Uniqueness of the solution of the filtering equations in spaces of measures

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Abstract

Nonlinear filtering is a pivotal problem that has attracted significant attention from mathematicians, statisticians, engineers, and various other scientific disciplines. The solution to this problem is governed by the so-called "filtering equations." In this paper, we investigate the uniqueness of solutions to these equations within measure spaces and introduce a novel, generalized framework for this analysis. Our approach provides new insights and extends the applicability of existing theories in the study of nonlinear filtering.

Keywords: Non-Linear Filtering, Measure Valued Processes, Stochastic Partial Differential Equations, Backward SPDEs, Uniqueness of Solutions.

1 Introduction

The objective of nonlinear filtering is to estimate an evolving dynamical system, modeled by a stochastic process X, referred to as the signal process. This signal process cannot be directly observed; instead, it is inferred through a related process Y, known as the observation process. The filtering problem involves determining the conditional distribution of the signal at the current time, based on the observation data accumulated up to that time. For a comprehensive treatment of the filtering problem and a historical overview of its extensive development over the past 80 years, starting with the foundational work of Kolmogorov, Krein, and Wiener, readers are encouraged to consult [4].

The conditional distribution of the signal given the observation is the solution of a nonlinear evolution equation. Moreover, it has a version which satisfies a linear evolution equations. In what follows we shall refer to these to equations as the filtering equations. The filtering equations have been studied at length by many different contributors to the filtering problem and in different frameworks. The most popular continuous time framework is that where the signal X satisfies a stochastic differential equation driven by a multi-dimensional Brownian motion denoted by V, and the observation Y satisfies an evolution equation of the form

$$dY_t = h(t, X_t)\,\mathrm{d}t + dW_t,$$

where the driving Brownian motion W is independent of X. In this case, the filtering equations are well understood and their solution is shown to be unique in suitably chosen spaces of measures, see, e.g., [1], for details.

In this set-up, the equations satisfied by the signal and the observation are asymmetric. There is no dependence of X on Y, and the observational noise W is chosen independent of the signal and with a coefficient that does not depend on either X or Y. Possibly a symmetric set-up would require the pair $\mathcal{X} := (X, Y)$ to satisfy a stochastic differential of the form

$$d\mathcal{X}_t = a(t, \mathcal{X}_t)\,\mathrm{d}t + b(t, \mathcal{X}_t)dB_t,$$

where $B := (V, W)$. However, perfect symmetry is not really possible. For example, should the diffusion observation coefficient depend on the signal, the solution will degenerate, see, e.g., [5] for details.

In this paper, we introduce a framework which is as close as possible to a symmetric one, see equations (2.1) and (2.2) below. This new framework represents a generalization of several existing ones, including those covered in [3] and [13]. A simplified version of the present set-up (where the signal and the observation noises are independent) was studied in [8].

In the new framework the coefficients of the stochastic differential equation satisfied by X depend on the pair (X, Y). Moreover X is driven by the pair (V, W) and not just by V. Put it differently the Brownian motion driving the signal is corellated with the Brownian motion driving the observation. Moreover we do not assume that the diffusion coefficient in the observation equation is invertible. In the degenerate case when the diffusion coefficient is equal to 0, the observation becomes independent of the signal: in this case the conditional distribution of the signal coincides with the (prior) distribution of the signal.

Within this framework we obtain the filtering equations and show that they have a unique solution within a suitably chosen space of measures. The following are contributions of this paper:

- We deduce the filtering equations for the signal and the observation processes satisfying equations (2.1)–(2.2) below. The coefficients of both equations depend on the signal-observation pair. Moreover, as opposed to frameworks treated elsewhere we do not assume an invertible diffusion term in the observation equation. We do this at the expense of assuming a special form for the observation function h, see (2.3) below.

- We show the equivalence of the uniqueness properties of the two filtering equations, namely the nonlinear Kushner–Stratonovich equation for the conditional distribution, and the linear Zakai equation for the "unnormalized conditional distribution", see equations (3.13) and, respectively, (3.12) below.

- We establish the uniqueness of the solution of the equation satisfied by the unnormalized conditional distribution of the signal in the space of measure valued processes.

Let us now explain the novelty of our proof of the uniqueness of the Zakai equation. The so-called "duality argument" is a standard method for proving uniqueness of the solution of a linear equation. Bensoussan applied the duality argument in [3] to show the uniqueness of the solution of the Zakai equation. In the framework treated in [3], the measure valued solution of the Zakai equation, which is a stochastic partial differential equation was paired with the (function valued) solution of a deterministic backward PDE.[1] The proof in [3] involves the use of an Itô's formula to deduce the evolution of the solution of a deterministic backward PDE integrated with respect to the measure valued process that is the solution of the Zakai equation. The same argument cannot be applied in the framework treated here since the coefficients of the operators appearing in the Zakai equation (hence also in the adjoint backward PDE) depend upon the current observation. As a result, the dual of the solution of the Zakai equation would now be the solution of a backward PDE starting at the time t and run backwards in the interval $[0, t]$ and, at any time $s \in (0, t)$, would be a function of the observations on the interval $[s, t]$, making it random and, moreover, anticipating at any time $s \in [0, t]$ the future of the observation process. As a result, the Itô formula can no longer be applied. In order to circumvent this difficulty, we replace the dual deterministic PDE by a backward Stochastic Partial Differential Equation" (or BSPDE for short). For this, we exploit recent results from Du, Meng [6] and Du, Tang , Zhang [7] which we need to adapt to our framework which requires a complex–valued BSPDE, equivalent to a system of two real valued BSPDEs, and construct Sobolev space valued solutions of our system of BSPDEs. Moreover, we exploit classical Sobolev embedding theorems, in order to deduce enough smoothness of the solution of the BSPDE so that it can be integrated against an arbitrary measure–valued solution to the Zakai equation.

The paper is structured as follows. In Section 2 we introduce the filtering framework and the two sets of assumptions that ensure the existence and uniqueness of the filtering equations (see **Assumption E** and **Asssumption U** below). In Section 3 we deduce the Kallianpur-Striebel formula (Proposition 3.5) which implies the existence of an unnormalized version of the conditional distribution of the signal. Next we deduce the filtering equation (3.13) for the conditional distribution of the signal, which is a non linear SPDE, as well as a linear equation (3.12) for the unnormalized version, see Theorem 3.11 below. Finally we show in Theorem 3.17 that equation (3.13) has a unique solution if and only if (3.12) has a unique solution. In Section 4 we introduce several results pertaining to a class of backward stochastic partial differential equations that are used in the subsequent section as well as a useful Itô type formula, see Theorem 4.1 and some preliminary Lemmas. The paper is concluded with Section 5 where the uniqueness of the solution of equation (3.12), respectively (3.13) is proved, see Theorem 5.5 below. Finally in the Appendix we recall the definition of the Moore–Penrose pseudo–inverse A^+ of a (possibly) rectangular matrix A, and prove that the mapping $A \mapsto A^+$ is measurable.

2 Framework

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space together with a filtration $(\mathcal{F}_t)_{t\geq 0}$ which satisfies the usual conditions[2]. We recall that to such a filtration we associate the σ–algebra $\mathcal{P}(\mathcal{F}_t)$ of progressively measurable subsets of $\Omega \times \mathbb{R}_+$, which is the class of sets $A \subset \Omega \times \mathbb{R}_+$ which are such that for all $t \geq 0$,

$$A \cap (\Omega \times [0, t]) \in \mathcal{F}_t \otimes \mathcal{B}_{[0,t]},$$

[1] The duality property is shown through the use of a collection of exponential martingales first introduced in the filtering framework by Krylov and Rozovskii.[10]

[2] The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions provided: a. $\mathcal{F}$ is complete i.e. $A \subset B$, $B \in \mathcal{F}$ and $\mathbb{P}(B) = 0$ implies that $A \in \mathcal{F}$ and $\mathbb{P}(A) = 0$, b. The filtration $\mathcal{F}_t$ is right continuous i.e. $\mathcal{F}_t = \mathcal{F}_{t+}$, and, c. $\mathcal{F}_0$ (and consequently all $\mathcal{F}_t$ for $t \geq 0$) contains all the $\mathbb{P}$-null sets.

where $\mathcal{B}_{[0,t]}$ denotes the σ–algebra of Borel measurable subsets of $[0,t]$. On $(\Omega, \mathcal{F}, \mathbb{P})$ we consider a $\mathcal{P}(\mathcal{F}_t)$-measurable[3] process (X,Y) with continuous paths. The process X is called the signal process and is assumed to take values in $\mathbb{R}^d$ (termed as the state space). The process Y is assumed to take values in $\mathbb{R}^{d'}$ and is called the observation process.

We will assume that the processes (X,Y) satisfy the following systems of stochastic differential equations

$$X_t = X_0 + \int_0^t f(s, X_s, Y_s)\,\mathrm{d}s + \int_0^t g(s, X_s, Y_s)\,\mathrm{d}V_s + \int_0^t \bar{g}(s, X_s, Y_s)\,\mathrm{d}W_s, \tag{2.1}$$

$$Y_t = Y_0 + \int_0^t h(s, X_s, Y_s)\,\mathrm{d}s + \int_0^t k(s, Y_s) dW_s, \tag{2.2}$$

where V and W are mutually independent ℓ (resp. ℓ') dimensional standard Brownian motions, and f,$g, \bar{g}, h, k$ satisfy suitable conditions so that the system (2.1)+(2.2) has a unique solution (see **Assumption E** below). In addition, we assume that

$$h(s,x,y) = h_1(s,y) + k(s,y)h_2(s,x,y). \tag{2.3}$$

Let $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^{d'})$ be the associated product Borel σ-algebra on $\mathbb{R}^d$ and, respectively, $\mathbb{R}^d \times \mathbb{R}^{d'}$ and let $b\mathcal{B}(\mathbb{R}^d)$ and $b\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^{d'})$ be the space of bounded $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^{d'})$, respectively, $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^{d'})$ measurable functions. Let A_s be the following differential operator

$$\begin{aligned} A_s \varphi(x) &= \sum_{i=1}^d f^i(s,x,Y_s)\,\partial_i \varphi(x) + \frac{1}{2}\sum_{i,j=1}^d a^{ij}(s,x,Y_s)\,\partial_i\partial_j\varphi(x), \\ a^{ij}(s,x,Y_s) &= \sum_{p=1}^{\ell} g^{ip}g^{jp}(s,x,Y_s) + \sum_{p=1}^{\ell'} \bar{g}^{ip}\bar{g}^{jp}(s,x,Y_s). \end{aligned}$$

We will impose the following sets of assumptions on the coefficients of the system (2.1)+(2.2):

Assumption E. The functions

$$\begin{aligned} f &: [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^{d'} \to \mathbb{R}^d \\ g &: [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^{d'} \to \mathbb{R}^{d\times\ell} \\ \bar{g} &: [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^{d'} \to \mathbb{R}^{d\times\ell'} \\ h &: [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^{d'} \to \mathbb{R}^{d'} \\ h_2 &: [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^{d'} \to \mathbb{R}^{\ell'} \\ h_1 &: [0,\infty) \times \mathbb{R}^{d'} \to \mathbb{R}^{d'} \\ k &: [0,\infty) \times \mathbb{R}^{d'} \to \mathbb{R}^{d'\times\ell'} \end{aligned}$$

have the following properties:

[3] This means that the mapping $(\omega, t) \mapsto (X_t(\omega), Y_t(\omega))$ is $\mathcal{P}(\mathcal{F}_t)$-measurable.

- $f, g, \bar{g}, h, h_1$ and k are locally Lipschitz in the (x, y) variables. In other words, for any $R > 0$, we have that

$$||f(t, x_1, y_1) - f(t, x_2, y_2)|| \leq K_R (||x_1 - x_2|| + ||y_1 - y_2||), \quad x_1, x_2 \in \mathbf{B}_R^d, \quad y_1, y_2 \in \mathbf{B}_R^{d'},$$

where $\mathbf{B}_R^d$ (resp. $\mathbf{B}_R^{d'}$) is the ball of centre 0 and radius R in $\mathbb{R}^d$ (resp. in $\mathbb{R}^{d'}$) and K_R is a constant which may depend upon R, but is independent of all variables, with a similar condition imposed on $g, \bar{g}, h, h_1$ and k.

- $f, g, \bar{g}, h, h_1, h_2$ and k satisfy a linear growth condition in the (x, y) variables. In other words,

$$||f(t, x, y)|| \leq K (1 + ||x|| + ||y||) \quad x_1, x_2 \in \mathbb{R}^d, \quad y_1, y_2 \in \mathbb{R}^{d'},$$

where K is a constant independent of all variables, with a similar condition imposed on $g, \bar{g}, h, h_1$ and k.

We also assume that X_0 and Y_0 have finite second moments, that is $\mathbb{E}[||X_0||^2 + ||Y_0||^2] < \infty$. The following is a classical result in the theory of stochastic differential equations, see e.g., [9] for a proof.

Remark 2.1. *Under **Assumption E**, the system (2.1)+(2.2) has a unique global solution. Moreover, for any $T > 0$, we have that*

$$\mathbb{E}\left[\sup_{s\in[0,T]} ||X_s||^2\right] + \mathbb{E}\left[\sup_{s\in[0,T]} ||Y_s||^2\right] < \infty. \tag{2.4}$$

Assumption U. The functions $f,\ g, \bar{g}, h_2$ are bounded on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^{d'}$ for arbitrary $T > 0$. The functions h_1 and k are bounded on $[0, T] \times \mathbb{R}^{d'}$ for arbitrary $T > 0$. Moreover, for some integer $n > \frac{d}{2} + 2$, all the partial derivatives of the functions $f,\ g, \bar{g}, h$ in the x variable with multi-index α, such that $|\alpha| \leq n$, are bounded on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^{d'}$ for arbitrary $T > 0$

Remark 2.2. *In the following, the evolution equation of the conditional distribution of the signal given the observation will be derived under **Assumption E**, whilst the uniqueness of its solution will be proved under the joint **Assumptions E+U**.*

Let $\{\mathcal{Y}_t,\ t \geq 0\}$ be the usual augmentation of the filtration associated with the process Y, viz

$$\mathcal{Y}_t = \bigcap_{\varepsilon>0} \sigma(Y_s,\ s \in [0, t + \varepsilon]) \vee \mathcal{N}, \quad \mathcal{Y} = \bigvee_{t\in\mathbb{R}_+} \mathcal{Y}_t. \tag{2.5}$$

where $\mathcal{N}$ is the class of all $\mathbb{P}$-null sets. Note that Y is $\mathcal{P}(\mathcal{F}_t)$-measurable, hence $\mathcal{Y}_t \subset \mathcal{F}_t$ for all $t \geq 0$.

Definition 2.3. *The filtering problem consists in determining the conditional distribution ς_t of the signal X at time t given the information accumulated from observing Y in the interval $[0, t]$; that is, for any $\varphi \in b\mathcal{B}(\mathbb{R}^d)$,*

$$\varsigma_t(\varphi) = \mathbb{E}[\varphi(X_t) \mid \mathcal{Y}_t]. \tag{2.6}$$

3 The Filtering Equations

In the following we deduce the evolution equation for ς_t. Let $\tilde{W} = \{\tilde{W}_t,\ t \geq 0\}$ be the process defined as

$$\tilde{W}_t = W_t + \int_0^t h_2(s, X_s, Y_s)ds.$$

We shall construct a new measure under which $\tilde{W}$ becomes a Brownian motion and ς has a representation in terms of an associated unnormalised version π. This π is then shown to satisfy a linear evolution equation which leads to the evolution equation for ς by an application of Itô's formula. Define $Z = (Z_t)_{t\geq 0}$ to be the exponential local martingale

$$Z_t = \exp\left(-\int_0^t h_2\left(s, X_s, Y_s\right)^\top dW_s - \frac{1}{2}\int_0^t \left|h_2\left(s, X_s, Y_s\right)\right|^2 ds\right). \tag{3.1}$$

We will work under the following additional assumption:

Assumption M. We assume that

$$\mathbb{E}\left[Z_t\right] = 1, \quad \forall t > 0. \tag{3.2}$$

In particular this implies that Z is a genuine martingale (not just a local martingale).

Remark 3.1. *There are several assumptions under which (3.2) holds, see e.g. Proposition 2.50 in [14]. The sufficient condition (ii) of that Proposition requires that for each $t > 0$, there exists $\gamma > 0$ such that*

$$\sup_{0\leq s\leq t} \mathbb{E}[\exp(\gamma|h_2(s, X_s, Y_s)|^2)] < \infty.$$

This condition is, in particular, satisfied if $h_2(t, X_t, Y_t)$ is a Gaussian process with locally bounded mean and variance, and also clearly if h_2 is bounded. It also follows that Assumption **U** *implies Assumption* **M**.

Let $\tilde{\mathbb{P}}$ be the probability measure defined on the field $\bigcup_{0\leq t<\infty} \mathcal{F}_t$ that is specified by its Radon–Nikodym derivative Z_t on each $\mathcal{F}_t$ with respect to the corresponding trace of $\mathbb{P}$; that is, for each $t \geq 0$:

$$\left.\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = Z_t.$$

$\tilde{\mathbb{P}}$ restricted to each $\mathcal{F}_t$ is equivalent to $\mathbb{P}$ since Z_t is a positive random variable[4].

Let $\tilde{Z} = \{\tilde{Z}_t,\ t \geq 0\}$ be the process defined as $\tilde{Z}_t = Z_t^{-1}$ for $t \geq 0$. Under $\tilde{\mathbb{P}}$, $\tilde{Z}_t$ satisfies the following stochastic differential equation,

$$\mathrm{d}\tilde{Z}_t = \tilde{Z}_t h_2(t, X_t, Y_t)^\top \,\mathrm{d}\tilde{W}_t = \sum_{j=1}^{\ell'} \tilde{Z}_t h_2^j(t, X_t, Y_t)\,\mathrm{d}\tilde{W}_t^j \tag{3.3}$$

[4]Note that we have not defined $\tilde{\mathbb{P}}$ on $\mathcal{F}_\infty$, where $\mathcal{F}_\infty = \bigvee_{t=0}^\infty \mathcal{F}_t = \sigma\left(\bigcup_{0\leq t<\infty} \mathcal{F}_t\right)$.

and since $\tilde{Z}_0 = 1$,

$$\tilde{Z}_t = \exp\left(\sum_{j=1}^{\ell'} \int_0^t h_2^j(s, X_s, Y_s)\,\mathrm{d}\tilde{W}_s^j - \frac{1}{2}\sum_{j=1}^{\ell'} \int_0^t h_2^j(s, X_s, Y_s)^2\,\mathrm{d}s\right), \tag{3.4}$$

then $\tilde{\mathbb{E}}[\tilde{Z}_t] = \mathbb{E}[\tilde{Z}_t Z_t] = 1$. So $\tilde{Z}$ is an $\mathcal{F}_t$–martingale under $\tilde{\mathbb{P}}$ and

$$\left.\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\tilde{\mathbb{P}}}\right|_{\mathcal{F}_t} = \tilde{Z}_t \quad \text{for } t \geq 0.$$

The probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are therefore equivalent on each σ-field $\mathcal{F}_t$ for any $t \geq 0$. The following proposition is a direct consequence of Girsanov's theorem:

Proposition 3.2. *If assumption* **M** *holds, then under* $\tilde{\mathbb{P}}$ *the process* $\tilde{W}$ *is a Brownian motion independent of* V*.*

Remark 3.3. *Since* $\mathbb{P}$ *and* $\tilde{\mathbb{P}}$ *are absolutely continuous with respect to each other, they have the same class of null sets* $\mathcal{N}$ *and therefore the (augmented) observation filtration is the same both under* $\mathbb{P}$ *and under* $\tilde{\mathbb{P}}$*.*

The following proposition is a consequence of the independent increments property of the process $\tilde{W}$ under $\tilde{\mathbb{P}}$.

Proposition 3.4. *Let* U *be an integrable* $\mathcal{F}_t$*-measurable random variable. Then we have*

$$\tilde{\mathbb{E}}[U \mid \mathcal{Y}_t] = \tilde{\mathbb{E}}[U \mid \mathcal{Y}]. \tag{3.5}$$

Proof. Under **Assumption E**, the process Y is the unique strong solution of the equation

$$Y_t = Y_0 + \int_0^t h_1(s, Y_s)\,\mathrm{d}s + \int_0^t k(s, Y_s) d\tilde{W}_s$$

driven by the Brownian motion $\tilde{W}$ (under $\tilde{\mathbb{P}}$) and we deduce from this that

$$\mathcal{Y} \subset \mathcal{Y}_t \vee \mathcal{F}^{t,\tilde{W}}$$

where $\mathcal{F}^{t,\tilde{W}} = \sigma\left(\tilde{W}_{t+s} - \tilde{W}_t| \;\; s > 0\right)$. Moreover $\mathcal{F}^{t,\tilde{W}}$ is independent of $\mathcal{F}_t \supseteq \mathcal{Y}_t$ under $\tilde{\mathbb{P}}$. It follows that since U is $\mathcal{F}_t$ measurable,

$$\tilde{\mathbb{E}}[U \mid \mathcal{Y}] = \tilde{\mathbb{E}}[\tilde{\mathbb{E}}[U \mid \mathcal{Y}_t \vee \mathcal{F}^{t,\tilde{W}}] \mid \mathcal{Y}] = \tilde{\mathbb{E}}[U \mid \mathcal{Y}_t].$$

□

In the following, the notation $\tilde{\mathbb{P}}(\mathbb{P})$-a.s. means that the result holds both $\tilde{\mathbb{P}}$-a.s. and $\mathbb{P}$-a.s.

Proposition 3.5 (Kallianpur–Striebel). *If assumption* $\mathbf{M}$ *holds, for every* $\varphi \in b\mathcal{B}(\mathbb{R}^d)$*, and* $t \in [0, \infty)$*,*

$$\varsigma_t(\varphi) = \frac{\tilde{\mathbb{E}}[\tilde{Z}_t \varphi(X_t) \mid \mathcal{Y}]}{\tilde{\mathbb{E}}[\tilde{Z}_t \mid \mathcal{Y}]} \qquad \tilde{\mathbb{P}}(\mathbb{P})\text{-a.s.} \tag{3.6}$$

Proof. Since $\varsigma_t(\varphi) = \mathbb{E}[\varphi(X_t)|\mathcal{Y}_t]$, for any $A \in \mathcal{Y}_t$,

$$\mathbb{E}[1_A \varsigma_t(\varphi)] = \mathbb{E}[1_A \varphi(X_t)]\,.$$

Consequently

$$\begin{aligned}
\tilde{\mathbb{E}}[1_A \varsigma_t(\varphi)\tilde{Z}_t] &= \tilde{\mathbb{E}}[1_A \varphi(X_t)\tilde{Z}_t] \\
\tilde{\mathbb{E}}[\tilde{\mathbb{E}}[1_A \varsigma_t(\varphi)\tilde{Z}_t|\mathcal{Y}_t]]] &= \tilde{\mathbb{E}}[\tilde{\mathbb{E}}[1_A \varphi(X_t)\tilde{Z}_t|\mathcal{Y}_t]]] \\
\tilde{\mathbb{E}}[1_A \varsigma_t(\varphi)\tilde{\mathbb{E}}[\tilde{Z}_t \mid \mathcal{Y}_t]] &= \tilde{\mathbb{E}}[1_A \tilde{\mathbb{E}}[\tilde{Z}_t \varphi(X_t) \mid \mathcal{Y}_t]].
\end{aligned}$$

It follows that $\tilde{\mathbb{E}}\left[1_A \left\{\varsigma_t(\varphi)\tilde{\mathbb{E}}[\tilde{Z}_t \mid \mathcal{Y}_t] - \tilde{\mathbb{E}}[\tilde{Z}_t\varphi(X_t) \mid \mathcal{Y}_t]\right\}\right] = 0$ for any $A \in \mathcal{Y}_t$. Since

$$\varsigma_t(\varphi)\tilde{\mathbb{E}}[\tilde{Z}_t \mathcal{Y}_t] - \tilde{\mathbb{E}}[\tilde{Z}_t \varphi(X_t)|\mathcal{Y}_t]$$

is $\mathcal{Y}_t$-measurable, we deduce that

$$\varsigma_t(\varphi)\tilde{\mathbb{E}}[\tilde{Z}_t \mid \mathcal{Y}_t] - \tilde{\mathbb{E}}[\tilde{Z}_t \varphi(X_t) \mid \mathcal{Y}_t] = 0$$

$\tilde{\mathbb{P}}$- almost surely. Hence (3.6) holds by Proposition 3.4. □

Let $\zeta = \{\zeta_t,\ t \geq 0\}$ be the process defined by

$$\zeta_t = \tilde{\mathbb{E}}[\tilde{Z}_t \mid \mathcal{Y}], \tag{3.7}$$

then as $\tilde{Z}_t$ is an $\mathcal{F}_t$-martingale under $\tilde{\mathbb{P}}$ and $\mathcal{Y}_s \subseteq \mathcal{F}_s$, it follows that for $0 \leq s < t$,

$$\tilde{\mathbb{E}}[\zeta_t \mid \mathcal{Y}_s] = \tilde{\mathbb{E}}[\tilde{Z}_t|\mathcal{Y}_s] = \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}[\tilde{Z}_t \mid \mathcal{F}_s] \mid \mathcal{Y}_s\right] = \tilde{\mathbb{E}}[\tilde{Z}_s \mid \mathcal{Y}_s] = \tilde{\mathbb{E}}[\tilde{Z}_s \mid \mathcal{Y}] = \zeta_s,$$

where the penultimate equality follows by Proposition 3.4. Therefore by Doob's regularization theorem (see e.g. Theorem 3.13 in [9]), since the filtration $\mathcal{Y}_t$ satisfies the usual conditions we can choose a càdlàg version of ζ_t which is a $\mathcal{Y}_t$-martingale. In what follows, assume that $\{\zeta_t, t \geq 0\}$ has been chosen to be this version. Given this version ζ, Proposition 3.5 suggests the following definition:

Definition 3.6. *Define the unnormalised conditional distribution of* X *to be the measure-valued process* $\pi = \{\pi_t,\ t \geq 0\}$ *given by* $\pi_t = \zeta_t \varsigma_t$ *for any* $t \geq 0$.

Notation. We shall denote by $\mathcal{M}_F(\mathbb{R}^d)$ the set of finite measures on $\mathbb{R}^d$, which we equip with the topology of weak convergence (i.e. $\mu_n \to \mu$ if $\langle \mu_n, \varphi\rangle \to \langle \mu, \varphi\rangle$, for all $\varphi \in C_b(\mathbb{R}^d)$), the set of continuous bounded functions on $\mathbb{R}^d$.

Lemma 3.7. *Under assumption* $\mathbf{M}$*, the process* $\{\pi_t,\ t \geq 0\}$ *is an* $\mathcal{M}_F(\mathbb{R}^d)$*–valued càdlàg and* $\mathcal{P}(\mathcal{Y}_t)$*-measurable process. Furthermore, for any* $t \geq 0$*,* $\varphi \in b\mathcal{B}(\mathbb{R}^d)$*,*

$$\pi_t(\varphi) = \tilde{\mathbb{E}}\left[\tilde{Z}_t \varphi(X_t) \mid \mathcal{Y}\right] \qquad \tilde{\mathbb{P}}(\mathbb{P})\text{-}a.s. \tag{3.8}$$

Proof. Both $\varsigma_t(\varphi)$ and ζ_t are $\mathcal{P}(\mathcal{Y}_t)$–measurable. By construction $\{\zeta_t,\ t \geq 0\}$ is also càdlàg. Moreover, there exists a suitable version of the process $\varsigma = \{\varsigma_t,\ t \geq 0\}$, so that ς_t is $\mathcal{P}(\mathcal{Y}_t)$-measurable probability measure-valued process for which (2.6) holds almost surely, see Theorem 2.24 in [1]. In addition, since $\mathcal{Y}_t$ is right-continuous, it follows that ς has a càdlàg version (see Corollary 2.26 in [1]). In the following, we take ς to be this version. Moreover, for any continuous bounded function φ, $\varsigma_t(\varphi)$ is the optional projection of $\varphi(X_t)$ with respect to the filtration $\mathcal{Y}_t$. Finally, since $\{\varsigma_t,\ t \geq 0\}$ is càdlàg and $\mathcal{P}(\mathcal{Y}_t)$ measurable, it follows that the process $\{\pi_t, t \geq 0\}$ is càdlàg and $\mathcal{P}(\mathcal{Y}_t)$-measurable.

For the second part, from Proposition 3.4 and Proposition 3.5 it follows that

$$\pi_t(\varphi) = \varsigma_t(\varphi)\tilde{\mathbb{E}}[\tilde{Z}_t \mid \mathcal{Y}] = \tilde{\mathbb{E}}[\tilde{Z}_t\varphi(X_t) \mid \mathcal{Y}] \qquad \tilde{\mathbb{P}}\text{-a.s.},$$

From (3.7), $\tilde{\mathbb{E}}[\tilde{Z}_t \mid \mathcal{Y}_t] = \zeta_t$ a.s. from which the result follows. ☐

Definition 3.6 gives us the following immediate corollary:

Corollary 3.8. *Under assumption* $\mathbf{M}$*, for every* $\varphi \in b\mathcal{B}(\mathbb{R}^d)$*,*

$$\varsigma_t(\varphi) = \frac{\pi_t(\varphi)}{\pi_t(\mathbf{1})} \qquad \forall t \in [0,\infty) \qquad \tilde{\mathbb{P}}(\mathbb{P})\text{-}a.s. \tag{3.9}$$

Remark 3.9. *The fact that the process* ς *has càdlàg paths is an application of the properties of the optional projection of a stochastic process, see [1] for further details. Whilst it is true that the optional projection of a process with càdlàg paths has càdlàg paths, it is not, in general, true that the optional projection of a continuous process is a continuous process. A counterexample is the Azema martingale, see Theorem 61, pp 180-182, [15].*[5]

Continuity of both ς *and* π *can be ensured if additional constraints are imposed. For example, if we have that*

$$\tilde{\mathbb{E}}[\sup_{t\in[0,T]} \tilde{Z}_t] < \infty, \tag{3.10}$$

for arbitrary $T > 0$*, then the process* ζ *is continuous by the (conditional) dominated convergence theorem. Similarly, the measure valued process* $\{\pi_t,\ t \geq 0\}$ *also has continuous paths in* $\mathcal{M}(\mathbb{R}^d)$*, the space of finite measures endowed with the weak topology. In turn this implies that also the process* $\{\varsigma_t,\ t \geq 0\}$ *has continuous paths in the same topology by (3.9).*

[5]We thank Martin Clark for pointing out this example to us.

The Kallianpur–Striebel formula explains the usage of the term unnormalised in the definition of π_t as the denominator $\pi_t(\mathbf{1})$ can be viewed as the normalizing factor. Below k^+ stands for the Moore Penrose pseudo-inverse of the matrix k (see the Appendix below for its definition). Since $(\omega, t) \mapsto k(t, Y_t(\omega))$ is $\mathcal{P}(\mathcal{Y}_t)$-measurable, and from Lemma 6.1 $k \mapsto k^+$ is measurable, we have that $k^+(t, Y_t)$ is $\mathcal{P}(\mathcal{Y}_t)$ measurable. Finally $h_2^\top$ stands for the transpose of h_2 in other words the row vector corresponding to h_2

Lemma 3.10. *For all $t \geq 0$, we have that*

$$\int_0^t \left(\left|\left|\pi_s(\bar{g})k^+k\,(s, Y_s)\right|\right|^2 + \left|\left|\pi_s(h_2^\top)k^+k\,(s, Y_s)\right|\right|^2 \right) \mathrm{d}s < \infty, \quad \tilde{\mathbb{P}}\text{−a.s.} \tag{3.11}$$

As a consequence, the stochastic integrals

$$t \mapsto \int_0^t \pi_s(\nabla\varphi_s\bar{g} + \varphi_s h_2^\top)k^+\,(s, Y_s)\,(\mathrm{d}Y_s - h_1(s, Y_s)\mathrm{d}s), \quad \textit{and}$$

$$t \mapsto \int_0^t \left(\varsigma_s(\nabla\varphi_s\bar{g} + \varphi_s h_2^\top) - \varsigma_s(\varphi)\varsigma_s(h_2^\top)\right) k^+\,(s, Y_s)\,(\mathrm{d}Y_s - \varsigma_s(h)\,\mathrm{d}s)$$

are well defined for any function $\varphi \in C_b^{1,2}([0, \infty) \times \mathbb{R}^d))$ and are local semi-martingales with almost surely continuous paths.

Proof. We only treat the first term in (3.11). The second can be dealt with in the same way. We first note that

$$\pi_s(\bar{g})k^+k\,(s, Y_s) = \varsigma_s(\bar{g})\pi_s(1)k^+k\,(s, Y_s)\,,$$

and for any $0 \leq s \leq t$,

$$\left\|\varsigma_s(\bar{g})\pi_s(1)k^+k\,(s, Y_s)\right\| \leq \sup_{0\leq s\leq t} \pi_s(1)\left\|k^+k\,(s, Y_s)\right\| \times \|\varsigma_s(\bar{g})\|$$

Since k^+k is locally bounded and the supremum on the interval $[0, t]$ of the process $\pi_s(1)$ (π_s has cádlág paths) and of the process $\|Y_s\|$ (Y has continuous paths) are finite $\tilde{P}$-a.s., clearly from (6.2) below,

$$\sup_{0\leq s\leq t} \pi_s(1)^2 \left\|k^+k\,(s, Y_s)\right\|^2 < \infty, \quad \tilde{P}-\text{ a.s.}$$

So it suffices to show that

$$\int_0^t \|\varsigma_s(\bar{g})\|^2\,ds < \infty \text{ a.s.}$$

This, in turn, is a consequence of the fact that for any entry $\bar{g}_{i,j}$ of the matrix $\bar{g}$, using Jensen's inequality,

$$\begin{aligned}
\mathbb{E}\int_0^t \left|\varsigma_s\left(\bar{g}_{i,j}\right)\right|^2 ds &= \mathbb{E}\int_0^t |\,\mathbb{E}\left[\bar{g}_{i,j}\,|\mathcal{Y}_s\right|^2|ds \\
&\leq \mathbb{E}\int_0^t \mathbb{E}\left[|\bar{g}_{i,j}|^2 \mid \mathcal{Y}_s\right] ds \\
&= \mathbb{E}\int_0^t \left|\bar{g}_{i,j}\left(s, X_s, Y_s\right)\right|^2 ds \\
&< \infty,
\end{aligned}$$

where the last inequality follows from the fact that all coefficients have at most linear growth and (2.4).

We next consider the two stochastic integrals. For the first one, we note that it can be rewritten as

$$\int_0^t \pi_s(\nabla\varphi_s \bar{g} + \varphi_s h_2^\top) k^+ k\,(s, Y_s)\,\mathrm{d}\tilde{W}_s,$$

and the second one equals

$$\int_0^t \left(\varsigma_s(\nabla\varphi_s \bar{g} + \varphi_s h_2^\top) - \varsigma_s(\varphi)\varsigma_s(h_2^\top)\right) k^+ k\,(s, Y_s) \left(\mathrm{d}\tilde{W}_s - \varsigma_s(h_2)\mathrm{d}s\right),$$

so that the result follows from the first part of the proof, combined with the fact that $\|k^+ k\,(s, Y_s)\| \leq C$, see (6.2) below. □

We are now in a position to establish the evolution equations for the processes π and ς in this set-up. Note that all stochastic integrals in the equations for π are well defined as per the above Lemma. Similarly all the deterministic integrals are well defined as the integrands are locally bounded. Similar arguments apply to the integrals in the equation for ς.

Theorem 3.11. *Under assumption **E**, the process π_t satisfies the following evolution equation*

$$\begin{aligned}\pi_t(\varphi_t) &= \pi_0(\varphi_0) + \int_0^t \pi_s\left(\partial_s\varphi_s + A_s\varphi_s\right)\,\mathrm{d}s \\ &\quad + \int_0^t \pi_s(\nabla\varphi_s \bar{g} + \varphi_s h_2^\top) k^+\,(s, Y_s)\left(\mathrm{d}Y_s - h_1\,(s, Y_s)\,\mathrm{d}s\right), \qquad \tilde{\mathbb{P}}\text{-a.s.}, \quad \forall t \geq 0 \qquad (3.12)\end{aligned}$$

for any function $\varphi \in C_b^{1,2}([0,\infty)\times\mathbb{R}^d)$[6]*. Moreover the conditional distribution ς_t satisfies the following evolution equation*

$$\begin{aligned}\varsigma_t(\varphi_t) &= \varsigma_0(\varphi_0) + \int_0^t \varsigma_s(\partial_s\varphi_s + A_s\varphi_s)\,\mathrm{d}s \\ &\quad + \int_0^t \left(\varsigma_s(\nabla\varphi_s \bar{g} + \varphi_s h_2^\top) - \varsigma_s(\varphi)\varsigma_s(h_2^\top)\right) k^+\,(s, Y_s)\left(\mathrm{d}Y_s - \varsigma_s(h)\,\mathrm{d}s\right) \qquad (3.13)\end{aligned}$$

for any function $\varphi \in C_b^{1,2}([0,\infty)\times\mathbb{R}^d)$.

Remark 3.12. *Assumption **E** includes the degenerate case $k = 0$. In this case, the observation process satisfies the evolution equation*

$$Y_t = Y_0 + \int_0^t h_1(s, Y_s)\,\mathrm{d}s. \qquad (3.14)$$

[6]The set $C_b^{1,2}([0,\infty)\times\mathbb{R}^d)$ is the set of functions $\varphi : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ that are once differentiable in the first variable and twice differentiable in the second variable and have all derivatives bounded.

It follows that Y being deterministic, it is independent of X. Hence both ς and π coincide with the law of the signal X. In particular, since $k^+ = 0^+ = 0$, equation (3.12) degenerates to

$$\pi_t(\varphi_t) = \pi_0(\varphi_0) + \int_0^t \pi_s \left(\partial_s \varphi_s + A_s \varphi_s\right) \mathrm{d}s \tag{3.15}$$

and so does equation (3.13).

Proof of Theorem 3.11.

We can re-write the equation satisfied by the process (X, Y, Z) as being driven by the pair of processes $(V, \tilde{W})$

$$\begin{aligned} X_t &= X_0 + \int_0^t [f(s, X_s, Y_s) - \bar{g} h_2(s, X_s, Y_s)] \mathrm{d}s + \int_0^t g(s, X_s, Y_s)\, \mathrm{d}V_s + \int_0^t \bar{g}(s, X_s, Y_s) \mathrm{d}\tilde{W}_s \\ Y_t &= Y_0 + \int_0^t h_1(s, Y_s)\, \mathrm{d}s + \int_0^t k(s, Y_s) \mathrm{d}\tilde{W}_s \\ \tilde{Z}_t &= 1 + \int_0^t \tilde{Z}_s \left(h_2(s, X_s, Y_s)\right)^\top \mathrm{d}\tilde{W}_s \end{aligned}$$

To ensure the integrability of the terms appearing in the following computations, we first approximate $\tilde{Z}_t$ with $\tilde{Z}_t^\varepsilon$ given by

$$\tilde{Z}_t^\varepsilon = \frac{\tilde{Z}_t}{1 + \varepsilon \tilde{Z}_t} = \frac{1}{\varepsilon} \frac{\varepsilon \tilde{Z}_t}{1 + \varepsilon \tilde{Z}_t} = \frac{1}{\varepsilon}\left(1 - \frac{1}{1 + \varepsilon \tilde{Z}_t}\right).$$

By Itô's formula, we deduce that[7]

$$\begin{aligned} \mathrm{d}\varphi(t, X_t) &= [(\partial_t + A_t)\varphi](t, X_t) \mathrm{d}t - \nabla\varphi(t, X_t)\, \bar{g} h_2(t, X_t, Y_t)] \mathrm{d}t \\ &\quad + (\nabla\varphi g)(t, X_t)\, \mathrm{d}V_t + \nabla\varphi \bar{g}(t, X_t)\, \mathrm{d}\tilde{W}_t \\ \mathrm{d}\tilde{Z}_t^\varepsilon &= \tilde{Z}_t (1 + \varepsilon \tilde{Z}_t)^{-2} \left(h_2(t, X_t, Y_t)\right)^\top \mathrm{d}\tilde{W}_t \\ &\quad - \varepsilon (1 + \varepsilon \tilde{Z}_t)^{-3} \tilde{Z}_t^2 \left(h_2(t, X_t, Y_t)\right)^\top h_2(t, X_t, Y_t)\, \mathrm{d}t \end{aligned}$$

[7] In the following $\nabla\varphi$ will denote the row vector $(\partial_1\varphi, ..., \partial_d\varphi)$.

Therefore

$$
\begin{aligned}
\mathrm{d}\varphi\left(t,X_t\right)\tilde{Z}_t^{\varepsilon} &= \tilde{Z}_t^{\varepsilon}[(\partial_t+A_t)\varphi](t,X_t)-\nabla\varphi\left(t,X_t\right)\bar{g}h_2(t,X_t,Y_t)]\mathrm{d}t \\
&+\tilde{Z}_t^{\varepsilon}\left((\nabla\varphi g)(t,X_t)\,\mathrm{d}V_t+\nabla\varphi\bar{g}\left(t,X_t\right)\mathrm{d}\tilde{W}_t\right) \\
&+\varphi\left(t,X_t\right)\tilde{Z}_t(1+\varepsilon\tilde{Z}_t)^{-2}\left(h_2(t,X_t,Y_t)\right)^{\top}\,\mathrm{d}\tilde{W}_t \\
&+\varphi\left(t,X_t\right)\left(-\varepsilon(1+\varepsilon\tilde{Z}_t)^{-3}\tilde{Z}_t^2\left(h_2(t,X_t,Y_t)\right)^{\top}h_2(t,X_t,Y_t)\,\mathrm{d}t\right) \\
&+\nabla\varphi\bar{g}\left(t,X_t\right)h_2(t,X_t,Y_t)\tilde{Z}_t(1+\varepsilon\tilde{Z}_t)^{-2}\mathrm{d}t \\
&= \tilde{Z}_t^{\varepsilon}[(\partial_t+A_t)\varphi](t,X_t)\mathrm{d}t-\nabla\varphi\left(t,X_t\right)\bar{g}h_2(t,X_t,Y_t)]\mathrm{d}t \\
&+\tilde{Z}_t^{\varepsilon}\nabla\varphi\bar{g}\left(t,X_t\right)k^{+}k\left(t,Y_t\right)\,\mathrm{d}\tilde{W}_t \\
&+\tilde{Z}_t^{\varepsilon}\nabla\varphi\bar{g}\left(t,X_t\right)\left(I-k^{+}k\left(t,Y_t\right)\right)\mathrm{d}\tilde{W}_t \\
&+\tilde{Z}_t^{\varepsilon}(\nabla\varphi g)(t,X_t)\,\mathrm{d}V_t \\
&+\varphi\left(t,X_t\right)\tilde{Z}_t^{\varepsilon}(1+\varepsilon\tilde{Z}_t)^{-1}\left(h_2(t,X_t,Y_t)\right)^{\top}\,k^{+}k\left(t,Y_t\right)\mathrm{d}\tilde{W}_t \\
&+\varphi\left(t,X_t\right)\tilde{Z}_t^{\varepsilon}(1+\varepsilon\tilde{Z}_t)^{-1}\left(h_2(t,X_t,Y_t)\right)^{\top}(I-k^{+}k\left(t,Y_t\right))\mathrm{d}\tilde{W}_t \\
&+\varphi\left(t,X_t\right)\left(-\varepsilon(1+\varepsilon\tilde{Z}_t)^{-1}\left(\tilde{Z}_t^{\varepsilon}\right)^2\left(h_2(t,X_t,Y_t)\right)^{\top}h_2(t,X_t,Y_t)\,\mathrm{d}t\right) \\
&+\nabla\varphi\bar{g}\left(t,X_t\right)h_2(t,X_t,Y_t)\tilde{Z}_t^{\varepsilon}(1+\varepsilon\tilde{Z}_t)^{-2}\mathrm{d}t \qquad (3.16)
\end{aligned}
$$

We next take the conditional expectation $\tilde{\mathbb{E}}(\cdot|\mathcal{Y})$ in this identity, as all the terms in (3.16) are square integrable over $[0,T]\times\Omega$. To show this we use repeatedly the fact that

$$
|\tilde{Z}_t^{\varepsilon}|=|\tilde{Z}_t(1+\varepsilon\tilde{Z}_t)^{-1}|\leq\varepsilon^{-1},\quad|(1+\varepsilon\tilde{Z}_t)^{-1}|\leq 1,
$$

that $\varphi\in C_b^{1,2}([0,\infty)\times\mathbb{R}^d))$, the linear growth of $f,g,\bar{g},h_2$ and k, and the square integrability of the processes X and Y.

We next take the conditional expectation $\tilde{\mathbb{E}}(\cdot|\mathcal{Y})$ in this identity, as all the terms in (3.16) are integrable over $[0,T]\times\Omega$. The conditional expectation operator $\tilde{\mathbb{E}}(\cdot|\mathcal{Y})$ commutes with the $\mathrm{d}t$ and the $\mathrm{d}\tilde{W}_t$ integrations, while $\tilde{\mathbb{E}}(\cdot|\mathcal{Y})$ of a stochastic integral w.r.t. dV_S and to $[I-k^{+}k(s,Y_s)]\,\mathrm{d}\tilde{W}_s$ is zero. The last claim is the content of the Lemma 3.13. We deduce that

$$
\begin{aligned}
\tilde{\mathbb{E}}[\tilde{Z}_t^{\varepsilon}\varphi(t,X_t)\mid\mathcal{Y}] &= \frac{\pi_0(\varphi)}{1+\varepsilon}+\int_0^t\tilde{\mathbb{E}}\left[\tilde{Z}_s^{\varepsilon}[(\partial_s+A_s)\varphi](s,X_s)-\nabla\varphi\left(s,X_s\right)\bar{g}h_2(s,X_s,Y_s)]\mid\mathcal{Y}\right]\,\mathrm{d}s \\
&+\int_0^t\tilde{\mathbb{E}}[\tilde{Z}_s^{\varepsilon}\nabla\varphi\bar{g}\left(s,X_s\right)|\mathcal{Y}]\,k^{+}k\left(s,Y_s\right)\mathrm{d}\tilde{W}_s \\
&+\int_0^t\tilde{\mathbb{E}}[\varphi\left(s,X_s\right)\tilde{Z}_s^{\varepsilon}(1+\varepsilon\tilde{Z}_s)^{-1}\left(h_2(s,X_s,Y_s)\right)^{\top}\,|\mathcal{Y}]\,k^{+}k\left(s,Y_s\right)\mathrm{d}\tilde{W}_s \\
&+\int_0^t\tilde{\mathbb{E}}\left[\varphi\left(s,X_s\right)\left(-\varepsilon(1+\varepsilon\tilde{Z}_s)^{-1}\left(\tilde{Z}_s^{\varepsilon}\right)^2\left(h_2(s,X_s,Y_s)\right)^{\top}h_2(s,X_s,Y_s)\right)\mid\mathcal{Y}\right]\,\mathrm{d}s \\
&+\int_0^t\tilde{\mathbb{E}}[\nabla\varphi\bar{g}\left(s,X_s\right)h_2(s,X_s,Y_s)\tilde{Z}_s^{\varepsilon}(1+\varepsilon\tilde{Z}_s)^{-2}\,|\mathcal{Y}]\,\mathrm{d}s \qquad (3.17)
\end{aligned}
$$

Using Proposition 3.4 we deduce from (3.17) by taking the limit as that ε tends to 0 that

$$\begin{aligned}\pi_t(\varphi_t) &= \pi_0(\varphi_0) + \int_0^t \pi_s\left(\partial_s\varphi_s + A_s\varphi_s - \nabla\varphi\bar{g}h_2 + \nabla\varphi\bar{g}h_2\right)\mathrm{d}s + \\ &+ \int_0^t \pi_s(\nabla\varphi_s\bar{g} + \varphi_s h_2^\top)k^+(s,Y_s)\left(\mathrm{d}Y_s - h_1(s,Y_s)\,\mathrm{d}s\right), \quad \tilde{\mathbb{P}}\text{-a.s.}, \quad \forall t \geq 0 \quad (3.18)\end{aligned}$$

In order to justify taking that the limit in (3.17) gives (3.18), we need to show that the integrands on the right hand side of (3.17) are uniformly bounded in ε by processes that are integrable over the product space $[0,t]\times\Omega$. First we have that

$$\begin{aligned}&\left|\tilde{Z}_s^\varepsilon[(\partial_s + A_s)\varphi](s,X_s) - \nabla\varphi(s,X_s)\,\bar{g}h_2(s,X_s,Y_s)]\right| \\ &\leq \tilde{Z}\,||[(\partial_s + A_s)\varphi](s,X_s) - \nabla\varphi(s,X_s)\,\bar{g}h_2(s,X_s,Y_s)]|| \\ &\leq c_1\tilde{Z}\left(1 + \sum_{i=1}^d \left|f^i(s,X_s,Y_s)\right| + \frac{1}{2}\sum_{i,j=1}^d \left|a^{ij}(s,X_s,Y_s)\right| + \sum_{i=1}^d\sum_{j=1}^{l'}\left|\bar{g}^{ij}(s,X_s,Y_s)\,h_2^j(s,X_s,Y_s)\right|\right) \\ &\leq c_2\tilde{Z}\left(1 + ||X_s||^2 + ||Y_s||^2\right),\end{aligned}$$

where we used the fact that $\varphi \in C_b^{1,2}([0,\infty)\times\mathbb{R}^d))$, the linear growth of $f, g, \bar{g}, h_2$ and k and denoted by $c_1 = c_1(\varphi) = ||\partial_s\varphi||_\infty + ||\nabla\varphi||_\infty + ||\nabla\nabla\varphi||_\infty$ and c_2 is a constant depending of c_1 and on the constant K from Assumption E. We then observe that

$$\int_0^t \tilde{\mathbb{E}}\left[c\tilde{Z}\left(1 + ||X_s||^2 + ||Y_s||^2\right)\right]\mathrm{d}s = c\int_0^t \mathbb{E}\left[\left(1 + ||X_s||^2 + ||Y_s||^2\right)\right]\mathrm{d}s < \infty,$$

which justifies that the integrand in the first term on the right hand side of (3.17) is dominated by an integrable bound independent of ε. This justifies the convergence of the first term in (3.17).

A similar argument applies to the last two terms of (3.17), using

$$\begin{aligned}\left|\varphi(s,X_s)\left(-\varepsilon(1+\varepsilon\tilde{Z}_s)^{-1}\left(\tilde{Z}_s^\varepsilon\right)^2(h_2(s,X_s,Y_s))^\top h_2(s,X_s,Y_s)\right)\right| &\leq c\tilde{Z}\left(1 + ||X_s||^2 + ||Y_s||^2\right) \\ \left|\nabla\varphi\bar{g}(s,X_s)h_2(s,X_s,Y_s)\tilde{Z}_s^\varepsilon(1+\varepsilon\tilde{Z}_s)^{-1}\right| &\leq c\tilde{Z}\left(1 + ||X_s||^2 + ||Y_s||^2\right)\end{aligned}$$

The convergence of the stochastic terms is harder. We combine them into a single term and re-write it as

$$M_t^\varepsilon = \int_0^t q_s^\varepsilon k^+(s,Y_s)\left(\mathrm{d}Y_s - h_1(s,Y_s)\,\mathrm{d}s\right) =: \int_0^t q_s^\varepsilon k^+k(s,Y_s)\,\mathrm{d}\tilde{W}_s, \quad t \geq 0, \qquad (3.19)$$

where

$$q_s^\varepsilon = \tilde{\mathbb{E}}[\tilde{Z}_s^\varepsilon\nabla\varphi\bar{g}(s,X_s) + \varphi\tilde{Z}_s^\varepsilon(1+\varepsilon\tilde{Z}_s)^{-1}(h_2(s,X_s,Y_s))^\top\,|\mathcal{Y}], \quad s \geq 0$$

Observe that M^ε is a square integrable martingale, however the intended limit

$$\begin{aligned}M_t &:= \int_0^t q_s k^+k(s,Y_s)\,\mathrm{d}\tilde{W}_s = \int_0^t \pi_s(\nabla\varphi_s\bar{g} + \varphi_s h_2^\top)k^+(s,Y_s)\left(\mathrm{d}Y_s - h_1(s,Y_s)\,\mathrm{d}s\right), \quad , \quad t \geq 0, \\ q_s &:= \pi_s(\nabla\varphi_s\bar{g} + \varphi_s h_2^\top), \quad s \geq 0\end{aligned}$$

is only a local martingale for any function $\varphi \in C_b^{1,2}([0,\infty)\times\mathbb{R}^d))$ with almost surely continuous paths which may not be square integrable, (see Lemma 3.10 for details). To begin the convergence argument, observe that for any $0 \le s \le t$ and a.s.

$$\lim_{\varepsilon\to\infty} \tilde{Z}_s^\varepsilon \nabla\varphi\bar{g}\,(s,X_s) + \varphi\tilde{Z}_s^\varepsilon(1+\varepsilon\tilde{Z}_s)^{-1}\,(h_2(s,X_s,Y_s))^\top \;= \tilde{Z}_s\nabla\varphi\bar{g}\,(s,X_s) + \varphi\tilde{Z}_s\,(h_2(s,X_s,Y_s))^\top$$

and

$$\left|\left|\tilde{Z}_s^\varepsilon \nabla\varphi\bar{g}\,(s,X_s) + \varphi\tilde{Z}_s^\varepsilon(1+\varepsilon\tilde{Z}_s)^{-1}\,(h_2(s,X_s,Y_s))^\top\right|\right| \le c\tilde{Z}_s\sqrt{\left(1+||X_s||^2+||Y_s||^2\right)}. \tag{3.20}$$

Therefore, since the term on the right hand side of (3.20) is $\tilde{\mathbb{P}}$ integrable we deduce by the dominated convergence theorem for conditional expectation that

$$\lim_{\varepsilon\to\infty} q_s^\varepsilon = q_s$$

$\tilde{P}$-almost surely and almost everywhere on the interval $[0,t]$. Also one deduces that

$$||q_s^\varepsilon|| \le c\varsigma_s(||\bar{g}||+||h_2||)\pi_s(1) \tag{3.21}$$

and since the term on the right hand side of (3.21) is a.s. bounded on the interval $[0,t]$ (the argument is similar to that used in Lemma 3.10), we deduce that (we use again the fact that $||k^+k|| \le C$)

$$0 \le \lim_{\varepsilon\to\infty}\int_0^t \left|\left|(q_s^\varepsilon - q_s)\,k^+k\,(s,Y_s)\right|\right|^2 ds \le \int_0^t ||q_s^\varepsilon - q_s||^2\, ds = 0$$

$\tilde{P}$-almost surely. This, in turn, implies that (for example by using Proposition B.41. in [1])

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}|M_t^\varepsilon - M_t| = \lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\left|\int_0^t (q_s^\varepsilon - q_s)\,k^+k\,(s,Y_s)\,\mathrm{d}\tilde{W}_s\right| = 0$$

in probability. The justification of the identity (3.12) is now complete.

To deduce that the conditional distribution of the signal ς_t satisfies (3.13), we first compute the evolution equation for the reciprocal of the mass process $\frac{1}{\pi_t(1)}$ which is

$$\begin{aligned}\frac{1}{\pi_t(1)} &= \frac{1}{\pi_0(1)} - \int_0^t \frac{\varsigma_s(h_2^\top)}{\pi_s(1)}k^+k\,(s,Y_s)\,\mathrm{d}\tilde{W}_s + \int_0^t \frac{\varsigma_s(h_2^\top)}{\pi_s(1)}k^+k\left(k^+k\right)^\top(s,Y_s)\,\varsigma_s(h_2)\mathrm{d}s \\ &= \frac{1}{\pi_0(1)} - \int_0^t \frac{\varsigma_s(h_2^\top)}{\pi_s(1)}k^+k\,(s,Y_s)\,\mathrm{d}\tilde{W}_s + \int_0^t \frac{\varsigma_s(h_2^\top)}{\pi_s(1)}k^+k\,(s,Y_s)\,\varsigma_s(h_2)\mathrm{d}s.\end{aligned}$$

To obtain the second identity we used that, see (6.1),

$$\left(k^+k\right)^\top = k^+k, \qquad kk^+k = k\,.$$

Finally we use Itô's formula to deduce that

$$
\begin{aligned}
\varsigma_t(\varphi_t) &= \frac{\pi_t(\varphi_t)}{\pi_t(1)} \\
&= \frac{\pi_0(\varphi_0)}{\pi_0(1)} + \int_0^t \frac{\pi_s\left(\partial_s\varphi_s + A_s\varphi_s\right)}{\pi_s(1)}\,\mathrm{d}s + \int_0^t \frac{\pi_s(\nabla\varphi_s\bar{g} + \varphi_s h_2^\top)}{\pi_s(1)} k^+ k\,(s, Y_s)\,\mathrm{d}\tilde{W}_s \\
&\quad - \int_0^t \frac{\pi_s(\varphi_s)\varsigma_s(h_2^\top)}{\pi_s(1)} k^+ k\,(s, Y_s)\,\mathrm{d}\tilde{W}_s + \int_0^t \frac{\pi_s(\varphi_s)\varsigma_s(h_2^\top)}{\pi_s(1)} k^+ k\,(s, Y_s)\,\varsigma_s(h_2)\mathrm{d}s \\
&\quad - \int_0^t \varsigma_s(\nabla\varphi_s\bar{g} + \varphi_s h_2^\top) k^+ k\,(s, Y_s)\,\varsigma_s(h_2)\mathrm{d}s \\
&= \varsigma_0(\varphi_0) + \int_0^t \varsigma_s(\partial_s\varphi_s + A_s\varphi_s)\,\mathrm{d}s \\
&\quad + \int_0^t \left(\varsigma_s(\nabla\varphi_s\bar{g} + \varphi_s h_2^\top) - \varsigma_s(\varphi)\varsigma_s(h_2^\top)\right) k^+\,(s, Y_s)\,(\mathrm{d}Y_s - \varsigma_s(h)\,\mathrm{d}s)\,.
\end{aligned}
$$

□

Lemma 3.13. *For any progressively measurable process κ assumed to be an m–dimensional row vector satisfying for any $t > 0$ $\tilde{\mathbb{E}}\int_0^t \|\kappa_s\|^2 ds < \infty$,*

$$
\tilde{\mathbb{E}}\left[\int_0^t \kappa_s[I - k^+ k(s, Y_s)]\,\mathrm{d}\tilde{W}_s \Big| \mathcal{Y}\right] = 0\,, \forall t > 0\,.
$$

Proof. Let $N = \{N_t,\ t \geq 0\}$ be the following process

$$
N_t = \int_0^t k(s, Y_s)\,\mathrm{d}\tilde{W}_s, \quad t \geq 0. \tag{3.22}
$$

We have

$$
Y_t = Y_0 + \int_0^t h_1(s, Y_s)\,\mathrm{d}s + N_t,\ t \geq 0\,. \tag{3.23}
$$

Denote by $\mathcal{N} = \{\mathcal{N}_t, t \geq 0\}$ the filtration generated by the process N. We prove that $\mathcal{Y}_0 \vee \mathcal{N}_t = \mathcal{Y}_t$. To do this we deduce from (3.23) that $\mathcal{N}_t \subset \mathcal{Y}_t$ and also that $\mathcal{Y}_0 \vee \mathcal{N}_t \subseteq \mathcal{Y}_t$. Next, if we denote by $\vartheta = \{\vartheta_t,\ t \geq 0\}$ the process $\vartheta_t = Y_t - N_t$, we get that

$$
\vartheta_t = Y_0 + \int_0^t h_1(s, \vartheta_s + N_s)\,\mathrm{d}s.
$$

and we deduce from the above that the process ϑ, which is the unique solution of an ODE with initial condition $\vartheta_0 = Y_0$, and whose coefficient depends upon the process N (note that h_1 is a Lipschitz function in the spatial variable), is $\mathcal{P}(\mathcal{Y}_0 \vee \mathcal{N}_t)$ measurable. From this we deduce that Y_t is measurable with respect to $\mathcal{Y}_0 \vee \mathcal{N}_t$ and therefore that $\mathcal{Y}_t \subseteq \mathcal{Y}_0 \vee \mathcal{N}_t$, which in turn, implies that $\mathcal{Y}_0 \vee \mathcal{N}_t = \mathcal{Y}_t$.

In the following we will use an argument first used by Krylov and Rozovsky in [10]. Let us denote by S_t the following set of uniformly bounded test random variables:

$$S_t = \left\{ \theta_t = \exp\left(i\bar{r}^\top Y_0 + i\int_0^t r_s^\top \, \mathrm{d}N_s + \frac{1}{2}\int_0^t \|k^T(s,Y_s)r_s\|^2 \, \mathrm{d}s \right), \ r \in L^\infty\left([0,t],\mathbb{R}^{d'}\right), \ \bar{r} \in \mathbb{R}^{d'} \right\}. \tag{3.24}$$

Then S_t is a total set in $L^1(\Omega, \mathcal{Y}_0 \vee \mathcal{N}_t, \tilde{\mathbb{P}}) \equiv L^1(\Omega, \mathcal{Y}_t, \tilde{\mathbb{P}})$. That is, if $a \in L^1(\Omega, \mathcal{Y}_t, \tilde{\mathbb{P}})$ and $\tilde{\mathbb{E}}[a\theta_t] = 0$, for all $\theta_t \in S_t$, then $a = 0$ $\tilde{\mathbb{P}}$-a.s. A proof of this result is similar with that of Lemma B.39, pp 355 in [1]. In addition, if $\theta_t \in S_t$, then

$$\theta_t = \exp\left(i\bar{r}^\top Y_0\right) + i\int_0^t \theta_s r_s^\top \, \mathrm{d}N_s = \exp\left(i\bar{r}^\top Y_0\right) + i\int_0^t \theta_s \, r_s^T k(s,Y_s) \, \mathrm{d}\tilde{W}_s.$$

In view of this result, all we need to prove is that for any $r \in L^\infty\left([0,t],\mathbb{R}^{d'}\right)$, $\bar{r} \in \mathbb{R}^{d'}$,

$$\tilde{\mathbb{E}}\left(\theta_t \int_0^t \kappa_s[I - k^+k(s,Y_s)] \, \mathrm{d}\tilde{W}_s \right) = 0\,.$$

Since $k[I - k^+k]^T = 0$, as a consequence of $(k^+k)^T = k^+k$ and $kk^+k = k$ (see (6.1)), we have

$$\begin{aligned} \tilde{\mathbb{E}}\left(\theta_t \int_0^t \kappa_s[I - k^+k(s,Y_s)] \, \mathrm{d}\tilde{W}_s \right) &= 0 + i\,\tilde{\mathbb{E}}\int_0^t \theta_s r_s^T k(s,Y_s)[I - k^+k(s,Y_s)]^T \kappa_s^T ds \\ &= 0. \end{aligned}$$

□

Remark 3.14. *In Lemma B.39 in [1], the process Y is chosen to be a Brownian motion starting from 0. In particular $Y_0 = 0$. However, the argument does not use the property that Y is a Brownian motion and can be extended to any martingale. The additional term $i\bar{r}^\top Y_0$ in the argument of θ_t takes care of the non-zero initial condition.*

The following lemma is an immediate consequence of Theorem 6 in Chapter V of [15].

Lemma 3.15. *Let ς be a solution of the evolution equation (3.13) and consider the following stochastic differential equation*

$$\begin{aligned} dj_t^\varsigma &= j_t^\varsigma \varsigma_t(h_2^\top)k^+\,(t,Y_t)\, dN_t, \\ &= j_t^\varsigma \varsigma_t(h_2^\top)k^+k(t,Y_t)\, \mathrm{d}\tilde{W}_t \quad j_0^\varsigma = 1. \end{aligned} \tag{3.25}$$

Then equation (3.25) has a unique solution $j^\varsigma = \{j_t^\varsigma, t \geq 0\}$ for any solution ς of the evolution equation (3.13).

We next prove.

Lemma 3.16. *Let ς be a solution of the evolution equation (3.13) starting from ς_0 which is a probability measure. Then ς_t is a probability measure valued process, P-almost surely. Moreover if j^ς is the corresponding solution of equation (3.25), then $j_t^\varsigma \varsigma_t$ is a solution of the evolution equation (3.12).*

Proof. From (3.13) we deduce that

$$\varsigma_t(1) - 1 = -\int_0^t \left(\varsigma_s(1) - 1)\varsigma_s(h_2^\top)\right) k^+ k(t, Y_s) \left(\mathrm{d}\tilde{W}_s - \varsigma_s(h_2) ds\right),$$

which implies that $a_t^0 := \varsigma_t(1) - 1$ is a solution of the linear equation

$$da_t = -a_t \varsigma_t(h_2^\top) k^+ k(t, Y_t) \left(\mathrm{d}\tilde{W}_t - \varsigma_t(h_2) dt\right). \tag{3.26}$$

But the fact that $a_t \equiv 0$ is the unique solution of that equation is a consequence of Theorem 6 in Chapter V of [15]. □

We can now establish the final result of this section.

Theorem 3.17. *Let $\pi_0 = \varsigma_0$ be a probability measure. Then uniqueness of a measure valued solution of the evolution equation (3.12) is equivalent to uniqueness of a measure valued solution of the evolution equation (3.13).*

Proof. Let us assume first that there exists a unique solution of the evolution equation (3.13). Let π^1 and π^2 be two solutions of the evolution equation (3.12) and let $\pi^1(1)$ and $\pi^2(1)$ be the corresponding total mass processes. From (3.12), we deduce that these processes satisfy the evolution equation (take $\varphi_t \equiv 1$ in (3.12))

$$\pi_t^i(1) = \pi_0^i(1) + \int_0^t \pi_s^i(h_2^\top) k^+ (s, Y_s) \left(\mathrm{d}Y_s - h_1(s, Y_s)\, \mathrm{d}s\right). \tag{3.27}$$

Define the normalized version of π^1 and π^2, $\varsigma^i \equiv \frac{\pi^i}{\pi^i(1)}$. Then both ς^1 and ς^2 satisfy the evolution equation (3.13), as follows from the argument in the last part of the proof of Theorem 3.11. It follows that $\varsigma^1 = \varsigma^2 = \varsigma$. Moreover from equation (3.27), we deduce that

$$\begin{aligned}\pi_t^i(1) &= 1 + \int_0^t \pi_s^i(1) \frac{\pi_s^i(h_2^\top)}{\pi_s^i(1)} k^+(s, Y_s)\, dN_t \\ &= 1 + \int_0^t \pi_s^i(1) \varsigma_t(h_2^\top) k^+(s, Y_s)\, dN_t\end{aligned}$$

In other words, both $\pi^1(1)$ and $\pi^2(1)$ are solutions of the equation (3.25) and therefore must coincide by Lemma 3.15. Hence

$$\pi^1 = \pi^1(1)\varsigma = \pi^2(1)\varsigma = \pi^2.$$

Hence the evolution equation (3.12) has a unique solution.

Now let us assume that the evolution equation (3.12) has a unique solution. Let ς^1 and ς^2 be two solutions of the evolution equation (3.13) and consider j^{ς^i}, $i = 1, 2$ the corresponding solutions of the equation (3.25). Then $j^{\varsigma^1}\varsigma^1$ and $j^{\varsigma^2}\varsigma^2$ are solutions of the evolution equation (3.12) by Lemma 3.16 and therefore must be equal, $j^{\varsigma^1}\varsigma^1 = j^{\varsigma^2}\varsigma^2 = \pi$. It follows that

$$j^{\varsigma^1} = j^{\varsigma^1}\varsigma^1(1) = \pi(1) = j^{\varsigma^2}\varsigma^2(1) = j^{\varsigma^2},$$

since ς^i are probability measures, again, by Lemma 3.16. In other words for both $i = 1, 2$, j^{ς^i} coincides with the total mass process of π. This gives us

$$\varsigma^1 = \frac{\pi}{j^{\varsigma^1}} = \frac{\pi}{j^{\varsigma^2}} = \varsigma^2$$

and hence the evolution equation (3.13) has a unique solution. □

4 The Backward SPDE approach to uniqueness

Following from Theorem 3.11, the process π_t satisfies the evolution equation

$$d\pi_t(\varphi_t) = \pi_t(\partial_t\varphi_t + A\varphi_t)dt + \pi_t(B^j\varphi_t)d\tilde{W}_t^j.$$

for any function $\varphi \in C_b^{1,2}([0,\infty) \times \mathbb{R}^d))$, where we use the convention of summation over repeated indices, and the notation

$$B_t^j\varphi_t := \big(\nabla\varphi_t[\bar{g}k^+k](t,\cdot,Y_t)\big)_j + ([h_2^Tk^+k](t,\cdot,Y_t))_j\varphi_t,\ 1 \leq j \leq m.$$

In the classical filtering problem (where Y_t does not appear in the coefficients), one associates to the Zakai equation a proper adjoint backward PDE, which allows to establish uniqueness of a measure valued solution to the Zakai equation (this is Bensoussan's approach to showing uniqueness of the solution of the Zakai equation, see [3] for details).

However, in our situation, this approach is not feasible because the backward partial differential equation would involve the observation process Y_t in its coefficients, resulting in a solution that is not adapted at each time t to the past of that process. To address this issue, we employ an adjoint backward stochastic partial differential equation (SPDE) instead of an adjoint backward PDE. The solution to the SPDE remains adapted at each time t to the past of the observation process. This approach will be developed in the next section. To facilitate this, we establish a new type of Itô formula, which is essential for leveraging the duality between the Zakai equation and an adjoint BSPDE.

Let $\mathcal{E}_T$ be the space of progressively measurable (with respect to the augmented filtration generated by $\tilde{W}$) processes $\{u_t,\ 0 \leq t \leq T\}$ such that

$$u \in C([0,T]; C_b^2(\mathbb{R}^d))$$

and

$$u_t = u_0 + \int_0^t \Sigma_s ds + \int_0^t \Lambda_s^j d\tilde{W}_s^j,\ 0 \leq t \leq T\,, \tag{4.1}$$

where Σ, Λ^j are progressively measurable $C_b(\mathbb{R}^d)$-valued processes such that

$$\Sigma \in L^1(0,T; C_b(\mathbb{R}^d)), \quad \Lambda^j \in L^2(0,T; C_b^2(\mathbb{R}^d))\,.$$

We denote by $\mathcal{U}$ the set of progressively measurable processes with values in $\mathcal{M}_F(\mathbb{R}^d)$ which satisfy for all $T > 0$

$$\sup_{0\leq t\leq T} \mu_t(\mathbf{1}) < \infty \ \text{ a.s.}$$

We shall say that $\mu \in \mathcal{U}$ solves the Zakai equation if for any $\varphi \in C_b^2(\mathbb{R}^d)$,

$$\mu_t(\varphi) = \mu_0(\varphi) + \int_0^t \mu_s(A_s\varphi)ds + \int_0^t \mu_s(B_s^j\varphi)d\tilde{W}_s^j,\ t \geq 0\,.$$

We will now establish a useful Itô formula.

Theorem 4.1. *For any $u \in \mathcal{E}_T$ of the form (4.1) and any $\mu \in \mathcal{U}$ that solves the Zakai equation we have (again with the convention of summation over repeated indices)*

$$\mu_t(u_t) = \mu_0(u_0) + \int_0^t \mu_s\left(A_s u_s + \Sigma_s + B_s^j \Lambda_s^j\right) ds + \int_0^t \mu_s\left(B_s^j u_s + \Lambda_s^j\right) d\tilde{W}_s^j. \tag{4.2}$$

Proof. We fix $t > 0$. For any $n \geq 1$, $0 \leq s \leq t$ and $x \in \mathbb{R}^d$, we let

$$\Lambda_n^j(s,x) = \sum_{p=0}^{n-1} \frac{n}{t} \int_{(\frac{p-1}{n}t)^+}^{\frac{p}{n}t} \Lambda^j(r,x) dr \mathbf{1}_{[\frac{p}{n}t, \frac{p+1}{n}t)}(s), \tag{4.3}$$

and define

$$u_n(t,x) = u_0(x) + \int_0^t \Sigma_s(x) ds + \int_0^t \Lambda_n^j(s,x) d\tilde{W}_s^j.$$

It is easy to see that $\Lambda_n^j \to \Lambda^j$ in $L^2((0,T); C_b^2(\mathbb{R}^d))$ a.s.. Consequently $u_n \to u$ in $C([0,T]; C_b^2(\mathbb{R}^d))$ in probability. Hence if we show (4.2) with (u, Λ^j) replaced by (u_n, Λ_n^j), the result will follow by taking the limit as $n \to \infty$. So from now on, we assume that $\Lambda^j = \Lambda_n^j$ is given by (4.3), and delete the index n. Now it suffices to prove (4.2) with $(0,t)$ replaced by (a,b), with for some $1 \leq k \leq n-1$, $\frac{k-1}{n}t \leq a < b \leq \frac{k}{n}t$. In other words, all we need to show is that

$$\begin{aligned}\mu_b(u(b)) = \mu_a(u(a)) &+ \int_a^b \mu_s\left(A_s u(s) + \Sigma(s) + B_s^j \Lambda^j(a)\right) ds \\ &+ \int_a^b \mu_s\left(B_s^j u(s) + \Lambda^j(a)\right) d\tilde{W}_s^j.\end{aligned}$$

Let now $a = s_0 < s_1 < \cdots < s_{n'} = b$, with $s_i = a + i\frac{b-a}{n'}$, where n' is an integer which will eventually tend to $+\infty$ (while n is kept fixed). We have

$$\begin{aligned}
&\mu_{s_{i+1}}(u(s_{i+1})) - \mu_{s_i}(u(s_i)) \\
&= \mu_{s_{i+1}}(u(s_i)) - \mu_{s_i}(u(s_i)) + \mu_{s_{i+1}}(u(s_{i+1}) - u(s_i)) \\
&= \int_{s_i}^{s_{i+1}} \mu_s(A_s u(s_i)) ds + \int_{s_i}^{s_{i+1}} \mu_s(B_s^j u(s_i)) d\tilde{W}_s^j \\
&\quad + \int_{s_i}^{s_{i+1}} \mu_{s_{i+1}}(\Sigma(s)) ds + \mu_{s_{i+1}}(\Lambda^j(a))(\tilde{W}_{s_{i+1}}^j - \tilde{W}_{s_i}^j) \\
&= \int_{s_i}^{s_{i+1}} \mu_s(A_s u(s_i)) ds + \int_{s_i}^{s_{i+1}} \mu_s(B_s^j u(s_i)) d\tilde{W}_s^j \\
&\quad + \int_{s_i}^{s_{i+1}} \mu_{s_{i+1}}(\Sigma(s)) ds + \mu_{s_i}(\Lambda^j(a))(\tilde{W}_{s_{i+1}}^j - \tilde{W}_{s_i}^j) \\
&\quad + (\tilde{W}_{s_{i+1}}^j - \tilde{W}_{s_i}^j) \int_{s_i}^{s_{i+1}} \mu_s(A_s \Lambda^j(a)) ds + (\tilde{W}_{s_{i+1}}^j - \tilde{W}_{s_i}^j) \int_{s_i}^{s_{i+1}} \mu_s(B^{j'} \Lambda^{j'}(a)) d\tilde{W}_s^{j'},
\end{aligned}$$

We wish to show that, as $n' \to \infty$,

$$\sum_{i=0}^{n'-1} \left[\mu_{s_{i+1}}(u(s_{i+1})) - \mu_{s_i}(u(s_i))\right]$$
$$\to \int_a^b \mu_s(A_s u(s) + \Sigma(s) + B_s^j \Lambda^j(a))ds + \int_a^b \mu_s(B_s^j u(s) + \Lambda^j(a))d\tilde{W}_s^j$$

in probability.

We first show that in probability, as $n' \to \infty$,

$$\sum_{i=0}^{n'-1} \left[\int_{s_i}^{s_{i+1}} \mu_s(A_s u(s_i))ds + \int_{s_i}^{s_{i+1}} \mu_s(B_s^j u(s_i))d\tilde{W}_s^j\right] \to \int_a^b \mu_s(A_s u(s))ds + \int_a^b \mu_s(B_s^j u(s))d\tilde{W}_s^j$$

This statement follows from the fact that, since $u \in C([0,T]; C_b^2(\mathbb{R}^d))$, and

$$\sup_i \sup_{s_i \le s < s_{i+1}} |\mu_s(A_s[u(s_i) - u(s)]]) \le \sup_s \mu_s(\mathbf{1}) \sup_i \sup_{s_i \le s < s_{i+1}} \|u(s_i) - u(s)\|_{C_b^2(\mathbb{R}^r)},$$
$$\sup_i \sup_{s_i \le s < s_{i+1}} |\mu_s(B_s^j[u(s_i) - u(s)]]) \le \sup_s \mu_s(\mathbf{1}) \sup_i \sup_{s_i \le s < s_{i+1}} \|u(s_i) - u(s)\|_{C_b^1(\mathbb{R}^r)},\ 1 \le j \le \ell'.$$

we have that in probability, as $n' \to \infty$,

$$\sum_{i=0}^{n'-1} \mathbf{1}_{[s_i, s_{i+1})}\mu_s(A_s u(s_i)) \to \mu_s(A_s u(s)) \quad \text{in } C([0,T]),$$
$$\sum_{i=0}^{n'-1} \mathbf{1}_{[s_i, s_{i+1})}\mu_s(B_s^j u(s_i)) \to \mu_s(B_s^j u(s)) \quad \text{in } C([0,T]),\ 1 \le j \le \ell'.$$

Secondly, since $\Lambda^j(a) \in C_b^2(\mathbb{R}^d))$, it follows from the Zakai equation that $s \to \mu_s(\Lambda^j(a))$ is continuous. Hence clearly

$$\sum_{i=0}^{n'-1} \mu_{s_i}(\Lambda^j(a))(\tilde{W}_{s_{i+1}}^j - \tilde{W}_{s_i}^j) \to \int_a^b \mu_s(\Lambda^j(a))d\tilde{W}_s.$$

Moreover, we know that $\mu_\cdot(B_j \Lambda^j(a)) \in L^\infty(0,T)$. Hence a classical arguments yields that

$$\sum_{i=0}^{n'-1}(\tilde{W}_{s_{i+1}}^j - \tilde{W}_{s_i}^j) \int_{s_i}^{s_{i+1}} \mu_s(B^k \Lambda^k(a))d\tilde{W}_s^k \to \int_a^b \mu_s(B^j \Lambda^j(a))ds.$$

Indeed, the limit is the joint quadratic variation on the interval $[a,b]$ of the two martingales $\tilde{W}_t$ and $\int_0^t \mu_s(B^k \Lambda^k(a))d\tilde{W}_s^k$.

We also note that

$$\left|\sum_{i=0}^{n'-1}(\tilde W^j_{s_{i+1}}-\tilde W^j_{s_i})\int_{s_i}^{s_{i+1}}\mu_s(A_s\Lambda^j(a))ds\right| \le \sup_i|\tilde W^j_{s_{i+1}}-\tilde W^j_{s_i}|\int_a^b|\mu_s(A_s\Lambda^j(a))|ds$$
$$\le C\sup_i|\tilde W^j_{s_{i+1}}-\tilde W^j_{s_i}|\int_a^b|\mu_s(\mathbf{1})|ds$$
$$\to 0\,.$$

Finally, we show that

$$\sum_{i=0}^{n'-1}\int_{s_i}^{s_{i+1}}\mu_{s_{i+1}}(\Sigma(s))ds \to \int_a^b \mu_s(\Sigma(s))ds\,. \tag{4.4}$$

We approximate Σ in $L^1(0,T;C_b(\mathbb{R}^d))$ by a sequence in $C([0,T];C_b^2(\mathbb{R}^d))$. For each $\varepsilon>0$, let $\Sigma_\varepsilon\in C([0,T];C_b^2(\mathbb{R}^d))$ be such that $\int_0^T\sup_x|\Sigma(t,x)-\Sigma_\varepsilon(t,x)|dt\le\varepsilon$. We have

$$\left|\sum_{i=0}^{n'-1}\int_{s_i}^{s_{i+1}}\mu_{s_{i+1}}(\Sigma(s))ds-\int_a^b\mu_s(\Sigma(s))ds\right| \le \left|\sum_{i=0}^{n'-1}\int_{s_i}^{s_{i+1}}\mu_{s_{i+1}}(\Sigma(s))ds-\sum_{i=0}^{n'-1}\int_{s_i}^{s_{i+1}}\mu_{s_{i+1}}(\Sigma_\varepsilon(s))ds\right|$$
$$+\left|\sum_{i=0}^{n'-1}\int_{s_i}^{s_{i+1}}[\mu_{s_{i+1}}(\Sigma_\varepsilon(s))-\mu_s(\Sigma_\varepsilon(s))]ds\right|$$
$$+\left|\int_a^b\mu_s(\Sigma_\varepsilon(s))ds-\int_a^b\mu_s(\Sigma(s))ds\right|$$

We observe that the first and the last term on the right hand side of the above inequality are bounded by $(b-a)\varepsilon\sup_{a\le s\le b}\mu_s(\mathbf{1})$. It thus remains to show that for each $\varepsilon>0$ fixed, the second term tends to 0, as $n'\to\infty$. This follows from the fact that, since μ_t solves the Zakai equation, for any $s_i\le s\le s_{i+1}$,

$$\mu_{s_{i+1}}(\Sigma_\varepsilon(s))-\mu_s(\Sigma_\varepsilon(s))=\int_s^{s_{i+1}}\mu_r(A_r\Sigma_\varepsilon(s))dr+\int_s^{s_{i+1}}\mu_r(B_r^j\Sigma_\varepsilon(s))d\tilde W_r^j\,.$$

We first note that

$$\left|\sum_{i=0}^{n'-1}\int_{s_i}^{s_{i+1}}\int_s^{s_{i+1}}\mu_r(A_r\Sigma_\varepsilon(s))drds\right|\le\sup_{a\le s\le b}\left(\mu_s(\mathbf{1})\|\Sigma_\varepsilon(s)\|_{C_b^2}\right)\times\sum_i(s_{i+1}-s_i)^2/2,$$

which tends to 0, as $n'\to\infty$, for any $\varepsilon>0$ fixed. Moreover, for any $M>0$, $\delta>0$,

$$\mathbb{P}\left(\left|\sum_{i=0}^{n'-1}\int_{s_i}^{s_{i+1}}ds\int_s^{s_{i+1}}\mu_r(B_r^j\Sigma_\varepsilon(s))d\tilde W_r^j\right|>\delta\right)\le\mathbb{P}\left(\sup_{a\le s\le b}(\mu_s(\mathbf{1})\|\Sigma_\varepsilon(s)\|_{C_b^1})>M\right)$$
$$+Ck\frac{M}{\delta}\sum_i(s_{i+1}-s_i)^{3/2}$$

For any $M>0$ and $\delta>0$ fixed, the second term on the right tends to 0 as $n'\to\infty$, while the first term tends to 0 as $M\to\infty$, with $\varepsilon>0$ fixed. (4.4) has been established. □

We will establish the above result with the same assumptions on Σ, and $\Lambda^j \in L^2((0,T); C_b^1(\mathbb{R}^d))$. However, the processes u, Σ and Λ^j will be given a Sobolev–space valued processus, and the fact that they take their values in $C_b^2(\mathbb{R}^d)$, $C_b(\mathbb{R}^d)$ and $C_b^1(\mathbb{R}^d)$ respectively, will be a consequence of classical Sobolev embedding theorems. Hence the result which will be useful to us is the following theorem, where $H^m := H^m(\mathbb{R}^d)$ (with $m \geq 0$ an integer) denotes the Sobolev space of square integrable functions whose distributional derivatives up to order m are all square integrable. In particular, $H^0 = L^2(\mathbb{R}^d)$. Moreover we will denote by $\mathcal{P}$ the σ-field of predictable subsets of $\mathbb{R}_+ \times \Omega$, and for any Hilbert space H, $L^2_{\mathcal{P}}(\Omega \times [0,T]; H)$ will denote the set of H valued processes which are $\mathcal{P}$ measurable, and square integrable with respect to the product measure $dt \times d\mathbb{P}$.

For the proof of Theorem 4.3 below, we need the following fundamental result on SPDEs:

Proposition 4.2. *Let for some $m \geq 0$ $u_0 \in L^2(\Omega; H^m)$, $f \in L^2_{\mathcal{P}}(\Omega \times [0,T]; H^{m-1})$ and for $1 \leq j \leq \ell'$, $g^j \in L^2_{\mathcal{P}}(\Omega \times [0,T]; H^m)$. Then the SPDE*

$$u_t = u_0 + \int_0^t [\Delta u_s + f_s] ds + \int_0^t g_s^j dW_s^j, \ t \geq 0$$

has a unique solution $u \in L^2_{\mathcal{P}}(\Omega \times [0,T]; H^{m+1}) \cap L^2(\Omega; C([0,T]; H^m))$. Moreover, the mapping $(f, g^1, \ldots, g^k) \mapsto u$ is continuous from $L^2_{\mathcal{P}}(\Omega \times [0,T]; H^{m-1}) \times (L^2_{\mathcal{P}}(\Omega \times [0,T]; H^m)^{\ell'}$ into $L^2_{\mathcal{P}}(\Omega \times [0,T]; H^{m+1}) \cap L^2(\Omega; C([0,T]; H^m))$.

Proof. The existence and uniqueness result in the case $m = 0$ is a particular case of Theorem 1.4 in [12] (see also [11]). The continuity follows readily from the estimates there. The result in the case $m \geq 1$ is deduced as follows. For any $1 \leq i \leq d$, $v_i := \partial u / \partial x_i$ is the solution of an equation to which the result for $m = 0$ can be applied. This establishes the result for $m = 1$. The result for $m > 1$ is obtained inductively by taking higher order derivatives. □

Theorem 4.3. *Suppose that, for some $m > d/2 + 2$, $u \in L^2_{\mathcal{P}}(\Omega \times [0,T]; H^m))$ and moreover for any $0 \leq t \leq T$,*

$$u(t) = u_0 + \int_0^t \Sigma(s) ds + \int_0^t \Lambda^j(s) d\tilde{W}^j(s),$$

where $u_0 \in L^2(\Omega; H^m)$ is $\mathcal{F}_0$ measurable, $\Sigma \in L^2_{\mathcal{P}}(\Omega \times (0,T), H^{m-2})$ and $\Lambda^j \in L^2_{\mathcal{P}}(\Omega \times (0,T); H^{m-1})$ for all $1 \leq j \leq \ell'$. Then the Itô formula in Theorem 4.1 still holds, i.e.

$$\mu_t(u_t) = \mu_0(u_0) + \int_0^t \mu_s \left(A_s u_s + \Sigma_s + B_s^j \Lambda_s^j \right) ds + \int_0^t \mu_s \left(B_s^j u_s + \Lambda_s^j \right) d\tilde{W}_s^j.$$

Proof. Let for all $1 \leq j \leq \ell'$ $\{\Lambda_n^j, \ n \geq 1\}$ denote a sequence in $L^2_{\mathcal{P}}(\Omega \times (0,T); H^m)$, such that $\Lambda_n^j \to \Lambda^j$ in $L^2_{\mathcal{P}}(\Omega \times (0,T); H^{m-1})$. Let moreover $\{u_n, \ n \geq 1\}$ (resp. $\{\Sigma_n, \ n \geq 1\}$) denote a sequence in $L^2_{\mathcal{P}}(\Omega \times (0,T), H^{m+1})$ (resp. in $L^2_{\mathcal{P}}(\Omega \times (0,T), H^{m-1})$), such that $u_n \to u$ in $L^2_{\mathcal{P}}(\Omega \times (0,T), H^m)$ (resp. $\Sigma_n \to \Sigma$ in $L^2_{\mathcal{P}}(\Omega \times (0,T), H^{m-2})$). We now define for each $n \geq 1$ v_n as the solution of the following SPDE (where Δ denotes the Laplace operator) :

$$v_n(t) = u_0 + \int_0^t [\Delta v_n(s) - \Delta u_n(s) + \Sigma_n(s)] ds + \int_0^t \Lambda_n^j(s) d\tilde{W}^j(s), \ 0 \leq t \leq T\,.$$

We shall now use repeatedly the results in Proposition 4.2. It is plain that this SPDE has a unique solution $v_n \in L^2_{\mathcal{P}}(\Omega \times (0,T); H^{m+1}) \cap L^2_{\mathcal{P}}(\Omega; C([0,T]; H^m))$, and as $n \to \infty$, $v_n \to \tilde{u}$ in $L^2_{\mathcal{P}}(\Omega \times (0,T); H^m)$, where $\tilde{u}$ is the unique solution in $L^2(\Omega \times (0,T); H^m)$ of the SPDE

$$\tilde{u}(t) = u_0 + \int_0^t [\Delta \tilde{u}(s) - \Delta u(s) + \Sigma(s)]ds + \int_0^t \Lambda^j(s) d\tilde{W}^j(s),\ 0 \le t \le T\,.$$

But $u \in L^2(\Omega \times (0,T); H^m)$ is a solution of that equation. Hence $\tilde{u} = u$. Now from Theorem 4.1, which we can use thanks to the Sobolev embedding, which in particular tells us that $H^m \subset C^2_b(\mathbb{R}^d)$,

$$\begin{aligned}\mu_t\left(v_n(t)\right) =& \mu_0\left(u_0\right) + \int_0^t \mu_s\left(A_s v_n(s) + \Sigma_n(s) + \Delta(v_n - u_n)(s) + B^j_s \Lambda^j_n(s)\right) ds \\ &+ \int_0^t \mu_s\left(B^j_s v_n(s) + \Lambda^j_n(s)\right) d\tilde{W}^j_s.\end{aligned}$$

Now we can take the limit in that identity as $n \to \infty$, which yields the result. Indeed, as $n \to \infty$, for any $t > 0$, $v_n(t) \to u(t)$ in $L^2(\Omega; H^{m-1})$,

$$Av_n + \Sigma_n + \Delta(v_n - u_n) + B^j \Lambda^j_n \to Au + \Sigma + B^j \Lambda^j \ \text{ in } L^2_{\mathcal{P}}(\Omega \times (0,T); H^{m-2})$$

for $1 \le j \le \ell'$,

$$B^j v_n + \Lambda^j_n \to B^j u + \Lambda^j \ \text{ in } L^2_{\mathcal{P}}(\Omega \times (0,T); H^{m-1}),$$

$H^{m-2}(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$ with continuous injection, and $\sup_{0 \le t \le T} \mu_t(\mathbf{1}) < \infty$. Combining those facts, we deduce that that as $n \to \infty$, the following convergences hold in probability:

$$\begin{gathered}\mu_t(v_n(t)) \to \mu_t(u(t)), \\ \mu_s(A_s v_n(s) + \Sigma_s + \Delta(v_n - u_n)(s) + B^j_s \Lambda^j_n(s)) \to \mu_s(A_s u_s + \Sigma_s + B^j_s \Lambda^j_s) \ \text{ in } L^2(0,T)\end{gathered}$$

and for $1 \le j \le k$,

$$\mu_s(B^j_s v_n(s) + \Lambda^j_n(s)) \to \mu_s(B^j_s u_s + \Lambda^j_s) \ \text{ in } L^2(0,T)\,.$$

The result follows. □

5 A system of BSPDEs

In the following we will make use of a complex valued $u \in \mathcal{E}_T$ of the form (4.1), which will be the solution of the BSPDE

$$du_t = -\left(Au_t + B^j v^j_t + ir^j_t B^j u_t + ir^j_t v^j_t\right) dt + v^j_t dW^j_t\,, \quad u_T = \varphi, \tag{5.1}$$

where again we adopt the convention of summation of the repeated index j from $j = 1$ to $j = \ell'$. Under the Assumptions $\mathbf{AA}_m$ to be specified below, as a result of Theorem 5.2, u will satisfy the assumptions of Theorem 4.3.

We write below the corresponding equations of the real, respectively, the imaginary part of u. In other words, assume that $u = u^1 + iu^2$. Then (u^1, u^2) satisfy the following system of BSPDEs

$$\begin{aligned} du_t^1 &= -\left(Au_t^1 + B^j v_t^{1,j} - r_t^j B^j u_t^2 - r_t^j v_t^{2,j}\right) dt + v_t^{1,j} dW_t^j\,, \quad u_T^1 = \varphi, \\ du_t^2 &= -\left(Au_t^2 + B^j v_t^{2,j} + r_t^j B^j u_t^1 + r_t^j v_t^{1,j}\right) dt + v_t^{2,j} dW_t^j\,, \quad u_T^2 = 0. \end{aligned} \tag{5.2}$$

We need to extend to the above system of BSPDEs the results from Du and Meng [6] and from Du, Tang and Zhang [7], which are established for a single BSPDE, of the same type. Note that the factor of dt in the u^1 (resp. u^2) equation involves second order derivatives of u^1 (resp. u^2) and first order derivatives of u^2 (resp. u^1), together with first order derivatives of v^1 (resp. v^2) and zero-th order derivatives of v^2 (resp. v^1). Hence the coupling between the two BSPDEs comes through terms of lower order, which is essential for our extension from the results for a single BSPDE to work.

We now first state and prove the extension of Theorem 2.3 from [6] to our system.

Theorem 5.1. *In addition to the assumptions **E** and **U**, let us suppose that for some $\kappa > 0$,*

$$gg^T(t,x) \geq \kappa I,\ \forall (t,x) \in [0,T] \times \mathbb{R}^d,\ a.s., \tag{5.3}$$

and for some integer $n \geq 1$, any multi-index α with $|\alpha| \leq n$,

$$ess\ sup_{\Omega\times[0,T]\times\mathbb{R}^d}(|D^\alpha a| + |D^\alpha f| + |D^\alpha \bar{g}| + |D^\alpha h_2|) \leq K. \tag{5.4}$$

Finally we assume that $\varphi \in H^{n+1}$.

Then the system of BSPDEs (5.2) *has a unique solution such that for $i = 1, 2$,*

$$u^i \in L^2_{\mathcal{P}}(\Omega \times [0,T]; H^{n+2}) \cap L^2(\Omega; L^\infty([0,T]; H^{n+1})), v^i \in L^2_{\mathcal{P}}(\Omega \times [0,T]; (H^{n+1})^{\otimes \ell'}).$$

Proof. We first need to extend Proposition 3.2 together with Theorem 2.1 from [6]. Let $V := H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$, $H = L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, so that if we identify H with its dual, V' is identified with $H^{-1}(\mathbb{R}^d) \times H^{-1}(\mathbb{R}^d)$. Referring to the notations in the proof of Proposition 3.2 in [6], we let

$$\mathcal{L} = \begin{pmatrix} A & -r^j B^j \\ r^j B^j & A \end{pmatrix}, \quad \mathcal{M}^j = \begin{pmatrix} B^j & -r^j \\ r^j & B^j \end{pmatrix}.$$

It is not hard to deduce from condition (5.3) that Assumption 3.1 in [6] is satisfied, namely there exists $\lambda, C > 0$ such that for any $u \in V$,

$$2\langle u, \mathcal{L}u\rangle + \sum_{j=1}^{\ell'} \|(\mathcal{M}^j)^T u\|_H^2 \leq -\lambda \|u\|_V^2 + C\|u\|_H^2,\ \|\mathcal{L}u\|_{V'} \leq C\|u\|_V\,.$$

It follows that the proofs of Proposition 3.2 and Theorem 2.1 from [6] are easily adapted to yield that provided the assumption (5.4) is satisfied with $n = 0$ and $\varphi \in L^2(\mathbb{R}^d)$, our system has a unique solution such that $(u^i, v^i) \in L^2(\Omega \times (0,T); H^1(\mathbb{R}^d) \times (L^2(\mathbb{R}^d))^{\otimes \ell'})$, $i = 1, 2$.

The rest of the proof follows exactly the lines of arguments in [6], with obvious adaptations. ☐

Assumptions $\mathbf{AA}_m$:

- <u>(smoothness and boundedness of the coefficients)</u> All coefficients are functions of $(\omega, t, x) \in \Omega \times [0,T] \times \mathbb{R}^d$, which are $\mathcal{P} \otimes \mathcal{B}_d$ measurable and, for some integer m, (5.4) is satisfied for any multi-index α with $|\alpha| \leq \sup\{1, m\}$, and for $|\alpha| \leq \sup\{2, m\}$ concerning the coefficients of the matrix a.

- <u>(Smoothness of the final condition)</u> The function $\varphi \in H^m$.

In the following we will use the notation

$$
\begin{aligned}
Q_t^{1,u,v} &= Au_t^1 + B^j v_t^{1,j} - r_t^j B^j u_t^2 - r_t^j v_t^{2,j}, \\
Q_t^{2,u,v} &= Au_t^2 + B^j v_t^{2,j} + r_t^j B^j u_t^1 + r_t^j v_t^{1,j}.
\end{aligned}
$$

Theorem 5.2. *Assume that for some integer m, $\boldsymbol{AA}_m$ is satisfied. Then the system of BSPDEs*

$$
\begin{aligned}
du_t^1 &= -Q_t^{1,u,v} dt + v_t^{1,j} dW_t^j, \\
du_t^2 &= -Q_t^{2,u,v} dt + v_t^{2,j} dW_t^j.
\end{aligned} \tag{5.5}
$$

with $(u_T^1, u_T^2) = (\varphi, 0)$ has a solution $((u^1, v^1), (u^2, v^2))$ such that for all $T > 0$, $u^i \in L^2_{\mathcal{P}}(\Omega; C_w([0,T]; H^m))$, $v^i \in L^2_{\mathcal{P}}(\Omega \times [0,T]; (H^{m-1})^{\otimes \ell'})$, $i = 1,2$ and we have for $i = 1,2$ (here $\|\cdot\|_m$ stands for the norm in H^m and $|||\cdot|||_m$ for the norm in $(H^m)^{\otimes \ell'}$)

$$
E\left[\sup_{t \leq T} ||u_t^i||_m^2\right] + \mathbb{E}\left[\int_0^T |||v_t^i + k^+ k \bar{g}^T \nabla u_t^i|||_m^2\right] \leq CE\, ||\varphi||_m^2 \tag{5.6}
$$

$$
E\left[\sup_{t \leq T} ||u_t^i||_m^2\right] + E\left[\int_0^T |||v_t^i|||_{m-1}^2\right] \leq CE\, ||\varphi||_m^2. \tag{5.7}
$$

Moreover, if $m > d/2$ then u is jointly continuous in (t,x) almost surely and if $m > d/2 + 2$ then (u,v) is a classical solution of (5.5).

Proof. The proof is almost identical to the proof of Theorem 2.1 and Corollary 2.2 in [7]. Let us first explain how (5.7) follows from (5.6). We first deduce from (5.6) that

$$
E\left[\sup_{t \leq T} ||u_t^i||_m^2\right] \leq CE\, ||\varphi||_m^2.
$$

This clearly implies that

$$
\mathbb{E}\int_0^T |||k^+ k \bar{g}^T \nabla u_t^i|||_{m-1}^2 \leq CE\, ||\varphi||_m^2.
$$

But (5.6) is also true for m replaced by $m-1$, which implies that

$$
E\int_0^T \int_0^T |||v_t^i + k^+ k \bar{g}^T \nabla u_t^i|||_{m-1}^2 \leq CE\, ||\varphi||_{m-1}^2,
$$

and (5.7) now follows from these three inequalities.

Let us now explain how we obtain (5.6) in the simple case where $m = 0$. Below $\|\cdot\|$ (resp. $|||\cdot|||$) stands for $\|\cdot\|_0$ (resp. $|||\cdot|||_0$), and $(\cdot,\cdot)$ stands for the scalar product in $H^0 = L^2(\mathbb{R}^d)$. Suppose we have a smooth enough solution of (5.5). Applying Itô's formula to compute $\|u^i_t\|^2$ and summing up the results for $i = 1$ and 2, we obtain

$$\|\varphi\|^2 = \|u^1_t\|^2 + \|u^2_t\|^2 + \int_t^T \{|||v^1_s|||^2 + |||v^2_s|||^2 - 2(Q^{i,u,v}_s, u^i_s)\}ds + 2\int_t^T (u^i_s, v^{i,j}_s)dW^j_s, \tag{5.8}$$

Then

$$\begin{aligned}
-2(Q^{i,u,v}_s, u^i_s) + \left(|||v^1_s|||^2 + |||v^2_s|||^2\right) &= -2\left(Au^1_s + B^j v^{1,j}_s - r^j_s B^j u^2_s - r^j_s v^{2,j}_s, u^1_s\right) \\
&\quad -2\left(Au^2_s + B^j v^{2,j}_s + r^j_s B^j u^1_s + r^j_s v^{1,j}_s, u^2_s\right) \\
&\quad + \left(|||v^1_s|||^2 + |||v^2_s|||^2\right) \\
&= -2\left(Au^1_s, u^1_s\right) - 2\left(Au^2_s, u^2_s\right) \\
&\quad -2\left(B^j v^{1,j}_s, u^1_s\right) - 2\left(B^j v^{2,j}_t, u^2_s\right) \\
&\quad +2r^j_s\left(B^j u^2_s, u^1_s\right) - 2r^j_s\left(B^j u^1_s, u^2_s\right) + 2r^j_s\left(v^{2,j}_s, u^1_s\right) - 2r^j_s\left(v^{1,j}_s, u^2_s\right) \\
&\quad + \left(|||v^1_s|||^2 + |||v^2_s|||^2\right).
\end{aligned}$$

By integration by parts, we deduce that

$$\begin{aligned}
&-2(Q^{i,u,v}_s, u^i_s) + \left(|||v^1_s|||^2 + |||v^2_s|||^2\right) \\
&\quad = ([\mathrm{div} f - \frac{1}{2}D^2 a]u^i_s, u^i_s) + (a^{j'l}\partial^{j'} u^i_s, \partial^l u^j_s) \\
&\qquad -2\left(\left(\nabla v^{1,j}_s[\bar{g}k^+k](t,\cdot,Y_t)\right)_j, u^1_s\right) - 2\left(([h_2^T k^+k](t,\cdot,Y_t))_j v^{1,j}_s, u^1_s\right) \\
&\qquad -2\left(\left(\nabla v^{2,j}_s[\bar{g}k^+k](t,\cdot,Y_t)\right)_j, u^2_s\right) - 2\left(([h_2^T k^+k](t,\cdot,Y_t))_j v^{2,j}_s, u^2_s\right) \\
&\qquad +2r^j_s\left(\left(\nabla u^2_s[\bar{g}k^+k](t,\cdot,Y_t)\right)_j, u^1_s\right) + 2r^j_s\left(([h_2^T k^+k](t,\cdot,Y_t))_j u^2_s, u^1_s\right) \\
&\qquad -2r^j_s\left(\left(\nabla u^1_s[\bar{g}k^+k](t,\cdot,Y_t)\right)_j, u^2_s\right) - 2r^j_s\left(([h_2^T k^+k](t,\cdot,Y_t))_j u^1_s, u^2_s\right) \\
&\qquad +2r^j_s\left(v^{2,j}_s, u^1_s\right) - 2r^j_s\left(v^{1,j}_s, u^2_s\right) \\
&\qquad + \left(|||v^1_s|||^2 + |||v^2_s|||^2\right) \\
&\quad \geq ([\mathrm{div} f - \frac{1}{2}D^2 a]u^i, u^i) + (gg^T\nabla u^i_s, \nabla u^i_s) + \sum_{i=1}^2 |||v^i_s + k^+k\bar{g}^T\nabla u^i_s|||^2 \\
&\quad +(\alpha^j v^{i,j}, u^i) \\
&\quad +2r^j_s\left(\left(\nabla u^2_s[\bar{g}k^+k](t,\cdot,Y_t)\right)_j, u^1_s\right) - 2r^j_s\left(\left(\nabla u^1_s[\bar{g}k^+k](t,\cdot,Y_t)\right)_j, u^2_s\right) \\
&\quad +2r^j_s\left(v^{2,j}_s, u^1_s\right) - 2r^j_s\left(v^{1,j}_s, u^2_s\right),
\end{aligned}$$

where

$$\alpha^j = 2\sum_p \left(\partial_p[\bar{g}k^+k](t,\cdot,Y_t)\right)_j - 2([h_2^T k^+k](t,\cdot,Y_t))_j.$$

In the above we have used the convention of summation over repeated indices and the notation $\mathrm{div} f = f^i_{x_i}$, and $D^2 a := \frac{\partial^2}{\partial x_i \partial x_j} a^{i,j}$ (here as an exception the repeated indices i and j are both summed from 1 to d), and the fact that k^+k is a projection operator, hence $\bar{g}\bar{g}^T \geq \bar{g}k^+k\bar{g}^T$, that

$$
\begin{aligned}
-2(Q^{i,u,v}_s, u^i_s) + |||v^1_s|||^2 + |||v^2_s|||^2 &\geq (gg^T \nabla u^i_s, \nabla u^i_s) + ([\mathrm{div} f - \frac{1}{2} D^2 a] u^i, u^i) + \sum_{i=1}^{2} |||v^i_s + k^+k\bar{g}^T \nabla u^i_s|||^2 \\
&\quad + (\alpha^{i,j} v^{i,j}, u^i) + (-1)^{3-i} 2 r^j_s ((k^+k\bar{g}^T)^j \nabla u^{3-i}_s + v^{3-i,j}_s, u^i_s) \\
&\geq \sum_{i=1}^{2} \left\{ \frac{1}{2} |||v^i_s + k^+k\bar{g}^T \nabla u^i_s|||^2 - C\|u^i_s\|^2 \right\},
\end{aligned}
\tag{5.9}
$$

Let us justify the last inequality. Since r^j_s is bounded, there exists a constant C such that

$$
(-1)^{3-i} 2 r^j_s ((k^+k\bar{g}^T)^j \nabla u^{3-i}_s + v^{3-i,j}_s, u^i_s) \geq -\frac{1}{4} |||(k^+k\bar{g}^T)^j \nabla u^{3-i}_s + v^{3-i,j}_s|||^2 - C\|u^i_s\|^2 .
$$

Moreover

$$
(\alpha^j v^j, u^i) = (v^{i,j}_s + (k^+k\bar{g}^T)^j \nabla u^i_s, \alpha^j u^i_s) - ((k^+k\bar{g}^T)^j \nabla u^i_s, \alpha^j u^i_s).
$$

We treat the first term on the right as in the last inequality, and by integration by parts the second term is bounded from below by $-C\|u^i_s\|^2$. Now assuming that we can take the expectation in (5.8) and that the expectation of the stochastic integral vanishes (this is not a serious difficulty, although it requires the use of stopping times), and combining the resulting identity with (5.9) and Gronwall's Lemma, we deduce (5.6) with $m = 0$, at least with the $\sup_t$ outside the expectation. (5.6) then follows using Doob's inequality. The reader may have noticed that the above argument requires that all coefficients have bounded first order partial derivatives, and the entries of the matrix a have also bounded second order partial derivatives.

Next we approximate our pair of BSPDEs with a system indexed by $\varepsilon > 0$, where the operator A has been replaced by $A_\varepsilon := A + \varepsilon\Delta$, where Δ stands for the Laplace operator. We can now invoke Theorem 5.1 to obtain the existence of a solution $(u^1_\varepsilon, u^2_\varepsilon)$ to our approximate system of BSPDEs. Clearly the above computations yield that $(u^i_\varepsilon, v^i_\varepsilon)$, $i = 1, 2$ satisfies the estimate (5.6) with $m = 0$ and a constant C which is independent of ε. Hence we can extract a subsequence such that each pair $(u^i_\varepsilon, v^i_\varepsilon)$ converges weakly $L^2_{\mathcal{P}}((0,T) \times \Omega; H^0 \times (H^0)^{\otimes \ell'})$, and it is not too hard to show that the limit still satisfies (5.6) for $m = 0$, and solves the system of BSPDEs (we take the limit in the equation written in weak form).

However, we are interested in more regular solutions. Mimicking the computations done in [7], we can extend the above computations to estimate the norms in H^m. Of course, this must be done sequentially w.r.t. m. It requires to take partial derivatives in our system of BSPDEs, which introduces a forcing term involving lower order derivatives, but it can be handled in our situation similarly as both in [7] and in [6]. As a result, we can take the limit weakly in $L^2_{\mathcal{P}}((0,T) \times \Omega; H^m \times (H^m)^{\otimes \ell'})$. The weak continuity of u^i with values in H^m follows by standard arguments. The result follows. □

Remark 5.3. *Theorem 2.1 in [7] also asserts the uniqueness of the corresponding equation. Whilst we don't need it for our purpose, the uniqueness of the solution of (5.5) can be shown in a similar manner and it is a consequence of* (5.6).

Theorem 5.4. *Let θ_t be the $\mathbb{C}$–valued solution of the SDE*

$$d\theta_t = i\theta_t r_t^j d\tilde{W}_t^j, \quad \theta_0 = 1\,, \tag{5.10}$$

*where r_t is an arbitrary element of $L^\infty([0,T];\mathbb{R}^{d'})$. Under the assumptions **E** and **U**, we get that, (u,v) denoting the solution of the BSPDE* (5.2) *and π_t a solution of the Zakai equation* (3.12) *which satisfies*

$$\mathbb{E}\left[\sup_{0\le t\le T}\pi_t(\mathbf{1})^2\right], \tag{5.11}$$

we have

$$d\theta_t\pi_t(u_t) = \theta_t\pi_t(B^j u_t + v_t^j + ir_t^j u_t)d\tilde{W}_t^j. \tag{5.12}$$

Moreover the process $\{\theta_t\pi_t(u_t),\ 0\le t\le T\}$ is a martingale, and therefore $\tilde{\mathbb{E}}\left[\theta_T\pi_T^1(\varphi)\right] = \tilde{\mathbb{E}}\left[\pi_0^1(\psi_0)\right]$.

Proof. Thanks to our assumption, it follows from Theorem 5.2 that we can apply Theorem 4.3, from which we deduce that

$$\begin{aligned} d\pi_t(u_t) &= \pi_t\left(Au_t - Au_t - B^j v_t^j - ir_t^j B^j u_t - ir_t^j v_t^j + B^j v_t^j\right)dt + \pi_t(B^j u_t + v_t^j)d\tilde{W}_t^j \\ &= -ir_t^j\pi_t\left(B^j u_t + v_t^j\right)dt + \pi_t(B^j u_t + v_t^j)d\tilde{W}_t^j \end{aligned} \tag{5.13}$$

The identity (5.12) follows from (5.10) and (5.13) by Itô's chain rule. The martingale property of the process $\{\theta_t\pi_t(u_t),\ 0\le t\le T\}$ will follow from the Burkholder–Davis–Gundy inequality for the first moment and the following estimate

$$\tilde{\mathbb{E}}\left[\sqrt{\int_0^T |\theta_t\pi_t(B^j u_t + v_t^j + ir_t^j u_t)|^2 ds}\right] < \infty \tag{5.14}$$

for $j = 1, ..., \ell'$ which we now establish. By Theorem 5.2, our current assumptions imply that for some $m > d/2+2$, $u \in L^2(\Omega\times[0,T];H^m(\mathbb{R}^d))$ and for $1\le j\le \ell'$, $v^j \in L^2(\Omega\times[0,T];H^{m-1}(\mathbb{R}^d))$. Consequently, thanks to the Sobolev embedding theorem, we deduce that

$$\tilde{\mathbb{E}}\int_0^T\left[\sup_x|\nabla u(t,x)|^2 + \sup_x|u(t,x)|^2 + \sum_{j=1}^m\sup_x|v^j(t,x)|^2\right]dt < \infty\,.$$

So, if we define $C^j(t,.) := B^j u_t + v_t^j + ir_t^j u_t$, we have that for $1\le j\le\ell'$,

$$\tilde{\mathbb{E}}\int_0^T\sup_x|C^j(t,x)|^2 dt < \infty\,. \tag{5.15}$$

We first note that for any $0\le t\le T$,

$$|\theta_t| = \exp\left(\frac{1}{2}\sum_{j=1}^{\ell'}\int_0^t|r_s^j|^2 ds\right)$$

is a deterministic quantity, whose supremum over $0 \le t \le T$ is finite. Hence we have

$$\int_0^T |\theta_t \pi_t(C^j(t,\cdot))|^2 dt \le \sup_{0\le t\le T} |\theta_t|^2 \sup_{0\le t\le T} \pi_t(\mathbf{1})^2 \int_0^T \sup_x |C^j(t,x)|^2 dt,$$

$$\tilde{\mathbb{E}}\left(\sqrt{\int_0^T |\theta_t \pi(C^j(t,\cdot))|^2 dt}\right) \le \sup_{0\le t\le T} |\theta_t| \tilde{\mathbb{E}}\left\{ \sup_{0\le t\le T} \pi_t(\mathbf{1}) \sqrt{\int_0^T \sup_x |C^j(t,x)|^2 dt} \right\}$$

$$\le \sup_{0\le t\le T} |\theta_t| \sqrt{\tilde{\mathbb{E}}\left(\sup_{0\le t\le T} \pi_t(\mathbf{1})^2\right)} \sqrt{\tilde{\mathbb{E}} \int_0^T \sup_x |C^j(t,x)|^2 dt},$$

and (5.14) now follows from (5.11) and (5.15). □

Theorem 5.5. *Under assumptions* **E** *and* **U***, there exists a unique solution of the equation* (3.12) *in the class of* $\mathcal{P}(\mathcal{Y}_t)$*-measurable measure valued processes satisfying* (5.11) *for any* $T > 0$*. Since the uniqueness of the solution of equation (3.12) is equivalent to that of equation (3.13) following from Theorem 3.17, we also deduce the uniqueness of the solution of equation (3.13).*

Proof. Assume that there are two solutions of the equation (3.12) denoted by π_1, π_2. We observe the following sequence of identities

$$\tilde{\mathbb{E}}\left[\theta_T \pi_T^1(\varphi)\right] = \tilde{\mathbb{E}}\left[\theta_0 \pi_0^1(u_0)\right] = \tilde{\mathbb{E}}\left[\theta_0 \pi_0^2(u_0)\right] = \tilde{\mathbb{E}}\left[\theta_T \pi_T^2(\varphi)\right]$$

and since both S_T is a total set, and φ is an arbitrary smooth function, it follows that $\pi_T^1 = \pi_T^2$. □

Finally we note that the unnormalized conditional distribution satisfies the condition (5.11). Indeed, if we let now π_t denote that unnormalized conditional distribution at time t, we deduce from Lemma 3.7 that $\pi_t(\mathbf{1}) = \tilde{Z}_t$, which is a $\tilde{\mathbb{P}}$ martingale, hence from Doob's inequality, it suffices to prove that $\tilde{\mathbb{E}}[|\tilde{Z}_t|^2] < \infty$ for all $t > 0$, which follows from (3.4) and the boundedness of h_2.

6 Appendix

In this Appendix, we recall the definition of the Moore–Penrose pseudo inverse of a possibly rectangular matrix, and prove that the map which to a matrix associates its pseudo–inverse is measurable.

In what follows, $A^+ \in \mathbb{R}^{\ell\times d}$ stands for the Moore-Penrose pseudo-inverse of the matrix $A \in \mathbb{R}^{d\times\ell}$. The Moore-Penrose pseudo-inverse A^+ of the matrix A is uniquely characterised by the following four properties

$$AA^+A = A, \quad A^+AA^+ = A^+, \quad \left(A^+A\right)^\top = A^+A, \qquad \left(AA^+\right)^\top = AA^+, \tag{6.1}$$

see [2] for details.

Moreover from the identities in (6.1) we deduce that A^+A is a projection onto the range of A^+A and $I - A^+A$ is a projection onto the orthogonal space of the range of A^+A. In particular we deduce that the norm of A^+A as a linear operator is bounded by 1. Since all norms on a finite dimensional space are equivalent, for any matrix norm $\|\cdot\|$, there exists a constant C which depends only upon ℓ and the particular choice of a norm on the set of $\ell\times\ell$ matrices such that

$$\|A^{+}A\|\leq C, \tag{6.2}$$

for any positive integers d, l and any $d\times\ell$ matrix A.

Next we give details of an explicit construction of the pseudo inverse of a matrix that will help us prove the measurability of the mapping $A\mapsto A^{+}$. In what follows we use the notation A for a generic matrix $A\in\mathbb{R}^{d\times\ell}$. We follow here the construction and the analysis in [2].

Let $0\leq r\leq\min(d,\ell)$ be the rank of A. If $r=0$ this means that $A=O^{d\times\ell}$ where $O^{d\times\ell}$ is the matrix with all entries null. In this degenerate case, $A^{+}=O^{\ell\times d}$ is the matrix with all null entries. If $r>0$, then A has a invertible minor of order r. Following Theorem 5 page 48 in [2], the pseudo-inverse is given by

$$A^{+}=G^{\top}\left(F^{\top}AG^{\top}\right)^{-1}F^{\top},$$

where G and F are matrices which appear in a full-rank factorization of A:

$$A=FG,\quad F\in\mathbb{R}^{d\times r},\quad G\in\mathbb{R}^{r\times\ell}.$$

The pair (F,G) is not unique, but we shall give one construction which is based upon a particular choice of an invertible $r\times r$ minor of A. We describe briefly the construction of a full-rank factorization of A (see also Section 4, in particular page 26 in [2]):

Let $1\leq c_1<c_2<\cdots<c_r\leq d\wedge\ell$ denote the ranks of the r columns containing all terms of one arbitrarily chosen invertible $r\times r$ minors of A. Let now P be the permutation matrix, which when applied to A on the right, makes the c_i–th column of A into the i–th column of AP, $1\leq i\leq r$. Let next P_1 denote the submatrix of P consisting of its first r columns, and $F=AP_1$. Finally let G be the unique $r\times\ell$ matrix such that $A=FG$. It is clear (see Lemma 1 page 26 of [2]) that the terms of the i–th column of G are the unique coefficients which express the i–th column of A as a linear combination of the elements of the basis of $R(A)$ given by the columns of F. We have $A=FG$, where F (resp. G) is a $d\times r$ (resp. $r\times\ell$) matrix of rank r.

We now want to prove the following lemma.

Lemma 6.1. *The mapping $A\mapsto A^{+}$ is measurable from $\mathbb{R}^{d\times\ell}$ into $\mathbb{R}^{\ell\times d}$.*

We note that the above mapping is clearly not continuous (in the case $d=\ell=1$, for $A\in\mathbb{R}$, $A^{+}=1/A$ if $A\neq 0$, and $0^{+}=0$, so if $A_n>0$, $A_n\to 0$, then $A_n^{+}\to+\infty$, while $(\lim_n A_n)^{+}=0$).

Proof. We first need to find a consistent way of identifying an invertible minor of order r of the matrix A. For this we introduce the following enumeration of the minors. The matrix A has $d\ell$ minors or order 1, $\begin{pmatrix} d\\ 2\end{pmatrix}\begin{pmatrix}\ell\\ 2\end{pmatrix}$ minors of order 2, ..., and $\begin{pmatrix}\max(d,\ell)\\ \min(d,\ell)\end{pmatrix}$ minors of order $\min(d,\ell)$. By convention we add a 'minor' of order 0 (to account for the case $r=0$) whose determinant is chosen to be 0. Next we consider

$$S(A)\in\mathbb{R}^{\theta},\qquad \theta:=1+d\ell+\begin{pmatrix} d\\ 2\end{pmatrix}\begin{pmatrix}\ell\\ 2\end{pmatrix}+\ldots+\begin{pmatrix}\max(d,\ell)\\ \min(d,\ell)\end{pmatrix},$$

$$S(A) := (0, ...),$$

where $S(A)$ is the list of the determinants of all the minors of A, and we choose a fixed arbitrary enumeration of all those minors. If $r > 0$, we denote by $m(A)$ the highest index in the enumeration of the minors for which the corresponding determinant is non-zero. Such an index exists and $m(A) > 1$. This means that

$$S(A) = (0, ..., d(A), 0, ..., 0),$$

where $d(A)$ is the $m(A)$-th entry of the vector $S(A)$, that is $d(A) = S(A)_{m(A)}$ is not zero and all subsequent entries (if any) $S(A)_{m(A)+1}$, $S(A)_{m(A)+2}$, ... are zero.

If $r = 0$, then $S(A)$ has all entries null

$$S(A) = (0, ..., ..., 0),$$

and $m(A) = 1$, by convention.

Note that the function $A \mapsto m(A)$ is an integer valued function. We split $\mathbb{R}^\theta$ into a finite collection of disjoint sets $\{H_n\}_{n=1}^{\theta}$ such that the index $m(A)$ stays constant if $S(A)$ takes values in H_n :

- $H_1 \subset \mathbb{R}^\theta$, $H_1 = \{(0, 0, 0, ..., 0) \in \mathbb{R}^\theta\}$. On this set $m(A) = 1$. In this case, since $S(A) = (0, 0, ..., 0)$, A is the null matrix,
- $H_2 \subset \mathbb{R}^{\theta}$, $H_2 = \{(0, a_2, 0, ..., 0) \in \mathbb{R}^{\theta},\ a_2 \neq 0\}$. On this set $m(A) = 2$. In this case, the first ranked minor of A is not zero and the determinants of all the higher ranked minors are zero.
- $H_3 \subset \mathbb{R}^{\theta}$, $H_3 = \{(0, a_2, a_3, ..., 0) \in \mathbb{R}^{\theta},\ a_3 \neq 0\}$. On this set $m(A) = 3$. In this case, the second ranked minor of the matrix A is not zero and the determinants of all the higher ranked minors are zero.
- ....
- $H_\theta \subset \mathbb{R}^{\theta}$, $H_\theta = \{(0, a_2, a_3, ..., a_\theta) \in \mathbb{R}^{\theta},\ a_\theta \neq 0\}$. On this set $m(A) = \theta$. In this case, the last minor of A in the list has a non zero determinant.

We distinguish two cases:

If $m(A)$ takes values in the set H_1 (in other words $m(A) = 1$ and $S(A) = (0, 0, ..., 0)$), then A is the null matrix $O^{d\times\ell}$ (in other words it is constant) on the preimage of this set

$$Q_1 = \{A \in \mathbb{R}^{d\times\ell} | S(A) = (0, 0, ..., 0) \in H_1\}.$$

If $m(A)$ takes values in each of the remaining sets $H_i, i = 2, ..., \theta$, then A has one fixed invertible minor on the preimage of each of those sets.

$$Q_i = \{A \in \mathbb{R}^{d\times\ell} | S(A) \in H_i\}, \quad i = 2, 3, ..., \theta.$$

As a result, on each set Q_i, the same (in most cases arbitrarily chosen) invertible $r \times r$ minor is selected. Let P_i be the permutation matrix presented above which, when applied to A on the right, moves the r columns containing elements of the selected invertible minor into the first r columns.

It is clear that those columns constitute a basis of $R(A)$, the range of A. Denote by $P_{i,1}$ the $\ell \times r$ matrix consisting of the first r columns of P_i. The matrix $P_{i,1}$ is constant on Q_i, hence $F_i = AP_{i,1}$ is a continuous function of A on Q_i. Next let G_i be the unique $r \times \ell$ matrix such that $A = F_i G_i$. The matrix G_i is a rational, hence continuous function of the entries of A. Therefore on each set Q_i the Moore-Penrose pseudo-inverse of A

$$A^+ = G_i^\top \left(F_i^\top A G_i^\top\right)^{-1} F_i^\top$$

is a continuous function of A. Finally, since $\{A \in Q_i\}$ is Borel, we get that

$$A^+ \;=\; O^{d\times\ell} 1_{A\in Q_1} + \sum_{i=2}^{\theta} G_i^\top \left(F_i^\top A G_i^\top\right)^{-1} F_i^\top 1_{\{A\in Q_i\}}$$

is a measurable function of A. □

References

[1] A. Bain, D. Crisan, Fundamentals of Stochastic Filtering, Stochastic Modelling and Applied Probability, Vol 60, Springer Verlag, 2008.

[2] Adi Ben-Israel, Thomas N.E. Greville, Generalized inverses : theory and applications; 2nd Edition, Springer Verlag; 2003.

[3] A. Bensoussan, Stochastic control of partially observable systems. Cambridge University Press, Cambridge, 1992.

[4] D. Crisan, B. Rozovsky, The Oxford handbook of nonlinear filtering, Oxford Univ. Press, Oxford, 2011.

[5] D. Crisan, M. Kouritzin, J. Xiong, Nonlinear filtering with signal dependent observation noise, Electron. J. Probab. 14, 1863-1883, 2009.

[6] K. Du, Q. Meng, A revisit of W_2^n-theory of super-parabolic backward stochastic partial differential equations in $\mathbb{R}^d$, Stoch. Proc. and Applic. **120**, 1996–2015, 2010.

[7] K. Du, S. Tang, Q. Zhang, $W^{m,p}$–solutions ($p \geq 2$) of linear degenerate backward stochastic partial differential equations in the whole space, J. of Differential Equ. **254**, 2877 –2904, 2013.

[8] D. J. Fotsa-Mbogne, E. Pardoux, Nonlinear filtering with degenerate noise. Electron. Commun. Probab. **22**, 2017.

[9] I. Karatzas, S. Shreve, Brownian motion and stochastic calculus, Second edition. Graduate Texts in Mathematics, 113, Springer-Verlag, New York, 1991.

[10] N.V. Krylov, B.L. Rozovskii, On the first integrals and Liouville equations for diffusion processes, Lecture Notes in Control and Info. Sciences **36**, pp. 117–125, Springer Verlag 1981.

[11] E. Pardoux, Equations aux dérivées partielles stochastiques non linéaries monotones. PhD thesis, Univ Paris XI, Orsay, 1975.

[12] E. Pardoux, Stochastic Partial Differential Equations and Filtering of Diffusion Processes, Stochastics **3**, 127–147, 1979.

[13] E. Pardoux, Nonlinear filtering and associated stochastic partial differential equations. École d'Été de Probabilités de Saint-Flour XIX-1989, 67-163, Lecture Notes in Math., 1464, Springer, Berlin, 1991.

[14] E. Pardoux, A. Rascanu, Stochastic differential equations, backward SDEs, partial differential equations. Stochastic Modelling and Applied Probability, 69. Springer, Cham, 2014.

[15] P.E. Protter, Stochastic Integration and Differential Equations, Stochastic Modelling and Applied Probability, Springer Berlin,, 2005.

[16] D. Revuz, M. Yor, Continuous martingales and Brownian motion. Third edition, 293. Springer-Verlag, Berlin, 1999.