t, x)}{\partial x_j}$$

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and for $i \in \mathcal{K}$, $\forall (t, x, y) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}$, the matrix $a_i(t, x, y) = (\sigma_i \sigma_i^*)(t, x, y)$ is supposed to be uniformly elliptic. Moreover, we give sufficient conditions which ensures existence and uniqueness of a solution of the system of forward–backward stochastic differential equations (FBSDE, $E^{0,\xi}$) [see Eq. (2.1)] whatever the time duration T may be.

It is well known that backward stochastic differential equations (BSDEs) are closely connected to PDEs. This link between PDEs and BSDEs has been used in two ways. In the first case, regularity results for solutions of PDEs have been used in the study of solutions of BSDEs and there have recently appeared some probabilistic methods for systems of PDEs, and an essential tool in this new method is the Malliavin Calculus. For example, Pardoux and Peng [7] prove that BSDEs provide a probabilistic formula for solutions of certain classes of quasilinear parabolic PDEs.

BSDEs with Poisson Process were first discussed by Tang and Li [12] and Wu [13]. Studying such equations, Barles *et al.* [1] generalized the result of Pardoux–Peng [7], and obtained a probabilistic interpretation of a solution of a parabolic integral–partial differential equation.

Soon after appeared fully coupled FBSDEs. This motivation comes originally from stochastic optimal control theory. In his Ph.D. Thesis Antonelli obtained the first result on the solvability of a FBSDE over a small time duration. Using the strong link between the FBSDEs and a quasilinear parabolic PDE, Ma *et al.* [5,6,11] in their so called “four step scheme” prove successfully an existence and uniqueness result for a FBSDE. But they require non-degeneracy assumption on the diffusion of the forward equation, non-randomness and rather stringent smoothness of the coefficients.

Recently in the same spirit, Delarue [2] proved a similar result under rather weaker assumptions than those required in the “four step scheme” and provided in a probabilistic manner a solution of a non-degenerate system of quasilinear parabolic PDEs. The key device of his method based on an iterative scheme is an efficient control of the length of the interval on which the result on small time duration holds, which ensures existence and uniqueness of solutions on arbitrary time intervals. For this to be true, the non-degeneracy of the forward diffusion coefficient is crucial.

All those systems of PDEs studied so far have the same linear second order differential operator on each line. Pardoux *et al.* [9] removed this restriction in the case of BSDEs and provided a stochastic representation of a viscosity solution of a system of semilinear parabolic PDEs with a different second order differential operator from one line to another. This is done by coupling the forward diffusion with a transmutation process driven by a finite sequence of Poisson Processes. We aim in this work to get a similar result in the case of non-degenerate parabolic quasilinear systems through the study of a corresponding system of FBSDEs following the method developed by Delarue [2].

The paper is organized as follows. In “FBSDE with poisson process in small time duration” section, we prove the existence and uniqueness of a solution of a system of FBSDE with respect to a Brownian motion and a finite sequence of Poisson Processes for a small time duration. We also present some properties of the solutions before extending this result to the case of random coefficients. Studying the case of smooth coefficients, we establish by a stochastic method our main result (existence and uniqueness under the non-degeneracy assumption) of the solution of the system of PDEs. Finally in “Global solution of the system of FBSDE” section, we extend the local solution of the FBSDE ($E^{0,\xi}$) obtained in the previous section to a global one.

FBSDE WITH POISSON PROCESS IN SMALL TIME DURATION

$\forall Q \in \mathbf{N}^*$, $|\cdot|$ stands for the Euclidian norm in \mathbf{R}^Q .

Let $k \geq 2$ be an integer and $\mathcal{K} = \{1, 2, \dots, k\}$, $L = \mathcal{K} - \{k\}$.

We consider a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P})$ such that the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ is generated by two mutually independent processes:

- a d -dimensional Brownian motion $(B_t)_{0 \leq t \leq T}$;
- a Poisson random measure N on $\mathbf{R}_+ \times L$ where L is the set of marks equipped with the field \mathcal{L} of all subsets of L such that $M((0, t] \times A) = N((0, t] \times A) - t\lambda \text{card } A$ is an \mathcal{F}_t -martingale for all $A \subset L$ and some fixed $\lambda > 0$.

Moreover, we consider a filtration $(\mathcal{G}_t)_{0 \leq t \leq T}$ satisfying the usual conditions such that $(B_t)_{0 \leq t \leq T}$ still a (\mathcal{G}_t) -Brownian motion and the filtration $(\mathcal{G}_t^0)_{0 \leq t \leq T}$ defined by $\forall t \in [0, T] \mathcal{G}_t^0 = \mathcal{G}_0 \vee \mathcal{F}_t$.

For $0 \leq t < s \leq T$, $i \in \mathcal{K}$ and $l \in L$, we define

$$N_s = N((0, s] \times L), \quad N_s(l) = N((0, s] \times \{l\}), \quad M_s(l) = N_s(l) - \lambda s$$

We define the Markov process $N_s^{t,i}$ by

$$N_s^{t,i} = i + \sum_{l=1}^{k-1} lN((t, s] \times \{l\}) \text{ mod}[k] \text{ and let } N_s^{0,i} = N_s^i$$

Let \mathcal{P} denote the σ -algebra of \mathcal{G}_t -predictable subsets of $\Omega \times [0, T]$.

For each $i \in \mathcal{K}$, we are given $\bar{b}_i \in \mathcal{C}([0, T] \times \mathbf{R}^d \times \mathbf{R}^k \times \mathbf{R}^d, \mathbf{R}^d)$, $\bar{f}_i \in \mathcal{C}([0, T] \times \mathbf{R}^d \times \mathbf{R}^k \times \mathbf{R}^d, \mathbf{R})$, $\sigma_i \in \mathcal{C}([0, T] \times \mathbf{R}^d \times \mathbf{R}, \mathbf{R}^d \times \mathbf{R}^d)$ and $h_i \in \mathcal{C}(\mathbf{R}^d, \mathbf{R})$.

We define the following continuous functions on their domains

$$\begin{aligned} \bar{b} : [0, T] \times \mathbf{R}^d \times \mathbf{R}^k \times \mathbf{R}^d \times \mathcal{K} &\rightarrow \mathbf{R}^d \\ \bar{f} : [0, T] \times \mathbf{R}^d \times \mathbf{R}^k \times \mathbf{R}^d \times \mathcal{K} &\rightarrow \mathbf{R} \\ \sigma : [0, T] \times \mathbf{R}^d \times \mathbf{R} \times \mathcal{K} &\rightarrow \mathbf{R}^d \times \mathbf{R}^d \\ h : \mathbf{R}^d \times \mathcal{K} &\rightarrow \mathbf{R} \end{aligned}$$

by putting $\mathbf{V}(t, x, y, u, z, i) \in [0, T] \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^k \times \mathbf{R}^d \times \mathcal{K}$, $\delta(x, u, z, i) = \delta_i(x, u, z)$ for $\delta = \bar{b}, \bar{f}$ and $\sigma(\text{resp } h)(t, x, y, i) = \sigma_i(t, x, y)(\text{resp } h_i(x))$.

Let $i \in \mathcal{K}$; we define the functions b_i, f_i and $\tilde{f}_i \in \mathcal{C}([0, T] \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^d)$ such that

$$\begin{aligned} \forall (t, x) \in [0, T] \times \mathbf{R}^d, \quad \forall u \in \mathbf{R}^k, \\ \tilde{f}_i(t, x, u_1, u_2, \dots, u_k, z) &= \tilde{f}_i(t, x, u_i, h^i, z) = f_i(t, x, u_i, h^i, z) + \lambda \sum_{l \in L} h^l(l) \\ \bar{b}_i(t, x, u_1, u_2, \dots, u_k, z) &= b_i(t, x, u_i, h^i, z) \\ \text{where } h_j^i &= \begin{cases} u_{i+j} - u_i & 1 \leq j \leq k - i \\ u_{i+j-k} - u_i & k - i + 1 \leq j \leq k - 1 \end{cases} \end{aligned}$$

For $i \in \mathcal{K}$ and $\xi \in \mathbf{R}^d$ a \mathcal{G}_0 -measurable random vector satisfying $\mathbf{E}(|\xi|^2) < \infty$, we intend to find a \mathcal{G}_t -progressively measurable quartet of processes $(\Theta_s^{0,\xi,i})_{0 \leq s \leq T} \triangleq (X_s^{0,\xi,i}, Y_s^{0,\xi,i}, H_s^{0,\xi,i}, Z_s^{0,\xi,i})_{0 \leq s \leq T}$ with values in $\mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^d$ solution of the problem

$$(E^{0,\xi}) \begin{cases} \forall s \in [0, T] \\ X_s^{0,\xi,i} = \xi + \int_0^s b(r, \Theta_r^{0,\xi,i}, N_r^i) dr + \int_0^s \sigma(r, X_r^{0,\xi,i}, Y_r^{0,\xi,i}, N_r^i) dB_r \\ Y_s^{0,\xi,i} = h_{N_t^i}(X_T^{0,\xi,i}) + \int_s^T f(r, \Theta_r^{0,\xi,i}, N_r^i) dr - \int_s^T Z_r^{0,\xi,i} dB_r - \int_s^T \sum_{l \in L} H_r^{0,\xi,i}(l) dM_r(l) \\ \mathbf{E} \int_0^T (|X_t^{0,\xi,i}|^2 + |Y_t^{0,\xi,i}|^2 + |Z_t^{0,\xi,i}|^2 + |H_t^{0,\xi,i}|^2) dt < \infty \end{cases} \quad (2.1)$$

where $\forall s \in [0, T], H_s^{0,\xi,i} = (H_s^{0,\xi,i}(1), H_s^{0,\xi,i}(2), \dots, H_s^{0,\xi,i}(k-1))$.

The superscript $0, \xi, i$ indicates the dependence of the solution on the initial date $(0, \xi, i)$ and will often be omitted for notational simplicity.

This section is devoted to the study of $(E^{0, \xi})$ in small time duration. In all of what follows, we assume that $T \leq 1$.

The Case of Non-smooth Coefficients

Existence and Uniqueness of Solutions

Let us first prove existence and uniqueness of solutions. The idea is based on a fixed point argument on a suitable space in the spirit of the method developed by Delarue [2], Théorème 1.1.

We say that the functions b, f, h and σ satisfy assumption **(H1)**, if there exist two constants K and Λ such that

(H1.1) $\forall t \in [0, T], \forall (x, y, u, z) \in \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^d, \forall i \in \mathcal{K}$ and $\forall (x', y', u', z') \in \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^d,$

$$\begin{aligned} |b_i(t, x, y, u, z) - b_i(t, x, y', u', z')| &\leq K(|y - y'| + |z - z'| + |u - u'|) \\ |f_i(t, x, y, u, z) - f_i(t, x', y, u', z')| &\leq K(|x - x'| + |z - z'| + |u - u'|) \\ |\sigma_i(t, x, y) - \sigma_i(t, x', y')|^2 &\leq K^2(|x - x'|^2 + |y - y'|^2) \\ |h_i(x) - h_i(x')| &\leq K|x - x'| \end{aligned}$$

(H1.2) $\forall t \in [0, T], \forall (x, y, u, z) \in \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^d, \forall i \in \mathcal{K}$ et $\forall (x', y') \in \mathbf{R}^d \times \mathbf{R},$

$$\begin{aligned} \langle x - x', b_i(t, x, y, u, z) - b_i(t, x', y, u, z) \rangle &\leq K|x - x'|^2 \\ (y - y')(f_i(t, x, y, u, z) - f_i(t, x, y', u, z)) &\leq K|y - y'|^2 \end{aligned}$$

(H1.3) $\forall t \in [0, T], \forall (x, y, u, z) \in \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^d, \forall i \in \mathcal{K},$

$$|b_i(t, x, y, u, z)| + |f_i(t, x, y, u, z)| + |\sigma_i(t, x, y)| + |h_i(x)| \leq \Lambda$$

THEOREM 2.1 Assume that **(H1)** holds and let $i \in \mathcal{K}$. Then for every \mathcal{G}_0 -measurable random vector ξ satisfying $\mathbf{E}|\xi|^2 < \infty$, a solution of $(E^{0, \xi})$ satisfies

- (i) $(X_s)_{0 \leq s \leq T}$ is continuous and $(Y_s)_{0 \leq s \leq T}$ has a càdlàg version.
- (ii) $\mathbf{E} \left(\sup_{0 \leq t \leq T} |X_t|^2 + \sup_{0 \leq t \leq T} |Y_t|^2 \right) < \infty$.

Moreover, there exists a constant $C_{K,\lambda}^{(i1)} > 0$ depending on K and λ such that for every $T \leq C_{K,\lambda}^{(i1)}$, the problem $(E^{0, \xi})$ admits a unique solution.

Proof As explained, our aim is to construct a contraction in a suitable space. To this end for $T > 0$ and $Q \in \mathbf{N}$, we consider the following sets

- $H_T^2(\mathbf{R}^Q)$ the space of $\{\mathcal{G}_t^0\}$ progressively measurable processes

$$\Psi : [0, T] \times \Omega \rightarrow \mathbf{R}^Q, \|\Psi\|^2 = \mathbf{E} \int_0^T |\Psi_t|^2 dt < \infty$$

- $S_T^2(\mathbf{R}^Q)$ the space of continuous $\{\mathcal{G}_t^0\}$ adapted processes

$$\Psi : [0, T] \times \Omega \rightarrow \mathbf{R}^Q, \|\Psi\|_2^2 = \mathbf{E} \left(\sup_{0 \leq t \leq T} |\Psi_t|^2 \right) < \infty$$

- $S_T^2(\mathbf{R}^Q)$ the space of $\{\mathcal{G}_t^0\}$ adapted càdlàg processes

$$\Psi : [0, T] \times \Omega \rightarrow \mathbf{R}^Q, \|\Psi\|_2^2 = \mathbf{E} \left(\sup_{0 \leq t \leq T} |\Psi_t|^2 \right) < \infty$$

- $[L^2(\mathcal{P} \otimes \mathcal{L})]^{k-1}$ the space of mappings $H : \Omega \times [0, T] \times L \rightarrow \mathbf{R}$ which are $\mathcal{P} \otimes \mathcal{L}$ -measurable such that

$$\|H\|_{[L^2(\mathcal{P} \otimes \mathcal{L})]^{k-1}}^2 = \mathbf{E} \sum_{l \in L} \int_0^T (H_t(l))^2 dt < \infty$$

- $\forall p \geq 1, \mathcal{B}_{[0,T]}^{2p}$ the space $S_T^2(\mathbf{R}^d) \times S_T^2(\mathbf{R}) \times [L^2(\mathcal{P} \otimes \mathcal{L})]^{k-1} \times H_T^2(\mathbf{R}^d)$ endowed with the norm

$$\begin{aligned} \|(X_s, Y_s, H_s, Z_s)\|_{\mathcal{B}_{[0,T]}^{2p}} &= \left(\mathbf{E} \sup_{0 \leq s \leq T} |X_s|^{2p} \right)^{\frac{1}{2p}} + \left(\mathbf{E} \sup_{0 \leq s \leq T} |Y_s|^{2p} \right)^{\frac{1}{2p}} \\ &+ \left(\mathbf{E} \left[\left(\int_0^T \sum_{l \in L} (H_s(l))^2 ds \right)^{\frac{1}{2}} \right]^p \right)^{\frac{1}{p}} + \left(\mathbf{E} \left[\left(\int_0^T |Z_s|^2 ds \right)^{\frac{1}{2}} \right]^p \right)^{\frac{1}{p}} \end{aligned}$$

Notice that $\mathcal{B}_{[0,T]}^{2p}$ endowed with this norm is a Banach space.

We define a map φ on $\mathcal{B}_{[0,T]}^2$ into itself as follows

$$\begin{aligned} \mathcal{B}_{[0,T]}^2 &\rightarrow \mathcal{B}_{[0,T]}^2 \\ (X, Y, H, Z) &\rightarrow (\bar{X}^{0,\xi,i}, \bar{Y}^{0,\xi,i}, \bar{H}^{0,\xi,i}, \bar{Z}^{0,\xi,i}) \end{aligned}$$

where the quartet $(\bar{\Theta}_s^{0,\xi,i})_{0 \leq s \leq T} \triangleq (\bar{X}_s^{0,\xi,i}, \bar{Y}_s^{0,\xi,i}, \bar{H}_s^{0,\xi,i}, \bar{Z}_s^{0,\xi,i})_{0 \leq s \leq T}$ is defined by

$$\begin{cases} \forall t \in [0, T] \\ \bar{X}_t^{0,\xi,i} = \xi + \int_0^t b(s, \bar{X}_s^{0,\xi,i}, Y_s, H_s, Z_s, N_s^i) ds + \int_0^t \sigma(s, \bar{X}_s^{0,\xi,i}, Y_s, N_s^i) dB_s \end{cases} \quad (2.2)$$

and

$$\begin{cases} \forall t \in [0, T] \\ \bar{Y}_t^{0,\xi,i} = h_{N_t^i}(\bar{X}_t^{0,\xi,i}) + \int_t^T f(s, \bar{\Theta}_s^{0,\xi,i}, N_s^i) ds - \int_t^T \bar{Z}_s^{0,\xi,i} dB_s - \int_t^T \sum_{l \in \mathcal{L}} \bar{H}_s^{0,\xi,i}(l) dM_s(l) \end{cases} \quad (2.3)$$

Note that the process $\bar{X}_s^{0,\xi,i}$ is a solution of a SDE with initial time 0 and initial values (ξ, i) (i initial value of the process N_s^i), whereas the triplet $(\bar{Y}_s^{0,\xi,i}, \bar{H}_s^{0,\xi,i}, \bar{Z}_s^{0,\xi,i})$ solves a BSDE with Poisson process.

In the sequel of the proof, we denote

$$\varphi(X, Y, H, Z) = (\bar{X}_s, \bar{Y}_s, \bar{H}_s, \bar{Z}_s) = \bar{\Theta}_s; \quad \varphi(U, V, G, W) = (\bar{U}_s, \bar{V}_s, \bar{G}_s, \bar{W}_s) = \bar{\Theta}'_s$$

Using Itô's formula, we get

$$\begin{aligned} |\bar{X}_s - \bar{U}_s|^2 &= 2 \int_0^s \langle \bar{X}_r - \bar{U}_r, b(r, \bar{X}_r, Y_r, Z_r, H_r, N_r^i) - b(r, \bar{U}_r, V_r, W_r, G_r, N_r^i) \rangle dr \\ &\quad + 2 \int_0^s \langle \bar{X}_r - \bar{U}_r, (\sigma(r, \bar{X}_r, Y_r, N_r^i) - \sigma(r, \bar{U}_r, V_r, N_r^i)) dB_r \rangle \\ &\quad + \int_0^s |\sigma(r, \bar{X}_r, Y_r, N_r^i) - \sigma(r, \bar{U}_r, V_r, N_r^i)|^2 dr \end{aligned}$$

Applying the Burkholder–Davis–Gundy inequality and assumption (H1.1), we find $\delta > 0$ such that

$$\begin{aligned} \mathbf{E} \sup_{0 \leq t \leq T} |\bar{X}_t - \bar{U}_t|^2 &\leq 2K\mathbf{E} \int_0^T |\bar{X}_r - \bar{U}_r| (|\bar{X}_r - \bar{U}_r| + |Y_r - V_r| + |Z_r - W_r| + |H_r - G_r|) dr \\ &\quad + 2\delta K\mathbf{E} \left[\int_0^T |\bar{X}_r - \bar{U}_r|^2 (|\bar{X}_r - \bar{U}_r|^2 + |Y_r - V_r|^2) dr \right]^{\frac{1}{2}} \\ &\quad + K^2 \mathbf{E} \int_0^T (|\bar{X}_r - \bar{U}_r|^2 + |Y_r - V_r|^2) dr \end{aligned}$$

so using standard estimates, we prove the existence of a constant γ_K depending on K such that

$$\begin{aligned} \mathbf{E} \sup_{0 \leq t \leq T} |\bar{X}_t - \bar{U}_t|^2 &\leq \gamma_K T^{1/2} \left[\mathbf{E} \sup_{0 \leq t \leq T} |\bar{X}_t - \bar{U}_t|^2 + \mathbf{E} \sup_{0 \leq t \leq T} |Y_t - V_t|^2 + \mathbf{E} \int_0^T |Z_r - W_r|^2 dr \right. \\ &\quad \left. + \mathbf{E} \sum_{l \in \mathcal{L}} \int_0^T ((H_r - G_r)(l))^2 dr \right] \end{aligned}$$

which implies

$$(1 - \gamma_K T^{1/2}) \mathbf{E} \sup_{0 \leq t \leq T} |\bar{X}_t - \bar{U}_t|^2 \leq \gamma_K T^{1/2} \left[\mathbf{E} \sup_{0 \leq t \leq T} |Y_t - V_t|^2 + \mathbf{E} \int_0^T |Z_r - W_r|^2 dr + \mathbf{E} \sum_{l \in L} \int_0^T ((H_r - G_r)(l))^2 dr \right] \tag{2.4}$$

Moreover, Itô's formula yields from Eq. (2.3), for all $t \in [0, T]$

$$\begin{aligned} |\bar{Y}_t - \bar{V}_t|^2 + \int_t^T |\bar{Z}_s - \bar{W}_s|^2 ds + \lambda \int_t^T \sum_{l \in L} ((\bar{H}_s - \bar{G}_s)(l))^2 ds + \sum_{t < s \leq T} (\Delta(\bar{Y}_s - \bar{V}_s))^2 \\ = |h_{N_T^i}(\bar{X}_T) - h_{N_T^i}(\bar{U}_T)|^2 + 2 \int_t^T (\bar{Y}_s - \bar{V}_s) (f(s, \bar{\Theta}_s, N_s^i) - f(s, \bar{\Theta}'_s, N_s^i)) ds \\ - 2 \int_t^T (\bar{Y}_s - \bar{V}_s) (\bar{Z}_s - \bar{W}_s) dB_s - 2 \int_t^T \sum_{l \in L} (\bar{Y}_{s-} - \bar{V}_{s-}) (\bar{H}_s - \bar{G}_s)(l) dM_s(l) \end{aligned} \tag{2.5}$$

Applying (H1.1) and (H1.2), we deduce

$$\begin{aligned} \mathbf{E} \int_0^T |\bar{Z}_s - \bar{W}_s|^2 ds + \lambda \mathbf{E} \int_0^T \sum_{l \in L} ((\bar{H}_s - \bar{G}_s)(l))^2 ds \leq K^2 \mathbf{E} \sup_{0 \leq t \leq T} |\bar{X}_t - \bar{U}_t|^2 \\ + 2K \left[\mathbf{E} \int_0^T |\bar{Y}_s - \bar{V}_s| |\bar{X}_s - \bar{U}_s| ds + \mathbf{E} \int_0^T |\bar{Y}_s - \bar{V}_s|^2 ds \right. \\ \left. + \mathbf{E} \int_0^T |\bar{Y}_s - \bar{V}_s| |\bar{Z}_s - \bar{W}_s| ds + \mathbf{E} \int_0^T |\bar{Y}_s - \bar{V}_s| |\bar{H}_s - \bar{G}_s| ds \right] \end{aligned}$$

Hence using standard estimates, we find a constant $\gamma_{(K,\lambda)}$ depending on K and λ such that

$$\begin{aligned} \mathbf{E} \int_0^T |\bar{Z}_s - \bar{W}_s|^2 ds + \lambda \mathbf{E} \int_0^T \sum_{l \in L} ((\bar{H}_s - \bar{G}_s)(l))^2 ds \\ \leq \gamma_{(K,\lambda)} \left[(1 + T) \mathbf{E} \sup_{0 \leq t \leq T} |\bar{X}_t - \bar{U}_t|^2 + T \mathbf{E} \sup_{0 \leq t \leq T} |\bar{Y}_t - \bar{V}_t|^2 \right] \end{aligned} \tag{2.6}$$

Furthermore, using Eq. (2.5) we obtain

$$\begin{aligned} \mathbf{E} \sup_{0 \leq t \leq T} |\bar{Y}_t - \bar{V}_t|^2 \leq \mathbf{E} (|h_{N_T^i}(\bar{X}_T) - h_{N_T^i}(\bar{U}_T)|^2) \\ + 2 \mathbf{E} \sup_{0 \leq t \leq T} \left(\left| \int_t^T (\bar{Y}_s - \bar{V}_s) (f(s, \bar{\Theta}_s, N_s^i) - f(s, \bar{\Theta}'_s, N_s^i)) ds \right| \right) \\ + 2 \mathbf{E} \sup_{0 \leq t \leq T} \left| \int_t^T (\bar{Y}_s - \bar{V}_s) (\bar{Z}_s - \bar{W}_s) dB_s \right| \\ + 2 \mathbf{E} \sup_{0 \leq t \leq T} \left| \int_t^T \sum_{l \in L} (\bar{Y}_{s-} - \bar{V}_{s-}) (\bar{H}_s - \bar{G}_s)(l) dM_s(l) \right| \end{aligned} \tag{2.7}$$

From the Burkholder–Davis–Gundy inequality, there exists a constant $C > 0$ which can change from line to line such that

$$\begin{aligned} 2\mathbf{E} \sup_{0 \leq t \leq T} \left| \int_t^T (\bar{Y}_s - \bar{V}_s)(\bar{Z}_s - \bar{W}_s) dB_s \right| &\leq C\mathbf{E} \left(\int_0^T |\bar{Y}_s - \bar{V}_s|^2 |\bar{Z}_s - \bar{W}_s|^2 ds \right)^{\frac{1}{2}} \\ &\leq \frac{1}{4}\mathbf{E} \sup_{0 \leq s \leq T} |\bar{Y}_s - \bar{V}_s|^2 + 4C^2\mathbf{E} \int_0^T |\bar{Z}_s - \bar{W}_s|^2 ds \end{aligned}$$

Similarly, using again the Burkholder–Davis–Gundy inequality for discontinuous processes and standard estimates where

$$\hat{M}_t(l) = (\bar{Y}_{s-} - \bar{V}_{s-})(\bar{H}_s - \bar{G}_s)(l) dM_s(l),$$

there exists another constant which we note C again changing from line to line such that

$$\begin{aligned} 2\mathbf{E} \sup_{0 \leq t \leq T} \left| \int_t^T \sum_{l \in \mathcal{L}} (\bar{Y}_{s-} - \bar{V}_{s-})(\bar{H}_s - \bar{G}_s)(l) dM_s(l) \right| \\ \leq C\mathbf{E} \left[\left(\sum \Delta \hat{M}_t(l)^2 \right)^{\frac{1}{2}} \right] = C\mathbf{E} \left[\int_0^T \sum_{l \in \mathcal{L}} (\bar{Y}_{s-} - \bar{V}_{s-})^2 ((\bar{H}_s - \bar{G}_s)(l))^2 dN_s(l) \right]^{\frac{1}{2}} \\ \leq \frac{1}{4}\mathbf{E} \sup_{0 \leq s \leq T} |\bar{Y}_s - \bar{V}_s|^2 + C^2\lambda\mathbf{E} \sum_{l \in \mathcal{L}} \int_0^T ((\bar{H}_s - \bar{G}_s)(l))^2 ds \end{aligned} \quad (2.8)$$

Hence, combining these two previous inequalities with Eq. (2.6), we deduce from Eq. (2.7)

$$\begin{aligned} \mathbf{E} \sup_{0 \leq s \leq T} |\bar{Y}_s - \bar{V}_s|^2 + \mathbf{E} \int_0^T |\bar{Z}_s - \bar{W}_s|^2 ds + \lambda\mathbf{E} \int_0^T \sum_{l \in \mathcal{L}} ((\bar{H}_s - \bar{G}_s)(l))^2 ds \\ \leq 2K\mathbf{E} \int_0^T |\bar{Y}_s - \bar{V}_s| (|\bar{X}_s - \bar{U}_s| + |\bar{Y}_s - \bar{V}_s| + |\bar{Z}_s - \bar{W}_s| + |\bar{H}_s - \bar{G}_s|) ds \\ + K^2\mathbf{E} \sup_{0 \leq s \leq T} |\bar{X}_s - \bar{U}_s|^2 + \frac{1}{2}\mathbf{E} \sup_{0 \leq s \leq T} |\bar{Y}_s - \bar{V}_s|^2 \\ + C\gamma_{(K,\lambda)} \left[(1+T)\mathbf{E} \sup_{0 \leq s \leq T} |\bar{X}_s - \bar{U}_s|^2 + T\mathbf{E} \sup_{0 \leq s \leq T} |\bar{Y}_s - \bar{V}_s|^2 \right] \end{aligned}$$

which implies using standard estimates and modifying $\gamma_{(K,\lambda)}$

$$\begin{aligned} \mathbf{E} \sup_{0 \leq s \leq T} |\bar{Y}_s - \bar{V}_s|^2 &\leq 2\gamma_{(K,\lambda)} \left[(1+T)\mathbf{E} \sup_{0 \leq s \leq T} |\bar{X}_s - \bar{U}_s|^2 + T\mathbf{E} \sup_{0 \leq s \leq T} |\bar{Y}_s - \bar{V}_s|^2 \right] \\ &+ C\gamma_{(K,\lambda)} \left[(1+T)\mathbf{E} \sup_{0 \leq s \leq T} |\bar{X}_s - \bar{U}_s|^2 + T\mathbf{E} \sup_{0 \leq s \leq T} |\bar{Y}_s - \bar{V}_s|^2 \right] \\ &+ \frac{1}{2}\mathbf{E} \sup_{0 \leq s \leq T} |\bar{Y}_s - \bar{V}_s|^2 \end{aligned} \quad (2.9)$$

hence putting $\delta_{(K,\lambda)} = 2(2\gamma_{(K,\lambda)} + C\gamma_{(K,\lambda)})$, we get

$$(1 - \delta_{(K,\lambda)}T)\mathbf{E} \sup_{0 \leq s \leq T} |\bar{Y}_s - \bar{V}_s|^2 \leq \delta_{(K,\lambda)}(1 + T)\mathbf{E} \sup_{0 \leq s \leq T} |\bar{X}_s - \bar{U}_s|^2 \quad (2.10)$$

and the following inequalities

$$\begin{aligned} & \mathbf{E} \int_0^T |\bar{Z}_s - \bar{W}_s|^2 ds + \mathbf{E} \sum_{l \in \mathcal{L}} \int_0^T ((\bar{H}_s - \bar{G}_s)(l))^2 ds \\ & \leq \gamma_{(K,\lambda)} \left[(1 + T)\mathbf{E} \sup_{0 \leq s \leq T} |\bar{X}_s - \bar{U}_s|^2 + T\mathbf{E} \sup_{0 \leq s \leq T} |\bar{Y}_s - \bar{V}_s|^2 \right] \end{aligned} \quad (2.11)$$

$$\begin{aligned} (1 - \gamma_K T^{1/2})\mathbf{E} \sup_{0 \leq s \leq T} |\bar{X}_s - \bar{U}_s|^2 & \leq \gamma_K T^{1/2} \left[\mathbf{E} \sup_{0 \leq s \leq T} |Y_s - V_s|^2 + \mathbf{E} \int_0^T |Z_s - W_s|^2 ds \right. \\ & \left. + \mathbf{E} \sum_{l \in \mathcal{L}} \int_0^T ((H_s - G_s)(l))^2 ds \right] \end{aligned} \quad (2.12)$$

This proves that there exists a constant $C_{(K,\lambda)}^{(i1)} > 0$ depending on K and λ such that for $T \leq C_{(K,\lambda)}^{(i1)}$, the map φ is a contraction from $\mathcal{B}_{[0,T]}^2$ into itself.

Consequently by the fixed point theorem, there exists a unique $\{\mathcal{G}_t^0\}_{0 \leq t \leq T}$ -progressively measurable quartet of processes with values in $(\mathbf{R}^d, \mathbf{R}, \mathbf{R}^{k-1}, \mathbf{R}^d)$ solution to (E^0, ξ) .

Note that this solution is obviously \mathcal{G}_T -progressively measurable. \square

Remark This theorem shows that for every $T \leq C_{(K,\lambda)}^{(i1)}$ and for every $x \in \mathbf{R}^d$ the problem $(E^{0,x})$ with x as initial value of the SDE admits a unique solution $\Theta_t^{0,x,i} \triangleq (X_t^{0,x,i}, Y_t^{0,x,i}, H_t^{0,x,i}, Z_t^{0,x,i})_{0 \leq t \leq T}$ which is a $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ -progressively measurable quartet of processes. In particular the \mathcal{F}_0 -progressively measurable random variable $Y_0 = Y_0^{0,x,i}$ is deterministic.

Now, we establish some properties of the solution, which will be useful in the sequel.

A Priori Estimates

THEOREM 2.2 Suppose that (b, f, h, σ) and $(\tilde{b}, \tilde{f}, \tilde{h}, \tilde{\sigma})$ are two quartets of functions satisfying **(H1)** with the same constants K and Λ . For $i \in \mathcal{K}$, let $\Theta_s^{0,\xi,i} = (X_s^{0,\xi,i}, Y_s^{0,\xi,i}, H_s^{0,\xi,i}, Z_s^{0,\xi,i})_{(0 \leq s \leq T)}$ (resp $\tilde{\Theta}_s^{0,\xi,i} = (\tilde{X}_s^{0,\xi,i}, \tilde{Y}_s^{0,\xi,i}, \tilde{H}_s^{0,\xi,i}, \tilde{Z}_s^{0,\xi,i})_{(0 \leq s \leq T)}$) be the associated solution to (b, f, h, σ) (resp $(\tilde{b}, \tilde{f}, \tilde{H}, \tilde{\sigma})$) with initial condition $(0, \xi, i)$ (resp $(0, \tilde{\xi}, i)$).

Then there exists two constants $C_{(K,\lambda)}^{(i2)}$ and $\beta_{(K,\lambda)}^{(i1)}$ depending on K and λ such that for $T \leq C_{(K,\lambda)}^{(i2)}$ and for every $A \in \mathcal{G}_0$, the following estimate holds

$$\begin{aligned} & \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |X_s - \tilde{X}_s|^2 \right) + \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |Y_s - \tilde{Y}_s|^2 \right) + \mathbf{E} \sum_{l \in \mathcal{L}} \int_0^T 1_A ((H_s - \tilde{H}_s)(l))^2 ds \\ & + \mathbf{E} \int_0^T 1_A |Z_s - \tilde{Z}_s|^2 ds \leq \beta_{(K,\lambda)}^{(i1)} \left[\mathbf{E}(1_A |\xi - \tilde{\xi}|^2) + \mathbf{E}(1_A |(h - \tilde{h})(X_T)|^2) \right. \\ & \left. + \mathbf{E} \int_0^T 1_A |(\sigma - \tilde{\sigma})(s, X_s, Y_s, N_s^i)|^2 ds + \mathbf{E} \left(\int_0^T 1_A (|b - \tilde{b}| + |f - \tilde{f}|)(s, \Theta_s^{0,\xi,i}, N_s^i) ds \right)^2 \right] \end{aligned} \quad (2.13)$$

Proof Let $A \in \mathcal{G}_0$. Using Itô's formula we have

$$\begin{aligned} \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |X_s - \tilde{X}_s|^2 \right) &\leq \mathbf{E}(1_A |\xi - \tilde{\xi}|^2) + \mathbf{E} \int_0^T 1_A |\sigma_{N_s^i}(s, X_s, Y_s) - \tilde{\sigma}_{N_s^i}(s, \tilde{X}_s, \tilde{Y}_s)|^2 ds \\ &\quad + 2\mathbf{E} \sup_{0 \leq t \leq T} \left(\int_0^t 1_A \langle X_s - \tilde{X}_s, b(s, \Theta_s^{0, \xi, i}, N_s^i) - \tilde{b}(s, \tilde{\Theta}_s^{0, \xi, i}, N_s^i) \rangle ds \right) \\ &\quad + 2\mathbf{E} \sup_{0 \leq t \leq T} \left(\int_0^t 1_A \langle X_s - \tilde{X}_s, (\sigma_{N_s^i}(s, X_s, Y_s) - \tilde{\sigma}_{N_s^i}(s, \tilde{X}_s, \tilde{Y}_s)) dB_s \rangle \right) \end{aligned}$$

Thanks to Burkholder–Davis–Gundy inequalities, assumption (H1) and standard estimates, there exists a constant γ_K depending on K such that

$$\begin{aligned} \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |\tilde{X}_s - X_s|^2 \right) &\leq \gamma_K \left[\mathbf{E}(1_A |\xi - \tilde{\xi}|^2) + \mathbf{E} \int_0^T 1_A (|\tilde{X}_s - X_s|^2 + |\tilde{Y}_s - Y_s|^2) ds \right. \\ &\quad + \mathbf{E} \int_0^T 1_A |\tilde{X}_s - X_s| (|\tilde{Z}_s - Z_s| + |\tilde{H}_s - H_s|) ds \\ &\quad + \mathbf{E} \int_0^T 1_A |(\tilde{\sigma} - \sigma)(s, X_s, Y_s, N_s^i)|^2 ds \\ &\quad \left. + \mathbf{E} \left(\int_0^T 1_A |\tilde{b} - b(s, \Theta_s^{0, \xi, i}, N_s^i)|^2 ds \right) \right] \quad (2.14) \end{aligned}$$

Furthermore, applying Itô's formula to Eq. (2.3), we have $\forall t \in [0, T]$,

$$\begin{aligned} &|\tilde{Y}_t - Y_t|^2 + \int_t^T |\tilde{Z}_s - Z_s|^2 ds + \lambda \sum_{l \in L} \int_t^T ((\tilde{H}_s - H_s)(l))^2 ds + \sum_{t < s \leq T} (\Delta(\tilde{Y}_s - Y_s))^2 \\ &= |\tilde{h}(\tilde{X}_T, N_T^i) - h(X_T, N_T^i)|^2 + 2 \int_t^T (\tilde{Y}_s - Y_s) \left(\tilde{f}(s, \tilde{\Theta}_s^{0, \xi, i}, N_s^i) - f(s, \Theta_s^{0, \xi, i}, N_s^i) \right) ds \\ &\quad - 2 \int_t^T (\tilde{Y}_s - Y_s) ((\tilde{Z}_s - Z_s) dB_s) - 2 \int_t^T \sum_{l \in L} (\tilde{Y}_s - Y_s) (\tilde{H}_s(l) - H_s(l)) dM_s(l) \quad (2.15) \end{aligned}$$

Hence, we deduce $\forall t \in [0, T]$,

$$\begin{aligned} &\mathbf{E} \int_t^T 1_A |\tilde{Z}_s - Z_s|^2 ds + \lambda \mathbf{E} \sum_{l \in L} \int_t^T 1_A ((\tilde{H}_s - H_s)(l))^2 ds \\ &\leq \mathbf{E} \left(1_A |\tilde{h}(\tilde{X}_T, N_T^i) - h(X_T, N_T^i)|^2 \right) \\ &\quad + 2\mathbf{E} \int_t^T 1_A (\tilde{Y}_s - Y_s) \left(\tilde{f}(s, \tilde{\Theta}_s^{0, \xi, i}, N_s^i) - f(s, \Theta_s^{0, \xi, i}, N_s^i) \right) ds \quad (2.16) \end{aligned}$$

Moreover from Burkholder–Davis–Gundy inequalities, there exists $\gamma > 0$, changing from line to line such that

$$\begin{aligned} 2\mathbf{E} \sup_{0 \leq t \leq T} \left| \int_t^T 1_A (\tilde{Y}_s - Y_s) ((\tilde{Z}_s - Z_s) dB_s) \right| &\leq 2\gamma \mathbf{E} \left(\int_0^T 1_A |\tilde{Y}_s - Y_s|^2 |\tilde{Z}_s - Z_s|^2 ds \right)^{\frac{1}{2}} \\ &\leq \frac{1}{4} \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |\tilde{Y}_s - Y_s|^2 \right) + \gamma \mathbf{E} \int_0^T 1_A |\tilde{Z}_s - Z_s|^2 ds \end{aligned} \quad (2.17)$$

and similarly to Eq. (2.9),

$$\begin{aligned} 2\mathbf{E} \sup_{0 \leq t \leq T} \left| \int_t^T \sum_{l \in L} 1_A (\tilde{Y}_{s^-} - Y_{s^-}) ((\tilde{H}_s - H_s)(l) dM_s(l)) \right| \\ \leq \frac{1}{4} \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |\tilde{Y}_s - Y_s|^2 \right) + \lambda \gamma \mathbf{E} \sum_{l \in L} \int_0^T 1_A ((\tilde{H}_s - H_s)(l))^2 ds \end{aligned} \quad (2.18)$$

Hence combining Eqs. (2.16), (2.17) and (2.18), we find from Eq. (2.15) a constant $\beta_{(K, \lambda)}$ depending on λ and K such that

$$\begin{aligned} \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |\tilde{Y}_s - Y_s|^2 \right) + \mathbf{E} \int_0^T 1_A |\tilde{Z}_s - Z_s|^2 ds + \mathbf{E} \sum_{l \in L} \int_0^T 1_A ((\tilde{H}_s - H_s)(l))^2 ds \\ \leq \beta_{(K, \lambda)} \left[\mathbf{E} \left(1_A \left| \tilde{h}_{N_T^i}(\tilde{X}_T) - h_{N_T^i}(X_T) \right|^2 \right. \right. \\ \left. \left. + \mathbf{E} \left(\sup_{0 \leq t \leq T} \int_t^T 1_A (\tilde{Y}_s - Y_s) \left(\tilde{f}(s, \tilde{\Theta}_s^{0, \xi, i}, N_s^i) - f(s, \Theta_s^{0, \xi, i}, N_s^i) \right) ds \right) \right) \right] \end{aligned} \quad (2.19)$$

Using assumptions (H1.1), (H1.2) and standard estimates, there exists $\tilde{\beta}_K > 0$ only depending on K satisfying

$$\begin{aligned} 2\mathbf{E} \left(\sup_{0 \leq t \leq T} \int_t^T 1_A (\tilde{Y}_s - Y_s) \left(\tilde{f}(s, \tilde{\Theta}_s^{0, \xi, i}, N_s^i) - f(s, \Theta_s^{0, \xi, i}, N_s^i) \right) ds \right) \\ \leq \tilde{\beta}_K \left[\mathbf{E} \int_0^T 1_A |\tilde{X}_s - X_s|^2 ds + \mathbf{E} \int_0^T 1_A |\tilde{Y}_s - Y_s| (|\tilde{Z}_s - Z_s| + |\tilde{H}_s - H_s|) ds \right. \\ \left. + \mathbf{E} \int_0^T 1_A |\tilde{Y}_s - Y_s|^2 ds + \mathbf{E} \left(\int_0^T 1_A |\tilde{f} - f|(s, \Theta_s^{0, \xi, i}, N_s^i) |ds \right)^2 \right] \end{aligned}$$

So, plugging this inequality into Eq. (2.19) and adding with Eq. (2.14), we obtain modifying $\beta_{(K,\lambda)}$

$$\begin{aligned} & \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |\tilde{X}_s - X_s|^2 \right) + \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |\tilde{Y}_s - Y_s|^2 \right) + \mathbf{E} \sum_{l \in L} \int_0^T 1_A ((\tilde{H}_s - H_s)(l))^2 ds \\ & + \mathbf{E} \int_0^T 1_A |\tilde{Z}_s - Z_s|^2 ds \leq \beta_{(K,\lambda)} \left[\mathbf{E}(1_A |\tilde{\xi} - \xi|^2) + \mathbf{E} \left(1_A |\tilde{h} N_T^i(\tilde{X}_T) - h_{N_T^i}(X_T)|^2 \right) \right] \\ & + \mathbf{E} \int_0^T 1_A (|\tilde{X}_s - X_s|^2) ds + \mathbf{E} \int_0^T 1_A |\tilde{Z}_s - Z_s| (|\tilde{X}_s - X_s| ds + |\tilde{Y}_s - Y_s|) ds \\ & + \mathbf{E} \int_0^T 1_A (|\tilde{Y}_s - Y_s|^2) ds + \mathbf{E} \int_0^T 1_A |\tilde{H}_s - H_s| (|\tilde{X}_s - X_s| + |\tilde{Y}_s - Y_s|) ds \\ & + \mathbf{E} \int_0^T 1_A |(\tilde{\sigma} - \sigma)(s, X_s, Y_s, N_s^i)|^2 ds + \mathbf{E} \left(\int_0^T 1_A (|\tilde{b} - b| + |\tilde{f} - f|)(s, \Theta_s^{0,\xi,i}, N_s^i) ds \right)^2 \end{aligned}$$

Hence using once again standard estimates, we prove the existence of two constants $\beta_{(K,\lambda)}^{(i1)}$ and $C_{(K,\lambda)}^{(i2)}$ depending on K and λ such that for $T \leq C_{(K,\lambda)}^{(i2)}$,

$$\begin{aligned} & \mathbf{E} \left(\sup_{0 \leq s \leq T} 1_A |\tilde{X}_s - X_s|^2 \right) + \mathbf{E} \left(\sup_{0 \leq s \leq T} 1_A |\tilde{Y}_s - Y_s|^2 \right) + \mathbf{E} \sum_{l \in L} \int_0^T 1_A ((\tilde{H}_s - H_s)(l))^2 ds \\ & + \mathbf{E} \int_0^T 1_A |\tilde{Z}_s - Z_s|^2 ds \leq \beta_{(K,\lambda)}^{(i1)} \left[\mathbf{E}(1_A |\tilde{\xi} - \xi|^2) + \mathbf{E}(1_A |(\tilde{h} N_T^i - h_{N_T^i})(X_T)|^2) \right] \quad (2.20) \\ & + \mathbf{E} \left(\int_0^T 1_A (|\tilde{b} - b| + |\tilde{f} - f|)(s, \Theta_s^{0,\xi,i}, N_s^i) ds \right)^2 + \mathbf{E} \int_0^T 1_A |(\tilde{\sigma} - \sigma)(s, X_s, Y_s, N_s^i)|^2 ds \end{aligned}$$

□

COROLLARY 2.1 Assume that **(H1)** holds and $T \leq C_{(K,\lambda)}^{(i2)}$. Then $\forall (t, i) \in [0, T] \times \mathcal{K}$ and for every $x \in \mathbf{R}^d$, the unique solution $\Theta_s^{t,x,i} = (X_s^{t,x,i}, Y_s^{t,x,i}, H_s^{t,x,i}, Z_s^{t,x,i})_{t \leq s \leq T}$ of the problem $(E^{t,x})$ extended on whole $[0, T]$ by putting $\forall s \in [0, t]$, $X_s^{t,x,i} = x$; $Y_s^{t,x,i} = Y_t^{t,x,i}$; $H_s^{t,x,i} = Z_s^{t,x,i} = 0$ satisfies

(i) $\exists C_{(K,\Lambda,\lambda)}^{(i1)}$ such that $\forall (t, x) \in [0, T] \times \mathbf{R}^d$,

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq s \leq T} |X_s^{t,x,i}|^2 + \mathbf{E} \sup_{0 \leq s \leq T} |Y_s^{t,x,i}|^2 + \mathbf{E} \sum_{l \in L} \int_0^T (H_s^{t,x,i}(l))^2 ds \\ & + \mathbf{E} \int_0^T |Z_s^{t,x,i}|^2 ds \leq C_{(K,\Lambda,\lambda)}^{(i1)} (1 + |x|^2) \quad (2.21) \end{aligned}$$

(ii) $\forall (t, x), (t', x') \in ([0, T] \times \mathbf{R}^d)^2$, $\exists C_{(K, \Lambda, \lambda)}^{(i2)}$ such that

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq s \leq T} |X_s^{t,x,i} - X_s^{t',x',i}|^2 + \mathbf{E} \sup_{0 \leq s \leq T} |Y_s^{t,x,i} - Y_s^{t',x',i}|^2 + \mathbf{E} \int_0^T |Z_s^{t,x,i} - Z_s^{t',x',i}|^2 ds \\ & + \mathbf{E} \sum_{l \in L} \int_0^T (H_s^{t,x,i}(l) - H_s^{t',x',i}(l))^2 ds \leq \beta_{(K, \lambda)}^{(i1)} |x - x'|^2 + C_{(K, \Lambda, \lambda)}^{(i2)} |t - t'| \end{aligned} \quad (2.22)$$

Proof (i) Let $x \in \mathbf{R}^d$ and assume $T \leq C_{(K, \lambda)}^{(i2)}$. The process $(\Theta_s^{t,x,i})_{0 \leq s \leq T}$ is a solution of the problem

$$\left\{ \begin{array}{l} \forall s \in [0, T] \\ X_s^{t,x,i} = x + \int_0^s 1_{[t, T]}(r) b(r, \Theta_r^{t,x,i}, N_r^{t,i}) dr + \int_0^s 1_{[t, T]}(r) \sigma(r, X_r, Y_r, N_r^{t,i}) dB_r \\ Y_s^{t,x,i} = h_{N_T^i}(X_T^{t,x,i}) + \int_s^T 1_{[t, T]}(r) f(r, \Theta_r^{t,x,i}, N_r^{t,i}) dr - \int_s^T Z_r dB_r - \int_s^T \sum_{l \in L} H_r(l) dM_r(l) \\ \mathbf{E} \int_0^T (|X_r|^2 + |Y_r|^2 + |Z_r|^2 + |H_r|^2) dr < +\infty \end{array} \right.$$

Applying Theorem 2.2 to the quartet of functions $(1_{[t, T]} f_{N_s^i}, 1_{[t, T]} g_{N_s^i}, 1_{[t, T]} \sigma_{N_s^i}, h_{N_T^i})$ and $(0, 0, 0, 0)$, we have

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq s \leq T} |X_s^{t,x,i}|^2 + \mathbf{E} \sup_{0 \leq s \leq T} |Y_s^{t,x,i}|^2 + \mathbf{E} \int_0^T |Z_s^{t,x,i}|^2 ds \\ & + \mathbf{E} \sum_{l \in L} \int_0^T (H_s^{t,x,i}(l))^2 ds \leq \beta_{(K, \lambda)}^{(i1)} \left[|x|^2 + \mathbf{E} \left(|h_{N_T^i}(0)|^2 \right) + \mathbf{E} \int_0^T |\sigma_{N_s^i}(s, 0, 0)|^2 ds \right. \\ & \left. + \mathbf{E} \left(\int_0^T (|b_{N_s^i}| + |f_{N_s^i}|)(s, 0, 0, 0, 0) \right)^2 \right] \end{aligned}$$

applying assumption (H1.3), there exists a constant $C_{(K, \Lambda, \lambda)}^{(i1)}$ depending on K, Λ and λ such that

$$\mathbf{E} \sup_{0 \leq s \leq T} |X_s^{t,x,i}|^2 + \mathbf{E} \sup_{0 \leq s \leq T} |Y_s^{t,x,i}|^2 + \mathbf{E} \int_0^T |Z_s^{t,x,i}|^2 ds + \mathbf{E} \sum_{l \in L} \int_0^T (H_s^{t,x,i}(l))^2 ds \leq C_{(K, \Lambda, \lambda)}^{(i1)} (1 + |x|^2)$$

(ii) Let $(t, x) \in [0, T] \times \mathbf{R}^d$ and $(t', x') \in [0, T] \times \mathbf{R}^d$, $t' < t$.

Noting that the two quartets of functions $(1_{[t, T]} b_{N_s^i}, 1_{[t, T]} f_{N_s^i}, 1_{[t, T]} \sigma_{N_s^i}, h_{N_T^i})$ and $(1_{[t', T]} b_{N_s^i}, 1_{[t', T]} f_{N_s^i}, 1_{[t', T]} \sigma_{N_s^i}, h_{N_T^i})$, satisfy **(H1)**, using of Theorem 2.2 we obtain

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq s \leq T} |X_s^{t,x,i} - X_s^{t',x',i}|^2 + \mathbf{E} \sup_{0 \leq s \leq T} |Y_s^{t,x,i} - Y_s^{t',x',i}|^2 + \mathbf{E} \int_0^T |Z_s^{t,x,i} - Z_s^{t',x',i}|^2 ds \\ & + \mathbf{E} \sum_{l \in L} \int_0^T (H_s^{t,x,i}(l) - H_s^{t',x',i}(l))^2 ds \leq \beta_{(K, \lambda)}^{(1)} \left[|x - x'|^2 + \mathbf{E} \left(|h_{N_T^i} - h_{N_T^i} \right)(X_T) \right]^2 \\ & + \mathbf{E} \int_0^T \left| \left(1_{[t', T]} \sigma_{N_r^i} - 1_{[t, T]} \sigma_{N_r^i} \right) (r, X_r^{t,x,i}, Y_r^{t,x,i}) \right|^2 dr \\ & + \mathbf{E} \left(\int_0^T \left(|1_{[t', T]} b_{N_r^i} - 1_{[t, T]} b_{N_r^i}| + |1_{[t', T]} f_{N_r^i} - 1_{[t, T]} f_{N_r^i}| \right) (r, \Theta_r^{t,x,i}) dr \right)^2 \end{aligned} \quad (2.23)$$

and from assumption (H1.3) there exists a constant $C_{\Lambda,\lambda}$ depending only on Λ and λ such that

$$\begin{aligned} \mathbf{E} \left| \left(h_{N_T^{t,i}} - h_{N_T^{t',i}} \right) (X_T) \right|^2 &= \mathbf{E} \left| \left(h_{N_T^{t,i}} - h_{N_T^{t',i}} \right) (X_T) \right|^2 1_{\{N_T^{t,i} \neq N_T^{t',i}\}} \\ &\leq \mathbf{E} \left| \left(h_{N_T^{t,i}} - h_{N_T^{t',i}} \right) (X_T) \right|^2 1_{\{\nu([t',t]) \neq 0\}} \\ &\quad \mathbf{E} \left| \left(h_{N_T^{t,i}} - h_{N_T^{t',i}} \right) (X_T) \right|^2 \\ &\leq 2\Lambda^2 \mathbf{P}(\nu([t',t]) \neq 0) \leq C_{\Lambda,\lambda} |t' - t| \end{aligned}$$

where ν is the law of the first jump between t and t' .

So, using the same argument in the two last integrals and assumption (H1.3), we easily find from Eq. (2.23) a constant $C_{(K,\Lambda,\lambda)}^{(i2)}$ such that

$$\begin{aligned} &\mathbf{E} \sup_{0 \leq s \leq T} \left| X_s^{t,x,i} - X_s^{t',x',i} \right|^2 + \mathbf{E} \sup_{0 \leq s \leq T} \left| Y_s^{t,x,i} - Y_s^{t',x',i} \right|^2 + \mathbf{E} \int_0^T \left| Z_s^{t,x,i} - Z_s^{t',x',i} \right|^2 ds \\ &+ \mathbf{E} \sum_{l \in L} \int_0^T \left(H_s^{t,x,i}(l) - H_s^{t',x',i}(l) \right)^2 ds \leq \beta_{(K,\lambda)}^{(i1)} |x - x'|^2 + C_{(K,\Lambda,\lambda)}^{(i2)} |t - t'| \end{aligned}$$

□

We deduce the following result which is an obvious consequence of the previous corollary.

COROLLARY 2.2 Assume that **(H1)** holds. Then there exists some constants $C_{(K,\Lambda,\lambda)}^{(1.1)}$, $C_{(K,\Lambda,\lambda)}^{(1.2)}$, $\beta_{(K,\lambda)}^{(1)}$ and $C_{(K,\lambda)}^{(2)}$ such that $\forall T \leq C_{(K,\lambda)}^{(2)}$, the map

$$\begin{aligned} \theta : [0, T] \times \mathbf{R}^d \times \mathcal{K} &\rightarrow \mathbf{R} \\ (t, x, i) &\rightarrow Y_t^{t,x,i} \end{aligned}$$

which defines also a mapping

$$\begin{aligned} \theta : [0, T] \times \mathbf{R}^d &\rightarrow \mathbf{R}^k \\ (t, x) &\rightarrow (\theta(t, x, 1), \theta(t, x, 2), \dots, \theta(t, x, k)) \end{aligned}$$

satisfies

$$(i) \quad \forall (t, x) \in [0, T] \times \mathbf{R}^d, \forall (t', x') \in [0, T] \times \mathbf{R}^d, \quad \forall i \in \mathcal{K},$$

$$|\theta(t, x, i)|^2 \leq C_{(K,\Lambda,\lambda)}^{(1.1)} (1 + |x|^2) \quad (2.24)$$

$$|\theta(t, x, i) - \theta(t', x', i)|^2 \leq \beta_{(K,\lambda)}^{(1)} |x - x'|^2 + C_{(K,\Lambda,\lambda)}^{(1.2)} |t - t'| \quad (2.25)$$

(ii) $\forall i \in \mathcal{K}$, $\forall (t, x) \in [0, T] \times \mathbf{R}^d$ and for every \mathcal{G}_t measurable random vector ξ with finite second moment, there exists a P -null set G_Y^{ξ} such that $\forall s \in [t, T]$,

$$\forall \omega \notin G_Y^{\xi} \quad \theta(s, X_s^{t,\xi}(\omega), N_s^{t,i}) = Y_s^{t,\xi,i}(\omega) \quad (2.26)$$

$$\forall l \in L \quad H_s^{t,x,i}(l) = \theta(s, X_s^{t,x,i}, N_s^{t,i} + l) - \theta(s, X_s^{t,x,i}, N_s^{t,i}) \quad (2.27)$$

$$f_{N_s^{t,i}}(s, X_s^{t,x,i}, Y_s^{t,x,i}, H_s^{t,x,i}, Z_s^{t,x,i}) = \bar{f}_{N_s^{t,i}}(s, X_s^{t,x,i}, \theta(s, X_s^{t,x,i}), Z_s^{t,x,i}) \quad (2.28)$$

$$b_{N_s^{t,i}}(s, X_s^{t,x,i}, Y_s^{t,x,i}, H_s^{t,x,i}, Z_s^{t,x,i}) = \bar{b}_{N_s^{t,i}}(s, X_s^{t,x,i}, \theta(s, X_s^{t,x,i}), Z_s^{t,x,i}) \quad (2.29)$$

Proof Putting

$$\begin{aligned} C_{(K,\Lambda,\lambda)}^{(1.1)} &= \max_{i=1,\dots,k} C_{(K,\Lambda,\lambda)}^{(i1)}; \beta_{(K,\lambda)}^{(1)} = \max_{i=1,\dots,k} \beta_{(K,\lambda)}^{(i1)} \\ C_{(K,\Lambda,\lambda)}^{(1.2)} &= \max_{i=1,\dots,k} C_{(K,\Lambda,\lambda)}^{(i2)}; C_{(K,\lambda)}^{(2)} = \min_{i=1,\dots,k} C_{(K,\lambda)}^{(i2)} \end{aligned}$$

then we easily get Eqs. (2.24) and (2.5) as immediate consequences of the previous corollary.

Let us recall the proof of Eq. (2.26) which is an adaptation of one given in Ref. [2]. We nevertheless include a complete proof for the convenience of the reader.

Let $(t, x) \in [0, T] \times \mathbf{R}^d, i \in \mathcal{K}$ and ξ a \mathcal{G}_t measurable random vector with finite second moment. Applying Theorem 2.2 (using the same coefficients and different initial values), we have

$$\forall s \in [t, T], \forall \varepsilon > 0 \quad \mathbf{E}\left(1_{\{|\xi-x|<\varepsilon\}} |Y_s^{t,x,i} - Y_s^{t,\xi,i}|^2\right) \leq \beta_{(K,\lambda)}^{(i1)} \mathbf{E}\left(1_{\{|\xi-x|<\varepsilon\}} |\xi - x|^2\right)$$

Hence

$$\forall \varepsilon > 0 \quad \mathbf{E}\left(1_{\{|\xi-x|<\varepsilon\}} |\theta(t, x, i) - Y_t^{t,\xi,i}|^2\right) \leq \beta_{(K,\lambda)}^{(1)} \mathbf{E}\left(1_{\{|\xi-x|<\varepsilon\}} |\xi - x|^2\right)$$

Furthermore, we have

$$\begin{aligned} &\mathbf{E}\left(1_{\{|\xi-x|<\varepsilon\}} |\theta(t, \xi, i) - Y_t^{t,\xi,i}|^2\right) \\ &\leq 2\left[\mathbf{E}\left(1_{\{|\xi-x|<\varepsilon\}} |\theta(t, \xi, i) - \theta(t, x, i)|^2\right) + \mathbf{E}\left(1_{\{|\xi-x|<\varepsilon\}} |\theta(t, x, i) - Y_t^{t,\xi,i}|^2\right)\right] \end{aligned}$$

Using the Lipschitz property, Eq. (2.25), we obtain

$$\mathbf{E}\left(1_{\{|\xi-x|<\varepsilon\}} |\theta(t, \xi, i) - \theta(t, x, i)|^2\right) \leq \beta_{(K,\lambda)}^{(1)} \mathbf{E}\left(1_{\{|\xi-x|<\varepsilon\}} |\xi - x|^2\right)$$

Hence

$$\forall i = 1, \dots, k \quad \mathbf{E}\left(1_{\{|\xi-x|<\varepsilon\}} |\theta(t, \xi, i) - Y_t^{t,\xi,i}|^2\right) \leq 4\beta_{(K,\lambda)}^{(1)} \mathbf{E}\left(1_{\{|\xi-x|<\varepsilon\}} |\xi - x|^2\right).$$

Choosing $\varepsilon = (1/m), \quad x = (p/m), \quad p \in \mathbf{Z}^d$, we have

$$\forall m \in \mathbf{N} \sum_{p \in \mathbf{Z}^d} \mathbf{E}\left(1_{\{|\xi-(p/m)|<\varepsilon\}} |\theta(t, \xi, i) - Y_t^{t,\xi,i}|^2\right) \leq \frac{4}{m^2} \beta_{(K,\lambda)}^{(1)} \sum_{p \in \mathbf{Z}^d} \mathbf{E}\left(1_{\{|\xi-(p/m)|_\infty < (1/m)\}}\right)$$

therefore

$$\mathbf{E} \left(\left| \theta(t, \xi, i) - Y_t^{t, \xi, i} \right|^2 \right) \leq \frac{2^{d+2}}{m^2} \beta_{(K, \lambda)}^{(1)} \quad \forall i = 1, \dots, k; \quad \forall m \in \mathbf{N}$$

We deduce

$$\text{p.s } \theta(t, \xi, i) = Y_t^{t, \xi, i} \quad \forall i = 1, \dots, k \tag{2.30}$$

Elsewhere $\forall s \in [t, T]$, $(\Theta_u^{t, \xi, i})_{s \leq u \leq T} \triangleq (X_u^{t, \xi, i}, Y_u^{t, \xi, i}, H_u^{t, \xi, i}, Z_u^{t, \xi, i})_{s \leq u \leq T}$ is solution of the problem

$$\begin{cases} \forall s \in [t, T] \\ X_u^{t, \xi, i} = X_s^{t, \xi, i} + \int_s^u b(r, \Theta_r^{t, \xi, i}, N_r^{s, N_s^{t, i}}) dr + \int_s^u \sigma(r, X_r, Y_r, N_r^{s, N_s^{t, i}}) dB_r \\ Y_u^{t, \xi, i} = h_{N_r^{s, N_s^{t, i}}} (X_T^{t, \xi, i}) + \int_u^T f(r, \Theta_r^{t, \xi, i}, N_r^{s, N_s^{t, i}}) dr - \int_u^T Z_r dB_r - \int_u^T \sum_{l \in L} H_r(l) dM_r(l) \\ \mathbf{E} \int_s^T (|X_r|^2 + |Y_r|^2 + |Z_r|^2 + |H_r|^2) dr < +\infty \end{cases}$$

hence using Eq. (2.30) and uniqueness of solutions of our system, we obtain

$$\forall s \in [t, T], \quad \theta(s, X_s^{t, \xi, i}, N_s^{t, i}) = Y_s^{s, X_s^{t, \xi, i}, N_s^{t, i}} = Y_s^{t, \xi, i} \quad \text{p.s}$$

Using Ref. [9], lemma 2.2 we get Eqs. (2.27), (2.28) and (2.29). □

In the next section, we extend our results to the case of random coefficients in order to get some regularity properties of the map θ given in the previous corollary.

However, the proofs being the same as in the previous section, we only give the statments of the results.

Extension to the Random Coefficients Case

We assume $\forall i \in \mathcal{K}$ the functions b_i, f_i, σ_i and h_i are, respectively, $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^d) \otimes \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R}^{k-1}) \otimes \mathcal{B}(\mathbf{R}^d) / \mathcal{B}(\mathbf{R}^d)$, $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^d) \otimes \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R}^{k-1}) \otimes \mathcal{B}(\mathbf{R}^d) / \mathcal{B}(\mathbf{R})$, $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^d) \otimes \mathcal{B}(\mathbf{R}) / \mathcal{B}(\mathbf{R}^{d \times d})$ and $\mathcal{G}_T^0 \otimes \mathbf{R}^d / \mathcal{B}(\mathbf{R}^d)$ measurable.

THEOREM 2.3 (SOLUTION IN SMALL TIME DURATION) Let $i \in \mathcal{K}$ and assume that the coefficients satisfy assumption **(H1)**. Then for every \mathcal{G}_0 measurable random vector ξ with finite second moment a solution of $(E^{0, \xi})$ satisfies

- (i) $(X_s)_{0 \leq s \leq T}$ is continuous and $(Y_s)_{0 \leq s \leq T}$ has a càdlàg version.
- (ii) $\mathbf{E} \left(\sup_{0 \leq t \leq T} |X_t|^2 + \sup_{0 \leq t \leq T} |Y_t|^2 \right) < \infty$.

Moreover, there exists a constant $\tilde{C}_{(K, \lambda)}^{(i1)}$ depending on K and λ such that for $T \leq \tilde{C}_{(K, \lambda)}^{(i1)}$, the problem $(E^{0, \xi})$ admits a unique solution.

THEOREM 2.4 (A PRIORI ESTIMATES). Let $i \in \mathcal{K}$, (b, f, h, σ) and $(\tilde{b}, \tilde{f}, \tilde{h}, \tilde{\sigma})$ two quartets of functions satisfying assumption **(H1)** with the same constants K and Λ .

Let $(\Theta_s^{0,\xi,i})_{(0 \leq s \leq T)}$ (resp $(\tilde{\Theta}_s^{0,\tilde{\xi},i})_{(0 \leq s \leq T)}$) the associated solution to (b, f, h, σ) (resp $(\tilde{b}, \tilde{f}, \tilde{h}, \tilde{\sigma})$) with initial condition $(0, \xi, i)$ (resp $(0, \tilde{\xi}, i)$).

There exists two constants $\tilde{C}_{(K,\lambda)}^{(i2)} \leq \tilde{C}_{(K,\lambda)}^{(i1)}$ and $\tilde{\beta}_{(K,\lambda)}^{(i1)}$ depending on K and λ such that for every $T \leq \tilde{C}_{(K,\lambda)}^{(i2)}$ and for every $A \in \mathcal{G}_0$, the following holds

$$\begin{aligned} & \mathbf{E}(1_A \sup_{0 \leq s \leq T} |X_s - \tilde{X}_s|^2) + \mathbf{E}(1_A \sup_{0 \leq s \leq T} |Y_s - \tilde{Y}_s|^2) + \mathbf{E} \sum_{l \in \mathcal{L}} \int_0^T 1_A |(H_s - \tilde{H}_s)(l)|^2 ds \\ & + \mathbf{E} \int_0^T 1_A |Z_s - \tilde{Z}_s|^2 ds \leq \tilde{\beta}_{(K,\lambda)}^{(i1)} \left[\mathbf{E}(1_A |\xi - \tilde{\xi}|^2) + \mathbf{E}(1_A |(h - \tilde{h})(X_T)|^2) \right. \\ & \left. + \mathbf{E} \int_0^T 1_A |\sigma - \tilde{\sigma}(s, X_s, Y_s, N_s^i)|^2 ds + \mathbf{E} \left(\int_0^T 1_A (|b - \tilde{b}| + |f - \tilde{f}|)(s, \Theta_s^{0,\xi,i}, N_s^i) ds \right)^2 \right] \end{aligned}$$

COROLLARY 2.3 $\forall T \leq \tilde{C}_{(K,\lambda)}^{(i2)}$, $\forall (t, i) \in [0, T] \times \mathcal{K}$ under the assumption **(H1)**, for every $x \in \mathbf{R}^d$ we define the process $(\Theta_s^{t,x,i})_{t \leq s \leq T}$ as the unique solution of the problem

$$\left\{ \begin{array}{l} \forall s \in [t, T] \\ X_s^{t,x,i} = x + \int_t^s b(r, \Theta_r^{t,x,i}, N_r^{t,i}) dr + \int_t^s \sigma(r, X_r, Y_r, N_r^{t,i}) dB_r \\ Y_s^{t,x,i} = h_{N_r^{t,i}}(X_s^{t,x,i}) + \int_t^s f(r, \Theta_r^{t,x,i}, N_r^{t,i}) dr - \int_t^s Z_r^{t,x,i} dB_r - \int_t^s \sum_{l \in \mathcal{L}} H_r^{t,x,i}(l) dM_r(l) \\ \mathbf{E} \int_t^T (|X_r|^2 + |Y_r|^2 + |H_r|^2 + |Z_r|^2) dr < +\infty \end{array} \right.$$

extended on $[0, T]$ by putting

$$\left\{ \begin{array}{l} \forall s \in [0, t], \quad X_s^{t,x,i} = x; Y_s^{t,x,i} = \mathbf{E}(Y_t^{t,x,i} / \mathcal{G}_s) \\ Y_t^{t,x,i} = Y_0^{t,x,i} + \int_0^t Z_s^{t,x,i} dB_u + \int_0^t \sum_{l \in \mathcal{L}} H_s^{t,x,i}(l) dM_r(l) \end{array} \right.$$

then

(i) $\exists \tilde{C}_{(K,\Lambda,\lambda)}^{(i1)}$ such that $\forall (t, x) \in [0, T] \times \mathbf{R}^d$,

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq s \leq T} |X_s^{t,x,i}|^2 + \mathbf{E} \sup_{0 \leq s \leq T} |Y_s^{t,x,i}|^2 + \mathbf{E} \sum_{l \in \mathcal{L}} \int_0^T (H_s^{t,x,i}(l))^2 ds + \mathbf{E} \int_0^T |Z_s^{t,x,i}|^2 ds \\ & \leq \tilde{C}_{(K,\Lambda,\lambda)}^{(i1)} (1 + |x|^2) \end{aligned}$$

(ii) $\forall ((t, x), (t', x')) \in ([0, T] \times \mathbf{R}^d)^2$, $\exists \tilde{C}_{(K,\Lambda,\lambda)}^{(i2)}$ such that

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq s \leq T} |X_s^{t,x,i} - X_s^{t',x',i}|^2 + \mathbf{E} \sup_{0 \leq s \leq T} |Y_s^{t,x,i} - Y_s^{t',x',i}|^2 + \mathbf{E} \sum_{l \in \mathcal{L}} \int_0^T (H_s^{t,x,i}(l) - H_s^{t',x',i}(l))^2 ds \\ & + \mathbf{E} \int_0^T |Z_s^{t,x,i} - Z_s^{t',x',i}|^2 ds \leq \tilde{\beta}_{(K,\lambda)}^{(i1)} |x - x'|^2 + \tilde{C}_{(K,\Lambda,\lambda)}^{(i2)} |t - t'| \end{aligned}$$

THEOREM 2.5 $\forall i \in \mathcal{K}, \forall p \geq 1$ there exists two constants $\beta_{(p,K,\lambda,\Lambda)}^{(i)}$ and $\beta_{(p,K,\lambda)}^{(i)}$ such that $\forall T \leq \tilde{C}_{(K,\lambda)}^{(i2)}$, and for every \mathcal{G}_0 -measurable random vector ξ such that $\mathbf{E}|\xi|^{2p} < \infty$,

(i) the process $(\Theta_s^{0,\xi,i})_{0 \leq s \leq T}$ solution of $(E^{0,\xi})$ satisfies

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq s \leq T} |X_s^{0,\xi,i}|^{2p} + \mathbf{E} \sup_{0 \leq s \leq T} |Y_s^{0,\xi,i}|^{2p} + \mathbf{E} \left(\int_0^T \sum_{l \in \mathcal{L}} (H_s^{0,\xi,i}(l))^2 ds \right)^p \\ & + \mathbf{E} \left(\int_0^T |Z_s^{0,\xi,i}|^2 ds \right)^p \leq \beta_{(p,K,\lambda,\Lambda)}^{(i)} (1 + \mathbf{E}|\xi|^{2p}) \end{aligned} \quad (2.31)$$

(ii) For every quartet $(\tilde{b}, \tilde{f}, \tilde{\sigma}, \tilde{h})$ [resp (b, f, σ, h)] satisfying **(H1)** with the constants K, Λ , for every $A \in \mathcal{G}_0$ and for every random vector $\tilde{\xi}$ (resp ξ) with finite $2p$ th moment we have

$$\begin{aligned} & \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |\tilde{X}_s - X_s|^{2p} \right) + \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |\tilde{Y}_s - Y_s|^{2p} \right) \\ & + \mathbf{E} \left(1_A \int_0^T |\tilde{Z}_s|^2 ds \right)^p + \mathbf{E} \left(1_A \int_0^T \sum_{l \in \mathcal{L}} ((\tilde{H}_s - H_s)(l))^2 ds \right)^p \\ & \leq \beta_{(p,K,\lambda)}^{(i)} \mathbf{E} \left[1_A \left(|\tilde{\xi} - \xi|^{2p} + \left| (\tilde{h}N_T^i - hN_T^i)(X_T) \right|^{2p} \right. \right. \\ & \left. \left. + \left(\int_0^T |\tilde{\sigma} - \sigma|^2(s, X_s, Y_s, N_s^i) ds \right)^p + \left(\int_0^T |\tilde{b} - b| + |\tilde{f} - f|(s, \Theta_s, N_s^i) ds \right)^{2p} \right) \right] \end{aligned} \quad (2.32)$$

where $(\tilde{X}_s, \tilde{Y}_s, \tilde{H}_s, \tilde{Z}_s)$ (resp (X_s, Y_s, H_s, Z_s)) is the solution of $(E^{0,\tilde{\xi}})$ [resp $(E^{0,\xi})$] associated to $(\tilde{b}, \tilde{f}, \tilde{\sigma}, \tilde{h})$ [resp (b, f, σ, h)] and with initial values $(0, \tilde{\xi}, i)$ [resp $(0, \xi, i)$].

Proof Let us recall that we assume $T \leq 1$. Using the same technique developed in Ref. [2], we suppose in a first step that the processes $(\Theta_s^{0,\xi,i})_{0 \leq s \leq T}$ and $(\tilde{\Theta}_s^{0,\tilde{\xi},i})_{0 \leq s \leq T}$ satisfy p -integrability conditions defined below

$$(L_p) \mathbf{E} \sup_{0 \leq s \leq T} |X_s^{0,\xi,i}|^{2p} + \mathbf{E} \sup_{0 \leq s \leq T} |Y_s^{0,\xi,i}|^{2p} + \mathbf{E} \left(\int_0^T \sum_{l \in \mathcal{L}} (H_s^{0,\xi,i}(l))^2 ds \right)^p + \mathbf{E} \left(\int_0^T |Z_s^{0,\xi,i}|^2 ds \right)^p < \infty$$

$$(\tilde{L}_p) \mathbf{E} \sup_{0 \leq s \leq T} |\tilde{X}_s^{0,\tilde{\xi},i}|^{2p} + \mathbf{E} \sup_{0 \leq s \leq T} |\tilde{Y}_s^{0,\tilde{\xi},i}|^{2p} + \mathbf{E} \left(\int_0^T \sum_{l \in \mathcal{L}} (\tilde{H}_s^{0,\tilde{\xi},i}(l))^2 ds \right)^p + \mathbf{E} \left(\int_0^T |\tilde{Z}_s^{0,\tilde{\xi},i}|^2 ds \right)^p < \infty$$

Let us prove the existence of a constant $\tilde{C}(p, K, \lambda)$ such that for every $T \leq \tilde{C}(p, K, \lambda)$ inequality Eq. (2.32) holds. For convenience in the proof, we omit the superscript $0, \tilde{\xi}, i$ and $0, \tilde{\xi}, i$

Let $A \in \mathcal{G}_0$. Thanks to Itô's formula with the function $\varphi_p(x) = x^{2p}$, $p \geq 1$, we have $\forall t \in [0, T]$,

$$\begin{aligned} \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |\tilde{X}_s - X_s|^{2p} \right) &\leq \mathbf{E}(1_A |\tilde{\xi} - \xi|^{2p}) + 2p \mathbf{E} \sup_{0 \leq t \leq T} \left(\int_0^t 1_A |\tilde{X}_s - X_s|^{2p-1} (\tilde{b}(s, \tilde{\Theta}_s, N_s^i) \right. \\ &\quad \left. - b(s, \Theta_s, N_s^i)) ds \right) + 2p \mathbf{E} \sup_{0 \leq t \leq T} \left(\int_0^t 1_A |\tilde{X}_s - X_s|^{2p-1} (\tilde{\sigma}(s, \tilde{X}_s, \tilde{Y}_s, N_s^i) \right. \\ &\quad \left. - \sigma(s, X_s, Y_s, N_s^i)) dB_s \right) + p(2p - 1) \mathbf{E} \int_0^T 1_A |\tilde{X}_s - X_s|^{2p-2} \\ &\quad \times |\tilde{\sigma}(s, \tilde{X}_s, \tilde{Y}_s, N_s^i) - \sigma(s, X_s, Y_s, N_s^i)|^2 ds \end{aligned}$$

Using Burkholder–Davis–Gundy inequalities and assumption (H1.1)–(H1.2), we obtain (with $\gamma > 0$)

$$\begin{aligned} \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |\tilde{X}_s - X_s|^{2p} \right) &\leq \mathbf{E}(1_A |\tilde{\xi} - \xi|^{2p}) \\ &\quad + 2pK \mathbf{E} \left[\int_0^T 1_A |\tilde{X}_s - X_s|^{2p-1} |(\tilde{b} - b)(s, \Theta_s, N_s^i)| ds \right] \\ &\quad + \int_0^T 1_A |\tilde{X}_s - X_s|^{2p-1} (|\tilde{X}_s - X_s| + |\tilde{Y}_s - Y_s| \\ &\quad + |\tilde{H}_s - H_s| + |\tilde{Z}_s - Z_s|) ds \\ &\quad + 2p\gamma \mathbf{E} \left(\int_0^T 1_A |\tilde{X}_s - X_s|^{4p-2} |\tilde{\sigma}(s, \tilde{X}_s, \tilde{Y}_s, N_s^i) \right. \\ &\quad \left. - \sigma(s, X_s, Y_s, N_s^i)|^2 ds \right)^{\frac{1}{2}} \\ &\quad + p(2p - 1) \mathbf{E} \int_0^T 1_A |\tilde{X}_s - X_s|^{2p-2} |\tilde{\sigma}(s, \tilde{X}_s, \tilde{Y}_s, N_s^i) \\ &\quad - \sigma(s, X_s, Y_s, N_s^i)|^2 ds \end{aligned}$$

Hence there exists a constant $c_{(p,K)}$ depending on p and K such that

$$\begin{aligned} \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |\tilde{X}_s - X_s|^{2p} \right) &\leq c_{(p,K)} \left[\mathbf{E}(1_A |\tilde{\xi} - \xi|^{2p}) \right. \\ &\quad + \mathbf{E} \int_0^T 1_A |\tilde{X}_s - X_s|^{2p-1} |(\tilde{b} - b)(s, \Theta_s, N_s^i)| ds \\ &\quad + \mathbf{E} \int_0^T 1_A |\tilde{X}_s - X_s|^{2p-1} (|\tilde{X}_s - X_s| \\ &\quad + |\tilde{Y}_s - Y_s| + |\tilde{H}_s - H_s| + |\tilde{Z}_s - Z_s|) ds \\ &\quad + \mathbf{E} \left(\int_0^T 1_A |\tilde{X}_s - X_s|^{4p-2} |(\tilde{\sigma} - \sigma)(s, X_s, Y_s, N_s^i)|^2 ds \right)^{\frac{1}{2}} \\ &\quad \left. + \mathbf{E} \left(\int_0^T 1_A |\tilde{X}_s - X_s|^{2p-2} |(\tilde{\sigma} - \sigma)(s, X_s, Y_s, N_s^i)|^2 ds \right) \right] \end{aligned}$$

which implies using Young inequalities and modifying $c_{(p,K)}$

$$\begin{aligned} \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |\tilde{X}_s - X_s|^{2p} \right) &\leq c_{(p,K)} \left[\mathbf{E}(1_A |\tilde{\xi} - \xi|^{2p}) + \mathbf{E} \int_0^T 1_A (|\tilde{X}_s - X_s|^{2p} + |\tilde{Y}_s - Y_s|^{2p}) ds \right. \\ &\quad + T^p \mathbf{E} \left(\int_0^T 1_A |\tilde{Z}_s - Z_s|^2 ds \right)^p + T^p \mathbf{E} \left(\sum_{l \in L} \int_0^T 1_A (\tilde{H}_s(l) - H_s(l))^2 ds \right)^p \\ &\quad \left. + \mathbf{E} \left(\int_0^T 1_A |(\tilde{b} - b)(s, \Theta_s, N_s^i)| ds \right)^{2p} + \mathbf{E} \left(\int_0^T 1_A |(\tilde{\sigma} - \sigma)(s, X_s, Y_s, N_s^i)|^2 ds \right)^p \right] \quad (2.33) \end{aligned}$$

Furthermore using the same function, from Itô's formula for discontinuous processes we have

$$\begin{aligned} |\tilde{Y}_t - Y_t|^{2p} &= |\tilde{Y}_T - Y_T|^{2p} + 2p \int_t^T (\tilde{Y}_{s-} - Y_{s-})^{2p-1} (\tilde{f}(s, \tilde{\Theta}_s, N_s^i) - f(s, \Theta_s, N_s^i)) ds \\ &\quad - p(2p-1) \int_t^T (\tilde{Y}_{s-} - Y_{s-})^{2p-2} |\tilde{Z}_s - Z_s|^2 ds \\ &\quad - 2p \int_t^T (\tilde{Y}_{s-} - Y_{s-})^{2p-1} (\tilde{Z}_s - Z_s) dB_s \\ &\quad - 2p \int_t^T \sum_{l \in L} (\tilde{Y}_{s-} - Y_{s-})^{2p-1} (\tilde{H}_s(l) - H_s(l)) dM_s(l) \\ &\quad - \sum_{t < s \leq T} \left[|\tilde{Y}_s - Y_s|^{2p} - |\tilde{Y}_{s-} - Y_{s-}|^{2p} - 2p(\tilde{Y}_{s-} - Y_{s-})^{2p-1} \Delta(\tilde{Y}_s - Y_s) \right] \quad (2.34) \end{aligned}$$

Putting $V_s = |\tilde{Y}_s - Y_s|^2 - |\tilde{Y}_{s^-} - Y_{s^-}|^2 = (2(\tilde{Y}_{s^-} - Y_{s^-}) + (\tilde{H}_s - H_s))(\tilde{H}_s - H_s)$, we deduce

$$\begin{aligned} & |\tilde{Y}_s - Y_s|^{2p} - |\tilde{Y}_{s^-} - Y_{s^-}|^{2p} - 2p(\tilde{Y}_{s^-} - Y_{s^-})^{2p-1} \Delta(\tilde{Y}_s - Y_s) \\ &= \left(|\tilde{Y}_{s^-} - Y_{s^-}|^2 + V_s \right)^p - \left(|\tilde{Y}_{s^-} - Y_{s^-}|^2 \right)^p - 2p(\tilde{Y}_{s^-} - Y_{s^-})^{2p-1} (\tilde{H}_s(l) - H_s(l)) \\ &= \left(|\tilde{Y}_{s^-} - Y_{s^-}|^2 + V_s \right)^p - \left(|\tilde{Y}_{s^-} - Y_{s^-}|^2 \right)^p - p \left(|\tilde{Y}_{s^-} - Y_{s^-}|^2 \right)^{p-1} V_s \\ &\quad + p(\tilde{Y}_{s^-} - Y_{s^-})^{2(p-1)} (\tilde{H}_s - H_s)^2 \end{aligned}$$

Hence substituting the last sum in Eq. (2.34) by the quantity

$$\begin{aligned} & \sum_{t < s \leq T} \left[\left(|\tilde{Y}_{s^-} - Y_{s^-}|^2 + V_s \right)^p - \left(|\tilde{Y}_{s^-} - Y_{s^-}|^2 \right)^p - p \left(|\tilde{Y}_{s^-} - Y_{s^-}|^2 \right)^{p-1} V_s \right] \\ & \quad + p \sum_{t < s \leq T} (\tilde{Y}_{s^-} - Y_{s^-})^{2(p-1)} (\Delta(\tilde{Y}_s - Y_s))^2 + \lambda p \int_t^T (\tilde{Y}_{s^-} - Y_{s^-})^{2(p-1)} |\tilde{H}_s - H_s|^2 ds \end{aligned}$$

we obtain $\forall t \in [0, T]$ and $A \in \mathcal{G}_0$,

$$\begin{aligned} & 1_A |\tilde{Y}_t - Y_t|^{2p} + p(2p - 1) \int_t^T 1_A (\tilde{Y}_{s^-} - Y_{s^-})^{2p-2} |\tilde{Z}_s - Z_s|^2 ds \\ & \quad + p \sum_{t < s \leq T} 1_A (\tilde{Y}_{s^-} - Y_{s^-})^{2(p-1)} (\Delta(\tilde{Y}_s - Y_s))^2 + \lambda p \int_t^T 1_A (\tilde{Y}_{s^-} - Y_{s^-})^{2(p-1)} |\tilde{H}_s - H_s|^2 ds \\ &= |\tilde{Y}_T - Y_T|^{2p} + 2p \int_t^T 1_A (\tilde{Y}_{s^-} - Y_{s^-})^{2p-1} (\tilde{f}(s, \tilde{\Theta}_s, N_s^i) - f(s, \Theta_s, N_s^i)) ds \\ & \quad - 2p \int_t^T 1_A (\tilde{Y}_{s^-} - Y_{s^-})^{2p-1} (\tilde{Z}_s - Z_s) dB_s \\ & \quad - 2p \int_t^T \sum_{l \in \mathcal{L}} 1_A (\tilde{Y}_{s^-} - Y_{s^-})^{2p-1} (\tilde{H}_s(l) - H_s(l)) dM_s(l) \\ & \quad - \sum_{t < s \leq T} 1_A \left[\left(|\tilde{Y}_{s^-} - Y_{s^-}|^2 + V_s \right)^p - \left(|\tilde{Y}_{s^-} - Y_{s^-}|^2 \right)^p - p \left(|\tilde{Y}_{s^-} - Y_{s^-}|^2 \right)^{p-1} V_s \right] \end{aligned}$$

Using the convexity property of φ_p , we prove that the last sum of the right hand side is positive. Therefore, there exists $c_p > 0$ such that $\forall t \in [0, T]$,

$$\begin{aligned} & \mathbf{E}(1_A |\tilde{Y}_t - Y_t|^{2p}) + \mathbf{E} \int_t^T 1_A |\tilde{Y}_s - Y_s|^{2(p-1)} (|\tilde{Z}_s - Z_s|^2 + \lambda |\tilde{H}_s - H_s|^2) ds \\ & \leq c_p \left[\mathbf{E} \int_t^T 1_A (\tilde{Y}_s - Y_s)^{2p-1} (\tilde{f}(s, \tilde{\Theta}_s, N_s^i) - f(s, \Theta_s, N_s^i)) ds + \mathbf{E} 1_A |\tilde{Y}_T - Y_T|^{2p} \right] \quad (2.35) \end{aligned}$$

Moreover since $\varphi_p'' > 0$ by the same argument, the last sum in Eq. (2.34) is positive.

Using the Burkholder–Davis–Gundy inequalities, we find a constant \tilde{c}_p only depending on p which can change line to line such that

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq t \leq T} \left| \int_t^T \sum_{l \in \mathcal{L}} 1_A (\tilde{Y}_s - Y_s)^{2p-1} (\tilde{H}_s(l) - H_s(l)) dM_s(l) \right| \\ & \leq \frac{1}{4} \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |\tilde{Y}_s - Y_s|^{2p} \right) + \lambda \tilde{c}_p \mathbf{E} \int_t^T \sum_{l \in \mathcal{L}} 1_A (\tilde{Y}_s - Y_s)^{2(p-1)} (\tilde{H}_s(l) - H_s(l))^2 ds \\ & \mathbf{E} \sup_{0 \leq t \leq T} \left| \int_t^T 1_A (\tilde{Y}_s - Y_s)^{2p-1} (\tilde{Z}_s - Z_s) dB_s \right| \leq \frac{1}{4} \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |\tilde{Y}_s - Y_s|^{2p} \right) \\ & \quad + \tilde{c}_p \mathbf{E} \int_t^T 1_A (\tilde{Y}_s - Y_s)^{2(p-1)} |\tilde{Z}_s - Z_s|^2 ds \end{aligned}$$

Hence using these two inequalities and modifying \tilde{c}_p if necessary we deduce from Eq. (2.34)

$$\begin{aligned} \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |\tilde{Y}_s - Y_s|^{2p} \right) & \leq \tilde{c}_p \left[\mathbf{E} (1_A |\tilde{Y}_T - Y_T|^{2p}) \right. \\ & \quad + \mathbf{E} \int_0^T 1_A |\tilde{Y}_s - Y_s|^{2(p-1)} (|\tilde{Z}_s - Z_s|^2 + \lambda |\tilde{H}_s - H_s|^2) ds \\ & \quad \left. + \int_0^T 1_A |\tilde{Y}_s - Y_s|^{2p-1} (\tilde{f}(s, \tilde{\Theta}_s, N_s^i) - f(s, \Theta_s, N_s^i)) ds \right] \quad (2.36) \end{aligned}$$

Combining Eqs. (2.35), (2.36) and using assumptions (H1.1)–(H1.2), we find a constant $c_{(p,K)}$ depending on p and K satisfying

$$\begin{aligned} & \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |\tilde{Y}_s - Y_s|^{2p} \right) + \mathbf{E} \int_0^T 1_A |\tilde{Y}_s - Y_s|^{2(p-1)} (|\tilde{Z}_s - Z_s|^2 + \lambda |\tilde{H}_s - H_s|^2) ds \\ & \leq c_{(p,K)} \left[\mathbf{E} (1_A |\tilde{Y}_T - Y_T|^{2p}) + \mathbf{E} \int_0^T 1_A (|\tilde{X}_s - X_s|^{2p} + |\tilde{Y}_s - Y_s|^{2p}) ds \right. \\ & \quad \left. + \mathbf{E} \int_0^T 1_A |\tilde{Y}_s - Y_s|^{2p-1} (|\tilde{Z}_s - Z_s| + |\tilde{H}_s - H_s|) ds + \mathbf{E} \left(\int_0^T 1_A |\tilde{f} - f|(s, \Theta_s, N_s^i) ds \right)^p \right] \end{aligned}$$

so, using standard estimates, we find $c_{(p,K,\lambda)}$ depending on p , λ and K such that

$$\begin{aligned} & \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |\tilde{Y}_s - Y_s|^{2p} \right) + \frac{1}{2} \mathbf{E} \int_0^T 1_A |\tilde{Y}_s - Y_s|^{2(p-1)} (|\tilde{Z}_s - Z_s|^2 + \lambda |\tilde{H}_s - H_s|^2) ds \\ & \leq c_{(p,K,\lambda)} \left[\mathbf{E} (1_A |\tilde{Y}_T - Y_T|^{2p}) + \mathbf{E} \int_0^T 1_A (|\tilde{X}_s - X_s|^{2p} + |\tilde{Y}_s - Y_s|^{2p}) ds \right] \\ & \quad + \mathbf{E} \left(\int_0^T 1_A |\tilde{f} - f|(s, \Theta_s, N_s^i) ds \right)^{2p} \quad (2.37) \end{aligned}$$

Furthermore, $\forall s \in [0, T]$,

$$\begin{aligned} & \int_s^T (\tilde{Z}_r - Z_r) dB_r + \int_s^T \sum_{l \in L} (\tilde{H}_r(l) - H_r(l)) dM_r(l) \\ &= (\tilde{Y}_T - Y_T) - (\tilde{Y}_s - Y_s) + \int_s^T (\tilde{f}(r, \tilde{\Theta}_r, N_r^i) - f(r, \Theta_r, N_r^i)) dr \end{aligned}$$

The Burkholder–Davis–Gundy inequality gives us a constant $\alpha_{p,\lambda}$ depending on p and λ such that

$$\begin{aligned} \mathbf{E} \left(\int_0^T (|\tilde{Z}_s - Z_s|^2 + |\tilde{H}_s - H_s|^2) ds \right)^p &\leq \alpha_{p,\lambda} \left[2 \mathbf{E} \sup_{0 \leq s \leq T} |\tilde{Y}_s - Y_s|^{2p} \right. \\ &\quad \left. + \mathbf{E} \left(\int_0^T |\tilde{f}(r, \tilde{\Theta}_r, N_r^i) - f(r, \Theta_r, N_r^i)| dr \right)^{2p} \right] \end{aligned}$$

Thanks to assumptions (H1.1)–(H1.2), there exists a constant $\alpha_{(p,K,\lambda)}$ depending on p , λ and K such that

$$\begin{aligned} \mathbf{E} \left(1_A \int_0^T (|\tilde{Z}_s - Z_s|^2 + |\tilde{H}_s - H_s|^2) ds \right)^p &\leq \alpha_{(p,K,\lambda)} \left[\mathbf{E} \int_0^T 1_A (|\tilde{X}_s - X_s|^{2p} + |\tilde{Y}_s - Y_s|^{2p}) ds \right. \\ &\quad \left. + \mathbf{E} \left(\int_0^T 1_A |\tilde{f} - f|(s, \Theta_s, N_s^i) ds \right)^{2p} \right. \\ &\quad \left. + T^p \mathbf{E} \left(\int_0^T 1_A (|\tilde{Z}_s - Z_s|^2 + |\tilde{H}_s - H_s|^2) ds \right)^p \right] \quad (2.38) \end{aligned}$$

Hence using Eqs. (2.33), (2.37) and (2.38), we prove the existence of two constants $\tilde{C}_{(p,K,\lambda)}$ and $\beta_{(p,K,\lambda)}^{(i)}$ depending on p , K and λ such that for $T \leq \tilde{C}_{(p,K,\lambda)}$, we have

$$\begin{aligned} & \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |\tilde{X}_s - X_s|^{2p} \right) + \mathbf{E} \left(1_A \sup_{0 \leq s \leq T} |\tilde{Y}_s - Y_s|^{2p} \right) \\ &+ \mathbf{E} \left(\int_0^T \sum_{l \in L} 1_A (\tilde{H}_s(l) - H_s(l))^2 ds \right)^p + \mathbf{E} \left(\int_0^T 1_A |\tilde{Z}_s - Z_s|^2 ds \right)^p \\ &\leq \beta_{(p,K,\lambda)}^{(i)} \mathbf{E} \left[1_A \left(|\tilde{\xi} - \xi|^{2p} + |(\tilde{h}_{N_T^i} - h_{N_T^i})(\tilde{X}_T)|^{2p} + \left(\int_0^T (|\tilde{\sigma} - \sigma|^2(s, X_s, Y_s, N_s^i)) ds \right)^p \right. \right. \\ &\quad \left. \left. + \left(\int_0^T (|\tilde{b} - b| + |\tilde{f} - f|)(s, X_s, Y_s, H_s, Z_s, N_s^i) ds \right)^{2p} \right) \right] \quad (2.39) \end{aligned}$$

For every process $(\Theta_s)_{0 \leq s \leq T}$ and $(\tilde{\Theta}_s)_{0 \leq s \leq T}$ satisfying (L_p) and (\tilde{L}_p) , thanks to theorem 2.3, the iterated scheme

$$\left\{ \begin{array}{l} X_s^{n+1} = \xi + \int_0^s b(r, X_r^{n+1}, Y_r^n, H_r^n, Z_r^n, N_r^i) dr + \int_0^s \sigma(r, X_r^{n+1}, Y_r^n, N_r^i) dB_r \\ Y_s^{n+1} = h_{N_T^i}(X_T^{n+1}) + \int_s^T f(r, \Theta_r^{n+1}, N_r^i) dr - \int_s^T Z_r^{n+1} dB_r - \int_s^T \sum_{l \in L} H_r^{n+1}(l) dM_r(l) \end{array} \right\}$$

satisfies $\|\Theta_r^n - \Theta_r\|_{B^2} \rightarrow 0$ as $n \rightarrow \infty$.

Hence choosing a quartet $(X_s^0, Y_s^0, H_s^0, Z_s^0)$ satisfying (L_p) conditions and using Pardoux–Buckdahn [8] we prove that the process $(\Theta_s^n)_{0 \leq s \leq T}$ verifies (L_p) .

Let us prove the existence of a constant $\gamma_{(p,K,\lambda)}$ such that for $T \leq \gamma_{(p,K,\lambda)}$ the sequence $(\Theta_s^n)_{s \in [0,T], n \in \mathbb{N}}$ satisfies

$$\|\Theta_r^n - \Theta_r^m\|_{B_{[0,T]}^{2p}} \rightarrow 0 \quad \text{whenever } m, n \rightarrow \infty$$

We assume $T \leq \tilde{C}_{(p,K,\lambda)}$ and let $n \in \mathbb{N}$ be fixed. Applying Eq. (2.39) to the previous iterated scheme, we obtain

$$\begin{aligned} & \mathbf{E} \left(\sup_{0 \leq s \leq T} |X_s^{n+2} - X_s^{n+1}|^{2p} \right) + \mathbf{E} \left(\sup_{0 \leq s \leq T} |Y_s^{n+2} - Y_s^{n+1}|^{2p} \right) \\ & + \mathbf{E} \left(\int_0^T \sum_{l \in L} (H_s^{n+2}(l) - H_s^{n+1}(l))^2 ds \right)^p + \mathbf{E} \left(\int_0^T |Z_s^{n+2} - Z_s^{n+1}|^2 ds \right)^p \\ & \leq \beta_{(p,K,\lambda)}^{(i)} \mathbf{E} \left[\left(\int_0^T |\sigma(s, X_s^{n+1}, Y_s^{n+1}, N_s^i) - \sigma(s, X_s^{n+1}, Y_s^n, N_s^i)|^2 ds \right)^p \right] \\ & + \left(\int_0^T (|b(s, X_s^{n+1}, Y_s^{n+1}, H_s^{n+1}, Z_s^{n+1}, N_s^i) - b(s, X_s^{n+1}, Y_s^n, H_s^n, Z_s^n, N_s^i)| ds)^{2p} \right) \end{aligned}$$

and with Schwarz' inequality, we deduce

$$\begin{aligned} & \mathbf{E} \left(\sup_{0 \leq s \leq T} |X_s^{n+2} - X_s^{n+1}|^{2p} \right) + \mathbf{E} \left(\sup_{0 \leq s \leq T} |Y_s^{n+2} - Y_s^{n+1}|^{2p} \right) \\ & + \mathbf{E} \left(\int_0^T \sum_{l \in L} (H_s^{n+2}(l) - H_s^{n+1}(l))^2 ds \right)^p + \mathbf{E} \left(\int_0^T |Z_s^{n+2} - Z_s^{n+1}|^2 ds \right)^p \\ & \leq K^{2p} (T^p + T^{2p}) \beta_{(p,K,\lambda)}^{(i)} \mathbf{E} \left[\sup_{0 \leq s \leq T} |Y_s^{n+1} - Y_s^n|^{2p} + \left(\int_0^T |Z_s^{n+1} - Z_s^n|^2 ds \right)^p \right] \\ & + \left(\int_0^T \sum_{l \in L} (H_s^{n+1}(l) - H_s^n(l))^2 ds \right)^p \end{aligned}$$

Therefore with a backward induction, there exists a constant $\gamma_{(p,K,\lambda)}$ such that the following series

$$\begin{aligned} & \left(\mathbf{E} \sup_{0 \leq s \leq T} |X_s^{n+1} - X_s^n|^{2p} \right)^{\frac{1}{2p}} + \left(\mathbf{E} \sup_{0 \leq s \leq T} |Y_s^{n+1} - Y_s^n|^{2p} \right)^{\frac{1}{2p}} \\ & + \left(\mathbf{E} \left(\int_0^T |Z_s^{n+1} - Z_s^n|^2 ds \right)^p \right)^{\frac{1}{2p}} + \left(\mathbf{E} \left(\int_0^T \sum_{l \in L} (H_s^{n+1}(l) - H_s^n(l))^2 ds \right)^p \right)^{\frac{1}{2p}} \end{aligned}$$

converges if $T \leq \gamma_{(p,K,\lambda)}$, which is enough to conclude.

Hence applying Eq. (2.32) to the quartet (X_s, Y_s, H_s, Z_s) and $(0, 0, 0, 0)$, we obtain

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq s \leq T} |X_s^{0,\xi,i}|^{2p} + \mathbf{E} \sup_{0 \leq s \leq T} |Y_s^{0,\xi,i}|^{2p} + \mathbf{E} \left(\int_0^T \sum_{l \in L} (H_s^{0,\xi,i}(l))^2 ds \right)^p + \mathbf{E} \left(\int_0^T |Z_s^{0,\xi,i}|^2 ds \right)^p \\ & \leq \beta_{(p,K,\lambda)}^{(i)} \mathbf{E} \left[|\xi|^{2p} + |h(0)|^{2p} + \left(\int_0^T |\sigma(s, 0, 0, N_s^i)|^2 ds \right)^p \right. \\ & \quad \left. + \left(\int_0^T |b(s, 0, 0, 0, 0, N_s^i)| + |f(s, 0, 0, 0, 0, N_s^i)| ds \right)^{2p} \right] \end{aligned}$$

and with the help of assumption (H1.3), we find a constant $\beta_{(p,K,\lambda,\Lambda)}^{(i)}$ such that for every $T \leq \gamma_{(p,K,\lambda)}$,

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq s \leq T} |X_s^{0,\xi,i}|^{2p} + \mathbf{E} \sup_{0 \leq s \leq T} |Y_s^{0,\xi,i}|^{2p} + \mathbf{E} \left(\sum_{l \in L} (H_s^{0,\xi,i}(l))^2 ds \right)^p + \mathbf{E} \left(\int_0^T |Z_s^{0,\xi,i}|^2 ds \right)^p \\ & \leq \beta_{(p,K,\lambda,\Lambda)}^{(i)} \mathbf{E}(1 + |\xi|^{2p}) \end{aligned}$$

Using the same method as in Ref. [2] theorem 4.5, we extend this result to $T \leq \tilde{C}_{(K,\lambda)}^{(i2)}$. \square

Taking a deterministic initial value, we get the following

COROLLARY 2.4 $\forall i \in \mathcal{K}, p \geq 1$, there exists a constant $\tilde{C}_{p,\lambda}^{k,\Lambda,i}$ depending on K, Λ, λ , and p such that for every $T \leq \tilde{C}_{(K,\lambda)}^{(i2)}$, we have

(i) For every $(t, x) \in [0, T] \times \mathbf{R}^d$

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq s \leq T} |X_s^{t,x,i}|^{2p} + \mathbf{E} \sup_{0 \leq s \leq T} |Y_s^{t,x,i}|^{2p} + \mathbf{E} \left(\int_0^T \sum_{l \in L} (H_s^{t,x,i}(l))^2 ds \right)^p + \mathbf{E} \left(\int_0^T |Z_s^{t,x,i}|^2 ds \right)^p \\ & \leq \beta_{(p,K,\lambda,\Lambda)}^{(i)} (1 + |x|^{2p}) \end{aligned} \tag{2.40}$$

(ii) For every $((t, x), (t', x')) \in ([0, T] \times \mathbf{R}^d)^2$

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq s \leq T} |X_s^{t,x,i} - X_s^{t',x',i}|^{2p} + \mathbf{E} \sup_{0 \leq s \leq T} |Y_s^{t,x,i} - Y_s^{t',x',i}|^{2p} \\ & + \mathbf{E} \left(\int_0^T \sum_{l \in L} \left((H_s^{t,x,i} - H_s^{t',x',i})(l) \right)^2 ds \right)^p + \mathbf{E} \left(\int_0^T |Z_s^{t,x,i} - Z_s^{t',x',i}|^2 ds \right)^p \\ & \leq \beta_{(p,K,\lambda)}^{(i)} |x - x'|^{2p} + \tilde{c}_{p,\lambda}^{K,A,i} |t - t'|^p \end{aligned} \quad (2.41)$$

□

In the sequel to this section, for all $i \in \mathcal{K}$, we consider the following system for the use of Malliavin Calculus.

$$(E^*) \begin{cases} \forall s \in [0, T] \\ X_s = \xi_s + \int_0^s b(r, \Theta_r^{0,\xi_s,i}, N_r^i) dr + \int_0^s \sigma(r, X_r, Y_r, N_r^i) dB_r \\ Y_s = h_{N_T^i}(X_T^{t,x,i}) + \int_s^T f(r, \Theta_r^{0,\xi_s,i}, N_r^i) dr - \int_s^T Z_r dB_r - \int_s^T \sum_{l \in L} H_r(l) dM_r(l) \\ \mathbf{E} \int_0^T (|X_t|^2 + |Y_t|^2 + |H_t|^2 + |Z_t|^2) dt < \infty \end{cases}$$

where $(\xi_s)_{0 \leq s \leq T}$ is continuous and \mathcal{G}_s^0 -adapted process.

Then we have the following

THEOREM 2.6 Assume that **(H1)** holds. Then $\forall i \in \mathcal{K}$, there exists a constant $C_{(K,\lambda)}^{*,i}$ depending on K and λ such that $\forall T \leq C_{(K,\lambda)}^{*,i}$, and for every \mathcal{G}_s^0 -adapted process $(\xi_s)_{0 \leq s \leq T}$ such that $\mathbf{E} \sup_{0 \leq s \leq T} |\xi_s|^2 < \infty$, (E^*) admits a unique solution $(\Theta_s)_{0 \leq s \leq T} = (X_s, Y_s, H_s, Z_s)_{0 \leq s \leq T}$ in \mathcal{B}^2 satisfying

- (i) $(X_s)_{0 \leq s \leq T}$ is continuous and $(Y_s)_{0 \leq s \leq T}$ has a càdlàg version.
- (ii) $\mathbf{E} \left(\sup_{0 \leq s \leq T} |X_s|^2 + \sup_{0 \leq s \leq T} |Y_s|^2 \right) < \infty$.

Moreover for every $p \geq 1$ there exists two constants $C_{(p,K,\lambda)}^{*,i} \leq C_{(K,\lambda)}^{*,i}$ and $\beta_{(p,K,\lambda)}^{*,i}$ depending on p , λ and K such that for every $T \leq C_{(p,K,\lambda)}^{*,i}$ and for every \mathcal{G}_s^0 adapted process $(\xi_s)_{0 \leq s \leq T}$ such that $\mathbf{E} \sup_{0 \leq s \leq T} |\xi_s|^{2p} < \infty$ the unique solution of (E^*) satisfies

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq s \leq T} |X_s|^{2p} + \mathbf{E} \sup_{0 \leq s \leq T} |Y_s|^{2p} + \mathbf{E} \left(\int_0^T \sum_{l \in L} (H_s(l))^2 ds \right)^p + \mathbf{E} \left(\int_0^T |Z_s|^2 ds \right)^p \\ & \leq \beta_{(p,K,\lambda)}^{*,i} \left(1 + \mathbf{E} \sup_{0 \leq s \leq T} |\xi_s|^{2p} \right) \end{aligned} \quad (2.42)$$

(ii) For every quartet $(\tilde{b}, \tilde{f}, \tilde{\sigma}, \tilde{h})$ satisfying **(H1)** with the same constants K, Λ and for every \mathcal{G}_s^0 adapted process $(\tilde{\xi}_s)_{0 \leq s \leq T}$ such that $\mathbf{E} \sup_{0 \leq s \leq T} |\tilde{\xi}_s|^{2p} < \infty$ we have

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq s \leq T} |\tilde{X}_s - X_s|^{2p} + \mathbf{E} \sup_{0 \leq s \leq T} |\tilde{Y}_s - Y_s|^{2p} \\ & + \mathbf{E} \left(\int_0^T \sum_{l \in \mathcal{L}} ((\tilde{H}_s - H_s)(l))^2 ds \right)^p + \mathbf{E} \left(\int_0^T |\tilde{Z}_s - Z_s|^2 ds \right)^p \\ & \leq \beta_{(p,K,\lambda)}^* \mathbf{E} \left[\sup_{0 \leq s \leq T} |\tilde{\xi}_s - \xi_s|^{2p} + |(\tilde{h}_{N_T^i} - h_{N_T^i})(X_T)|^{2p} \right. \\ & \quad \left. + \left(\int_0^T |\tilde{\sigma} - \sigma|^2(s, X_s, Y_s, N_s^i) ds \right)^p + \left(\int_0^T |\tilde{b} - b| + |\tilde{f} - f|(s, \Theta_s, N_s^i) ds \right)^{2p} \right] \quad (2.43) \end{aligned}$$

where $(\tilde{X}_s, \tilde{Y}_s, \tilde{H}_s, \tilde{Z}_s)$ is the solution of (E^*) associated to $(\tilde{b}, \tilde{f}, \tilde{\sigma}, \tilde{h})$ with initial values $(0, \tilde{\xi}_s, i)$.

Proof The proof is the same as in theorem 2.5 with some alterations to the forward equation.

We suppose that the processes (Θ_s) and $(\tilde{\Theta}_s)$ are solutions of (E^*) and satisfy the (L^p) conditions.

The process defined by $\forall s \in [0, T], X_s^* = X_s - \xi_s$ solves the SDE

$$X_s^* = \int_0^s b^* \left(r, X_r^*, Y_r, H_r, Z_r, N_r^i \right) dr + \int_0^s \sigma(r, X_r, Y_r, N_r^i) dB_r$$

where the function b^* defined by

$$\forall (t, x, y, u, z) \in [0, T] \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^d,$$

$$\forall i \in \mathcal{K} \quad b_i^*(t, x, y, u, z) = b_i(t, x + \xi_s, y, u, z)$$

satisfies assumption **(H1)**. Then using the same technique as in the proof of Eq. (2.33), we find a constant $\alpha_{(p,K)}$ depending on p and K such that

$$\begin{aligned} & \mathbf{E} \left(\sup_{0 \leq s \leq T} |\tilde{X}_s^* - X_s^*|^{2p} \right) \leq \alpha_{(p,K)} \left[\mathbf{E} \int_0^T 1_A \left(|\tilde{X}_s^* - X_s^*|^{2p} + |\tilde{Y}_s - Y_s|^{2p} \right) ds \right. \\ & \quad \left. + T^p \mathbf{E} \left(\int_0^T 1_A |\tilde{Z}_s - Z_s|^2 ds \right)^p + T^p \mathbf{E} \left(\sum_{l \in \mathcal{L}} \int_0^T 1_A (\tilde{H}_s(l) - H_s(l))^2 ds \right)^p \right. \\ & \quad \left. + \mathbf{E} \left(\int_0^T 1_A |\tilde{b} - b^*(s, X_s^*, Y_s, H_s, Z_s, N_s^i)| ds \right)^{2p} + \mathbf{E} \left(\int_0^T 1_A |(\tilde{\sigma} - \sigma)(s, X_s, Y_s, N_s^i)|^2 ds \right)^p \right] \end{aligned}$$

Using standard computations and modifying $\alpha_{(p,K)}$, we deduce

$$\begin{aligned} \mathbf{E} \sup_{0 \leq s \leq T} |\tilde{X}_s - X_s|^{2p} &\leq \alpha_{p,K} \left[\mathbf{E} \sup_{0 \leq s \leq T} |\tilde{\xi}_s - \xi_s|^{2p} + T \mathbf{E} \left(\sup_{0 \leq s \leq T} |\tilde{X}_s - X_s|^{2p} + \sup_{0 \leq s \leq T} |\tilde{Y}_s - Y_s|^{2p} \right) \right] \\ &\quad + T^p \mathbf{E} \left(\int_0^T \sum_{l \in L} ((\tilde{H}_s - H_s)(l))^2 ds \right)^p + T^p \mathbf{E} \left(\int_0^T |\tilde{Z}_s - Z_s|^2 ds \right)^p \\ &\quad + \mathbf{E} \left(\int_0^T |\tilde{b} - b|(s, \Theta_s, N_s^i) ds \right)^{2p} + \mathbf{E} \left(\int_0^T |(\tilde{\sigma} - \sigma)(s, \Theta_s, N_s^i)|^2 ds \right)^p \end{aligned}$$

Similarly to the proof of theorem 2.5, we find for any $i \in \mathcal{K}$ a constant $C_{(p,K,\lambda)}^{*,i}$ such that for $T \leq C_{(p,K,\lambda)}^{*,i}$, inequality Eq. (2.43) holds.

Therefore for an $i \in \mathcal{K}$ fixed, applying this inequality with $p = 1$ to the following iterated procedure

$$\begin{cases} \forall s \in [0, T], \\ X_s^{n+1} = \xi_s + \int_0^s b(r, X_r^{n+1}, Y_r^n, H_r^n, Z_r^n, N_r^i) dr + \int_0^s \sigma(r, X_r^{n+1}, Y_r^n, N_r^i) dB_r \\ Y_s^{n+1} = h_{N_r^i}(X_T^{n+1}) + \int_s^T f(r, \Theta_r^{n+1}, N_r^i) dr - \int_s^T Z_r^{n+1} dB_r - \int_s^T \sum_{l \in L} H_r^{n+1}(l) dM_r(l) \end{cases}$$

we prove existence and uniqueness of solution to (E^*) in a small time duration. Similarly to theorem 2.5, we show the two last inequalities.

The Case of Smooth Coefficients

In what follows we reinforce our assumptions on the deterministic coefficients of the problem $(E^{0,x})$ in order to establish some differentiability properties of the map θ defined in corollary 2.2.

Regularity of Θ

We introduce $D : L^2(\Omega) \rightarrow L^2(\Omega \times [0, T], \mathbf{R}^d)$ the Malliavin derivative operator with respect to the Brownian motion and the following space

$$\mathbf{D}^{1,2} = \left\{ \xi \in L^2(\Omega) / \mathbf{E}(|\xi|^2) + \mathbf{E} \int_0^T |D_r \xi|^2 dr < \infty \right\}$$

For a differentiable function in \mathbf{R}^Q g , g'_x stand for its partial derivative with respect to x . We say that the coefficients b, f, σ and h satisfy assumption **(H2)** if

(H2.1) **(H1.3)** holds.

(H2.2) All the functions are twice continuously differentiable with respect to x, y, u and z .

(H2.2) All the functions and their derivatives up to order two are K Lipschitz with respect to x, y, u and z .

We claim the following

THEOREM 2.7 Assume that **(H2)** holds. Then for all $i \in \mathcal{K}$ there exists a constant $C_{(K,\lambda)}^{*,i}$ such that $\forall T \leq C_{(K,\lambda)}^{*,i}$, $\forall (t, x) \in [0, T] \times \mathbf{R}^d$ the process $(\Theta_s^{t,x,i})_{t \leq s \leq T}$ solution of $(E^{t,x})$ satisfies

- (i) $\forall s \in [t, T]$, $X_s^{t,x,i} \in (\mathbf{D}^{1,2})^d$, $Y_s^{t,x,i} \in \mathbf{D}^{1,2}$, $H_s^{t,x,i} \in (\mathbf{D}^{1,2})^{k-1}$, $Z_s^{t,x,i} \in (\mathbf{D}^{1,2})^d$
- (ii) $\forall r \in [0, T] \setminus (t, s]$, $D_r X_s^{t,x,i} = D_r Y_s^{t,x,i} = D_r H_s^{t,x,i} = D_r Z_s^{t,x,i} = 0$
- (iii) $\forall j \in \{1, \dots, d\}$, the process $(D_r^j \Theta_s^{t,x,i})_{r \leq s \leq T} \triangleq (D_r^j X_s^{t,x,i}, D_r^j Y_s^{t,x,i}, D_r^j H_s^{t,x,i}, D_r^j Z_s^{t,x,i})_{r \leq s \leq T}$ is the unique solution of the problem

$$\left\{ \begin{array}{l} D_r^j X_s = \sigma^{(j)}(r, X_r, Y_r, N_r^{t,i}) + \int_r^s \mathcal{B}^{t,x,i}(u, D_r^j X_u, D_r^j Y_u, D_r^j H_u, D_r^j Z_u, N_u^{t,i}) du \\ \quad + \int_r^s \Sigma^{t,x,i}(u, D_r^j X_u, D_r^j Y_u, N_u^{t,i}) dB_u \\ D_r^j Y_s = \mathcal{H}^{t,x,i}(D_r^j X_T, N_T^{t,i}) - \int_s^T F^{t,x,i}(u, D_r^j X_u, D_r^j Y_u, D_r^j H_u, D_r^j Z_u, N_u^{t,i}) du \\ \quad - \int_s^T D_r^j Z_u dB_u - \int_s^T \sum_{l \in \mathcal{L}} D_r^j H_u(l) dM_u(l) \\ \mathbf{E} \int_r^T (|D_r^j X_u|^2 + |D_r^j Y_u|^2 + |D_r^j H_u|^2 + |D_r^j Z_u|^2) du < \infty \end{array} \right.$$

where the functions $\mathcal{B}^{t,x,i}$, $\Sigma^{t,x,i}$, $F^{t,x,i}$ and $\mathcal{H}^{t,x,i}$ are defined by

$$\forall (r, \tilde{x}, v, w, q) \in [0, T] \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^d$$

$$\begin{aligned} \mathcal{B}^{t,x,i}(r, \tilde{x}, v, w, q, N_r^{t,i}) &= b'_x(r, \Theta_r^{t,x,i}, N_r^{t,i})\tilde{x} + b'_y(r, \Theta_r^{t,x,i}, N_r^{t,i})v + b'_u(r, \Theta_r^{t,x,i}, N_r^{t,i})w \\ &\quad + b'_z(r, \Theta_r^{t,x,i}, N_r^{t,i})q \end{aligned}$$

$$\begin{aligned} F^{t,x,i}(r, \tilde{x}, v, w, q, N_r^{t,i}) &= f'_x(r, \Theta_r^{t,x,i}, N_r^{t,i})\tilde{x} + f'_y(r, \Theta_r^{t,x,i}, N_r^{t,i})v + f'_u(r, \Theta_r^{t,x,i}, N_r^{t,i})w \\ &\quad + f'_z(r, \Theta_r^{t,x,i}, N_r^{t,i})q \end{aligned}$$

$$\Sigma^{t,x,i}(r, \tilde{x}, v, N_r^{t,i}) = \sigma'_x(r, X_r^{t,x,i}, Y_r^{t,x,i}, N_r^{t,i})\tilde{x} + \sigma'_y(r, X_r^{t,x,i}, Y_r^{t,x,i}, N_r^{t,i})v$$

$$\mathcal{H}^{t,x,i}(\tilde{x}, N_T^{t,i}) = h'(X_T^{t,x,i}, N_T^{t,i})\tilde{x}$$

and $\sigma^{(j)}$ is the j -th column of the matrix σ .

- (iv) $\forall j = 1, \dots, d$, the process $\{D_s^j Y_s^{t,x,i}, s \in [t, T]\}$ is a version of $\{(Z_s^{t,x,i})^j, t \leq s \leq T\}$.

Proof Let $i \in \mathcal{K}$ be fixed. Thanks to Theorem 2.5, there exists a constant $\tilde{C}_{K,\lambda}^{i,3}$ such that for $T \leq \tilde{C}_{K,\lambda}^{i,3}$ the sequence (we omit the superscript “ t, x, i ” for a sake of simplicity) defined by

$$\left\{ \begin{array}{l} \forall s \in [t, T] \\ X_s^{n+1} = x + \int_t^s b(r, X_r^{n+1}, Y_r^n, H_r^n, Z_r^n, N_r^{t,i}) dr + \int_t^s \sigma(r, X_r^{n+1}, Y_r^n, N_r^{t,i}) dB_r \\ Y_s^{n+1} = h_{N_T^n}(X_T^{n+1}) + \int_s^T f(r, \Theta_r^{n+1}, N_r^{t,i}) dr - \int_s^T Z_r^{n+1} dB_r - \int_s^T \sum_{l \in \mathcal{L}} H_r^{n+1} dM_r(l) \end{array} \right.$$

satisfies

$$\sup_{n \in \mathbb{N}} \left[\mathbf{E} \sup_{t \leq s \leq T} |X_s^n|^4 + \mathbf{E} \sup_{t \leq s \leq T} |Y_s^n|^4 + \mathbf{E} \left(\int_t^T |Z_s^n|^2 ds \right)^2 + \mathbf{E} \left(\int_t^T \sum_{l \in \mathcal{L}} (H_s^n(l))^2 ds \right)^2 \right] < \infty$$

and

$$\begin{aligned} & \mathbf{E} \sup_{t \leq s \leq T} |X_s^m - X_s^n|^4 + \mathbf{E} \sup_{t \leq s \leq T} |Y_s^m - Y_s^n|^4 + \mathbf{E} \left(\int_t^T |Z_s^m - Z_s^n|^2 ds \right)^2 \\ & + \mathbf{E} \left(\int_t^T \sum_{l \in \mathcal{L}} (H_s^m(l) - H_s^n(l))^2 ds \right)^2 \rightarrow 0 \end{aligned} \tag{2.44}$$

whenever $n, m \rightarrow \infty$.

For $n \in \mathbb{N}$, we consider the following property (\mathcal{P}_n)

$$(\mathcal{P}_n) : \left\{ \begin{array}{l} \forall s \in [t, T], X_s^n \in (\mathbf{D}^{1,2})^d, Y_s^n \in \mathbf{D}^{1,2}, H_s^n \in L^2([t, T], (\mathbf{D}^{1,2})^{k-1}), \\ Z_s^n \in L^2([t, T], (\mathbf{D}^{1,2})^d) \\ \text{There exists a version of } (D_r \Theta_s^n) = (D_r X_s^n, D_r Y_s^n, D_r H_s^n, D_r Z_s^n)_{0 \leq r \leq T, t \leq s \leq T} \text{ such that} \\ \sup_{0 \leq r \leq T} \left[\mathbf{E} \sup_{t \leq s \leq T} (|D_r X_s^n|^2 + |D_r Y_s^n|^2) \right. \\ \left. + \mathbf{E} \left(\int_t^T \sum_{l \in \mathcal{L}} |D_r H_s^n(l)|^2 ds \right)^2 + \mathbf{E} \left(\int_t^T |D_r Z_s^n|^2 ds \right)^2 \right] < \infty \end{array} \right.$$

Using Pardoux–Pradeilles–Rao [9], Proposition 3.1, we prove $(\mathcal{P}_n) \Rightarrow (\mathcal{P}_{n+1})$. Therefore choosing a quartet (X^0, Y^0, H^0, Z^0) satisfying (\mathcal{P}_0) then $\forall n \in \mathbb{N}$, (\mathcal{P}_n) holds. Moreover,

$$\forall r \in [0, T](t, s], D_r X_s^n = 0; D_r Y_s^n = 0; D_r H_s^n = 0; D_r Z_s^n = 0.$$

and $\forall r \in [t, T], (D_r \Theta_s^{t, X, n+1})_{r \leq s \leq T}$ is solution of the system, $\forall j = 1, \dots, d$

$$\left\{ \begin{array}{l} D_r^j X_s^{n+1} = \sigma^{(j)}(r, X_r^{n+1}, Y_r^n, N_r^{t,i}) + \int_r^s \mathcal{B}^{n,i}(u, D_r^j X_u^{n+1}, D_r^j Y_u^n, D_r^j H_u^n, D_r^j Z_u^n, N_u^{t,i}) du \\ \quad + \int_r^s \Sigma^{n,i}(u, D_r^j X_u^{n+1}, D_r^j Y_u^n, N_u^{t,i}) dB_u \\ D_r^j Y_s^{n+1} = \mathcal{H}^{n,i}(D_r^j X_T^{n+1}, N_T^{t,i}) - \int_s^T F^{n,i}(u, D_r^j \Theta_u^{n+1}, N_u^{t,i}) du \\ \quad - \int_s^T D_r^j Z_u^{n+1} dB_u - \int_s^T \sum_{l \in \mathcal{L}} D_r^j H_u^{n+1}(l) dM_u(l) \end{array} \right.$$

where for $n \in \mathbf{N}$, $\mathcal{B}^{n,i}$, $F^{n,i}$, $\Sigma^{n,i}$ and $\mathcal{H}^{n,i}$ are defined by $\forall (r, \tilde{x}, v, w, q) \in [0, T] \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^d$

$$\begin{aligned} \mathcal{B}^{n,i}(r, \tilde{x}, v, w, q, N_r^{t,i}) &= b'_x(r, X_r^{n+1}, Y_r^n, H_r^n, Z_r^n, N_r^{t,i})\tilde{x} + b'_y(r, X_r^{n+1}, Y_r^n, H_r^n, Z_r^n, N_r^{t,i})v \\ &\quad + b'_u(r, X_r^{n+1}, Y_r^n, H_r^n, Z_r^n, N_r^{t,i})w + b'_z(r, X_r^{n+1}, Y_r^n, H_r^n, Z_r^n, N_r^{t,i})q \\ F^{n,i}(r, \tilde{x}, v, w, q, N_r^{t,i}) &= f'_x(r, \Theta_r^{n+1}, N_r^{t,i})\tilde{x} + f'_y(r, \Theta_r^{n+1}, N_r^{t,i})v + f'_u(r, \Theta_r^{n+1}, N_r^{t,i})w \\ &\quad + f'_z(r, \Theta_r^{n+1}, N_r^{t,i})q \\ \Sigma^{n,i}(r, \tilde{x}, v, N_r^{t,i}) &= \sigma'_x(r, X_r^{n+1}, Y_r^n, N_r^{t,i})\tilde{x} + \sigma'_y(r, X_r^{n+1}, Y_r^n, N_r^{t,i})v \\ \mathcal{H}^{n,i}(\tilde{x}, N_T^{t,i}) &= h'(X_T^{n+1}, N_T^{t,i})\tilde{x} \end{aligned}$$

Noting that the functions $\mathcal{B}^{n,i}$, $F^{n,i}$, $\Sigma^{n,i}$ and $\mathcal{H}^{n,i}$ are K -Lipschitz with respect to (r, \tilde{x}, v, w, q) , thanks to Theorem 2.6, there exists two constants which we note again $\tilde{C}_{(K,\lambda)}^{i,3}$ and $\tilde{\beta}_{(K,\lambda)}^i$ such that for $T \leq \tilde{C}_{(K,\lambda)}^{i,3}$, $\forall r \in [t, T]$,

$$\begin{aligned} &\mathbf{E} \sup_{r \leq s \leq T} |D_r^j X_s^{n+1}|^4 + \mathbf{E} \sup_{r \leq s \leq T} |D_r^j Y_s^{n+1}|^4 + \mathbf{E} \left(\int_r^T \sum_{l \in \mathcal{L}} (D_r^j H_s^{n+1}(l))^2 ds \right)^2 \\ &+ \mathbf{E} \left(\int_r^T |D_r^j Z_s^{n+1}|^2 ds \right)^2 \leq \tilde{\beta}_{(K,\lambda)}^i \left[\mathbf{E} \sup_{r \leq s \leq T} |\sigma^{(j)}(r, X_r^{n+1}, Y_r^n, N_r^{t,i})|^4 \right. \\ &+ \mathbf{E} \left(\int_r^T |\Sigma^{n,i}(u, 0, D_r^j Y_u^n, N_u^{t,i}) - \Sigma^{n,i}(u, 0, 0, N_u^{t,i})|^2 du \right)^2 \\ &\left. + \mathbf{E} \left(\int_r^T |\mathcal{B}^{n,i}(u, 0, D_r^j Y_u^n, D_r^j H_u^n, D_r^j Z_u^n, N_u^{t,i}) - \mathcal{B}^{n,i}(u, 0, 0, 0, 0, N_u^{t,i})| du \right)^4 \right] \end{aligned}$$

Using the Lipschitz property of the coefficients and (H1.3), we deduce $\forall r \in [t, T]$

$$\begin{aligned} &\mathbf{E} \sup_{r \leq s \leq T} |D_r^j X_s^{n+1}|^4 + \mathbf{E} \sup_{r \leq s \leq T} |D_r^j Y_s^{n+1}|^4 + \mathbf{E} \left(\int_r^T \sum_{l \in \mathcal{L}} (D_r^j H_s^{n+1}(l))^2 ds \right)^2 \\ &+ \mathbf{E} \left(\int_r^T |D_r^j Z_s^{n+1}|^2 ds \right)^2 \leq \tilde{\beta}_{(K,\lambda)}^i \left[\Lambda^4 + K^4 T^2 \mathbf{E} \sup_{r \leq s \leq T} |D_r^j Y_s^n|^4 \right. \\ &\left. + K^4 T^2 \left(\mathbf{E} \sup_{r \leq s \leq T} |D_r^j Y_s^n|^4 + \mathbf{E} \left(\int_r^T \sum_{l \in \mathcal{L}} (D_r^j H_s^n(l))^2 ds \right)^2 + \mathbf{E} \left(\int_r^T |D_r^j Z_s^n|^2 ds \right)^2 \right) \right] \end{aligned}$$

Hence applying an induction and modifying $\tilde{C}_{(K,\lambda)}^{i,3}$ we show that for $T \leq \tilde{C}_{(K,\lambda)}^{i,3}$, the following series

$$\begin{aligned} & \sup_{0 \leq r \leq T} \mathbf{E} \sup_{r \leq s \leq T} |D_r^j X_s^n|^4 + \sup_{0 \leq r \leq T} \mathbf{E} \sup_{r \leq s \leq T} |D_r^j Y_s^n|^4 + \sup_{0 \leq r \leq T} \mathbf{E} \left(\int_r^T \sum_{l \in L} (D_r^j H_s^n(l))^2 ds \right)^2 \\ & + \sup_{0 \leq r \leq T} \mathbf{E} \left(\int_r^T |D_r^j Z_s^n|^2 ds \right)^2 \end{aligned}$$

converges. Which implies for $T \leq \tilde{C}_{(K,\lambda)}^{i,3}$,

$$\begin{aligned} & \sup_{n \in \mathbb{N}} \left[\sup_{0 \leq r \leq T} \mathbf{E} \sup_{r \leq s \leq T} |D_r^j X_s^n|^4 + \sup_{0 \leq r \leq T} \mathbf{E} \sup_{r \leq s \leq T} |D_r^j Y_s^n|^4 \right. \\ & \left. + \sup_{0 \leq r \leq T} \mathbf{E} \left(\int_r^T \sum_{l \in L} (D_r^j H_s^n(l))^2 ds \right)^2 + \sup_{0 \leq r \leq T} \mathbf{E} \left(\int_r^T |D_r^j Z_s^n|^2 ds \right)^2 \right] < \infty \end{aligned}$$

Similarly since the derivatives of functions are K -Lipschitz and Eq. (2.44) holds, we deduce using again theorem 2.6, for all $T \leq \tilde{C}_{(K,\lambda)}^{i,3}$ (modifying $\tilde{C}_{(K,\lambda)}^{i,3}$ if necessary)

$$\begin{aligned} & \sup_{0 \leq r \leq T} \mathbf{E} \sup_{r \leq s \leq T} |D_r^j X_s^n - D_r^j X_s^m|^4 + \sup_{0 \leq r \leq T} \mathbf{E} \sup_{r \leq s \leq T} |D_r^j Y_s^n - D_r^j Y_s^m|^4 \\ & + \sup_{0 \leq r \leq T} \mathbf{E} \left(\int_r^T \sum_{l \in L} (D_r^j H_s^n(l) - D_r^j H_s^m(l))^2 ds \right)^2 + \sup_{0 \leq r \leq T} \mathbf{E} \left(\int_r^T |D_r^j Z_s^n - D_r^j Z_s^m|^2 ds \right)^2 \rightarrow 0 \end{aligned}$$

whenever $m, n \rightarrow \infty$

Hence we deduce (i), (ii) and (iii).

Furthermore (iv) is proved using Proposition 3.1 of Pardoux–Pradeilles–Rao [9]. \square

Elsewhere since the functions $\mathcal{B}^{t,x,i}$, $F^{t,x,i}$, $\Sigma^{t,x,i}$ and $\mathcal{H}^{t,x,i}$ are K -Lipschitz with respect to \tilde{x}, v, w, q , the following result can be proved analogously to Theorem 2.3.

THEOREM 2.8 Assume that **(H2)** holds and let $i \in \mathcal{K}$. Then $\forall T \leq \tilde{C}_{(K,\lambda)}^{(i,2)}$, $\forall (t, x) \in [0, T] \times \mathbf{R}^d$, $\forall j \in \{1, \dots, d\}$, the system

$$\left\{ \begin{array}{l} \forall s \in [t, T] \\ \partial^j X_s^{t,x,i} = e_j + \int_t^s \mathcal{B}^{t,x,i}(u, \partial^j X_u, \partial^j Y_u, \partial^j H_u, \partial^j Z_u, N_u^{t,i}) du + \int_t^s \Sigma^{t,x,i}(u, \partial^j X_u, \partial^j Y_u, N_u^{t,i}) dB_u \\ \partial^j Y_s^{t,x,i} = \mathcal{H}^{t,x,i}(\partial^j X_T, N_T^{t,i}) - \int_s^T F^{t,x,i}(u, \partial^j X_u, \partial^j Y_u, \partial^j H_u, \partial^j Z_u, N_u^{t,i}) du - \int_s^T \partial^j Z_u dB_u \\ \quad - \int_s^T \sum_{l \in L} \partial^j H_u(l) dM_u(l) \\ \mathbf{E} \int_t^T (|\partial^j X_u|^2 + |\partial^j Y_u|^2 + |\partial^j H_u|^2 + |\partial^j Z_u|^2) du < \infty \end{array} \right.$$

admits a unique solution $(\partial\Theta_s^{t,x,i})_{t \leq s \leq T} \triangleq (\partial X_s^{t,x,i}, \partial Y_s^{t,x,i}, \partial H_s^{t,x,i}, \partial Z_s^{t,x,i})_{t \leq s \leq T}$, where e_j is the j^{th} vector of the canonical basis of \mathbf{R}^d .

From uniqueness of solutions of this system we deduce the following

COROLLARY 2.5 Assume that **(H2)** holds. Let $i \in \mathcal{K}$ and $T \leq \tilde{C}_{(K,\lambda)}^{(i2)}$. Then $\mathbf{V}(t, x) \in [0, T] \times \mathbf{R}^d$, $\forall t \leq r \leq s \leq T$, we have

$$\begin{aligned} D_r X_s^{t,x,i} &= (\partial X_s^{t,x,i})(\partial X_r^{t,x,i})^{-1} \sigma(r, X_r^{t,x,i}, Y_r^{t,x,i}, N_r^{t,i}) \\ D_r Y_s^{t,x,i} &= (\partial Y_s^{t,x,i})(\partial X_r^{t,x,i})^{-1} \sigma(r, X_r^{t,x,i}, Y_r^{t,x,i}, N_r^{t,i}) \\ D_r H_s^{t,x,i} &= (\partial H_s^{t,x,i})(\partial X_r^{t,x,i})^{-1} \sigma(r, X_r^{t,x,i}, Y_r^{t,x,i}, N_r^{t,i}) \\ D_r Z_s^{t,x,i} &= (\partial Z_s^{t,x,i})(\partial X_r^{t,x,i})^{-1} \sigma(r, X_r^{t,x,i}, Y_r^{t,x,i}, N_r^{t,i}). \end{aligned}$$

If \tilde{h} is a function of $x \in \mathbf{R}^d$, for $\rho \in \mathbf{R} - \{0\}$, let

$$\Delta_\rho^j \tilde{h}(x) = \frac{1}{\rho} (\tilde{h}(x + \rho e_j) - \tilde{h}(x)), \quad 1 \leq j \leq d$$

PROPOSITION 2.1 Assume **(H2)** holds and $i \in \mathcal{K}$ be fixed. For $T \leq \tilde{C}_{(K,\lambda)}^{(i2)}$, $(t, x) \in [0, T] \times \mathbf{R}^d$ we consider the processes $(\Theta_s^{t,x,i})_{(t \leq s \leq T)}$ and $(\partial\Theta_s^{t,x,i})_{(t \leq s \leq T)}$ extended on $[0, T]$ as in corollary 2.3.

Then $\forall p \geq 1$, $\forall j = 1, \dots, d$,

(i) There exists a constant $\gamma_{(p,K,\lambda)}^{(i1)}$ depending on p, K and λ such that $\mathbf{V}(t, x) \in [0, T] \times \mathbf{R}^d$

$$\begin{aligned} &\mathbf{E} \sup_{0 \leq s \leq T} \left(\left| \Delta_\rho^j X_s^{t,x,i} \right|^{2p} + \left| \partial^j X_s^{t,x,i} \right|^{2p} \right) + \mathbf{E} \sup_{0 \leq s \leq T} \left(\left| \Delta_\rho^j Y_s^{t,x,i} \right|^{2p} + \left| \partial^j Y_s^{t,x,i} \right|^{2p} \right) \\ &+ \mathbf{E} \left(\sum_{l \in \mathcal{L}} \int_0^T \left(\left| \Delta_\rho^j H_s^{t,x,i} \right|^2 + \left| \partial^j H_s^{t,x,i} \right|^2 \right) (l) ds \right)^p \\ &+ \mathbf{E} \left(\int_0^T \left(\left| \Delta_\rho^j Z_s^{t,x,i} \right|^2 + \left| \partial^j Z_s^{t,x,i} \right|^2 \right) ds \right)^p \leq \gamma_{(p,K,\lambda)}^{(i1)} \end{aligned} \tag{2.45}$$

(ii) There exists a constant $\lambda_{(p,K,\lambda)}^{(i2)}$ depending on p, K and λ such that $\mathbf{V}(t, x, \rho) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^*$

$$\begin{aligned} &\mathbf{E} \sup_{0 \leq s \leq T} \left| \Delta_\rho^j X_s^{t,x,i} - \partial^j X_s^{t,x,i} \right|^{2p} + \mathbf{E} \sup_{0 \leq s \leq T} \left| \Delta_\rho^j Y_s^{t,x,i} - \partial^j Y_s^{t,x,i} \right|^{2p} \\ &+ \mathbf{E} \left(\sum_{l \in \mathcal{L}} \int_0^T \left(\left(\Delta_\rho^j H_s^{t,x,i} - \partial^j H_s^{t,x,i} \right) (l) \right) ds \right)^p + \mathbf{E} \left(\int_0^T \left| \Delta_\rho^j Z_s^{t,x,i} - \partial^j Z_s^{t,x,i} \right|^2 ds \right)^p \\ &\leq \gamma_{(p,K,\lambda)}^{(i2)} |\rho|^{2p} \end{aligned} \tag{2.46}$$

(iii) There exists two constants $\gamma_{(p,K,\lambda)}^{(i3)}$ depending on p, K, λ and $\gamma_{(p,K,\lambda,\Lambda)}^{(i4)}$ depending on p, K, λ and Λ such that $\forall((t, x), (t', x')) \in ([0, T] \times \mathbf{R}^d)^2$

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq s \leq T} \left| \partial^j X_s^{t,x,i} - \partial^j X_s^{t',x',i} \right|^{2p} + \mathbf{E} \sup_{0 \leq s \leq T} \left| \partial^j Y_s^{t,x,i} - \partial^j Y_s^{t',x',i} \right|^{2p} \\ & + \mathbf{E} \left(\sum_{l \in L} \int_0^T \left((\partial^j H_s^{t,x,i} - \partial^j H_s^{t',x',i})(l) \right)^2 ds \right)^p \\ & + \mathbf{E} \left(\int_0^T \left| \partial^j Z_s^{t,x,i} - \partial^j Z_s^{t',x',i} \right|^2 ds \right)^p \leq \gamma_{(p,K,\lambda)}^{(i3)} |x - x'|^{2p} + \gamma_{(p,K,\lambda)}^{(i4)} |t - t'|^p \quad (2.47) \end{aligned}$$

Proof: Let $i \in \mathcal{K}$ and $T \leq \tilde{C}_{(K,\lambda)}^{(i2)}$.

We consider the following functions defined for all $(r, \tilde{x}, v, w, q) \in [0, T] \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^d$ by

$$\mathcal{B}^{t,x,\rho}(r, \tilde{x}, v, w, q, N_r^{t,i}) = \int_0^1 \left(b'_x \left(\zeta_{r,\alpha}^{t,x,\rho} \right) \tilde{x} + b'_y \left(\zeta_{r,\alpha}^{t,x,\rho} \right) v + b'_u \left(\zeta_{r,\alpha}^{t,x,\rho} \right) w + b'_z \left(\zeta_{r,\alpha}^{t,x,\rho} \right) q \right) d\alpha$$

$$F^{t,x,\rho}(r, \tilde{x}, v, w, q, N_r^{t,i}) = \int_0^1 \left(f'_x \left(\zeta_{r,\alpha}^{t,x,\rho} \right) \tilde{x} + f'_y \left(\zeta_{r,\alpha}^{t,x,\rho} \right) v + f'_u \left(\zeta_{r,\alpha}^{t,x,\rho} \right) w + f'_z \left(\zeta_{r,\alpha}^{t,x,\rho} \right) q \right) d\alpha$$

$$\sum^{t,x,\rho}(r, \tilde{x}, v, N_r^{t,i}) = \int_0^1 \left(\sigma'_x \left(\zeta_{r,\alpha}^{t,x,\rho} \right) \tilde{x} + \sigma'_y \left(\zeta_{r,\alpha}^{t,x,\rho} \right) v \right) d\alpha$$

$$\mathcal{H}^{t,x,\rho}(\tilde{x}, N_T^{t,i}) = \int_0^1 \left(h' \left(X_T^{t,x,i} + \alpha \rho \Delta_\rho^j X_T^{t,x,i}, N_T^{t,i} \right) \tilde{x} \right) d\alpha$$

$$\text{where } \zeta_{r,\alpha}^{t,x,\rho} = \left(r, X_r + \alpha \rho \Delta_\rho^j X_r^{t,x,i}, Y_r + \alpha \rho \Delta_\rho^j Y_r^{t,x,i}, H_r + \alpha \rho \Delta_\rho^j H_r^{t,x,i}, Z_r + \alpha \rho \Delta_\rho^j Z_r^{t,x,i}, N_r^{t,i} \right)$$

$$\tilde{\zeta}_{r,\alpha}^{t,x,\rho} = \left(r, X_r + \alpha \rho \Delta_\rho^j X_r^{t,x,i}, Y_r + \alpha \rho \Delta_\rho^j Y_r^{t,x,i}, N_r^{t,i} \right)$$

Note that for all $(t, x, \rho, i) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^* \times \mathcal{K}$, $\forall j = 1, \dots, d$, the quartet of processes $\left(\Delta_\rho^j \Theta_r^{t,x,i} \right)_{(0 \leq s \leq T)} \triangleq \left(\Delta_\rho^j X_s^{t,x,i}, \Delta_\rho^j Y_s^{t,x,i}, \Delta_\rho^j H_s^{t,x,i}, \Delta_\rho^j Z_s^{t,x,i} \right)_{(0 \leq s \leq T)}$ is solution of the system

$$\left\{ \begin{aligned} & \Delta_\rho^j X_s^{t,x,i} = e_j + \int_0^s \mathbf{1}_{[t,T]}(r) \mathcal{B}^{t,x,\rho} \left(r, \Delta_\rho^j X_r, \Delta_\rho^j Y_r, \Delta_\rho^j H_r, \Delta_\rho^j Z_r, N_r^{t,i} \right) dr \\ & \quad + \int_0^s \mathbf{1}_{[t,T]}(r) \sum^{t,x,\rho} \left(r, \Delta_\rho^j X_r, \Delta_\rho^j Y_r, N_r^{t,i} \right) dB_r \\ & \Delta_\rho^j Y_s^{t,x,i} = \mathcal{H}^{t,x,\rho} \left(\Delta_\rho^j X_T, N_T^{t,i} \right) + \int_s^T \mathbf{1}_{[t,T]}(r) F^{t,x,\rho} \left(r, \Delta_\rho^j X_r, \Delta_\rho^j Y_r, \Delta_\rho^j H_r, \Delta_\rho^j Z_r, N_r^{t,i} \right) dr \\ & \quad - \int_s^T \Delta_\rho^j Z_r dB_r - \int_s^T \sum_{l \in L} \Delta_\rho^j H_r(l) dM_r(l) \\ & \mathbf{E} \int_0^T \left(\left| \Delta_\rho^j X_r \right|^2 + \left| \Delta_\rho^j Y_r \right|^2 + \left| \Delta_\rho^j H_r \right|^2 + \left| \Delta_\rho^j Z_r \right|^2 \right) dr < \infty \end{aligned} \right.$$

and the two quartets of functions $(1_{[t,T]}(r)\mathcal{B}^{t,x,\rho}, 1_{[t,T]}(r)F^{t,x,\rho}, 1_{[t,T]}(r)\sum^{t,x,\rho}, \mathcal{H}^{t,x,\rho})$ and $(\mathcal{B}^{t,x,i}, F^{t,x,i}, \sum^{t,x,i}, \mathcal{H}^{t,x,i})$ satisfy assumption **(H2)**, then applying Eq. (2.31) to the associated solutions, we easily deduce Eq. (2.45).

Furthermore, using once again theorem 2.5, we obtain $\forall p \geq 1$,

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq s \leq T} \left| \Delta_\rho^j X_s^{t,x,i} - \partial^j X_s^{t,x,i} \right|^{2p} + \mathbf{E} \sup_{0 \leq s \leq T} \left| \Delta_\rho^j Y_s^{t,x,i} - \partial^j Y_s^{t,x,i} \right|^{2p} \\ & + \mathbf{E} \left(\sum_{l \in L} \int_0^T \left((\Delta_\rho^j H_s^{t,x,i} - \partial^j H_s^{t,x,i})(l) \right)^2 ds \right)^p + \mathbf{E} \left(\int_0^T \left| \Delta_\rho^j Z_s^{t,x,i} - \partial^j Z_s^{t,x,i} \right|^2 ds \right)^p \\ & \leq \beta_{(\rho,K,\lambda)}^{(i)} \left[\mathbf{E} \left| (\mathcal{H}^{t,x,\rho} - \mathcal{H}^{t,x,i})(\partial^j X_T, N_T^{t,i}) \right|^{2p} + \mathbf{E} \left(\int_t^T \left| \sum^{t,x,\rho} - \sum^{t,x,i} \right|^2 (s, \partial^j X_s, \partial^j Y_s, N_s^{t,i}) ds \right)^p \right. \\ & \left. + \mathbf{E} \left(\int_t^T (|\mathcal{B}^{t,x,\rho} - \mathcal{B}^{t,x,i}| + |F^{t,x,\rho} - F^{t,x,i}|)(s, \partial^j X_s, \partial^j Y_s, \partial^j H_s, \partial^j Z_s, N_s^{t,i}) ds \right)^{2p} \right] \end{aligned}$$

Since $\forall i \in \mathcal{K}$, $h'(\cdot, i)$ is K -Lipschitz, there exists a constant c_p depending on p such that

$$\begin{aligned} & \left| (\mathcal{H}^{t,x,\rho} - \mathcal{H}^{t,x,i})(\partial^j X_T, N_T^{t,i}) \right|^{2p} = \left| \partial^j X_T \left(\int_0^1 \left(h' \left(X_T + \alpha \rho \Delta_\rho^j X_T, N_T^{t,i} \right) - h' \left(X_T, N_T^{t,i} \right) \right) d\alpha \right) \right|^{2p} \\ & \leq c_p K^{2p} |\rho|^{2p} \left(\left| \Delta_\rho^j X_T \right| \left| \partial^j X_T \right| \right)^{2p} \end{aligned}$$

Using the same argument in the two last integrals, we prove the existence of a constant $\gamma_{(\rho,K,\lambda)}^{(i)}$ depending on p , λ and K satisfying

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq s \leq T} \left| \Delta_\rho^j X_s^{t,x,i} - \partial^j X_s^{t,x,i} \right|^{2p} + \mathbf{E} \sup_{0 \leq s \leq T} \left| \Delta_\rho^j Y_s^{t,x,i} - \partial^j Y_s^{t,x,i} \right|^{2p} \\ & + \mathbf{E} \left(\sum_{l \in L} \int_0^T \left(\Delta_\rho^j H_s^{t,x,i} - \partial^j H_s^{t,x,i} \right)(l) ds \right)^p + \mathbf{E} \left(\int_0^T \left| \Delta_\rho^j Z_s^{t,x,i} - \partial^j Z_s^{t,x,i} \right|^2 ds \right)^p \\ & \leq \gamma_{(\rho,K,\lambda)}^{(i)} |\rho|^{2p} \left[\mathbf{E} \left(\left| \Delta_\rho^j X_T \right| \left| \partial^j X_T \right| \right)^{2p} + \mathbf{E} \left(\int_0^T \left(\left| \Delta_\rho^j X_s \right|^2 + \left| \Delta_\rho^j Y_s \right|^2 \right) \left(\left| \partial^j X_s \right|^2 + \left| \partial^j Y_s \right|^2 \right) ds \right)^{2p} \right. \\ & \left. + \mathbf{E} \left(\int_0^T \left(\left| \Delta_\rho^j X_s \right| + \left| \Delta_\rho^j Y_s \right| + \left| \Delta_\rho^j H_s \right| + \left| \Delta_\rho^j Z_s \right| \right) \times \left(\left| \partial^j X_s \right| + \left| \partial^j Y_s \right| + \left| \partial^j H_s \right| + \left| \partial^j Z_s \right| \right) ds \right)^{2p} \right] \end{aligned}$$

Whence thanks to standard estimates and inequality Eq. (2.31), we deduce Eq. (2.46).

Using once again theorem 2.5, we have $\forall (t, x), (t', x') \in ([0, T] \times \mathbf{R}^d)^2 \forall j = 1, \dots, d$

$$\begin{aligned} & \mathbf{E} \sup_{0 \leq s \leq T} \left| \partial^j X_s^{t,x,i} - \partial^j X_s^{t',x',i} \right|^{2p} + \mathbf{E} \sup_{0 \leq s \leq T} \left| \partial^j Y_s^{t,x,i} - \partial^j Y_s^{t',x',i} \right|^{2p} \\ & + \mathbf{E} \left(\sum_{l \in \mathcal{L}} \int_0^T \left((\partial^j H_s^{t,x,i} - \partial^j H_s^{t',x',i})(l) \right)^2 ds \right)^p + \mathbf{E} \left(\int_0^T \left| \partial^j Z_s^{t,x,i} - \partial^j Z_s^{t',x',i} \right|^2 ds \right)^p \\ & \leq \beta_{(p,K,\lambda)}^{(i)} \left[|t - t'|^p + \mathbf{E} \left(\left| X_T^{t,x} - X_T^{t',x'} \right| \left| \partial^j X_T \right| \right)^{2p} \right. \\ & + \mathbf{E} \left(\int_{t'}^T \left(\left| X_s^{t,x} - X_s^{t',x'} \right|^2 + \left| Y_s^{t,x} - Y_s^{t',x'} \right|^2 \right) \left(\left| \partial^j X_s^{t,x} \right|^2 + \left| \partial^j Y_s^{t,x} \right|^2 \right) ds \right)^p \\ & + \mathbf{E} \left(\int_{t'}^T \left(\left| X_s^{t,x} - X_s^{t',x'} \right| + \left| Y_s^{t,x} - Y_s^{t',x'} \right| + \left| H_s^{t,x} - H_s^{t',x'} \right| + \left| Z_s^{t,x} - Z_s^{t',x'} \right| \right) \right. \\ & \quad \left. \times \left(\left| \partial^j X_s^{t,x} \right| + \left| \partial^j Y_s^{t,x} \right| + \left| \partial^j H_s^{t,x} \right| + \left| \partial^j Z_s^{t,x} \right| \right) ds \right)^{2p} \end{aligned}$$

Thanks to standard estimates and corollary 2.3, we find two constants $\gamma_{(p,K,\lambda)}^{(i3)}$ and $\tilde{\gamma}_{(p,K,\lambda,\Lambda)}^{(i4)}$ satisfying Eq. (2.47). \square

COROLLARY 2.6 Assume that **(H2)** holds and let $i \in \mathcal{K}$ be fixed. Then for every $T \leq \tilde{C}_{(K,\lambda)}^{(i2)}$, the map θ is twice differentiable with respect to x and for $j = 1, \dots, d$, the functions θ , $(\partial\theta/\partial x_j)_j$ and $(\partial\theta/\partial x_m \partial x_j)_{m,j}$ are continuous on $[0, T] \times \mathbf{R}^d$.

Proof : Let $i \in \mathcal{K}$, $T \leq \tilde{C}_{(K,\lambda)}^{(i2)}$ and $(t, x) \in [0, T] \times \mathbf{R}^d$. Inequality Eq. (2.25) implies θ is Lipschitz continuous with respect to x and we easily deduce from Eq. (2.46)

$$\forall j = 1, \dots, d, \quad \lim_{\rho \rightarrow 0} \Delta_\rho^j Y_t^{t,x,i} = \partial^j Y_t^{t,x,i}$$

hence $\theta(t, \cdot, i)$ is differentiable with respect to x and its partial derivatives are defined by

$$\forall j = 1, \dots, d, \quad \frac{\partial \theta}{\partial x_j}(t, x, i) = \partial^j Y_t^{t,x,i}$$

Moreover, using Eq. (2.47), we deduce $\partial\theta/\partial x_j(t, \cdot, i) = \partial^j Y_t^{t,\cdot,i}$ is continuous and $\gamma_{p,K,\lambda}^{(i3)}$ Lipschitz.

With assumptions **(H2)**, applying the same method as Ref. [3], corollary 5.4 to the derivatives of the functions, we can prove that for $T \leq \tilde{C}_{(K,\lambda)}^{(i2)}$ and $\forall j = 1, \dots, d$, $\partial\theta(t, \cdot, i)/\partial x_j$ is continuously differentiable with respect to x . \square

COROLLARY 2.7 Let $i \in \mathcal{K}$ and $T \leq \tilde{C}_{(K,\lambda)}^{(i2)}$. Then $\forall (t, x) \in [0, T] \times \mathbf{R}^d$, $\forall t \leq s \leq T$,

$$Z_s^{t,x,i} = \nabla_x \theta_{N_s^{t,i}}(s, X_s^{t,x,i}) \tilde{\sigma}_{N_s^{t,i}}(s, X_s^{t,x,i})$$

with $\tilde{\sigma}_{N_s^{t,i}}(s, X_s^{t,x,i}) = \sigma_{N_s^{t,i}}(s, X_s^{t,x,i}, \theta_{N_s^{t,i}}(s, X_s^{t,x,i}))$.

Proof: It is an obvious consequence of corollary 2.5 and theorem 2.7. \square

We are now in position to prove our main result.

Solution of a Parabolic System of PDEs

In this section, we are interested in the system of PDEs (\mathcal{E}) and we need to specify the Lipschitz constant of σ and h .

We say that the coefficients b, σ, f and h satisfy assumption **(H3)** if **(H2)** holds with \tilde{k} as Lipschitz constant of σ and h and moreover, the following non-degeneracy condition is satisfied

(H3.1) $\exists \delta > 0$ such that $\forall i \in \mathcal{K}, \forall v \in \mathbf{R}^d, \forall (t, x, y) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}, \langle v, a_i(t, x, y)v \rangle \geq \delta |v|^2$.

Let us recall that

$$\theta(t, x) = (\theta(t, x, 1), \theta(t, x, 2), \dots, \theta(t, x, k)), \quad (t, x) \in [0, T] \times \mathbf{R}^d \tag{2.48}$$

We claim the following

PROPOSITION 2.2 Assume **(H3)** is in force. Then there exists a constant $\tilde{C}_{(K,\lambda)}^{(2)}$ such that for all $T \leq \tilde{C}_{(K,\lambda)}^{(2)}$, the map $\theta \in \mathcal{C}^{1,2}([0, T] \times \mathbf{R}^d, \mathbf{R}^k)$ and is solution to (ε).

Moreover, there exists some constants C (depending on Λ and T) and Γ (depending on $\Lambda, \tilde{k}, \delta, d$ and T) such that

$$\sup_{(t,x) \in [0,T] \times \mathbf{R}^d} |\theta(t, x)| \leq C \tag{2.49}$$

$$\sup_{(t,x) \in [0,T] \times \mathbf{R}^d} |\nabla_x \theta(t, x)| \leq \Gamma. \tag{2.50}$$

Proof Let us define $\tilde{C}_{(K,\lambda)}^{(2)} = \min_{i \in \mathcal{K}} \tilde{C}_{(K,\lambda)}^{(i2)}$ and we assume that $T \leq \tilde{C}_{(K,\lambda)}^{(2)}$. For some $t \in [0, T]$ we have $\forall s \in [t, T]$ and $i \in \mathcal{K}$,

$$\theta(s, x, i) - \theta(t, x, i) = \theta(s, x, i) - \theta(s, X_s^{t,x,i}, N_s^{t,i}) + \theta(s, X_s^{t,x,i}, N_s^{t,i}) - \theta(t, x, i)$$

Applying Itô's formula to the function $\theta(s, \cdot, \cdot)$, and putting $\theta(s, x, i) = \theta_i(s, x)$ we obtain $\forall s \in [t, T]$,

$$\begin{aligned} \theta_{N_s^{t,i}}(s, X_s^{t,x,i}) &= \theta_i(s, x) + \int_t^s \nabla_x \theta_{N_r^{t,i}}(s, X_r) b(r, \Theta_r^{t,x,i}, N_r^{t,i}) \, dr \\ &\quad + \int_t^s \nabla_x \theta_{N_r^{t,i}}(s, X_r) \sigma(r, X_r, Y_r, N_r^{t,i}) \, dB_r \\ &\quad + \frac{1}{2} \int_t^s Tr \left(\partial_{xx}^2 \theta_{N_r^{t,i}}(s, X_r) \right) (\sigma \sigma^*) (r, X_r, Y_r, N_r^{t,i}) \, dr \\ &\quad + \int_t^s \sum_{l \in \mathcal{L}} \left[\left(\theta_{N_{r^-}^{t,i} + l} - \theta_{N_{r^-}^{t,i}} \right) (s, X_r^{t,x,i}) \right] \, dN_r(l) \end{aligned}$$

Furthermore, we have $\forall s \in [t, T]$,

$$\begin{aligned} \theta_{N_s^{t,i}}(s, X_s^{t,x,i}) - \theta_i(t, x) &= Y_s^{t,x,i} - Y_t^{t,x,i} \\ &= - \int_t^s f(r, \Theta_r^{t,x,i}, N_r^{t,i}) \, dr + \int_t^s Z_r^{t,x,i} \, dB_r + \int_t^s \sum_{l \in \mathcal{L}} H_r^{t,x,i}(l) \, dM_r(l) \end{aligned}$$

Hence we deduce $\forall s \in [t, T]$

$$\begin{aligned} \theta_i(s, x) - \theta_i(t, x) &= - \int_t^s \left[\frac{1}{2} \text{Tr} \left(\partial_{xx}^2 \theta_{N_r^{t,i}}(s, X_r) \right) (\sigma \sigma^*) (r, X_r, Y_r, N_r^{t,i}) \right. \\ &\quad + \nabla_x \theta_{N_r^{t,i}}(s, X_r) b(r, \Theta_r^{t,x,i}, N_r^{t,i}) + f(r, \Theta_r^{t,x,i}, N_r^{t,i}) \\ &\quad + \lambda \sum_{l \in \mathcal{L}} \left(\left(\theta_{N_r^{t,i}+l} - \theta_{N_r^{t,i}} \right) (s, X_r^{t,x,i}) \right) \text{d}r \\ &\quad + \int_t^s \left(Z_r^{t,x,i} - \nabla_x \theta_{N_r^{t,i}}(s, X_r) \right) \sigma(r, X_r, Y_r, N_r^{t,i}) \text{d}B_r \\ &\quad + \int_t^s \sum_{l \in \mathcal{L}} \left[H_r^{t,x,i}(l) - \left(\theta_{N_r^{t,i}+l} - \theta_{N_r^{t,i}} \right) (s, X_r^{t,x,i}) \right] \text{d}M_r(l) \end{aligned}$$

So, considering a subdivision $t = t_0 < t_1 < \dots < t_n = T$ of $[t, T]$, we deduce

$$\begin{aligned} \theta_i(T, x) - \theta_i(t, x) &= \sum_{j=0}^{n-1} \theta_i(t_{j+1}, x) - \theta_i(t_j, x) \\ &= - \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left[\frac{1}{2} \text{Tr} \left(\partial_{xx}^2 \theta_{N_r^{t_j,i}}(t_{j+1}, X_r^{t_j,x,i}) \right) (\sigma \sigma^*) (r, X_r^{t_j,x,i}, Y_r^{t_j,x,i}, N_r^{t_j,i}) \right. \\ &\quad + \nabla_x \theta_{N_r^{t_j,i}}(t_{j+1}, X_r^{t_j,x,i}) b(r, \Theta_r^{t_j,x,i}, N_r^{t_j,i}) \\ &\quad + f(r, \Theta_r^{t_j,x,i}, N_r^{t_j,i}) + \lambda \sum_{l \in \mathcal{L}} \left(\theta_{N_r^{t_j,i}+l} - \theta_{N_r^{t_j,i}} \right) (t_{j+1}, X_r^{t_j,x,i}) \left. \right] \text{d}r \\ &\quad + \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left(\nabla_x \theta_{N_r^{t_j,i}}(r, X_r) - \nabla_x \theta_{N_r^{t_j,i}}(t_{j+1}, X_r) \right) \sigma(r, X_r^{t_j,x,i}, Y_r^{t_j,x,i}, N_r^{t_j,i}) \text{d}B_r \\ &\quad + \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \sum_{l \in \mathcal{L}} \left[\left(\theta_{N_r^{t_j,i}+l} - \theta_{N_r^{t_j,i}} \right) (r, X_r^{t_j,x,i}) \right. \\ &\quad \left. - \left(\theta_{N_r^{t_j,i}+l} - \theta_{N_r^{t_j,i}} \right) (t_{j+1}, X_r^{t_j,x,i}) \right] \text{d}M_r(l) \end{aligned}$$

Since for $i \in \mathcal{K}$, $(X_s^{t,x,i})_{0 \leq t \leq T, 0 \leq s \leq T, x \in R^d}$ is continuous and $(Y_s^{t,x,i})_{0 \leq t \leq T, 0 \leq s \leq T, x \in R^d}$ is a càdlàg process, hence choosing a subdivision $\{t_0^n, t_1^n, \dots, t_n^n\}_{n \in \mathbb{N}}$ of $[t, T]$ such that $\lim_{n \rightarrow \infty} \sup_{0 \leq j \leq n-1} (t_{j+1}^n - t_j^n) = 0$, we obtain from corollary 2.2 and corollary 2.7

$$\theta(t, x, i) = h_i(x) + \int_t^T \left(L^i \theta_i(r, x) + \bar{f}_i(r, x, \theta(r, x), \nabla_x \theta_i(r, x) \sigma_i(r, x, \theta_i(r, x))) \right) \text{d}r$$

This is enough to conclude that θ is solution of (\mathcal{E}) . Consequently, considering each line of the system of PDEs (\mathcal{E}) we note that $\forall i \in \mathcal{K}$, $\theta(\cdot, \cdot, i)$ solves the parabolic quasilinear

equations

$$(\tilde{\mathcal{E}}) \begin{cases} \mathbf{V}(t, x) \in [0, T] \times \mathbf{R}^d, \\ \frac{\partial \tilde{u}}{\partial t}(t, x) + L_i^{(1)} \tilde{u}(t, x) + \tilde{f}_i^{(1)}(t, x, \tilde{u}(t, x), \nabla_x \tilde{u}(t, x) \sigma_i(t, x, \tilde{u}(t, x))) = 0 \\ \tilde{u}(T, x) = h_i(x) \end{cases}$$

where $\bar{b}_i^{(1)}$ and $\bar{f}_i^{(1)}$ are defined by $\mathbf{V}(t, x, y, z) \in [0, T] \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^d$

$$\begin{aligned} \bar{f}_i^{(1)}(t, x, y, z) &= \bar{f}_i(t, x, \theta_1(t, x), \theta_2(t, x), \dots, \theta_{i-1}(t, x), y, \theta_{i+1}(t, x), \dots, \theta_k(t, x), z) \\ \bar{b}_i^{(1)}(t, x, y, z) &= b_i(t, x, \theta_1(t, x), \theta_2(t, x), \dots, \theta_{i-1}(t, x), y, \theta_{i+1}(t, x), \dots, \theta_k(t, x), z) \end{aligned}$$

and $L_i^{(1)}$ is defined from L^i by substituting \bar{b}_i by $\bar{b}_i^{(1)}$. Using theorem 7.1, chapter VII of Ladyzenskaja *et al.* [4] théorème 7.1, under assumptions **(H3)** the quasilinear parabolic PDEs $(\tilde{\mathcal{E}})$ have a unique bounded solution $\tilde{u} \in C^{1,2}([0, T] \times \mathbf{R}^d, \mathbf{R})$ with bounded derivatives. Therefore $\forall i \in \mathcal{K}$,

$$\forall j, m \in \{1, \dots, d\}^2, \quad \frac{\partial \theta_i}{\partial x_j} \quad \text{and} \quad \frac{\partial^2 \theta_i}{\partial x_j \partial x_m} \quad \text{are bounded on } \mathbf{R}^d.$$

Let us prove Eqs. (2.49) and (2.50). For $i \in \mathcal{K}$, we define for $(t, x) \in [0, T] \times \mathbf{R}^d$

$$\begin{aligned} \tilde{B}_i(t, x) &= \tilde{B}(t, x, i) = \bar{b}_i(t, x, \theta(t, x), \nabla_x \theta_i(t, x) \sigma_i(t, x, \theta_i(t, x))) \\ \Sigma_i(t, x) &= \Sigma(t, x, i) = \sigma_i(t, x, \theta_i(t, x)) \end{aligned}$$

Let $i \in \mathcal{K}$ and ξ be a \mathcal{G}_t measurable random vector with finite second moment. We consider $X_s^{t, \xi, i}$ the diffusion process defined by $\forall s \in [t, T]$,

$$X_s^{t, \xi, i} = \xi + \int_t^s \tilde{B}(r, X_r, N_r^{t, i}) dr + \int_t^s \Sigma(r, X_r, N_r^{t, i}) dB_r$$

and the following processes $\forall s \in [t, T]$,

$$\begin{aligned} Y_s^{t, \xi, i} &= \theta_{N_s^{t, i}}(s, X_s^{t, \xi, i}); \quad Z_s^{t, \xi, i} = \nabla_x \theta_{N_s^{t, i}}(s, X_s^{t, \xi, i}) \sigma_{N_s^{t, i}}(s, X_s^{t, \xi, i}, Y_s^{t, \xi, i}) \\ \forall l \in L, \quad H_s^{t, \xi, i}(l) &= \left(\theta_{N_{s^-}^{t, i} + l} - \theta_{N_{s^-}^{t, i}} \right) (s, X_s^{t, \xi, i}) \end{aligned}$$

Since $\theta_{N_s^{t, i}} \in C^{1,2}([0, T] \times \mathbf{R}^d, \mathbf{R})$, Itô's formula yields for all $s \in [t, T]$,

$$\begin{aligned} \theta_{N_s^{t, i}}(s, X_s^{t, \xi, i}) &= \theta_{N_T^{t, i}}(T, X_T^{t, \xi, i}) - \int_s^T \frac{\partial}{\partial t} \theta_{N_r^{t, i}}(r, X_r^{t, \xi, i}) dr - \int_s^T \nabla_x \theta_{N_r^{t, i}}(r, X_r) \tilde{B}(r, X_r, N_r^{t, i}) dr \\ &\quad - \int_s^T \nabla_x \theta_{N_r^{t, i}}(r, X_r) \Sigma(r, X_r, N_r^{t, i}) dB_r - \frac{1}{2} \int_s^T Tr \left(\partial_{xx}^2 \theta_{N_r^{t, i}}(r, X_r) \right) (\Sigma \Sigma^*) (r, X_r, N_r^{t, i}) dr \\ &\quad - \int_s^T \sum_{l \in L} \left[\left(\theta_{N_{r^-}^{t, i} + l} - \theta_{N_{r^-}^{t, i}} \right) (r, X_r^{t, \xi, i}) \right] dN_r(l) \end{aligned}$$

which implies $\forall s \in [t, T]$

$$\begin{aligned} \theta_{N_s^{t,i}}(s, X_s^{t,\xi,i}) &= \theta_{N_T^{t,i}}(T, X_T^{t,\xi,i}) - \int_s^T \left[\frac{\partial}{\partial t} \theta_{N_r^{t,i}}(r, X_r^{t,\xi,i}) + L^{N_r^{t,i}} \theta_{N_r^{t,i}}(r, X_r^{t,\xi,i}) \right] dr \\ &\quad - \int_s^T Z_r^{t,\xi,i} dB_r - \int_s^T \sum_{l \in \mathcal{L}} \left[(\theta_{N_{r-}^{t,i}+l} - \theta_{N_{r-}^{t,i}})(r, X_r^{t,\xi,i}) \right] dN_r(l) \end{aligned}$$

Hence using (E) we deduce, $\forall s \in [t, T]$

$$\begin{aligned} Y_s^{t,\xi,i} &= h_{N_T^{t,i}}(X_T^{t,\xi,i}) + \int_s^T \left(\bar{f}_{N_r^{t,i}}(r, X_r, \theta(r, X_r), Z_r) + \lambda \sum_{l \in \mathcal{L}} H_r^{t,\xi,i}(l) \right) dr \\ &\quad - \int_s^T Z_r^{t,\xi,i} dB_r - \int_s^T \sum_{l \in \mathcal{L}} H_r^{t,\xi,i}(l) dM_r(l) \end{aligned}$$

Therefore, the \mathcal{G}_s -progressively measurable quartet of processes $(X_s^{t,\xi,i}, Y_s^{t,\xi,i}, H_s^{t,\xi,i}, Z_s^{t,\xi,i})_{(t \leq s \leq T)}$ with values in $\mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^d$ solves the FBSDE with transmutation associated to f, b, σ, h .

For some $a \in \mathbf{R}$, applying Itô's formula for discontinuous processes to the semimartingale $(e^{as}|Y_s|^2)_{t \leq s \leq T}$, we obtain $\forall s \in [t, T]$

$$\begin{aligned} e^{as}|Y_s|^2 &+ \int_s^T e^{ar}|Z_r|^2 dr + \lambda \int_s^T e^{ar}|H_r|^2 dr + \sum_{s < r \leq T} e^{ar}(\Delta Y_r)^2 \leq e^{aT}|Y_T|^2 \\ &- a \int_s^T e^{ar}|Y_r|^2 dr + 2 \int_s^T e^{ar} Y_r f(r, \Theta_r, N_r^{t,i}) dr \\ &- 2 \int_s^T e^{ar} Y_r Z_r dB_r - 2 \int_s^T \sum_{l \in \mathcal{L}} e^{ar} Y_r H_r(l) dM_r(l) \end{aligned}$$

Using assumption (H1.3), we deduce $\forall s \in [t, T]$

$$\begin{aligned} e^{as}|Y_s|^2 &+ \int_s^T e^{ar}|Z_r|^2 dr + \lambda \int_s^T e^{ar}|H_r|^2 dr \leq e^{aT}|Y_T|^2 + \int_s^T e^{ar} [\Lambda + (\Lambda - a)|Y_r|^2] dr \\ &- 2 \int_s^T e^{ar} Y_r Z_r dB_r - 2 \int_s^T \sum_{l \in \mathcal{L}} e^{ar} Y_r H_r(l) dM_r(l) \end{aligned}$$

Choosing $a = \Lambda$ and taking the conditional expectation given \mathcal{G}_t , for such $i \in \mathcal{K}$ we find a constant C_i , only depending on Λ and T , such that

$$\forall (t, x) \in [0, T] \times \mathbf{R}^d \quad |\theta_i(t, x)| \leq C_i$$

Using the same argument as in Delarue [3], lemma 2.1, we prove the existence of a constant Γ_i depending only on $C_i, \tilde{k}, \delta, \Lambda$ and d such that

$$\forall (t, x) \in [0, T] \times \mathbf{R}^d \quad |\nabla_x \theta_i(t, x)| \leq \Gamma_i$$

We have established the proposition with $C = \max_{i=1, \dots, k} C_i$ and $\Gamma = \max_{i=1, \dots, k} \Gamma_i$ □

In the following, we build a regular solution of the system of PDEs (\mathcal{E}) whatever the time duration T may be.

Let $\gamma = \tilde{C}_{(K,\Lambda)}^{(2)}$, $m = E(T/\gamma) + 1$ and the partition $(t_j)_{0 \leq j \leq m}$ of $[0, T]$ defined by

$$t_0 = 0, \quad \forall j \geq 1, \quad t_j = T - (m - j)\gamma$$

COROLLARY 2.8 Assume that **(H3)** holds. Then $\forall T > 0$ the system of PDEs (\mathcal{E}) admits a unique solution \tilde{u} and there exists two constants \tilde{C} (depending on Λ and T) and \tilde{I} (depending on $\Lambda, \tilde{k}, \delta, d$ and T) such that

$$\sup_{(t,x) \in [0,T] \times \mathbf{R}^d} |\tilde{u}(t,x)| \leq \tilde{C} \tag{2.51}$$

$$\sup_{(t,x) \in [0,T] \times \mathbf{R}^d} |\nabla_x \tilde{u}(t,x)| \leq \tilde{I} \tag{2.52}$$

Proof Using proposition 2.2, we build $\tilde{u} : [T - \gamma, T] \times \mathbf{R}^d \rightarrow \mathbf{R}^k$ solution of the system

$$(\mathcal{E}) \begin{cases} \forall (t,x) \in [T - \gamma, T] \times \mathbf{R}^d, \quad \forall i \in \mathcal{K} \\ \frac{\partial u_i}{\partial t}(t,x) + L^i u_i(t,x) + \tilde{f}_i(t,x, u(t,x), \nabla_x u_i(t,x) \sigma_i(t,x, u_i(t,x))) = 0 \\ u_i(T,x) = h_i(x) \end{cases}$$

and there exists two constants \tilde{C} (depending on Λ and T) and \tilde{I} (depending on $\Lambda, \tilde{k}, \delta, d$ and T) such that

$$\sup_{(t,x) \in [0,T] \times \mathbf{R}^d} |\tilde{u}(t,x)| \leq \tilde{C} \quad \text{and} \quad \sup_{(t,x) \in [0,T] \times \mathbf{R}^d} |\nabla_x \tilde{u}(t,x)| \leq \tilde{I}$$

Moreover, the first and second derivatives of \tilde{u} with respect to x are Lipschitz and the function $\tilde{u}(T - \gamma, \cdot)$ is \tilde{I} Lipschitz.

Considering the system

$$(\mathcal{E}) \begin{cases} \forall (t,x) \in [T - 2\gamma, T - \gamma] \times \mathbf{R}^d, \quad \forall i \in \mathcal{K} \\ \frac{\partial u_i}{\partial t}(t,x) + L^i u_i(t,x) + \tilde{f}_i(t,x, u(t,x), \nabla_x u_i(t,x) \sigma_i(t,x, u_i(t,x))) = 0 \\ u_i(T,x) = \tilde{u}(T - \gamma, x) \end{cases}$$

we can build another solution satisfying the same properties as \tilde{u} with two constants which we note again \tilde{C} and \tilde{I} .

Therefore, we can extend \tilde{u} on $[T - 2\gamma, T]$, so by induction we construct a bounded solution of (\mathcal{E}) whose gradient is uniformly bounded on $[0, T] \times \mathbf{R}^d$.

For the proof of uniqueness we consider v another solution of (\mathcal{E}) satisfying Eqs. (2.51) and (2.52).

We associate to v a family of solutions $(X_s^{v,t,x,i})_{(t,x) \in [0,T] \times \mathbf{R}^d}$ of the SDE

$$\begin{aligned} X_s^{v,t,x,i} &= x + \int_t^s \bar{b}(r, X_r, v(r, X_r), (\nabla_x v \sigma)(r, X_r, Y_r, N_r^{t,i}), N_r^{t,i}) dr \\ &\quad + \int_t^s \sigma(r, X_r, v(r, X_r, N_r^{t,i}), N_r^{t,i}) dB_r \end{aligned}$$

Considering the processes

$$\forall s \in [t, T], \quad Y_s^{v,t,x,i} = v_{N_s^{t,i}}(s, X_s^{v,t,x,i}); \quad Z_s^{v,t,x,i} = \nabla_x v_{N_s^{t,i}}(s, X_s^{v,t,x,i}) \sigma_{N_s^{t,i}}(s, X_s^{v,t,x,i}, Y_s^{v,t,x,i})$$

$$\forall l \in L, \quad H_s^{v,t,x,i}(l) = \left(v_{N_s^{t,i}+l} - v_{N_s^{t,i}} \right)(s, X_s^{v,t,x,i})$$

we deduce thanks to Itô's formula,

$$\forall s \in [t, T],$$

$$Y_s^{v,t,x,i} = h_{N_s^{t,i}}(X_T^{v,t,x,i}) + \int_s^T f(r, X_r^{v,t,x,i}, Y_r^{v,t,x,i}, H_r^{v,t,x,i}, Z_r^{v,t,x,i}, N_r^{t,i}) dr - \int_s^T Z_r^{v,t,x,i} dB_r$$

$$- \int_s^T \sum_{l \in L} H_r^{v,t,x,i}(l) dM_r(l)$$

Therefore, the quartet $(X_r^{v,t,x,i}, Y_r^{v,t,x,i}, H_r^{v,t,x,i}, Z_r^{v,t,x,i})$ solves the FBSDE with transmutation process $(E^{t,x})$ with initial values t,x,i . Hence thanks to Theorem 2.1 we have

$$\forall (t, x) \in [T - \gamma, T] \tilde{u}(t, x) = v(t, x)$$

Then using again an induction we prove $\tilde{u} = v$ sur $[0, T] \times \mathbf{R}^d$. □

Comment: The previous corollary gives us a regular solution of our non-degenerate system of quasilinear parabolic PDEs whose gradient is uniformly bounded by a quantity which is independent of the Lipschitz constant of the drifts of the FBSDE $(E^{0,\xi})$.

Thanks to this result, we have a control on the Lipschitz constant of θ_i , $i \in \mathcal{K}$. Indeed θ_i plays the role of h_i in the latter application of Theorem 2.1 on a appropriate subdivision of $[0, T]$ in order to extended the solution in small time duration to a global one. This is what we do in the next section.

GLOBAL SOLUTION OF THE SYSTEM OF FBSDE

In order to extend the local solution of the FBSDE, we remove regularity required in the previous section and assume that the coefficients satisfy the following assumptions **(H4)**:

(H4.1) b and f satisfy **(H1)**.

(H4.2) σ and h satisfy (H1.3) and there exists a constant $\tilde{k} > 0$ such that

$$\forall i \in \mathcal{K}, \quad \forall t \in [0, T], \quad \forall ((x, y), (x', y')) \in (\mathbf{R}^d \times \mathbf{R})^2$$

$$|\sigma_i(t, x, y) - \sigma_i(t, x', y')|^2 \leq \tilde{k}^2(|x - x'|^2 + |y - y'|^2)$$

$$|h_i(x) - h_i(x')| \leq \tilde{k}|x - x'|$$

(H4.3) The function σ satisfies **(H3.1)**.

We have the following

PROPOSITION 3.1 Assume that **(H4)** hold and $T \leq \tilde{C}_{(\mathcal{K}, \lambda)}^{(2)}$. Then there exists a sequence of bounded C^∞ functions $(\bar{b}^{(n)}, \bar{f}^{(n)}, \sigma^{(n)}, h^{(n)})_{n \in \mathbf{N}}$ with bounded derivatives of every order such

that

$$\begin{aligned} \forall (t, x, u, z) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^k \times \mathbf{R}^d, \\ (\bar{b}^{(n)}, \bar{f}^{(n)}, \sigma^{(n)}, h^{(n)})(t, x, u, z) \xrightarrow{n \rightarrow +\infty} (\bar{b}, \bar{f}, \sigma, h)(t, x, u, z) \end{aligned}$$

Moreover, if $(X_s^{t,n,x,i}, Y_s^{t,n,x,i}, H_s^{t,n,x,i}, Z_s^{t,n,x,i})_{0 \leq s \leq T}$ stands for the solution of $(E^{0,x})$ associated to $(b^{(n)}, f^{(n)}, \sigma^{(n)}, h^{(n)})$, the map θ_n defined by

$$\begin{aligned} \theta_n : [0, T] \times \mathbf{R}^d \times \mathcal{K} &\rightarrow \mathbf{R} \\ (t, x, i) &\rightarrow Y_i^{t,n,x,i} \end{aligned}$$

also define a function $\theta_n : [0, T] \times \mathbf{R}^d \rightarrow \mathbf{R}^k$: solution of the system of PDEs

$$\begin{cases} \forall (t, x) \in [0, T] \times \mathbf{R}^d, \quad \forall i \in \mathcal{K} \\ \frac{\partial u_i}{\partial t}(t, x) + L_n^i u_i(t, x) + \bar{f}_i^{(n)}(t, x, u(t, x), \nabla_x u_i(t, x) \sigma_i^{(n)}(t, x, u_i(t, x))) = 0 \\ u_i(T, x) = h_i^{(n)}(x) \end{cases} \quad (3.1)$$

satisfying $\forall n \in \mathbf{N}$,

$$\sup_{(t,x) \in [0,T] \times \mathbf{R}^d} |\theta_n(t, x)| \leq \tilde{C} \quad (3.1)$$

$$\sup_{(t,x) \in [0,T] \times \mathbf{R}^d} |\nabla_x \theta_n(t, x)| \leq \tilde{\Gamma} \quad (3.3)$$

Moreover,

$$\theta_n \rightarrow \theta \quad \text{as } n \rightarrow +\infty \quad (3.4)$$

where θ is the map defined in Corollary 2.2.

Proof There exists a regularization sequence of bounded C^∞ functions $(\bar{b}^{(n)}, \bar{f}^{(n)}, \sigma^{(n)}, h^{(n)})_{n \in \mathbf{N}}$ with bounded derivatives such that for every $n \in \mathbf{N}$, the quartet $(b^{(n)}, f^{(n)}, \sigma^{(n)}, h^{(n)})$ satisfies **(H4)** with the constants $K + 4\Lambda, \tilde{k}, 2\Lambda$ and $(\delta/2)$ (we refer to Ref. [3], proposition 2.2).

Moreover, since $(b^{(n)}, f^{(n)}, \sigma^{(n)}, h^{(n)})$ satisfy also **(H3)**, by virtue of Proposition 2.2, there exist two constants still noted \tilde{C} and $\tilde{\Gamma}$ and not depending on n such that for all $n \in \mathbf{N}$, the map θ_n is a solution of Eq. (3.1) which satisfies Eqs. (3.2) and (3.3).

Furthermore applying Theorem 2.2 to the two quartets of functions $(b^{(n)}, f^{(n)}, \sigma^{(n)}, h^{(n)})$ and (b, f, σ, h) , we deduce from the Lebesgue dominated convergence theorem $\theta_n \rightarrow \theta$ as $n \rightarrow +\infty$. \square

Let $\tilde{K} = \max(\tilde{k}, \Lambda, \tilde{\Gamma})$, $\gamma = \tilde{C}_{(\tilde{K}, \lambda)}^{(2)}$. For any arbitrary $T > 0$, let $m = E(T/\gamma) + 1$ and consider the subdivision $(t_j)_{0 \leq j \leq m}$ of $[0, T]$ defined by

$$t_0 = 0, \quad \forall j \geq 1, \quad t_j = T - (m - j)\gamma$$

PROPOSITION 3.2 Assume that **(H4)** holds. Then $\forall T > 0$, there exists a map $\theta : [0, T] \times \mathbf{R}^d \times \mathcal{K} \rightarrow \mathbf{R}$ satisfying

- (i) $\forall (t, x, i) \in [0, T] \times \mathbf{R}^d \times \mathcal{K}, \quad |\theta(t, x, i)| \leq \tilde{C}$
- (ii) $\forall (t, i) \in [0, T] \times \mathcal{K}, \quad \forall (x, x') \in (\mathbf{R}^d)^2, \quad |\theta(t, x, i) - \theta(t, x', i)| \leq \tilde{L}|x - x'|$
- (iii) $\forall x \in \mathbf{R}^d, \quad \forall i \in \mathcal{K}, \quad \theta(T, x, i) = h_i(x)$
- (iv) $\forall j \in 0, \dots, m-1, \forall t \in [t_j, t_{j+1}], \forall i \in \mathcal{K}$, for every \mathcal{G}_t -measurable random vector ξ with finite second moment, the problem

$$\begin{cases} \forall s \in [t, t_{j+1}], \\ X_s = \xi + \int_t^s b(r, \Theta_r^{(j)}, N_r^{t,i}) dr + \int_t^s \sigma(r, X_r, Y_r, N_r^{t,i}) dB_r \\ Y_s = \theta(t_{j+1}, X_{t_{j+1}}, N_{t_{j+1}}^{t,i}) + \int_s^{t_{j+1}} f(r, \Theta_r^{(j)}, N_r^{t,i}) dr - \int_s^{t_{j+1}} Z_r dB_r - \int_s^{t_{j+1}} \sum_{l \in L} H_r(l) dM_r(l) \end{cases}$$

has a unique solution $\Theta_s^{(j)} = (X_s^{(j)}, Y_s^{(j)}, H_s^{(j)}, Z_s^{(j)})_{t \leq s \leq t_{j+1}}$ in $\mathcal{B}_{[t, t_{j+1}]}$ such that $\forall s \in [t, t_{j+1}]$,

$$Y_s^{(j)} = \theta(s, X_s^{(j)}, N_s^{t,i}) \quad (3.5)$$

$$\forall l \in L, \quad H_s^{(j)}(l) = \theta(s, X_s^{(j)}, N_{s^-}^{t,i} + l) - \theta(s, X_s^{(j)}, N_{s^-}^{t,i}) \quad (3.6)$$

$$Z_s^{(j)} = \nabla_x \theta(s, X_s^{(j)}, N_s^{t,i}) \sigma(s, X_s^{(j)}, Y_s^{(j)}, N_s^{t,i}) \quad (3.7)$$

Proof We build θ by a backward induction.

$\forall t \in [t_{m-1}, T]$, thanks to Theorem 2.1, $\forall i \in \mathcal{K}$, the problem

$$\begin{cases} \forall s \in [t, T], \\ X_s = \xi + \int_t^s b(r, \Theta_r^{(m-1)}, N_r^{t,i}) dr + \int_t^s \sigma(r, X_r, Y_r, N_r^{t,i}) dB_r \\ Y_s = h_{N_T^{t,i}}(X_T) + \int_s^T f(r, \Theta_r^{(m-1)}, N_r^{t,i}) dr - \int_s^T Z_r dB_r - \int_s^T \sum_{l \in L} H_r(l) dM_r(l) \end{cases} \quad (3.8)$$

has a unique solution $(\Theta_s^{(m-2)})_{t_{m-1} \leq s \leq T} \triangleq (X_s^{(m-1)}, Y_s^{(m-1)}, H_s^{(m-1)}, Z_s^{(m-1)})_{t_{m-1} \leq s \leq T}$ in $\mathcal{B}_{[t_{m-1}, T]}^2$ and applying Corollary 2.2, the map θ defined by

$$\begin{aligned} \theta : [t_{m-1}, T] \times \mathbf{R}^d \times \mathcal{K} &\rightarrow \mathbf{R} \\ (t, x, i) &\rightarrow Y_t^{t,x,i} \end{aligned}$$

satisfies the following equalities for all $t \in [t_{m-1}, T]$,

$$\theta(T, x, i) = h_i(x) \quad (3.9)$$

$$\forall s \in [t, T], \quad Y_s^{(m-1)} = \theta(s, X_s^{(m-1)}, N_s^{t,i}) \quad (3.10)$$

$$\forall l \in L, \quad H_s^{(m-1)}(l) = \theta(s, X_s^{(m-1)}, N_{s^-}^{t,i} + l) - \theta(s, X_s^{(m-1)}, N_{s^-}^{t,i}) \quad (3.11)$$

Furthermore using the previous Proposition, the map

$$\begin{aligned} \theta_n : [t_{m-1}, T] \times \mathbf{R}^d \times \mathcal{K} &\rightarrow \mathbf{R} \\ (t, x, i) &\rightarrow Y_t^{t,n,x,i} \end{aligned}$$

defines a function $\theta_n \in C^{1,2}([t_{m-1}, T] \times \mathbf{R}^d, \mathbf{R}^k)$, a solution of Eq. (3.1) and which satisfies

$$\sup_{(t,x) \in [t_{m-1}, T] \times \mathbf{R}^d} |\theta_n(t, x)| \leq \tilde{C} \tag{3.12}$$

$$\sup_{(t,x) \in [t_{m-1}, T] \times \mathbf{R}^d} |\nabla_x \theta_n(t, x)| \leq \tilde{I} \tag{3.13}$$

Let $n \rightarrow +\infty$, we deduce easily (i) and (ii). Therefore the proposition holds on $[t_{m-1}, T]$.

$\forall t \in [t_{m-2}, t_{m-1}], \forall i \in \mathcal{K}$, using once again Theorem 2.1, for every \mathcal{G}_t —measurable random vector ξ with finite second moment, the problem

$$\begin{cases} \forall s \in [t, t_{m-1}], \\ X_s = \xi + \int_t^s b(r, \Theta_r^{(m-2)}, N_r^{t,i}) dr + \int_t^s \sigma(r, X_r, Y_r, N_r^{t,i}) dB_r \\ Y_s = \theta(t_{m-1}, X_{t_{m-1}}^{(m-2)}, N_{t_{m-1}}^{t,i}) + \int_s^{t_{m-1}} f(r, \Theta_r^{(m-2)}, N_r^{t,i}) dr - \int_s^{t_{m-1}} Z_r dB_r - \int_s^{t_{m-1}} \sum_{l \in L} H_r(l) dM_r(l) \end{cases}$$

admits a unique solution $(\Theta_s^{(m-2)}) \triangleq (X_s^{(m-2)}, Y_s^{(m-2)}, H_s^{(m-2)}, Z_s^{(m-2)})_{t_{m-2} \leq s \leq t_{m-1}}$ in $\mathcal{B}_{[t_{m-2}, t_{m-1}]}$. Hence using the same method, there exists a map which we note again θ

$$\begin{aligned} \theta : [t_{m-2}, t_{m-1}] \times \mathbf{R}^d \times \mathcal{K} &\rightarrow \mathbf{R} \\ (t, x, i) &\rightarrow Y_t^{t,x,i} \end{aligned}$$

satisfying for all $t \in [t_{m-2}, t_{m-1}]$,

$$|\theta(t, x, i) - \theta(t, x', i)| \leq \tilde{I}|x - x'| \tag{3.14}$$

$$|\theta(t, x, i)| \leq \tilde{C} \tag{3.15}$$

$$\forall s \in [t, T], \quad Y_s^{(m-1)} = \theta(s, X_s^{(m-1)}, N_s^{t,i}) \tag{3.16}$$

$$\forall l \in L, \quad H_s^{(m-1)}(l) = \theta(s, X_s^{(m-1)}, N_s^{t,i} + l) - \theta(s, X_s^{(m-1)}, N_s^{t,i}) \tag{3.17}$$

So, we can define θ on $[t_{m-2}, T] \times \mathbf{R}^d \times \mathcal{K} \rightarrow \mathbf{R}$.

Therefore using an induction we build $\theta : [0, T] \times \mathbf{R}^d \times \mathcal{K}$. □

COROLLARY 3.1 Under the assumptions **(H4)** and the statements of proposition 3.2, for every $(t, i) \in [0, T] \times \mathcal{K}$ and ξ a \mathcal{G}_t —measurable random vector with finite second moment, every solution $(\Theta_s^{t,\xi,i})_{(t \leq s \leq T)}$ of the problem $(E^{t,\xi})$ satisfies

$$\forall n \leq j \leq m - 1,$$

$$\begin{aligned} \mathbf{E} \sup_{\tilde{i}_j \leq s \leq \tilde{i}_{j+1}} \left| X_s^{t,\xi,i} - X_s^{\tilde{i}_j, X_{\tilde{i}_j}, N_{\tilde{i}_j}^{t,i}} \right|^2 &= \mathbf{E} \sup_{\tilde{i}_j \leq s \leq \tilde{i}_{j+1}} \left| Y_s^{t,\xi,i} - Y_s^{\tilde{i}_j, X_{\tilde{i}_j}, N_{\tilde{i}_j}^{t,i}} \right|^2 = \mathbf{E} \int_{\tilde{i}_j}^{\tilde{i}_{j+1}} \left| Z_s^{t,\xi,i} - Z_s^{\tilde{i}_j, X_{\tilde{i}_j}, N_{\tilde{i}_j}^{t,i}} \right|^2 ds \\ &= \mathbf{E} \int_{\tilde{i}_j}^{\tilde{i}_{j+1}} \sum_{l \in L} \left(H_s^{t,\xi,i}(l) - H_s^{\tilde{i}_j, X_{\tilde{i}_j}, N_{\tilde{i}_j}^{t,i}}(l) \right)^2 ds = 0 \end{aligned}$$

where n is the unique integer such that $t \in [t_n, t_{n+1}]$ and $(\tilde{t}_j)_{n \leq j \leq m}$ is the sequence defined by

$$\tilde{t}_j = t \quad \text{si } j = n; \quad \tilde{t}_j = t_j \quad \text{si } j > n$$

Proof If $n = m - 1$, using proposition 3.2 the result is an easy consequence of the uniqueness of solution on $[t_{m-1}, T]$.

If $n \leq m - 2$, and $(X_s, Y_s, H_s, Z_s)_{t \leq s \leq T}$ is the unique solution of the problem, then $(X_s, Y_s, H_s, Z_s)_{t_{m-1} \leq s \leq T}$ is solution of the system

$$\begin{cases} \forall s \in [t_{m-1}, T] \\ X_s^{t, \xi, i} = X_{t_{m-1}} + \int_{t_{m-1}}^s b(r, \Theta_r^{t, \xi, i}, N_r^{t_{m-1}, i_1}) dr + \int_{t_{m-1}}^s \sigma(r, X_r, Y_r, N_r^{t_{m-1}, i_1}) dB_r \\ Y_s^{t, \xi, i} = h_{N_r^{t_{m-1}, i_1}}(X_T) + \int_s^T f(r, \Theta_r^{t, \xi, i}, N_r^{t_{m-1}, i_1}) dr - \int_s^T Z_r^{t, \xi, i} dB_r - \int_s^T \sum_{l \in L} H_r^{t, \xi, i}(l) dM_r(l) \\ \mathbf{E} \int_{t_{m-1}}^T \left(|X_r^{t, \xi, i}|^2 + |Y_r^{t, \xi, i}|^2 + |H_r^{t, \xi, i}|^2 + |Z_r^{t, \xi, i}|^2 \right) dr < +\infty \end{cases}$$

where $i_1 = N_{t_{m-1}}^{t, i}$.

From uniqueness of solution and Proposition 3.2, we deduce

$$\begin{aligned} \mathbf{E} \sup_{t_{m-1} \leq s \leq T} \left| X_s^{t, \xi, i} - X_s^{t_{m-1}, X_{t_{m-1}}, i_1} \right|^2 &= \mathbf{E} \sup_{t_{m-1} \leq s \leq T} \left| Y_s - Y_s^{t_{m-1}, X_{t_{m-1}}, i_1} \right|^2 \\ &= \mathbf{E} \sum_{l \in L} \int_{t_{m-1}}^T \left(H_s(l) - H_s^{t_{m-1}, X_{t_{m-1}}, i_1}(l) \right)^2 ds \\ &= \mathbf{E} \int_{t_{m-1}}^T \left| Z_s - Z_s^{t_{m-1}, X_{t_{m-1}}, i_1} \right|^2 ds = 0 \end{aligned}$$

and $\forall s \in [t_{m-1}, T]$, $Y_s = \theta_{N_s^{t_{m-1}, i_1}}(s, X_s)$, in particular

$$Y_{t_{m-1}} = \theta_{i_1}(t_{m-1}, X_{t_{m-1}}) = \theta_{N_{t_{m-1}}^{t, i}}(t_{m-1}, X_{t_{m-1}})$$

hence the process $(X_s, Y_s, H_s, Z_s)_{t \leq s \leq t_{m-1}}$ solves the system

$$\begin{cases} \forall s \in [t, t_{m-1}] \\ X_s^{t, \xi, i} = \xi + \int_t^s b(r, \Theta_r^{t, \xi, i}, N_r^{t, i}) dr + \int_t^s \sigma(r, X_r, Y_r, N_r^{t, i}) dB_r \\ Y_s^{t, \xi, i} = \theta_{N_{t_{m-1}}^{t, i}}(t_{m-1}, X_{t_{m-1}}) + \int_s^{t_{m-1}} f(r, \Theta_r^{t, \xi, i}, N_r^{t, i}) dr - \int_s^{t_{m-1}} Z_r^{t, \xi, i} dB_r \\ \quad - \int_s^{t_{m-1}} \sum_{l \in L} H_r^{t, \xi, i}(l) dM_r(l) \\ \mathbf{E} \int_t^{t_{m-1}} \left(|X_r^{t, \xi, i}|^2 + |Y_r^{t, \xi, i}|^2 + |H_r^{t, \xi, i}|^2 + |Z_r^{t, \xi, i}|^2 \right) dr < +\infty \end{cases}$$

analogously, we prove

$$\begin{aligned} \mathbf{E} \sup_{t_{m-2} \leq s \leq t_{m-1}} \left| X_s^{t, \xi, i} - X_s^{t_{m-2}, X_{t_{m-2}}, i_2} \right|^2 &= \mathbf{E} \sup_{t_{m-2} \leq s \leq t_{m-1}} \left| Y_s - Y_s^{t_{m-2}, X_{t_{m-2}}, i_2} \right|^2 \\ &= \mathbf{E} \sum_{l \in L} \int_{t_{m-2}}^{t_{m-1}} \left(H_s(l) - H_s^{t_{m-2}, X_{t_{m-2}}, i_2}(l) \right)^2 ds \\ &= \mathbf{E} \int_{t_{m-2}}^{t_{m-1}} \left| Z_s - Z_s^{t_{m-2}, X_{t_{m-2}}, i_2} \right|^2 ds = 0 \end{aligned}$$

and in particular

$$Y_{t_{m-2}} = \theta_{t_2}(t_{m-2}, X_{t_{m-2}}) = \theta_{N_{t_{m-2}}^{t_2, i_1}}(t_{m-2}, X_{t_{m-2}}) \quad \text{where } i_2 = N_{t_{m-1}}^{t_2, i_1}.$$

Applying an induction we get the desired result. \square

THEOREM 3.1 (GLOBAL SOLUTION) Under the assumptions **(H4)**, $\forall T > 0$, $\forall i \in \mathcal{K}$, for every \mathcal{G}_0 -measurable random vector ξ with finite second moment the system $(E^{0, \xi})$ has a unique solution.

Proof Assume that **(H4)** holds and $T > 0$. Let $i \in \mathcal{K}$ be fixed and ξ a \mathcal{G}_0 -measurable random vector with finite second moment.

With the previous subdivision of $[0, T]$, we define for all $j \in \{0, \dots, m-1\}$,

$$n_0 = i; \quad \text{and } \forall 1 \leq j \leq m, \quad n_j = N_{t_j}^{t_{j-1}, n_{j-1}}$$

Thanks to proposition 3.2, the system

$$\begin{cases} \forall s \in [0, t_1], \\ X_s = \xi + \int_0^s b(r, \Theta_r^{(0)}, N_r^{0,i}) dr + \int_0^s \sigma(r, X_r, Y_r, N_r^{0,i}) dB_r \\ Y_s = \theta(t_1, X_{t_1}^{(0)}, N_{t_1}^{0,i}) + \int_s^{t_1} f(r, \Theta_r^{(0)}, N_r^{0,i}) dr - \int_s^{t_1} Z_r dB_r - \int_s^{t_1} \sum_{l \in L} H_r(l) dM_r(l) \end{cases}$$

admits a unique solution $(\Theta_s^{0, \xi, i}) = (X_s^{(0)}, Y_s^{(0)}, H_s^{(0)}, Z_s^{(0)})_{0 \leq s \leq t_1}$ in $\mathcal{B}_{[0, t_1]}^2$ satisfying

$$Y_{t_1}^{(0)} = \theta(t_1, X_{t_1}^{(0)}, N_{t_1}^{0,i})$$

$$\forall l \in L, \quad H_{t_1}^{(0)}(l) = \theta(t_1, X_{t_1}^{(0)}, N_{t_1-}^{0,i} + l) - \theta(t_1, X_{t_1}^{(0)}, N_{t_1-}^{0,i})$$

Using the same result on $[t_1, t_2]$, we know that the system

$$\begin{cases} \forall s \in [t_1, t_2], \\ X_s = X_{t_1}^{(0)} + \int_{t_1}^s b(r, \Theta_r^{(1)}, N_r^{t_1, n_1}) dr + \int_{t_1}^s \sigma(r, X_r, Y_r, N_r^{t_1, n_1}) dB_r \\ Y_s = \theta(t_2, X_{t_2}^{(1)}, N_{t_2}^{t_1, n_1}) + \int_s^{t_2} f(r, \Theta_r^{(1)}, N_r^{t_1, n_1}) dr - \int_s^{t_2} Z_r dB_r - \int_s^{t_2} \sum_{l \in L} H_r(l) dM_r(l) \end{cases}$$

admits a unique solution $(\Theta_s^{t_1, X_{t_1}^{(0)}, n_1}) \triangleq (X_s^{(1)}, Y_s^{(1)}, H_s^{(1)}, Z_s^{(1)})_{t_1 \leq s \leq t_2}$ in $\mathcal{B}_{[t_1, t_2]}^2$ and applying the previous result, we have $\forall s \in [t_1, t_2]$,

$$Y_s^{(1)} = \theta(s, X_s^{(1)}, N_s^{t_1, n_1}).$$

$$\forall l \in L, \quad H_s^{(1)}(l) = \theta(s, X_s^{(1)}, N_s^{t_1, n_1} + l) - \theta(s, X_s^{(1)}, N_s^{t_1, n_1}).$$

Hence we can deduce

$$X_{t_1}^{(1)} = X_{t_1}^{(0)}$$

$$Y_{t_1}^{(1)} = \theta(t_1, X_{t_1}^{(1)}, n_1) = \theta(t_1, X_{t_1}^{(0)}, n_1) = Y_{t_1}^{(0)}.$$

$$\forall l \in L, \quad H_{t_1}^{(1)}(l) = \theta(t_1, X_{t_1}^{(1)}, n_1 + l) - \theta(t_1, X_{t_1}^{(1)}, n_1) = H_{t_1}^{(0)}(l)$$

So, applying again this argument on $[t_2, t_3]$, the system

$$\begin{cases} \forall s \in [t_2, t_3], \\ X_s = X_{t_2}^{(1)} + \int_{t_2}^s b(r, \Theta_r^{(2)}, N_r^{t_2, n_2}) dr + \int_{t_2}^s \sigma(r, X_r, Y_r, N_r^{t_2, n_2}) dB_r \\ Y_s = \theta(t_3, X_{t_3}^{(2)}, N_{t_3}^{t_2, n_2}) + \int_s^{t_3} f(r, \Theta_r^{(2)}, N_r^{t_2, n_2}) dr - \int_s^{t_3} Z_r dB_r - \int_s^{t_3} \sum_{l \in L} H_r(l) dM_r(l) \end{cases}$$

admits a unique solution $(\Theta_s^{t_2, X_{t_2}^{(1)}, n_2}) = (X_s^{(2)}, Y_s^{(2)}, H_s^{(2)}, Z_s^{(2)})_{t_2 \leq s \leq t_3}$ dans $\mathcal{B}_{[t_2, t_3]}^2$ and applying again the previous result we get $\forall s \in [t_2, t_3]$, $\theta(s, X_s^{(2)}, N_s^{t_2, n_2}) = Y_s^{(2)}$ and we deduce

$$\begin{aligned} X_{t_2}^{(2)} &= X_{t_2}^{(1)} \\ Y_{t_2}^{(2)} &= \theta(t_2, X_{t_2}^{(2)}, n_2) = \theta(t_2, X_{t_2}^{(1)}, n_2) = Y_{t_2}^{(1)}. \\ \forall l \in L, \quad H_{t_2}^{(2)}(l) &= \theta(t_2, X_{t_2}^{(2)}, n_2 + l) - \theta(t_2, X_{t_2}^{(1)}, n_2) = H_{t_2}^{(1)}(l) \end{aligned}$$

Using an induction we are able to build a quartet $(\Theta_s^{(j)}) = (X_s^{(j)}, Y_s^{(j)}, H_s^{(j)}, Z_s^{(j)})_{j=0, \dots, m-1, s \in [t_j, t_{j+1}]}$ solution in $\mathcal{B}_{[t_j, t_{j+1}]}^2$ of the system

$$(\tilde{E}) \begin{cases} \forall s \in [t_j, t_{j+1}], \\ X_s = X_{t_j}^{(j-1)} + \int_{t_j}^s b(r, \Theta_r^{(j)}, N_r^{t_j, n_j}) dr + \int_{t_j}^s \sigma(r, X_r, Y_r, N_r^{t_j, n_j}) dB_r \\ Y_s = \theta(t_{j+1}, X_{t_{j+1}}^{(j)}, N_{t_{j+1}}^{t_j, n_j}) + \int_s^{t_{j+1}} f(r, \Theta_r^{(j)}, N_r^{t_j, n_j}) dr - \int_s^{t_{j+1}} Z_r dB_r - \int_s^{t_{j+1}} \sum_{l \in L} H_r(l) dM_r(l) \end{cases}$$

which satisfies

$$X_{t_j}^{(j)} = X_{t_j}^{(j-1)}; \quad Y_{t_j}^{(j)} = \theta(t_j, X_{t_j}^{(j-1)}, n_j) = Y_{t_j}^{(j-1)}; \quad H_{t_j}^{(j)} = H_{t_j}^{(j-1)}.$$

Therefore, the quartet of \mathcal{G}_s -measurable process $(\Theta_s)_{0 \leq s \leq T} \triangleq (X_s, Y_s, H_s, Z_s)_{0 \leq s \leq T}$ defined by putting

$$\forall j \in \{0, \dots, m-1\}, \quad \forall s \in [t_j, t_{j+1}], \quad X_s = X_s^{(j)}, \quad Y_s = Y_s^{(j)}, \quad H_s = H_s^{(j)}, \quad Z_s = Z_s^{(j)}$$

is solution of $(E^{0, \xi})$.

For the proof of uniqueness, let us consider the process $(\tilde{\Theta}_s)_{0 \leq s \leq T}$ as another solution of $(E^{0, \xi})$. Using Corollary 3.1, we have

$$\mathbf{E} \sup_{0 \leq s \leq t_1} |\tilde{X}_s - X_s|^2 = \mathbf{E} \sup_{0 \leq s \leq t_1} |\tilde{Y}_s - Y_s|^2 = \int_0^{t_1} \sum_{l \in L} (\tilde{H}_s(l) - H_s(l))^2 ds = \int_0^{t_1} |\tilde{Z}_s - Z_s|^2 ds = 0$$

Hence we deduce

$$\tilde{X}_{t_1} = X_{t_1}.$$

Using a backward induction similarly to the construction of the solution $(\Theta_s)_{0 \leq s \leq T}$, we prove that

$$\forall t \in [0, T], \quad \tilde{X}_t = X_t; \quad \tilde{Y}_t = Y_t; \quad \tilde{H}_t = H_t; \quad \tilde{Z}_t = Z_t$$

□

COROLLARY 3.2 Assume that **(H3)** holds and for $1 \leq i \leq k$, let $(Y_s^{t,x,i}, t \leq s \leq T)$ be the unique solution of $(E^{t,x})$ given by Theorem 3.1. Then the map θ defined by

$$\theta(t, x) = (Y_t^{t,x,1}, Y_t^{t,x,2}, \dots, Y_t^{t,x,k}), \quad (t, x) \in [0, T] \times \mathbf{R}^d \quad (3.18)$$

coincides with the unique solution of the system of PDEs (\mathcal{E}) given by Corollary 2.8.

Proof Let us consider the previous subdivision of $[0, T]$ and θ defined by Eq. (3.18). Let n be the unique integer such that $t \in [t_n, t_{n+1}]$. Thanks to the assumptions **(H3)** and Proposition 2.2, the function θ is a solution of (\mathcal{E}) on $[t_n, t_{n+1}]$. Then using Corollary 2.8, we deduce $\forall t \in [t_n, t_{n+1}], \quad \theta(t, x) = \tilde{u}(t, x)$.

Using an induction, we complete the proof. \square

Moreover, we have the following result whose proof is an adaptation of the ones given in Refs. [9,10]. However, for the notion of viscosity solution to make sense we need to assume that b does not depend on z .

COROLLARY 3.3 Under the assumptions **(H4)** if moreover b does not depend on z , the map θ defined in the previous corollary provides a viscosity solution of the system of quasilinear PDEs (\mathcal{E}) .

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