The effect of competition on the height and length of the forest of genealogical trees of a large population

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Abstract

We consider a population generating a forest of genealogical trees in continuous time, with m roots (the number of ancestors). In order to model competition within the population, we superimpose to the traditional Galton-Watson dynamics (births at constant rate μ, deaths at constant rate λ) a death rate which is γ times the size of the population alive at time t raised to some power $\alpha > 0$ ($\alpha = 1$ is a case without competition). If we take the number of ancestors at time 0 to be equal to $[xN]$, weight each individual by the factor $1/N$, choose adequately μ, λ and γ as functions of N, then the population process converges as N goes to infinity to a Feller SDE with a negative polynomial drift. The genealogy in the continuous limit is described by a real tree (in the sense of Aldous [1]). In both the discrete and the continuous case, we study the height and the length of the genealogical tree, as an (increasing) function of the initial population. We show that the expectation of the height of the tree remains bounded as the size of the initial population tends to infinity iff $\alpha > 1$, while the expectation of the length of the tree remains bounded as the size of the initial population tends to infinity iff $\alpha > 2$.

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Introduction

Consider a Galton–Watson binary branching process in continuous time with m ancestors at time $t = 0$, in which each individual gives birth to children at a constant rate μ, and dies after an exponential time with parameter λ. Suppose we superimpose deaths due to competition. For instance, we might decide to add to each individual a death rate equal to γ times the number of presently alive individuals in the population, which amounts to add a global death rate equal to $\gamma(X_t^m)^2$, if X_t^m denotes the total number of alive individuals at time t. It is rather clear that the process which describes the evolution of the total population, which is not a branching process (due to the interactions between branches, created by the competition term), goes extinct in finite tim a. s.

If we consider this population with $m = [Nx]$ ancestors at time $t = 0$, weight each individual with the factor $1/N$, and choose $\mu_N = 2N + \theta$, $\lambda_N = 2N$ and $\gamma_N = \gamma/N$, then it is shown in Le, Pardoux and Wakolbinger [4] that the "total population mass process" converges weakly to the solution of the Feller SDE with logistic drift

$$dZ_t^x = \left[\theta Z_t^x - \gamma(Z_t^x)^2\right] dt + 2\sqrt{Z_t^x}dW_t,\ Z_0^x = x.$$

This equation has been studied in Lambert [3], who shows in particular that the population goes extinct in finite time a. s.

There is a natural way of describing the genealogical tree of the discrete population. The notion of genealogical tree is discussed for this limiting continuous population as well in [4] and [6], in terms of continuous random trees in the sense of Aldous [1]. Clearly that forest of trees is finite a.s., and one can define the height H^m and the length L^m of the discrete forest of genealogical trees, as well as the height of the continuous "forest of genealogical trees", equal to the lifetime T^x of the process Z^x, and the length of the same forest of trees, given as $S^x := \int_0^{T^x} Z_t^x dt$.

Let us now generalize the above models, both in the discrete and in the continuous case, replacing in the first case the death rate $\gamma(X_t^m)^2$ by $\gamma(X_t^m)^\alpha$ and in the second case the drift term $-\gamma(Z_t^x)^2$ by $-\gamma(Z_t^x)^\alpha$, for some $\alpha > 0$. In the case $\alpha = 1$, there is no competition, we are back to branching processes, both discrete and continuous. The case $0 < \alpha < 1$ corresponds to a situation where an increase of the population size reduces the per capita death rate, by allowing for an improvement of the living conditions (one can argue that this a reasonable model, at least for moderate population size compared to the available resources). The case $\alpha > 1$ is the case of competition, where an increase of the population size increases the per capita death rate, because for instance of the limitation of available resources.

The main result of this paper is the following

Theorem 0.1 *Both* $\mathbb{E}[\sup_m H^m] < \infty$ *and* $\mathbb{E}[\sup_x T^x] < \infty$ *if* $\alpha > 1$*, while* $H^m \to \infty$ *as* $m \to \infty$ *and* $T^x \to \infty$ *as* $x \to \infty$ *a. s. if* $\alpha \leq 1$*. Both* $\mathbb{E}[\sup_m L^m] < \infty$ *and* $\mathbb{E}[\sup_x S^x] < \infty$ *if* $\alpha > 2$*, while* $L^m \to \infty$ *as* $m \to \infty$ *and* $S^x \to \infty$ *as* $x \to \infty$ *a. s. if* $\alpha \leq 2$*.*

Note that the monotonicity in α is not a surprise, $\sup_m H^m = \infty$ a. s. when $\alpha = 1$ follows rather easily from the branching property, $\mathbb{E}[\sup_m H^m] < \infty$ and $\mathbb{E}[\sup_m L^m] = \infty$ in case $\alpha = 2$ follow from results in [5], and again in the case $\alpha = 2$, $\sup_x S^x = \infty$ has been established in [4]. The main novelty of our results concerns the case $\alpha > 2$, which we discovered while trying to generalize the quadratic competition term.

Our theorem necessitates to define in a consistent way the population processes jointly for all initial population sizes, i. e. we will need to define the two–parameter processes $\{X^m_t,\ t \geq 0, m \geq 1\}$ and $\{Z^x_t,\ t \geq 0, x > 0\}$. One of the objectives of this paper is also to prove that the renormalized discrete two–parameter processes converge weakly, for an appropriate topology, towards $\{Z^x_t,\ t \geq 0, x > 0\}$.

The paper is organized as follows. In the first section we present the discrete model, which provides the coupling for different values n of the initial size of the population. We describe in section 2 the renormalized model for large population sizes. We then construct in section 3 a random field indexed t and x in the case of the continuous model, for which we precise the laws. After that we establish the convergence of the renormalized discrete random field to the continuous random field model in section 4. We finally study the finiteness of the supremum over the initial population size of the height and of the length of the forest of genealogical trees in the discrete case in section 5, and in the continuous case in section 6.

1 The discrete model

We first present the discrete model. As declared in the introduction, we consider a continuous time $\mathbb{Z}_+$–valued population process $\{X^m_t,\ m \geq 1\}$, which starts at time zero from the initial condition $X^m_0 = m$, i. e. m is the number of ancestors of the whole population. The process X^m_t evolves as follows. Each individual, independently of the others, spawns a child at a constant rate μ, and dies either “from natural death” at constant rate λ, or from the competition pressure, which results in a total additional death rate equal at time t to $\gamma(X^m_t)^\alpha$ (in fact it will be a quantity close to that one, see below). This description is valid for one initial condition m. But it is not sufficiently precise to describe the joint evolution of $\{(X_t^m, X_t^n),\ t \geq 0\}$, with say $1 \leq m < n$. We must precise the effect of the competition upon the death rate of each individual. In order to be consistent, we need to introduce a non–symmetric picture of the effect of the competition term, exactly as it was first introduced in [4] in the case $\alpha = 2$, in order to describe the exploration process of the genealogical tree. The idea is that the progeny X_t^m of the m "first" ancestors should not feel the competition due to the progeny $X_t^n - X_t^m$ of the $n - m$ "additional" ancestors which is present in the population X_t^n. One way to do so is to model the effect of the competition in the following asymmetric way. We order the ancestors from left to right, this order being passed to their progeny. This means that the forest of genealogical trees of the population is a planar forest of trees, where the ancestor of the population X_t^1 is placed on the far left, the ancestor of $X_t^2 - X_t^1$ immediately on his right, etc... Moreover, we draw the genealogical trees in such a way that distinct branches never cross. This defines in a non–ambiguous way an order from left to right within the population alive at each time t. Now we model the competition as each individual being "under attack" from his contemporaries located on his left in the planar tree. The "competition death rate" of a given individual i at time t is defined a $\gamma[\mathcal{L}_i(t)^\alpha - (\mathcal{L}_i(t) - 1)^\alpha]$, if $\mathcal{L}_i(t)$ denotes the number of alive individuals at time t, who are located at his left on the planar tree. Note that this rate, as a function of $\mathcal{L}_i(t)$, is decreasing if $0 < \alpha < 1$, constant if $\alpha = 1$, and increasing if $\alpha > 1$. Of course, conditionally upon $\mathcal{L}_i(\cdot)$, the occurence of a "competition death event" for individual i is independent of the other birth/death events and of what happens to the other individuals.

The resulting total death rate endured by the population X_t^m at time t is then

$$\gamma \sum_{k=2}^{X_t^m} [(k-1)^\alpha - (k-2)^\alpha] = \gamma (X_t^m - 1)^\alpha,$$

which is a reasonable approximation of $\gamma (X_t^m)^\alpha$.

As a result, $\{X_t^m,\ t \geq 0\}$ is a continuous time $\mathbb{Z}_+$–valued Markov process, which evolves as follows. If $X_t^m = 0$, then $X_s^m = 0$ for all $s \geq t$. While at state $k \geq 1$, the process

$$X_t^m \text{ jumps to } \begin{cases} k+1, & \text{at rate } \mu k; \\ k-1, & \text{at rate } \lambda k + \gamma (k-1)^\alpha. \end{cases}$$

The above description specifies the joint evolution of all $\{X_t^m,\ t \geq 0\}_{m \geq 0}$, or in other words of the two–parameter process $\{X_t^m,\ t \geq 0, m \geq 0\}$. Let us rephrase it in more mathematical terms.

In the case $\alpha = 1$, for each fixed $t > 0$, $\{X_t^m,\ m \geq 1\}$ is an independent increments process. In the case $\alpha \neq 1$, $\{X_t^m,\ m \geq 1\}$ is not a Markov chain for fixed t. That is to say, the conditional law of X_t^{n+1} given X_t^n differs from its conditional law given $(X_t^1, X_t^2, \dots, X_t^n)$. The intuitive reason for that is that the additional information carried by $(X_t^1, X_t^2, \dots, X_t^{n-1})$ gives us a clue as to the level of competition which the progeny of the $n+1$st ancestor had to suffer, between time 0 and time t.

However, $\{X_\cdot^m,\ m \geq 0\}$ is a Markov chain with values in the space $D([0,\infty); \mathbb{Z}_+)$ of càdlàg functions from $[0,\infty)$ into $\mathbb{Z}_+$, which starts from 0 at $m = 0$. Consequently, in order to describe the law of the whole process, that is of the two–parameter process $\{X_t^m,\ t \geq 0, m \geq 0\}$, it suffices to describe the conditional law of $X_\cdot^n$, given $\{X_\cdot^{n-1}\}$. We now describe that conditional law for arbitrary $0 \leq m < n$. Let $V_t^{m,n} := X_t^n - X_t^m$, $t \geq 0$. Conditionally upon $\{X_\cdot^\ell,\ \ell \leq m\}$, and given that $X_t^m = x(t)$, $t \geq 0$, $\{V_t^{m,n},\ t \geq 0\}$ is a $\mathbb{Z}_+$–valued time inhomogeneous Markov process starting from $V_0^{m,n} = n - m$, whose time–dependent infinitesimal generator $\{Q_{k,\ell}(t),\ k, \ell \in \mathbb{Z}_+\}$ is such that its off–diagonal terms are given by

$$\begin{aligned}
Q_{0,\ell}(t) &= 0, \quad \forall \ell \geq 1, \quad \text{and for any } k \geq 1,\\
Q_{k,k+1}(t) &= \mu k,\\
Q_{k,k-1}(t) &= \lambda k + \gamma(x(t) + k - 1)^\alpha,\\
Q_{k,\ell}(t) &= 0, \quad \forall \ell \notin \{k-1, k, k+1\}.
\end{aligned}$$

The reader can easily convince himself that this description of the conditional law of $\{X_t^n - X_t^m,\ t \geq 0\}$, given $X_\cdot^m$ is prescribed by what we have said above, and that $\{X_\cdot^m,\ m \geq 0\}$ is indeed a Markov chain.

2 Renormalized discrete model

We consider a family of models like in the previous section, indexed by $N \in \mathbb{N}$. We choose the number of ancestors to be $m = \lfloor Nx \rfloor$, for some fixed $x > 0$, the birth rate to be $\mu_N = 2N + \theta$, for some $\theta > 0$, the "natural death rate" to be $\lambda_N = 2N$, and the competition death parameter to be $\gamma_N = \gamma / N^{\alpha-1}$. We now weight each individual by a factor N^{-1}, which means that we want to study the limit, as $N \to \infty$, of the "reweighted total mass population" process $Z_t^{N,x} := X_t^{\lfloor Nx \rfloor}/N$. The process $\{Z_t^{N,x},\ t \geq 0\}$ is a $\mathbb{Z}_+/N$–valued continuous time Markov process which starts from $Z_0^{N,x} = \lfloor Nx \rfloor / N$, is such that if $Z_t^{N,x} = 0$, then $Z_s^{N,x} = 0$, for all $s \geq t$, and while at state k/N, $k \geq 1$,

$$Z^{N,x} \text{ jumps to } \begin{cases} (k+1)/N, & \text{at rate } 2Nk + k\theta;\\ (k-1)/N, & \text{at rate } 2Nk + \gamma N \left(\frac{k-1}{N}\right)^\alpha. \end{cases}$$

Clearly there exist three mutually independent standard Poisson processes P_1, P_2 and P_3 such that

$$\begin{aligned}X_t^{\lfloor Nx \rfloor} &= \lfloor Nx \rfloor + P_1\left(\int_0^t (2N+\theta)X_r^{\lfloor Nx \rfloor} dr\right) - P_2\left(2N\int_0^t X_r^{\lfloor Nx \rfloor} dr\right) \\ &\quad - P_3\left(\gamma N\int_0^t \left[\frac{X_r^{\lfloor Nx \rfloor}-1}{N}\right]^\alpha dr\right).\end{aligned}$$

Consequently there exists a martingale $M^{N,x}$ such that

$$\begin{aligned}Z_t^{N,x} &= Z_0^{N,x} + \int_0^t \left\{\theta Z_r^{N,x} - \gamma(Z_r^{N,x}-1/N)^\alpha\right\} dr + M_t^{N,x}, \quad \text{with} \\ \langle M^{N,x}\rangle_t &= \int_0^t \left\{4Z_r^{N,x} + \frac{\theta}{N}Z_r^{N,x} + \frac{\gamma}{N}(Z_r^{N,x}-1/N)^\alpha\right\} dr.\end{aligned} \tag{2.1}$$

Now for $0<x<y$, let $V_t^{N,x,y} := Z_t^{N,y} - Z_t^{N,x}$. It it not too hard to show that there exists tree further standard Poisson processes P_4, P_5 and P_6, such that the six Poisson processes $P_1, P_2, P_3, P_4, P_5, P_6$ are mutually independent, and moreover

$$\begin{aligned}V_t^{N,x,y} &= V_0^{N,x,y} + N^{-1}P_4\left(\int_0^t (2N^2+\theta N)V_r^{N,x,y} dr\right) - N^{-1}P_5\left(2N^2\int_0^t V_r^{N,x,y} dr\right) \\ &\quad - N^{-1}P_6\left(\gamma N\int_0^t \left[(Z_r^{N,x}+V_r^{N,x,y}-1/N)^\alpha - (Z_r^{N,x}-1/N)^\alpha\right] dr\right),\end{aligned}$$

from which we deduce that there exists a martingale $M^{N,x,y}$ such that

$$\begin{aligned}V_t^{N,x,y} &= V_0^{N,x,y} + \int_0^t \left\{\theta V_r^{N,x,y} - \gamma\left[(Z_r^{N,x}+V_r^{N,x,y}-1/N)^\alpha - (Z_r^{N,x}-1/N)^\alpha\right]\right\} dr + M_t^{N,x,y}, \\ \langle M^{N,x,y}\rangle_t &= \int_0^t \left\{4V_r^{N,x,y} + \frac{\theta}{N}V_r^{N,x,y} + \frac{\gamma}{N}\left[(Z_r^{N,x}+V_r^{N,x,y}-1/N)^\alpha - (Z_r^{N,x}-1/N)^\alpha\right]\right\} dr\end{aligned} \tag{2.2}$$

and moreover

$$\langle M^{N,x,y}, M^{N,x}\rangle_t \equiv 0. \tag{2.3}$$

The formulas for $\langle M^{N,x}\rangle_t$ and $\langle M^{N,x,y}\rangle_t$, as well as (2.3), rely on the following lemma, for the statement of which we need to introduce some notations. Let

$$\begin{aligned}
M_t^{N,1} &= N^{-1}P_1\left((2N^2+\theta N)\int_0^t Z_r^{N,x}dr\right) - \int_0^t (2N+\theta)Z_r^{N,x}dr,\\
M_t^{N,2} &= N^{-1}P_2\left(2N^2\int_0^t Z_r^{N,x}dr\right) - 2N\int_0^t Z_r^{N,x}dr,\\
M_t^{N,3} &= N^{-1}P_3\left(\gamma N\int_0^t \left[Z_r^{N,x}-1/N\right]^\alpha dr\right) - \gamma\int_0^t \left[Z_r^{N,x}-1/N\right]^\alpha dr,\\
M_t^{N,4} &= N^{-1}P_4\left(\int_0^t (2N^2+\theta N)V_r^{N,x,y}dr\right) - \int_0^t (2N+\theta)V_r^{N,x,y}dr,\\
M_t^{N,5} &= N^{-1}P_5\left(2N^2\int_0^t V_r^{N,x,y}dr\right) - 2N\int_0^t V_r^{N,x,y}dr,\\
M_t^{N,6} &= N^{-1}P_6\left(\gamma N\int_0^t \left[(Z_r^{N,x}+V_r^{N,x,y}-1/N)^\alpha-(Z_r^{N,x}-1/N)^\alpha\right]dr\right)\\
&\quad -\gamma\int_0^t \left[(Z_r^{N,x}+V_r^{N,x,y}-1/N)^\alpha-(Z_r^{N,x}-1/N)^\alpha\right]dr.
\end{aligned}$$

Lemma 2.1 *For any $1\le i\ne j\le 6$, the martingales $M^{N,i}$ and $M^{N,j}$ are orthogonal, in the sense that*

$$\langle M^{N,i}, M^{N,j}\rangle \equiv 0.$$

PROOF: All we have to show is that $M^{N,i}$ and $M^{N,j}$ have a. s. no common jump time. In other words we need to show that

$$P_i\left(\int_0^t \varphi_i(r)dr\right) \quad \text{and} \quad P_j\left(\int_0^t \varphi_j(r)dr\right)$$

have no common jump time, where

$$\varphi_i(r) = f_i(Z_r^{N,x}, V_r^{N,x,y}) \quad \text{and } \varphi_j(r) = f_j(Z_r^{N,x}, V_r^{N,x,y}),$$

for some functions f_i and f_j from $(\mathbb{Z}_+/N)^2$ into $\mathbb{R}_+$.

Let

$$\begin{aligned}
A_i(t) &= \int_0^t \varphi_i(r)dr, \quad \eta_i(t) = \inf\{s>0,\ A_i(s)>t\},\\
A_j(t) &= \int_0^t \varphi_j(r)dr, \quad \eta_j(t) = \inf\{s>0,\ A_j(s)>t\}.
\end{aligned}$$

Suppose the Lemma is not true, i.e. for some jump time T_k^i of P_i and some jump time T_ℓ^j of P_j, $\eta_i(T_k^i) = \eta_j(T_\ell^j)$. Let $S = \eta_i(T_{k-1}^i)\vee\eta_j(T_{\ell-1}^j)$. On the interval $[A^i(S), T^i_k)$, $\varphi_i(r)$ depends upon the jump times $T^i_1, \dots T^i_{k-1}$ of P_i, the jump times $T^j_1, \dots, T^j_{\ell-1}$ of P_j, plus upon some of the jump times of the other four Poisson processes, which are independent of (P_i, P_j). The same is true for $\varphi_j(r)$ on the interval $[A^j(S), T^j_\ell)$. It is now easy to show that conditionally upon those values of φ_i and φ_j, the two random variables $\eta_i(T^i_k) - S$ and $\eta_j(T^j_\ell) - S$ are independent, and their laws are absolutely continuous. Consequently $\mathbb{P}(\eta_i(T^i_k) = \eta_j(T^j_\ell)) = 0$. ■

3 The continuous model

We now define an $\mathbb{R}_+$–valued two–parameter stochastic process $\{Z^x_t,\ t \ge 0, x \ge 0\}$ which such that for each fixed $x > 0$, $\{Z^x_t, t \ge 0\}$ is continuous process, solution of the SDE

$$dZ^x_t = [\theta Z^x_t - \gamma (Z^x_t)^\alpha]\, dt + 2\sqrt{Z^x_t} dW_t,\ Z^x_0 = x, \tag{3.1}$$

where $\theta \in \mathbb{R}$, $\gamma > 0$, $\alpha > 0$, and $\{W_t,\ t \ge 0\}$ is a standard scalar Brownian motion. Similarly as in the discrete case, the process $\{Z^x_\cdot,\ x \ge 0\}$ is a Markov process with values in $C([0, \infty), \mathbb{R}_+)$, the space of continuous functions from $[0, \infty)$ into $\mathbb{R}_+$, starting from 0 at $x = 0$. The transition probabilities of this Markov process are specified as follows. For any $0 < x < y$, $\{V^{x,y}_t := Z^y_t - Z^x_t,\ t \ge 0\}$ solves the SDE

$$dV^{x,y}_t = [\theta V^{x,y}_t - \gamma \{(Z^x_t + V^{x,y}_t)^\alpha - (Z^x_t)^\alpha\}]\, dt + 2\sqrt{V^{x,y}_t} dW'_t,\ V^{x,y}_0 = y - x, \tag{3.2}$$

where the standard Brownian motion $\{W'_t,\ t \ge 0\}$ is independent from the Brownian motion W which drives the SDE (3.1) for Z^x_t. It is an easy exercise to show that $Z^y_t = Z^x_t + V^{x,y}_t$ solves the same equation as Z^x_t, with the initial condition $Z^y_0 = y$, and a different driving standard Brownian motion. Moreover we have that whenever $0 \le x < y$, $Z^x_t \le Z^y_t$ for all $t \ge 0$, a. s., and in the case $\alpha = 1$, the increment of the mapping $x \to Z^x_t$ are in dependent, for each $t > 0$. Moreover, the conditional law of $Z^y_\cdot$, given that $Z^x_t = z(t)$, $t \ge 0$, is the law of the sum of z plus the solution of (3.2) with Z^x_t replaced by $z(t)$.

4 Convergence as $N \to \infty$

The aim of this section is to prove the convergence in law as $N \to \infty$ of the two–parameter process $\{Z^{N,x}_t,\ t \ge 0, x \ge 0\}$ defined in section 2 towards the process $\{Z_t^x,\ t \geq 0, x \geq 0\}$ defined in section 3. We need to make precise the topology for which this convergence will hold. We note that the process $Z_t^{N,x}$ (resp. Z_t^x) is a Markov processes indexed by x, with values in the space of càdlàg (resp. continuous) functions of t $D(([0,\infty);\mathbb{R}_+)$ (resp. $C(([0,\infty);\mathbb{R}_+))$ (note that the trajectories have compact support – the population process goes extinct in finite time – except in the cases $\alpha < 1, \theta > 0$ and $\alpha = 1, \theta > \gamma$). So it will be natural to consider a topology of functions of x, with values in functions of t.

The second point to notice is that for each fixed x, the process $t \to Z_t^{N,x}$ is càdlàg, constant between its jumps, with jumps of size $\pm N^{-1}$, while the limit process $t \to Z_t^x$ is continuous. On the other hand, both $Z_t^{N,x}$ and Z_t^x are discontinuous as functions of x. $x \to Z_\cdot^x$ has countably many jumps on any compact interval, but the mapping $x \to \{Z_t^x,\ t \geq \varepsilon\}$, where $\varepsilon > 0$ is arbitrary, has finitely many jumps on any compact interval, and it is constant between its jumps. It will be simpler to discuss the convergence of the sequence $\tilde{Z}_t^{N,x}$, which is defined as follows : for each fixed $x > 0$, $t \to \tilde{Z}_t^{N,x}$ is the piecewise linear interpolation of $t \to Z_t^{N,x}$, which interpolates linearly the latter between its jumps. From the size of the jumps of $Z_t^{N,x}$, clearly $\sup_{t>0,x>0} |\tilde{Z}_t^{N,x} - Z_t^{N,x}| \leq 1/N$, and essentially the two processes converge for the same topology, but it will be simpler to state the result for processes which are continuous in t. We will prove

Theorem 4.1 *As $N \to \infty$,*

$$\{\tilde{Z}_t^{N,x},\ t \geq 0, x \geq 0\} \Rightarrow \{Z_t^x,\ t \geq 0, x \geq 0\}$$

in $D([0,\infty); C([0,\infty);\mathbb{R}_+))$, equipped with the Skohorod topology of the space of càdlàg functions of x, with values in the space $C([0,\infty);\mathbb{R}_+)$ equipped with the topology of locally uniform convergence.

We first establish tightness for fixed x.

4.1 Tightness of $Z^{N,x}$, x fixed

Let us prove the tightness of the sequence $\left\{Z^{N,x},\ N \geq 0\right\}$. For this end, we first establish some a priori estimates.

Lemma 4.2 $\forall\ T > 0$, *there exists a constant $C_1 > 0$ such that :*

$$\sup_{N \geq 1} \sup_{0 \leq t \leq T} \mathbb{E}\left\{ Z_t^{N,x} + \int_0^t (Z_r^{N,x})^\alpha dr \right\} \leq C_1.$$

It follows from this and the expression for $\langle M^{N,x} \rangle$ that the local martingale $M^{N,x}$ is in fact a square integrable martingale. We then have

Lemma 4.3 *$\forall\, T > 0$, there exists a constant $C_2 > 0$ such that :*

$$\sup_{N\geq 1}\sup_{0\leq t\leq T}\mathbb{E}\left\{\left(Z_t^{N,x}\right)^2 + \int_0^t \left(Z^{N,x}\right)^{\alpha+1} dr\right\} \leq C_2.$$

The proof of those two lemmas is obtained easily using equation (2.1), elementary stochastic calculus, Gronwall and Fatou's lemmas.
We want to check the tightness of the sequence $\left\{Z^{N,x},\ N \geq 0\right\}$ using the Aldous criterion. Let $\{\tau_N,\ N \geq 1\}$ be a sequence of stopping time with values in $[0, T]$. We deduce from Lemma 4.2

Proposition 4.4 *For any $T > 0$ and η, $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\sup_{N\geq 1}\sup_{0\leq a\leq\delta}\mathbb{P}\left(\int_{\tau_N}^{(\tau_N+a)\wedge T} Z_r^{N,x} dr \geq \eta\right) \leq \epsilon.$$

Proof: We have that

$$\begin{aligned}\sup_{0\leq a\leq\delta}\mathbb{P}\left(\int_{\tau_N}^{(\tau_N+a)\wedge T} Z_r^{N,x} dr \geq \eta\right) &\leq \sup_{0\leq a\leq\delta}\frac{1}{\eta}\mathbb{E}\int_{\tau_N}^{(\tau_N+a)\wedge T} Z_r^{N,x} dr\\ &\leq \frac{\delta}{\eta}\sup_{0\leq t\leq T}\mathbb{E}(Z_t^{N,x})\\ &\leq C_1\frac{\delta}{\eta}.\end{aligned}$$

Hence the result with $\delta = \epsilon\eta/C_1$. ∎

We also deduce from Lemma 4.3

Proposition 4.5 *For any $T > 0$ and η, $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\sup_{N\geq 1}\sup_{0\leq a\leq\delta}\mathbb{P}\left(\int_{\tau_N}^{(\tau_N+a)\wedge T} (Z_r^{N,x})^\alpha dr \geq \eta\right) \leq \epsilon.$$

Proof: For any $M > 0$, we have

$$\int_{\tau_N}^{(\tau_N+a)\wedge T} (Z_r^{N,x})^\alpha dr \leq M^\alpha a + M^{-1}\int_0^T (Z_r^{N,x})^{\alpha+1} dr.$$

This implies that

$$\begin{aligned}\sup_{N\geq 1}\sup_{0\leq a\leq\delta}\mathbb{P}\left(\int_{\tau_N}^{(\tau_N+a)\wedge T} (Z_r^{N,x})^\alpha dr \geq \eta\right) &\leq \sup_{N\geq 1}\sup_{0\leq a\leq\delta}\eta^{-1}\mathbb{E}\left(\int_{\tau_N}^{(\tau_N+a)\wedge T} (Z_r^{N,x})^\alpha dr\right)\\ &\leq \frac{M^\alpha\delta}{\eta} + \frac{C_2}{M\eta}\end{aligned}$$

The result follows by choosing first $M = 2C_2/\epsilon\eta$, and then $\delta = \epsilon\eta/2M^\alpha$. ∎

From (2.1), Propositions 4.4 and 4.5, we deduce

Proposition 4.6 *For each fixed $x > 0$, the sequence of processes $\left\{Z^{N,x},\ N \geq 1\right\}$ is tight in $D([0,\infty))$.*

4.2 Proof of Theorem 4.1

Theorem 4.1 will be a consequence of the two next Propositions

Proposition 4.7 *For any $n \in \mathbb{N}$, $0 \leq x_1 < x_2 < \cdots < x_n$,*

$$\left(Z^{N,x_1}, Z^{N,x_2}, \cdots, Z^{N,x_n}\right) \Rightarrow \left(Z^{x_1}, Z^{x_2}, \cdots, Z^{x_n}\right)$$

as $N \to \infty$, for the topology of locally uniform convergence in t.

Clearly, the same result holds with Z replaced by $\tilde{Z}$.
Proof: We prove the statement in the case $n = 2$ only. The general statement can be proved in a very similar way. For $0 \leq x_1 < x_2$, we consider the process $(Z^{N,x_1}, V^{N,x_1,x_2})$, using the notations from section 2. The process $(Z^{N,x_1}, V^{N,x_1,x_2})$ is tight, as a consequence of Proposition 4.6, and thanks to (2.1), (2.2) and (2.3), any weak limit (Z^{x_1}, V^{x_1,x_2}) of a subsequence of $\left\{U^{N,x_1,x_2}, N \geq 1\right\}$ is the unique weak solution of the pair of coupled SDEs (3.1) and (3.2). ∎

Proposition 4.8 *There exists a constant C, which depends only upon θ and T, such that for any $0 \leq x < y < z$, which are such that $y - x \leq 1$, $z - y \leq 1$,*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |Z_t^{N,y} - Z_t^{N,x}|^2 \times \sup_{0\leq t\leq T} |Z_t^{N,z} - Z_t^{N,y}|^2\right] \leq C|z - x|^2.$$

Proof: For any $0 \leq x < y < z$, we have

$$\begin{aligned}
\sup_{0\leq t\leq T} |Z_t^{N,y} - Z_t^{N,x}|^2 &= \sup_{0\leq t\leq T} (V_t^{N,x,y})^2 \leq \sup_{0\leq t\leq T} (U_t^{N,y,x})^2 \\
\sup_{0\leq t\leq T} |Z_t^{N,z} - Z_t^{N,y}|^2 &= \sup_{0\leq t\leq T} (V_t^{N,z,y})^2 \leq \sup_{0\leq t\leq T} (U_t^{N,z,y})^2,
\end{aligned}$$

where $U_t^{N,x,y}$ and $U_t^{N,z,y}$ are mutually independent branching processes, with in particular

$$U_t^{N,x,y} = y - x + \theta \int_0^t U_r^{N,x,y} dr + \tilde{M}_t^{N,x,y},$$

with $\tilde{M}^{N,x,y}$ a local martingale such that $\langle \tilde{M}^{N,x,y}\rangle_t = (4+\frac{\theta}{N})\int_0^t U_r^{N,x,y}dr$. Consequently

$$\mathbb{E}\left[\sup_{0\le s\le t}\left(U_t^{N,y,x}\right)^2\right] \le 3|y-x|^2 + 3\theta^2 t\int_0^t \mathbb{E}\left[\sup_{0\le r\le s}\left(U_r^{N,y,x}\right)^2\right]ds + 3\mathbb{E}\left[\sup_{0\le s\le t}\left(\tilde{M}_s^{N,x,y}\right)^2\right]$$
$$\le 3|y-x|^2 + 3\theta^2 t\int_0^t \mathbb{E}\left[\sup_{0\le r\le s}\left(U_r^{N,y,x}\right)^2\right]ds + 3C\mathbb{E}\langle \tilde{M}^{N,x,y}\rangle_t$$

But clearly

$$\mathbb{E}[U_s^{N,y,x}] = |x-y|\exp(\theta s),$$

hence

$$\mathbb{E}\langle \tilde{M}^{N,x,y}\rangle_T \le C(\theta,T)|x-y|.$$

Note that since $|x-y|\le 1$, $|x-y|^2\le |x-y|$. The above computations, combined with Gronwall's Lemma, lead to

$$\mathbb{E}\left[\sup_{0\le t\le T}\left(U_t^{N,y,x}\right)^2\right] \le C'(\theta,T)|x-y|.$$

We obtain similarly

$$\mathbb{E}\left[\sup_{0\le t\le T}\left(U_s^{N,z,y}\right)^2\right] \le C'(\theta,T)|z-y|.$$

Since moreover the two random processes $U_t^{N,y,x}$ and $U_t^{N,z,y}$ are independent, the Proposition follows from the above computations. ■

Proof of Theorem 4.1 We will show that for any $T>0$,

$$\{\tilde{Z}_t^{N,x},\ 0\le t\le T,\ x\ge 0\} \Rightarrow \{Z_t^x,\ 0\le t\le T,\ x\ge 0\}$$

in $D([0,\infty);C([0,T],\mathbb{R}_+))$. From Theorems 13.1 and 16.8 in [2], since from Proposition 4.7, for all $n\ge 1$, $0<x_1<\cdots<x_n$,

$$(\tilde{Z}_\cdot^{N,x_1},\ldots,\tilde{Z}_\cdot^{N,x_n}) \Rightarrow (Z_\cdot^{x_1},\ldots,Z_\cdot^{x_n})$$

in $C([0,T];\mathbb{R}^n)$, it suffices to show that for all all $\bar{x}>0$, ε, $\eta>0$, there exists $N_0\ge 1$ and $\delta>0$ such that for all $N\ge N_0$,

$$\mathbb{P}(w_{\bar{x},\delta}(\tilde{Z}^N)\ge\varepsilon)\le\eta, \tag{4.1}$$

where for a function $(x,t)\to z(x,t)$, with the notation $\|z(x,\cdot)\| = \sup_{0\le t\le T}|z(x,t)|$,

$$w_{\bar{x},\delta}(z) = \sup_{0\le x_1\le x\le x_2\le\bar{x},x_2-x_1\le\delta} \inf\left\{\|z(x,\cdot)-z(x_1,\cdot)\|, \|z(x_2,\cdot)-z(x,\cdot)\|\right\}.$$

Since clearly

$$|w_{\bar{x},\delta}(Z) - w_{\bar{x},\delta}(\tilde{Z})| \le 2/N,$$

it suffices in fact to establish (4.1) with $\tilde{Z}^N$ replaced by Z^N. But from the proof of Theorem 13.5 in [2], (4.1) for Z^N follows from Proposition 4.8. ■

5 Height and length of the genealogical tree in the discrete case

5.1 Height of the discrete tree

We consider the two–parameter $\mathbb{Z}_+$–valued stochastic process $\{X_t^m,\ t \geq 0, m \geq 1\}$ defined in section 1, and define the height and length of its genealogical tree by

$$H^m = \inf\{t > 0,\ X_t^m = 0\}, \quad L^m = \int_0^{H^m} X_t^m dt, \quad \text{for } m \geq 1.$$

We shall occasionally write $X_t^{\alpha,m}$ when we want to emphasize the dependence upon the value of α. We first prove the

Proposition 5.1 *If* $0 < \alpha \leq 1$*, then*

$$\sup_{m\geq 1} H^m = +\infty \qquad a.\ s.$$

PROOF: Since for any $0 < \alpha < 1$, $j \geq 2$, $(j-1)^\alpha - (j-2)^\alpha < 1$, it is not hard to couple the two–parameter processes $\{X_t^{\alpha,m},\ t \geq 0, m \geq 1\}$ and $\{X_t^{1,m},\ t \geq 0, m \geq 1\}$ in such a way that $X_t^{\alpha,m} \geq X_t^{1,m}$, for all $m \geq 1$, $t \geq 0$, a. s.. Consequently it suffices to prove the Proposition in the case $\alpha = 1$.

But in that case $\{X_t^m,\ t \geq 0\}$ is the sum of m mutually independent copies of $\{X_t^1,\ t \geq 0\}$. Hence H^m is the sup of m independent copies of H^1, and the result follows from the fact that $\mathbb{P}(H^1 > t) > 0$, for all $t > 0$. ■

We now prove the

Theorem 5.2 *If* $\alpha > 1$*, then*

$$\mathbb{E}\left[\sup_{m\geq 1} H^m\right] < \infty.$$

PROOF: Since $m \to H^m$ is a. s. increasing, it suffices to prove that there exists a constant $C > 0$ such that

$$\mathbb{E}[H^m] \leq C, \quad \text{for any } m \geq 1.$$

We first show that $\lim_{m\to\infty} \mathbb{E}[H_1^m] < \infty$, where

$$H_1^m = \inf\{s \geq 0; X_s^m = 1\}.$$

It suffices to prove this result in the case $\lambda = 0$, which implies the result in the case $\lambda > 0$.

Proposition 5.3 *For $\alpha > 1$, $\lambda = 0$, $\forall\ m \geq 1$, $\mathbb{E}\left(H_1^m\right)$ is given by*

$$\mathbb{E}\left(H_1^m\right) = \sum_{k=2}^{m} \frac{1}{\gamma(k-1)^\alpha} \sum_{n=0}^{\infty} \frac{\mu^n}{\gamma^n} \frac{1}{[k(k+1)\cdots(k+n-1)]^{\alpha-1}}.$$

PROOF: Define $u_m = \mathbb{E}\left(H_1^m\right)$. It is clear that $u_1 = 0$. The waiting time of X^m at state k is an exponential variable with mean $\dfrac{1}{\mu k + \gamma(k-1)^\alpha}$, and either X^m jumps from k to $k-1$ with probability $\dfrac{\gamma(k-1)^\alpha}{\mu k + \gamma(k-1)^\alpha}$ or either from k to $k+1$ with probability $\dfrac{\mu k}{\mu k + \gamma(k-1)^\alpha}$. We then have the recursive formula for u_m for any $m \geq 1$.

$$u_m = \frac{1}{\mu m + \gamma(m-1)^\alpha} + \frac{\gamma(m-1)^\alpha}{\mu m + \gamma(m-1)^\alpha} u_{m-1} + \frac{\mu m}{\mu m + \gamma m^\alpha} u_{m-1}.$$

If we define $w_m = u_m - u_{m-1}$, we obtain, for any $n \geq 0$, the following relation.

$$w_m = \frac{[(m-1)!]^{\alpha-1}}{\gamma(m-1)^\alpha} \left(\sum_{k=0}^{n-1} \frac{\mu^k}{\gamma^k} \frac{1}{[(m+k-1)!]^{\alpha-1}} + \frac{\mu^n}{\gamma^n} \frac{m+n-1}{[(m+n-2)!]^{\alpha-1}} w_{m+n} \right)$$

Define the random variable τ_{m+n} by

$$\tau_{m+n} = \inf\left\{t \geq 0;\ X_t^{m+n} = m+n-1\right\}.$$

We have that $w_{m+n} = \mathbb{E}(\tau_{m+n})$. Let R_{m+n} be the number of births which occur before Z^{m+n} reaches the value $m+n-1$, starting from $m+n$. For any $k \geq 0$ we have

$$\mathbb{P}\left(R_{m+n} = k\right) \leq a_k \left(\frac{\mu(m+n)}{\gamma(m+n-1)^\alpha} \right)^k,$$

where a_k is the cardinal of the set of binary trees with $k+1$ leaves. It is called a Catalan number and is given by

$$a_k = \frac{1}{k+1} \binom{2k}{k}, \quad a_k \sim \frac{4^k}{k^{3/2}\sqrt{\pi}}.$$

Moreover we have that

$$\mathbb{E}\left(\tau_{m+n} | R_{m+n} = k\right) \leq \frac{2k+1}{\gamma(m+n-1)^\alpha}.$$

Finally, with $c = \sup_{k\geq 1}(2k+1)4^{-k}a_k/\gamma$,

$$\mathbb{E}\left(\tau_{m+n}\right) \leq \frac{c}{(m+n-1)^\alpha}\sum_{k=1}^{\infty}\left(\frac{\mu}{\gamma}\frac{4(m+n)}{(m+n-1)^\alpha}\right)^k.$$

For large n, we have $\frac{\mu}{\gamma}\frac{4(m+n)}{(m+n-1)^\alpha} < \frac{1}{2}$. This implies that there exists another constant C such that $\mathbb{E}\left(\tau_{m+n}\right) \leq C(m+n-1)^{-\alpha}$, and $\lim_{n\to\infty} w_{m+n} = 0$. We then deduce that

$$w_m = \frac{[(m-1)!]^{\alpha-1}}{\gamma(m-1)^\alpha}\left(\sum_{k=0}^{\infty}\frac{\mu^k}{\gamma^k}\frac{1}{[(m+k-1)!]^{\alpha-1}}\right)$$

Consequently, for $\alpha > 1$, we have

$$u_m = \sum_{k=1}^{m} w_k = \sum_{k=2}^{m}\frac{1}{\gamma(k-1)^\alpha}\sum_{n=0}^{\infty}\left(\frac{\mu}{\gamma}\right)^n\frac{1}{[k(k+1)\cdots(k+n-1)]^{\alpha-1}}$$

■

End of the proof of Theorem 5.2

Furthermore, for $0 \leq \ell \leq n-1$, we have

$$\frac{1}{[k(k+1)...(k+n-1)]^{\alpha-1}} \leq \frac{(k+\ell)^{\ell(\alpha-1)}}{[k(k+1)\cdots(k+\ell-1)]^{\alpha-1}}\frac{1}{(k+\ell)^{n(\alpha-1)}}$$

Let $K = \lfloor(2\mu/\gamma)^{1/(\alpha-1)}\rfloor$. We conclude that

$$\begin{aligned}
&\text{If } K \geq 3, \quad u_m \leq \frac{2}{\gamma}\left(\sum_{k=2}^{K-1}\left(\frac{K^K}{k-1}\right)^{\alpha-1} + \frac{1}{(\alpha-1)(K-2)^{\alpha-1}}\right),\\
&\text{if } K \leq 2, \quad u_m \leq \frac{2}{\gamma}\frac{\alpha}{\alpha-1}.
\end{aligned}$$

In all cases, $\sup_{m\geq 1}\mathbb{E}[H_1^m] < \infty$.

Finally starting from 1 at time H_1^m, the probability p that X_t^m hits zero before hitting 2 is $\frac{\lambda}{\mu+\lambda}$. Let G be a random variable defined as follows. Let X_t^1 start from 1 at time 0. If X_t^1 hits zero before hitting 2, then $G = 1$. If not, we wait until X_t^1 goes back to 1. This time is less than $T_1 + H_1^2$, where T_1 is an exponential random variable with mean $1/(\lambda+\mu)$, which is independent of G. If starting again from 1 at that time, if X_t^1 reaches 0 before 2, we stop and $G = 2$. If not, we continue and so on. The random variable G is geometric with parameter p and independent of H_1^m. Clearly we have that

$$H^m \le H_1^m + GH_1^2 + \sum_{i=1}^{G} T_i,$$

We conclude that $\sup_{m\ge 1} \mathbb{E}[H^m] < \infty$. ∎

We have proved in particular that (in the terminology used in coalescent theory) the population process comes down from infinity if $\alpha > 1$. This means that if the population starts with an infinite number of individuals at time $t = 0$, instantaneously the population becomes finite, that is for all $t > 0$, $X_t^\infty < \infty$.

5.2 Length of the discrete tree

Define now

$$A_t^m := \int_0^t X_r^m dr, \quad \eta_t^m = \inf\{s > 0;\ A_s^m > t\}.$$

We consider the process $U^m := X^m \circ \eta^m$. Let S^m be the stopping time defined by

$$S^m = \inf\{r > 0; U_r^n = 0\}.$$

Note that $S^m = L^m$, the length of the genealogical forest of trees of the population X^m, since we have $S^m = \int_0^{H^m} X_r^m dr$. The process X^m can be expressed using two mutually independent standard Poisson processes, as

$$X_t^m = m + P_1\left(\int_0^t \mu X_r^m dr\right) - P_2\left(\int_0^t [\lambda X_r^m + \gamma(X_r^m - 1)^\alpha]\, dr\right),$$

Consequently the process $U^m = Z^m \circ \eta^m$ satisfies

$$U_t^m = m + P_1(\mu t) - P_2\left(\int_0^t \left[\lambda + \gamma(U_r^m)^{-1}(U_r^m - 1)^\alpha\right] dr\right).$$

On the interval $[0, S^m)$, $U_t^m \ge 1$, and consequently we have the two inequalities

$$\begin{aligned} m - P_2\left(\int_0^t \left[\lambda U_r^m + \gamma(U_r^m - 1)^{\alpha-1}\right] dr\right) &\le U_t^m \\ &\le m + P_1\left(\int_0^t \mu U_r^m dr\right) - P_2\left(\int_0^t \left[\frac{\gamma}{2}(U_r^m - 1)^{\alpha-1}\right] dr\right) \end{aligned}$$

The following result is now a consequence of Proposition 5.1 and Theorem 5.2

Theorem 5.4 *If $\alpha \leq 2$, then*

$$\sup_{m\geq 0} L^m = \infty \quad a.\ s..$$

If $\alpha > 2$, then

$$\mathbb{E}\left[\sup_{m\geq 0} L^m\right] < \infty.$$

6 Height and length of the continuous tree

Now we study the same quantities in the continuous model. We first need to establish some preliminary results on SDEs with infinite initial condition.

6.1 SDE with infinite initial condition

Let $f : \mathbb{R}_+ \to \mathbb{R}$ be locally Lipschitz and such that

$$\lim_{x\to\infty} \frac{|f(x)|}{x^\alpha} = 0. \tag{6.1}$$

Theorem 6.1 *Let $\alpha > 1$, $\gamma > 0$ and f satisfy the assumption* (6.1)*. Then there exists a minimal $X \in C((0, +\infty); \mathbb{R})$ which solves*

$$\begin{cases} dX_t = [f(X_t) - \gamma(X_t)^\alpha]\mathbf{1}_{\{X_t\geq 0\}}dt + dW_t; \\ X_t \to \infty, \text{ as } t \to 0. \end{cases} \tag{6.2}$$

Moreover, if $T_0 := \inf\{t > 0,\ X_t = 0\}$, then $\mathbb{E}[T_0] < \infty$.

Proof: A priori estimate Setting $V_t = X_t - W_t$, the result is equivalent to the existence of a minimal $V \in C((0, +\infty); \mathbb{R})$ solution of the ODE

$$\begin{cases} \dfrac{dV_t}{dt} = f(V_t + W_t) - \gamma(V_t + W_t)^\alpha; \\ V_t \to \infty, \text{ as } t \to 0. \end{cases} \tag{6.3}$$

Let first

$$\begin{aligned} M &= \inf\{x > 0;\ |f(x)| \leq \gamma x^\alpha/2\}, \\ \tau &= \inf\{t > 0,\ W_t \notin [-M, 2M]\}. \end{aligned}$$

Suppose there exists a solution $\{V_t,\ t \geq 0\}$ to the ODE (6.3). Then the following random time is positive a. s.

$$S := \inf\{t > 0,\ V_t < 2M\}.$$

Now on the time interval $[0, \tau \wedge S]$,

$$
\begin{aligned}
-\frac{V_t}{2} &\leq W_t \leq V_t, \\
M \leq \frac{V_t}{2} &\leq V_t + W_t \leq 2V_t, \\
-\frac{3\gamma}{2}(V_t + W_t)^\alpha &\leq f(V_t + W_t) - \gamma(V_t + W_t)^\alpha \leq -\frac{1}{2}\gamma(V_t + W_t)^\alpha, \\
-3\gamma 2^{\alpha-1} V_t^\alpha &\leq \frac{dV_t}{dt} \leq -\frac{\gamma}{2^{\alpha+1}} V_t^\alpha, \\
\frac{\gamma(\alpha-1)}{2^{\alpha+1}} &\leq \frac{d}{dt}\left[(V_t)^{-(\alpha-1)}\right] \leq 3\gamma(\alpha-1)2^{\alpha-1}, \\
\frac{\gamma(\alpha-1)}{2^{\alpha+1}} t &\leq \frac{1}{(V_t)^{\alpha-1}} \leq 3\gamma(\alpha-1)2^{\alpha-1} t, \\
\frac{c_{\alpha,\gamma}}{t^{1/(\alpha-1)}} &\leq V_t \leq \frac{C_{\alpha,\gamma}}{t^{1/(\alpha-1)}},
\end{aligned}
$$

where

$$
c_{\alpha,\gamma} = \frac{1}{2}[3\gamma(\alpha-1)]^{-1/(\alpha-1)}, \quad C_{\alpha,\gamma} = \frac{2^{\frac{\alpha+1}{\alpha-1}}}{[\gamma(\alpha-1)]^{1/(\alpha-1)}}.
$$

Proof of Existence To each $x > 0$, we associate the unique solution X^x of equation (6.2), but with the initial condition $X_0^x = x$. Clearly $x \leq y$ implies that $X_t^x \leq X_t^y$ for all $t \geq 0$ a. s. Consider an increasing sequence $x_n \to \infty$, the corresponding increasing sequence of processes $\{X_t^{x_n},\ t \geq 0\}_{n\geq 1}$, and define $V_t^n := X_t^{x_n} - W_t$, $S_n = \inf\{t > 0,\ V_t^n < 2M\}$. Note that S_n is increasing. A minor modification of the computations in the first part of this proof shows that for $0 \leq t \leq S_n$,

$$
\frac{1}{(c_{\alpha,\gamma}^{\alpha-1} t + x_n^{-(\alpha-1)})^{1/(\alpha-1)}} \leq V_t^n \leq \frac{1}{(C_{\alpha,\gamma}^{\alpha-1} t + x_n^{-(\alpha-1)})^{1/(\alpha-1)}}.
$$

It readily follows that $V_t := \lim_{n\to\infty} V_t^n$ solves (6.3), while $X_t := V_t + W_t$ solves (6.2). Those solutions do not depend upon the choice of a particular sequence $x_n \to \infty$, and the thus constructed solution is clearly the minimal solution of (6.2).

Proof of $\mathbb{E}[T_0] < \infty$; Step1 We first show that $S \wedge \tau$ is bounded, and $V_{S\wedge\tau}$ is integrable.

For that sake, start noting that $V_{S\wedge\tau} \geq 2M$. It then follows from one of the inequalities obtained in the first part of the proof that

$$
S \wedge \tau \leq \left(\frac{C_{\alpha,\gamma}}{2M}\right)^{\alpha-1}.
$$

On the set $S < \tau$, $V_{S\wedge\tau} = 2M$. Consequently

$$\begin{aligned} V_{S\wedge\tau} &= 2M\mathbf{1}_{\{S<\tau\}} + V_\tau \mathbf{1}_{\{\tau\le S\}} \\ &\le 2M + C_{\alpha,\gamma}\tau^{-1/(\alpha-1)}. \end{aligned}$$

We now show that

$$\mathbb{E}[V_{S\wedge\tau}] < \infty. \tag{6.4}$$

From the last inequality,

$$\mathbb{E}[V_{S\wedge\tau}] \le 2M + C_{\alpha,\gamma}\mathbb{E}\left[\tau^{-1/(\alpha-1)}\right].$$

We need to compute (below $W_t^* = \sup_{0\le s\le t}|W_s|$)

$$\begin{aligned} \mathbb{E}\left[\tau^{-1/(\alpha-1)}\right] &= \int_0^\infty \mathbb{P}\left(\tau^{-1/(\alpha-1)} > t\right) dt \\ &= \int_0^\infty \mathbb{P}\left(\tau < t^{-(\alpha-1)}\right) dt \\ &\le \int_0^\infty \mathbb{P}\left(W^*_{t^{-(\alpha-1)}} \ge 2M\right) dt \\ &\le 4\int_0^\infty \mathbb{P}\left(W_{t^{-(\alpha-1)}} \ge 2M\right) dt \\ &\le 4\int_0^\infty \mathbb{P}\left(W_1 \ge 2Mt^{(\alpha-1)/2}\right) dt \\ &\le 4\sqrt{e}\int_0^\infty \exp\left(-2Mt^{(\alpha-1)/2}\right) dt \\ &< \infty, \end{aligned}$$

where we have used for the fourth inequality the following Chebycheff inequality

$$\mathbb{P}(W_1 > A) = \mathbb{P}\left(e^{W_1-1/2} > e^{A-1/2}\right) \le \frac{\sqrt{e}}{e^A}.$$

(6.4) follows.

Proof of $\mathbb{E}[T_0] < \infty$; Step 2 Now we turn back to the X equation, and we show in this step that X comes down to level M in time which has finite expectation. Since $V_{S\wedge\tau} \ge 2M$ and $-M \le W_{S\wedge\tau} \le 2M$,

$$M \le X_{S\wedge\tau} \le V_{S\wedge\tau} + 2M.$$

In order to simplify notations, let us write $\xi := X_{S\wedge\tau}$. Until X_t reaches the level M,

$$f(X_t) - \gamma(X_t)^\alpha \le -\frac{\gamma}{2}(X_t)^\alpha \le -1,$$

from the choice of M. Conseqently, if $T_M = \inf\{t > 0, X_t \leq M\}$, $X_{S\wedge\tau+r} \leq Y_r$ for all $0 \leq r \leq T_M - S \wedge \tau$ a. s., where Y solves

$$dY_r = -dr + dW_r, \quad Y_0 = \xi,$$

where W is a standard Brownian motion independent of ξ. Let $R_M := \inf\{r > 0,\ Y_r \leq M\}$. Clearly $T_M \leq S \wedge \tau + R_M$. So since $S \wedge \tau$ is bounded, Step 2 will follow if we show that $\mathbb{E}(R_M) < \infty$. But the time taken by Y to descend a given level is linear in that level. So, given that $\mathbb{E}(\xi) < \infty$, it suffices to show that the time needed for Y to descend a distance one is integrable, which is easy, since if $\tau := \inf\{t > 0,\ W_t - t \leq -1\}$, for $t \geq 2$,

$$\begin{aligned}\mathbb{P}(\tau > t) &\leq \mathbb{P}(W_t > t - 1)\\ &\leq \mathbb{P}(W_t > t/2)\\ &\leq \mathbb{P}(W_1 > \sqrt{t}/2)\\ &\leq \sqrt{e}\exp(-\sqrt{t}/2).\end{aligned}$$

Proof of $\mathbb{E}[T_0] < \infty$; Step 3 For proving that $\mathbb{E}[T_0] < \infty$, it remains to show that the time taken by X to descend from M to 0 is integrable, which we now establish. Given any fixed $T > 0$, let p denote the probability that starting from M at time $t = 0$, X hit zero before time T. Clearly $p > 0$. Let α be a geometric random variable with success probability p, which is defined as follows. Let X start from M at time 0. If X hits zero before time T, then $\alpha = 1$. If not, we look at the position X_T of X at time T. If $X_T > M$, we want until X goes back to M. The time needed is bounded by the integrable random variable η, which is the time needed for X to descend to M, when starting from $+\infty$. If however $X_T \leq M$, we start afresh from there, since the probability to reach zero in less than T is greater than or equal to p, for all starting points in the interval $(0, M]$. So either at time T, or at time $T + \eta$, we start again from a level which is less than or equal to M. If zero is reached during the next interval of length T, then $\alpha = 2$. Repeating this procedure, we see that the time needed to reach 0, starting from M, is bounded by

$$\alpha T + \sum_{i=1}^{\alpha} \eta_i,$$

where the r. v.'s η_i are i. i. d., with the same law as η, globally independent of α. Now the total time needed to descend from $+\infty$ to 0 is bounded by

$$\alpha T + \sum_{i=0}^{\alpha} \eta_i,$$

whose expectation is $T/p + (1 + 1/p)\mathbb{E}(\eta) < \infty$. ∎

6.2 Height of the continuous tree

We consider again the process $\{Z_t^x,\ t \ge 0\}$ solution of (3.1) :

$$dZ_t^x = [\theta Z_t^x - \gamma(Z_t^x)^\alpha]\,dt + 2\sqrt{Z_t^x}dW_t,\ Z_0^x = x,$$

with $\theta \ge 0$, $\gamma > 0$, $\alpha > 0$, and define $T^x = \inf\{t > 0,\ Z_t^x = 0\}$.

We first prove

Theorem 6.2 *If $0 < \alpha < 1$, $0 < \mathbb{P}(T^x = \infty) < 1$ if $\theta > 0$, while $T^x < \infty$ a. s. if $\theta = 0$.*

If $\alpha = 1$, $T^x < \infty$ a. s. if $\gamma \ge \theta$, while $0 < \mathbb{P}(T^x = \infty) < 1$ if $\gamma < \theta$.

If $\alpha > 1$, $T^x < \infty$ a. s.

Proof: Clearly, if $\theta > 0$ and $\alpha < 1$, then for large values of Z_t^x, the nonlinear term $-\gamma(Z_t^x)^\alpha$ is negligible with respect to the linear term θZ_t^x, hence the process behaves as in the super critical branching case : both extinction in finite time, and infinite time survival happen with positive probability. If however $\theta = 0$, then the process goes extinct in finite time a. s., since on the interval $[1, \infty)$ the process is bounded from above by the Brownian motion with constant negative drift (equal to $-\gamma$), which comes back to 1 as many times as necessary, until it hits 0, hence $T^x < \infty$ a. s.

In case $\alpha = 1$ we have a continuous branching process, whose behavior is well–known.

In case $\alpha > 1$, the non linear term $-\gamma(Z_t^x)^\alpha$ dominates for large values of Z_t^x, hence the process comes back to 1 as many times as necessary, until it hits 0, hence $T^x < \infty$ a. s. ∎

We now establish the large x behaviour of T^x.

Theorem 6.3 *It $\alpha \le 1$, then $T^x \to \infty$ a. s., as $x \to \infty$.*

Proof: The result is equivalent to the fact that the time to reach 1, starting from x, goes to ∞ as $x \to \infty$. But when $Z_t^x \ge 1$, a comparison of SDEs for various values of α shows that it suffices to consider the case $\alpha = 1$. But in that case, T^n is the maximum of the extinction times of n mutually independent copies of Z_t^1, hence the result. ∎

Theorem 6.4 *If $\alpha > 1$, then $\mathbb{E}\left[\sup_{x>0} T^x\right] < \infty$.*

Proof: It follows from the Itô formula that the process $Y_t^x := \sqrt{Z_t^x}$ solves the SDE

$$dY_t^x = \left[\frac{\theta}{2}Y_t^x - \frac{\gamma}{2}(Y_t^x)^{2\alpha-1} - \frac{1}{8Y_t^x}\right]dt + dW_t,\ Y_0^x = \sqrt{x}.$$

By a well–known comparison theorem, $Y_t^x \le U_t^x$, where U_t^x solves

$$dU_t^x = \left[\frac{\theta}{2}U_t^x - \frac{\gamma}{2}(U_t^x)^{2\alpha-1}\right] dt + dW_t,\ U_0^x = \sqrt{x}.$$

The result now follows readily from Theorem 6.1, since $\alpha > 1$ implies that $2\alpha - 1 > 1$. ■

6.3 Length of the continuous tree

Recall that in the continuous case, the length of the genealogical tree is given as

$$S^x = \int_0^{T^x} Z_t^x dt.$$

For fixed values of x, S^x is finite iff T^x is finite (remind that $T^x = \infty$ requires that $Z_t^x \to \infty$ as $t \to \infty$), hence the result of Theorem 6.2 translates immediately into a result for S^x. We next consider the limit of S^x as $x \to \infty$. Consider the additive functional

$$A_t = \int_0^t Z_s^x ds,\ t \ge 0,$$

and the associated time change

$$\eta(t) = \inf\{s > 0,\ A_s > t\}.$$

We now define $U_t^x = Z^x \circ \eta(t)$, $t \ge 0$. It is easily seen that the process U^x solves the SDE

$$dU_t^x = \left[\theta - \gamma(U_t^x)^{\alpha-1}\right] dt + 2dW_t,\ U_0^x = x. \tag{6.5}$$

Let $\tau^x := \inf\{t > 0,\ U_t^x = 0\}$. It follows from the above that $\eta(\tau^x) = T^x$, hence $S^x = \tau^x$.

We have the following results.

Theorem 6.5 *If $\alpha \le 2$, then $S^x \to \infty$ a. s. as $x \to \infty$.*

Proof: $\alpha \le 2$ means $\alpha - 1 \le 1$. The same argument as in Theorem 6.3 implies that it suffices to consider the case $\alpha = 2$. But in that case equation (6.5) has the explicit solution

$$U_t^x = e^{-\gamma t}x + \int_0^t e^{-\gamma(t-s)}[\theta ds + 2dW_s],$$

hence

$$S^x = \inf\left\{t > 0,\ \int_0^t e^{\gamma s}(\theta ds + 2dW_s) \le -x\right\},$$

which clearly goes to infinity, as $x \to \infty$. ■

Theorem 6.6 *If* $\alpha > 2$*, then* $\mathbb{E}\left[\sup_{x>0} S^x\right] < \infty$.

PROOF: This theorem follows readily from Theorem 6.1 applied to the U^x–equation (6.5), since $\alpha > 2$ means $\alpha - 1 > 1$. ■

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