Spatial SIR epidemic model with varying infectivity without movement of individuals: Law of Large Numbers

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Abstract. In this work, we use a new approach to study the spread of an infectious disease. Indeed, we study a SIR epidemic model with variable infectivity, where the individuals are distributed over a compact subset D of $\mathbb{R}^d$. We define empirical measures which describe the evolution of the state (susceptible, infectious, recovered) of the individuals in the various locations, and the total force of infection in the population. In our model, the individuals do not move. We establish a law of large numbers for these measures, as the population size tends to infinity.

1. Introduction

Epidemic models using ordinary differential equations have been the subject of much research in recent years. Anderson and Britton [2], Britton and Pardoux [6] have shown that these models are limits, when the population size tends towards infinity, of stochastic Markovian models. In particular, the Markovian nature of this model implies that the duration of infection is exponentially distributed, which is unrealistic for most epidemics.

As a result, models with non-exponential infection durations have attracted some interest, see in particular [14] and [17]. Kermack and McKendrick [10] also considered that the infectivity should be a function which varies with the time since infection. The duration of infection is the time taken by this function to vanish out definitively; its law is completely arbitrary. In [7], the authors have established the law of large numbers for the SIR model with variable infectivity, where the infectivity varies from one individual to another and depends upon the time elapsed since infection. They assume that the infectivity function has a finite number of jumps, and satisfy an assumption of uniform continuity between jumps. In [8], the same law of large numbers is established under a weaker assumption: infectivity functions have their trajectories in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, and are bounded by a constant. However, in the various models studied above, the authors ignore the fact that a population extends over a spatial region. Yet, spatial heterogeneity, habitat connectivity and movement rates play an important role in the evolution of infectious diseases. Both deterministic and stochastic models have been used to understand the importance of the movement of individuals in a population on the spread of infectious diseases, on the persistence or extinction of an endemic disease, for example [1], [9] and [12]. Some Markovian models in this framework have been studied in [4]. They studied a stochastic SIR compartmental epidemic model for a population which moves on a torus ($\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$) according to Stochastic Differential Equations driven by independent Brownian motions. They define sequences of empirical measures that describe the evolution of the positions of susceptible, infected and recovered individuals. They establish large-population approximations of these sequences of measures. In [16], the authors consider a population distributed in the space $\mathbb{R}^d$ whose individuals are characterized by: a position and an infection state, interact and move in $\mathbb{R}^d$. An epidemic model combining spatial structure and variable infectivity would be more realistic.

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This is the focus of our work. As a result, we are considering a population distributed over a compact subset D of $\mathbb{R}^d$; and use the same type of arguments as in [8]. We define sequences of empirical measures which describe the evolution of the positions of susceptible, infected and recovered individuals, and establish the law of large numbers for these measures. In this paper, we restrict us to the case where the individuals do not move. Note however that infectious individuals can infect susceptible individuals located far away. This is a way to take into account movements of individuals (daily from home to work, or occasionally for vacation and visits to the family), without modeling those movements explicitly.

The same model, with diffusive movement of the individuals will be considered in another publication.

1.1. **Notations.** We note

- $\mathcal{M}$: The set of finite positive measures on D which we equip with the weak convergence topology;
- $\mathbb{D} := \mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$: The space of càdlàg functions defined on $\mathbb{R}_+$ with values in $\mathbb{R}_+$;
- $\mathbb{D}_{\mathcal{M}}$: The space of càdlàg functions defined on $\mathbb{R}_+$ with values in $\mathcal{M}$.
- $\forall \varphi \in C_b(D)$, $\forall \mu \in \mathcal{M}$, $(\mu, \varphi) = \displaystyle\int_D \varphi(y)\mu(dy)$;
- $C(y, l_y, \alpha) = \{z \in \mathbb{R}^d : z \neq y \text{ and } (z - y, l_y) \geq \alpha\|z - y\|, \alpha \in [0; 1)\}$.

c and C denote positive constants that can change from line to line.

2. Model description

The epidemic model studied here is the SIR model in the spatial framework with variable infectivity; the letters S, I and R represent the different states of an individual (”susceptible”,”infected” and ”recovered” respectively). The SIR model states that a susceptible individual can become infected, and finally recovered when he/she recovers from the disease. In our spatial model, an individual is characterized by its state $E \in \{S, I, R\}$ and its position X, a continuous variable with values in D which is a compact subset of $\mathbb{R}^d$. To simplify the mathematical description, we identify the S, I and R states as 0, 1 and 2 respectively. The space of individuals is therefore $D \times \{0, 1, 2\}$. We consider a population of fixed size N; and we assume that at time t=0 the population is divided into three subsets: those susceptible , there are $S^N(0)$ of them, those infected, there are $I^N(0)$ of them, and those removered, there are $R^N(0)$ of them i.e $S^N(0) + I^N(0) + R^N(0) = N$. We denote by $\{X^i, i \in \{\mathfrak{S}, \mathfrak{I}, \mathfrak{R}\}\}$ the positions of the individuals at time $t = 0$, where $(\mathfrak{S}, \mathfrak{I}, \mathfrak{R})$ forms a partition of $\mathbf{B} := \{1, \cdots, N\}$ with $Card(\mathfrak{S}) = S^N(0)$, $Card(\mathfrak{I}) = I^N(0)$ and $Card(\mathfrak{R}) = R^N(0)$. Now let us consider $\{\lambda_{-j}, j \geq 1\}$ and $\{\lambda_j, j \geq 1\}$ two mutually independent sequences of i.i.d random elements of $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$. $\lambda_{-j}(t)$ is the infectivity at time t of the $j-th$ initially infected individual and $\lambda_j(t)$ is the infectivity at time t after its infection of the $j-th$ initially susceptible individual. We assume that there exists a deterministic constant $\lambda^* > 0$ such that $0 \leq \lambda_j(t) \leq \lambda^*$ a.s, for all $j \in \mathbb{Z}^*$ and $t \geq 0$, with the convention: $\forall j \in \mathbb{Z}^*$, $\lambda_j(t) = 0$ if $t < 0$ and we shall use the notations $\overline{\lambda}^0(t) = \mathbb{E}(\lambda_{-1}(t))$ and $\overline{\lambda}(t) = \mathbb{E}(\lambda_1(t))$. It is natural that an infected individual is more likely to infect a close neighbor than a more distant one. While these different transmission behaviors are averaged in homogeneous SIR models, in our model we use an infection rate that depends on the relative positions of the two parties. The infection rate between two positions will be given by the function K defined on $D \times D$ with values in $[0, 1]$. A susceptible individual i becomes infected (in other words, his/her state changes from 0 to 1) at time t at rate

$$\frac{1}{N^{1-\gamma}}\left[\sum_{j\in\mathfrak{I}}\frac{K(X^i,X^j)}{\left[\sum_{\ell\in\mathbf{B}}K(X^\ell,X^j)\right]^\gamma}\lambda_{-j}(t)+\sum_{j\in\mathfrak{S}}\frac{K(X^i,X^j)}{\left[\sum_{\ell\in\mathbf{B}}K(X^\ell,X^j)\right]^\gamma}\lambda_j(t-\tau_j^N)\right],\quad \gamma\in[0,1] \tag{2.1}$$

where τ_j^N is the infection time of the initially susceptible individual j . At time $t\geq 0$, we define the following measures:

$$\begin{aligned}
\mu_t^{S,N} &= \sum_{i\in\mathfrak{S}}\mathbf{1}_{E_t^{i,N}=0}\delta_{X^i}=\sum_{i\in\mathfrak{S}}\mathbf{1}_{E_0^i=0}\delta_{X^i}-\sum_{i\in\mathfrak{S}}\mathbf{1}_{t\geq\tau_i^N}\delta_{X^i}=\mu_0^{S,N}-\sum_{i\in\mathfrak{S}}\mathbf{1}_{t\geq\tau_i^N}\delta_{X^i}\\
\mu_t^{I,N} &= \sum_{i\in\mathfrak{I}}\mathbf{1}_{E_t^i=1}\delta_{X^i}+\sum_{i\in\mathfrak{S}}\mathbf{1}_{E_t^{i,N}=1}\delta_{X^i}\\
&= \mu_0^{I,N}-\sum_{i\in\mathfrak{I}}\mathbf{1}_{\eta_{-i}\leq t}\delta_{X^i}+\sum_{i\in\mathfrak{S}}\mathbf{1}_{t\leq\tau_i^N}\delta_{X^i}-\sum_{i\in\mathfrak{S}}\mathbf{1}_{t\geq\tau_i^N+\eta_i}\delta_{X^i}\\
\mu_t^{R,N} &= \sum_{i\in\mathfrak{R}}\mathbf{1}_{E_0^i=2}\delta_{X^i}+\sum_{i\in\mathfrak{I}}\mathbf{1}_{E_t^i=2}\delta_{X^i}+\sum_{i\in\mathfrak{S}}\mathbf{1}_{E_t^{i,N}=2}\delta_{X^i}\\
&= \mu_0^{R,N}+\sum_{i\in\mathfrak{I}}\mathbf{1}_{\eta_{-i}\leq t}\delta_{X^i}+\sum_{i\in\mathfrak{S}}\mathbf{1}_{\tau_i^N+\eta_i\leq t}\delta_{X^i}\\
\mu_t^{N} &= \mu_t^{S,N}+\mu_t^{I,N}+\mu_t^{R,N}=\sum_{i\in\mathfrak{S}}\mathbf{1}_{E_0^i=0}\delta_{X^i}+\sum_{i\in\mathfrak{I}}\mathbf{1}_{E_0^i=1}\delta_{X^i}+\sum_{i\in\mathfrak{R}}\mathbf{1}_{E_0^i=2}\delta_{X^i}\\
&= \sum_{i\in\mathbf{B}}\delta_{X^i}:=\mu^N\\
\mu_t^{\mathfrak{F},N} &= \sum_{i\in\mathfrak{I}}\lambda_{-i}(t)\delta_{X^i}+\sum_{i\in\mathfrak{S}}\lambda_i(t-\tau_i^N)\delta_{X^i}.
\end{aligned} \tag{2.2}$$

where

- $\mu_t^{S,N}$ is the empirical measure of susceptible individuals at time t in a population of size N;
- $\mu_t^{\mathfrak{F},N}$ is the empirical measure of the total force of infection;
- $\mu_t^{I,N}$ is the empirical measure of infected individuals at time t in a population of size N;
- $\mu_t^{R,N}$ is the empirical measure of individuals recovered at time t in a population of size N;
- μ^N is the empirical measure of the total population, which does not depend upon t

Now, we define $\overline{\mu}_t^{S,N}:=\frac{1}{N}\mu_t^{S,N}$; $\overline{\mu}_t^{I,N}:=\frac{1}{N}\mu_t^{I,N}$; $\overline{\mu}_t^{R,N}:=\frac{1}{N}\mu_t^{R,N}$; $\overline{\mu}^{N}:=\frac{1}{N}\mu^{N}$ and $\overline{\mu}_t^{\mathfrak{F},N}:=\frac{1}{N}\mu_t^{\mathfrak{F},N}$. We rewrite (2.1) as follows

$$\mathbf{1}_{E_t^{i,N}=0}\int_D\frac{K(X^i,y)}{\left[\int_D K(z,y)\overline{\mu}^N(dz)\right]^\gamma}\overline{\mu}_t^{\mathfrak{F},N}(dy),\quad \gamma\in[0,1].$$

Let us discuss the role of our parameter $\gamma\in[0,1]$. In the case $\gamma=1$, if we sum the above expression over i, and fix the parameter y, we man think that the obtained ratio is the probability that an individual encountered from the position y is susceptible. The reason for the values $\gamma<1$ is to allow the abundance of individuals around y to play a role in the rate of infections from y.

Let η_j be the random variable defined by $\eta_j:=\sup\{t>0,\ \lambda_j(t)>0\}\quad\forall j\in\mathbb{Z}^*$. The two sequences of random variables $\{\eta_{-j},j\geq 1\}$ and $\{\eta_j,j\geq 1\}$ are i.i.d and globally independent of each other. $F(t):=\mathbb{P}(\eta_1\leq t)$ and $F_0(t):=\mathbb{P}(\eta_{-1}\leq t)$ are the distribution functions of η_j for

$j \in \mathbb{Z}_+$ and for $j \in \mathbb{Z}_-$, respectively. For $1 \le i \le S^N(0)$, consider a counting process $A_i^N(t)$, which takes the value 0 when individual i is not yet infected at time t, and takes the value 1 when the latter has been infected by time t. Thus, $\tau_i^N := \inf\{t > 0,\ A_i^N(t) = 1\}$. We define A_i^N as follows :

$$A_i^N(t) = \int_0^t \int_0^\infty \mathbf{1}_{A_i^N(s^-)=0} \mathbf{1}_{u \le \overline{\Gamma}^N(s,X^i)} P^i(ds,du),$$

where $\overline{\Gamma}^N(t,x) = \int_D \frac{K(x,y)}{\left[\int_D K(z,y)\overline{\mu}^N(dz)\right]^\gamma} \overline{\mu}_t^{\mathfrak{F},N}(dy)$, $\gamma \in [0;1]$ and the $\{P^i,\ i \ge 1\}$ are standard Poisson random measures on $\mathbb{R}_+^2$ which are mutually independent.

The next proposition follows readily from our model

Proposition 2.1 *For all $\varphi \in C_b(D)$, $\left\{\overline{\mu}_t^{S,N}, \overline{\mu}_t^{\mathfrak{F},N}, \overline{\mu}_t^{I,N}, \overline{\mu}_t^{R,N},\ t \ge 0\right\}$ satisfies*

$$\begin{cases} (\overline{\mu}_t^{S,N},\varphi) = (\overline{\mu}_0^{S,N},\varphi) - \frac{1}{N}\sum_{i\in\mathfrak{S}} \varphi(X^i) A_i^N(t) \\ (\overline{\mu}_t^{\mathfrak{F},N},\varphi) = \frac{1}{N}\sum_{i=1}^{I^N(0)} \lambda_{-i}(t)\varphi(X^i) + \frac{1}{N}\sum_{i=1}^{S^N(0)} \lambda_i(t-\tau_i^N)\varphi(X^i) \\ (\overline{\mu}_t^{I,N},\varphi) = (\overline{\mu}_0^{I,N},\varphi) + \frac{1}{N}\sum_{i\in\mathfrak{S}} \varphi(X^i)A_i^N(t) - \frac{1}{N}\sum_{i\in\mathfrak{I}} \varphi(X^{-i})\mathbf{1}_{\eta_{-i}\le t} \\ \qquad\qquad - \frac{1}{N}\sum_{i\in\mathfrak{S}} \varphi(X^i)\int_0^t \mathbf{1}_{\eta_i \le t-s} dA_i^N(s) \\ (\overline{\mu}_t^{R,N},\varphi) = (\overline{\mu}_0^{R,N},\varphi) + \frac{1}{N}\sum_{i\in\mathfrak{I}} \varphi(X^{-i})\mathbf{1}_{\eta_{-i}\le t} + \frac{1}{N}\sum_{i\in\mathfrak{S}} \varphi(X^i)\int_0^t \mathbf{1}_{\eta_i\le t-s} dA_i^N(s) \end{cases} \tag{2.3}$$

3. Law of large numbers of measures

In this section, we determine the limits of the empirical measures defined in section 2 when the population size tends to infinity. Necessary intermediate results are established; they are summarized in lemmas and propositions. In what follows, we are given a probability measure $\overline{\mu}$ on D, with the density $\overline{\mu}(z)$.

Assumption 3.1 *In the following, we assume that:*

- $\exists \alpha, r > 0, \forall y \in \partial D\ \exists l_y$ *such that* $C(y,l_y,\alpha) \cap B(y,r) \subseteq D$ *and* $\int_{C(y,l_y,\alpha)\cap B(y,r)} dz = m > 0$
- *The function K is continous on $D \times D$ and $\forall x, y \in D$, $\exists r, \underline{c} > 0$ such that* $\|x-y\| \le r \implies K(x,y) \ge \underline{c}$
- $\sup_{x\in D} \overline{\mu}(x) < \infty$*; and* $\inf_{x\in D} \overline{\mu}(x) > 0$
- $\overline{S}^N(0) := \frac{S^N(0)}{N} \to \overline{S}(0)$*; and* $I^N(0) := \frac{I^N(0)}{N} \to \overline{I}(0)$*;*
- $\left(X^i, i \in \mathfrak{S}\right)$ *i.i.d with density function π_S, and have same law as $X_{\mathfrak{S}}$; $\left(X^i, i\in\mathfrak{I}\right)$ i.i.d with density function π_I, and have same law as $X_{\mathfrak{I}}$; and $\left(X^i, i\in\mathfrak{R}\right)$ i.i.d with density function π_R, and have same law as $X_{\mathfrak{R}}$;*
- $\left(X^i, i\in\mathfrak{S}\right)$, $\left(X^i, i\in\mathfrak{I}\right)$, *and* $\left(X^i, i\in\mathfrak{R}\right)$ *are independent.*

Theorem 3.1 *Under assumption 3.1, the sequence $(\overline{\mu}^{S,N}, \overline{\mu}^{\mathfrak{F},N}, \overline{\mu}^{I,N}, \overline{\mu}^{R,N})_{N\ge1}$ converges in probability in $\mathbb{D}^4_{\mathcal{M}}$ to $(\overline{\mu}^S, \overline{\mu}^{\mathfrak{F}}, \overline{\mu}^I, \overline{\mu}^R)$ such that for all $\varphi \in C_b(D)$, $\left\{(\overline{\mu}_t^S,\varphi), (\overline{\mu}_t^{\mathfrak{F}},\varphi), (\overline{\mu}_t^I,\varphi), (\overline{\mu}_t^R,\varphi), t\ge 0\right\}$*

satisfies

$$\begin{cases} (\overline{\mu}_t^S, \varphi) = (\overline{\mu}_0^S, \varphi) - \displaystyle\int_0^t \int_D \varphi(x)\overline{\Gamma}(s,x)\overline{\mu}_s^S(dx)ds, \\ (\overline{\mu}_t^{\mathfrak{F}}, \varphi) = \overline{\lambda}^0(t)(\overline{\mu}_0^I, \varphi) + \displaystyle\int_0^t \overline{\lambda}(t-s) \int_D \varphi(x)\overline{\Gamma}(s,x)\overline{\mu}_s^S(dx)ds, \\ (\overline{\mu}_t^I, \varphi) = (\overline{\mu}_0^I, \varphi)F_0^c(t) + \displaystyle\int_0^t F^c(t-s) \int_D \varphi(x)\overline{\Gamma}(s,x)\overline{\mu}_s^S(dx)ds, \\ (\overline{\mu}_t^R, \varphi) = (\overline{\mu}_0^R, \varphi) + (\overline{\mu}_0^I, \varphi)F_0(t) + \displaystyle\int_0^t F(t-s) \int_D \varphi(x)\overline{\Gamma}(s,x)\overline{\mu}_s^S(dx)ds, \\ \overline{\Gamma}(t,x) = \displaystyle\int_D \frac{K(x,y)}{\left[\int_D K(z,y)\overline{\mu}(dz)\right]^\gamma}\overline{\mu}_t^{\mathfrak{F}}(dy), \quad \gamma \in [0;1], \\ \overline{\mu}_0^S(dx) = \overline{S}(0)\pi_S(x)dx; \ \overline{\mu}_0^I(dx) = \overline{I}(0)\pi_I(x)dx; \ \overline{\mu}_0^R(dx) = \overline{R}(0)\pi_R(x)dx, \\ \qquad \overline{\mu}_t = \overline{\mu}_t^S + \overline{\mu}_t^I + \overline{\mu}_t^R = \overline{\mu}, \\ \overline{\mu}_t(dx) = \overline{\mu}(dx) = \overline{\mu}_0^S(dx) + \overline{\mu}_0^I(dx) + \overline{\mu}_0^R(dx), \forall t \geq 0. \end{cases} \tag{3.1}$$

We will first establish the next result

Proposition 3.1 *For all $t \in [0;T]$, the above system admits an unique solution $\left(\overline{\mu}_t^S, \overline{\mu}_t^{\mathfrak{F}}, \overline{\mu}_t^I, \overline{\mu}_t^R\right)$ which is absolutely continuous with respect to the Lebesgue measure, with the densities $\left(\overline{\mu}^S(t,.), \overline{\mu}^{\mathfrak{F}}(t,.), \overline{\mu}^I(t,.), \overline{\mu}^R(t,.)\right)$ satisfying for all $x \in D$*

$$\begin{cases} \overline{\mu}^S(t,x) = \overline{\mu}^S(0,x) - \displaystyle\int_0^t \overline{\Gamma}(s,x)\overline{\mu}^S(s,x)ds \\ \overline{\mu}^{\mathfrak{F}}(t,x) = \overline{\lambda}^0(t)\mu^I(0,x) + \displaystyle\int_0^t \overline{\lambda}(t-s)\overline{\Gamma}(s,x)\overline{\mu}^S(s,x)ds. \\ \overline{\mu}^I(t,x) = \mu^I(0,x)F_0^c(t) + \displaystyle\int_0^t F^c(t-s)\overline{\Gamma}(s,x)\overline{\mu}^S(s,x)ds \\ \overline{\mu}^R(t,x) = \mu^R(0,x) + \mu^I(0,x)F_0(t) + \displaystyle\int_0^t F(t-s)\overline{\Gamma}(s,x)\overline{\mu}^S(s,x)ds \\ \overline{\Gamma}(t,x) = \displaystyle\int_D \frac{K(x,y)}{\left[\int_D K(z,y)\overline{\mu}(dz)\right]^\gamma}\overline{\mu}^{\mathfrak{F}}(t,y)dy, \quad \gamma \in [0;1] \\ \overline{\mu}^S(0,x) = \overline{S}(0)\pi_S(x); \overline{\mu}^I(0,x) = \overline{I}(0)\pi_I(x); \overline{\mu}^R(0,x) = \overline{R}(0)\pi_R(x); \\ \qquad \overline{\mu}(x) = \overline{\mu}^S(0,x) + \overline{\mu}^I(0,x) + \overline{\mu}^R(0,x). \end{cases} \tag{3.2}$$

Admitting for a moment the first part of Proposition 3.1, we first establish the following a priori estimates.

Proposition 3.2 *Let $T > 0$, and let $(\overline{\mu}^S, \overline{\mu}^{\mathfrak{F}})$ be a solution of the first two equations of (3.2). Then there exists a positive constant C such that:*

- $\forall t \in [0;T]$, $\|\overline{\mu}^S(t,.)\|_\infty \leq C$;
- $\inf\limits_{y \in D} \displaystyle\int_D K(z,y)\overline{\mu}(z)dz > \frac{1}{C}$;
- $\forall t \in [0;T]$; $\|\overline{\mu}^{\mathfrak{F}}(t,.)\|_\infty \leq C$.

Proof. Let $t \in [0;T]$ and $\forall x \in D$.

$$\overline{\mu}^S(t,x) \leq \overline{\mu}^S(0,x) \leq \overline{\mu}(x)$$

$$\|\overline{\mu}^S(t,.)\|_\infty \leq \|\overline{\mu}(.)\|_\infty := C. \tag{3.3}$$

$$\int_D K(z,y)\overline{\mu}(dz) = \int_D K(z,y)\overline{\mu}(z)dz \geq \inf_{z\in D}\overline{\mu}(z)\int_D K(z,y)dz$$

$$\geq \inf_{z\in D}\overline{\mu}(z)\int_{D\cap B(y,r)} K(z,y)dz \geq \underline{c}\inf_{z\in D}\overline{\mu}(z)\int_{D\cap B(y,r)} dz$$

$$\int_D K(z,y)\overline{\mu}(dz) \geq c\int_{C(y,l_y,\alpha)\cap B(y,r)} dz = cm.$$

We have shown that $\inf_{y\in D}\int_D K(z,y)\overline{\mu}(dz) > 0$. Next

$$\overline{\mu}^{\mathfrak{F}}(t,x) = \overline{\lambda}^0(t)\mu^I(0,x) + \int_0^t \overline{\lambda}(t-s)\overline{\Gamma}(s,x)\overline{\mu}^S(s,x)ds$$

$$\overline{\mu}^{\mathfrak{F}}(t,x) \leq \lambda^*\overline{\mu}(x) + \lambda^*\int_0^t \overline{\Gamma}(s,x)\overline{\mu}^S(s,x)ds$$

$$\|\overline{\mu}^{\mathfrak{F}}(t,.)\|_\infty \leq \lambda^* C + \lambda^*\int_0^t \|\overline{\Gamma}(s,.)\overline{\mu}^S(s,.)\|_\infty ds. \tag{3.4}$$

$$\overline{\mu}^S(t,x)\overline{\Gamma}(t,x) = \int_D \frac{K(x,y)\overline{\mu}^S(t,x)}{\left[\int_D K(z,y)\overline{\mu}(z)dz\right]^\gamma}\overline{\mu}^{\mathfrak{F}}(t,y)dy$$

$$\leq \frac{C}{(cm)^\gamma}\|\overline{\mu}^{\mathfrak{F}}(t,.)\|_\infty\int_D K(x,y)dy$$

Since K is continuous and D compact, $\sup_{x\in D}\int_D K(x,y)dy < \infty$. Thus,

$$\|\overline{\mu}^S(t,.)\overline{\Gamma}(t,.)\|_\infty \leq C\|\overline{\mu}^{\mathfrak{F}}(t,.)\|_\infty \tag{3.5}$$

From (3.4) and (3.5), we deduce that $\forall t \in [0;T]$

$$\|\overline{\mu}^{\mathfrak{F}}(t,.)\|_\infty \leq \lambda^* C + \lambda^* C\int_0^t \|\overline{\mu}^{\mathfrak{F}}(s,.)\|_\infty ds,$$

which combined with Gronwall's inequality yields

$$\|\overline{\mu}^{\mathfrak{F}}(t,.)\|_\infty \leq \lambda^* C e^{\lambda^* CT}, \quad \forall t \in [0;T] \tag{3.6}$$

□

Proof. ***Proposition** 3.1*. We first show that for all $t \geq 0$ any solution of (3.1), $\left(\overline{\mu}_t^S, \overline{\mu}_t^{\mathfrak{F}}, \overline{\mu}_t^I, \overline{\mu}_t^R\right)$ is absolutely continuous with respect to the Lebesgue measure, and the densities $\left(\overline{\mu}^S(t,.), \overline{\mu}^{\mathfrak{F}}(t,.), \overline{\mu}^I(t,.), \overline{\mu}^R(t,.)\right)$ verify (3.2).

From the first equation of (3.1), $\overline{\mu}_t^S \leq \overline{\mu}_0^S$. Since $\overline{\mu}_0^S$ is absolutely continuous, $\overline{\mu}_t^S$ has the same property, and we denote its density by $\overline{\mu}^S(t,x)$.

From the third equation of (3.1), $\overline{\mu}_t^I \leq \overline{\mu}_0^I + \int_0^t \overline{\Gamma}(s,.)\overline{\mu}_s^S ds$, thus $\overline{\mu}_t^I$ is absolutely continuous , since $\overline{\mu}_0^I$ is absolutely continuous, as well as $\overline{\mu}_s^S$ for all s. The same argument applies to $\overline{\mu}_t^{\mathfrak{F}}$ and $\overline{\mu}_t^R$.The system of equation (3.2) now follows readily from (3.1).

We will verify that $\left(\overline{\mu}^S(t,.), \overline{\mu}^{\mathfrak{F}}(t,.), \overline{\mu}^I(t,.), \overline{\mu}^R(t,.)\right)$ is unique. For that sake, it suffices to show that the solution $(\overline{\mu}^S(t,.), \overline{\mu}^{\mathfrak{F}}(t,.))$ of the first two equations of the system is unique. The first two equations of the system (3.2) constitute the following system

$$\begin{cases} \overline{\mu}^S(t,x) = \overline{\mu}^S(0,x) - \displaystyle\int_0^t \int_D \frac{K(x,y)}{\left[\int_D K(z,y)\overline{\mu}(dz)\right]^\gamma} \overline{\mu}^{\mathfrak{F}}(s,y)\overline{\mu}^S(s,x)dyds \\ \overline{\mu}^{\mathfrak{F}}(t,x) = \overline{\lambda}^0(t)\mu^I(0,x) + \displaystyle\int_0^t \overline{\lambda}(t-s) \int_D \frac{K(x,y)}{\left[\int_D K(z,y)\overline{\mu}(dz)\right]^\gamma} \overline{\mu}^{\mathfrak{F}}(s,y)\overline{\mu}^S(s,x)dyds. \\ \overline{\mu}^S(0,x) = \overline{S}(0)\pi_S(x); \overline{\mu}^I(0,x) = \overline{I}(0)\pi_I(x), \text{and} \quad \overline{\mu}^R(0,x) = \overline{R}(0)\pi_{\mathfrak{R}}(x) \\ \quad \overline{\mu}(x) = \overline{\mu}^S(0,x) + \overline{\mu}^I(0,x) + \overline{\mu}^R(0,x). \end{cases}$$

Let $(f_1(t,.), g_1(t,.))$ and $(f_2(t,.), g_2(t,.))$ be two solutions of the above system with the same initial condition.

On the one hand

$$\begin{aligned} f_1(t,x) - f_2(t,x) &= \int_0^t (f_2(s,x) - f_1(s,x)) \int_D \frac{K(x,y)}{\left[\int_D K(z,y)\overline{\mu}(dz)\right]^\gamma} g_2(s,y)dyds \\ &\quad + \int_0^t f_1(s,x) \int_D \frac{K(x,y)}{\left[\int_D K(z,y)\overline{\mu}(dz)\right]^\gamma} (g_2(s,y) - g_1(s,y))\, dyds \\ \|f_1(t,.) - f_2(t,.)\|_\infty &\leq \left(\frac{C}{c}\right)^\gamma \int_0^t \|f_2(s,.) - f_1(s,.)\|_\infty \|g_2(s,.)\|_\infty \left\| \int_D K(.,y)dy \right\|_\infty ds \\ &\quad + \left(\frac{C}{c}\right)^\gamma \int_0^t \|g_2(s,.) - g_1(s,.)\|_\infty \|f_1(s,.)\|_\infty \left\| \int_D K(.,y)dy \right\|_\infty ds \\ \|f_1(t,.) - f_2(t,.)\|_\infty &\leq C \int_0^t \left(\|f_2(s,.) - f_1(s,.)\|_\infty + \|g_2(s,.) - g_1(s,.)\|_\infty\right) ds \end{aligned} \tag{3.7}$$

Moreover,

$$\begin{aligned} g_1(t,x) - g_2(t,x) &= \int_0^t \overline{\lambda}(t-s)(f_1(s,x) - f_2(s,x)) \int_D \frac{K(x,y)}{\left[\int_D K(z,y)\overline{\mu}(dz)\right]^\gamma} g_1(s,y)dyds \\ &\quad + \int_0^t \overline{\lambda}(t-s) f_1(s,x) \int_D (g_1(s,y) - g_2(s,y)) \frac{K(x,y)}{\left[\int_D K(z,y)\overline{\mu}(dz)\right]^\gamma} dyds \\ |g_1(t,x) - g_2(t,x)| &\leq C \int_0^t \|f_1(s,.) - f_2(s,.)\|_\infty \left\| \int_D \frac{K(.,y)}{\left[\int_D K(z,y)\overline{\mu}(dz)\right]^\gamma} dy \right\|_\infty ds \\ &\quad + C \int_0^t \|g_1(s,.) - g_2(s,.)\|_\infty \left\| \int_D \frac{K(.,y)}{\left[\int_D K(z,y)\overline{\mu}(dz)\right]^\gamma} dy \right\|_\infty ds \\ \|g_1(t,.) - g_2(t,.)\|_\infty &\leq C \int_0^t \left(\|f_1(s,.) - f_2(s,.)\|_\infty + \|g_1(s,.) - g_2(s,.)\|_\infty\right) ds \end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we have

$$\|g_1(t,.) - g_2(t,.)\|_\infty + \|f_1(t,.) - f_2(t,.)\|_\infty \leq C \int_0^t \left(\|g_1(s,.) - g_2(s,.)\|_\infty + \|f_1(s,.) - f_2(s,.)\|_\infty\right) ds$$

Using Gronwall's inequality, we obtain

$$\|g_1(t,.) - g_2(t,.)\|_\infty + \|f_1(t,.) - f_2(t,.)\|_\infty = 0.$$

□

Let Φ be the continuous function for $\mathbb{R}_+$ into $\mathbb{R}_+$ defined by $\Phi(x) := \left(x \vee \frac{c}{2}\right)^\gamma$.
Now, define a variant $\tilde{A}_i^N$ of the process A_i^N by:

$$\tilde{A}_i^N(t) := \int_0^t \int_0^\infty \mathbf{1}_{\tilde{A}_i^N(s^-)=0} \mathbf{1}_{u \le \tilde{\Gamma}^N(s,X^i)} P^i(ds,du)$$

$$\tilde{\Gamma}^N(t,x) := \frac{1}{N}\sum_{j\in\mathfrak{I}} \frac{K(x,X^j)}{\Phi\left(\int_D K(z,X^j)\overline{\mu}^N(dz)\right)}\lambda_{-j}(t) + \frac{1}{N}\sum_{j\in\mathfrak{S}} \frac{K(x,X^j)}{\Phi\left(\int_D K(z,X^j)\overline{\mu}^N(dz)\right)}\lambda_j(t-\tilde{\tau}_j^N),$$

The variant $\left(\tilde{\mu}^{S,N}, \tilde{\mu}^{I,N}, \tilde{\mu}^{R,N}, \tilde{\mu}^{\mathfrak{F},N}\right)$ of $\left(\overline{\mu}^{S,N}, \overline{\mu}^{I,N}, \overline{\mu}^{R,N}, \overline{\mu}^{\mathfrak{F},N}\right)$ verifies, for all $t \in [0;T]$, $\varphi \in C_b(D)$

$$(\tilde{\mu}_t^{S,N},\varphi) = (\overline{\mu}_0^{S,N},\varphi) - \frac{1}{N}\sum_{i\in\mathfrak{S}} \varphi(X^i)\tilde{A}_i^N(t)$$

$$(\tilde{\mu}_t^{I,N},\varphi) = (\overline{\mu}_0^{I,N},\varphi) + \frac{1}{N}\sum_{i\in\mathfrak{S}} \varphi(X^i)\tilde{A}_i^N(t) - \frac{1}{N}\sum_{i\in\mathfrak{I}} \varphi(X^i)\mathbf{1}_{\eta_{-i}\le t} - \frac{1}{N}\sum_{i\in\mathfrak{S}} \varphi(X^i)\int_0^t \mathbf{1}_{\eta_i \le t-s} d\tilde{A}_i^N(s)$$

$$(\tilde{\mu}_t^{R,N},\varphi) = (\overline{\mu}_0^{R,N},\varphi) + \frac{1}{N}\sum_{i\in\mathfrak{I}} \varphi(X^i)\mathbf{1}_{\eta_{-i}\le t} + \frac{1}{N}\sum_{i\in\mathfrak{S}} \varphi(X^i)\int_0^t \mathbf{1}_{\eta_i \le t-s} d\tilde{A}_i^N(s)$$

$$(\tilde{\mu}_t^{\mathfrak{F},N},\varphi) = \frac{1}{N}\sum_{i\in\mathfrak{I}} \lambda_{-i}(t)\varphi(X^i) + \frac{1}{N}\sum_{i\in\mathfrak{S}} \lambda_i(t-\tilde{\tau}_i^N)\varphi(X^i), \quad \tilde{\tau}_i^N := \inf\{t\ge 0; \tilde{A}_i^N(t)=1\}$$

Theorem 3.2 *Under assumption 3.1, the sequence* $(\tilde{\mu}^{S,N}, \tilde{\mu}^{\mathfrak{F},N}, \tilde{\mu}^{I,N}, \tilde{\mu}^{R,N})_{N\ge 1}$ *converges in probability in* $\mathbb{D}^4_{\mathcal{M}}$ *to* $(\tilde{\mu}^S, \tilde{\mu}^{\mathfrak{F}}, \tilde{\mu}^I, \tilde{\mu}^R)$ *such that for all* $\varphi \in C_b(D)$, $\left\{(\tilde{\mu}_t^S,\varphi), (\tilde{\mu}_t^{\mathfrak{F}},\varphi), (\tilde{\mu}_t^I,\varphi), (\tilde{\mu}_t^R,\varphi), t\in[0;T]\right\}$ *satisfies*

$$(\tilde{\mu}_t^S,\varphi) = (\overline{\mu}_0^S,\varphi) - \int_0^t \int_D \varphi(x)\tilde{\Gamma}(s,x)\tilde{\mu}_s^S(dx)ds \tag{3.9}$$

$$(\tilde{\mu}_t^{\mathfrak{F}},\varphi) = \overline{\lambda}^0(t)(\overline{\mu}_0^I,\varphi) + \int_0^t \int_D \overline{\lambda}(t-s)\varphi(x)\tilde{\Gamma}(s,x)\tilde{\mu}_s^S(dx)ds. \tag{3.10}$$

$$(\tilde{\mu}_t^I,\varphi) = (\overline{\mu}_0^I,\varphi)F_0^c(t) + \int_0^t F^c(t-s)\int_D \varphi(x)\tilde{\Gamma}(s,x)\tilde{\mu}_s^S(dx)ds \tag{3.11}$$

$$(\tilde{\mu}_t^R,\varphi) = (\overline{\mu}_0^R,\varphi) + (\overline{\mu}_0^I,\varphi)F_0(t) + \int_0^t F(t-s)\int_D \varphi(x)\tilde{\Gamma}(s,x)\tilde{\mu}_s^S(dx)F(du)ds \tag{3.12}$$

$$\tilde{\Gamma}(t,x) = \int_D \frac{K(x,y)}{\Phi\left(\int_D K(z,y)\overline{\mu}(dz)\right)}\tilde{\mu}_t^{\mathfrak{F}}(dy). \tag{3.13}$$

By the equation (3.9), we have the formula

$$\begin{aligned}\tilde{\mu}_t^S &= \overline{\mu}_0^S - \int_0^t \tilde{\Gamma}(s,.)\tilde{\mu}_s^S ds \\ &= \overline{\mu}_0^S \exp\left(-\int_0^t \tilde{\Gamma}(s,.)ds\right).\end{aligned}$$

For all $\varphi \in C_b(D)$,

$$\begin{aligned}(\tilde{\mu}_t^S,\varphi) &= \left(\overline{\mu}_0^S \exp\left(-\int_0^t \tilde{\Gamma}(s,.)ds\right), \varphi\right) \\ &= \int_{\mathbb{T}^2} \varphi(x)\exp\left(-\int_0^t \tilde{\Gamma}(s,x)ds\right)\overline{\mu}_0^S(dx) \\ &= \overline{S}(0)\int_{\mathbb{T}^2} \varphi(x)\exp\left(-\int_0^t \tilde{\Gamma}(s,x)ds\right)\pi_S(x)dx\end{aligned}$$

$$(\tilde{\mu}_t^S, \varphi) = \overline{S}(0)\mathbb{E}\left(\varphi(X_{\mathfrak{S}})\exp\left(-\int_0^t \tilde{\Gamma}(s, X_{\mathfrak{S}})ds\right)\right)$$

Lemma 3.1 *For all $t \in [0;T]$, the system (3.9)–(3.12) admits an unique solution $\left(\tilde{\mu}_t^S, \tilde{\mu}_t^{\mathfrak{F}}, \tilde{\mu}_t^I, \tilde{\mu}_t^R\right)$ which is absolutely continuous with respect to the Lebesgue measure of densities $\left(\tilde{\mu}^S(t,.), \tilde{\mu}^{\mathfrak{F}}(t,.), \tilde{\mu}^I(t,.), \tilde{\mu}^R(t,.)\right)$.*

Proof. We use the same arguments as in the proof of Proposition 3.1, with the system (3.9)–(3.12) instead of (3.1) □

Now, we associate to the pair (X, P), where X and P are independent, X is an D–valued r.v. whose law is π_S, and P is a standard Poisson random measure on $\mathbb{R}_+^2$, the process $\tilde{A}(t)$ defined by

$$\begin{aligned}
\tilde{A}(t) &= \int_0^t \int_0^\infty \mathbf{1}_{\tilde{A}(s^-)=0}\mathbf{1}_{u \le \tilde{\mathfrak{G}}(s,X)} P(ds,du), \ t \ge 0 \ \text{ with} \\
\tilde{\mathfrak{G}}(t,x) &:= \overline{I}(0)\mathbb{E}\left(\frac{K(x, X_{\mathfrak{I}})}{\Phi\left(\int_D K(z, X_{\mathfrak{I}})\overline{\mu}(dz)\right)}\lambda_{-1}(t)\right) \\
&+ \overline{S}(0)\mathbb{E}\left(\frac{K(x, X_{\mathfrak{S}})}{\Phi\left(\int_D K(z, X_{\mathfrak{S}})\overline{\mu}(dz)\right)}\lambda_1(t-\tilde{\tau})\right) \\
\text{where} \quad & \tilde{\tau} := \inf\{t > 0; \tilde{A}(t) = 1\}.
\end{aligned} \tag{3.14}$$

Note that $X_{\mathfrak{I}}$ is a r.v. independent of λ_{-1}, whose law has the density π_I, and $X_{\mathfrak{S}}$ is a r.v. independent of $(\lambda_1, \tilde{\tau})$, whose law has the density π_S.

Lemma 3.2 *Equation (3.14) has a unique solution $(\tilde{A}(t), t \ge 0)$, such that $\tilde{\mathfrak{G}}(t,x) = \tilde{\Gamma}(t,x)$ given by (3.13).*

Proof. Given an $\mathcal{M}-$valued function $(m_t, t \ge 0)$, we consider the $\{0,1\}-$valued increasing process associated with the m measure $\tilde{A}^{(m)}(t)$ defined by

$$\tilde{A}^{(m)}(t) = \int_0^t \int_0^\infty \mathbf{1}_{\tilde{A}^{(m)}(s^-)=0}\mathbf{1}_{u \le \left(m_s, \frac{K(X,.)}{\Phi\left(\int_D K(z,.)\overline{\mu}(dz)\right)}\right)} P(ds,du).$$

Define $\tilde{\tau}^{(m)} = \inf\{t > 0, \tilde{A}^{(m)}(t) = 1\}$; also $\forall \varphi \in C_b(D)$,

$$(\tilde{\mu}_t^{\mathfrak{F},m}, \varphi) = \overline{\lambda}^0(t)\overline{I}(0)\mathbb{E}\left[\varphi(X_{\mathfrak{I}})\right] + \overline{S}(0)\mathbb{E}\left[\varphi(X_{\mathfrak{S}})\overline{\lambda}(t-\tilde{\tau}^{(m)})\right]$$

We note that if $m = \tilde{\mu}^{\mathfrak{F},m}$, then $\tilde{A}^m$ is a solution of (3.14) .
Indeed, if $m = \tilde{\mu}^{\mathfrak{F},m}$, then

$$\begin{aligned}
\left(m_t, \frac{K(x,.)}{\Phi\left(\int_D K(z,.)\overline{\mu}(dz)\right)}\right) &= \overline{\lambda}^0(t)\overline{I}(0)\mathbb{E}\left[\frac{K(x, X_{\mathfrak{I}})}{\Phi\left(\int_D K(z, X_{\mathfrak{I}})\overline{\mu}(dz)\right)}\right] \\
&+ \overline{S}(0)\mathbb{E}\left[\frac{K(x, X_{\mathfrak{S}})}{\Phi\left(\int_D K(z, X_{\mathfrak{S}})\overline{\mu}(dz)\right)}\overline{\lambda}(t-\tilde{\tau}^{(m)})\right]
\end{aligned}$$

Thus, by definition of the process $(\tilde{A}^m(t), t \ge 0)$, it is a solution of (3.14) . It therefore suffices to show the existence and uniqueness of m^* such that $m^* = \tilde{\mu}^{\mathfrak{F},m^*}$. As $(\tilde{\mu}^S, \tilde{\mu}^{\mathfrak{F}})$ verifying (3.9)-(3.10) exists and is unique, it suffices to show that $m = \tilde{\mu}^{\mathfrak{F},m}$ if and only if $(\tilde{\mu}^{S,m}, \tilde{\mu}^{\mathfrak{F},m})$ verifies (3.9)-(3.10), where $(\tilde{\mu}_t^{S,m}, \varphi) = \overline{S}(0)\mathbb{E}\left[\varphi(X)\exp\left\{-\int_0^t \left(m_s, \frac{K(X,.)}{\Phi\left(\int_D K(z,.)\overline{\mu}(dz)\right)}\right)ds\right\}\right]$.
Let $\varphi \in C_b(D)$,

$$(\tilde{\mu}_t^{\mathfrak{F},m}, \varphi) = \overline{\lambda}^0(t)\overline{I}(0)\mathbb{E}\left[\varphi(X_{\mathfrak{I}})\right] + \overline{S}(0)\mathbb{E}\left[\varphi(X_{\mathfrak{S}})\overline{\lambda}(t-\tilde{\tau}^{(m)})\right]$$

$$= \overline{\lambda}^0(t)\overline{I}(0)\int_D \varphi(x)\pi_I(x)dx + \overline{S}(0)\mathbb{E}\left[\varphi(X_{\mathfrak{S}})\overline{\lambda}(t-\tilde{\tau}^{(m)})\right]$$
$$= \overline{\lambda}^0(t)\int_D \varphi(x)\overline{\mu}_0^I(dx) + \overline{S}(0)\mathbb{E}\left[\varphi(X_{\mathfrak{S}})\overline{\lambda}(t-\tilde{\tau}^{(m)})\right]$$

We have

$$\begin{aligned}
\mathbb{E}\left[\varphi(X_{\mathfrak{S}})\overline{\lambda}(t-\tilde{\tau}^{(m)})\right] &= \mathbb{E}\int_0^t \varphi(X_{\mathfrak{S}})\overline{\lambda}(t-s)d\tilde{A}^{(m)}(s)\\
&= \mathbb{E}\int_0^t\int_0^\infty \overline{\lambda}(t-s)\varphi(X_{\mathfrak{S}})\mathbf{1}_{\tilde{A}^{(m)}(s^-)=0}\mathbf{1}_{u\le\left(m_s,\frac{K(X_{\mathfrak{S}},.)}{\Phi\left(\int_D K(z,.)\overline{\mu}(dz)\right)}\right)}P(ds,du)\\
&= \mathbb{E}\int_0^t \overline{\lambda}(t-s)\varphi(X_{\mathfrak{S}})\mathbf{1}_{\tilde{A}^{(m)}(s)=0}\left(m_s,\frac{K(X_{\mathfrak{S}},.)}{\Phi\left(\int_D K(z,.)\overline{\mu}(dz)\right)}\right)ds\\
&= \mathbb{E}\int_0^t \overline{\lambda}(t-s)\varphi(X_{\mathfrak{S}})\mathbb{P}\left(\tilde{A}^{(m)}(s)=0|X_{\mathfrak{S}}\right)\left(m_s,\frac{K(X_{\mathfrak{S}},.)}{\Phi\left(\int_D K(z,.)\overline{\mu}(dz)\right)}\right)ds
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}\left(A^{(m)}(s)=0|X_{\mathfrak{S}}\right) &= \mathbb{P}\left(\int_0^s\int_0^\infty \mathbf{1}_{u\le\left(m_r,\frac{K(X_{\mathfrak{S}},.)}{\Phi\left(\int_D K(z,.)\overline{\mu}(dz)\right)}\right)}P(dr,du)=0\Big|X_{\mathfrak{S}}\right)\\
&= e^{-\int_0^s\left(m_r,\frac{K(X_{\mathfrak{S}},.)}{\Phi\left(\int_D K(z,.)\overline{\mu}(dz)\right)}\right)dr}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[\varphi(X_{\mathfrak{S}})\lambda(t-\tilde{\tau}^{(m)})\right] = \mathbb{E}\int_0^t \overline{\lambda}(t-s)\varphi(X_{\mathfrak{S}})&\left(m_s,\frac{K(X_{\mathfrak{S}},.)}{\Phi\left(\int_D K(z,.)\overline{\mu}(dz)\right)}\right)\\
&\times e^{-\int_0^s\left(m_r,\frac{K(X_{\mathfrak{S}},.)}{\Phi\left(\int_{\mathbb{T}^2} K(z,.)\overline{\mu}(dz)\right)}\right)dr}ds.
\end{aligned}$$

$$\begin{aligned}
(\tilde{\mu}_t^{\mathfrak{F},m},\varphi) = \overline{\lambda}^0(t)(\overline{\mu}_0^I,\varphi) + \mathbb{E}\int_0^t\int_D \overline{\lambda}(t-s)\overline{S}(0)\varphi(X_{\mathfrak{S}})\left(m_s,\frac{K(X_{\mathfrak{S}},.)}{\Phi\left(\int_D K(z,.)\overline{\mu}(dz)\right)}\right)\\
\times e^{-\int_0^s\left(m_r,\frac{K(X_{\mathfrak{S}},.)}{\Phi\left(\int_D K(z,.)\overline{\mu}(dz)\right)}\right)dr}ds.
\end{aligned}$$

$m=\tilde{\mu}^{\mathfrak{F},m}$ is equivalent to

$$(\tilde{\mu}_t^{\mathfrak{F},m},\varphi) = \overline{\lambda}^0(t)(\overline{\mu}_0^I,\varphi) + \int_0^t\int_D\int_D \overline{\lambda}(t-s)\varphi(x)\frac{K(x,y)}{\Phi\left(\int_D K(z,y)\overline{\mu}(dz)\right)}\tilde{\mu}_s^{\mathfrak{F},m}(dy)\tilde{\mu}_s^{S,m}(dx)ds,$$

which says that $(\tilde{\mu}^{S,m},\tilde{\mu}^{\mathfrak{F},m})$ is a solution of (3.9)-(3.10). Consequently, $m=\tilde{\mu}^{\mathfrak{F},m}$ is equivalent to the fact that $(\tilde{\mu}^{S,m},\tilde{\mu}^{\mathfrak{F},m})$ is a solution of (3.9)-(3.10). □

For each $1\le i\le S^N(0)$, we define the process $(\tilde{A}_i(t),\tilde{\tau}_i,t\ge 0)$, solution (3.14) with (X,P) replaced by (X^i,P^i)), and we remark that the $\{(\tilde{A}_i^N(\cdot)-\tilde{A}_i(\cdot)),\ i\ge 1\}$ are identically distributed.

Lemma 3.3 *For all $T>0$ and $\varphi\in C_b(D)$,* $\mathbb{E}\left(\sup_{0\le t\le T}\left|\tilde{A}_i^N(t)-\tilde{A}_i(t)\right|\right)\xrightarrow[N\to\infty]{}0.$

Proof. Let $\varphi \in C_b(D)$ and $t \in [0;T]$, we have

$$\left|\tilde{A}_i^N(t) - \tilde{A}_i(t)\right| \leq \int_0^t \int_{\tilde{\Gamma}^N(s,X^i)\wedge\tilde{\Gamma}(s,X^i)}^{\tilde{\Gamma}^N(s,X^i)\vee\tilde{\Gamma}(s,X^i)} P^i(ds,du)$$

$$\sup_{0\leq t\leq T}\left|\tilde{A}_i^N(t) - \tilde{A}_i(t)\right| \leq \int_0^T \int_{\tilde{\Gamma}^N(s,X^i)\wedge\tilde{\Gamma}(s,X^i)}^{\tilde{\Gamma}^N(s,X^i)\vee\tilde{\Gamma}(s,X^i)} P^i(ds,du)$$

$$\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\tilde{A}_i^N(t) - \tilde{A}_i(t)\right|\right) \leq \int_0^T \mathbb{E}\left(\left|\tilde{\Gamma}^N(s,X^i) - \tilde{\Gamma}(s,X^i)\right|\right) ds. \tag{3.15}$$

We have,

$$\left|\tilde{\Gamma}^N(t,X^i) - \tilde{\Gamma}(t,X^i)\right| \leq \left|\frac{1}{N}\sum_{j\in\mathfrak{I}}\frac{K(X^i,X^j)}{\Phi\left(\int_D K(z,X^j)\overline{\mu}^N(dz)\right)}\lambda_{-j}(t) - \overline{\lambda}^0(t)\overline{I}(0)\mathbb{E}\left(\frac{K(X^i,X_{\mathfrak{I}})}{\Phi\left(\int_D K(z,X_{\mathfrak{I}})\overline{\mu}(dz)\right)}\Big|X^i\right)\right|$$
$$+\left|\frac{1}{N}\sum_{j\in\mathfrak{S}}\frac{K(X^i,X^j)}{\Phi\left(\int_D K(z,X^j)\overline{\mu}^N(dz)\right)}\lambda_j(t-\tilde{\tau}_j^N) - \overline{S}(0)\mathbb{E}\left(\frac{K(X^i,X_{\mathfrak{S}})\overline{\lambda}(t-\tilde{\tau})}{\Phi\left(\int_D K(z,X_{\mathfrak{S}})\overline{\mu}(dz)\right)}\Big|X^i\right)\right|.$$

We obtain

$$\mathbb{E}\left(\left|\tilde{\Gamma}^N(t,X^i) - \tilde{\Gamma}(t,X^i)\right|\right) \leq \Upsilon_1(t) + \Upsilon_2(t) \tag{3.16}$$

where

$$\Upsilon_1(t) = \mathbb{E}\left|\frac{1}{N}\sum_{j\in\mathfrak{I}}\frac{K(X^i,X^j)}{\Phi\left(\int_D K(z,X^j)\overline{\mu}^N(dz)\right)}\lambda_{-j}(t) - \overline{\lambda}^0(t)\overline{I}(0)\mathbb{E}\left(\frac{K(X^i,X_{\mathfrak{I}})}{\Phi\left(\int_D K(z,X_{\mathfrak{I}})\overline{\mu}(dz)\right)}\Big|X^i\right)\right|$$

$$\Upsilon_2(t) = \mathbb{E}\left|\frac{1}{N}\sum_{j\in\mathfrak{S}}\frac{K(X^i,X^j)}{\Phi\left(\int_D K(z,X^j)\overline{\mu}^N(dz)\right)}\lambda_j(t-\tau_j^N) - \overline{S}(0)\mathbb{E}\left(\frac{K(X^i,X_{\mathfrak{S}})\overline{\lambda}(t-\tilde{\tau})}{\Phi\left(\int_D K(z,X_{\mathfrak{S}})\overline{\mu}(dz)\right)}\Big|X^i\right)\right|$$

On the one hand,

$$\Upsilon_1(t) \leq \mathbb{E}\left(\frac{1}{N}\sum_{j\in\mathfrak{I}}K(X^i,X^j)\lambda_{-j}(t)\left|\frac{1}{\Phi\left(\int_D K(z,X^j)\overline{\mu}^N(dz)\right)} - \frac{1}{\Phi\left(\int_D K(z,X^j)\overline{\mu}(dz)\right)}\right|\right)$$
$$+\mathbb{E}\left|\frac{1}{N}\sum_{j\in\mathfrak{I}}\left[\frac{K(X^i,X^j)}{\Phi\left(\int_D K(z,X^j)\overline{\mu}(dz)\right)}\lambda_{-j}(t) - \overline{\lambda}^0(t)\mathbb{E}\left(\frac{K(X^i,X_{\mathfrak{I}})}{\Phi\left(\int_D K(z,X_{\mathfrak{I}})\overline{\mu}(dz)\right)}\Big|X^i\right)\right]\right|$$
$$+\overline{\lambda}^0(t)\left|\overline{I}^N(0) - \overline{I}(0)\right|\mathbb{E}\left(\frac{K(X^i,X_{\mathfrak{I}})}{\Phi\left(\int_D K(z,X_{\mathfrak{I}})\overline{\mu}(dz)\right)}\right)$$
$$= \Upsilon_{1,1}(t) + \Upsilon_{1,2}(t) + \Upsilon_{1,3}(t). \tag{3.17}$$

In the computations which follow, we decompose $\overline{\mu}^N = \frac{1}{N}\delta_{X^j} + \overline{\mu}_j^N$, so that $\overline{\mu}_j^N$ and X^j are independent.

$$\Upsilon_{1,1}(t) \leq \left(\frac{2C}{c}\right)^{2\gamma}\|K\|_\infty\lambda^*\mathbb{E}\left(\frac{1}{N}\sum_{j\in\mathfrak{I}}\left|\Phi\left(\int_D K(z,X^j)\overline{\mu}^N(dz)\right) - \Phi\left(\int_D K(z,X^j)\overline{\mu}(dz)\right)\right|\right)$$
$$= C\mathbb{E}\left(\frac{1}{N}\sum_{j\in\mathfrak{I}}\left|\frac{1}{N}K(X^j,X^j) + \int_D K(z,X^j)\overline{\mu}_j^N(dz) - \int_D K(z,X^j)\overline{\mu}(dz)\right|\right)$$

$$
\begin{aligned}
&\le \frac{C}{N}\mathbb{E}\left(\frac{1}{N}\sum_{j\in\mathfrak{I}}K(X^j,X^j)\right)+\mathbb{E}\left(\frac{1}{N}\sum_{j\in\mathfrak{I}}\left|\int_D K(z,X^j)[\overline{\mu}_j^N(dz)-\overline{\mu}(dz)]\right|\right)\\
&\le \frac{C}{N}\int_D K(y,y)\pi_I(y)dy+\int_D \mathbb{E}\left|\int_D K(z,y)\left(\overline{\mu}_j^N(dz)-\overline{\mu}(dz)\right)\right|\pi_I(y)dy
\end{aligned}
$$

$$
\Upsilon_{1,1}(t)\le \frac{C\|K\|_\infty}{N}+\int_D \mathbb{E}\left|\int_D K(z,y)\left(\overline{\mu}_j^N(dz)-\overline{\mu}(dz)\right)\right|\pi_I(y)dy.
$$

$$
\begin{aligned}
\Upsilon_{1,2}(t)&\le \frac{1}{N}\left[\mathbb{E}\left\{\sum_{j\in\mathfrak{I}}\left[\frac{K(X^i,X^j)}{\Phi\left(\int_D K(z,X^j)\overline{\mu}(dz)\right)}\lambda_{-j}(t)-\overline{\lambda}^0(t)\mathbb{E}\left(\frac{K(X^i,X_{\mathfrak{I}})}{\Phi\left(\int_D K(z,X_{\mathfrak{I}})\overline{\mu}(dz)\right)}\Big|X^i\right)\right]\right\}^2\right]^{\frac12}\\
&=\frac{1}{N}\left[\sum_{j\in\mathfrak{I}}\mathbb{E}\left[\frac{K(X^i,X^j)}{\Phi\left(\int_D K(z,X^j)\overline{\mu}(dz)\right)}\lambda_{-j}(t)-\overline{\lambda}^0(t)\mathbb{E}\left(\frac{K(X^i,X_{\mathfrak{I}})}{\Phi\left(\int_D K(z,X_{\mathfrak{I}})\overline{\mu}(dz)\right)}\Big|X^i\right)\right]^2\right]^{\frac12}\\
&\le \frac{1}{N}\left[\sum_{j\in\mathfrak{I}}\mathbb{E}\left(\frac{K(X^i,X^j)}{\Phi\left(\int_D K(z,X^j)\overline{\mu}(dz)\right)}\lambda_{-j}(t)\right)^2\right]^{\frac12}\\
&\le \frac{C}{N}\left[\sum_{j\in\mathfrak{I}}\mathbb{E}\left(K^2(X^i,X^j)\right)\right]^{\frac12}\\
&\le CN^{\frac{-1}{2}}\left[\int_D\int_D K^2(x,y)\pi_S(x)\pi_I(y)dydx\right]^{\frac12}
\end{aligned}
$$

$$
\Upsilon_{1,2}(t)\le C\|K\|_\infty N^{\frac{-1}{2}}.
$$

$$
\begin{aligned}
\Upsilon_{1,3}(t)&\le\left(\frac{2C}{c}\right)^{2\gamma}\overline{\lambda}^0(t)\left|\overline{I}^N(0)-\overline{I}(0)\right|\mathbb{E}\left(K(X^i,X_{\mathfrak{I}})\right)\\
&=\left(\frac{2C}{c}\right)^{2\gamma}\overline{\lambda}^0(t)\left|\overline{I}^N(0)-\overline{I}(0)\right|\int_D\int_D K(x,y)\pi_S(x)\pi_I(y)dxdy
\end{aligned}
$$

$$
\Upsilon_{1,3}(t)\le C\left|\overline{I}^N(0)-\overline{I}(0)\right|
$$

From (3.17), we obtain

$$
\Upsilon_1(t)\le\frac{C\|K\|_\infty}{N}+\int_D\mathbb{E}\left|\int_D K(z,y)\left(\overline{\mu}_j^N(dz)-\overline{\mu}(dz)\right)\right|\pi_I(y)dy+C\|K\|_\infty N^{\frac{-1}{2}}+C\mathbb{E}\left(\left|\overline{I}^N(0)-\overline{I}(0)\right|\right)
\tag{3.18}
$$

Moreover,

$$
\begin{aligned}
\Upsilon_2(t)\le\ &\mathbb{E}\left(\frac{1}{N}\sum_{j=1}^{S^N(0)}K(X^i,X^j)\lambda_j(t-\tilde{\tau}_j^N)\left|\frac{1}{\Phi\left(\int_D K(z,X^j)\overline{\mu}^N(dz)\right)}-\frac{1}{\Phi\left(\int_D K(z,X^j)\overline{\mu}(dz)\right)}\right|\right)\\
&+\mathbb{E}\left|\frac{1}{N}\sum_{j=1}^{S^N(0)}\frac{K(X^i,X^j)}{\Phi\left(\int_D K(z,X^j)\overline{\mu}(dz)\right)}\lambda_j(t-\tilde{\tau}_j^N)-\frac{1}{N}\sum_{j=1}^{S^N(0)}\frac{K(X^i,X^j)}{\Phi\left(\int_D K(z,X^j)\overline{\mu}(dz)\right)}\lambda_j(t-\tilde{\tau}_j)\right|\\
&+\mathbb{E}\left|\frac{1}{N}\sum_{j=1}^{S^N(0)}\left[\frac{K(X^i,X^j)}{\Phi\left(\int_D K(z,X^j)\overline{\mu}(dz)\right)}\lambda_j(t-\tilde{\tau}_j)-\mathbb{E}\left(\frac{K(X^i,X_{\mathfrak{S}})\lambda(t-\tilde{\tau})}{\Phi\left(\int_D K(z,X_{\mathfrak{S}})\overline{\mu}(dz)\right)}\Big|X^i\right)\right]\right|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\overline{S}^N(0)-\overline{S}(0)\right|\mathbb{E}\left(\frac{K(X^i,X_{\mathfrak{S}})\lambda(t-\tilde{\tau})}{\Phi\left(\int_D K(z,X_{\mathfrak{S}})\overline{\mu}(dz)\right)}\right)\\
&=\Upsilon_{2,1}(t)+\Upsilon_{2,2}(t)+\Upsilon_{2,3}(t)+\Upsilon_{2,4}(t)
\end{aligned}
\tag{3.19}
$$

We recall the decomposition $\overline{\mu}^N=\frac{1}{N}\delta_{X^j}+\overline{\mu}^N_j$.

$$
\begin{aligned}
\Upsilon_{2,1}(t)&\le C\mathbb{E}\left(\frac{1}{N}\sum_{j=1}^{S^N(0)}\left|\int_D K(z,X^j)\overline{\mu}^N(dz)-\int_D K(z,X^j)\overline{\mu}(dz)\right|\right)\\
&=C\mathbb{E}\left(\frac{1}{N}\sum_{j=1}^{S^N(0)}\left|\frac{1}{N}K(X^j,X^j)+\int_D K(z,X^j)\overline{\mu}^N_j(dz)-\int_D K(z,X^j)\overline{\mu}(dz\right|)\right)\\
&\le\frac{C\|K\|_\infty}{N}+\int_D\mathbb{E}\left|\int_D K(z,y)\left(\overline{\mu}^N_j(dz)-\overline{\mu}(dz)\right)\right|\pi_S(y)dy.\\
\Upsilon_{2,2}(t)&\le\frac{1}{N}\mathbb{E}\left[\sum_{j=1}^{S^N(0)}\frac{K(X^i,X^j)}{\Phi\left(\int_D K(z,X^j)\overline{\mu}(dz)\right)}\left|\lambda_j(t-\tilde{\tau}^N_j)-\lambda_j(t-\tilde{\tau}_j)\right|\right]\\
&\le C\mathbb{E}\left(\mathbf{1}_{\left\{t\ge\tilde{\tau}^N_j\wedge\tilde{\tau}_j;\tilde{\tau}^N_j\neq\tilde{\tau}_j\right\}}\right)\\
&\le C\int_0^t\mathbb{E}\left(|\tilde{\Gamma}^N(s,X^j)-\tilde{\Gamma}(s,X^j)|\right)ds.\\
\Upsilon_{2,3}(t)&\le\frac{1}{N}\left[\sum_{j=1}^{I^N(0)}\mathbb{E}\left[\frac{K(X^i,X^j)\lambda_j(t-\tilde{\tau}_j)}{\Phi\left(\int_D K(z,X^j)\overline{\mu}(dz)\right)}-\mathbb{E}\left(\frac{K(X^i,X_{\mathfrak{S}})\overline{\lambda}(t-\tilde{\tau})}{\Phi\left(\int_D K(z,X_{\mathfrak{S}})\overline{\mu}(dz)\right)}\Big|X^i\right)\right]^2\right]^{\frac{1}{2}}\\
&\le CN^{\frac{-1}{2}}\left[\int_D\int_D K^2(x,y)\pi_S(x)\pi_S(y)dydx\right]^{\frac{1}{2}}\\
\Upsilon_{2,3}(t)&\le C\|K\|_\infty N^{\frac{-1}{2}}.\\
\Upsilon_{2,4}(t)&\le C\left|\overline{S}^N(0)-\overline{S}(0)\right|
\end{aligned}
$$

From (3.19), we obtain

$$
\begin{aligned}
\Upsilon_2(t)\le{}&\frac{C\|K\|_\infty}{N}+\int_D\mathbb{E}\left|\int_D K(z,y)\left(\overline{\mu}^N_j(dz)-\overline{\mu}(dz)\right)\right|\pi_S(y)dy+C\|K\|_\infty N^{\frac{-1}{2}}\\
&+C\left|\overline{S}^N(0)-\overline{S}(0)\right|+C\int_0^t\mathbb{E}\left(\left|\tilde{\Gamma}^N(s,X^j)-\tilde{\Gamma}(s,X^j)\right|\right)ds.
\end{aligned}
\tag{3.20}
$$

From (3.16), (3.18) and (3.20), we obtain

$$
\begin{aligned}
\mathbb{E}\left(\left|\tilde{\Gamma}^N(t,X^i)-\tilde{\Gamma}(t,X^i)\right|\right)\le{}&\frac{2C\|K\|_\infty}{N}+\int_D\mathbb{E}\left|\int_D K(z,y)\left(\overline{\mu}^N_j(dz)-\overline{\mu}(dz)\right)\right|\pi_I(y)dy+2C\|K\|_\infty N^{\frac{-1}{2}}\\
&\int_D\mathbb{E}\left|\int_D K(z,y)\left(\overline{\mu}^N_j(dz)-\overline{\mu}(dz)\right)\right|\pi_S(y)dy+C\left|\overline{S}^N(0)-\overline{S}(0)\right|\\
&+C\left|\overline{I}^N(0)-\overline{I}(0)\right|+C\int_0^t\mathbb{E}\left(\left|\tilde{\Gamma}^N(s,X^j)-\tilde{\Gamma}(s,X^j)\right|\right)ds.
\end{aligned}
\tag{3.21}
$$

Applying Gronwall's inequality to (3.21), $\mathbb{E}\left(\left|\tilde{\Gamma}^N(t,X^i)-\tilde{\Gamma}(t,X^i)\right|\right)\xrightarrow[N\to\infty]{}0.$

Combining this result with (3.15), we deduce that $\mathbb{E}\left(\sup_{0\le t\le T}\left|\tilde{A}_i^N(t-\tilde{A}_i(t)\right|\right)\xrightarrow[N\to\infty]{}0.$ □

Proof. ***Theorem*** *3.2*. Let $\varphi\in C_b(D)$ and $T>0$ be arbitrary..

$$(\tilde{\mu}_t^{S,N},\varphi)=(\overline{\mu}_0^{S,N},\varphi)-\frac{1}{N}\sum_{i\in\mathfrak{S}}\varphi(X^i)\tilde{A}_i^N(t).$$

According to Law of Large Numbers, the sequence $(\overline{\mu}_0^{S,N},\varphi)$ converges to $(\overline{\mu}_0^{S},\varphi)$ *a.s.*
Using Lemma 3.3, as $N\to\infty$

$$\sup_{0\le t\le T}\left|\frac{1}{N}\sum_{i\in\mathfrak{S}}\varphi(X^i)\tilde{A}_i^N(t)-\frac{1}{N}\sum_{i\in\mathfrak{S}}\varphi(X^i)\tilde{A}_i(t)\right|\xrightarrow{\mathbb{P}}0,\quad\text{where}$$

$$\frac{1}{N}\sum_{i\in\mathfrak{S}}\varphi(X^i)\tilde{A}_i(t)=\frac{1}{N}\sum_{i=1}^{S^N(0)}\int_0^t\int_0^\infty\mathbf{1}_{\tilde{A}_i(s^-)=0}\mathbf{1}_{u\le\tilde{\Gamma}(s,X^i)}\varphi(X^i)P^i(ds,du).\quad\text{In addition,}$$

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le T}\left|\int_0^t\int_0^\infty\mathbf{1}_{\tilde{A}_i(s^-)=0}\mathbf{1}_{u\le\tilde{\Gamma}(s,X^i)}\varphi(X^i)P^i(ds,du)\right|\right)&=\mathbb{E}\left(\int_0^T\int_0^\infty\mathbf{1}_{\tilde{A}_i(s^-)=0}\mathbf{1}_{u\le\tilde{\Gamma}(s,X_{\mathfrak{S}})}|\varphi(X_{\mathfrak{S}})|P(ds,du)\right)\\&\le C\|\varphi\|_\infty T\end{aligned}$$

Combinig Lemma 3.3 with the Law of Large Numbers in $\mathbb{D}(\mathbb{R}_+,\mathbb{R})$ we deduce that, locally uniformly in t

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{S^N(0)}\int_0^t\varphi(X^i)d\tilde{A}_i^N(s)&\xrightarrow{a.s.}\overline{S}(0)\mathbb{E}\left(\int_0^t\int_0^\infty\mathbf{1}_{\tilde{A}(s^-)=0}\mathbf{1}_{u\le\tilde{\Gamma}(s,X_{\mathfrak{S}})}\varphi(X_{\mathfrak{S}})P(ds,du)\right)\\&=\overline{S}(0)\mathbb{E}\left(\int_0^t\mathbf{1}_{A(s^-)=0}\tilde{\Gamma}(s,X_{\mathfrak{S}})\varphi(X_{\mathfrak{S}})ds\right)\\&=\overline{S}(0)\mathbb{E}\int_0^t\mathbb{P}\left(\tilde{A}(s^-)=0|X_{\mathfrak{S}}\right)\varphi(X_{\mathfrak{S}})\tilde{\Gamma}(s,X_{\mathfrak{S}})ds\\&=\int_0^t\overline{S}(0)\mathbb{E}\left(\varphi(X_{\mathfrak{S}})\tilde{\Gamma}(s,X_{\mathfrak{S}})e^{-\int_0^s\tilde{\Gamma}(r,X_{\mathfrak{S}})dr}\right)ds\\&=\int_0^t\int_D\varphi(x)\tilde{\Gamma}(s,x)\tilde{\mu}_s^S(dx)ds,\end{aligned}$$

hence $(\tilde{\mu}_t^{S,N},\varphi)\to(\tilde{\mu}_t^{S},\varphi)$ in probability locally uniformly in t, where

$$(\tilde{\mu}_t^{S},\varphi)=(\overline{\mu}_0^{S},\varphi)-\int_0^t\int_D\varphi(x)\tilde{\Gamma}(s,x)\tilde{\mu}_s^S(dx)ds.$$

Next,

$$(\tilde{\mu}_t^{\mathfrak{F},N},\varphi)=\frac{1}{N}\sum_{i=1}^{I^N(0)}\lambda_{-i}(t)\varphi(X^i)+\frac{1}{N}\sum_{i=1}^{S^N(0)}\lambda_i(t-\tilde{\tau}_i^N)\varphi(X^i).$$

On the one hand,

$$\frac{1}{N}\sum_{i=1}^{I^N(0)}\lambda_{-i}(t)\varphi(X^{-i})\xrightarrow{a.s}\overline{I}(0)\mathbb{E}\left(\lambda_{-1}(t)\varphi(X_{\mathfrak{I}})\right)=\overline{\lambda}^0(t)\int_D\overline{I}(0)\varphi(x)\pi_I(x)dx=\overline{\lambda}^0(t)(\overline{\mu}_0^I,\varphi)$$

where $\overline{\mu}_0^I(dx)=\overline{I}(0)\pi_I(x)dx$

Moreover, $\displaystyle\frac{1}{N}\sum_{i=1}^{S^N(0)}\lambda_i(t-\tilde{\tau}_i^N)\varphi(X^i)=\frac{1}{N}\sum_{i=1}^{S^N(0)}\int_0^t\lambda_i(t-s)\varphi(X^i)d\tilde{A}_i^N(s)$.
According to a variant of Lemma 3.3, as $N\to\infty$
$\displaystyle\sup_{t\le T}\left|\frac{1}{N}\sum_{i\in\mathfrak{S}}\int_0^t\lambda_i(t-s)\varphi(X^i)d\tilde{A}_i^N(s)-\frac{1}{N}\sum_{i\in\mathfrak{S}}\int_0^t\lambda_i(t-s)\varphi(X^i)d\tilde{A}_i(s)\right|\xrightarrow{\mathbb{P}}0$. Indeed,
$\displaystyle\sup_{t\le T}\left|\frac{1}{N}\sum_{i\in\mathfrak{S}}\int_0^t\lambda_i(t-s)\varphi(X^i)\left(d\tilde{A}_i^N(s)-d\tilde{A}_i(s)\right)\right|\le\frac{\lambda^*\|\varphi\|_\infty}{N}\sum_{i\in\mathfrak{S}}\mathbb{E}\left(\sup_{0\le t\le T}\left|\tilde{A}_i^N(t)-\tilde{A}_i(t)\right|\right)$ where,
$\displaystyle\frac{1}{N}\sum_{i\in\mathfrak{S}}\int_0^t\lambda_i(t-s)\varphi(X^i)d\tilde{A}_i(s)=\frac{1}{N}\sum_{i=1}^{S^N(0)}\int_0^t\int_0^\infty\lambda_i(t-s)\varphi(X^i)\mathbf{1}_{\tilde{A}_i(s^-)=0}\mathbf{1}_{u\le\tilde{\Gamma}(s,X^i)}P^i(ds,du)$. In addition,

$$\mathbb{E}\left(\sup_{0\le t\le T}\left|\int_0^t\int_0^\infty\lambda_i(t-s)\varphi(X^i)\mathbf{1}_{\tilde{A}_i(s^-)=0}\mathbf{1}_{u\le\tilde{\Gamma}(s,X^i)}P^i(ds,du)\right|\right)\le\lambda^*C\|\varphi\|_\infty T.$$

Hence combining the above result with the Law of Large Numbers in $\mathbb{D}(\mathbb{R}_+,\mathbb{R})$, we deduce that, in probability, locally uniformly in t

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{S^N(0)}\int_0^t\lambda_i(t-s)\varphi(X^i)d\tilde{A}_i(s)&\to\overline{S}(0)\mathbb{E}\left(\int_0^t\lambda(t-s)\int_0^\infty\mathbf{1}_{\tilde{A}(s^-)=0}\mathbf{1}_{u\le\tilde{\Gamma}(s,X_{\mathfrak{S}})}\varphi(X_{\mathfrak{S}})P(ds,du)\right)\\&=\overline{S}(0)\int_0^t\overline{\lambda}(t-s)\mathbb{E}\left(\mathbf{1}_{A(s^-)=0}\tilde{\Gamma}(s,X_{\mathfrak{S}})\varphi(X_{\mathfrak{S}})\right)ds\\&=\overline{S}(0)\int_0^t\overline{\lambda}(t-s)\mathbb{E}\left[\mathbb{P}\left(\tilde{A}(s^-)=0|X_{\mathfrak{S}}\right)\varphi(X_{\mathfrak{S}})\tilde{\Gamma}(s,X_{\mathfrak{S}})\right]ds\\&=\int_0^t\overline{\lambda}(t-s)\overline{S}(0)\mathbb{E}\left(\varphi(X_{\mathfrak{S}})\tilde{\Gamma}(s,X_{\mathfrak{S}})e^{-\int_0^s\tilde{\Gamma}(r,X_{\mathfrak{S}})dr}\right)ds\\&=\int_0^t\overline{\lambda}(t-s)\int_D\varphi(x)\tilde{\Gamma}(s,x)\tilde{\mu}_s^S(dx)ds\end{aligned}$$

We thus obtain that $(\tilde{\mu}_t^{\mathfrak{F},N},\varphi)\to(\tilde{\mu}_t^{\mathfrak{F}},\varphi)$ in probability locally uniformly in t, where

$$(\tilde{\mu}_t^{\mathfrak{F}},\varphi)=\overline{\lambda}^0(t)(\overline{\mu}_0^I,\varphi)+\int_0^t\overline{\lambda}(t-s)\int_D\varphi(x)\tilde{\Gamma}(s,x)\tilde{\mu}_s^S(dx)ds.$$

Now,

$$(\tilde{\mu}_t^{I,N},\varphi)=(\overline{\mu}_0^{I,N},\varphi)+\frac{1}{N}\sum_{i\in\mathfrak{S}}\varphi(X^i)\tilde{A}_i^N(t)-\frac{1}{N}\sum_{i\in\mathfrak{I}}\varphi(X^i)\mathbf{1}_{\eta_{-i}\le t}-\frac{1}{N}\sum_{i\in\mathfrak{S}}\varphi(X^i)\int_0^t\mathbf{1}_{\eta_i\le t-s}d\tilde{A}_i^N(s)$$

According to the Law of Large Numbers:

$$(\overline{\mu}_0^{I,N},\varphi)\xrightarrow{a.s.}(\overline{\mu}_0^I,\varphi).$$

$$\frac{1}{N}\sum_{i\in\mathfrak{I}}\varphi(X^i)\mathbf{1}_{\eta_{-i}\le t}\xrightarrow{a.s.}\overline{I}(0)\mathbb{E}\left(\varphi(X_{\mathfrak{I}})\mathbf{1}_{\eta_{-1}\le t}\right)=\overline{I}(0)F_0(t)\int_D\varphi(x)\pi_I(x)dx=F_0(t)(\overline{\mu}_0^I,\varphi)$$

According to the previous results,

$$\frac{1}{N}\sum_{i\in\mathfrak{S}}\varphi(X^i)\tilde{A}_i^N(t)\xrightarrow{\mathbb{P}}\int_0^t\int_D\varphi(x)\tilde{\Gamma}(s,x)\tilde{\mu}_s^S(dx)ds$$

According to a variant of Lemma 3.3, as $N\to\infty$

$\sup_{t\le T}\left|\frac{1}{N}\sum_{i\in\mathfrak{S}}\varphi(X^i)\int_0^t\mathbf{1}_{\eta_i\le t-s}d\tilde{A}_i^N(s)-\frac{1}{N}\sum_{i\in\mathfrak{S}}\varphi(X^i)\int_0^t\mathbf{1}_{\eta_i\le t-s}d\tilde{A}_i(s)\right|\xrightarrow{\mathbb{P}}0$. Indeed,

$$\sup_{t\le T}\left|\frac{1}{N}\sum_{i\in\mathfrak{S}}\varphi(X^i)\int_0^t\mathbf{1}_{\eta_i\le t-s}d\tilde{A}_i^N(s)-\frac{1}{N}\sum_{i\in\mathfrak{S}}\varphi(X^i)\int_0^t\mathbf{1}_{\eta_i\le t-s}d\tilde{A}_i(s)\right|\le\frac{\|\varphi\|_\infty}{N}\sum_{i=1}^{S^N(0)}\mathbb{E}\left(\sup_{0\le t\le T}\left|\tilde{A}_i^N(t)-\tilde{A}_i(t)\right|\right)$$

where, $\frac{1}{N}\sum_{i\in\mathfrak{S}}\varphi(X^i)\int_0^t\mathbf{1}_{\eta_i\le t-s}d\tilde{A}_i(s)=\frac{1}{N}\sum_{i=1}^{S^N(0)}\int_0^t\int_0^\infty\mathbf{1}_{\eta_i\le t-s}\varphi(X^i)\mathbf{1}_{\tilde{A}_i(s^-)=0}\mathbf{1}_{u\le\tilde{\Gamma}(s,X^i)}P^i(ds,du)$.
In addition

$$\mathbb{E}\left(\sup_{0\le t\le T}\left|int_0^t\int_0^\infty\mathbf{1}_{\eta_i\le t-s}\varphi(X^i)\mathbf{1}_{\tilde{A}_i(s^-)=0}\mathbf{1}_{u\le\tilde{\Gamma}(s,X^i)}P^i(ds,du)\right|\right)\le C\|\varphi\|_\infty T\,.$$

Applying the Law of Large Numbers in $\mathbb{D}\left(\mathbb{R}_+,\mathbb{R}\right)$, we deduce that, locally uniformly in t

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{S^N(0)}\int_0^t\mathbf{1}_{\eta_i\le t-s}\varphi(X^i)d\tilde{A}_i(s)&\xrightarrow{a.s.}\overline{S}(0)\mathbb{E}\left(\int_0^t\mathbf{1}_{\eta_1\le t-s}\int_0^\infty\mathbf{1}_{\tilde{A}(s^-)=0}\mathbf{1}_{u\le\tilde{\Gamma}(s,X_{\mathfrak{S}})}\varphi(X_{\mathfrak{S}})P(ds,du)\right)\\&=\overline{S}(0)\int_0^t\mathbb{P}(\eta_1\le t-s)\mathbb{E}\left(\mathbf{1}_{A(s^-)=0}\tilde{\Gamma}(s,X_{\mathfrak{S}})\varphi(X_{\mathfrak{S}})\right)ds\\&=\overline{S}(0)\int_0^tF(t-s)\mathbb{E}\left[\mathbb{P}\left(\tilde{A}(s^-)=0|X_{\mathfrak{S}}\right)\varphi(X_{\mathfrak{S}})\tilde{\Gamma}(s,X_{\mathfrak{S}})\right]ds\\&=\int_0^tF(t-s)\overline{S}(0)\mathbb{E}\left(\varphi(X_{\mathfrak{S}})\tilde{\Gamma}(s,X_{\mathfrak{S}})e^{-\int_0^s\tilde{\Gamma}(r,X_{\mathfrak{S}})dr}\right)ds\\&=\int_0^tF(t-s)\int_D\varphi(x)\tilde{\Gamma}(s,x)\tilde{\mu}_s^S(dx)ds\,.\end{aligned}$$

Combining the above results, we deduce that $(\tilde{\mu}_t^{N,I},\varphi)\to(\tilde{\mu}_t^I,\varphi)$ in probability locally uniformly in t, where

$$(\tilde{\mu}_t^I,\varphi)=(\overline{\mu}_0^I,\varphi)+\int_0^t\int_D\varphi(x)\tilde{\Gamma}(s,x)\tilde{\mu}_s^S(dx)ds-(\overline{\mu}_0^I,\varphi)F_0(t)-\int_0^tF(t-s)\int_D\varphi(x)\tilde{\Gamma}(s,x)\tilde{\mu}_s^S(dx)ds,$$

which can be rewritten as:

$$(\tilde{\mu}_t^I,\varphi)=(\overline{\mu}_0^I,\varphi)F_0^c(t)+\int_0^tF^c(t-s)\int_D\varphi(x)\tilde{\Gamma}(s,x)\tilde{\mu}_s^S(dx)ds$$

We argue analogously about $\tilde{\mu}^{R,N}$ and obtain that

$$(\tilde{\mu}_t^R,\varphi)=(\overline{\mu}_0^R,\varphi)+(\overline{\mu}_0^I,\varphi)F_0(t)+\int_0^tF(t-s)\int_D\varphi(x)\tilde{\Gamma}(s,x)\tilde{\mu}_s^S(dx)ds.$$

□

Proof. ***Theorem*** *3.1*. Let us define $\Omega_N := \left\{\omega, \inf_{y\in D}\int_D K(z,y)\overline{\mu}^N(dz) > \frac{c}{2}\right\}$. We remark that on Ω_N, $\left(\tilde{\mu}^{S,N}, \tilde{\mu}^{\mathfrak{F},N}, \tilde{\mu}^{I,N}, \tilde{\mu}^{R,N}\right) = \left(\overline{\mu}^{S,N}, \overline{\mu}^{\mathfrak{F},N}, \overline{\mu}^{I,N}, \overline{\mu}^{R,N}\right)$, and for all $t \in \mathbb{R}_+$, $\left(\overline{\mu}^S_t, \overline{\mu}^{\mathfrak{F}}_t, \overline{\mu}^I_t, \overline{\mu}^R_t\right)$, solution of the system of Theorem 3.1, is also a solution of the system of Theorem 3.2.
Clearly Theorem 3.1 will follow from Theorem 3.2 if we prove that $\mathbb{P}(\Omega_N) \to 1$, as $N \to \infty$. We have

$$\begin{aligned}\Omega_N^c &\subseteq \left\{\omega, \left|\inf_{y\in D}\int_D K(z,y)\overline{\mu}^N(dz) - \inf_{y\in D}\int_D K(z,y)\overline{\mu}(dz)\right| > \frac{c}{2}\right\}\\ &\subseteq \left\{\omega, \sup_{y\in D}\left|\int_D K(z,y)\overline{\mu}^N(dz) - \int_D K(z,y)\overline{\mu}(dz)\right| > \frac{c}{2}\right\}.\end{aligned}$$

Let $n \geq 1, y_1, \cdots, y_n \in D$ be such that for any $y \in D$, $\exists 1 \leq i \leq n$ such that

$$\sup_{z\in D}|K(z,y) - K(z,y_i)| \leq \frac{c}{6}.$$

Then

$$\sup_{y\in D}\left|\int_D K(z,y)\overline{\mu}^N(dz) - \int_D K(z,y)\overline{\mu}(dz)\right| \leq \frac{c}{3} + \sup_{1\leq i\leq n}\left|\int_D K(z,y_i)\overline{\mu}^N(dz) - \int_D K(z,y_i)\overline{\mu}(dz)\right|,$$

and consequently

$$\Omega_N^c \subseteq \left\{\omega, \sup_{1\leq i\leq n}\left|\int_D K(z,y_i)\overline{\mu}^N(dz) - \int_D K(z,y_i)\overline{\mu}(dz)\right| > \frac{c}{6}\right\},$$

from which it follows clearly that

$$\mathbb{P}(\Omega_N^c) \to 0, \quad \text{as} \quad N \to \infty.$$

□

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