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Probability Theory and Related Fields

Forward-backward stochastic differential equations and quasilinear parabolic PDEs

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Abstract. This paper studies, under some natural monotonicity conditions, the theory (existence and uniqueness, a priori estimate, continuous dependence on a parameter) of forward–backward stochastic differential equations and their connection with quasilinear parabolic partial differential equations. We use a purely probabilistic approach, and allow the forward equation to be degenerate.

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1. Introduction

Let $(\Omega, \mathscr{F}, P)$ be a probability space, $T > 0$, and $\{B(t),\ 0 \le t \le T\}$ a d-dimensional Brownian motion, whose natural filtration, completed with the class of P-null sets of $\mathscr{F}$, is denoted by $\{\mathscr{F}_t\}$.

Consider the following forward–backward stochastic differential equation (in short FBSDE)

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$$\begin{cases} X(t) = x + \int_0^t f(s, X(s), Y(s), Z(s))\, ds \\ \qquad\quad + \int_0^t \sigma(s, X(s), Y(s), Z(s))\, dB(s), \\ Y(t) = h(X(T)) + \int_t^T g(s, X(s), Y(s), Z(s))\, ds \\ \qquad\quad - \int_t^T Z(s)\, dB(s), t \in [0, T] \ , \end{cases} \tag{1.1}$$

for which we are seeking an $\mathscr{F}_t$-adapted solution $\{(X(t), Y(t), Z(t)),\ 0 \le t \le T\}$ with values in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$, satisfying

$$E\int_0^T \|Z(t)\|^2\, dt < \infty \ .$$

Here, the functions $f\colon \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^n$, $g\colon \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^m$, $\sigma\colon \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^{m\times d}$ and $h\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ are continuous with respect to $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$, and satisfy the following properties:

(A1) There exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that for all $t, x, x_1, x_2, y, y_1, y_2, z$, and a.s.,

$$\langle f(t, x_1, y, z) - f(t, x_2, y, z), x_1 - x_2\rangle \le \lambda_1 |x_1 - x_2|^2,$$

$$\langle g(t, x, y_1, z) - g(t, x, y_2, z), y_1 - y_2\rangle \le \lambda_2 |y_1 - y_2|^2 \ .$$

(A2) The function f is uniformly Lipschitz continuous with respect to (y, z), at most linearly growing in x, and g is uniformly Lipschitz continuous with respect to (x, z), at most linearly growing in y. In other wor ds, there exist $k, k_i > 0, i = 1, 2, 3, 4$, such that for all $t, x, x_1, x_2, y, y_1, y_2, z, z_1, z_2$, and a.s.,

$$|f(t, x, y_1, z_1) - f(t, x, y_2, z_2)| \le k_1|y_1 - y_2| + k_2\|z_1 - z_2\|,$$

$$|f(t, x, y, z)| \le |f(t, 0, y, z)| + k(1 + |x|),$$

$$|g(t, x_1, y, z_1) - g(t, x_2, y, z_2)| \le k_3|x_1 - x_2| + k_4\|z_1 - z_2\|,$$

$$|g(t, x, y, z)| \le |g(t, x, 0, z)| + k(1 + |y|) \ .$$

(A3) The function σ is uniformly Lipschitz continuous with respect to (x, y, z). That is, there exist $k_i, i = 5, 6, 7$, such that for all $t, x_1, x_2, y_1, y_2, z_1, z_2$, and a.s.,

$$\|\sigma(t, x_1, y_1, z_1) - \sigma(t, x_2, y_2, z_2)\|^2 \le k_5^2|x_1 - x_2|^2 + k_6^2|y_1 - y_2|^2 + k_7^2\|z_1 - z_2\|^2 \ .$$

(A4) The function h is uniformly Lipschitz continuous in x. That is, there exists k_8 such that for all x_1, x_2, and a.s.,

$$|h(x_1) - h(x_2)| \le k_8|x_1 - x_2| \ .$$

(A5) The processes $f(\cdot, x, y, z)$, $g(\cdot, x, y, z)$ and $\sigma(\cdot, x, y, z)$ are $\mathscr{F}_t$-adapted, and the random variable $h(x)$ is $\mathscr{F}_T$-measurable, for all (x, y, z). Moreover, the following holds:

$$E\int_0^T |f(s,0,0,0)|^2\,ds + E\int_0^T |g(s,0,0,0)|^2\,ds + E\int_0^T \|\sigma(s,0,0,0)\|^2\,ds + E|h(0)|^2 < \infty \ .$$

In the above, we have used –and shall use in the following– the notations $|\cdot|$ and $\|\cdot\|$ to denote the square-root of the sum of squares of the components of a vector and a matrix, respectively, when the underlying content in both notations is a vector and a matrix, respectively.

Equations such as (1.1) are used in mathematical economics (see Antonelli [1], Duffie and Epstein [6], for example), and in mathematical finance (see El Karoui, Peng and Quenez [7]). When the coefficients involved are deterministic and the coefficient σ is independent of z, it is closely related with the following system of quasilinear parabolic partial differential equations (in short PDEs):

$$\begin{cases} \dfrac{\partial u_k}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{i,j=1}^n a_{ij}(t,x,u)\dfrac{\partial^2 u_k}{\partial x_i \partial x_j} + \langle f(t,x,u,\nabla u\sigma(t,x,u)), \nabla u_k\rangle \\ + g_k(t,x,u,\nabla u\sigma(t,x,u)) = 0, \quad k = 1,\ldots,m,\ t \in (0,T),\ x \in \mathbb{R}^n, \\ u_k(T,x) = h_k(x), \quad k = 1,\ldots,m,\ x \in \mathbb{R}^n \ . \end{cases} \tag{1.2}$$

with $a_{ij}(t,x,u) = (\sigma\sigma^*(t,x,u))_{ij}$, $1 \le i, j \le n$. Knowing properties of the system of PDEs (1.2), we can derive some results on the FBSDE (1.1); on the other hand, from the knowledge of the FBSDE (1.1), we can derive some results on the system of PDEs (1.2). Our main interest is to study the PDEs with the help of the FBSDE (1.1). So we are concerned with a probabilistic method for studying the FBSDE (1.1), rather than a PDE approach as exploited in Ma, Yong [12] and Ma, Protter and Yong [11].

When the forward equation does not depend on the backward component $(Y(\cdot), Z(\cdot))$, or the backward equation does not depend on the forward component $X(\cdot)$, the FBSDE (1.1) can be solved rather easily. That is, e.g. for the former case, we can first solve the forward equation, which determines the process $X(\cdot)$, and then solve the backward equation, with the process $X(\cdot)$ known.

The general theory of backward stochastic differential equations was first established by Pardoux and Peng [14], and is by now well known. The reader is referred to Pardoux and Peng [14], [15] and the expository paper Pardoux [13] for the theory and the application of the FBSDE (1.1), when the forward equation is completely decoupled from the backward equation. The difficulty of solving the general FBSDE (1.1) lies in the coupling between the forward and backward equations, which leads to a circular dependence in the solution of the forward and backward equations. In order to attack this difficulty, we construct mappings, which are based on the circle, consider the circular relation as a result of a fixed point of those mappings, and we use a monotonicity assumption as a technical condition to ensure the existence of a fixed point.

What we mean here by coupling is the fact that both the solution of the forward and backward equation appear in the coefficients (including the terminal condition) of the backward and forward equation.

The proof of existence and uniqueness of a forward (resp. backward) stochastic differential equation can be based on the fixed point theorem, by using a change of norm which consists in multiplying the solution by $\exp(\lambda t)$, $|\lambda|$ large enough, with $\lambda < 0$ (resp. $\lambda > 0$), see Feyel [8] in the forward case. This change of norm is equivalent to adding λI to the drift of the equation. Our proof of existence and uniqueness of the FBSDE (1.1) is based on the same idea. Our monotonicity assumption (A1), together with the lower bound (3.7) or (3.15) on the quantity $\lambda_1 + \lambda_2$, is necessary to resolve the contradiction between the necessity of choosing a negatively large $\lambda \in \mathbb{R}$ for the forward component, and a positively large $\lambda \in \mathbb{R}$ for the backward component. Note that the drift in the backward equation is in fact $-g$.

FBSDEs like (1.1) were first considered by Antonelli [1]. In this work, the coefficients f, σ, g are independent of the variable z and satisfy Lipschitz type conditions, and he could obtain only a local existence and uniqueness result for the FBSDE (1.1). To the authors' knowledge, there are two classes of global existence and uniqueness results for the FBSDE (1.1). One is given by Ma and Yong [12], Ma, Protter and Yong [11], via a PDE approach, under an assumption of nondegeneracy of the forward equation. The other is given by Hu and Peng [10], Peng and Wu [19], based on stochastic Hamiltonian systems, under a monotonicity condition which is different from ours.

While a first version of the present paper had already been circulated, the authors received a preprint from Yong [22], who generalizes the results of [10] and [19], by introducing a more flexible (but rather complicated) type of monotonicity condition.

We feel that our monotonicity condition is both simple and very natural. It appears to be similar, also not identical, to one version of Yong's condition, but our conditions are very simple to check, and our method of proof is quite different from those in the papers Hu and Peng [10], Peng and Wu [19], and

Yong [22]. Also, our way of connecting FBSDEs with quasilinear PDEs is new, as far as we know.

The paper is organized as follows. Section 2 is devoted to the proof of some essential estimates, which are gathered in four lemmas, and the construction of two mappings. Existence and uniqueness of a solution to the FBSDE (1.1) is proved in Section 3 under three sets of assumptions. Section 4 is devoted to the continuous dependence of the solution upon a parameter. Finally, the connection with quasilinear PDEs is established in Section 5.

2. Preliminary: fundamental estimates

In this section, we shall present four lemmas, which will be frequently used in later analysis. The proof of them involves only a combination of Itô's formula, Gronwall's inequality, classical martingale inequalities, as well as some elementary algebraic inequalities, and is left to the reader.

Let $\mathbb{H}$ be an Euclidean space. We denote by $M^2(0, T; \mathbb{H})$ the set of those $\mathbb{H}$-valued $\mathscr{F}_t$-progressively measurable processes $\{u(t), 0 \leq t \leq T\}$ which are such that

$$\|u(\cdot)\| := \left(E \int_0^T |u(s)|^2 \, ds \right)^{1/2} < \infty \ .$$

For $\forall \lambda \in \mathbb{R}$, let

$$|u(\cdot)\|_\lambda := \left(E \int_0^T \exp(-\lambda s)|u(s)|^2 \, ds \right)^{1/2} < \infty \ .$$

Obviously, the two norms $\| \cdot \|_\lambda$ and $\| \cdot \|$ are equivalent.

Before stating and proving our lemmas, let us first make a simple but important remark. Whenever (X, Y, Z) solves the FBSDE (1.1),

$$\begin{cases} X(t) = x + \int_0^t f(s, X(s), Y(s), Z(s)) \, ds \\ \qquad\qquad + \int_0^t \sigma(s, X(s), Y(s), Z(s)) \, dB(s), \\ Y(t) = Y(0) - \int_0^t g(s, X(s), Y(s), Z(s)) \, ds \\ \qquad\qquad + \int_0^t Z(s) \, dB(s), t \in [0, T] \ , \end{cases}$$

It then follows easily from the facts that $Y(0)$ is deterministic (since it is $\mathscr{F}_0$-measurable) hence square integrable and $E \int_0^T \|Z(t)\|^2 \, dt < \infty$, and the assumptions (A2), (A5), using standard estimates, including Schwarz's and Burkholder–Davis–Gundy's inequalities, that

$$E\left(\sup_{0\le t\le T}|X(t)|^2+\sup_{0\le t\le T}|Y(t)|^2\right)<\infty\ .$$

In particular, any solution (X, Y, Z) of the FBSDE (1.1) belongs to the space $M^2(0, T; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d})$.

Lemma 2.1. *Let the assumptions* (A1)–(A3) *and* (A5) *be satisfied. Let* $(X(\cdot),\ Y(\cdot),\ Z(\cdot)) \in M^2(0,T;\mathbb{R}^n)\ \times M^2(0,T;\mathbb{R}^m)\times M^2(0,T;\mathbb{R}^{m\times d})$ *satisfy the forward equation in* (1.1). *Then, for all* $\lambda\in\mathbb{R}$, $\varepsilon, C_1, C_2>0$,

$$\begin{aligned}
&\exp(-\lambda t)E|X(t)|^2+\bar\lambda_1\int_0^t \exp(-\lambda s)E|X(s)|^2\,ds\\
&\le (k_1C_1+k_6^2(1+\varepsilon))\int_0^t \exp(-\lambda s)E|Y(s)|^2\,ds\\
&\quad+(k_2C_2+k_7^2(1+\varepsilon))\int_0^t \exp(-\lambda s)E|Z(s)|^2\,ds\\
&\quad+|x|^2+\frac{1}{\varepsilon}\int_0^t \exp(-\lambda s)E|f(s,0,0,0)|^2\,ds\\
&\quad+\left(1+\frac{1}{\varepsilon}\right)\int_0^t \exp(-\lambda s)E\|\sigma(s,0,0,0)\|^2\,ds\ ,
\end{aligned}\tag{2.1}$$

where $\bar\lambda_1:=\lambda-2\lambda_1-k_1C_1^{-1}-k_2C_2^{-1}-k_5^2(1+\varepsilon)-\varepsilon$. *Furthermore, we have*

$$\begin{aligned}
&\exp(-\lambda t)E|X(t)|^2\\
&\le (k_1C_1+k_6^2(1+\varepsilon))\int_0^t \exp(-\bar\lambda_1(t-s))\exp(-\lambda s)E|Y(s)|^2\,ds\\
&\quad+(k_2C_2+k_7^2(1+\varepsilon))\int_0^t \exp(-\bar\lambda_1(t-s))\exp(-\lambda s)E|Z(s)|^2\,ds\\
&\quad+\exp(-\bar\lambda_1 t)|x|^2\\
&\quad+\frac{1}{\varepsilon}\int_0^t \exp(-\bar\lambda_1(t-s))\exp(-\lambda s)E|f(s,0,0,0)|^2\,ds\\
&\quad+\left(1+\frac{1}{\varepsilon}\right)\int_0^t \exp(-\bar\lambda_1(t-s))\exp(-\lambda s)E\|\sigma(s,0,0,0)\|^2\,ds\ .
\end{aligned}\tag{2.2}$$

Remark 2.1. We can deduce from (2.2) the following inequality

$$\|X(\cdot)\|_\lambda^2 \le \frac{1-\exp(-\bar{\lambda}_1 T)}{\bar{\lambda}_1}\Bigg[(k_1C_1 + k_6^2(1+\varepsilon))\|Y(\cdot)\|_\lambda^2 + (k_2C_2 + k_7^2(1+\varepsilon))\|Z(\cdot)\|_\lambda^2 + |x|^2 + \frac{1}{\varepsilon}\|f(\cdot,0,0,0)\|_\lambda^2 + \left(1+\frac{1}{\varepsilon}\right)\|\sigma(\cdot,0,0,0)\|_\lambda^2\Bigg] . \tag{2.3}$$

Moreover, if $\bar{\lambda}_1 \ge 0$, we deduce from (2.1) that

$$\exp(-\lambda T)E|X(T)|^2 \le \Bigg[(k_1C_1 + k_6^2(1+\varepsilon))\|Y(\cdot)\|_\lambda^2 + (k_2C_2 + k_7^2(1+\varepsilon))\|Z(\cdot)\|_\lambda^2 + |x|^2 + \frac{1}{\varepsilon}\|f(\cdot,0,0,0)\|_\lambda^2 + \left(1+\frac{1}{\varepsilon}\right)\|\sigma(\cdot,0,0,0)\|_\lambda^2\Bigg] . \tag{2.4}$$

Lemma 2.2. *Assume that* (A1), (A2) *and* (A4), (A5) *are satisfied. Let* $(X(\cdot), Y(\cdot), Z(\cdot)) \in M^2(0,T;\mathbb{R}^n) \times M^2(0,T;\mathbb{R}^m) \times M^2(0,T;\mathbb{R}^{m\times d})$ *satisfy the backward equation in* (1.1).
Then, for all $\lambda \in \mathbb{R}$, ε, C_3, $C_4 > 0$,

$$\exp(-\lambda t)E|Y(t)|^2 + \bar{\lambda}_2\int_t^T \exp(-\lambda s)E|Y(s)|^2\,ds + (1-k_4C_4) \times \int_t^T \exp(-\lambda s)E|Z(s)|^2\,ds \le k_8^2(1+\varepsilon)\exp(-\lambda T)E|X(T)|^2 + k_3C_3\int_t^T \exp(-\lambda s)E|X(s)|^2\,ds + \left(1+\frac{1}{\varepsilon}\right)\exp(-\lambda T)E|h(0)|^2 + \frac{1}{\varepsilon}\int_t^T \exp(-\lambda s)E|g(s,0,0,0)|^2\,ds \ , \tag{2.5}$$

where $\bar{\lambda}_2 := -\lambda - 2\lambda_2 - k_3C_3^{-1} - k_4C_4^{-1} - \varepsilon$. *Furthermore, we have*

$$\exp(-\lambda t)E|Y(t)|^2 + (1-k_4C_4)\int_t^T \exp(-\bar{\lambda}_2(s-t))\exp(-\lambda s)E|Z(s)|^2\,ds$$

$$
\begin{aligned}
&\leq k_8^2(1+\varepsilon)\exp(-\bar{\lambda}_2(T-t))\exp(-\lambda T)E|X(T)|^2 \\
&\quad + k_3C_3\int_t^T \exp(-\bar{\lambda}_2(s-t))\exp(-\lambda s)E|X(s)|^2\,ds \\
&\quad + \left(1+\frac{1}{\varepsilon}\right)\exp(-\bar{\lambda}_2(T-t))\exp(-\lambda T)E|h(0)|^2 \\
&\quad + \frac{1}{\varepsilon}\int_t^T \exp(-\bar{\lambda}_2(s-t))\exp(-\lambda s)E|g(s,0,0,0)|^2\,ds \quad . \qquad (2.6)
\end{aligned}
$$

Remark 2.2. We can deduce from (2.6) that whenever $0 < C_4 < k_4^{-1}$,

$$
\begin{aligned}
&\|Y(\cdot)\|_\lambda^2 \\
&\leq \frac{1-\exp(-\bar{\lambda}_2 T)}{\bar{\lambda}_2}\Bigg[k_8^2(1+\varepsilon)\exp(-\lambda T)E|X(T)|^2 + k_3C_3\|X(\cdot)\|_\lambda^2 \\
&\quad + \left(1+\frac{1}{\varepsilon}\right)\exp(-\lambda T)E|h(0)|^2 + \frac{1}{\varepsilon}\|g(\cdot,0,0,0)\|_\lambda^2\Bigg] \quad . \qquad (2.7)
\end{aligned}
$$

Furthermore, if $\bar{\lambda}_2 \geq 0$, we deduce from (2.5) that

$$
\begin{aligned}
&E\|Z(\cdot)\|_\lambda^2 \\
&\leq \frac{1}{1-k_4C_4}\Bigg[k_8^2(1+\varepsilon)\exp(-\lambda T)E|X(T)|^2 + k_3C_3\|X(\cdot)\|_\lambda^2 \\
&\quad + \left(1+\frac{1}{\varepsilon}\right)\exp(-\lambda T)E|h(0)|^2 + \frac{1}{\varepsilon}\|g(\cdot,0,0,0)\|_\lambda^2\Bigg] \quad . \qquad (2.8)
\end{aligned}
$$

Lemma 2.3. *Let the assumptions* (A1)–(A3) *and* (A5) *be satisfied. Let* $X_i(\cdot)$ *be the solution of the forward equation in* (1.1), *corresponding to* $(Y(\cdot), Z(\cdot)) = (Y_i(\cdot), Z_i(\cdot)) \in M^2(0,T;\mathbb{R}^m) \times M^2(0,T;\mathbb{R}^{m\times d})$, $i = 1, 2$. *Then, for all* $\lambda \in \mathbb{R}$, $C_1, C_2 > 0$,

$$
\begin{aligned}
&\exp(-\lambda t)E|X_1(t)-X_2(t)|^2 + \bar{\lambda}_1\int_0^t \exp(-\lambda s)E|X_1(s)-X_2(s)|^2\,ds \\
&\leq (k_1C_1+k_6^2)\int_0^t \exp(-\lambda s)E|Y_1(s)-Y_2(s)|^2\,ds \\
&\quad + (k_2C_2+k_7^2)\int_0^t \exp(-\lambda s)E|Z_1(s)-Z_2(s)|^2\,ds \quad , \qquad (2.9)
\end{aligned}
$$

where $\bar{\lambda}_1 := \lambda - 2\lambda_1 - k_1 C_1^{-1} - k_2 C_2^{-1} - k_5^2$. *Moreover*,

$$\begin{aligned}
&\exp(-\lambda t) E|X_1(t) - X_2(t)|^2 \\
&\quad \leq (k_1 C_1 + k_6^2) \int_0^t \exp(-\bar{\lambda}_1 (t-s)) \exp(-\lambda s) E|Y_1(s) - Y_2(s)|^2 \, ds \\
&\qquad + (k_2 C_2 + k_7^2) \int_0^t \exp(-\bar{\lambda}_1 (t-s)) \exp(-\lambda s) E|Z_1(s) - Z_2(s)|^2 \, ds \ .
\end{aligned} \tag{2.10}$$

Remark 2.3. We can deduce from (2.10) the following inequality

$$\begin{aligned}
&\|X_1(\cdot) - X_2(\cdot)\|_\lambda^2 \\
&\quad \leq \frac{1 - \exp(-\bar{\lambda}_1 T)}{\bar{\lambda}_1} [(k_1 C_1 + k_6^2) \|Y_1(\cdot) - Y_2(\cdot)\|_\lambda^2 \\
&\qquad + (k_2 C_2 + k_7^2) \|Z_1(\cdot) - Z_2(\cdot)\|_\lambda^2] \ .
\end{aligned} \tag{2.11}$$

Furthermore, if $\bar{\lambda}_1 \geq 0$, we deduce from (2.1) that

$$\begin{aligned}
&\exp(-\lambda T) E|X_1(T) - X_2(T)|^2 \\
&\quad \leq (k_1 C_1 + k_6^2) \|Y_1(\cdot) - Y_2(\cdot)\|_\lambda^2 \\
&\qquad + (k_2 C_2 + k_7^2) \|Z_1(\cdot) - Z_2(\cdot)\|_\lambda^2 \ .
\end{aligned} \tag{2.12}$$

Lemma 2.4. *Assume that* (A1), (A2) *and* (A4), (A5) *are satisfied. Let* $(Y_i(\cdot), Z_i(\cdot))$ *be the solution of the backward equation in* (1.1), *corresponding to* $X(\cdot) = X_i(\cdot) \in M^2(0, T; \mathbb{R}^n)$, $i = 1, 2$. *Then, for all* $\lambda \in \mathbb{R}$, C_3, $C_4 > 0$,

$$\begin{aligned}
&\exp(-\lambda t) E|Y_1(t) - Y_2(t)|^2 + \bar{\lambda}_2 \int_t^T \exp(-\lambda s) E|Y_1(s) - Y_2(s)|^2 \, ds \\
&\qquad + (1 - k_4 C_4) \int_t^T \exp(-\lambda s) E|Z_1(s) - Z_2(s)|^2 \, ds \\
&\quad \leq k_8^2 \exp(-\lambda T) E|X_1(T) - X_2(T)|^2 \\
&\qquad + k_3 C_3 \int_t^T \exp(-\lambda s) E|X_1(s) - X_2(s)|^2 \, ds \ ,
\end{aligned} \tag{2.13}$$

where $\bar{\lambda}_2 := -\lambda - 2\lambda_2 - k_3 C_3^{-1} - k_4 C_4^{-1}$. *Moreover,*

$$\begin{aligned}
&\exp(-\lambda t) E|Y_1(t) - Y_2(t)|^2 \\
&\quad + (1 - k_4 C_4) \int_t^T \exp(-\bar{\lambda}_2(s-t)) \exp(-\lambda s) E|Z_1(s) - Z_2(s)|^2 \, ds \\
&\le k_8^2 \exp(-\bar{\lambda}_2(T-t)) \exp(-\lambda T) E|X_1(T) - X_2(T)|^2 \\
&\quad + k_3 C_3 \int_t^T \exp(-\bar{\lambda}_2(s-t)) \exp(-\lambda s) E|X_1(s) - X_2(s)|^2 \, ds \ .
\end{aligned} \tag{2.14}$$

Remark 2.4. Suppose moreover that $0 < C_4 < k_4^{-1}$. Then, we can deduce from (2.14) the following inequality

$$\begin{aligned}
&\|Y_1(\cdot) - Y_2(\cdot)\|_\lambda^2 \\
&\quad \le \frac{1 - \exp(-\bar{\lambda}_2 T)}{\bar{\lambda}_2} [k_8^2 \exp(-\lambda T) E|X_1(T) - X_2(T)|^2 \\
&\qquad + k_3 C_3 \|X_1(\cdot) - X_2(\cdot)\|_\lambda^2] \ .
\end{aligned} \tag{2.15}$$

If moreover $\bar{\lambda}_2 \ge 0$, then we deduce from (2.13) that

$$\begin{aligned}
&\|Z_1(\cdot) - Z_2(\cdot)\|_\lambda^2 \\
&\quad \le \frac{1}{1 - k_4 C_4} [k_8^2 \exp(-\lambda T) E|X_1(T) - X_2(T)|^2 \\
&\qquad + k_3 C_3 \|X_1(\cdot) - X_2(\cdot)\|_\lambda^2] \ .
\end{aligned} \tag{2.16}$$

From now on, C_4 will always be assumed to satisfy $0 < C_4 < k_4^{-1}$.

Finally, before closing this section, let us introduce two maps Γ_1 and Γ_2. Note that the forward equation in the FBSDE (1.1) induces in a natural way a map $M_1 : M^2(0, T; \mathbb{R}^m) \times M^2(0, T; \mathbb{R}^{m \times d}) \to M^2(0, T; \mathbb{R}^n)$, which to each

$$(Y(\cdot), Z(\cdot)) \in M^2(0, T; \mathbb{R}^m) \times M^2(0, T; \mathbb{R}^{m \times d})$$

associates $M_1(Y(\cdot), Z(\cdot))$, the unique solution of the forward equation:

$$\begin{aligned}
X(t) = x &+ \int_0^t f(s, X(s), Y(s), Z(s)) \, ds \\
&+ \int_0^t \sigma(s, X(s), Y(s), Z(s)) \, dB(s), t \in [0, T] \ .
\end{aligned} \tag{2.17}$$

Similarly, the backward equation in the FBSDE (1.1) also induces in a natural way a map M_2 (see Darling and Pardoux [5]) from $M^2(0, T; \mathbb{R}^n)$ into $M^2(0, T; \mathbb{R}^m) \times M^2(0, T; \mathbb{R}^{m\times d})$, which to each $X(\cdot) \in M^2(0, T; \mathbb{R}^n)$ associates $M_2(X(\cdot))$, the unique adapted solution $(\bar{Y}(\cdot), \bar{Z}(\cdot))$ of the backward equation:

$$\bar{Y}(t) = h(X(T)) + \int_t^T g(s, X(s), \bar{Y}(s), \bar{Z}(s))\, ds - \int_t^T \bar{Z}(s)\, dB(s), t \in [0, T] \ . \tag{2.18}$$

Define the map Γ_1 as the composition $\Gamma_1 := M_2 \circ M_1$, and the map Γ_2 as the composition $\Gamma_2 := M_1 \circ M_2$. It can be proved that Γ_1 maps $M^2(0, T; \mathbb{R}^m) \times M^2(0, T; \mathbb{R}^{m\times d})$ into itself, and Γ_2 maps $M^2(0, T; \mathbb{R}^n)$ into itself.

For $(Y_i(\cdot), Z_i(\cdot)) \in M^2(0, T; \mathbb{R}^m) \times M^2(0, T; \mathbb{R}^{m\times d})$, let $X_i(\cdot) := M_1(Y_i(\cdot), Z_i(\cdot))$ and $(\bar{Y}_i(\cdot), \bar{Z}_i(\cdot)) := \Gamma_1((Y_i(\cdot), Z_i(\cdot)))$, $i = 1, 2$.

Set

$$\begin{aligned} &a := \|X_1 - X_2\|_\lambda^2, \quad A := \exp(-\lambda T) E|X_1(T) - X_2(T)|^2, \\ &b := \|Y_1 - Y_2\|_\lambda^2, \qquad c := \|Z_1 - Z_2\|_\lambda^2, \\ &\bar{b} := \|\bar{Y}_1 - \bar{Y}_2\|_\lambda^2, \qquad \bar{c} := \|\bar{Z}_1 - \bar{Z}_2\|_\lambda^2 \ . \end{aligned} \tag{2.19}$$

Then, from Lemmas 2.3 and 2.4, we have

$$\begin{aligned} &a \le \frac{1 - \exp(-\bar{\lambda}_1 T)}{\bar{\lambda}_1}[(k_1 C_1 + k_6^2) b + (k_2 C_2 + k_7^2) c], \\ &A \le [1 \vee \exp(-\bar{\lambda}_1 T)][(k_1 C_1 + k_6^2) b + (k_2 C_2 + k_7^2) c], \\ &\bar{b} \le \frac{1 - \exp(-\bar{\lambda}_2 T)}{\bar{\lambda}_2}[k_8^2 A + k_3 C_3 a], \\ &\bar{c} \le \frac{k_8^2 \exp(-\bar{\lambda}_2 T) A + k_3 C_3 [1 \vee \exp(-\bar{\lambda}_2 T)] a}{(1 - k_4 C_4)[1 \wedge \exp(-\bar{\lambda}_2 T)]} \ . \end{aligned} \tag{2.20}$$

Further, when $\bar{\lambda}_1 > 0$ and $\bar{\lambda}_2 > 0$, we have from (2.20) and (2.16) that

$$\begin{aligned} &A \le [(k_1 C_1 + k_6^2) b + (k_2 C_2 + k_7^2) c], \\ &\bar{c} \le \frac{k_8^2 A + k_3 C_3 a}{1 - k_4 C_4} \ . \end{aligned} \tag{2.21}$$

Remark 2.5. The use of the techniques of equivalent norms and the contraction mapping theorem for the existence and uniqueness proof of ordinary and stochastic differential equations seems to be first due to Feyel [8]. The application of these techniques to the backward equations appears in Tang [21], Tang and Li [20], El Karoui, Peng and Quenez [7], Barles, Buckdahn and Pardoux [2]. In the next section, the reader will see that the equivalent norm technique plays an elegant role of exploiting our monotonicity conditions to establish existence and uniqueness for the FBSDE.

3. Existence and uniqueness results

We are going to give three results of existence and uniqueness. Each result will require an upper bound on the quantity $\lambda_1 + \lambda_2$ in terms of the k_i's. The first result assumes in addition that the coupling between the forward and backward components of the FBSDE (1.1) is weak (relative to the lengh T of the time interval $[0, T]$). The second assumes that the final condition for the backward equation is a (possibly random) constant, and the third one that the diffusion coefficient of the forward equation does not depend on $Z(\cdot)$.

In this section, $\bar{\lambda}_1$ and $\bar{\lambda}_2$ denote the quantities defined in Lemma 2.3 and Lemma 2.4.

3.1. The case of the forward equation being weakly coupled with the backward equation

The difficulty of solving the FBSDE (1.1) lies in the coupling between the forward and backward equations. It is natural to think that when the coupling is sufficiently weak, the FBSDE (1.1) should be solvable. The following assertion gives a precise statement corresponding to the above guess.

Theorem 3.1. *Let the conditions* (A1)–(A5) *be satisfied. Then there exists an* $\varepsilon_0 > 0$, *which depends on* $k_3, k_4, k_5, k_8, \lambda_1, \lambda_2, T$, *such that when* $k_1, k_2, k_6, k_7 \in [0, \varepsilon_0)$, *there exists a unique adapted solution* (X, Y, Z) *to the* FBSDE (1.1). *Further, if* $\lambda_1 + \lambda_2 < -(k_5^2 + k_4^2)/2$, *there is an* $\varepsilon_1 > 0$, *which depends on* $k_3, k_4, k_5, k_8, \lambda_1, \lambda_2$ *and is independent of* T, *such that when* $k_1, k_2, k_6, k_7 \in [0, \varepsilon_1)$, *there exists a unique adapted solution* (X, Y, Z) *to the* FBSDE (1.1).

Proof of Theorem 3.1. Consider the map Γ_1. It is enough to show that the map Γ_1 is a contraction for some equivalence norm $\|\cdot\|_\lambda$. We have

$$
\begin{aligned}
&\bar{b}+\bar{c}\\
&\quad \le k_8^2\left(\frac{1-\exp(-\bar{\lambda}_2 T)}{\bar{\lambda}_2}+\frac{\exp(-\bar{\lambda}_2 T)}{(1-k_4C_4)(1\wedge\exp(-\bar{\lambda}_2 T))}\right)A\\
&\qquad +k_3C_3\left(\frac{1-\exp(-\bar{\lambda}_2 T)}{\bar{\lambda}_2}+\frac{1\vee\exp(-1\bar{\lambda}_2 T)}{(1-k_4C_4)(1\wedge\exp(-\bar{\lambda}_2 T))}\right)a\\
&\quad \le \left(\frac{1-\exp(-\bar{\lambda}_2 T)}{\bar{\lambda}_2}+\frac{1\vee\exp(-\bar{\lambda}_2 T)}{(1-k_4C_4)(1\wedge\exp(-\bar{\lambda}_2 T))}\right)\\
&\qquad \times(k_8^2A+k_3C_3a)\\
&\quad \le \left(\frac{1-\exp(-\bar{\lambda}_2 T)}{\bar{\lambda}_2}+\frac{1\vee\exp(-\bar{\lambda}_2 T)}{(1-k_4C_4)(1\wedge\exp(-\bar{\lambda}_2 T))}\right)\\
&\qquad \times[(k_1C_1+k_6^2)b+(k_2C_2+k_7^2)c]\\
&\qquad \times\left(k_8^2[1\vee\exp(-\bar{\lambda}_1 T)]+k_3C_3\frac{1-\exp(-\bar{\lambda}_1 T)}{\bar{\lambda}_1}\right)\ . \qquad (3.1)
\end{aligned}
$$

Now the first assertion is immediate.

Note that

$$
\begin{aligned}
\bar{\lambda}_1 &= \lambda-2\lambda_1-k_1C_1^{-1}-k_2C_2^{-1}-k_5^2,\\
\bar{\lambda}_2 &= -\lambda-2\lambda_2-k_3C_3^{-1}-k_4C_4^{-1}\ .
\end{aligned}
\qquad (3.2)
$$

If

$$
2\lambda_1+2\lambda_2<-k_4^2-k_5^2\ , \qquad (3.3)
$$

we can choose $\lambda\in\mathbb{R}$, $\tilde{C}_i=C_i/k_i$, $i=1,2,3,4$, such that

$$
\bar{\lambda}_1>0,\quad \bar{\lambda}_2>0,\quad 1-k_4^2\tilde{C}_4>0\ . \qquad (3.4)
$$

Then, we have

$$
\begin{aligned}
\bar{b} &\le \frac{k_8^2A+k_3^2\tilde{C}_3a}{\bar{\lambda}_2},\\
\bar{c} &\le \frac{k_8^2A+k_3^2\tilde{C}_3a}{1-k_4^2\tilde{C}_4},\\
\bar{b}+\bar{c} &\le \left[\frac{1}{\bar{\lambda}_2}+\frac{1}{1-k_4^2\tilde{C}_4}\right][(k_1^2\tilde{C}_1+k_6^2)b+(k_2^2\tilde{C}_2+k_7^2)c]\left[k_8^2+\frac{k_3^2\tilde{C}_3}{\bar{\lambda}_1}\right]\ ,
\end{aligned}
\qquad (3.5)
$$

and

$$\begin{aligned}\bar{\lambda}_1 &= \lambda - 2\lambda_1 - \tilde{C}_1^{-1} - \tilde{C}_2^{-1} - k_5^2,\\ \bar{\lambda}_2 &= -\lambda - 2\lambda_2 - \tilde{C}_3^{-1} - \tilde{C}_4^{-1}\ .\end{aligned} \tag{3.6}$$

This completes the second assertion. □

Remark 3.1. If $k_8 = 0$, we consider the map Γ_2 and the proof simplifies (see the proof of Theorem 3.2 below). Further, when $k_2 = k_4 = k_7 = 0$, this is essentially the very case considered by Antonelli [1].

3.2. The case where h is a constant

When the backward equation of the FBSDE (1.1) does not couple with the forward part in the terminal condition, that is $k_8 = 0$, the problem turns out to be simpler.

Theorem 3.2. *Let the conditions* (A1)–(A5) *be satisfied. Assume that* $k_8 = 0$ *and that there exist* $C_i > 0$, $i = 1, 2, 3, 4$, $C_4 < k_4^{-1}$, $\theta > 0$ *such that*

$$\begin{aligned}\lambda_1 + \lambda_2 < -\frac{1}{2}\Bigg\{ & k_3 C_3 \left[\frac{k_2 C_2 + k_7^2}{1 - k_4 C_4} + \frac{k_1 C_1 + k_6^2}{\theta}\right] \\ & + k_1 C_1^{-1} + k_2 C_2^{-1} + k_3 C_3^{-1} + k_4 C_4^{-1} + k_5^2 + \theta \Bigg\}\ .\end{aligned} \tag{3.7}$$

Then there exists a unique adapted solution to the FBSDE (1.1).

Proof of Theorem 3.2. First, choose

$$\lambda = -(2\lambda_2 + k_3 C_3^{-1} + k_4 C_4^{-1} + \theta)\ . \tag{3.8}$$

Then

$$\begin{aligned}&\bar{\lambda}_2 = \theta > 0,\\ &\bar{\lambda}_1 = -(2\lambda_1 + 2\lambda_2 + k_1 C_1^{-1} + k_2 C_2^{-1} + k_3 C_3^{-1} + k_4 C_4^{-1} + k_5^2 + \theta)\ .\end{aligned} \tag{3.9}$$

Consider the map Γ_2. For $X(\cdot) \in M^2(0, T; \mathbb{R}^n)$, $\Gamma_2(X(\cdot))$ is the solution $\bar{X}(\cdot)$ of

$$\begin{aligned}\bar{X}(t) = x &+ \int_0^t f(s, \bar{X}(s), \bar{Y}(s), \bar{Z}(s))\, ds \\ &+ \int_0^t \sigma(s, \bar{X}(s), \bar{Y}(s), \bar{Z}(s))\, dB(s),\ t \in [0, T]\end{aligned} \tag{3.10}$$

where $(\bar{Y}(\cdot), \bar{Z}(\cdot))$ is the solution of

$$\begin{aligned}\bar{Y}(t) = h(X(T)) + \int_t^T g(s, X(s), \bar{Y}(s), \bar{Z}(s))\,ds \\ - \int_t^T \bar{Z}(s)\,dB(s),\ t \in [0, T]\ .\end{aligned} \tag{3.11}$$

Note that Γ_2 maps $M^2(0, T; \mathbb{R}^n)$ into itself.

Set

$$\begin{aligned}&a := \|X_1 - X_2\|_\lambda^2,\ \bar{a} := \|\bar{X}_1 - \bar{X}_2\|_\lambda^2, \\ &\bar{b} := \|\bar{Y}_1 - \bar{Y}_2\|_\lambda^2, \\ &\bar{c} := \|\bar{Z}_1 - \bar{Z}_2\|_\lambda^2\ .\end{aligned} \tag{3.12}$$

Then, from Lemmas 2.3 and 2.4, we have

$$\begin{aligned}\bar{a} &\le \frac{1 - \exp(-\bar{\lambda}_1 T)}{\bar{\lambda}_1}[(k_1C_1 + k_6^2)\bar{b} + (k_2C_2 + k_7^2)\bar{c}], \\ \bar{b} &\le \frac{1 - \exp(-\bar{\lambda}_2 T)}{\bar{\lambda}_2}k_3C_3a, \\ \bar{c} &\le \frac{k_3C_3}{(1 - k_4C_4)}a\ .\end{aligned} \tag{3.13}$$

Hence, recalling the first equality of (3.9), we have

$$\bar{a} \le \frac{1 - \exp(-\bar{\lambda}_1 T)}{\bar{\lambda}_1}k_3C_3\left(\frac{k_1C_1 + k_6^2}{\theta} + \frac{k_2C_2 + k_7^2}{1 - k_4C_4}\right)a\ . \tag{3.14}$$

Since (3.7) holds, then $\bar{\lambda}_1 > 0$ and moreover the map Γ_2 is a contraction and thus possesses a unique fixed point. □

3.3. The case of the diffusion coefficient σ being independent of its third argument z

Theorem 3.3. *Let the conditions* (A1)–(A5) *be satisfied. Assume that* $k_7 = 0$ *and that there exist* $C_i > 0, i = 1, 3, 4, C_4 < k_4^{-1}, \theta > 0, \alpha > 0$ *such that*

$$\lambda_1 + \lambda_2 < -\frac{1}{2}\left\{(1+\alpha)\left[k_1C_1 + k_6^2 + \frac{k_2^2}{\alpha(1-k_4C_4)}\right]\left(k_8^2 + \frac{k_3C_3}{\theta}\right) + k_1C_1^{-1} + k_3C_3^{-1} + k_4C_4^{-1} + k_5^2 + \theta\right\} . \tag{3.15}$$

Then there exists a unique adapted solution to the FBSDE (1.1).

Proof of Theorem 3.3. Choose

$$\lambda = 2\lambda_1 + k_1C_1^{-1} + k_2C_2^{-1} + k_5^2 + \theta \ . \tag{3.16}$$

Then

$$\bar\lambda_1 = \theta > 0,$$

$$\bar\lambda_2 = -2\lambda_1 - 2\lambda_2 - k_1C_1^{-1} - k_2C_2^{-1} - k_3C_3^{-1} - k_4C_4^{-1} - k_5^2 - \theta \ . \tag{3.17}$$

Consider the map Γ_1. It is enough to prove that Γ_1 is a contraction.

Since $k_7 = 0$, we have from (2.20) and (2.21) that

$$\begin{aligned} a &\le \frac{1-\exp(-\bar\lambda_1 T)}{\bar\lambda_1}[(k_1C_1 + k_6^2)b + k_2C_2c], \\ A &\le (k_1C_1 + k_6^2)b + k_2C_2c, \\ \bar b &\le \frac{1-\exp(-\bar\lambda_2 T)}{\bar\lambda_2}(k_8^2A + k_3C_3a), \\ \bar c &\le \frac{1}{1-k_4C_4}(k_8^2A + k_3C_3a) \ , \end{aligned} \tag{3.18}$$

where $\bar\lambda_2 > 0$. Therefore,

$$\begin{aligned} \bar b &\le \frac{1}{\bar\lambda_2}\left(k_8^2 + k_3C_3\frac{1-\exp(-\bar\lambda_1 T)}{\bar\lambda_1}\right)\left[(k_1C_1 + k_6^2)b + k_2C_2c\right], \\ \bar c &\le \frac{1}{1-k_4C_4}\left(k_8^2 + k_3C_3\frac{1-\exp(-\bar\lambda_1 T)}{\bar\lambda_1}\right)\left[(k_1C_1 + k_6^2)b + k_2C_2c\right] \ . \end{aligned} \tag{3.19}$$

Set

$$\gamma := \alpha \frac{1 - k_4 C_4}{\bar{\lambda}_2} \quad . \tag{3.20}$$

We have

$$\bar{b} + \gamma \bar{c} \le \frac{1+\alpha}{\bar{\lambda}_2} \left(k_8^2 + k_3 C_3 \frac{1 - \exp(-\bar{\lambda}_1 T)}{\bar{\lambda}_1} \right) \times (k_1 C_1 + k_6^2) \left[b + \frac{k_2 C_2}{\alpha(1 - k_4 C_4)(k_1 C_1 + k_6^2)} \bar{\lambda}_2 \gamma c \right] \quad . \tag{3.21}$$

Choose

$$C_2^{-1} = \frac{k_2 \bar{\lambda}_2}{\alpha(1 - k_4 C_4)(k_1 C_1 + k_6^2)} \quad . \tag{3.22}$$

Combining first (3.17) and (3.22), then (3.21) and (3.22), we deduce

$$\bar{\lambda}_2 \left(1 + \frac{k_2^2}{\alpha(1 - k_4 C_4)(k_1 C_1 + k_6^2)} \right) = -2\lambda_1 - 2\lambda_2 - k_1 C_1^{-1}$$

$$-k_3 C_3^{-1} - k_4 C_4^{-1} - k_5^2 - \theta,$$

$$\bar{b} + \gamma \bar{c} \le (b + \gamma c) \frac{1+\alpha}{\bar{\lambda}_2} \left(k_8^2 + k_3 C_3 \frac{1 - \exp(-\theta T)}{\theta} \right) (k_1 C_1 + k_6^2) \quad . \tag{3.23}$$

Since (3.15) holds, the mapping Γ_1 is a contraction under some appropriate norm, and thus it has a unique fixed point. □

4. Continuous dependence of the solution on a parameter

4.1. A priori estimates

Theorem 4.1. *Assume that* (A1)–(A5) *be satisfied. Further, assume that one of the following two sets of conditions hold*:

1) $k_7 = 0$ *and the inequality* (3.15) *holds*;

2) $k_8 = 0$ *and the inequality* (3.7) *holds.*

Then, if $(X(\cdot), Y(\cdot), Z(\cdot))$ *solves the* FBSDE (1.1), *then there is a constant* C, *which depends on* k_i, $i = 1, ..., 8$, *and* λ_1, λ_2, T, *such that*

$$E \sup_{0\le t\le T} |X(t)|^2 + E \sup_{0\le t\le T} |Y(t)|^2 + \|Z(\cdot)\|^2$$
$$\le C\left(|x|^2 + E|h(0)|^2 + E\int_0^T |f(t,0,0,0)|^2\,dt\right.$$
$$\left. + E\int_0^T |g(t,0,0,0)|^2\,dt + E\int_0^T \|\sigma(t,0,0,0)\|^2\,dt\right) \quad . \tag{4.1}$$

Proof of Theorem 4.1. We shall prove that for some $\lambda \in \mathbb{R}$,

$$\|X(\cdot)\|_\lambda^2 + \|Y(\cdot)\|_\lambda^2 + \|Z(\cdot)\|_\lambda^2$$
$$\le C\left(\exp(-\lambda T)(|x|^2 + E|h(0)|^2) + \|f(\cdot,0,0,0)\|_\lambda^2\right.$$
$$\left. + \|g(\cdot,0,0,0)\|_\lambda^2 + \|\sigma(\cdot,0,0,0)\|_\lambda^2\right) \quad . \tag{4.2}$$

The result will then follow from Burkholder's inequality.

We prove (4.2) under the first set of conditions only; the proof of (4.2) under the other set of conditions is similar.

Let $\bar{\lambda}_1$ and $\bar{\lambda}_2$ be defined as in Lemma 2.1 and Lemma 2.2, respectively. Choose

$$\lambda = 2\lambda_1 + k_1C_1^{-1} + k_2C_2^{-1} + k_5^2(1+\varepsilon) + \varepsilon + \theta \quad . \tag{4.3}$$

Then

$$\bar{\lambda}_1 = \theta > 0,$$
$$\bar{\lambda}_2 = -2\lambda_1 - 2\lambda_2 - k_1C_1^{-1} - k_2C_2^{-1} - k_3C_3^{-1} \tag{4.4}$$
$$-k_4C_4^{-1} - k_5^2(1+\varepsilon) - 2\varepsilon - \theta \quad .$$

Write

$$a := \|X(\cdot)\|_\lambda^2, \quad A := \exp(-\lambda T)E|X(T)|^2, \quad b := \|Y(\cdot)\|_\lambda^2,$$
$$c := \|Z(\cdot)\|_\lambda^2 \tag{4.5}$$

and

$$a_0 := |x|^2 + \frac{1}{\varepsilon}\|f(\cdot,0,0,0)\|_\lambda^2 + \left(1+\frac{1}{\varepsilon}\right)\|\sigma(\cdot,0,0,0)\|_\lambda^2,$$
$$b_0 := \left(1+\frac{1}{\varepsilon}\right)\exp(-\lambda T)E|h(0)|^2 + \frac{1}{\varepsilon}\|g(\cdot,0,0,0)\|^2 \quad . \tag{4.6}$$

From Remarks 2.1 and 2.2, we have

$$
\begin{aligned}
a &\le \frac{1-\exp(-\bar{\lambda}_1 T)}{\bar{\lambda}_1}[(k_1C_1 + k_6^2(1+\varepsilon))b + k_2C_2c + a_0],\\
A &\le (k_1C_1 + k_6^2(1+\varepsilon))b + k_2C_2c + a_0,\\
b &\le \frac{1-\exp(-\bar{\lambda}_2 T)}{\bar{\lambda}_2}[k_8^2(1+\varepsilon)A + k_3C_3a + b_0],\\
c &\le \frac{1}{1-k_4C_4}[k_8^2(1+\varepsilon)A + k_3C_3a + b_0]\ ,
\end{aligned} \tag{4.7}
$$

provided $\bar{\lambda}_2 \ge 0$. Therefore,

$$
\begin{aligned}
b &\le \frac{1}{\bar{\lambda}_2}\left(k_8^2(1+\varepsilon) + k_3C_3\frac{1-\exp(-\bar{\lambda}_1 T)}{\bar{\lambda}_1}\right)\\
&\quad \times \left[(k_1C_1 + k_6^2(1+\varepsilon))b + k_2C_2c\right],\\
&\quad + \frac{1}{\bar{\lambda}_2}\left(k_8^2(1+\varepsilon)a_0 + \frac{k_3C_3}{\bar{\lambda}_1}a_0 + b_0\right),\\
c &\le \frac{1}{1-k_4C_4}\left(k_8^2(1+\varepsilon) + k_3C_3\frac{1-\exp(-\bar{\lambda}_1 T)}{\bar{\lambda}_1}\right)\\
&\quad \times \left[(k_1C_1 + k_6^2(1+\varepsilon))b + k_2C_2c\right]\\
&\quad + \frac{1}{1-k_4C_4}\left(k_8^2(1+\varepsilon)a_0 + \frac{k_3C_3}{\bar{\lambda}_1}a_0 + b_0\right)\ .
\end{aligned} \tag{4.8}
$$

Set

$$
\gamma := \alpha\frac{1-k_4C_4}{\bar{\lambda}_2}\ . \tag{4.9}
$$

We have (noting the first equality of (3.5))

$$
\begin{aligned}
b + \gamma c &\le \frac{1+\alpha}{\bar{\lambda}_2}\left(k_8^2(1+\varepsilon) + k_3C_3\frac{1-\exp(-\bar{\lambda}_1 T)}{\bar{\lambda}_1}\right)\\
&\quad \times (k_1C_1 + k_6^2(1+\varepsilon))\left[b + \frac{k_2C_2}{\alpha(1-k_4C_4)(k_1C_1 + k_6^2(1+\varepsilon))}\bar{\lambda}_2\gamma c\right]
\end{aligned}
$$

$$+\frac{1+\alpha}{\bar{\lambda}_2}\left(k_8^2(1+\varepsilon)a_0+\frac{k_3C_3}{\bar{\lambda}_1}a_0+b_0\right) \quad . \tag{4.10}$$

Choose

$$C_2^{-1}=\frac{k_2\bar{\lambda}_2}{\alpha(1-k_4C_4)[k_1C_1+k_6^2(1+\varepsilon)]} \quad . \tag{4.11}$$

We have

$$\begin{aligned}
&\bar{\lambda}_2\left(1+\frac{k_2^2}{\alpha(1-K_4C_4)[k_1C_1+k_6^2(1+\varepsilon)]}\right)\\
&\quad=-2\lambda_1-2\lambda_2-k_1C_1^{-1}-k_3C_3^{-1}\\
&\qquad-k_4C_4^{-1}-k_5^2(1+\varepsilon)-2\varepsilon-\theta,\\
b+\gamma c \le\ &(b+\gamma c)\frac{1+\alpha}{\bar{\lambda}_2}\left(k_8^2(1+\varepsilon)+k_3C_3\frac{1-\exp(-\theta T)}{\theta}\right)\\
&\times[k_1C_1+k_6^2(1+\varepsilon)]\\
&+\frac{1+\alpha}{\bar{\lambda}_2}\left(k_8^2(1+\varepsilon)a_0+\frac{k_3C_3}{\theta}a_0+b_0\right) \quad ,
\end{aligned} \tag{4.12}$$

Since (3.15) holds, we can choose a sufficiently small $\varepsilon > 0$ such that $\bar{\lambda}_2 > 0$ and

$$\mu := \frac{1+\alpha}{\bar{\lambda}_2}\left(k_8^2(1+\varepsilon)+k_3C_3\frac{1-\exp(-\theta T)}{\theta}\right)[k_1C_1+k_6^2(1+\varepsilon)] < 1 \quad . \tag{4.13}$$

Hence,

$$b+\gamma c \le \frac{1+\alpha}{\bar{\lambda}_2(1-\mu)}\left(k_8^2(1+\varepsilon)a_0+\frac{k_3C_3}{\theta}a_0+b_0\right) \quad , \tag{4.14}$$

which, together with the first inequality of (4.7), immediately gives (4.2). □

4.2. A continuous dependence theorem

Let $\{x(\alpha),\ f(\alpha,\cdot),\ g(\alpha,\cdot),\ \sigma(\alpha,\cdot),\ h(\alpha,\cdot),\ \alpha\in \mathbb{R}\}$ be a family of boundary conditions and coefficients of FBSDEs (1.1), which satisfy, uniformly in the parameter α, the assumptions (A1)–(A5) and one of the following two sets of conditions: 1) $k_7 = 0$ and the inequality (3.15) holds; 2) $k_8 = 0$ and the inequality (3.7) holds. According to Theorems 3.2 and 3.3, the solutions of the corresponding FBSDEs exist uniquely. We denote the solution by $(X^\alpha, Y^\alpha, Z^\alpha)$.

Theorem 4.2. *Assume that the family of boundary conditions and coefficients is continuous at $\alpha = 0$ in the following sense*:

$$\begin{aligned}
&\lim_{\alpha\to 0} |x(\alpha) - x| = 0,\\
&\lim_{\alpha\to 0} E|h(\alpha, X^0(T)) - h(0, X^0(T))|^2 = 0,\\
&\lim_{\alpha\to 0} \| f(\alpha, \cdot, X^0(\cdot), Y^0(\cdot), Z^0(\cdot))\\
&\qquad\qquad - f(0, \cdot, X^0(\cdot), Y^0(\cdot), Z^0(\cdot))\| = 0,\\
&\lim_{\alpha\to 0} \|\sigma(\alpha, \cdot, X^0(\cdot), Y^0(\cdot), Z^0(\cdot))\\
&\qquad\qquad -\sigma(0, \cdot, X^0(\cdot), Y^0(\cdot), Z^0(\cdot))\| = 0,\\
&\lim_{\alpha\to 0} \|g(\alpha, \cdot, X^0(\cdot), Y^0(\cdot), Z^0(\cdot))\\
&\qquad\qquad -g(0, \cdot, X^0(\cdot), Y^0(\cdot), Z^0(\cdot))\| = 0 \ .
\end{aligned} \tag{4.15}$$

Then,

$$\begin{aligned}
&\lim_{\alpha\to 0} E \sup_{0\le t\le T} |X^\alpha(t) - X^0(t)|^2 = 0,\\
&\lim_{\alpha\to 0} E \sup_{0\le t\le T} |Y^\alpha(t) - Y^0(t)|^2 = 0,\\
&\lim_{\alpha\to 0} \|Z^\alpha(\cdot) - Z^0(\cdot)\| = 0 \ .
\end{aligned} \tag{4.16}$$

Further, *if the coefficients are Lipschitz in the parameter α in the following sense*:

$$\begin{aligned}
&|x(\alpha_1) - x(\alpha_2)| \le C|\alpha_1 - \alpha_2|,\\
&E|h(\alpha_1, X^{\alpha_2}(T)) - h(\alpha_2, X^{\alpha_2}(T))|^2 \le C|\alpha_1 - \alpha_2|^2,\\
&\| f(\alpha_1, \cdot, X^{\alpha_2}(\cdot), Y^{\alpha_2}, Z^{\alpha_2}(\cdot))\\
&\quad - f(\alpha_2, \cdot, X^{\alpha_2}(\cdot), Y^{\alpha_2}(\cdot), Z^{\alpha_2}(\cdot))\| \le C|\alpha_1 - \alpha_2|,\\
&\|\sigma(\alpha_1, \cdot, X^{\alpha_2}(\cdot), Y^{\alpha_2}(\cdot), Z^{\alpha_2}(\cdot))\\
&\quad -\sigma(\alpha_2, \cdot, X^{\alpha_2}(\cdot), Y^{\alpha_2}(\cdot), Z^{\alpha_2}(\cdot))\| \le C|\alpha_1 - \alpha_2|,\\
&\|g(\alpha_1, \cdot, X^{\alpha_2}(\cdot), Y^{\alpha_2}(\cdot), Z^{\alpha_2}(\cdot))\\
&\quad -g(\alpha_2, \cdot, X^{\alpha_2}(\cdot), Y^{\alpha_2}(\cdot), Z^{\alpha_2}(\cdot))\| \le C|\alpha_1 - \alpha_2| \ .
\end{aligned} \tag{4.17}$$

Then,

$$
\begin{aligned}
&E \sup_{0\le t\le T} |X^{\alpha_1}(t) - X^{\alpha_2}(t)|^2 \le C|\alpha_1 - \alpha_2|^2,\\
&E \sup_{0\le t\le T} |Y^{\alpha_1}(t) - Y^{\alpha_2}(t)|^2 \le C|\alpha_1 - \alpha_2|^2,\\
&\|Z^{\alpha_1}(\cdot) - Z^{\alpha_2}(\cdot)\| \le C|\alpha_1 - \alpha_2| \ .
\end{aligned}
\tag{4.18}
$$

Proof of Theorem 4.2. Write

$$
\delta X^\alpha = X^\alpha - X^0,\ \delta Y^\alpha = Y^\alpha - Y^0,\ \delta Z^\alpha = Z^\alpha - Z^0 \ , \tag{4.19}
$$

and

$$
\begin{aligned}
F^\alpha(r,x,y,z) &= f(\alpha, r, x + X^0(r), y + Y^0(r), z + Z^0(r))\\
&\quad - f(0, r, X^0(r), Y^0(r), Z^0(r)),\\
G^\alpha(r,x,y,z) &= g(\alpha, r, x + X^0(r), y + Y^0(r), z + Z^0(r))\\
&\quad - g(0, r, X^0(r), Y^0(r), Z^0(r)),\\
\Sigma^\alpha(r,x,y,z) &= \sigma(\alpha, r, x + X^0(r), y + Y^0(r), z + Z^0(r))\\
&\quad - \sigma(0, r, X^0(r), Y^0(r), Z^0(r)),\\
H^\alpha(x) &= h(\alpha, x + X^0(T)) - h(0, X^0(T)), \quad \delta x^\alpha = x(\alpha) - x \ .
\end{aligned}
\tag{4.20}
$$

Then, $(\delta X^\alpha, \delta Y^\alpha, \delta Z^\alpha)$ solves the following FBSDE:

$$
\begin{cases}
X(t) = \delta x^\alpha + \int_0^t F^\alpha(r, X(r), Y(r), Z(r))\, dr\\
\qquad\qquad + \int_0^t \Sigma^\alpha(r, X(r), Y(r), Z(r))\, dB(r),\\
Y(t) = H^\alpha(X(T)) + \int_t^T G^\alpha(r, X(r), Y(r), Z(r))\, dr\\
\qquad\qquad - \int_t^T Z(r)\, dB(r), t \in [0,T] \ .
\end{cases}
\tag{4.21}
$$

The first assertion of the theorem is an immediate result of Theorem 4.1. The second assertion is proved similarly. □

From this theorem, we immediately deduce the

Corollary 4.1. *Let the assumptions of* Theorem 4.1 *hold. If* $(X^{t_i,x_i}, Y^{t_i,x_i}, Z^{t_i,x_i})$ *is the adapted solution of* (1.1) *corresponding to the initial point* (t_i, x_i), $i = 1, 2$, *then we have*

$$
\begin{aligned}
&E \sup_{t_1 \vee t_2 \le s \le T} |X^{t_1,x_1}(s) - X^{t_2,x_2}(s)|^2 \\
&\quad \le C\left(|x_1 - x_2|^2 + (1 + |x_1|^2 \vee |x_2|^2)|t_1 - t_2|\right), \\
&E \sup_{t_1 \vee t_2 \le s \le T} |Y^{t_1,x_1}(s) - Y^{t_2,x_2}(s)|^2 \\
&\quad \le C\left(|x_1 - x_2|^2 + (1 + |x_1|^2 \vee |x_2|^2)|t_1 - t_2|\right), \\
&\int_{t_1 \vee t_2}^{T} E|Z^{t_1,x_1}(s) - Z^{t_2,x_2}(s)|^2\, ds \\
&\quad \le C\left(|x_1 - x_2|^2 + (1 + |x_1|^2 \vee |x_2|^2)|t_1 - t_2|\right) .
\end{aligned}
\tag{4.22}
$$

5. Connection with quasilinear parabolic PDEs

It is classical that a system of first order semilinear PDEs can be solved via the method of characteristic curves (see Courant and Hilbert [3]). The well known Feynman-Kac formula gives a probabilistic interpretation for linear second order PDEs of elliptic or parabolic types, and has been generalized to the case of systems of semilinear second order PDEs by Peng [17], [18], Pardoux and Peng [15], Barles, Buckdahn and Pardoux [2], Darling and Pardoux [5] and Pardoux, Pradeilles, Rao [16], see also Pardoux [13], with the help of the theory of BSDEs. This section can be viewed as a continuation of such a theme, and will exploit the above theory of FBSDEs in order to provide a probabilistic formula for the solution of a quasilinear PDE of parabolic type. Our approach to this topic seems to be new.

We assume that the functions f, g, σ, h are deterministic, $k_7 = 0$ and the inequality (3.15) holds. We first consider the case $m = 1$.

For $\varphi \in C^{1,2}([0,T) \times \mathbb{R}^n)$, $s \in [0,T]$, $\tilde{x} \in \mathbb{R}^n$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, we define

$$
\begin{aligned}
(L\varphi)(s, \tilde{x}, y, z) := \frac{1}{2}\sum_{i,j=1}^{n} a_{i,j}(s, \tilde{x}, y)\frac{\partial^2 \varphi}{\partial x_i \partial x_j}(s, \tilde{x}) \\
+\langle f(s, \tilde{x}, y, z), \nabla\varphi(s, \tilde{x})\rangle
\end{aligned}
$$

with $a_{ij}(s, \tilde{x}, y) = (\sigma\sigma^*(s, \tilde{x}, y))_{ij}$, $1 \le i, j \le n$.

For each $(t, x) \in [0, T] \times \mathbb{R}^n$, let $\{(X^{t,x}(s), Y^{t,x}(s), Z^{t,x}(s)),\ t \le s \le T\}$ denote the unique solution, given by Theorem 3.3, of the FBSDE (below, s runs from t to T):

$$\begin{cases} X^{t,x}(s) = x + \int_t^s f(r, X^{t,x}(r), Y^{t,x}(r), Z^{t,x}(r))\,dr \\ \qquad\qquad + \int_t^s \sigma(r, X^{t,x}(r), Y^{t,x}(r))\,dB(r), \\ \\ Y^{t,x}(s) = h(X^{t,x}(T)) + \int_s^T g(r, X^{t,x}(r), Y^{t,x}(r), Z^{t,x}(r))\,dr \\ \qquad\qquad - \int_s^T Z^{t,x}(r)\,dB(r) \quad . \end{cases} \tag{5.1}$$

We shall prove in this section that the function $u(t, x) := Y^{t,x}(t)$, $(t, x) \in [0, T] \times \mathbb{R}^n$, is a viscosity solution of the following backward quasilinear second-order parabolic PDE:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) + (Lu)(t, x, u(t, x), \nabla u(t, x)\sigma(t, x, u(t, x))) \\ \\ \quad +g(t, x, u(t, x), \nabla u(t, x)\sigma(t, x, u(t, x))) = 0, \quad t \in [0, T), x \in \mathbb{R}^n, \\ \\ u(T, x) = h(x), \quad x \in \mathbb{R}^n \quad . \end{cases} \tag{5.2}$$

Let us recall the definition of a viscosity solution for the PDE (5.2) (see Crandall, Ishii, Lions [4], Fleming and Soner [9]).

Definition 5.1. *Let $u \in C([0, T] \times \mathbb{R}^n)$ satisfy $u(T, x) = h(x)$, $x \in \mathbb{R}^n$. u is called a viscosity subsolution (resp. supersolution) of the* PDE (5.2) *if, whenever $\varphi \in C^{1,2}([0, T) \times \mathbb{R}^n)$, and $(t, x) \in [0, T) \times \mathbb{R}^n$ is a local minimum (resp. maximum) of $\varphi - u$, we have*

$$\frac{\partial \varphi}{\partial t}(t, x) + (L\varphi)(t, x, u(t, x), \nabla\varphi(t, x)\sigma(t, x, u(t, x))) \\ + g(t, x, u(t, x), \nabla\varphi(t, x)\sigma(t, x, u(t, x))) \geq 0$$

(*resp.*

$$\frac{\partial \varphi}{\partial t}(t, x) + (L\varphi)(t, x, u(t, x), \nabla\varphi(t, x)\sigma(t, x, u(t, x))) \\ +g(t, x, u(t, x), \nabla\varphi(t, x)\sigma(t, x, u(t, x))) \leq 0.$$

u is called a viscosity solution of the PDE (5.2) *if it is both a viscosity sub- and super-solution.*

We now prove the

Theorem 5.1. *Assume that the functions f, σ, g, h are deterministic, globally continuous, and that they satisfy* (A1)–(A4) *with $k_7 = 0$, and* (3.15). *Then, the function u defined by $u(t, x) := Y^{t,x}(t)$, $(t, x) \in [0, T] \times \mathbb{R}^n$, is continuous and it is a viscosity solution of the* PDE (5.2).

Proof of Theorem 5.1. The continuity of u is a consequence of Corollary 4.1. We only show that u is a viscosity subsolution of the PDE (5.2). A similar argument would show that it is a viscosity supersolution.

We first note that from the uniqueness result for the FBSDE (5.1), we can infer that for any $t \le s \le T$,

$$Y^{t,x}(s) = Y^{s,X^{t,x}(s)}(s) = u(s, X^{t,x}(s)) \ .$$

Let $\varphi \in C^{1,2}([0,T) \times \mathbb{R}^n)$, and $(t,x) \in [0,T) \times \mathbb{R}^n$ be a local minimum of $\varphi - u$. We assume w.l.o.g. that $\varphi(t,x) = u(t,x)$.

We now assume that

$$\frac{\partial \varphi}{\partial t}(t,x) + (L\varphi)(t,x,u(t,x),\nabla\varphi(t,x)\sigma(t,x,u(t,x))) \\ + g(t,x,u(t,x),\nabla\varphi(t,x)\sigma(t,x,u(t,x))) < 0 \ ,$$

and we will obtain a contradiction.

It follows from the above that there exists $0 < \alpha < T - t$ such that for all $(s,y) \in [t,T] \times \mathbb{R}^n$ satisfying $t \le s \le t + \alpha$, $|x - y| \le \alpha$,

$$u(s,y) \le \varphi(s,y),$$

$$\frac{\partial \varphi}{\partial s}(s,y) + (L\varphi)(s,y,u(s,y),\nabla\varphi(s,y)\sigma(s,y,u(s,y))) \\ + g\,(s,y,u(s,y),\nabla\varphi(s,y)\sigma(s,y,u(s,y))) < 0 \ .$$

Let now τ denote the stopping time

$$\tau \stackrel{\triangle}{=} \inf\{s > t : \ |X^{t,x}(s) - x| \ge \alpha\} \wedge (t + \alpha) \ .$$

We first note that the pair of processes

$$(\bar{Y}(s), \bar{Z}(s)) := (Y^{t,x}(s \wedge \tau), \mathbf{1}_{[t,\tau]}(s) Z^{t,x}(s)), \quad t \le s \le t + \alpha$$

is the solution of the BSDE

$$\bar{Y}(s) = u(\tau, X^{t,x}(\tau)) + \int_{s\wedge\tau}^{\tau} g(r, X^{t,x}(r), u(r, X^{t,x}(r)), \bar{Z}(r))\, dr \\ - \int_s^{t+\alpha} \bar{Z}(r)\, dB(r) \ .$$

Next, it follows from Itô's formula that the pair of processes

$$(\hat{Y}(s), \hat{Z}(s)) := (\varphi(s \wedge \tau, X^{t,x}(s \wedge \tau)), \mathbf{1}_{[t,\tau]}(s) \nabla\varphi(s, X^{t,x}(s)) \\ \sigma(s, X^{t,x}(s), u(s, X^{t,x}(s)))), \ t \le s \le t + \alpha$$

is the solution of the BSDE

$$\hat{Y}(s) = \varphi(\tau, X^{t,x}(\tau)) - \int_{s\wedge\tau}^{\tau} \left[\frac{\partial\varphi}{\partial t}(r, X^{t,x}(r)) \right.$$
$$\left. + \ (L\varphi)(r, X^{t,x}(r), u(r, X^{t,x}(r)), \bar{Z}(r)) \right] dr$$
$$- \int_s^{t+\alpha} \hat{Z}(r)\, dB(r), \quad t \le s \le t+\alpha \ .$$

Define

$$\hat{\beta}(r) := - \left[\frac{\partial\varphi}{\partial t}(r, X^{t,x}(r)) + (L\varphi)(r, X^{t,x}(r), u(r, X^{t,x}(r)), \hat{Z}(r)) \right.$$
$$\left. + g(r, X^{t,x}(r), u(r, X^{t,x}(r)), \hat{Z}(r)) \right] ,$$

$$\bar{\beta}(r) = - \left[\frac{\partial\varphi}{\partial t}(r, X^{t,x}(r)) + (L\varphi)(r, X^{t,x}(r), u(r, X^{t,x}(r)), \bar{Z}(r)) \right.$$
$$\left. + \ g(r, X^{t,x}(r), u(r, X^{t,x}(r)), \bar{Z}(r)) \right] .$$

We note that for some $c > 0$,

$$|\bar{\beta}(r) - \hat{\beta}(r)| \le c\|\bar{Z}(r) - \hat{Z}(r)\| \ .$$

Hence there exists a bounded $\mathscr{F}_t$-adapted process $\{\gamma(r)\}$ with values in $\mathbb{R}^d$ such that

$$\bar{\beta}(r) - \hat{\beta}(r) = \langle \gamma(r), \bar{Z}(r) - \hat{Z}(r) \rangle \ .$$

Define $(\tilde{Y}(s), \tilde{Z}(s)) = (\hat{Y}(s) - \bar{Y}(s), \hat{Z}(s) - \bar{Z}(s))$. We have

$$\tilde{Y}(s) = \varphi(\tau, X^{t,x}(\tau)) - u(\tau, X^{t,x}(\tau))$$
$$+ \int_{s\wedge\tau}^{\tau} [\hat{\beta}(r) + \langle \gamma(r), \tilde{Z}(r) \rangle]\, dr - \int_{s\wedge\tau}^{\tau} \tilde{Z}(r)\, dB(r) \ .$$

Hence (see the proof of Theorem 1.6 in Pardoux [13])

$$\tilde{Y}(t) = E\left[\Gamma_{t,\tau} \tilde{Y}(\tau) + \int_t^{\tau} \Gamma_{t,s} \hat{\beta}(s)\, ds \right] ,$$

where

$$\Gamma_{t,s} = \exp\left(\int_t^s \langle\gamma(r),\, dB(r)\rangle - \frac{1}{2}\int_t^s |\gamma(r)|^2\, dr\right) \quad .$$

Now from the choice of α and τ, a.s.

$$u(\tau, X^{t,x}(\tau)) \le \varphi(\tau, X^{t,x}(\tau)), \quad 0 < \hat{\beta}(r) \text{ on the interval } [t,\tau], \quad \tau > t \quad .$$

Consequently $\tilde{Y}(t) > 0$, i.e. $u(t,x) < \varphi(t,x)$, which contradicts an earlier assumption. □

The same proof which we gave extends easily to systems of quasilinear second order PDEs of parabolic type. However, for the notion of viscosity solution to make sense, we need to make two restrictions on the dependence of the coefficients f and g upon the variable z:

(a) f does not depend on z.

(b) $\forall 1 \le k \le m$, the k-th coordinate g_k of g depends only on the k-th row of the matrix z.

Hence the system of quasilinear parabolic PDEs takes the form:

$$\begin{cases} \frac{\partial u_k}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t,x,u)\frac{\partial^2 u_k}{\partial x_i \partial x_j} + \langle f(t,x,u), \nabla u_k\rangle \\ + g_k(t,x,u,\nabla u_k \sigma(t,x,u)) = 0, \quad k = 1,\ldots,m,\, t \in (0,T),\, x \in \mathbb{R}^n, \\ u_k(T,x) = h_k(x), \quad k = 1,\ldots,m,\, x \in \mathbb{R}^n \quad . \end{cases}$$

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