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A path–valued Markov process indexed by the ancestral mass

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Abstract. A family of Feller branching diffusions Z^x, $x \geq 0$, with nonlinear drift and initial value x can, with a suitable coupling over the *ancestral masses* x, be viewed as a path-valued process indexed by x. For a coupling due to Dawson and Li, which in case of a linear drift describes the corresponding Feller branching diffusion, and in our case makes the path-valued process Markovian, we find an SDE solved by Z, which is driven by a random point measure on excursion space. In this way we are able to identify the infinitesimal generator of the path-valued process. We also establish path properties of $x \mapsto Z^x$ using various couplings of Z with classical Feller branching diffusions.

1. Introduction

Consider the SDE

$$Z_t^x = x + \int_0^t f(Z_s^x)ds + 2\int_0^t \int_0^{Z_s^x} W(ds, d\xi), \quad t \geq 0,\, x \geq 0 \tag{1.1}$$

where $W(\cdot, \cdot)$ denotes a two–dimensional white noise, i.e. a generalized zero mean Gaussian random field on $\mathbb{R}_+^2$, whose covariance operator is the identity operator on $L^2(\mathbb{R}_+^2)$. Our assumptions on the function $f : \mathbb{R}_+ \to \mathbb{R}$ will be specified below.

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For any fixed $x > 0$, the solution of (1.1) has the same law as the solution of the simpler SDE

$$Z_t^x = x + \int_0^t f(Z_s^x)ds + 2\int_0^t \sqrt{Z_s^x}dB_s,$$

where B is a standard scalar Brownian motion. However, the formulation (1.1), which follows Dawson and Li (2012), is the simplest and most natural way to specify a coupling which in the case of a linear drift renders the branching property of Z, i.e. the independence of Z^x and $Z^{x+y} - Z^x$ for all $x, y > 0$. For a quadratic f, this same coupling is discussed also in Pardoux and Wakolbinger (2011a) Sec. 2, but there without appealing to the representation (1.1).

When f is linear, i.e. $f(z) = \theta z$, the solution Z_t^x, $t \geq 0$, is a continuous state branching process with ancestral mass x. This process models the evolution of the (continuum scaling limit of the) size of a population when the reproduction dynamics of the various individuals are mutually independent.

In the terminology of a (pre-limiting) individual-based description, the possible nonlinearities of f model an impact of the current population size on the individual reproduction dynamics, and in this way go along with an interaction in the individuals' reproductive behavior. If the interaction is of the type of competition for rare resources, then an increase in the population size decreases the individual birth rate and/or increases the death rate. This means that $f(z)/z$ should be decreasing, and $f(z)$ should be negative for z large enough. On the other hand, specifically for moderate values of z, $f(z)/z$ might be increasing. This is the case in the presence of the so–called Allee effect, where there is a negative growth rate for small population sizes z and a positive growth rate for larger population sizes (as long as the population size does not exceed a certain carrying capacity).

In previous publications, we obtained several results on the solution of equation (1.1) (for f of the form $f(z) = \theta z - \gamma z^2$ or for more general f). In particular we discussed

(i) its approximation by finite population models (Le et al. (2013), Ba and Pardoux (2015)),

(ii) the extension of the second Ray–Knight theorem and a description of the forest of genealogical trees of the population whose total size follows (1.1) (Le et al. (2013), Ba and Pardoux (2015), Pardoux and Wakolbinger (2011a)),

(iii) the effect of competition on the asymptotic extinction time and total mass of the forest of trees for large population size (Le and Pardoux (2015)).

In this paper, we study the solution of (1.1) as a path-valued process indexed by the mass x of the ancestral population.

In the case where f is linear, much is known about the $\mathbb{R}_+$–valued process $\{Z_t^x,\ x > 0\}$ for any $t > 0$ fixed, as well as about the $C(\mathbb{R}_+; \mathbb{R}_+)$–valued process $\{Z_\cdot^x,\ x > 0\}$. Those have independent increments, which reflects the independence of the progenies of various ancestors in a branching process. Moreover, for any fixed $t > 0$, $x \mapsto Z_t^x$ is an increasing process which has a.s. finitely many jumps on any finite time interval, and is constant between its jumps. On the other hand, the path-valued process $x \mapsto Z_\cdot^x$ has infinitely many jumps on any interval of positive Lebesgue measure. Also, for linear f explicit formulas for the law of the random variables Z_t^x are available.

In the presence of a nonlinear drift f, the situation is more complicated, and our paper contributes to its investigation. Our first step is to show that $x \mapsto Z_t^x$ is, for fixed $t > 0$, again an increasing process which increases only by jumps, whose number is finite on any compact interval. Now the process $x \mapsto Z^x_t$ in general does no longer have independent increments. We strongly suspect that (for fixed t) it is not a Markov process. However, the path-valued process $x \mapsto Z^x_\cdot$ *is* Markovian, and it is the objective of this paper to write an SDE driven by a random point measure for the process $x \mapsto Z^x_\cdot$, and to identify its infinitesimal generator, by writing a martingale problem formulation of the path–valued SDE which it solves, see Theorem 4.12 and Corollary 4.13 below.

Let us now specify our standing assumptions on the nonlinear function f. We assume that $f \in C(\mathbb{R}_+;\mathbb{R})$, $f(0) = 0$, and f satisfies in addition the following three assumptions

$$f(a+b) - f(a) \le \theta b, \quad \text{for some } \theta \ge 0 \text{ and all } a, b > 0\,; \tag{1.2}$$

f is $\frac{1}{2}$–Hölder continuous, i.e. for all $M > 0$ there exists a $C_M < \infty$ such that

$$|f(a+b) - f(a)| \le C_M\sqrt{b} \text{ for all } a \in [0,M] \text{ and } b \in [0,1]\,; \tag{1.3}$$

$$\int_1^\infty \exp\left(-\frac{1}{2}\int_1^u \frac{f(r)}{r}dr\right)du = +\infty. \tag{1.4}$$

The first assumption is crucial for equation (1.1) to be well–posed, see Ba and Pardoux (2015). It also implies that $f(z) \le \theta z$ for all $z > 0$, which will be important in the next section. The second assumption implies that $b^{-1/2}[f(a+b) - f(a)]$ remains bounded while $b \to 0$, which will be essential for our Girsanov transformations below. Finally, it is shown in Ba and Pardoux (2015) that the third assumption implies that, for all $x > 0$, the random path $Z^x_\cdot$ hits zero in finite time a.s., which will be important in many of our arguments below. For technical as well as for conceptual reasons we want to study models of populations which go extinct in finite time. This is why we assume that $f(0) = 0$, and not just that $f(0) \ge 0$. In particular, we do not consider populations with immigration (except as an auxiliary construction in some proofs below).

Note that a sufficient condition for (1.4) to hold is that there exists $z_0 > 0$ such that $f(z) \le 2$, for all $z \ge z_0$. Clearly, a wide variety of functions f satisfy our assumptions.

The paper is organized as follows. In section 2, we establish the basic properties of the solution of (1.1), recalling in particular the existence and uniqueness result from Dawson and Li (2012).

Section 3 is devoted to comparison with a supercritical Feller diffusion Y, with supercriticality parameter θ, the same real number which appears in the assumption (1.2). We first prove a basic and easy comparison theorem between Z and Y. Next we construct another coupling of the two processes, for which a much stronger comparison holds. This permits us to deduce that Z increases only where Y increases, in particular $x \mapsto Z^x|_{[\delta,\infty)}$ is constant between its jumps for all $\delta > 0$.

Section 4 is devoted to establishing a path–valued SDE satisfied by $\{Z^x_\cdot,\ x > 0\}$, and deducing the exact form of the generator of that Markov process. Here again we shall consider a pair (Y, Z). However the process Y will then be a critical Feller diffusion, and in this case there will be no comparison between Z and Y. Instead, we shall exploit Girsanov's theorem and write the Radon–Nikodym derivative of the law of Z with respect to that of Y. This will be our key ingredient for the identification of the generator of the process $Z^x_\cdot$.

2. Basic results

It follows from Theorem 2.1 in Dawson and Li (2012) that for any $x > 0$, (1.1) has a unique continuous non–negative solution. Note in particular that the assumptions of that theorem are satisfied here, since we can decompose $f(a) = \theta a + [f(a) - \theta a]$, where $a \mapsto \theta a$ is Lipschitz, while $a \mapsto f(a) - \theta a$ is continuous and non increasing.

Since we are interested in the two-parameter process $\{Z^x_t,\ t \geq 0,\ x > 0\}$, we need to make sure that we can choose an appropriate version.

Lemma 2.1. *The mapping $\xi \mapsto Z^\xi_\cdot$ is continuous in probability.*

Proof : Let $x, y > 0$. Theorem 2.2 in Dawson and Li (2012) implies that

$$\mathbb{P}(Z^{x+y}_s - Z^x_s \geq 0, \forall s \geq 0) = 1\,. \tag{2.1}$$

Consequently,

$$Z^{x+y}_t - Z^x_t = y + \int_0^t [f(Z^{x+y}_s) - f(Z^x_s)]ds + 2\int_0^t \int_{Z^x_s}^{Z^{x+y}_s} W(ds, d\xi). \tag{2.2}$$

Taking the expectation in this identity, and exploiting (1.2) and Gronwall's Lemma, we infer that

$$\mathbb{E}[Z^{x+y}_t - Z^x_t] \leq y \exp(\theta t). \tag{2.3}$$

Let M_t denote the last term on the right of (2.2). The quadratic variation of this martingale is given by

$$\langle M, M\rangle_t = 4\int_0^t (Z^{x+y}_s - Z^x_s)ds.$$

A consequence of (2.2) and (1.2) is that for any $t > 0$:

$$\sup_{0\leq s\leq t}(Z^{x+y}_s - Z^x_s) \leq y + \theta\int_0^t (Z^{x+y}_s - Z^x_s)ds + \sup_{0\leq s\leq t}|M_s|. \tag{2.4}$$

Taking the expectation in (2.4), we deduce from the Burkholder–Davis–Gundy and Schwarz inequalities and (2.3) that there exists a constant $c > 0$ such that

$$\mathbb{E}\left[\sup_{0\leq s\leq t}(Z^{x+y}_s - Z^x_s)\right] \leq y(1 + e^{\theta t}) + c\sqrt{y\theta^{-1}e^{\theta t}},$$

from which the result follows, again in view of (2.1). ♢

Definition 2.2. We will denote by E the space of continuous functions u from $[0, +\infty)$ into itself, which are such that whenever $\zeta(u) := \inf\{t > 0,\ u(t) = 0\}$ is finite, then $u(t) = 0$, for any $t \geq \zeta(u)$. We equip E with the topology of uniform convergence on compacts.

The following result is similar to Theorem 3.6 in Dawson and Li (2012).

Lemma 2.3. *There exists a version of the mapping $\xi \mapsto Z^\xi_\cdot$ which is a.s. increasing and càdlàg with values in E.*

PROOF : Let $\{\tilde{Z}^\xi,\ \xi > 0\}$ denote an arbitrary collection of processes, such that $\tilde{Z}^\xi$ solves the SDE (1.1) for any $\xi > 0$. The fact that $\xi \mapsto \tilde{Z}^\xi_\cdot$ is a.s. increasing from $\mathbb{Q}_+$ into $C([0,+\infty))$ follows from (2.1). Now for any $x > 0$ we define

$$Z^x = \lim_{\xi_n \downarrow x,\ \xi_n \in \mathbb{Q}_+} \tilde{Z}^{\xi_n}.$$

By monotonicity, the sequence converges a.s., and it follows from Lemma 2.1 that, for any $x > 0$, $Z^x = \tilde{Z}^x$ a.s., hence Z^x solves (1.1). The result follows. ♢

As we will see below, the mapping $\xi \mapsto Z^\xi_\cdot$ does have discontinuities with positive probability.

3. Connection with a supercritical Feller diffusion

In this section, $\{Y^x_t,\ t \ge 0,\ x > 0\}$ stands for a Feller branching diffusion with supercriticality parameter θ, starting from an ancestral mass $x > 0$. More precisely, for a given space–time white noise W, and $\theta > 0$ being the parameter that enters condition (1.2) on f, we write Y^x for the solution of

$$Y^x_t = x + \theta \int_0^t Y^x_s ds + 2 \int_0^t \int_0^{Y^x_s} W(ds, d\xi). \tag{3.1}$$

Let Z^x be the solution of (1.1), with f satisfying conditions (1.2), (1.3) and (1.4). The two equations (3.1) and (1.1) with the same W describe one possible coupling of the two random fields $\{Y^x_t,\ t \ge 0,\ x > 0\}$ and $\{Z^x_t,\ t \ge 0,\ x > 0\}$.

Proposition 3.1. *For each $x > 0$,*

$$\mathbb{P}(Z^x_t \le Y^x_t,\ \forall t \ge 0) = 1.$$

PROOF : Since $f(x) \le \theta x$, this is immediate from the comparison theorem (Theorem 2.2) in Dawson and Li (2012). ♢

We now construct yet another coupling which will allow to derive distributional properties of Z that are required in the sequel. For each $t > 0$, $x > 0$, let

$$D_t = \{\xi > 0;\ Y^\xi_t > Y^{\xi-}_t\},\ \text{and}$$
$$A^x_t(Z) = \cup_{\xi \le x,\ \xi \in D_t} (Y^{\xi-}_t, Y^{\xi-}_t + Z^\xi_t - Z^{\xi-}_t].$$

Note that the random set A^x_t depends upon the copy of Z, in particular upon the chosen coupling of Y and Z. Note also that the Lebesgue measure of the set $A^x_t(Z)$ equals Z^x_t. We have the

Theorem 3.2. *There exists a random field $\{\tilde{Z}^x_t,\ x > 0, t \ge 0\}$ such that $t \mapsto \tilde{Z}^x_t$ is continuous, $x \mapsto \tilde{Z}^x_t$ is right–continuous, $\{\tilde{Z}^x_t,\ x > 0, t \ge 0\}$ has the same law as $\{Z^x_t,\ x > 0, t \ge 0\}$ (the solution of (1.1)), $\{\tilde{Z}^x_t,\ x > 0, t \ge 0\}$ solves the SDE*

$$\tilde{Z}^x_t = x + \int_0^t f(\tilde{Z}^x_s) ds + 2 \int_0^t \int_{A^x_s(\tilde{Z})} W(ds, d\xi), \tag{3.2}$$

and moreover for all $x, y > 0$,

$$\mathbb{P}(\tilde{Z}^{x+y}_t - \tilde{Z}^x_t \le Y^{x+y}_t - Y^x_t,\ \forall t \ge 0) = 1. \tag{3.3}$$

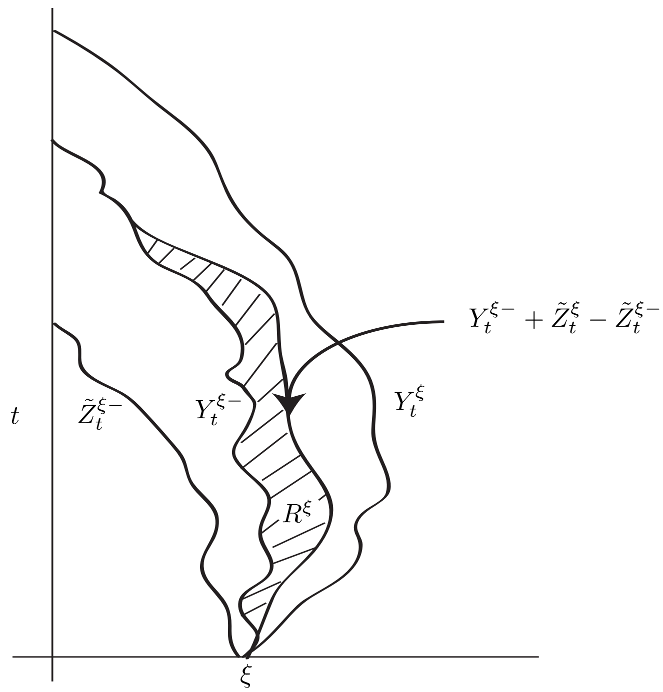

FIGURE 3.1. The region R^ξ of the noise W that drives $\tilde{Z}^\xi - \tilde{Z}^{\xi-}$ (which is shaded in the picture) is contained in the region of the noise that drives $Y^\xi - Y^{\xi-}$ (which is the one between $Y^{\xi-}$ and Y^ξ). In particular, R^ξ does not intersect the region to the left of $Y^{\xi-}$, which would be the case if $\tilde{Z}$ were replaced by Z.

PROOF : For a solution $\tilde{Z}$ of (3.2), the equality in law between $\{\tilde{Z}_t^x,\ x > 0, t \geq 0\}$ and $\{Z_t^x,\ x > 0, t \geq 0\}$ follows from the fact that the Lebesgue measure of $A_t^x(\tilde{Z})$ equals $\tilde{Z}_t^x$. We now construct a solution of (3.2).

For each $k, n \geq 1$, let $x_n^k := 2^{-n}k$. For each $n \geq 1$, we now define $\{Z_t^{n,x},\ t \geq 0\}$. For $0 < x \leq x_n^1$, we require that $\{Z_t^{n,x},\ t \geq 0\}$ solves

$$Z_t^{n,x} = x + \int_0^t f(Z_s^{n,x})ds + 2\int_0^t \int_0^{Z_s^{n,x}} W(ds, d\xi).$$

And for $k \geq 2$, we define recursively $\{Z_t^{n,x},\ t \geq 0\}$ for $x_n^{k-1} < x \leq x_n^k$ as the solution of

$$\begin{aligned} Z_t^{n,x} - Z_t^{n,x_n^{k-1}} =& x - x_n^{k-1} + \int_0^t \left[f(Z_s^{n,x}) - f(Z_s^{n,x_n^{k-1}}) \right] ds \\ &+ 2\int_0^t \int_{Y_s^{x_n^{k-1}}}^{Y_s^{x_n^{k-1}} + Z_s^{n,x} - Z_s^{n,x_n^{k-1}}} W(ds, d\xi). \end{aligned}$$

From (1.2) together with (3.1) and Theorem 2.2 in Dawson and Li (2012) it follows that for all $k \geq 1$ and $x_n^{k-1} < x \leq x_n^k$,

$$Z_t^{n,x} - Z_t^{n,x_n^{k-1}} \leq Y_t^x - Y_t^{x_n^{k-1}} \quad \text{a.s. for all } t \geq 0\,. \tag{3.4}$$

Moreover, the law of $\{Z_t^{n,x},\, x>0, t\ge 0\}$ is the same as that of $\{Z_t^x,\, x>0,\, t\ge 0\}$, the solution of (1.1).

Recall that for each $t>0$, $x\mapsto Y_t^x$ has finitely many jumps on any compact interval, and is constant between its jumps, and if $0<s<t$,

$$\{x,\, Y_t^x \ne Y_t^{x-}\}\subset \{x,\, Y_s^x\ne Y_s^{x-}\}. \tag{3.5}$$

Let us now fix $\delta, M>0$. For almost any realization of Y, the mapping $x\mapsto Y_\delta^x$ has only finitely many jumps on $(0,M]$. Let n be so large that there is at most one of those jumps in each interval $(k2^{-n},(k+1)2^{-n}]$, for $k\le M2^n-1$. Then for each x that belongs to an interval $(k2^{-n},(k+1)2^{-n}]$ which contains no jump of $x\mapsto Y_\delta^x$, and for any $n'>n$, we have $Z_t^{n',x}=Z_t^{n,x}$ for any $t\ge\delta$.

Since δ and M are arbitrary positive reals, we have shown that

$$\tilde Z_t^x := \text{ a.s. } \lim_{n\to\infty} Z_t^{n,x} \tag{3.6}$$

exists for all $t\ge 0$, $x>0$. The thus constructed random field $\{\tilde Z_t^x,\, t\ge 0, x>0\}$ has the same law as the solution of the SDE (1.1), and satisfies (3.3) and hence also

$$\{x,\, \tilde Z_t^x\ne \tilde Z_t^{x-}\}\subset\{x,\, Y_t^x\ne Y_t^{x-}\} \tag{3.7}$$

for all $t>0$. We still have to show that $\tilde Z$ satisfies (3.2). It is plain that for any $\delta>0$,

$$\tilde Z_t^x=\tilde Z_\delta^x+\int_\delta^t f(\tilde Z_s^x)ds+2\int_\delta^t\int_{A_s^x(\tilde Z)} W(ds,d\xi).$$

In order to deduce that $\tilde Z$ satisfies (3.2), it remains to show that $\tilde Z_\delta^x\to x$ a.s., as $\delta\to 0$, which follows readily from the equality of the laws of $\tilde Z$ and Z. ♢

Corollary 3.3. *For any $t>0$, $x\mapsto Z_t^x$ has finitely many jumps on any compact interval, and is constant between these jumps.*

Proof : The assertion follows from the fact that $\tilde Z$ possesses that property, as a consequence of (3.3) and the properties of Y. ♢

From the properties of the map $x\mapsto Z^x$, we infer that $x\mapsto \zeta^x:=\zeta(Z^x)$ (recall Definition 2.2) is increasing and right continuous, constant between its jumps, with a.s. finitely many jumps on any compact subinterval of $(0,+\infty)$, and a sufficient condition is given in Le and Pardoux (2015) for the limit ζ^∞ to be a.s. finite.

We have moreover

Corollary 3.4. *For any $s>0$,*

$$\mathbb{P}\left(\bigcup_{t>s}\{x,\ Z_t^x\ne Z_t^{x-}\}\subset\{x,\ Z_s^x\ne Z_s^{x-}\}\ \textit{for all}\ x>0\right)=1.$$

Proof : Let us first fix $t>s$ and $x>0$. We have

$$Z_t^x-Z_t^{x-}=Z_s^x-Z_s^{x-}+\int_s^t[f(Z_r^x)-f(Z_r^{x-})]dr+2\int_s^t\int_{Z_r^{x-}}^{Z_r^x}W(dr,d\xi).$$

Consequently, taking the conditional expectation given $Z_s^x-Z_s^{x-}$, and using both (1.2) and Gronwall's Lemma, we obtain

$$\mathbb{E}[Z_t^x-Z_t^{x-}|Z_s^x-Z_s^{x-}]\le[Z_s^x-Z_s^{x-}]\exp(\theta(t-s))\quad\text{a.s.}$$

This shows that $Z_t^x = Z_t^{x-}$ a.s. on the event $\{Z_s^x = Z_s^{x-}\}$.

For all $M > 0$ it follows from (3.7) and Theorem 3.2 that $\{0 < x \le M;\ Z_s^x \neq Z_s^{x-}\}$ is a.s. a random finite set. Let $0 < V_1 < V_2$ be $\sigma\{Z_r^y,\ y > 0, r \le s\}$-measurable and such that $Z_s^x - Z_s^{x-} = 0$ for all $V_1 \le x < V_2$. From the above argument, for any $x \in [V_1, V_2)$, $t \ge s$, $Z_t^x - Z_t^{x-} = 0$ a.s. Since, for $0 < y < x$, $Z_t^x - Z_t^y$ is continuous in t and right continuous both in x and in y,

$$\mathbb{P}(Z_t^x - Z_t^{x-} = 0,\ \text{for all } V_1 \le x < V_2,\ t \ge s) = 1.$$

The result follows from the fact that the set $\{0 < x \le M;\ Z_s^x - Z_s^{x-} = 0\}$ is a.s. a finite union of intervals of the form $[V_1, V_2)$, and $M > 0$ was arbitrary. ♢

Remark 3.5. We believe that the coupling constructed in Theorem 3.2 is interesting in its own right. In the rest of this paper we shall exploit its Corollary 3.3.

4. An SDE for the path-valued Markov process

Again, let Z^x be the solution of (1.1), with f satisfying conditions (1.2), (1.3) and (1.4). From now on, the process Y^x will be the solution of

$$Y_t^x = x + 2\int_0^t \int_0^{Y_s^x} W(ds, d\xi). \tag{4.1}$$

We shall use the notation

$$F(a, b) = f(a + b) - f(a).$$

Let $x, y > 0$, and define

$$V_t^{x,y} = Z_t^{x+y} - Z_t^x, \quad U_t^{x,y} = Y_t^{x+y} - Y_t^x, \quad t \ge 0. \tag{4.2}$$

We can couple these stochastic processes, by representing them as solutions of

$$V_t^{x,y} = y + \int_0^t F(Z_s^x, V_s^{x,y})ds + 2\int_0^t \int_{Y_s^x}^{Y_s^x + V_s^{x,y}} W(ds, d\xi), \tag{4.3}$$

$$U_t^{x,y} = y + 2\int_0^t \int_{Y_s^x}^{Y_s^x + U_s^{x,y}} W(ds, d\xi), \tag{4.4}$$

with a W different from (but having the same distribution as) the one appearing in (1.1) and (4.1), and leading to a pair (U, V) that has the same marginal distributions as the ones specified by (4.2).

We now define a Girsanov–Radon–Nikodym derivative, which will play an essential role in the sequel. For $z \in E$, $t > 0$ and U as in (4.4) (or, as we will need it later, also for some other $\mathbb{R}_+$–valued continuous semimartingale U with quadratic variation $d\langle U\rangle_s = 4U_s ds$), we define

$$L_t(z, U) = \exp\left(\frac{1}{4}\int_0^t \frac{F(z(s), U_s)}{U_s} dU_s - \frac{1}{8}\int_0^t \frac{F(z(s), U_s)^2}{U_s} ds\right), \tag{4.5}$$

where we use the convention $\frac{F(z,0)}{0} = 0$. It follows from (1.3) that for any $M < \infty$, $a \in [0, M]$ and $b \in [0, 1]$,

$$\frac{|F(a, b)|}{\sqrt{b}} \le C_M; \tag{4.6}$$

hence $L_t(z,U)$ is a well–defined random variable. We shall also consider $L_t(Z,U)$, where z is replaced by the process Z, solution of (1.1) with some initial condition $Z_0 = x$. Note that whenever we consider $L_t(Z,U)$, the processes Z and U will always be mutually independent.

Finally $L(Z,U)$ (resp. $L(z,U)$) will be defined by

$$L(Z,U) = L_\infty(Z,U) = L_\zeta(Z,U) \quad (\text{resp. } L(z,U) = L_\infty(z,U) = L_\zeta(z,U)), \quad (4.7)$$

where $\zeta = \zeta(U) = \inf\{t > 0,\ U_t = 0\}$. We shall consider the r.v. $L(Z,U)$ (or $L(z,U)$) only when $\zeta < \infty$ a.s., which e.g. is the case if U solves (4.4); hence the above quantities are well defined.

We have

Proposition 4.1. *For V and U as in (4.3), (4.4), the law of $\{V_t^{x,y},\ 0 \le t \le \zeta\}$ is absolutely continuous with respect to the law of $\{U_t^{x,y},\ 0 \le t \le \zeta\}$, and the Radon–Nikodym derivative is $L(Z^x, U^{x,y})$.*

Proof : For simplicity, we suppress the superindices x and y. We consider the filtration $\{\mathcal{F}_t, t \ge 0\}$ defined by $\mathcal{F}_t := \sigma((Z_s, U_s) : 0 \le s \le t \wedge \zeta)$ and introduce the local martingale

$$L_t = \exp\left(\frac{1}{4}\int_0^{t\wedge\zeta} \frac{F(Z_s,U_s)}{U_s} dU_s - \frac{1}{8}\int_0^{t\wedge\zeta} \frac{F(Z_s,U_s)^2}{U_s} ds\right),$$

where again $\zeta = \inf\{t > 0,\ U_t = 0\}$ is the extinction time of U. Define, for each $n \ge 1$, $T_n = \inf\{t > 0,\ \int_0^t U_s^{-1} F^2(Z_s,U_s)ds > n\} \wedge \zeta$. It is plain that the sequence of events $A_n = \{T_n = \zeta\}$ is increasing. Thus from the fact that $\zeta < \infty$ $\mathbb{P}$ a.s. together with the assumption (1.3) it follows that $\mathbb{P}(\bigcup_n A_n) = 1$. Moreover for any fixed $n \ge 1$, $(L_{t\wedge T_n})_{t\ge 0}$ is a uniformly integrable martingale, and if we define $\mathbb{Q}_n$ on $\mathcal{F}_{T_n}$ by

$$\frac{d\mathbb{Q}_n|_{\mathcal{F}_{T_n}}}{d\mathbb{P}|_{\mathcal{F}_{T_n}}} = L_{T_n},$$

we have that the law of $(U_{t\wedge T_n})_{t\ge 0}$ under $\mathbb{Q}_n$ equals the law of the process $(V_{t\wedge T_n})_{t\ge 0}$. It follows (see e.g. Proposition 3.5 in Ba and Pardoux (2015)) that there exists a unique probability measure $\mathbb{Q}$ on $\mathcal{F}_\zeta = \sigma(\cup_n \mathcal{F}_{T_n})$ such that, for each $n \ge 1$, its restriction to $\mathcal{F}_{T_n}$ coincides with $\mathbb{Q}_n$. It remains to show that $\mathbb{Q} \ll \mathbb{P}$, and that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = L_\zeta.$$

For this purpose, let $A \in \mathcal{F}_\zeta$ and $n \ge 1$. Clearly $A \cap A_n \in \mathcal{F}_{T_n}$ and

$$\begin{aligned}\mathbb{Q}(A \cap A_n) &= \mathbb{E}_{\mathbb{P}}(\mathbf{1}_{A\cap A_n} L_{T_n}) \\ &= \mathbb{E}_{\mathbb{P}}(\mathbf{1}_{A\cap A_n} L_\zeta).\end{aligned}$$

We have not only $\mathbb{P}(\bigcup_n A_n) = 1$, but also $\mathbb{Q}(\bigcup_n A_n) = 1$ (indeed condition (1.4) implies that $Z^{x+y} = Z^x + V^{x,y}$ goes extinct in finite time a.s., hence also $V = V^{x,y}$ has this property). Thus, by letting $n \to \infty$ in the above equality, we deduce from the monotone convergence theorem that

$$\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}(\mathbf{1}_A L_\zeta).$$

Since $L_\zeta = L(Z^x, U^{x,y})$, the proposition is proved. ♢

Let us define for each $x > 0$ the sigma–field $\mathcal{G}^x = \sigma\{Z_t^\xi,\, 0 < \xi \le x,\, t \ge 0\}$. As a corollary of Proposition 4.1 and of the independence of Z^x and U^x we obtain for an arbitrary $t > 0$,

$$\mathbb{E}(V_t^{x,y}|\mathcal{G}^x) = \mathbb{E}(L(Z^x, U^{x,y})U_t^{x,y}|\mathcal{G}^x). \tag{4.8}$$

In order to achieve the goal of deriving an SDE for the path-valued process $Z^x, x \ge 0$, and in view of Proposition 4.1, we want to take the limit as $y \to 0$ in the expression

$$\frac{1}{y}\mathbb{E}(L(Z^x, U^{x,y})U_t^{x,y}|\mathcal{G}^x).$$

Note that the law of $U^{x,y}$ is that of the unique solution of the SDE

$$U_t = y + 2\int_0^t \sqrt{U_s}\, dB_s. \tag{4.9}$$

In particular that law (which is a probability measure on E) does not depend on x; we denote it by $\mathbf{P}_y$. For any $t > 0$, $A \in \sigma\{U_r,\ r \ge 0\}$, $y > 0$, let

$$\mathbf{Q}_{y,t}(A) = y^{-1}\mathbf{E}_y(U_t; A). \tag{4.10}$$

For $t > 0$ we now write $\mathcal{F}_t := \sigma\{U_r,\ 0 \le r \le t\}$. The next result is known, tracing back to Lamperti and Ney (1968) Theorem 1 (see also Roelly-Coppoletta and Rouault (1989), Theorem 2, Li (2000) Theorem 4.1 , and Lambert (2007) Theorem 4.1 for more general versions). We give a proof, which is short, for the convenience of the reader.

Proposition 4.2. *For any fixed $t > 0$ and $y > 0$, the process $U = \{U_r, 0 \le r \le t\}$ is, under $\mathbf{Q}_{y,t}$, a Feller process with immigration. More precisely, U solves under $\mathbf{Q}_{y,t}$ the SDE*

$$U_r = y + 4r + 2\int_0^r \sqrt{U_s}d\bar{B}_s,\ 0 \le r \le t, \tag{4.11}$$

where $\bar{B}$ is a $\mathbf{Q}_{y,t}$–standard Brownian motion.

Proof : Denoting by ζ the extinction time of U, we have

$$\frac{U_{t\wedge\zeta}}{y} = \exp(\log U_{t\wedge\zeta} - \log U_0),$$
$$\log U_{t\wedge\zeta} - \log U_0 = 2\int_0^{t\wedge\zeta} \frac{dB_s}{\sqrt{U_s}} - 2\int_0^{t\wedge\zeta}\frac{ds}{U_s},$$

hence

$$\frac{d\mathbf{Q}_{y,t}|_{\mathcal{F}_t}}{d\mathbf{P}_y|_{\mathcal{F}_t}} = \exp\left(2\int_0^{t\wedge\zeta} \frac{dB_s}{\sqrt{U_s}} - 2\int_0^{t\wedge\zeta}\frac{ds}{U_s}\right),$$

which by Girsanov's theorem implies the result, if we let $\bar{B}_r = B_r - 2\int_0^r (U_s)^{-1/2}ds$.

Note that we can apply Girsanov's theorem here, since $\mathbb{E}_{\mathbf{P}_y} U_t = y$ implies that

$$\mathbb{E}_{\mathbf{P}_y}\left[\exp\left(2\int_0^{t\wedge\zeta} \frac{dB_s}{\sqrt{U_s}} - 2\int_0^{t\wedge\zeta}\frac{ds}{U_s}\right)\right] = 1.$$

◇

As a corollary to Proposition 4.2 (and the Markov property) we get that for $t>0$, the process $\{U_r, r\geq 0\}$ under $\mathbf{Q}_{y,t}$ solves the SDE

$$U_r = y + 4\,(r\wedge t) + 2\int_0^r \sqrt{U_s}d\bar{B}_s, \quad r\geq 0. \tag{4.12}$$

It is immediate that the limit of $\mathbf{Q}_{y,t}$ exists as $y\to 0$. We will denote this limit by $\mathbf{Q}_{0,t}$, and note that it is the law of $\{U_r,\ r\geq 0\}$, the solution of

$$U_r = 4(r\wedge t) + 2\int_0^r \sqrt{U_s}d\bar{B}_s,\ r\geq 0.$$

For $y\geq 0$, we denote by $\mathbf{Q}_{y,\infty}$ the law of the process $U=\{U_r, r\geq 0\}$, that satisfies (4.12) with $t=\infty$. It is well known (see e.g. Lambert (2007)) that $\mathbf{Q}_{y,\infty}$ is the law of a critical Feller process conditioned to never die out, hence for $y>0$ we have

$$\mathbf{Q}_{y,\infty}(U_s>0 \quad \forall s\geq 0) = 1, \tag{4.13}$$

whereas for $y=0$ we observe that

$$\mathbf{Q}_{0,\infty}(U_s>0 \quad \forall s>0) = 1. \tag{4.14}$$

For $t<\infty$, $r>0$ and $z\in E$, from $L_r(z,U)$ and $L(z,U)$ defined as in (4.5) and (4.7) we obtain the $\mathbf{Q}_{y,t}$-a.e. defined measurable functions $u\mapsto L_r(z,u)$ and $u\mapsto L(z,u)$. Under $\mathbf{Q}_{y,\infty}$, we shall consider only $L_r(z,U)$, since $\zeta=+\infty$ $\mathbf{Q}_{y,\infty}$ a.s.

For any $t>0$ and any partition $\mathcal{P}=\{x_k,\ k=0,1,\ldots\}$ with $0=x_0<x_1<\cdots$ we have that

$$\begin{aligned} Z_t^{x_\ell} &= \sum_{k=1}^{\ell}(x_k-x_{k-1})\mathbb{E}\left(\frac{Z_t^{x_k}-Z_t^{x_{k-1}}}{x_k-x_{k-1}}\Big| Z^{x_{k-1}}\right) + M_t^{x_\ell,\mathcal{P}} \\ &= \sum_{k=1}^{\ell}(x_k-x_{k-1})\int_E L(Z^{x_{k-1}},u)\mathbf{Q}_{x_k-x_{k-1},t}(du) + M_t^{x_\ell,\mathcal{P}}, \end{aligned}$$

where we have used (4.8) and (4.10), and where we define

$$M_t^{x_\ell,\mathcal{P}} = \sum_{k=1}^{\ell}\left[Z_t^{x_k} - Z_t^{x_{k-1}} - \mathbb{E}(Z_t^{x_k}-Z_t^{x_{k-1}}|\mathcal{G}^{x_{k-1}})\right].$$

For every $n\in\mathbb{N}$, we consider the partition $\mathcal{P}_n := \{x_k^n = k2^{-n},\ k\geq 1\}$. It follows from the above arguments that if x is a dyadic number and n is large enough, then, with $M^{x,n} := M^{x,\mathcal{P}_n}$,

$$Z_t^x = \sum_{k=1}^{x2^n} 2^{-n}\int_E L(Z^{(k-1)2^{-n}},u)\mathbf{Q}_{2^{-n},t}(du) + M_t^{x,n}. \tag{4.15}$$

Our aim is to show convergence of the right hand side as $n\to\infty$, leading to

$$Z_t^x = \int_{[0,x]\times E} L(Z^\xi,u)\mathbf{Q}_{0,t}(du)d\xi + M_t^x, \tag{4.16}$$

where $\{M_t^x,\ x\geq 0\}$ is a $\mathcal{G}^x$–martingale. To this purpose we start by proving

Lemma 4.3. *For any $y\geq 0$, $t>0$ and $z\in E$,*

$$\int_E L(z,u)\mathbf{Q}_{y,t}(du) = \int_E L_t(z,u)\mathbf{Q}_{y,\infty}(du). \tag{4.17}$$

PROOF : Since $\mathbf{Q}_{y,\infty}$ and $\mathbf{Q}_{y,t}$ coincide when restricted to $\mathcal{F}_t$, and since $L_t(z,\cdot)$ is $\mathcal{F}_t$-measurable, the right hand side of (4.17) equals

$$\int_E L_t(z,u)\mathbf{Q}_{y,t}(du). \tag{4.18}$$

It thus remains to show that this is equal to the left hand side of (4.17). Because of (4.12), the process $(U_s)_{s\geq t}$ under $\mathbf{Q}_{y,t}$ is a driftless Feller diffusion. Thus the exponential martingale

$$(M_s)_{s\geq t} := (L_s(z,U)/L_t(z,U))_{s\geq t}$$

constitutes a process of Girsanov densities with respect to $\mathbf{Q}_{y,t}$, with the property that U under the transformed measure satisfies the SDE

$$dU_s = F(z_s,U_s)ds + 2\sqrt{U_s}d\tilde{B}_s, \quad s\geq t. \tag{4.19}$$

for a standard Brownian motion $\tilde{B}$. From this we conclude in precisely the same manner as in the proof of Proposition 4.1 that $\mathbb{E}_{\mathbf{Q}_{y,t}}[M_\zeta] = 1$, where ζ denotes the extinction time of U. This implies that $\mathbb{E}_{\mathbf{Q}_{y,t}}[M_\zeta|\mathcal{F}_t] = 1$ a.s., hence $\mathbb{E}_{\mathbf{Q}_{y,t}}[L(z,U)] = \mathbb{E}_{\mathbf{Q}_{y,t}}[L_\zeta(z,U)] = E_{\mathbf{Q}_{y,t}}[L_t(z,U)]$, that is, the l.h.s. of (4.17) indeed equals (4.18). ♢

For $y\geq 0$, $z\in E$ and $\tilde{B}$ a standard Brownian motion, consider the SDE

$$V_r = y + 4r + \int_0^r F(z_s,V_s)ds + 2\int_0^r \sqrt{V_s}d\tilde{B}_s,\ r\geq 0, \tag{4.20}$$

and denote the law of its solution by $\tilde{\mathbf{Q}}^z_{y,\infty}$.

Lemma 4.4. *For any $y\geq 0$, $t>0$ and $z\in E$,*

$$\int_E L_t(z,u)\mathbf{Q}_{y,\infty}(du) = \int_E \exp\left(\int_0^t u_s^{-1}F(z_s,u_s)\,ds\right)\tilde{\mathbf{Q}}^z_{y,\infty}(du).$$

PROOF : Let $y\geq 0$, $t>0$ and $z\in E$ be fixed throughout the proof. We recall that $\mathbf{Q}_{y,\infty}$ has been defined as the law of the solution U of the SDE

$$U_r = y + 4r + 2\int_0^r \sqrt{U_s}d\bar{B}_s,\ r\geq 0. \tag{4.21}$$

We note that

$$L_t(z,U) = G_t(z,U)\ e^{\int_0^t U_s^{-1}F(z_s,U_s)ds}, \tag{4.22}$$

with G_t being defined as

$$G_t = G_t(z,U) = \exp\left(\frac{1}{2}\int_0^t \frac{F(z_s,U_s)}{\sqrt{U_s}}d\bar{B}_s - \frac{1}{8}\int_0^t \frac{F^2(z_s,U_s)}{U_s}\,ds\right).$$

It is plain that $(G_r, r\geq 0)$ is a local martingale under $\mathbf{Q}_{y,\infty}$. In order to conclude that $\tilde{\mathbf{Q}}^z_{y,\infty}|\mathcal{F}_t$ has density G_t w.r.to $\mathbf{Q}_{y,\infty}|\mathcal{F}_t$, and thus to infer the assertion of the lemma, it will be sufficient to ensure the Girsanov condition

$$\mathbb{E}_{\mathbf{Q}_{y,\infty}}[G_t] = 1. \tag{4.23}$$

For this we proceed similarly as in the proof of Proposition 4.1 and define for $n=0,1,2,\ldots$

$$T_n := \inf\{r : \int_0^r U_s^{-1}F^2(z_s,U_s)\,ds \geq n\}\ , \quad T_\infty := \lim_{n\to\infty} T_n.$$

Since $(G_{t\wedge T_n}, n = 0, 1, 2, \ldots)$ is a martingale, we can define a measure $\tilde{\mathbf{Q}}_y$ on $\mathcal{F}_{t\wedge T_\infty}$ whose restriction to $\mathcal{F}_{t\wedge T_n}$ is given by

$$\frac{d\tilde{\mathbf{Q}}_y|\mathcal{F}_{t\wedge T_n}}{d\mathbf{Q}_{y,\infty}|\mathcal{F}_{t\wedge T_n}} = G_{t\wedge T_n}.$$

Then for all $n \geq 1$, since $\{T_n \geq t\} \in \mathcal{F}_{t\wedge T_n}$, we have

$$\tilde{\mathbf{Q}}_y(\{T_n \geq t\}) = \mathbb{E}_{\mathbf{Q}_{y,\infty}}(\mathbf{1}_{\{T_n\geq t\}} G_{t\wedge T_n}) = \mathbb{E}_{\mathbf{Q}_{y,\infty}}(\mathbf{1}_{\{T_n\geq t\}} G_t).$$

Letting $n \to \infty$ we obtain

$$\tilde{\mathbf{Q}}_y(\{T_\infty \geq t\}) = \mathbb{E}_{\mathbf{Q}_{y,\infty}}(\mathbf{1}_{\{T_\infty\geq t\}} G_t). \tag{4.24}$$

Because of

$$\{T_n \geq t\} = \{T_n \wedge t = t\} = \{\int_0^t U_s^{-1} F^2(z_s, U_s) ds \leq n\},$$

we have

$$\{T_\infty \geq t\} = \{\int_0^t U_s^{-1} F^2(z_s, U_s) ds < \infty\}. \tag{4.25}$$

Under the measure $\mathbf{Q}_{y,\infty}$ the process U solves (4.11), hence it does not explode in finite time. Thus, from (1.3),

$$\mathbf{Q}_{y,\infty}(T_\infty \geq t) = 1.$$

On the other hand, under the measure $\tilde{\mathbf{Q}}_y$, the process U is the solution of

$$U_r = y + 4r + \int_0^r F(z_s, U_s) ds + 2\int_0^r \sqrt{U_s} d\tilde{B}_s \tag{4.26}$$

up to the time $T_\infty \wedge t$. Thanks to (1.2) the solution of (4.26) does not explode in finite time, consequently we deduce again from (1.3) that

$$\tilde{\mathbf{Q}}_y(T_\infty \geq t) = 1.$$

Thus (4.24) simplifies to (4.23). ♢

We now prove the

Proposition 4.5. *For all $z \in E$, $y \geq 0$, the mapping $t \mapsto \varphi(t) := \int_E L_t(z, u)\mathbf{Q}_{y,\infty}(du)$ is continuous on $(0, \infty)$, satisfies $0 \leq \varphi(t) \leq \exp(\theta t)$, and it converges to 1 as $t \to 0$.*

Proof : Fix $z \in E$ and $y \geq 0$. Let U be an E-valued random variable with distribution $\mathbf{Q}_{y,\infty}$. Then, for all $z \in E$, the mapping $t \to L_t(z, U)$ is a.s. continuous. From (4.22) and (1.2), we infer that $0 \leq L_t(z, U) \leq G_t(z, U)e^{\theta t}$, hence for any $b > 0$, the uniform integrability of the family of random variables $\{L_t(z, U),\ 0 \leq t \leq b\}$ follows from that of $\{G_t(z, U),\ 0 \leq t \leq b\}$, which in turn follows from its martingale property established in Lemma 4.4. This implies the claimed continuity. The fact that the integral equals 1 at $t = 0$ follows from the fact that $L_0(z, u) = 1$. Finally, the fact that $\int_E L_t(z, u)\mathbf{Q}_{y,\infty}(du) \leq \exp(\theta t)$ follows readily from Lemma 4.4 and (1.2). ♢

Proposition 4.6. *For any $t, y \geq 0$, the mapping*

$$z \mapsto \int_E L_t(z,u)\mathbf{Q}_{y,\infty}(du)$$

is continuous from E into $[0, e^{\theta t}]$.

Proof : (a) Let us first check that the mappings $z \mapsto G_t(z,.)$ and $z \mapsto L_t(z,.)$ and both are continuous in $\mathbf{Q}_{y,\infty}$-probability. Clearly, for any $s > 0$ the mapping $z \mapsto (U_s^{-1}F(z(s),U_s), U_s^{-1/2}F(z(s),U_s))$ is $\mathbf{Q}_{y,\infty}$ a.s. continuous from E into $\mathbb{R}^2$.
(i) We start by considering the case $y > 0$, and recall (4.13) as well as (4.6). Hence, if $z_n(s) \to z(s)$ uniformly in $s \in [0,t]$, then

$$\int_0^t \left| U_s^{-1/2}F(z(s),U_s) - U_s^{-1/2}F(z_n(s),U_s)\right|^2 ds \to 0 \quad \mathbf{Q}_{y,\infty} \text{ a.s.} \tag{4.27}$$

and

$$\int_0^t U_s^{-1}F(z_n(s),U_s)ds \to \int_0^t U_s^{-1}F(z(s),U_s)ds \quad \mathbf{Q}_{y,\infty} \text{ a.s.} \tag{4.28}$$

(Note that (4.28) follows from (4.6) and dominated convergence, since due to (4.13) we have $\int_0^t U_s^{-1/2}ds < \infty \quad \mathbf{Q}_{y,\infty}$ a.s.) From (4.27) and (4.28) we conclude that $G_t(z_n,U) \to G_t(z,U)$ and $L_t(z_n,U) \to L_t(z,U)$, both in $\mathbf{Q}_{y,\infty}$-probability.
(ii) It remains to treat the case $y = 0$. Then $U_0 = 0$, and (4.14) holds. Let $T = \inf\{0 \leq s \leq 1,\ U_s \geq 1\}$. Consider again a sequence z_n in E such that $z_n(s) \to z(s)$ uniformly in $[0,t]$. We let $M = \sup_{n\geq 1}\sup_{0\leq s\leq t} z_n(s)$, and C_M be the associated constant appearing in (1.3). Then whenever $0 < s \leq T \wedge t$,

$$|U_s^{-1/2}F(z_n(s),U_s)\mathbf{1}_{\{U_s\leq 1\}}| \leq C_M,$$

hence by bounded convergence

$$\int_0^{t\wedge T} \left| U_s^{-1/2}F(z(s),U_s) - U_s^{-1/2}F(z_n(s),U_s)\right|^2 ds \to 0 \quad \mathbf{Q}_{0,\infty} \text{ a.s.} \tag{4.29}$$

as $n \to \infty$. The convergence (4.27) for $y = 0$ now follows from (4.29) and the above part 1 of the proof, with the strong Markov property applied to the stopping time T and $y = 1$. From this we conclude that $G_t(z_n,U) \to G_t(z,U)$ in $\mathbf{Q}_{0,\infty}$-probability. To obtain (4.28) from (4.6) and dominated convergence also in the case $y = 0$, we observe, using (4.11) and Itô's formula applied to $\sqrt{U_s}$, that

$$\int_0^t U_s^{-1/2}ds = \frac{2}{3}[\sqrt{U_t} - B_t] < \infty \quad \mathbf{Q}_{0,\infty} \text{ a.s.}$$

On the other hand from (4.6) and (4.14) we have for all $s > 0$

$$\frac{|F(z_n(s),U_s)|}{U_s}\mathbf{1}_{\{U_s\leq 1\}} \leq \frac{C_M}{\sqrt{U_s}} \quad \mathbf{Q}_{0,\infty} \text{ a.s.},$$

thus we conclude by Lebesgue's dominated convergence theorem that

$$\int_0^{t\wedge T} U_s^{-1}F(z_n(s),U_s)ds \to \int_0^{t\wedge T} U_s^{-1}F(z(s),U_s)ds \quad \mathbf{Q}_{0,\infty} \text{ a.s.}$$

This together with (4.27) for $y = 1$ implies (4.27) also for $y = 0$.

We have thus established (4.27) and (4.28) for $y \geq 0$, which yields the claimed continuity of the mapping $z \mapsto L_t(z,.)$ in $\mathbf{Q}_{y,\infty}$-probability.
(b) In order to conclude the proof we need uniform integrability of the family

$L_t(z_n,.)$, $z \in E$, with respect to $\mathbf{Q}_{y,\infty}$, where $z_n, z \in E$ and $z_n \to z$. This can be established in the very same manner as we did in the proof of Proposition 4.5, after observing that $0 \le L_t(z_n,.) \le G_t(z_n,.)e^{\theta t}$, and that $G_t(z_n,.)$ not only converges in $\mathbf{Q}_{y,\infty}$-probability towards $G_t(z,.)$ due to part (a), but also is uniformly integrable because of $\mathbb{E}_{\mathbf{Q}_{y,\infty}}[G_t(z_n,U)] = 1$ for all $n \ge 1$, and $\mathbb{E}_{\mathbf{Q}_{y,\infty}}[G_t(z,U)] = 1$. $\diamondsuit$

Combining this result with Lemma 2.3, we deduce

Corollary 4.7. *For any $t, y \ge 0$, the mapping*

$$x \mapsto \int_E L_t(Z^x, u)\mathbf{Q}_{y,\infty}(du)$$

is a.s. càdlàg from $\mathbb{R}_+$ into $\mathbb{R}_+$.

We are now in a position to establish

Proposition 4.8. *For any $t, x \ge 0$,*

$$\sum_{k=1}^{\lfloor x2^n \rfloor} 2^{-n} \int_E L_t(Z^{(k-1)2^{-n}}, u)\mathbf{Q}_{2^{-n},\infty}(du) \to \int_{[0,x]\times E} L_t(Z^{\xi}, u)\mathbf{Q}_{0,\infty}(du)d\xi$$

in probability, as $n \to \infty$.

Proof : We first show that

$$\sum_{k=1}^{\lfloor x2^n \rfloor} 2^{-n} \int_E L_t(Z^{(k-1)2^{-n}}, u) \left[\mathbf{Q}_{2^{-n},\infty}(du) - \mathbf{Q}_{0,\infty}(du)\right] \to 0 \tag{4.30}$$

as $n \to \infty$. To this purpose we define for each $n \ge 1$ and $z \in E$

$$H_n(z) = \int_E L_t(z, u) \left[\mathbf{Q}_{2^{-n},\infty}(du) - \mathbf{Q}_{0,\infty}(du)\right].$$

It follows from Lemma 4.4 that

$$\begin{aligned} H_n(z) = \mathbb{E}\Bigg[&\exp\left(\int_0^t (\mathcal{V}_s^{2^{-n}}(z))^{-1} F(z_s, \mathcal{V}_s^{2^{-n}}(z))ds\right) \\ &- \exp\left(\int_0^t (\mathcal{V}_s^{0}(z))^{-1} F(z_s, \mathcal{V}_s^{0}(z))ds\right)\Bigg], \end{aligned} \tag{4.31}$$

where $\mathcal{V}^{2^{-n}}(z)$ (resp. $\mathcal{V}^0(z)$) denotes the solution of the SDE (4.20) with $y = 2^{-n}$ (resp. with $y = 0$). For $\xi \in [0, x]$ we put

$$h_n(\xi) = \sum_{k=1}^{\lfloor x2^n \rfloor} H_n(Z^{(k-1)2^{-n}})\mathbf{1}_{[(k-1)2^{-n}, k2^{-n})}(\xi).$$

Whenever $\xi \in [(k-1)2^{-n}, k2^{-n})$, we briefly write

$$\xi_n := \lfloor \xi 2^n \rfloor 2^{-n} = (k-1)2^{-n}, \tag{4.32}$$

hence as $n \to \infty$, $\xi_n \to \xi$ and $Z^{\xi_n} \to Z^{\xi-}$ a.s. Also, h_n can be rewritten as

$$h_n(\xi) = H_n(Z^{\xi_n}).$$

From (1.2), the expression in the expectation on the right hand side of (4.31) is bounded in absolute value by $2\exp(\theta t)$. Hence we infer from Lemma 4.9 below and Lebesgue's dominated convergence theorem that for all $\xi > 0$

$$H_n(Z^{\xi_n}) \to 0 \text{ in probability as } n \to \infty.$$

Since the left hand side of (4.30) equals $\int_0^x h_n(\xi)d\xi$, and since $|H_n(z)| \le 2\exp(\theta t)$, the assertion (4.30) follows by dominated convergence. It thus remains to show that

$$\sum_{k=1}^{\lfloor x2^n \rfloor} 2^{-n} \int_E L_t(Z^{\xi_n}, u)\mathbf{Q}_{0,\infty}(du) \to \int_{[0,x]\times E} L_t(Z^{\xi}, u)\mathbf{Q}_{0,\infty}(du)d\xi$$

a.s, as $n \to \infty$. This follows readily from Corollary 4.7 together with the elementary fact that for any right–continuous mapping $\xi \mapsto \mathcal{A}(\xi)$ from $[0, +\infty)$ into $[0, 1]$, and any dyadic $x > 0$, as $n \to \infty$, one has

$$2^{-n} \sum_{k=1}^{\lfloor x2^n \rfloor} \mathcal{A}((k-1)2^{-n}) \to \int_{[0,x]} \mathcal{A}(\xi)d\xi. \qquad \diamondsuit$$

We finally establish the following result

Lemma 4.9. *Let $\xi > 0$ and ξ_n be as in (4.32), $\mathcal{V}^{n,n} := \mathcal{V}^{2^{-n}}(Z^{\xi_n})$, $\mathcal{V}^n := \mathcal{V}^0(Z^{\xi_n})$. Then*

$$\int_0^t (\mathcal{V}_s^{n,n})^{-1} F(Z_s^{\xi_n}, \mathcal{V}_s^{n,n})ds - \int_0^t (\mathcal{V}_s^n)^{-1} F(Z_s^{\xi_n}, \mathcal{V}_s^n)ds \to 0$$

in probability, as $n \to \infty$.

Proof : We first note that $\mathcal{V}_s^{n,n} > 0$ a.s. for all $s \ge 0$, while $\mathcal{V}_s^n > 0$ for all $s > 0$, but $\mathcal{V}_0^n = 0$. These facts follow from our assumption (1.3), which implies that when those solutions get close to zero, their drift is bigger than 2. Consequently, since for any s the mapping $v \mapsto v^{-1}F(Z_s^{\xi_n}, v)$ is locally bounded and continuous away from $v = 0$, we conclude that for any $0 < \delta \le 1$, both

$$\int_\delta^t (\mathcal{V}_s^{n,n})^{-1} F(Z_s^{\xi_n}, \mathcal{V}_s^{n,n})ds \to \int_\delta^t (\tilde{\mathcal{V}}_s)^{-1} F(Z_s^{\xi-}, \tilde{\mathcal{V}}_s)ds, \text{ and}$$
$$\int_\delta^t (\mathcal{V}_s^n)^{-1} F(Z_s^{\xi_n}, \mathcal{V}_s^n)ds \to \int_\delta^t (\tilde{\mathcal{V}}_s)^{-1} F(Z_s^{\xi-}, \tilde{\mathcal{V}}_s)ds$$

a.s., where $\tilde{\mathcal{V}}$ denotes the solution of the SDE (4.20) with $y = 0$ and $z = Z^{\xi-}$. Since for any $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{P}\left(\left|\int_0^t (\mathcal{V}_s^{n,n})^{-1} F(Z_s^{\xi_n}, \mathcal{V}_s^{n,n})ds - \int_0^t (\mathcal{V}_s^n)^{-1} F(Z_s^{\xi_n}, \mathcal{V}_s^n)ds\right| > 3\varepsilon\right) \\
&\quad \le \mathbb{P}\left(\left|\int_\delta^t (\mathcal{V}_s^{n,n})^{-1} F(Z_s^{\xi_n}, \mathcal{V}_s^{n,n})ds - \int_\delta^t (\mathcal{V}_s^n)^{-1} F(Z_s^{\xi_n}, \mathcal{V}_s^n)ds\right| > \varepsilon\right) \\
&\quad + \mathbb{P}\left(\left|\int_0^\delta (\mathcal{V}_s^{n,n})^{-1} F(Z_s^{\xi_n}, \mathcal{V}_s^{n,n})ds\right| > \varepsilon\right) \\
&\quad + \mathbb{P}\left(\left|\int_0^\delta (\mathcal{V}_s^n)^{-1} F(Z_s^{\xi_n}, \mathcal{V}_s^n)ds\right| > \varepsilon\right),
\end{aligned}$$

the Lemma will follow from the fact that for any $\varepsilon > 0$, $\eta > 0$, there exists $0 < \delta \le 1$ small enough such that

$$\begin{aligned} \mathbb{P}\left(\left|\int_0^\delta (\mathcal{V}_s^{n,n})^{-1} F(Z_s^{\xi_n}, \mathcal{V}_s^{n,n}) ds\right| > \varepsilon\right) \le \eta, \\ \mathbb{P}\left(\left|\int_0^\delta (\mathcal{V}_s^{n})^{-1} F(Z_s^{\xi_n}, \mathcal{V}_s^{n}) ds\right| > \varepsilon\right) \le \eta. \end{aligned} \tag{4.33}$$

It is plain that $\mathcal{V}_s^n \le \mathcal{V}_s^{n,n} \le \overline{V}_s$, where $\overline{V}$ solves the SDE

$$\overline{V}_t = 1 + (4+\theta)t + 2\int_0^t \sqrt{\overline{V}_s} dB_s,\ t \ge 0, \tag{4.34}$$

and for any $\eta > 0$, there exists $M > 0$ such that

$$\mathbb{P}\left(\sup_{0\le s\le \delta} \overline{V}_s > M\right) \le \frac{\eta}{2}.$$

In order to show (4.33) we consider the events $\Omega_M = \{\sup_{0\le s\le 1, n\ge 1} Z_s^{\xi_n} \le M\}$. From (1.3) we infer $\sup_{0\le u\le M;\ 0\le z\le M} u^{-1/2}|F(z,u)| \le c_M$ for some finite constant c_M depending on M. Thus, if we let $\tau_M = \inf\{s > 0,\ \overline{V}_s > M\}$, it follows from Itô's formula that, writing $\mathcal{V}_s$ for either $\mathcal{V}_s^{n,n}$ or $\mathcal{V}_s^n$,

$$\int_0^{\delta\wedge\tau_M} \frac{ds}{\sqrt{\mathcal{V}_s}} = \frac{2}{3}\left[\sqrt{\mathcal{V}_{\delta\wedge\tau_M}} - \sqrt{\mathcal{V}_0} - \frac{1}{2}\int_0^{\delta\wedge\tau_M} \frac{F(Z_s^{\xi_n}, \mathcal{V}_s)}{\sqrt{\mathcal{V}_s}} ds - \tilde{B}_{\delta\wedge\tau_M}\right],$$

hence we have on Ω_M

$$0 \le \int_0^{\delta\wedge\tau_M} \frac{ds}{\sqrt{\mathcal{V}_s}} \le \frac{2}{3}\left[\sqrt{\overline{V}_{\delta\wedge\tau_M}} + \delta\frac{c_M}{2} - \tilde{B}_{\delta\wedge\tau_M}\right]. \tag{4.35}$$

Now for both $\mathcal{V} = \mathcal{V}^{n,n}$ and $\mathcal{V} = \mathcal{V}^n$,

$$\begin{aligned} &\mathbb{P}\left(\left|\int_0^\delta \mathcal{V}_s^{-1} F(Z_s^{\xi_n}, \mathcal{V}_s) ds\right| > \varepsilon\right) \\ &\le \mathbb{P}(\Omega_M) + \mathbb{P}(\tau_M < \delta) + \mathbb{P}\left(\left|\int_0^{\delta\wedge\tau_M} \mathcal{V}_s^{-1} F(Z_s^{\xi_n}, \mathcal{V}_s) ds\right| > \varepsilon; \Omega_M\right) \\ &\le \mathbb{P}(\Omega_M) + \frac{\eta}{2} + \mathbb{P}\left(\int_0^{\delta\wedge\tau_M} \frac{ds}{\sqrt{\mathcal{V}_s}} > c_M^{-1}\varepsilon; \Omega_M\right). \end{aligned}$$

Finally, since $\mathbb{P}(\Omega_M) \to 0$ as $M \to \infty$, by choosing first M sufficiently large and then δ sufficiently small, (4.33) follows readily from the previous estimate and (4.35). $\diamondsuit$

Guided by (4.16) we now define

$$M_t^x := Z_t^x - \int_{[0,x]\times E} L_t(Z^\xi, u)\mathbf{Q}_{0,t}(du)d\xi, \quad t \ge 0,\ x \ge 0. \tag{4.36}$$

Remark 4.10. Due to Lemma 4.3 and Proposition 4.5, also the second summand on the r.h.s. of (4.36) is a.s. continuous in t for all x, hence (4.36) lifts to an identity for continuous-path valued processes (indexed by x).

Lemma 4.11. *For any fixed $t > 0$, $\{M_t^x,\ x > 0\}$, defined by (4.36), is a càdlàg $(\mathcal{G}^x)$-martingale.*

Proof : The first term on the right hand side of (4.36) is càdlàg, and the second is continuous. As for the martingale property, we first note that M_t^x is integrable (since Z_t^x is integrable, and the last term in the right hand side of (4.36) takes values in $[0, xe^{\theta t}]$ due to Proposition 4.5) and $\mathcal{G}^x$-measurable for all $x > 0$. We next show that if $0 < a < x$ are two dyadic real numbers, then

$$\mathbb{E}[M_t^x - M_t^a | \mathcal{G}^y] = 0. \tag{4.37}$$

Indeed, there exists $m \in \mathbb{N}$ such that am and xm are integers. Then for any $n \geq m$, $M_t^{x,n}$ and $M_t^{a,n}$ defined by (4.15) satisfy

$$\mathbb{E}[M_t^{x,n} - M_t^{a,n} | \mathcal{G}^y] = 0. \tag{4.38}$$

It follows from the above arguments that $M_t^{x,n} \to M_t^x$ and $M_t^{a,n} \to M_t^a$ a.s. as $n \to \infty$. Moreover $M_t^{x,n}$ is the difference of the integrable r.v. Z_t^x which does not depend upon n, and a nonnegative r.v. which depends upon n and is uniformly bounded by $xe^{\theta t}$. Consequently the convergence holds also in L^1. The same is true for the sequence $M_t^{a,n}$. Hence (4.37) follows from (4.38).

Suppose now that x and a are arbitrary positive real numbers, satisfying again $0 < a < x$. Let x_n (resp. a_n) be a decreasing sequence of dyadic reals, such that $x_n \to x$ (resp. $a_n \to y$), and with $a_n < x_n$ for all $n \geq 1$. It is plain that $M_t^{x_n} \to M_t^x$ and $M_t^{a_n} \to M_t^a$ a.s. and in L^1. Moreover for each $n \geq 1$,

$$\mathbb{E}[M_t^{x_n} - M_t^{a_n} | \mathcal{G}^x] = \mathbb{E}\left\{\mathbb{E}[M_t^{x_n} - M_t^{a_n} | \mathcal{G}^{x_n}] | \mathcal{G}^x\right\} = 0,$$

hence taking the limit as $n \to \infty$ in that identity, we deduce that (4.37) holds true for any $0 < a < x$. The result is established. ♢

We now show that for any $A \in \mathcal{E}$, the Borel field of E, any $t > 0$,

$$\int_A \mathbf{Q}_{0,t}(du) = \int_A u(t) Q(du), \tag{4.39}$$

where the σ–finite measure Q on $(E, \mathcal{E})$ is the excursion measure of Feller's critical diffusion (4.9), in the sense of Pitman and Yor, see formula (3a) in Pitman and Yor (1982), with the scale function s being chosen as $s(y) = y$. To see (4.39), first note that for all $\Phi \in C_b(E)$,

$$\lim_{y\to 0} \frac{1}{y} \mathbb{E}_y[\Phi(U)] = \int_E \Phi(u) Q(du).$$

Since $\frac{1}{y}\mathbb{E}_y[U_t] = 1$ for all $y > 0$, this implies by a uniform integrability argument that

$$\lim_{y\to 0} \frac{1}{y} \mathbb{E}_y[\Phi(U) U_t] = \int_E \Phi(u) u(t) Q(du).$$

By the definition of $\mathbf{Q}_{y,t}$ the l.h.s is

$$\lim_{y\to 0} \int_E \Phi(u) \mathbf{Q}_{y,t}(du),$$

which is $\int_E \Phi(u) \mathbf{Q}_{0,t}(du)$ in view of (4.11). This proves (4.39).

The measures $Q \circ u_t^{-1}$, $t > 0$, constitute an entrance law of of the Feller diffusion (4.9). This entrance law, which also figures in formula (3.2) of Pitman and Yor (1982), is given by

$$Q \circ u_t^{-1} = (2t)^{-1}\mathrm{Exp}((2t)^{-1}), t > 0. \tag{4.40}$$

Indeed, it is readily checked from formula (4.12) that the distribution of u_t under $\mathbf{Q}_{y,t}$ is Gamma$(2, 2t)$, which is the size-biasing of Exp$((2t)^{-1})$. On the other hand, it is immediate from (4.39) that $Q_{0,t} \circ u_t$ is the size-biasing of $Q \circ u_t^{-1}$. From this the claim (4.40) is immediate. Let us also note that our probabilities $\mathbf{Q}_{y,\infty}$ are the "upward diffusions" $P_y^{\uparrow}$ of Pitman and Yor (1982).

Combining (4.36), (4.39) and Remark 4.10, we immediately arrive at our main result

Theorem 4.12. *The path–valued process $\{Z^x_\cdot,\ x > 0\}$ admits the decomposition*

$$Z^x = \int_{[0,x]\times E} uL(Z^\xi, u)Q(du)d\xi + M^x, \tag{4.41}$$

where M^x is a $C([0, +\infty); \mathbb{R})$–valued càdlàg martingale (if $C([0, +\infty); \mathbb{R})$ is equipped with the topology of uniform convergence on compacts).

We know that $x \mapsto Z^x$ arises as a sum of excursions, as was stated above in Corollary 3.3. Call $N_Z(d\xi, du)$ the corresponding point process, which is such that for all $x > 0$,

$$Z^x = \int_{[0,x]\times E} uN_Z(d\xi, du).$$

The above statement shows that the predictable intensity measure of N_Z is

$$L(Z^\xi, u)Q(du)d\xi.$$

Intuitively (and somewhat informally stated) this means that, given $(Z^\xi)_{0\le\xi<x}$, the predicted increment of Z in the next bit dx of ancestral mass is a Poisson point process with intensity measure $L(Z^{x-}, u)Q(du)\,dx$. This is made precise by the following statement, which was conjectured in the case of a logistic drift in Pardoux and Wakolbinger (2011b):

Corollary 4.13. *For bounded $g : \mathbb{R}_+ \to \mathbb{R}_+$ and $z \in E$, put $\Phi_g(z) := e^{-\langle g,z\rangle}$. Then, for this class of functions,*

$$A\Phi_g(z) := \Phi_g(z) \int_E \left(e^{-\langle g,u\rangle} - 1\right) L(z, u)Q(du)$$

gives the generator of Z in the sense that for all $g : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\Phi_g(Z^x) - \Phi_g(Z^0) - \int_{[0,x]\times E} A\Phi_g(Z^\xi)d\xi,\ x \ge 0 \qquad \textit{is a martingale.} \tag{4.42}$$

Proof : The validity of (4.42) can be seen by writing

$$\begin{aligned}\Phi_g(Z^x) - \Phi_g(Z^0) &= \int_{[0,x]\times E} (\Phi_g(Z^{\xi-} + u) - \Phi_g(Z^{\xi-}))N(d\xi, du)\\ &= \int_{[0,x]\times E} e^{-\langle g,Z^{\xi-}\rangle}(e^{-\langle g,u\rangle} - 1)N(d\xi, du).\end{aligned}$$

The same line of arguments that led to (4.41) now shows that the r.h.s. equals

$$\int_{[0,x]\times E} e^{-\langle g, Z^{\xi-}\rangle}(e^{-\langle g,u\rangle}-1)L(Z^{\xi-},u)Q(du)d\xi + M_g^x,\ x\geq 0,$$

for some real-valued càdlàg martingale $\{M_g^x,\ x>0\}$, which yields (4.42). $\diamondsuit$

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