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STOCHASTIC EPIDEMIC MODEL FOR MALARIA: THE LAW OF LARGE NUMBERS

Othmane BAGHDADI[1] and Étienne PARDOUX[2]

[1]Mohammed First University, Oujda, Morocco

[2]Aix Marseille Univ., CNRS, I2M, Marseille, France

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Abstract. We study an individual-based stochastic host–vector epidemic model for malaria, in which humans can experience repeated infections over their lifetime. In contrast to classical Ross–Macdonald or compartmental SIR/SEIR models, each infection episode is characterised by a random time-dependent infectivity profile: after infection, a human host transmits parasites to susceptible mosquitoes according to a random infectivity function of the time since infection, while recovered hosts gradually regain susceptibility according to a random susceptibility function. On the vector side, susceptible mosquitoes become infected through contact with infectious humans and then contribute to transmission until death, under a demographic regime that combines birth and mortality processes.

We analyse the large-population asymptotic behaviour of this coupled host–vector system and prove a functional law of large numbers (FLLN) by constructing a sequence of i.i.d. auxiliary processes. The limiting dynamics are described by a nonlinear deterministic system of renewal-type integral equations that generalises both the classical Kermack–McKendrick age-of-infection framework and standard malaria models. The solution of the limiting deterministic system depends upon the law of the infectivity function only through its mean, but it depends upon the expectations of more complex functions of the susceptibility function.

1. Introduction.

Vector-borne diseases remain a major global health burden. They arise when an infection is transmitted indirectly between hosts through a biological vector—most often blood-feeding insects such as mosquitoes, but also ticks, flies, fleas, or freshwater snails, depending on the pathogen. Beyond their wide ecological diversity, vector-borne diseases collectively account for a substantial share of infectious morbidity and mortality worldwide and continue to spread under the combined effects of environmental change and globalization.

From a modelling perspective, vector transmission requires coupling two interacting populations (hosts and vectors), typically with distinct epidemiological structures and time scales. In many mosquito-borne infections, vectors do not recover and remain infectious until death, so that an SI/SEI description is natural on the vector side, while hosts may follow SIRS/SEIRS -type dynamics. Because vector lifespans are often much shorter than host lifespans, demographic turnover in the vector population (births and deaths) is an essential modeling component, and it is common

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to incorporate explicit vector demography and, when appropriate, time-scale separation arguments. Historically, the foundations of mathematical epidemiology for vector-borne diseases trace back to Ross' pioneering work on malaria [13], which led to the celebrated insight that reducing vector density below a threshold can prevent sustained transmission (often referred to as Ross mosquito theorem). Mathematical models of malaria and other mosquito-borne diseases have developed substantially beyond the original Ross framework, including detailed host-vector models and broad reviews of mosquito-borne pathogen transmission [2, 10, 12]. On the other hand, Kermack and McKendrick introduced a general framework for infectious disease dynamics in which infectivity depends on the infection age (time since infection) [7], and later works in their series also accounted for progressive loss of immunity [8, 9]. These formulations naturally lead to integral or partial differential equation models rather than ordinary differential equations, but they capture more realistic within-host and between-host heterogeneities in infectiousness and immunity. A complementary and now classical viewpoint is that deterministic compartmental systems can be obtained as law-of-large-numbers limits of finite-population stochastic epidemic models. Recent developments have shown that Kermack–McKendrick-type infection-age models can arise as limits of *non-Markovian* stochastic systems, thereby providing a probabilistic foundation for varying infectivity and waning immunity beyond the Markovian setting. This probabilistic perspective is also crucial to describe early epidemic stages and extinction phases, which are inherently driven by stochasticity and cannot be captured by purely deterministic models [6, 5, 4, 3].

The goal of this work is to extend the non-Markovian, varying-infectivity and waning-immunity framework of [6] to a Ross-type vector-borne setting. In contrast with classical malaria models where human immunity loss is often assumed instantaneous, we incorporate a *progressive* waning of host immunity, and we allow both host and vector infectivities to depend upon infection age and to vary across individuals. On the vector side, we explicitly include demography through births and deaths. Our main contributions are (i) the formulation of a finite-population stochastic host–vector model with infection-age-dependent infectivity and progressive immunity decay, and (ii) the derivation of the corresponding deterministic limit as the population sizes of both vectors and hosts tend to infinity.

Notation. Throughout the paper, all the random variables and processes are defined on a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We use $\xrightarrow[N\to+\infty]{\text{a.s.}}$ to denote almost sure convergence as the parameter $N \to \infty$. We use $\mathbf{1}_{\{\cdot\}}$ for the indicator function. Let $D = D(\mathbb{R}_+;\mathbb{R})$ be the space of $\mathbb{R}$-valued càdlàg functions defined on $\mathbb{R}_+$, with convergence in D meaning convergence in the Skorohod J_1 topology (see, e.g., [1, Chapter 3]). We also define $D_+ = D(\mathbb{R}_+, \mathbb{R}_+)$.

2. Model Description.

We model malaria transmission between a host population of size N and a vector population of size N^V and we shall let $N, N^V \to \infty$ while $N^V/N = \kappa \in (0,\infty)$. Transmission occurs in both directions:

$$\text{host} \longrightarrow \text{vector}, \qquad \text{vector} \longrightarrow \text{host}.$$

2.1. Host Infectivity and Susceptibility Profiles.

Let

$$(\lambda^0, \gamma^0) \qquad \text{and} \qquad (\lambda, \gamma)$$

be random elements of

$$D_+^2 := D(\mathbb{R}_+;\mathbb{R}_+)^2.$$

Let $\bar{S}_H(0) > 0$, $\bar{I}_H(0) > 0$, and $\bar{R}_H(0) \geq 0$ be such that

$$\bar{S}_H(0) + \bar{I}_H(0) + \bar{R}_H(0) = 1.$$

Let $\{(\lambda_k^0, \gamma_k^0),\ 1 \leq k \leq N\}$ be an i.i.d. sequence of random elements of D_+^2, each having the law of (λ^0, γ^0).

For every $1 \leq k \leq N$ and $t \geq 0$, we define

$$(\lambda_{k,0}(t), \gamma_{k,0}(t)) = \begin{cases} (0,1), & \text{with probability } \bar{S}_H(0), \\ (\lambda_k^0(t), \gamma_k^0(t)), & \text{with probability } \bar{I}_H(0), \\ (0, \gamma_k^0(\eta_k^0 + \xi_k + t)), & \text{with probability } \bar{R}_H(0), \end{cases} \tag{1}$$

where $\eta_k^0 := \sup\{t > 0,\ \lambda_k^0(t) > 0\}$, and $\xi_k \geq 0$ is an arbitrary random variable, in such a way that the sequence $\{(\lambda_{k,0}, \gamma_{k,0})_{1 \leq k \leq N}\}$ is i.i.d.

Next, for $1 \leq k \leq N$ and $i \geq 1$,

$$(\lambda_{k,i}, \gamma_{k,i})$$

are i.i.d. copies of (λ, γ), independent of the previous family.

- $(\lambda_{k,0}, \gamma_{k,0})$ characterises the epidemiological status of individual k at time 0, while for $i \geq 1$:
- $\lambda_{k,i}(t)$ denotes the infectivity of host k towards vectors (host $\to$ vector) at infection-age t after its i-th (re)infection;
- $\gamma_{k,i}(t)$ denotes the susceptibility of host k to vector-borne infection (vector $\to$ host) at infection-age t after its i-th (re)infection.

It clearly follows from (1) that

$$\bar{I}_H(0) = \mathbb{P}\big(\eta_{k,0} > 0\big).$$

This quantity is the probability that a randomly chosen individual in the population is initially infected.

2.2. Host Infection Counting Process and Age Since Infection.

For each host k, the counting process of infection events is

$$B_k^N(t) := \text{number of infection events experienced by host } k \text{ in } (0,t].$$

The time since its most recent infection is

$$\mathfrak{a}_k^N(t) := t - \sup\{s \in (0,t] : B_k^N(s) = B_k^N(s^-) + 1\},$$

with $\sup \varnothing = 0$.

The current infectivity and susceptibility of host k at time t are

$$\lambda_k(t) := \lambda_{k,B_k^N(t)}\big(\mathfrak{a}_k^N(t)\big), \qquad \gamma_k(t) := \gamma_{k,B_k^N(t)}\big(\mathfrak{a}_k^N(t)\big).$$

Note that whenever $B_k^N(t) = 0$, $\mathfrak{a}_k^N(t) = t$.

2.3. Vector Infectivity Profiles (Vector $\to$ Host).

Let

$$\lambda^{V,0} \qquad \text{and} \qquad \lambda^V$$

be random elements of D_+, representing vector infectivity towards hosts.

Let $\bar{S}_V(0) > 0$ and $\bar{I}_V(0) > 0$ be such that

$$\bar{S}_V(0) + \bar{I}_V(0) = 1.$$

Let $\{\lambda_\ell^{V,0},\ 1 \leq \ell \leq N_V\}$ be an i.i.d. sequence of random elements of D_+, each having the law of $\lambda^{V,0}$.

For every $1 \le \ell \le N_V$ and $t \ge 0$, we define

$$\lambda^V_{\ell,0}(t) = \begin{cases} 0, & \text{with probability } \bar{S}_V(0), \\ \lambda^{V,0}_\ell(t), & \text{with probability } \bar{I}_V(0). \end{cases}$$

Next for $\ell \in \{1, \dots, N_V\}$,

$$\lambda^V_{\ell,1}$$

are i.i.d. copies of λ^V, independent of the previous family and of all host processes.

The process $\lambda^V_{\ell,0}$ contains the information about the epidemiological status of vector ℓ at time 0, while $\lambda^V_{\ell,1}(t)$ describes the infectivity of the initially susceptible vector ℓ towards hosts (vector $\to$ host) at time t after its infection. More precisely, if $\tau^{N_V}_\ell$ denotes the time of infection of ℓ, then its infectivity at time $t > \tau^{N_V}_\ell$ is

$$\lambda^V_{\ell,1}\big(t - \tau^{N_V}_\ell\big).$$

Remark 2.1. The following two probabilistic conditions hold:

1. There exists $t_0 > 0$ such that, for every $t \in [0, t_0]$,
$$\mathbb{P}\big(\gamma(t) < 1\big) > 0.$$
2. For every $t > 0$, $1 \le k \le N$,
$$\mathbb{P}\big(\lambda_{k,0}(t) < \lambda^*\big) > 0.$$

Justification of 1. As a consequence of Assumption 2.4 below, there exists $t_0 > 0$ such that

$$\mathbb{P}\big(\gamma(t) = 0, 0 \le t \le t_0\big) > 0,$$

and therefore

$$\mathbb{P}\big(\gamma(t) < 1\big) > 0, \qquad \forall\, t \in [0, t_0],$$

which establishes condition 1.

Justification of 2. Assume by contradiction that there exists $t > 0$ such that

$$\mathbb{P}\big(\lambda_{k,0}(t) = \lambda^*\big) = 1.$$

This would imply that the baseline infection intensity almost surely attains its maximal value at time t, meaning that the proportion of initially infected individuals equals one at that time. By monotonicity of the infection process, this would in turn imply that the proportion of infected individuals was already equal to one at time $t = 0$. Consequently, the initial proportion of susceptible individuals would be zero, which contradicts the standing assumptions of the model. Hence,

$$\mathbb{P}\big(\lambda_{k,0}(t) < \lambda^*\big) > 0$$

must hold for every $t > 0$, hence condition 2.

Vector demography (births and deaths). We need to take into account the demography of vectors. At time t, susceptible vectors are borns at rate $N_V \mu_V(t)$ and each vector, whether infected or susceptible, dies at rate $\mu_V(t)$. For each vector ℓ which is not present in the population at time 0, we denote by α_ℓ its birth time and by σ_ℓ its death time.

Assumption 2.2. We assume that $\mu_V(t)$, which is both the birth and death rate of the vectors is locally integrable, i.e.,

$$\mu_V \in L^1_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}_+)\,.$$

We model the birth of vectors through the birth point process

$$\mathcal{A}^{N_V}(\mathrm{d}s) := \sum_{i > S^{N_V}(0)} \delta_{\alpha_i}(\mathrm{d}s),$$

and vector removal through the death point process

$$\mathcal{M}^{N_V}(\mathrm{d}s) := \sum_{i \geq 1} \delta_{\sigma_i}(\mathrm{d}s).$$

The associated counting processes are

$$A^{N_V}(t) = \mathcal{A}^{N_V}([0,t]) = \sum_{i > S^{N_V}(0)} \mathbf{1}_{\{\alpha_i \leq t\}}, \qquad M^{N_V}(t) = \mathcal{M}^{N_V}([0,t]) = \sum_{i \geq 1} \mathbf{1}_{\{\sigma_i \leq t\}}.$$

Hence the vector population size evolves as

$$N_V(t) = N_V + A^{N_V}(t) - M^{N_V}(t).$$

Remark 2.3 (Deaths due to the disease)**.** We have not explicitly modeled the fact that the disease is likely to kill both vectors and humans. Concerning the vectors, we could easily increase the rate of death of infected mosquitos (compared to the rate of death of susceptible mosquitos).

Concerning the humans, we can decide that with a certain probability, $\gamma \equiv 0$. This means that in that case the concerned human will no longer contribute to the epidemic, and we could decide that those humans die at the end of the infected period.

2.4. **Average infectivity and susceptibility.**

Assumption 2.4. We assume that, almost surely, the host infectivity and susceptibility profiles satisfy

$$0 \leq \lambda^0(t),\, \lambda(t) \leq \lambda^*, \qquad 0 \leq \gamma^0(t),\, \gamma(t) \leq 1,$$

and that the host susceptibility becomes positive only when its infectivity vanishes:

$$\sup\{t \geq 0 : \lambda^0(t) > 0\} \leq \inf\{t \geq 0 : \gamma^0(t) > 0\},$$
$$\sup\{t \geq 0 : \lambda(t) > 0\} \leq \inf\{t \geq 0 : \gamma(t) > 0\}.$$

In addition, we impose analogous assumptions for the vector infectivity profile:

$$0 \leq \lambda^{V,0}(t),\, \lambda^V(t) \leq \lambda^*,$$

We also assume that, for all $t < 0$,

$$\lambda^0(t) = \lambda(t) = \gamma^0(t) = \gamma(t) = \lambda^{V,0}(t) = \lambda^V(t) = 0.$$

For host. The quantities

$$\overline{\mathfrak{F}}_1^N(t) := \frac{1}{N} \sum_{k=1}^{N} \lambda_{k, B_k^N(t)}\big(\mathfrak{a}_k^N(t)\big).$$

$$\overline{\mathfrak{S}}^N(t) := \frac{1}{N} \sum_{k=1}^{N} \gamma_{k, B_k^N(t)}\big(\mathfrak{a}_k^N(t)\big).$$

are respectively the average infectivity and the average susceptibility in the human population.

For vectors. The quantities

$$\overline{\mathfrak{F}}_2^{N_V}(t) := \frac{1}{N_V}\left(\sum_{i=1}^{I^{N_V}(0)} \lambda_{i,0}^V(t)\mathbf{1}_{\sigma_i>t} + \sum_{j=1}^{S^{N_V}(0)} \lambda_{j,1}^V\big(t-\tau_j^{N_V}\big)\mathbf{1}_{\sigma_j>t} + \sum_{\alpha_k\in(0,t]} \lambda_{\alpha_k,1}^V\big(t-\tau_{\alpha_k}^{N_V}\big)\mathbf{1}_{\sigma_k>t}\right).$$

where in the last formula, σ_k denotes the death time of the individual born at time α_k, and

$$\bar{S}_V^{N_V}(t) := \frac{1}{N_V} S_V^{N_V}(t).$$

are respectively the average infectivity in the vector population and the proportion of susceptible vectors, where $S_V^{N_V}(t)$ denotes the number of susceptible vectors in the vector population at time t.
Above $\{0 < \alpha_1 < \alpha_2 < \cdots,\}$ are the successive birth times of vectors. Since a.s. none of the α_k is an integer, there is no ambiguity in our notations. Of course, we assume that the random functions $\lambda_{1,1}^V, \lambda_{2,1}^V, \dots, \lambda_{S^{N_V}(0),1}^V \lambda_{\alpha_1,1}^V, \lambda_{\alpha_2,1}^V, \dots$ are i.i.d. The vector population at time $t = 0$ is

$$N_V = I^{N_V}(0) + S^{N_V}(0),$$

partitioned into $I^{N_V}(0)$ initially infected and $S^{N_V}(0)$ initially susceptible.

A susceptible host k gets infected by vectors at the instantaneous rate

$$\Upsilon_k^N(t) := \kappa\gamma_{k,B_k^N(t)}(\mathfrak{a}_k^N(t))\,\overline{\mathfrak{F}}_2^{N_V}(t),$$

and each susceptible vector ℓ becomes infected by biting infectious hosts with intensity

$$\frac{1}{\kappa}\overline{\mathfrak{F}}_1^N(t).$$

2.5. **Construction of the Counting Processes B_k^N and $C_\ell^{N_V}$.** Let $\big(\lambda_{k,i}, \gamma_{k,i}\big)_{i\geq 0,\ 1\leq k\leq N}$, $\big(\lambda_{\ell,i}^V\big)_{i\in\{0,1\},\ \ell\geq 1}$, be a collection of independent random variables as introduced above, and $\big(Q_k\big)_{1\leq k\leq N_V}$ (resp. $\big(Q_k^V\big)_{1\leq k\leq N}$) be a collection of independent and identically distributed standard Poisson random measures (PRMs) on $\mathbb{R}_+^2$, which are independent of the above random functions. See below section 5.1 for some basic facts about Poisson random measures.

For each host $k \in \{1,\dots,N\}$, the counting process of its successive infection events is given as

$$B_k^N(t) = \int_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{u\leq \Upsilon_k^N(r-)\}}\, Q_k(\mathrm{d}r,\mathrm{d}u), \qquad t\geq 0, \tag{2}$$

By construction, $(B_k^N(t))_{t\geq 0}$ is a càdlàg, integer-valued, nondecreasing process with jumps of size 1.

Similarly, let $\big(Q_\ell^V\big)_{\ell\geq 1}$ be an i.i.d. family of standard Poisson random measures on $\mathbb{R}_+^2$, independent of everything else. For each $\ell \in \{1,\dots,N_V\}$, the counting process of infection events is defined by

$$C_\ell^{N_V}(t) = \int_0^t \int_0^\infty \mathbf{1}_{\{C_\ell^{N_V}(s^-)=0\}}\,\mathbf{1}_{\left\{u\leq \frac{\overline{\mathfrak{F}}_1^N(s^-)}{\kappa}\right\}}\, Q_\ell^V(\mathrm{d}s,\mathrm{d}u), \qquad t\geq 0, \tag{3}$$

and, for a vector born at time α_k,

$$C_{\alpha_k}^{N_V}(t) = \int_{\alpha_k}^t \int_0^\infty \mathbf{1}_{\{C_{\alpha_k}^{N_V}(s^-)=0\}}\,\mathbf{1}_{\left\{u\leq \frac{\overline{\mathfrak{F}}_1^N(s^-)}{\kappa}\right\}}\, Q_{S^{N_V}(0)+k}^V(\mathrm{d}s,\mathrm{d}u).$$

Each vector can be infected at most once. Note that $t \mapsto C_\ell^{N_V}(t)$ is non decreasing and $\{0,1\}$-valued, and that whenever the vector ℓ is initially infected, then by convention $C_\ell^{N_V}(0) = 1$

2.6. **Infectious Periods for Hosts.** We now define the random infectious periods associated with the infectivity profiles of hosts and vectors.

Host infectious periods. The infectious periods for hosts are defined by

$$\eta_0 := \sup\{t \geq 0 : \lambda^0(t) > 0\}, \qquad \eta := \sup\{t \geq 0 : \lambda(t) > 0\},$$

with the convention $\sup \varnothing = 0$. Thus η_0 (resp. η) represents the duration during which an initially infected host (resp. a newly infected host) can infect vectors.

For each host $k \in \{1, \dots, N\}$ and infection index $i \geq 0$, we define the individual infectious period

$$\eta_{k,i} := \sup\{t \geq 0 : \lambda_{k,i}(t) > 0\}.$$

By construction, $\eta_{k,0}$ has the same distribution as η_0, and $\eta_{k,i}$ (for $i \geq 1$) has the same distribution as η.

2.7. **Numbers of Infected and Non-Infected Hosts.** We now define the indicators of current infection status using the previously introduced infection counting processes and infectious periods.

A host k is *currently infected (infectious)* at time t if it has experienced at least one infection before t and its time since the last infection is strictly smaller than the corresponding infectious period. We thus define the indicator

$$I_k^N(t) := \mathbf{1}_{\left\{ \mathfrak{a}_k^N(t) < \eta_{k, B_k^N(t)} \right\}},$$

where again $B_k^N(t)$ is the infection counting process of host k, see (2), $\mathfrak{a}_k^N(t)$ is its age since last infection, and $\eta_{k,B_k^N(t)}$ is the infectious period associated with its most recent infection. In the case $B_k^N(t) = 0$, we have $\mathfrak{a}_k^N(t) = t$, and individual k is infected at time t if it was initially infected and $\eta_{k,0} > t$.

The total number of (currently) infected hosts at time t is

$$I_H^N(t) := \sum_{k=1}^{N} I_k^N(t),$$

and the number of non-infected (non-infectious) hosts is

$$U_H^N(t) := N - I_H^N(t).$$

Here, $U_H^N(t)$ groups together susceptible and immune hosts that are not currently infectious.

2.8. **Markovian SIRS–SI host–vector model as a special case.** In this subsection we show how the classical SIRS–SI host–vector model can be recovered as a particular case of the general stochastic framework introduced above, by choosing specific infectivity and susceptibility profiles for hosts and vectors.

Host side: SIRS structure. We assume that each host, once infected, goes through three successive phases:

- an *infectious* phase whose duration follows an exponential distribution with parameter $\mu_H > 0$,
- an *immune* phase whose duration follows an exponential distribution with parameter $\theta_H > 0$,
- followed by a fully susceptible phase, until the next infection.

We denote by $\beta_{HV} > 0$ the transmission rate from an infectious host to vectors, and we neglect host heterogeneity by taking identical profiles for all infection episodes. At the level of the prototype functions, this corresponds to choosing

$$\lambda(t) := \beta_{HV}\,\mathbf{1}_{\{0\le t<\xi_H\}}, \qquad t \ge 0, \quad \text{where } \xi_H \sim \mathrm{Exp}(\mu_H).$$

and

$$\gamma(t) := \mathbf{1}_{\{t\ge \xi_H+\zeta_H\}},\ t\ge 0,\ \text{ where } \zeta_H \sim \mathrm{Exp}(\theta_H),\ \text{ and } \zeta_H \text{ and } \xi_H \text{ are independent.}$$

Note that β_{HV}, μ_H, and θ_H are identical for all hosts, whereas the random variables ξ_H and ζ_H are i.i.d. across hosts.

Vector side: SI structure with demography. For the vectors, we consider a simple SI structure: after infection, vectors remain infectious until they die and never return to a susceptible state. We denote by $\beta_{VH} > 0$ the transmission rate from an infectious host to susceptible vectors.

In the general framework, this corresponds to choosing a vector infectivity profile of the form

$$\lambda^V(t) := \beta_{VH}\,\mathbf{1}_{\{\alpha\le t<\sigma\}}, \qquad t \ge 0,$$

α and σ denote respectively the birth and death time. Vectors who are alive at time 0 are supposed be born at time 0. Susceptible vectors are born at rate $\mu_V N_V > 0$, and die naturally at rate $\mu_V > 0$. Equivalently, each vector has an i.i.d. lifetime $\sigma - \alpha \sim \mathrm{Exp}(\mu_V)$. Denoting by $S_V(t)$ and $I_V(t)$ the numbers of susceptible and infected vectors, infection occurs at rate $\beta_{VH}\, I_H(t)/N_H$ and infected vectors remain infectious until death. The following deterministic model can be justified in this Markovian context :

$$\frac{dS_H}{dt} = -\kappa\beta_{HV} I_V S_H + \theta R_H, \qquad \frac{dI_H}{dt} = \kappa\beta_{HV} I_V S_H - \mu I_H, \qquad \frac{dR_H}{dt} = \mu I_H - \theta R_H$$

$$\frac{dS_V}{dt} = \mu_V - \frac{\beta_{VH}}{\kappa} I_H S_V - \mu_V S_V, \qquad \frac{dI_V}{dt} = \frac{\beta_{VH}}{\kappa} I_H S_V - \mu_V I_V.$$

3. **Functional Law of Large Numbers and Limit Integral System.** In this section we state the functional law of large numbers (FLLN) for the scaled host–vector processes and introduce the deterministic limit through a *two-dimensional* system of integral equations. The key point is that, in the large-population limit, the vector layer can be expressed explicitly as a functional of the host-to-vector renormalised force of infection noted by y in this section, so that the core limit dynamics closes on two unknowns.

Let us first formulate an assumption, which is in fact a consequence of the formulation of our model.

Assumption 3.1. As $N \to \infty$,

$$\frac{S_H(0)}{N} \longrightarrow \overline{S}_H(0), \quad \frac{I_H(0)}{N} \longrightarrow \overline{I}_H(0), \quad \frac{R_H(0)}{N} \longrightarrow \overline{R}_H(0)\,.$$

As $N_V \to \infty$,

$$\frac{I_V^{N_V}(0)}{N_V} \longrightarrow \overline{I}_V(0), \qquad \frac{S_V^{N_V}(0)}{N_V} \longrightarrow \overline{S}_V(0).$$

3.1. **Mean infectivities and deterministic inputs.** We denote by $\bar{I}_H(0)$ and $\bar{I}_V(0)$ the initial fractions of infected hosts and vectors, and by $\bar{S}_V(0)$ the initial fraction of susceptible vectors. We set the mean infectivities and the (conditional) survival functions

$$\begin{aligned}
\bar{\lambda}_0(t) &:= \mathbb{E}\big[\lambda_0(t)\big], & \bar{\lambda}(t) &:= \mathbb{E}\big[\lambda(t)\big],\\
\bar{\lambda}^{V,0}(t) &:= \mathbb{E}\big[\lambda^{V,0}(t)\big], & \bar{\lambda}^{V}(t) &:= \mathbb{E}\big[\lambda^{V}(t)\big],\\
F_0^c(t) &:= \mathbb{P}\big(\eta_0 > t \mid \eta_0 > 0\big), & F^c(t) &:= \mathbb{P}(\eta > t).
\end{aligned}$$

3.2. **Vector layer as a functional of y.** Given a nonnegative function y on $\mathbb{R}_+$ representing the renormalised host force of infection, define the induced fraction of susceptible vectors

$$\mathcal{S}(y)(t) = \bar{S}_V(0)\,\exp\left(-\int_0^t \big[\frac{y(r)}{\kappa}+\mu_V(r)\big]\,\mathrm{d}r\right)+\int_0^t \mu_V(s)\,\exp\left(-\int_s^t \big[\frac{y(r)}{\kappa}+\mu_V(r)\big]\,\mathrm{d}r\right)\mathrm{d}s, \tag{4}$$

and the induced vector-to-host force of infection

$$\mathcal{V}(y)(t) = \bar{\lambda}_0^V(t)\,\exp\Big(-\int_0^t \mu_V(u)\,\mathrm{d}u\Big)+\frac{1}{\kappa}\int_0^t \bar{\lambda}^V(t-s)\,\exp\Big(-\int_s^t \mu_V(u)\,\mathrm{d}u\Big)\,\mathcal{S}(y)(s)\,y(s)\,\mathrm{d}s. \tag{5}$$

$\mathcal{S}(y)(t)$ and $\mathcal{V}(y)(t)$ represent, respectively, the susceptible fraction and the renormalised vector-to-host force of infection in the vector population at time t as a functions of the renormalised host force of infection between time 0 and time t.

3.3. **Two-dimensional limit system.** To describe the deterministic limit of the FLLN, we introduce a two-dimensional system of integral equations. Observe that the host layer depends on the vector layer only through $\mathcal{V}(y)$ defined in (5). We thus look for a solution $(x, y) \in D_+^2$ of the closed system

$$\begin{cases}
x(t) = \mathbb{E}\Big[\gamma_0(t)\,\exp\Big(-\kappa\displaystyle\int_0^t \gamma_0(r)\,\mathcal{V}(y)(r)\,\mathrm{d}r\Big)\Big]\\
\qquad\qquad + \kappa\displaystyle\int_0^t \mathbb{E}\Big[\gamma(t-s)\exp\Big(-\kappa\int_s^t \gamma(r-s)\,\mathcal{V}(y)(r)\,\mathrm{d}r\Big)\Big]\,x(s)\,\mathcal{V}(y)(s)\,\mathrm{d}s,\\
y(t) = \bar{\lambda}_0(t) + \kappa\displaystyle\int_0^t \bar{\lambda}(t-s)\,x(s)\,\mathcal{V}(y)(s)\,\mathrm{d}s.
\end{cases} \tag{6}$$

In (6), the only random ingredients from the microscopic model enter through the expectations defining the mean infectivities $(\bar{\lambda}_0, \bar{\lambda}, \bar{\lambda}_0^V, \bar{\lambda}^V)$ and through the two laws of γ_0 and γ. While the infectivity functions enter (5) and the second line of (6) only through their means, the expectation of rather complicated functionals of $\gamma_0(t)$ and $\gamma(t)$ and their integrals appear in the first equation of (6).

It should be noted that the term $\exp\Big(-\kappa\int_0^t \gamma_0(r)\mathcal{V}(y)(r)\,dr\Big)$ is the conditional expectation, given the random function γ_0, that an individual having this susceptibility function is not infected on the interval $(0, t]$. Similarly, for an individual who gets infected at time s, $\exp\Big(-\kappa\int_s^t \gamma(r-s)\mathcal{V}(y)(r)\,dr\Big)$ is the conditional expectation, given the function γ, that this individual is not reinfected between time s and time t.

This is reminiscent of the fact that, in the formula for $\overline{\mathfrak{S}}^N(t)$ in Section 2.4, as soon as $B_k^N(t)$ jumps by one, i.e. the individual k gets (re)infected, then $\gamma_{k,B_k^N(t)}$ jumps down to 0.

3.4. **Link with the Markovian model.** We consider the system defined in Subsection 2.8, with the initial condition such that $\bar{R}_H(0)=0$ (assumed here for simplicity), the other initial conditions being >0, and such that $\bar{S}_H(0)+\bar{I}_H(0)=1$, $\bar{S}_V(0)+\bar{I}_V(0)=1$. We consider now the system (6), (5), (4), in the particular case where, with ξ and η being two independent r.v.s with $\xi\sim Exp(\mu)$ and $\eta\sim Exp(\theta)$,

$$\begin{aligned}
\lambda(t)&=\beta_{HV}\mathbf{1}_{0\le t\le\xi},\\
\lambda^V(t)&=\beta_{VH},\\
(\lambda_0(t),\gamma_0(t))&=\begin{cases}(0,1), & \text{with probability } \bar{S}_H(0),\\ (\beta_{HV}\mathbf{1}_{0\le t\le\xi},\mathbf{1}_{\xi+\eta\le t}), & \text{with probability } \bar{I}_H(0),\end{cases}\\
\gamma(t)&=\mathbf{1}_{\xi+\eta\le t}\,.
\end{aligned}$$

As result,

$$\begin{aligned}
\overline{\lambda}(t)&=\beta_{HV}e^{-\mu t},\\
\mathbb{E}\left[\gamma_0(t)\exp\left(-\kappa\int_0^t\gamma_0(r)\mathcal{V}(y)(r)dr\right)\right]&=\bar{S}_H(0)\exp\left(-\kappa\int_0^t\mathcal{V}(y)(r)dr\right)\\
&\quad+\bar{I}_H(0)\mathbb{E}\left[\mathbf{1}_{\xi+\eta\le t}\exp\left(-\kappa\int_{\xi+\eta}^t\mathcal{V}(y)(r)dr\right)\right]\\
\mathbb{E}\left[\gamma(t-s)\exp\left(-\kappa\int_0^t\gamma(r-s)\mathcal{V}(y)(r)dr\right)\right]&=\mathbb{E}\left[\mathbf{1}_{\xi+\eta\le t-s}\exp\left(-\kappa\int_{s+\xi+\eta}^t\mathcal{V}(y)(r)dr\right)\right]
\end{aligned}$$

Lemma 3.2. *The r.v. $\xi+\eta$ is positive, with the density*

$$\frac{\mu\theta}{\mu-\theta}(e^{-\theta t}-e^{-\mu t})\,.$$

Proof of Lemma 3.2 . We first note that we might assume that $\frac{1}{\theta}>\frac{1}{\mu}$ (in most diseases, the recovery period is significantly longer than the infectious period), hence $\mu>\theta$.

$$\begin{aligned}
\mathbb{P}(\xi+\eta>t)&=\mathbb{P}(\xi>t)+\mathbb{P}(\eta>t-\xi,\xi\le t)\\
&=e^{-\mu t}+\mu\int_0^t e^{-\theta(t-s)}e^{-\mu s}ds\\
&=e^{-\mu t}+\frac{\mu}{\mu-\theta}(e^{-\theta t}-e^{-\mu t})\\
&=\frac{\mu}{\mu-\theta}e^{-\theta t}-\frac{\theta}{\mu-\theta}e^{-\mu t},
\end{aligned}$$

hence the formula for the density. □

The correspondence of notations between the general model and the Markov model is $x(t)=\bar{S}_H(t)$, $y(t)=\beta_{HV}\bar{I}_H(t)$, $\mathcal{S}(y)(t)=\bar{S}_V(t)$ and $\mathcal{V}(y)(t)=\beta_{VH}\bar{I}_V(t)$.

We will show that if (x, y) is the unique solution of (6), then the corresponding quantities of the Markov model satisfy the system of ODEs which appear at the end of section 2.8.

Let us start with the second equation in (6), which here becomes

$$
\begin{aligned}
y(t) &= \bar{I}_H(0)\beta_{HV}e^{-\mu t} + \kappa\beta_{HV}\int_0^t e^{-\mu(t-s)}x(s)\mathcal{V}(y)(s)ds \\
\frac{dy}{dt}(t) &= \kappa\beta_{HV}x(t)\mathcal{V}(y)(t) - \mu y(t)\,,
\end{aligned}
$$

which, divided by β_{HV}, gives precisely the ODE for $\bar{I}_H(t)$.

Let us next differentiate (4). We obtain

$$
\frac{d\mathcal{S}(y)}{dt}(t) = \mu_V - \left(\frac{y(t)}{\kappa} + \mu_V\right)\mathcal{S}(y)(t)\,,
$$

which is the ODE for $\bar{S}_V$. In the present case, (5) reads

$$
\begin{aligned}
\mathcal{V}(y)(t) &= \bar{I}_V(0)\beta_{VH}e^{-\mu_V t} + \frac{\beta_{VH}}{\kappa}\int_0^t e^{-\mu_V(t-s)}\mathcal{S}(y)(s)y(s)ds \\
\frac{d\mathcal{V}(y)}{dt}(t) &= \frac{\beta_{VH}}{\kappa}\mathcal{S}(y)(t)y(t) - \mu_V\mathcal{V}(y)(t)\,,
\end{aligned}
$$

which, divided by β_{VH}, gives the ODE for $\bar{I}_V$.

It remains to verify that $x(t) = \bar{S}_H(t)$ satisfies the first ODE.

Let us first rewrite the first equation of (6) in the Markov model. From some of the above formula, (6) in the present case reads

$$
\begin{aligned}
x(t) &= \bar{S}_H(0)\exp\left(-\kappa\int_0^t \mathcal{V}(y)(r)dr\right) + \bar{I}_H(0)\mathbb{E}\left[\mathbf{1}_{\xi+\eta\le t}\exp\left(-\kappa\int_{\xi+\eta}^t \mathcal{V}(y)(r)dr\right)\right] \\
&\quad + \kappa\int_0^t \mathbb{E}\left[\mathbf{1}_{\xi+\eta\le t-s}\exp\left(-\kappa\int_{s+\xi+\eta}^t \mathcal{V}(y)(r)dr\right)\right]x(s)\mathcal{V}(y)(s)ds \\
&= \bar{S}_H(0)\exp\left(-\kappa\int_0^t \mathcal{V}(y)(r)dr\right) + \bar{I}_H(0)\int_0^t \exp\left(-\kappa\int_s^t \mathcal{V}(y)(r)dr\right)\frac{\mu\theta}{\mu-\theta}(e^{-\theta s} - e^{-\mu s})ds \\
&\quad + \kappa\int_0^t\int_0^{t-s}\exp\left(-\kappa\int_{s+r}^t \mathcal{V}(y)(u)du\right)x(s)\mathcal{V}(y)(s)\frac{\mu\theta}{\mu-\theta}(e^{-\theta r} - e^{-\mu r})drds \\
\frac{dx}{dt}(t) &= -\kappa\mathcal{V}(y)(t)x(t) + \bar{I}_H(0)\frac{\mu\theta}{\mu-\theta}(e^{-\theta t} - e^{-\mu t}) + \kappa\frac{\mu\theta}{\mu-\theta}\int_0^t (e^{-\theta(t-s)} - e^{-\mu(t-s)})x(s)\mathcal{V}(y)(s)ds,
\end{aligned}
$$

which is the ODE for $\bar{S}_H$, provided we show that

$$
\bar{R}_H(t) = \bar{I}_H(0)\frac{\mu}{\mu-\theta}(e^{-\theta t} - e^{-\mu t}) + \kappa\frac{\mu}{\mu-\theta}\int_0^t (e^{-\theta(t-s)} - e^{-\mu(t-s)})x(s)\mathcal{V}(y)(s)ds\,.
$$

Indeed, applying the Duhamel formula first to the $\bar{I}_H$ equation, next to the $\bar{R}_H$ equation and recalling that $\bar{R}_H(0) = 0$, we have:

$$
\begin{aligned}
\bar{I}_H(s) &= \bar{I}_H(0)e^{-\mu s} + \kappa\int_0^s e^{-\mu(s-r)}x(r)\mathcal{V}(y)(r)dr, \\
\bar{R}_H(t) &= \bar{I}_H(0)\mu e^{-\theta t}\int_0^t e^{-(\mu-\theta)s}ds + \kappa\mu\int_0^t e^{-\theta(t-s)}\int_0^s e^{-\mu(s-r)}x(r)\mathcal{V}(y)(r)drds \\
&= \bar{I}_H(0)\frac{\mu}{\mu-\theta}(e^{-\theta t} - e^{-\mu t}) + \kappa\mu e^{-\theta t}\int_0^t e^{\mu r}\int_r^t e^{-(\mu-\theta)s}ds\; x(r)\mathcal{V}(y)(r)dr,
\end{aligned}
$$

from which the result follows by the explicit integration of the ds integral.

3.5. **Well-posedness and FLLN statement.** Our first result establishes existence and uniqueness for the reduced limit system (6).

Theorem 3.3 (Existence and uniqueness). *Assume that Assumption 2.4 holds. Then the system (6) admits a unique solution $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}}_1) \in D_+^2$. Moreover, if $t \mapsto \mathbb{E}[\lambda_0(t)]$ is continuous and $t \mapsto \gamma_0(t)$ is continuous in probability, then $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}}_1) \in C(\mathbb{R}_+)^2$.*

We now state the functional law of large numbers. Let

$$(\overline{\mathfrak{F}}_1^N, \overline{\mathfrak{S}}^N, \bar{S}_V^{N_V}, \overline{\mathfrak{F}}_2^{N_V})$$

be the scaled host–vector processes in the (N, N_V)-population model (defined in Subsection 2.4).

Theorem 3.4 (FLLN). *Under Assumptions 2.4, 2.2 and 3.1, as $N, N_V \to \infty$ while $\frac{N_v}{N} = \kappa \in (0, \infty)$ is fixed,*

$$(\overline{\mathfrak{S}}^N, \overline{\mathfrak{F}}_1^N) \longrightarrow (\overline{\mathfrak{S}}, \overline{\mathfrak{F}}_1) \qquad \textit{in } D_+^2,$$

where $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}}_1)$ is the unique solution of (6) and moreover,

$$(\bar{S}_V^{N_V}, \overline{\mathfrak{F}}_2^{N_V}) \longrightarrow (\mathcal{S}(\overline{\mathfrak{F}}_1), \mathcal{V}(\overline{\mathfrak{F}}_1)) \qquad \textit{in } D_+^2.$$

Corollary 3.5. Given the solution $\left(\overline{\mathfrak{S}}, \overline{\mathfrak{F}}_1\right)$ of (6),

$$(\overline{U}^N, \overline{I}^N) \xrightarrow[N \to +\infty]{\mathbb{P}} (\overline{U}, \overline{I}) \qquad \text{in } D_+^2,$$

With $(\overline{U}^N, \overline{I}^N) = (\frac{I_H^N(t)}{N}, \frac{U_H^N(t)}{N})$ and $(\overline{U}, \overline{I})$ is given by

$$\begin{aligned} \overline{U}(t) = \mathbb{E}\Big[\mathbf{1}_{t \geq \eta_0} \exp\Big(- \kappa \int_0^t \gamma_0(r)\, \mathcal{V}(\overline{\mathfrak{F}}_1)(r)\, dr\Big)\Big] \\ + \kappa \int_0^t \mathbb{E}\Big[\mathbf{1}_{t-s \geq \eta} \exp\Big(- \kappa \int_s^t \gamma(r-s)\, \mathcal{V}(\overline{\mathfrak{F}}_1)(r)\, dr\Big)\Big]\, \overline{\mathfrak{S}}(s)\, \mathcal{V}(\overline{\mathfrak{F}}_1)(s)\, ds, \end{aligned} \tag{7}$$

$$\overline{I}(t) = \overline{I}_H(0)\, F_0^c(t) + \kappa \int_0^t F^c(t-s)\, \overline{\mathfrak{S}}(s)\, \mathcal{V}(\overline{\mathfrak{F}}_1)(s)\, ds. \tag{8}$$

Theorem 3.3 and Theorem 3.4 will be proved in the next two sections, and the proof of Corollary 3.5 will follow immediately thereafter.

4. **Proof of Theorem 3.3.** Before turning to the proof of theorem 3.3, let us point out the fact that the coefficients of our system of Volterra type integral equations are non linear, and rather complex, notably due to the appearance of the functional $\mathcal{V}(y)$ (which itself involves the functional $\mathcal{S}(y)$). Our proof will follow the following route. We shall first prove an priori estimate of the solution, more precisely that whenever (x, y) solves (6), then x, y, $\mathcal{S}(y)$ and $\mathcal{V}(y)$ satisfy some uniform bound, at least on any bounded time interval $[0, T]$, $T < \infty$. The second step will consist in exploiting those uniform bounds, and a type of locally Lipschitz property of the coefficients, in order to prove uniqueness. Finally, a Picard iteration scheme will allow us to prove existence. However, since we need a priori bounds on the approximating sequence, and since the bounds we have are only valid for a solution of the limiting equation, we will define a Picard approximation for the solution of

a system where both the solution and the approximating sequence are "forced" to satisfy the wished bounds, and finally prove that the solution of the modified system is in fact a solution of our original system.

Let us first establish the following lemma, which will be crucial for our a priori bounds.

Lemma 4.1 (Mass conservation)**.** *Assume that the pair (x, y) is a solution of the system (6). Then, for every $t \geq 0$, we have*

$$\mathbb{E}\left[\exp\left(-\kappa \int_0^t \gamma_0(r)\,\mathcal{V}(y)(r)\,\mathrm{d}r\right)\right] + \kappa \int_0^t \mathbb{E}\left[\exp\left(-\kappa \int_s^t \gamma(r-s)\,\mathcal{V}(y)(r)\,\mathrm{d}r\right)\right] x(s)\,\mathcal{V}(y)(s)\,\mathrm{d}s = 1. \tag{9}$$

Proof of Lemma 4.1. We note as soon as (x, y) solves (6),

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbb{E}\left[\exp\Big(-\kappa \int_0^t \gamma_0(r)\,\mathcal{V}(y)(r)\,\mathrm{d}r\Big)\right] + \kappa \int_0^t \mathbb{E}\left[\exp\Big(-\kappa \int_s^t \gamma(r-s)\,\mathcal{V}(y)(r)\,\mathrm{d}r\Big)\right] x(s)\,\mathcal{V}(y)(s)\,\mathrm{d}s\right)\\
&\quad = \left(\mathbb{E}\left[-\gamma_0(t)\,\exp\Big(-\kappa \int_0^t \gamma_0(r)\,\mathcal{V}(y)(r)\,\mathrm{d}r\Big)\right] + x(t)\right.\\
&\qquad \left. -\kappa \int_0^t \mathbb{E}\left[\gamma(t-s)\,\exp\Big(-\kappa \int_s^t \gamma(r-s)\,\mathcal{V}(y)(r)\,\mathrm{d}r\Big)\right] x(s)\,\mathcal{V}(y)(s)\,\mathrm{d}s\right)\kappa\,\mathcal{V}(y)(t) = 0.
\end{aligned}$$

Hence the left-hand side of (9) is equal to its value at $t = 0$, which is 1. □

A priori estimates of the solution. Let $(x, y) \in D$ be a solution of the system (6). From Assumption 2.4 we have that $0 \leq \gamma_0(t) \leq 1$ and $0 \leq \gamma(t-s) \leq 1$ for all $0 \leq s \leq t \leq T$. Hence, the combination of the first equation of (6) and Lemma 4.1 implies that

$$0 \leq x(t) \leq 1, \qquad \forall t \geq 0.$$

Moreover, we have

$$\begin{aligned}
\mathcal{S}(y)(t) &= \bar{S}_V(0)\,\exp\left(-\int_0^t \big(\frac{y(r)}{\kappa} + \mu_V(r)\big)\,\mathrm{d}r\right) + \int_0^t \mu_V(s)\,\exp\left(-\int_s^t \big(\frac{y(r)}{\kappa} + \mu_V(r)\big)\,\mathrm{d}r\right)\,\mathrm{d}s\\
&\leq \bar{S}_V(0)\,\exp\left(-\int_0^t \big(\frac{y(r)}{\kappa} + \mu_V(r)\big)\,\mathrm{d}r\right) + \int_0^t \big(\frac{y(s)}{\kappa} + \mu_V(s)\big)\,\exp\left(-\int_s^t \big(\frac{y(r)}{\kappa} + \mu_V(r)\big)\,\mathrm{d}r\right)\,\mathrm{d}s\\
&\leq \bar{S}_V(0)\,\exp\left(-\int_0^t \big(\frac{y(r)}{\kappa} + \mu_V(r)\big)\,\mathrm{d}r\right) + 1 - \exp\left(-\int_0^t \big(\frac{y(r)}{\kappa} + \mu_V(r)\big)\,\mathrm{d}r\right),
\end{aligned}$$

hence, since $\bar{S}_V(0) \leq 1$,

$$\mathcal{S}(y)(t) \leq 1, \qquad \forall t \geq 0. \tag{10}$$

In addition, by Assumption 2.4, if $\lambda(t) > 0$ (resp. $\lambda_0(t) > 0$), then $\gamma(s) = 0$ (resp. $\gamma_0(s) = 0$) for all $0 \leq s \leq t$. From the second line of (6), we deduce that for any $t \geq 0$,

$$\begin{aligned}
y(t) &= \mathbb{E}\left[\lambda_0(t)\,\exp\left(-\kappa \int_0^t \gamma_0(r)\,\mathcal{V}(y)(r)\,\mathrm{d}r\right)\right]\\
&\qquad + \kappa \int_0^t \mathbb{E}\left[\lambda(t-s)\,\exp\left(-\kappa \int_s^t \gamma(r-s)\,\mathcal{V}(y)(r)\,\mathrm{d}r\right)\right] x(s)\,\mathcal{V}(y)(s)\,\mathrm{d}s\\
&\leq \lambda^*,
\end{aligned}$$

where we have again exploited Lemma 4.1.

Finally, we can deduce that for any $T > 0, 0 \le t \le T$:

$$\begin{aligned}\mathcal{V}(y)(t) &= \bar{I}_V(0)\,\bar{\lambda}_0^V(t)\,\exp\Big(-\int_0^t \mu_V(u)\,\mathrm{d}u\Big)\\ &\quad+\frac{1}{\kappa}\int_0^t \bar{\lambda}^V(t-s)\,\exp\Big(-\int_s^t \mu_V(u)\,\mathrm{d}u\Big)\,\mathcal{S}(y)(s)\,y(s)\,\mathrm{d}s\\ &\le \lambda^* + \frac{(\lambda^*)^2}{\kappa}T =: C_T. \end{aligned} \tag{11}$$

Uniqueness of the solution to the integral system. We prove that the integral system admits at most one solution on any bounded interval $[0, T]$.

For a *càdlàg* function $f : \mathbb{R}_+ \to \mathbb{R}$, define

$$\|f\|_t := \sup_{0\le r\le t} |f(r)|, \qquad t > 0.$$

Let (x, y) and $(\tilde{x}, \tilde{y})$ be two elements of D_+^2 solutions of the system on $[0, T]$ with the same initial conditions. Set

$$\Delta x(t) := x(t) - \tilde{x}(t), \qquad \Delta y(t) := y(t) - \tilde{y}(t), \qquad t \in [0, T].$$

Moreover, we write

$$\begin{aligned} V(t) &:= \mathcal{V}(y)(t), \qquad \tilde{V}(t) := \mathcal{V}(\tilde{y})(t), \qquad \Delta V(t) := V(t) - \tilde{V}(t),\\ S(t) &:= \mathcal{S}(y)(t), \qquad \tilde{S}(t) := \mathcal{S}(\tilde{y})(t), \qquad \Delta S(t) := S(t) - \tilde{S}(t), \end{aligned}$$

For all $t \in [0, T]$ we have,

$$\Delta y(t) = \kappa \int_0^t \bar{\lambda}(t-s)\,\Big(x(s)\,V(s) - \tilde{x}(s)\,\tilde{V}(s)\Big)\,\mathrm{d}s.$$

Using the bounds $0 \le x, \tilde{x} \le 1$ and (11), we obtain

$$|\Delta y(t)| \le \lambda^*\kappa \int_0^t \Big(C_T|\Delta x(s)| + |\Delta V(s)|\Big)\,\mathrm{d}s, \qquad t \in [0, T]. \tag{12}$$

we have for all $t \in [0, T]$,

$$\Delta V(t) = \frac{1}{\kappa}\int_0^t \bar{\lambda}^V(t-s)\,e^{-\int_s^t \mu_V(u)\,\mathrm{d}u}\,\Big(S(y)(s)y(s) - \tilde{S}(y)(s)\tilde{y}(s)\Big)\,\mathrm{d}s.$$

using $0 \le \tilde{x}_V \le 1$, the bound $\bar{\lambda}^V \le \lambda^*$, and (10) we obtain

$$|\Delta V(t)| \le \frac{\lambda^*}{\kappa} max(\lambda^*, 1) \int_0^t \Big(|\Delta S(s)| + |\Delta y(s)|\Big)\,\mathrm{d}s. \tag{13}$$

Since the exponential is 1- lipschitizian on $\mathbb{R}_-$, we get, for all $t \in [0, T]$,

$$\begin{aligned}|\Delta S(t)| &\le \frac{\bar{S}_V(0)}{\kappa}\int_0^t |\Delta y(r)|\,\mathrm{d}r + \frac{1}{\kappa}\int_0^t \mu_V(s)exp(-\int_s^t \mu_V(r)dr)\int_s^t |\Delta y(r)|\,\mathrm{d}r\,\mathrm{d}s\\ &\le \frac{1}{\kappa}\Big(\bar{S}_V(0) + \int_0^t \mu_V(s)exp(-\int_s^t \mu_V(r)dr)\,\mathrm{d}s\Big)\int_0^t |\Delta y(r)|\,\mathrm{d}r\\ &\le \frac{\bar{S}_V(0)+1}{\kappa}\int_0^t |\Delta y(r)|\,\mathrm{d}r.\end{aligned}$$

Consequently, setting

$$C_2 := \frac{\bar{S}_V(0)+1}{\kappa},$$

$$\|\Delta S\|_t \le C_2 \int_0^t \|\Delta y\|_s \, \mathrm{d}s, \qquad t \in [0,T]. \tag{14}$$

Combining (13) and (14), we obtain :

$$\|\Delta V\|_t \le C_3(T) \int_0^t \|\Delta y\|_s \, \mathrm{d}s, \qquad t \in [0,T], \tag{15}$$

with

$$C_3(T) = \frac{\lambda^*}{\kappa} \max(\lambda^*, 1)\,(1 + C_2 T).$$

Now we decompose x as

$$x(t) = A(V)(t) + B(x,V)(t), \qquad t \in [0,T],$$

where,

$$A(V)(t) := \mathbb{E}\Big[\gamma_0(t)\, \exp\Big(-\kappa \int_0^t \gamma_0(r)\, V(r)\, \mathrm{d}r\Big)\Big],$$

$$B(x,V)(t) := \kappa \int_0^t \mathbb{E}\Big[\gamma(t-s)\, \exp\Big(-\kappa \int_s^t \gamma(r-s)\, V(r)\, \mathrm{d}r\Big)\Big]\; x(s)\, V(s)\, \mathrm{d}s.$$

For every $t \in [0,T]$, we have

$$\begin{aligned} \big|A(V)(t) - A(\widetilde{V})(t)\big| &\le \kappa \int_0^t \big|V(r) - \widetilde{V}(r)\big|\, \mathrm{d}r \\ &\le \kappa C_3(T) \int_0^t \|\Delta y\|_r \, \mathrm{d}r, \end{aligned} \tag{16}$$

We next estimate $B(x,V) - B(\tilde{x},\tilde{V})$. We add and subtract the quantity

$$\kappa \mathbb{E}\Big[\gamma(t-s)\, \exp\Big(-\kappa \int_s^t \gamma(r-s)\, V(r)\, \mathrm{d}r\Big)\Big]\; x(s)\, \tilde{V}(s),$$

to obtain

$$\big|B(x,V)(t) - B(\tilde{x},\tilde{V})(t)\big| \le \kappa \int_0^t \Big(C_3(T)\, |\Delta x(s)| + |\Delta y(s)|\Big)\, \mathrm{d}s. \tag{17}$$

Combining (17) and (12), we conclude that :

$$\|\Delta x\|_t \le \kappa\, C_4(T) \int_0^t (\|\Delta x\|_s + \|\Delta V\|_s)\, \mathrm{d}s. \tag{18}$$

for some constant $C_4(T) > 0$ depending only on λ^* and T.

Combining (15), (16), (17) and (18), we deduce that there exists a constant $K > 0$ such that, for all $t \in [0,T]$,

$$D(t) := \|\Delta x\|_t + \|\Delta y\|_t \le K \int_0^t D(s)\, \mathrm{d}s.$$

Since the two solutions have the same initial conditions, we have $D(0) = 0$. By Grönwall's lemma it follows that $D(t) = 0$ for all $t \in [0,T]$, that is,

$$\Delta x(t) = \Delta y(t) = 0 \quad \text{for all } t \in [0,T].$$

Consequently, the two solutions coincide, which proves the uniqueness of the solution to the integral system on $[0,T]$, for all $T > 0$, hence on $\mathbb{R}_+$.

Existence for the integral system via Picard iteration.

1) Operators and truncated system.
Host operators. For bounded measurable functions $x, y : [0,T] \to \mathbb{R}_+$ and $V : [0,T] \times D \to \mathbb{R}_+$ define

$$F(t;x,y) := \mathbb{E}\Big[\gamma_0(t)\exp\Big(-\kappa\int_0^t \gamma_0(r)\,V(r,y)\,dr\Big)\Big] + \kappa\int_0^t \mathbb{E}\Big[\gamma(t-s)\exp\Big(-\kappa\int_s^t \gamma(r-s)\,V(r,y)\,dr\Big)\Big]\,x(s)\,V(s,y)\,ds, \tag{19}$$

$$G(t;x,y) := \bar\lambda_0(t) + \kappa\int_0^t \bar\lambda(t-s)\,x(s)\,V(s,y)\,ds. \tag{20}$$

Truncations. Define the 1-Lipschitz truncation maps

$$\Pi_1(u) := u \wedge 1, \qquad \Pi_2(u) := u \wedge \lambda^*$$

Truncated system. We consider the truncated fixed-point problem on $[0,T]$:

$$\begin{cases} x(t) = \Pi_1\Big(F(t;x,y)\Big), \\ y(t) = \Pi_2\Big(F(t;x,y)\Big), \end{cases} \tag{21}$$

2) Picard iteration and uniform bounds.
Picard scheme. We initialize

$$x^{(0)}(t) = 0, \qquad y^{(0)}(t) = 0, \qquad t \in [0,T],$$

and, for $n \geq 0$, set

$$\begin{cases} x^{(n+1)}(t) := \Pi_1\Big(F(t;x^{(n)},y^{(n)})\Big), \\ y^{(n+1)}(t) := \Pi_2\Big(F(t;x^{(n)},y^{(n)})\Big), \end{cases} \tag{22}$$

Uniform bounds. By construction for all $n \geq 0$ and $t \in [0,T]$,

$$0 \leq x^{(n)}(t) \leq 1, \qquad 0 \leq y^{(n)}(t) \leq \lambda^*, \tag{23}$$

3) Contraction-type estimate and convergence.
Differences. For $n \geq 0$ define, for $t \in [0,T]$,

$$\delta_x^{(n)}(t) := |x^{(n+1)}(t) - x^{(n)}(t)|, \quad \delta_y^{(n)}(t) := |y^{(n+1)}(t) - y^{(n)}(t)|,$$

$$\delta_V^{(n)}(s) = V(t,y^{(n+1)}) - V(t,y^{(n)})$$

Set

$$\Delta_n(t) := \delta_x^{(n)}(t) + \delta_y^{(n)}(t). \tag{24}$$

Exploiting the boundedness of the sequences $x^{(n)}$ and $y^{(n)}$ (thanks to the truncation), from estimates analogous to those used in the proof of uniqueness, we obtain that for any $T > 0$, there exists C_T such that, for $n \geq 1$,

$$\sup_{0\leq t\leq T} \Delta_n(t) \leq \frac{(C_T T)^n}{n!} \sup_{0\leq t\leq T} \Delta_0(t),$$

hence $\sum_{n\geq 0}\sup_{t\leq T}\Delta_n(t) < \infty$. Therefore $(x^{(n)}, y^{(n)})$ is Cauchy in $(D([0,T]))^2$ for the sup norm and converges uniformly to some $(x,y) \in (D([0,T]))^2$.

4) Passage to the limit and solution of the truncated system. By the uniform bounds (23), all integrands in (19)–(20) with (x, y) replaced by $(x^{(n)}, y^{(n)})$ are dominated by constants (depending on T only). Hence, by dominated convergence, we may pass to the limit in (22) and obtain that (x, y) satisfies the truncated system (21) on $[0, T]$ for all $T > 0$, hence on $\mathbb{R}_+$.

Assume that (x, y) is a solution of the truncated system. Then

$$F(0, x, y) = \mathbb{E}\big[\gamma_0(0)\big], \qquad G(0, x, y) = \bar{\lambda}_0(0).$$

Hence

$$x(0) = F(0, x, y) \wedge 1, \qquad y(0) = G(0, x, y) \wedge \lambda^*,$$

satisfy $x(0) < 1$ and $y(0) < \lambda^*$ see Remark 2.1. By the right continuity of the map $t \to (F(t; x, y), G(t; x, y))$ there exists $t_1 > 0$ such that, for any $0 \le t \le t_1$,

$$x(t) = F(t, x, y) < 1, \qquad \text{and} \quad y(t) = G(t, x, y) < \lambda^*.$$

Consequently, the solution of (21) is also a solution of (6) on $[0, t_1]$.

Define the (possible) first exit time from the truncated domain by

$$\tau := \inf\Big\{t \ge 0 : \ \big(F(t, x, y) - 1\big) \vee \big(G(t, x, y) - \lambda_*\big) \ge 0\Big\}.$$

Assume that $\tau < +\infty$. Then, for every $t \in [0, \tau)$,

$$F(t, x, y) < 1 \quad \text{and} \quad G(t, x, y) < \lambda^*,$$

and hence (x, y) is a solution of (6) on $[0, \tau)$.

Moreover, the mapping

$$t \longmapsto \mathbb{E}\left[\exp\left(-\kappa \int_0^t \gamma_0(r)\, V(r, y)\, dr\right)\right] + \kappa \int_0^t \mathbb{E}\left[\exp\left(-\kappa \int_s^t \gamma(r-s)\, y_V(r)\, dr\right)\right] x(s)\, V(s, y)\, ds$$

is continuous and from Lemma 4.1 it is equals 1 on $[0, \tau)$. Hence

$$\mathbb{E}\left[\exp\left(-\kappa \int_0^\tau \gamma_0(r)\, V(r, y)\, dr\right)\right] + \kappa \int_0^\tau \mathbb{E}\left[\exp\left(-\kappa \int_s^\tau \gamma(r-s)\, V(r, y)\, dr\right)\right] x(s)\, V(s, y)\, ds = 1.$$

So, we have $x(\tau) = 1$ if and only if

$$\gamma_0(\tau) = 1 \text{ a.s., } \forall\, t \ge 0, \qquad \text{and} \qquad \gamma(r-s) = 1 \text{ a.s., } \forall\, r \in [s, \tau].$$

Likewise, $y(t) = \lambda^*$ if and only if

$$\lambda_0(\tau) = \lambda^* \text{ a.s., } \forall\, t \ge 0, \qquad \text{and} \qquad \lambda(\tau - s) = \lambda^* \text{ a.s., } \forall\, s \in [0, \tau].$$

Using Remark 2.1, we infer that

$$x(\tau) < 1 \qquad \text{and} \qquad y(\tau) < \lambda^*.$$

Hence

$$F(\tau, x, y) < 1 \quad \text{and} \quad G(\tau, x, y) < \lambda^*,$$

which contradicts the definition of τ.

So, necessarily $\tau = +\infty$. Consequently, for every $t \ge 0$, the solution of (21) is also a solution of (6), which proves the existence of a solution to (6) We now prove the continuity of the solutions $t \mapsto \big(\overline{\mathfrak{F}}_1(t), \overline{\mathfrak{S}}(t)\big)$ under the additional assumption. To this end, we prove the continuity of $t \mapsto \big(F(t, x, y),\, G(t, x, y)\big)$.

1. **Continuity of $t \mapsto F(t, x, y)$**

- **Continuity of the first expectation term in $F(t,x,y)$.**

$\mathbb{E}\Big[\gamma_0(t)\exp\Big(-\kappa\int_0^t \gamma_0(r)\,\mathcal{V}(y)(r)\,dr\Big)\Big]$ is continuous and takes its values in $[0,1]$.

Indeed, $\gamma_0(t)$ is continuous in probability and takes its values in $[0,1]$, hence the result by Lebesgue's dominated convergence theorem.

- **Continuity of the integral term in $F(t,x,y)$.**

For $t\in[0,T]$, define

$$I(t) := \int_0^t \Phi(t,s)\,ds, \qquad \Phi(t,s) := \Psi(t,s)\,x(s)\,\mathcal{V}(y)(s),$$

where

$$\Psi(t,s) := \mathbb{E}\Big[\gamma(t-s)\exp\Big(-\kappa\int_s^t \gamma(r-s)\,\mathcal{V}(y)(r)\,dr\Big)\Big], \qquad 0\le s\le t\le T.$$

Fix $t\in[0,T]$ and let $(t_n)_{n\ge1}\subset[0,T]$ be such that $t_n\to t$. Choose $T^\star > t\vee\sup_n t_n$ We have, for all admissible (t,s),

$$|\Phi(t,s)| \le |\Psi(t,s)|\,|x(s)\mathcal{V}(y)(s)| \le \mathbb{E}[|\gamma(t-s)|]\,|x(s)\mathcal{V}(y)(s)| \le C_T.$$

For each n,

$$\begin{aligned}|I(t)-I(t_n)| &= \left|\int_0^t \Phi(t,s)\,ds - \int_0^{t_n}\Phi(t_n,s)\,ds\right|\\ &\le \left|\int_{t\wedge t_n}^{t\vee t_n}\Phi(t\vee t_n,s)\,ds\right| + \left|\int_0^{t\wedge t_n}\big(\Phi(t\vee t_n,s)-\Phi(t\wedge t_n,s)\big)\,ds\right|\\ &=: |A_n|+|B_n|. \end{aligned} \tag{25}$$

First

$$|A_n| \le \int_{t\wedge t_n}^{t\vee t_n} C_T\,ds = C_T\,|t-t_n| \xrightarrow[n\to\infty]{} 0.$$

We have

$$|B_n| \le \left|\int_0^t [\Phi(t\vee t_n,s)-\Phi(t\wedge t_n,s)]ds\right|.$$

But the argument used above shows that for any s fixed, $t\mapsto\Psi(t,s)$ is continuous, and the same is true with Ψ replaced by Φ. Moreover $\Phi(t,s)$ is locally bounded, hence the convergence $|B_n|\to0$ as $n\to\infty$ follows readily by dominated convergence.

2. **Continuity of $t\mapsto G(t,x,y)$**

Fix $t\in[0,T]$ and let $t_n\to t$. Proceed as above:

$$G(t_n)-G(t) = \big(\bar\lambda_0(t_n)-\bar\lambda_0(t)\big)+\int_0^{t_n}\bar\lambda(t_n-s)x(s)\mathcal{V}(y)(s)\,ds-\int_0^t\bar\lambda(t-s)x(s)\mathcal{V}(y)(s)\,ds.$$

The first term tends to 0 by continuity of $\bar\lambda_0$. For the integral term, one repeats the decomposition (25) and uses the bound $|\bar\lambda(t-s)x(s)\mathcal{V}(y)(s)|\le C_T\lambda^*$. Hence G is continuous.

5. **Proof of Theorem 3.4.** This proof section is organized as follows. We shall first give some background on Poisson Random Measures, for the convenience of (at least some of) the readers. Next we shall establish two Lemmas in the second subsection. Lemma 5.1 will provide an explicit representation of the unique solution to system (6) in terms of some modified counting processes; see (28) and (34). Next, Lemma 5.2 is used to characterize two key quantities associated to the vectors, namely the force of infection and the proportion of susceptible vectors. Finally, in the last subsection, we show that the approximation errors arising from replacing the counting processes introduced in Subsection 2.5 by those constructed in (28) and (34) converge to 0 as N and N_V tend to infinity.

5.1. **Basic facts about Poisson Random Measures.** The content of this subsection is mainly for readers not very familiar with stochastic processes and stochastic epidemic models. In our model, infections happen at a certain rate. Let us talk about infections of hosts. Each host is infected at a time–varying rate, say $\mu(t)$, which is the product of the susceptibility of the host, and N^{-1} times the total force of infection of the vectors. This means that the next infection event of a given host is the first point of a time-changed Poisson process, that is $P(\int_0^t \mu(s)ds)$, where P is a rate 1 Poisson process (i.e. P has independent increments, and for $0 \le s < t$, $P(t) - P(s) \simeq Poi(t-s)$). We prefer rather to write this process in the equivalent form

$$\int_0^t \int_0^\infty \mathbf{1}_{u \le \mu(s^-)} Q(ds, du),$$

where Q is a standard Poisson Randon Measure (abbreviated below PRM) on $\mathbb{R}_+^2$ (standard means “with mean measure the Lebesgue measure”). Q is the sum of Dirac measures located at random points of $\mathbb{R}_+^2$, such that for disjoint sets $A_1, \ldots, A_n$, $Q(A_1), \ldots, Q(A_n)$ are independent, and $\mathbb{E}[Q(A)] = \int_A dx$. Those assumptions imply that $Q(A) \simeq Poi(\int_A dx)$. The reason for $\mu(s^-)$ (and not $\mu(s)$) is that $\mu(\cdot)$ is right continuous, and at the time of infection, the susceptibility of the individual is zero, unlike the susceptibility just before the infection.

If μ were deterministic, then the number of infections on the interval $(s,t]$ would be $Poi(\int_s^t \mu(r)dr)$. In particular, the probability that there is no infection on that interval would be $\exp\left(-\int_s^t \mu(r)dr\right)$.

Let now $(E, \mathcal{E})$ be a measurable space, equipped with a measure ν. The above PRM Q can be written as $\sum_{i\ge1} \delta_{(s_i,u_i)}$. Suppose that we are given a sequence $\{e_i,\ i \ge 1\}$ of i.i.d. random points in E, whose law is ν, and which is globally independent of Q. Then $Q' = \sum_{i\ge1} \delta_{(s_i,u_i,e_i)}$ is a PRM on $\mathbb{R}_+^2 \times E$, with mean mesure $dsdu\nu(de)$. Clearly Q is the projection of Q' on the first two coordinates.

Let Q be a PRM on $\mathbb{R} \times E$ with mean measure $ds\nu(de)$. Suppose now that on our probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we are given an increasing collection $\mathcal{F}_t$, $t \ge 0$ of sub-sigma algebras of $\mathcal{F}$, which is such for each t, each measurable set $A \subset [0,t] \times E$, $Q(A)$ if $\mathcal{F}_t$ measurable, and $\mathcal{F}_t$ and $Q|_{(t,+\infty)\times E}$ are independent. We denote by $\mathcal{F}^p$ the sigma-algebra on $\mathbb{R}_+ \times \Omega$ generated by the sets of the form $(s,t] \times A$ for $0 \le s < t$ and $A \in \mathcal{F}_s$.

Let now G be a random function of $(t,e) \in \mathbb{R}_+ \times E$, which is assumed to be $\mathcal{F}^p \otimes \mathcal{E}$ measurable, and such that for all $t > 0$, $E \int_0^t \int_E G^2(s,e) ds\nu(de) < \infty$. Then one can define the integral $\int_0^t\int_E G(s,e)Q(ds,de)$ and moreover

$$\mathbb{E}\int_0^t\int_E G(s,e)Q(ds,de) = \mathbb{E}\int_0^t\int_E G(s,e)ds\nu(de),$$
$$\mathbb{E}\left(\left|\int_0^t\int_E G(s,e)[Q(ds,de)-ds\nu(de)]\right|^2\right) = \mathbb{E}\int_0^t\int_E G^2(s,e)ds\nu(de)\,.$$

5.2. **Two lemmas.** Let $(Q_k)_{1\le k\le N}$ and $(Q^V_\ell)_{1\le \ell\le N_V}$ be i.i.d. standard Poisson random measures on $\mathbb{R}^2_+$, which are mutually independent.
For the host population, the model can be written as follows:

$$\overline{\mathfrak{F}}_1^N(t) := \frac{1}{N}\sum_{k=1}^N \lambda_{k,B_k^N(t)}\big(\mathfrak{a}_k^N(t)\big), \tag{26}$$

$$\overline{\mathfrak{S}}^N(t) := \frac{1}{N}\sum_{k=1}^N \gamma_{k,B_k^N(t)}\big(\mathfrak{a}_k^N(t)\big), \tag{27}$$

where

$$B_k^N(t) := \int_0^t\int_0^\infty \mathbf{1}_{\{u\le \Upsilon_k^N(s^-)\}}\, Q_k(\mathrm{d}s,\mathrm{d}u). \tag{28}$$

Moreover we have, $\overline{\mathfrak{F}}_2^{N_V}(t)$ being defined a few lines below,

$$\Upsilon_k^N(t) := \kappa\,\gamma_{k,B_k^N(t)}\big(\mathfrak{a}_k^N(t)\big)\,\overline{\mathfrak{F}}_2^{N_V}(t). \tag{29}$$

For vectors the model can be written as :

$$\bar{S}^{N_V}(t) = \frac{1}{N^V}\Big(\sum_{\ell=1}^{S^{N_V}(0)} \mathbf{1}_{\{t<\sigma_\ell\}}\mathbf{1}_{\{C_\ell^{N_V}(t)=0\}} + \sum_{k>S^{N_V}(0):\,\alpha_k\in(0,t]} \mathbf{1}_{\{\alpha_k\le t<\sigma_k\}}\mathbf{1}_{\{C_{\alpha_k}^{N_V}(t)=0\}}\Big), \tag{30}$$

with again σ_k denoting the death time of the individual born at time α_k.

$$\begin{aligned}\overline{\mathfrak{F}}_2^{N_V}(t) = \frac{1}{N^V}\Big(&\sum_{\ell=1}^{I^{N_V}(0)} \lambda_\ell^0(t)\,\mathbf{1}_{\{t<\sigma_\ell\}} + \sum_{\ell=1}^{S^{N_V}(0)} \lambda_\ell\big(t-\tau_\ell^{N_V}\big)\,\mathbf{1}_{\{t<\sigma_\ell\}} \\ &+ \sum_{\substack{k>S^{N_V}(0)\\ \alpha_k\in(0,t]}} \lambda_k^V\big(t-\tau_k^{N_V}\big)\,\mathbf{1}_{\{\alpha_k\le t<\sigma_k\}}\Big),\end{aligned} \tag{31}$$

where $\tau_i^{N_V}$ and $\tau_k^{N_V}$ are, respectively, the jump times of the processes

$$C_i^{N_V}(t) = \int_0^t\int_0^\infty \mathbf{1}_{\{C_i^{N_V}(s^-)=0\}}\,\mathbf{1}_{\left\{u\le \frac{\overline{\mathfrak{F}}_1^N(s^-)}{\kappa}\right\}}\,Q_i^V(ds,du),$$

and

$$C_{\alpha_k}^{N_V}(t) = \int_{\alpha_k}^t\int_0^\infty \mathbf{1}_{\{C_{\alpha_k}^{N_V}(s^-)=0\}}\,\mathbf{1}_{\left\{u\le \frac{\overline{\mathfrak{F}}_1^N(s^-)}{\kappa}\right\}}\,Q_k^V(ds,du).$$

We now define the processes $\{B_k\}$ and $\{C_\ell\}$ using the *same* PRMs $\{Q_k\}$ and $\{Q_\ell^V\}$.

For hosts,

$$B_k(t) = \int_0^t \int_0^\infty \mathbf{1}_{\{u \le \Upsilon_k(s^-)\}}\, Q_k(\mathrm{d}s, \mathrm{d}u), \quad \text{with} \quad \Upsilon_k(s) = \kappa\, \gamma_{B_k(s)}\big(\mathfrak{a}_k(s)\big)\, \mathcal{V}(\overline{\mathfrak{F}}_1)(s), \tag{32}$$

Where the *age* of infection $\mathfrak{a}_k$ (or elapsed time since the last jump of B_k) is defined by

$$\mathfrak{a}_k(t) := t - \sup\Big\{ s \in (0,t] : B_k(s) = B_k(s^-) + 1 \Big\}. \tag{33}$$

Then $\{(B_k, \mathfrak{a}_k)\}_{1 \le k \le N}$ is an i.i.d. family.
For vectors,

$$\begin{cases} C_\ell(t) = \displaystyle\int_0^t \int_0^\infty \mathbf{1}_{C_\ell(s^-)=0}\, \mathbf{1}_{u \le \frac{\overline{\mathfrak{F}}_1(s)}{\kappa}}\, Q_\ell(ds, du), \\ C_{\alpha_k}(t) = \displaystyle\int_{\alpha_k}^t \int_0^\infty \mathbf{1}_{C_{\alpha_k}(s^-)=0}\, \mathbf{1}_{u \le \frac{\overline{\mathfrak{F}}_1(s)}{\kappa}}\, Q_{S^{N_V}(0)+k}(ds, du). \end{cases} \tag{34}$$

Lemma 5.1. *Equation* (32) *admits a unique càdlàg solution $B_k(t)$. Moreover, the associated pair :*

$$\Big(\mathbb{E}\big[\gamma_{B_k(t)}(\mathfrak{a}_k(t))\big],\, \mathbb{E}\big[\lambda_{B_k(t)}(\mathfrak{a}_k(t))\big]\Big).$$

is the unique solution to the limiting system (6).

Proof of Lemma 5.1 . Note that we make the mild abuse of notation which consists in writing $\lambda_{B_k(t)}, \gamma_{B_k(t)}$ instead of $\lambda_{k,B_k(t)}, \gamma_{k,B_k(t)}$. Existence and uniqueness is quite elementary. We leave its verification to the reader.
For each host k we work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathbb{P})$ supporting the Poisson random measure Q_k and the mutually independent marks $(\lambda_{k,i}, \gamma_{k,i})_{i\ge0}$. We denote by $(\mathcal{F}_t)_{t\ge0}$ the filtration generated by these sources of randomness, i.e.

$$\mathcal{F}_t := \sigma\Big((\lambda_{k,i}, \gamma_{k,i})_{i \le B_k(t)},\, Q_k|_{[0,t]\times\mathbb{R}_+} \Big).$$

Let τ_i denote the i-th jump time of the counting process B_k, namely

$$\tau_i := \inf\{t \ge 0 :\; B_k(t) \ge i\}, \qquad i \ge 1,$$

with the convention $\inf \varnothing = +\infty$. Then $(\tau_i)_{i\ge1}$ is a sequence of $(\mathcal{F}_t)_{t\ge0}$-stopping times, and the restriction $Q_k|_{[\tau_i,t]\times\mathbb{R}_+}$ is independent of $\mathcal{F}_{\tau_i}$. Hence,

$$\mathbb{P}(B_k(t) = i \,|\, \mathcal{F}_{\tau_i}) = \exp\bigg(-\int_{\tau_i}^t \gamma_i(r - \tau_i)\, \mathcal{V}(\overline{\mathfrak{F}}_1)(s)\, \mathrm{d}r \bigg).$$

We shall denote below by ϱ the law of γ,

Step 1: Proof of $\mathbb{E}\big[\gamma_{B_k(t)}(\mathfrak{a}(t))\big] = \overline{\mathfrak{S}}_1(t)$.

$$\gamma_{B_k(t)}(\zeta(t)) = \gamma_0(t)\mathbf{1}_{\{B_k(t)=0\}} + \sum_{i\ge1} \gamma_i(t-\tau_i)\mathbf{1}_{\{B_k(t)=i\}},$$

(recall that $\gamma_i(s) = 0$ if $s < 0$, see Assumption 2.4)

$$\mathbb{E}\big[\gamma_0(t)\,\mathbf{1}_{\{B_k(t)=0\}}\big] = \mathbb{E}[\gamma_0(t)\,\mathbb{P}(B_k(t)=0\,|\,\gamma_0)] = \mathbb{E}\bigg[\gamma_0(t)\exp\bigg(-\kappa\int_0^t \gamma_0(r)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\bigg)\bigg].$$

$$\begin{aligned}
\mathbb{E}\Bigg[\sum_{i\ge 1}\gamma_i(t-\tau_i)\,\mathbf{1}_{\{B_k(t)=i\}}\Bigg] &= \sum_{i\ge 1}\mathbb{E}\Bigg[\gamma(t-\tau_i)\exp\Big(-\kappa\int_{\tau_i}^t \gamma(r-\tau_i)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\Bigg]\\
&= \sum_{i\ge 1}\mathbb{E}\Bigg[\int_D \gamma(t-\tau_i)\exp\Big(-\kappa\int_{\tau_i}^t \gamma(r-\tau_i)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\,\varrho(d\gamma)\Bigg]\\
&= \mathbb{E}\Bigg[\int_0^t \mathbb{E}\Bigg[\gamma(t-s)\exp\Big(-\kappa\int_s^t \gamma(r-s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\Bigg]\,dB_k(s)\Bigg]\\
&= \kappa\mathbb{E}\Bigg[\int_0^t \gamma(t-s)\exp\Big(-\int_s^t \gamma(r-s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\,\mathbb{E}\big[\gamma_{B_k(s)}(\mathfrak{a}(s))\big]\,\mathcal{V}(\overline{\mathfrak{F}}_1)(s)\,ds\Bigg].
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\big[\gamma_{B_k(t)}(\mathfrak{a}(t))\big] &= \mathbb{E}\Bigg[\gamma_0(t)\exp\Big(-\kappa\int_0^t \gamma_0(r)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\Bigg]\\
&\quad+\mathbb{E}\Bigg[\int_0^t \gamma(t-s)\exp\Big(-\kappa\int_s^t \gamma(r-s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\,\kappa\mathbb{E}\big[\gamma_{B_k(s)}(\mathfrak{a}(s))\big]\,\mathcal{V}(\overline{\mathfrak{F}}_1)(s)\,ds\Bigg].
\end{aligned}$$

This proves that $\mathbb{E}\big[\gamma_{B_k(t)}(\mathfrak{a}(t))\big]=\overline{\mathfrak{S}}(t)$, as a consequence of the uniqueness of a solution to the above linear integral equation, given that $\mathcal{V}(\overline{\mathfrak{F}}_1)(t)\le C_T$ for all $0\le t\le T$.

Step 2: Proof of $\mathbb{E}\big[\lambda_{B_k(t)}(\mathfrak{a}(t))\big]=\overline{\mathfrak{F}}_1(t)$.

$$\begin{aligned}
\lambda_{B_k(t)}\big(\zeta(t)\big) &= \lambda_0(t)\,\mathbf{1}_{\{B_k(t)=0\}}+\sum_{i\ge 1}\lambda_i(t-\tau_i)\,\mathbf{1}_{\{B_k(t)=i\}}\\
&= \lambda_0(t)+\sum_{i\ge 1}\lambda_i(t-\tau_i)\,.
\end{aligned}$$

Indeed, $\lambda_i(t-\tau_i)=0$ if $t<\tau_i$, and also if $B_k(t)>i$, see Assumption 2.4. We have

$$\begin{aligned}
\mathbb{E}\big[\lambda_{B_k(t)}\big(\zeta(t)\big)\big] &= \mathbb{E}[\lambda_0(t)]+\mathbb{E}\Bigg[\sum_{i\ge 1}\lambda_i\big(t-\tau_i\big)\Bigg]\\
&= \bar\lambda_0(t)+\mathbb{E}\Bigg[\sum_{i\ge 1}\bar\lambda\big(t-\tau_i\big)\Bigg]\\
&= \bar\lambda_0(t)+\mathbb{E}\Bigg[\int_0^t \bar\lambda(t-s)\,\mathrm{d}B_k(s)\Bigg]\\
&= \bar\lambda_0(t)+\int_0^t \bar\lambda(t-s)\,\kappa\mathbb{E}\big[\gamma_{B_k(s)}\big(\zeta(s)\big)\big]\;\mathcal{V}(\overline{\mathfrak{F}}_1)(s)\,\mathrm{d}s\\
&= \bar\lambda_0(t)+\kappa\int_0^t \bar\lambda(t-s)\,\overline{\mathfrak{S}}(s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(s)\,\mathrm{d}s,
\end{aligned}$$

where we have used the result of step 1. Hence the result. □

We now define $\widetilde{\mathfrak{S}}^N(t)$ and $\widetilde{\mathfrak{F}}_1^N(t)$ by replacing the sequence $\{(B_k^N,\mathfrak{a}_k^N)_{1\le k\le N}\}$ by $\{(B_k,\mathfrak{a}_k)_{1\le k\le N}\}$ in the formulas (27) and (26). An immediate Corollary of the strong law of large numbers in D (see [11]) and Lemma 5.1 is that

$$(\widetilde{\mathfrak{S}}^N, \widetilde{\mathfrak{F}}_1^N) \to (\overline{\mathfrak{S}}, \overline{\mathfrak{F}}_1)$$

a.s. in D^2.

We next define $\widetilde{S}^{N_V}(t)$ and $\widetilde{\mathfrak{F}}_2^{N_V}(t)$ by replacing $C_i^{N_V}$ with C_i in the formula (30) for $\bar{S}^{N_V}(t)$ and the formula (31) for $\overline{\mathfrak{F}}_2^{N_V}(t)$.

Lemma 5.2. *As $N, N_V \to \infty$, with $\frac{N_V}{N} = \kappa$,*

$$\widetilde{S}^{N_V}(t) \to \mathcal{S}(\overline{\mathfrak{F}}_1)(t) \quad \textit{and} \quad \widetilde{\mathfrak{F}}_2^{N_V}(t) \to \mathcal{V}(\overline{\mathfrak{F}}_1)(t).$$

a.s. in D

Proof of Lemma 5.2. Step 1: Convergence of $\widetilde{S}^{N_V}(t)$

We have :

$$\begin{aligned}\mathbb{E}[C_i(t)] &= \mathbb{E}\int_0^t \int_0^\infty \mathbf{1}_{C(s^-)=0} \mathbf{1}_{u \le \frac{\overline{\mathfrak{F}}_1(s^-)}{\kappa}} Q_i^V(ds, du) \\ &= \frac{1}{\kappa}\int_0^t \exp\left(-\frac{1}{\kappa}\int_0^s \overline{\mathfrak{F}}_1(r)dr\right)\overline{\mathfrak{F}}_1(s)ds,\end{aligned}$$

We introduce the following two quantities:

$$\widetilde{S}_1^{N_V}(t) := \frac{1}{N^V}\sum_{i=1}^{S^{N_V}(0)} \mathbf{1}_{\{t<\sigma_i\}} \mathbf{1}_{\{C_i(t)=0\}}, \qquad \widetilde{S}_2^{N_V}(t) := \frac{1}{N_V}\sum_{k\ge 1} \mathbf{1}_{\alpha_k \le t < \sigma(\alpha_k)} \mathbf{1}_{C_{\alpha_k}(t)=0}$$

On the event $\{\alpha_i < t\}$

$$\mathbb{P}(C_{\alpha_i}(t) = 0 \,|\, \alpha_i) = \exp\left(-\frac{1}{\kappa}\int_{\alpha_i}^t \overline{\mathfrak{F}}_1(r)\,\mathrm{d}r\right),$$

Similarly, the survival probability up to time t is given by

$$\mathbb{P}(t < \sigma_i | \alpha_i) = \exp\left(-\int_{\alpha_i}^t \mu_V(r)\,dr\right).$$

Since the infection and death mechanisms are independent, we obtain for $i < S^{V_N}(0)$

$$\mathbb{P}(t < \sigma_i, C_{\alpha_i}(t) = 0) = \exp\left(-\int_0^t \left[\mu_V(r) + \frac{\overline{\mathfrak{F}}_1(r)}{\kappa}\right]dr\right).$$

By linearity of the expectation,

$$\mathbb{E}[\widetilde{S}_1^{N_V}(t)] = \bar{S}^{N_V}(0)\,\exp\left(-\int_0^t \left[\mu_V(r) + \frac{\overline{\mathfrak{F}}_1(r)}{\kappa}\right]dr\right).$$

By the law of large numbers in D (see [11]), we have that, as $N_V \to \infty$,

$$\tilde{S}_1^{N_V}(t) \to \bar{S}^V(0)\exp\left(-\int_0^t \left[\mu_V(r) + \frac{\overline{\mathfrak{F}}_1(r)}{\kappa}\right]dr\right)$$

a.s. in D.

Next, $\{\alpha_k,\ k\geq 1\}$ denoting the points of a Poisson process with intensity $N_V\mu_V(t)$ on $\mathbb{R}_+$, we consider :

$$\begin{aligned}\widetilde{S}_2^{N_V}(t) :&= \frac{1}{N^V}\sum_{k\geq 1}\mathbf{1}_{\{\alpha_k\leq t<\sigma_k(\alpha_k)\}}\,\mathbf{1}_{\{C_{\alpha_k}(t)=0\}}.\\ &= \frac{1}{N_V}\int_{[0,t]\times\mathbb{R}_+^3}\mathbf{1}_{u\leq N_v\mu_V(s)}\mathbf{1}_{t-s<\sigma}\mathbf{1}_{t-s<\tau}Q_b(ds,du,d\sigma,d\tau),\end{aligned}$$

where Q_b is a PRM on $\mathbb{R}_+^4$ with mean measure

$$ds\,du\,\mathrm{Exp}_{\mu_V(s+\cdot)}(d\sigma)\,\mathrm{Exp}_{\kappa^{-1}\overline{\mathfrak{F}}_1(s+\cdot)}(d\tau).$$

$$\mathbb{E}\Big[\widetilde{S}_2^{N_V}(t)\Big] = \int_0^t \mu_V(s)\,\mathbb{E}\big[\mathbf{1}_{\{t<\sigma(s)\}}\mathbf{1}_{\{C_s(t)=0\}}\big]\,\mathrm{d}s.$$

It is clear that :

$$\mathbb{E}\Big[\widetilde{S}_2^{N_V}(t)\Big] = \int_0^t \mu_V(s)\,\exp\bigg(-\int_s^t\Big(\mu_V(r)+\frac{\overline{\mathfrak{F}}_1(r)}{\kappa}\Big)\,\mathrm{d}r\bigg)\,\mathrm{d}s.$$

Now the wished convergence a.s. in D of $\tilde{S}_2^{N_V}$ towards

$$\int_0^{\cdot}\mu_V(s)\exp\left(-\int_s^t\left[\mu_V(r)+\kappa^{-1}\overline{\mathfrak{F}}_1(r)\right]dr\right)$$

follows again from Rao's [11] LLN in D, if we note that

$$\begin{aligned}&\frac{1}{N_V}\int_{[0,t]\times\mathbb{R}_+^3}\mathbf{1}_{u\leq N_v\mu_V(s)}\mathbf{1}_{t-s<\sigma}\mathbf{1}_{t-s<\tau}Q_b(ds,du,d\sigma,d\tau)\\ &\qquad= \frac{1}{N_V}\sum_{k=1}^{N_V}\int_{[0,t]\times\mathbb{R}_+^3}\mathbf{1}_{(k-1)\mu_V(s)<u\leq k\mu_V(s)}\mathbf{1}_{t-s<\sigma}\mathbf{1}_{t-s<\tau}Q_b(ds,du,d\sigma,d\tau),\end{aligned}$$

and from the well-known properties of Poisson random measures, the above sum is a sum of i.i.d. r.v.'s.

Hence $\widetilde{S}^{N_V}(t)\to\mathcal{S}\big(\overline{\mathfrak{F}}_1\big)(t)$ a.s. in D

Step 2: Convergence of $\widetilde{\mathfrak{F}}_2^{N_V}(t)$

We next consider the three terms $\tilde{\mathfrak{F}}_{2,0}^{N_V}(t)$, $\tilde{\mathfrak{F}}_{2,1}^{N_V}(t)$ and $\tilde{\mathfrak{F}}_{2,2}^{N_V}(t)$.

We have

$$\tilde{\mathfrak{F}}_{2,0}^{N_V}(t) := \frac{1}{N_V}\sum_{j=1}^{I^{N_V}(0)}\lambda_j^{V,0}(t)\mathbf{1}_{t<\sigma_{-j}}\ ,$$

where $\sigma_{-1},\sigma_{-2},\ldots$ stand for the death times of the initially infected individuals.

We clearly have the following a.s. convergence in D, as $N_V\to\infty$,

$$\tilde{\mathfrak{F}}_{2,0}^{N_V}(t)\to \bar{I}_V(0)\bar{\lambda}_0^V(t)\exp\left(-\int_0^t\mu_V(r)dr\right)\,.$$

Concerning the second term, as $N_V\to\infty$,

$$\widetilde{\mathfrak{F}}_{2,1}^{N_V}(t) = \frac{1}{N^V}\sum_{i=1}^{S^{N_V}(0)}\lambda_i^V\big(t-\tau_i^V\big)\,\mathbf{1}_{\{t<\sigma_i\}} = \frac{1}{N^V}\sum_{i\geq 1}\int_0^t\lambda_i^V(t-s)\mathbf{1}_{\{t<\sigma_i\}}\,C_i(ds)$$

$$\longrightarrow \bar{S}_V(0)\,\mathbb{E}\Big[\int_0^t \lambda_1^V(t-s)\mathbf{1}_{\{s<\sigma_i\}}\exp\Big(-\int_0^s \frac{\overline{\mathfrak{F}}_1(r)}{\kappa}\,dr\Big)\frac{\overline{\mathfrak{F}}_1(s)}{\kappa}\,ds\Big]$$
$$= \bar{S}_V(0)\int_0^t \bar{\lambda}^V(t-s)\,\exp\Big(-\int_s^t \mu_V(r)\,dr\Big)\exp\Big(-\int_0^s \mu_V(r)+\frac{\overline{\mathfrak{F}}_1(r)}{\kappa}\,dr\Big)\frac{\overline{\mathfrak{F}}_1(s)}{\kappa}\,ds,$$

For the last term in $\tilde{\mathfrak{F}}_{2,2}^{N_V}(t)$, we will need to introduce the following formalism.

Let $Q_b'(\mathrm{d}s,\mathrm{d}u,\mathrm{d}\sigma,\mathrm{d}\tau,\mathrm{d}\lambda)$ be a Poisson random measure (PRM) on $\mathbb{R}_+^4\times D$, with mean (intensity) measure (ν_V denoting the law of λ^V) :

$$\bar{Q}_b'(\mathrm{d}s,\mathrm{d}u,\mathrm{d}\sigma,\mathrm{d}\tau,\mathrm{d}\lambda)=\mathrm{d}u\,\mathrm{d}s\;\mathrm{Exp}_{\mu_V(s+\cdot)}(\mathrm{d}\sigma)\;\mathrm{Exp}_{\frac{\overline{\mathfrak{F}}_1(t)}{\kappa}(s+\cdot)}(\mathrm{d}\tau)\;\nu_V(\mathrm{d}\lambda).$$

Note that the PRM Q_b introduced in Step 1 of the proof is the projection of Q_b' on $\mathbb{R}_+^4$.

Now,

$$\widetilde{\mathfrak{F}}_{2,2}^{N_V}(t)=\frac{1}{N_V}\sum_{\substack{i>S^{N_V}(0)\\ \alpha_i\in(0,t]}}\lambda_i^V\big(t-\tau_i^V\big)\,\mathbf{1}_{\{\alpha_i\le t<\sigma(\alpha_i)\}},$$

and

$$\frac{1}{N_V}\sum_{\substack{i>S^{N_V}(0)\\ \alpha_i\in(0,t]}}\lambda_i^V(t-\tau_i^V)\,\mathbf{1}_{\{\alpha_i\le t<\sigma_i\}}=\frac{1}{N^V}\int_{[0,t]\times\mathbb{R}_+^3\times D}\mathbf{1}_{\{u\le N_V\mu_V(s)\}}\,\lambda^V(t-\tau-s)\,\mathbf{1}_{\{\tau\le t-s<\sigma\}}\,Q_b'(\mathrm{d}s,\mathrm{d}u,\mathrm{d}\sigma,\mathrm{d}\tau,\mathrm{d}\lambda).$$

We have :

$$\mathbb{E}\left[\frac{1}{N_V}\int_{[0,t]\times\mathbb{R}_+^3\times D}\mathbf{1}_{\{u\le N_V\mu_V(s)\}}\,\lambda^V(t-s-\tau)\,\mathbf{1}_{\{\tau\le t-s<\sigma\}}\,Q_b'(\mathrm{d}s,\mathrm{d}u,\mathrm{d}\sigma,\mathrm{d}\tau,\mathrm{d}\lambda)\right]$$
$$=\int_0^t\int_s^t \mu_V(s)\,\bar{\lambda}^V(t-r)\,\exp\Big(-\int_s^t\mu_V(v)\,\mathrm{d}v\Big)\,\exp\Big(-\int_s^r\frac{\overline{\mathfrak{F}}_1(u)}{\kappa}\,\mathrm{d}u\Big)\,\frac{\overline{\mathfrak{F}}_1(r)}{\kappa}\,\mathrm{d}r\,\mathrm{d}s.$$

Again, we easily obtain that, as $N_V\to\infty$, a.s. in D,

$$\tilde{\mathfrak{F}}_{2,2}^{N_V}(t)\to\int_0^t\int_s^t\mu_V(s)\bar{\lambda}^V(t-r)\exp\left(-\int_s^t\mu_V(v)dv\right)\frac{\overline{\mathfrak{F}}_1(r)}{\kappa}\exp\left(-\int_s^r\frac{\overline{\mathfrak{F}}_1(u)}{\kappa}du\right)drds\,.$$

When combinig (4) and (5) we see that $\mathcal{V}(y)$ is the sum of three terms.We have shown that $\mathbb{E}(\widetilde{\mathfrak{F}}_{2,0}^{N_V}(t)),\mathbb{E}(\widetilde{\mathfrak{F}}_{2,1}^{N_V}(t)$ and $\mathbb{E}(\widetilde{\mathfrak{F}}_{2,2}^{N_V}(t))$ are respectively equal to the first, the second and the third term in $\mathcal{V}(\overline{\mathfrak{F}}_1)$. Finally, we have shown that $\widetilde{\mathfrak{F}}_2^{N_V}(t)\to\mathcal{V}\big(\overline{\mathfrak{F}}_1\big)(t)$ a.s. in D. □

5.3. **Completing the proof of Theorem 3.4.** Now, we introduce the *disagreement counting process*

$$\Delta_k^H(t):=\int_0^t\int_0^\infty\mathbf{1}_{\Big\{\min\big(\Upsilon_k^N(s^-),\Upsilon_k(s^-)\big)<u\le\max\big(\Upsilon_k^N(s^-),\Upsilon_k(s^-)\big)\Big\}}\,Q_k(ds,du).$$

By construction of the coupling, we have

$$\big|B_k^N(t)-B_k(t)\big|\le\Delta_k^H(t),\qquad t\ge0.$$

Moreover, since t $\to\Delta_k^H(t)$ is nondecreasing

$$\mathbb{E}\Big[\sup_{r\in[0,t]}\big|B_k^N(r)-B_k(r)\big|\Big]\le\mathbb{E}\big[\Delta_k^H(t)\big]$$

$$= \int_0^t \mathbb{E}\big[\big|\Upsilon_k^N(s) - \Upsilon_k(s)\big|\big]\, ds. \tag{35}$$

Recall that

$$\Upsilon_k^N(t) := \kappa\, \gamma_{k,B_k^N(t)}\big(\mathfrak{a}_k^N(t)\big)\, \overline{\mathfrak{F}}_2^{N_V}(t), \qquad \Upsilon_k(t) := \kappa\, \gamma_{k,B_k(t)}\big(\mathfrak{a}_k(t)\big)\, \overline{\mathfrak{F}}_2(t).$$

We have $\gamma_{k,i} \le 1$ for all i, and for all $t \ge 0, 0 \le \overline{\mathfrak{F}}_2^{N_V}(t), \overline{\mathfrak{F}}_2(t) \le \lambda^*$. Then, for all $t \ge 0$,

$$\mathbb{E}\big[\big|\Upsilon_k^N(t) - \Upsilon_k(t)\big|\big] \le \mathbb{E}\Big[\big|\Upsilon_k^N(t) - \Upsilon_k(t)\big|\, \mathbf{1}_{\{B_k^N(t)=B_k(t),\, \mathfrak{a}_k^N(t)=\mathfrak{a}_k(t)\}}\Big] + \lambda^*\, \mathbb{P}\big(B_k^N(t) \ne B_k(t) \text{ or } \mathfrak{a}_k^N(t) \ne \mathfrak{a}_k(t)\big). \tag{36}$$

On the event $\{B_k^N(t) = B_k(t),\, \mathfrak{a}_k^N(t) = \mathfrak{a}_k(t)\}$ we have $\gamma_{k,B_k^N(t)}(\mathfrak{a}_k^N(t)) = \gamma_{k,B_k(t)}(\mathfrak{a}_k(t)) \le 1$, hence

$$\mathbb{E}\Big[\big|\Upsilon_k^N(t) - \Upsilon_k(t)\big|\, \mathbf{1}_{\{B_k^N(t)=B_k(t),\, \mathfrak{a}_k^N(t)=\mathfrak{a}_k(t)\}}\Big] \le \kappa\, \mathbb{E}\Big[\big|\overline{\mathfrak{F}}_2^{N_V}(t) - \overline{\mathfrak{F}}_2(t)\big|\Big]. \tag{37}$$

$$\begin{aligned}
\mathbb{E}\Big[\big|\overline{\mathfrak{F}}_2^{N_V}(t) - \overline{\mathfrak{F}}_2(t)\big|\Big] \le\; & \mathbb{E}\left[\left|\frac{1}{N^V}\sum_{j=1}^{I^{N_V}(0)}\Big(\lambda_j^{V,0}(t)\mathbf{1}_{\{t<\sigma_j\}} - \mathbb{E}\big[\lambda_j^{V,0}(t)\mathbf{1}_{\{t<\sigma_j\}}\big]\Big)\right|\right] + \lambda^*\, \mathbb{E}\Big[\big|\overline{I}^{N_V}(0) - \overline{I}(0)\big|\Big] \\
&+ \mathbb{E}\left[\left|\frac{1}{N^V}\sum_{j=1}^{S^{N_V}(0)}\Big(\lambda_j^V(t-\tau_j^{N_V})\mathbf{1}_{\{t<\sigma_j\}} - \mathbb{E}\big[\lambda_j^V(t-\tau_j^V)\mathbf{1}_{\{t<\sigma_j\}}\big]\Big)\right|\right] + \lambda^*\, \mathbb{E}\Big[\big|\overline{S}^{N_V}(0) - \overline{S}(0)\big|\Big] \\
&+ \mathbb{E}\left[\left|\frac{1}{N^V}\sum_{\substack{k > I^{N_V}(0)+S^{N_V}(0) \\ \alpha_k \in (0,t]}}\Big(\lambda_k^V(t-\tau_k^{N_V})\mathbf{1}_{\{\alpha_k \le t < \sigma(\alpha_k)\}} - \lambda_k^V(t-\tau_k^V)\mathbf{1}_{\{\alpha_k \le t < \sigma_k\}}\Big)\right|\right] \\
&+ \mathbb{E}\left[\frac{1}{N_V}\left|\int_{[0,t]\times\mathbb{R}_+^3\times D} \mathbf{1}_{\{u \le N_V\mu_V(s)\}}\, \lambda^V(t-s-\tau)\, \mathbf{1}_{\{\tau \le t-s<\sigma\}}\, \big(Q_b' - \overline{Q_b'}\big)(\mathrm{d}s, \mathrm{d}u, \mathrm{d}\sigma, \mathrm{d}\tau, \mathrm{d}\lambda)\right|\right],
\end{aligned} \tag{38}$$

where $\overline{Q_b'}$ is the mean measure of the Poisson Random Measure Q_b'.

Lemma 5.3. *As $N_V \to \infty$,*

$$\mathbb{E}\left[\frac{1}{N_V}\left|\int_{[0,t]\times\mathbb{R}_+^3\times D} \mathbf{1}_{\{u \le N_V\mu_V(s)\}}\, \lambda^V(t-\tau-s)\, \mathbf{1}_{\{\tau \le t-s<\sigma\}}\, \big(Q_b' - \overline{Q_b'}\big)(\mathrm{d}s, \mathrm{d}u, \mathrm{d}\sigma, \mathrm{d}\tau, \mathrm{d}\lambda)\right|\right] \xrightarrow[N_V\to\infty]{} 0,$$

Proof of Lemma 5.3. Define

$$f_{N_V}(s,u,\sigma,\tau) := \mathbf{1}_{\{u \le N_V\mu_V(s)\}}\, \lambda^V(t-\tau-s)\, \mathbf{1}_{\{\tau \le t-s<\sigma\}},$$

and

$$M_{N_V}(t) := \int_{[0,t]\times\mathbb{R}_+^3\times D} f_{N_V}(s,u,\sigma,\tau)\, \big(Q_b' - \overline{Q_b'}\big)(\mathrm{d}s, \mathrm{d}u, \mathrm{d}\sigma, \mathrm{d}\tau, \mathrm{d}\lambda).$$

So we have

$$\mathbb{E}\big[|M_{N_V}(t)|^2\big] = \int_{[0,t]\times\mathbb{R}_+^3\times D} f_{N_V}(s,u,\sigma,\tau)^2\, \overline{Q_b'}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}\sigma, \mathrm{d}\tau, \mathrm{d}\lambda).$$

by Cauchy–Schwarz,

$$\mathbb{E}\left[\frac{1}{N_V}|M_{N_V}(t)|\right] \le \frac{1}{N_V}\Big(\mathbb{E}[|M_{N_V}(t)|^2]\Big)^{1/2}.$$

Therefore,

$$\mathbb{E}\big[|M_{N_V}(t)|^2\big] \le (\lambda^*)^2 \int_{[0,t]\times\mathbb{R}_+^3\times D} \mathbf{1}_{\{u \le N_V\mu_V(s)\}}\, \overline{Q_b'}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}\sigma, \mathrm{d}\tau, \mathrm{d}\lambda)$$

$$= (\lambda^*)^2 \int_0^t \big(N_V \mu_V(s)\big)\,\mathrm{d}s = (\lambda^*)^2 N_V \int_0^t \mu_V(s)\,\mathrm{d}s,$$

Consequently,

$$\mathbb{E}\left[\frac{1}{N_V}|M_{N_V}(t)|\right] \le \frac{1}{N_V}\,\lambda^* \sqrt{N_V\int_0^t \mu_V(s)\,\mathrm{d}s} = \frac{\lambda^*}{\sqrt{N_V}}\sqrt{\int_0^t \mu_V(s)\,\mathrm{d}s} \xrightarrow[N_V\to\infty]{} 0,$$

□

It follows from our assumptions that as $N_V \to +\infty$,

$$\begin{aligned}
&\mathbb{E}\Big[\big|\overline{I}^{N_V}(0) - \overline{I}(0)\big|\Big] + \mathbb{E}\Big[\big|\overline{S}^{N_V}(0) - \overline{S}(0)\big|\Big]\\
&\quad + \mathbb{E}\left[\frac{1}{N_V}\left|\int_{[0,t]\times\mathbb{R}_+^3\times D} \mathbf{1}_{\{u\le N_V\mu_V(s)\}}\,\lambda^V(t-\tau-s)\,\mathbf{1}_{\{\tau\le t-s<\sigma\}}\,\big(Q_b' - \overline{Q_b'}\big)(\mathrm{d}s,\mathrm{d}u,\mathrm{d}\sigma,\mathrm{d}\tau,\mathrm{d}\lambda)\right|\right]\\
&\quad \xrightarrow[N_V\to\infty]{} 0.
\end{aligned} \tag{39}$$

For each $t \ge 0$, the $\lambda_{j,0}^V(t)$ are i.i.d. and bounded, and globally independent of $I^N(0)$, hence, from the classical strong law of large numbers,

$$\mathbb{E}\left[\left|\frac{1}{N_V}\sum_{j=1}^{I^{N_V}(0)}\Big(\lambda_{j,0}^V(t)\mathbf{1}_{\{t<\sigma_j\}} - \mathbb{E}\big[\lambda_{j,0}^V(t)\mathbf{1}_{\{t<\sigma_j\}}\big]\Big)\right|\right] \xrightarrow[N_V\to\infty]{} 0. \tag{40}$$

Next

$$\begin{aligned}
&\mathbb{E}\left[\left\|\frac{1}{N_V}\sum_{j=1}^{S^{N_V}(0)}\Big(\lambda_j^V\big(t-\tau_j^{N_V}\big)\mathbf{1}_{\{t<\sigma_j\}} - \mathbb{E}\big[\lambda_j^V\big(t-\tau_j^V\big)\mathbf{1}_{\{t<\sigma_j\}}\big]\Big)\right\|\right]\\
&\le \mathbb{E}\left[\left\|\frac{1}{N}\sum_{j=1}^{S^{N_V}(0)}\Big(\lambda_j^V\big(t-\tau_j^{N_V}\big) - \lambda_j^V\big(t-\tau_j^V\big)\Big)\mathbf{1}_{\{t<\sigma_j\}}\right\|\right] \qquad (41)\\
&\quad + \mathbb{E}\left[\left\|\frac{1}{N}\sum_{j=1}^{S^{N_V}(0)}\Big(\lambda_j^V\big(t-\tau_j^V\big)\mathbf{1}_{\{t<\sigma_j\}} - \mathbb{E}\big[\lambda_j^V\big(t-\tau_j^V\big)\mathbf{1}_{\{t<\sigma_j\}}\big]\Big)\right\|\right].
\end{aligned}$$

By the same argument as that used for establishing (40), we get that, as $N_V \to \infty$,

$$\mathbb{E}\left[\left\|\frac{1}{N_V}\sum_{j=1}^{S^{N_V}(0)}(\lambda_j^V(t-\tau_j^V)\mathbf{1}_{\{t<\sigma_j\}} - \mathbb{E}[\lambda^V(t-\tau_j^V)\mathbf{1}_{\{t<\sigma_j\}}])\right\|\right] \to 0. \tag{42}$$

In order to bound the first term on the right of (41), we first note that

$$|\lambda_j^V(t-\tau_j^{N_V}) - \lambda_j^V(t-\tau_j^V)| \le \lambda^*\mathbf{1}_{\tau_j^{V,N_V}\wedge t\neq\tau_j^V\wedge t}$$

$$\begin{aligned}
\mathbb{E}[|\lambda_j^V(t-\tau_j^{N_V}) - \lambda_j^V(t-\tau_j^V)|] &\le \lambda^*\mathbb{P}(\tau_j^{N_V}\wedge t \neq \tau_j^V\wedge t)\\
&= \lambda^*\mathbb{P}\left(\sup_{0\le r\le t}|C_j^N(r) - C_j(t)| \ge 1\right)
\end{aligned}$$

$$\leq \lambda^* \mathbb{E}\left[\sup_{0\leq r\leq t} |C_j^N(r) - C_j(t)|\right] \tag{43}$$

Let

$$\begin{aligned}\varepsilon_{N_V}(t) := \mathbb{E}\Big[\big|\overline{I}^{N_V}(0) - \overline{I}(0)\big|\Big] + \mathbb{E}\Big[\big|\overline{S}^{N_V}(0) - \overline{S}(0)\big|\Big] \\ + \mathbb{E}\left[\frac{1}{N_V}\left|\int_{[0,t]\times\mathbb{R}_+^3\times D} \mathbf{1}_{\{u\leq N_V\mu_V(s)\}}\,\lambda(t-\tau)\,\mathbf{1}_{\{\tau\leq t<\sigma\}}\,\big(Q_b' - \overline{Q_b'}\big)(\mathrm{d}s,\mathrm{d}u,\mathrm{d}\sigma,\mathrm{d}\tau,\mathrm{d}\lambda)\right|\right]. \\ + \mathbb{E}\left[\left\|\frac{1}{N_V}\sum_{j=1}^{I^{N_V}(0)}\big(\lambda_j^0(t) - \overline{\lambda}^0(t)\big)\right\|\right] \\ + \mathbb{E}\left[\left\|\frac{1}{N_V}\sum_{j=1}^{S^{N_V}(0)}\Big(\lambda_j^V(t-\tau_j^V)\mathbf{1}_{\{t<\sigma_j\}} - \mathbb{E}\Big[\overline{\lambda}^V(t-\tau_j^V)\mathbf{1}_{\{t<\sigma_j\}}\Big]\Big)\right\|\right]\end{aligned}$$

.

We have that for all $t\geq 0$, $\varepsilon_{N_V}(t)\to 0$ as $N_V\to\infty$, and moreover $\varepsilon_N(t)\leq 2+3\lambda^*$, hence also $\int_0^t \varepsilon_{N_V}(s)ds\to 0$ as $N_V\to\infty$. Combining the above inequalities, we have that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\big|B_k^N(t) - B_k(t)\big|\right] \leq \int_0^T \varepsilon_{N_V}(r)\,dr + \int_0^t \frac{1}{N_V}\,\mathbb{E}\left[\sum_{j=1}^{S^{N_V}(0)}\sup_{0\leq s\leq r}\big|C_j^{N_V}(s) - C_j(s)\big|\right]dr \\ + \int_0^t \frac{1}{N_V}\,\mathbb{E}\left[\sum_{j\geq S^{N_V}(0)+1}\sup_{0\leq s\leq r}\big|C_j^{N_V}(s) - C_j(s)\big|\right]dr \\ + \int_0^t \mathbb{E}\left[\sup_{0\leq r'\leq u}\big|B_k^N(r') - B_k(r')\big|\right]du, \qquad \forall t\leq T.\end{aligned} \tag{44}$$

On the other hand, define

$$\Delta_k^V(t) := \int_0^t \int_{\overline{\mathfrak{F}}_1^N(s^-)\wedge\overline{\mathfrak{F}}_1(s^-)}^{\overline{\mathfrak{F}}_1^N(s^-)\vee\overline{\mathfrak{F}}_1(s^-)} Q_k^V(ds,du).$$

we have

$$\big|C_k^{N_V}(t) - C_k(t)\big| \leq \Delta_k^V(t), \qquad t\geq 0.$$

In particular since t $\to \Delta_k^V$ is nondecreasing,

$$\begin{aligned}\mathbb{E}\left[\sup_{r\in[0,t]}\big|C_k^{N_V}(r) - C_k(r)\big|\right] &\leq \mathbb{E}\big[\Delta_k^V(t)\big] \\ &= \int_0^t \mathbb{E}\Big[\big|\overline{\mathfrak{F}}_1^N(s^-) - \overline{\mathfrak{F}}_1(s^-)\big|\Big]\,ds.\end{aligned} \tag{45}$$

$$\mathbb{E}\left[\left|\overline{\mathfrak{F}}_1^N(t) - \overline{\mathfrak{F}}_1(t)\right|\right] = \kappa\,\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N\Big(\lambda_{j,B_j^N(t)}\big(\mathfrak{a}_j^N(t)\big) - \mathbb{E}\big[\lambda_{1,B_1(t)}(\mathfrak{a}_1(t))\big]\Big)\right|\right]$$

$$\leq \kappa\, \mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^{N}\Big(\lambda_{j,B_j^N(t)}\big(\mathfrak{a}_j^N(t)\big) - \lambda_{j,B_j(t)}\big(\mathfrak{a}_j(t)\big)\Big)\right|\right]$$
$$+ \kappa\, \mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^{N}\big(\lambda_{j,B_j(t)}\big(\mathfrak{a}_j(t)\big) - \mathbb{E}\big[\lambda_{1,B_1(t)}\big(\mathfrak{a}_1(t)\big)\big]\big)\right|\right]. \quad (46)$$

Since $\big((\lambda_{k,i})_i,\, B_k,\, \mathfrak{a}_k\big)_k$ are i.i.d., the family $\big(\lambda_{k,B_k(t)}(\mathfrak{a}_k(t))\big)_k$ is i.i.d. Hence, by Cauchy–Schwarz,

$$\begin{aligned}
\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^{N}\Big(\lambda_{j,B_j(t)}(\mathfrak{a}_j(t)) - \mathbb{E}\big[\lambda_{1,B_1(t)}(\mathfrak{a}_1(t))\big]\Big)\right|\right] &\leq \frac{1}{N}\left(\mathbb{E}\left[\left(\sum_{j=1}^{N}\Big(\lambda_{j,B_j(t)}(\mathfrak{a}_j(t)) - \mathbb{E}\big[\lambda_{j,B_j(t)}(\mathfrak{a}_j(t))\big]\Big)\right)^2\right]\right)^{1/2} \\
&= \frac{1}{N}\left(\sum_{j=1}^{N}\mathbb{E}\Big[\Big(\lambda_{j,B_j(t)}(\mathfrak{a}_j(t)) - \mathbb{E}\big[\lambda_{j,B_j(t)}(\mathfrak{a}_j(t))\big]\Big)^2\Big]\right)^{1/2} \\
&\leq \frac{\lambda^*}{\sqrt{N}}.
\end{aligned} \quad (47)$$

. In addition, as $(B_j^N(t), \mathfrak{a}_j^N(t), B_j(t), \mathfrak{a}_j(t))_{1\leq j\leq N}$ are exchangeable, we have

$$\begin{aligned}
\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^{N}\Big(\lambda_{j,B_j^N(t)}\big(\mathfrak{a}_j^N(t)\big) - \lambda_{j,B_j(t)}\big(\mathfrak{a}_j(t)\big)\Big)\right|\right] &= \mathbb{E}\Bigg[\Bigg|\frac{1}{N}\sum_{j=1}^{N}\Big(\lambda_{j,B_j^N(t)}\big(\mathfrak{a}_j^N(t)\big) - \lambda_{j,B_j(t)}\big(\mathfrak{a}_j(t)\big)\Big) \\
&\qquad \times \mathbf{1}_{\{B_j(t)\neq B_j^N(t) \text{ or } \mathfrak{a}_j(t)\neq \mathfrak{a}_j^N(t)\}}\Bigg|\Bigg] \\
&\leq \frac{\lambda^*}{N}\sum_{j=1}^{N}\mathbb{P}\big(B_j(t)\neq B_j^N(t) \text{ or } \mathfrak{a}_j(t)\neq \mathfrak{a}_j^N(t)\big) \\
&= \lambda^*\,\mathbb{P}\big(B_k(t)\neq B_k^N(t) \text{ or } \mathfrak{a}_k(t)\neq \mathfrak{a}_k^N(t)\big).
\end{aligned}$$

On the other hand, since

$$\{B_k^N(t)\neq B_k(t) \text{ or } \mathfrak{a}_k^N(t)\neq \mathfrak{a}_k(t)\} \subset \left\{\sup_{r\in[0,t]}\big|B_k^N(r) - B_k(r)\big| \geq 1\right\},$$

$$\mathbb{P}\big(B_k^N(t)\neq B_k(t) \text{ or } \mathfrak{a}_k^N(t)\neq \mathfrak{a}_k(t)\big) \leq \mathbb{E}\left[\sup_{r\in[0,t]}\big|B_k^N(r) - B_k(r)\big|\right]. \quad (48)$$

Therefore,

$$\mathbb{E}\left[\left|\overline{\mathfrak{F}}_1^N(t) - \overline{\mathfrak{F}}_1(t)\right|\right] \leq \kappa\,\frac{\lambda^*}{\sqrt{N}} + \lambda^*\mathbb{E}\left[\sup_{r\in[0,t]}\big|B_k^N(r) - B_k(r)\big|\right]. \quad (49)$$

we deduce that for any $t\geq 0$,

$$\mathbb{E}\left[\sup_{r\in[0,t]}\big|C_k^{N_V}(r) - C_k(r)\big|\right] \leq \frac{\lambda^*}{\sqrt{N}}\,t + 2\lambda^*\int_0^t \mathbb{E}\left[\sup_{r\in[0,t]}\big|B_k^N(r) - B_k(r)\big|\right]dt. \quad (50)$$

Lemma 5.4. *Fix $T>0$ and $k\in\mathbb{N}$. For $t\in[0,T]$, define*

$$b_k^N(t) := \mathbb{E}\left[\sup_{0\leq r\leq t}\big|B_k^N(r) - B_k(r)\big|\right], \qquad c_k^{N_V}(t) := \mathbb{E}\left[\sup_{0\leq r\leq t}\big|C_k^{N_V}(r) - C_k(r)\big|\right].$$

Then, for all $t \in [0,T]$,

$$b_k^N(t) + c_k^{N_V}(t) \;\le\; a_{N_V}(t) \;+\; K_{N_V} \int_0^t \big(b_k^N(u) + c_k^{N_V}(u)\big)\,du,$$

where

$$a_{N_V}(t) := \int_0^t \varepsilon_{N_V}(r)\,dr + \frac{\lambda^*}{\sqrt{N_V}}\,t, \qquad K_{N_V} := \big(1+2\lambda^*\big) + \frac{S^{N_V}(0)}{N_V}.$$

In particular, by Gronwall's inequality,

$$\sup_{t\in[0,T]} \big(b_k^N(t) + c_k^{N_V}(t)\big) \;\le\; a_{N_V}(T)\exp\!\big(K_{N_V}T\big).$$

Proof. From (50) we have :

$$c_k^{N_V}(t) \le \frac{\lambda^*}{\sqrt{N_V}}\,t + 2\lambda^* \int_0^t b_k^N(u)\,du. \tag{51}$$

From (44) we have

$$b_k^N(t) \le \int_0^t \varepsilon_{N_V}(r)\,dr + \frac{S^{N_V}(0)}{N_V}\int_0^t c_k^{N_V}(r)\,dr + \int_0^t b_k^N(u)\,du. \tag{52}$$

Let $y_k^{N,N_V}(t) := b_k^N(t) + c_k^{N_V}(t)$. Adding (50) and (52) gives

$$y_k^{N,N_V}(t) \le \int_0^t \varepsilon_{N_V}(r)\,dr + \frac{\lambda^*}{\sqrt{N_V}}\,t + \int_0^t b_k^N(u)\,du + 2\lambda^*\int_0^t b_k^N(u)\,du + \frac{S^{N_V}(0)}{N_V}\int_0^t c_k^{N_V}(r)\,dr.$$

Since $\int_0^t b_k^N \le \int_0^t y_k^{N,N_V}$ and $\int_0^t c_k^{N_V} \le \int_0^t y_k^{N,N_V}$, we obtain

$$y_k^{N,N_V}(t) \le \left(\int_0^t \varepsilon_{N_V}(r)\,dr + \frac{\lambda^*}{\sqrt{N_V}}\,t\right) + \Big((1+2\lambda^*) + \frac{S^{N_V}(0)}{N_V}\Big)\int_0^t y_k^{N,N_V}(u)\,du.$$

This is precisely an integral Gronwall inequality, hence

$$y_k^{N,N_V}(t) \le a_{N_V}(t)\exp(K_{N_V}t) \le a_{N_V}(T)\exp(K_{N_V}T), \qquad t\in[0,T].$$

This concludes the proof of Theorem 3.4. ☐

Proof of Corollary 3.5. We keep the same process $B_k(t)$ defined in (32) with its jumps $(\tau_i)_{i\ge1}$. It follows from the proof of the theorem 3.4 that, as $N\to\infty$,

$$\frac{1}{N}\sum_{k=1}^N \Big(\mathbf{1}_{\mathfrak{a}_k^N(t)\ge\eta_{k,B_k^N(t)}} - \mathbf{1}_{\mathfrak{a}_k(t)\ge\eta_{k,B_k(t)}}\Big) \to 0,$$

$$\frac{1}{N}\sum_{k=1}^N \Big(\mathbf{1}_{\mathfrak{a}_k^N(t)<\eta_{k,B_k^N(t)}} - \mathbf{1}_{\mathfrak{a}_k(t)<\eta_{k,B_k(t)}}\Big) \to 0,$$

in probability, locally uniformly in t.

Moreover, since $\big(\mathfrak{a}_k(t), \eta_{k,B_k(t)}\big)_{k\ge1}$ is a collection of i.i.d. random variables in D^2, the law of large numbers in D^2 [[11], Theorem 1] yields that, a.s. in D^2, as $N\to\infty$,

$$\left(\frac{1}{N}\sum_{k=1}^N \mathbf{1}_{\mathfrak{a}_k(t)\ge\eta_{k,B_k(t)}},\; \frac{1}{N}\sum_{k=1}^N \mathbf{1}_{\mathfrak{a}_k(t)<\eta_{k,B_k(t)}}\right) \to \Big(\mathbb{E}\big[\mathbf{1}_{\mathfrak{a}_k(t)\ge\eta_{k,B_k(t)}}\big],\; \mathbb{E}\big[\mathbf{1}_{\mathfrak{a}_k(t)<\eta_{k,B_k(t)}}\big]\Big).$$

We decompose:

$$\overline{U}^N(t) = \frac{1}{N}\sum_{k=1}^N \Big(\mathbf{1}_{\mathfrak{a}_k^N(t)\geq \eta_{k,B_k^N(t)}} - \mathbf{1}_{\mathfrak{a}_k(t)\geq \eta_{k,B_k(t)}}\Big) + \frac{1}{N}\sum_{k=1}^N \mathbf{1}_{\mathfrak{a}_k(t)\geq \eta_{k,B_k(t)}},$$

$$\overline{I}^N(t) = \frac{1}{N}\sum_{k=1}^N \Big(\mathbf{1}_{\mathfrak{a}_k^N(t)< \eta_{k,B_k^N(t)}} - \mathbf{1}_{\mathfrak{a}_k(t)< \eta_{k,B_k(t)}}\Big) + \frac{1}{N}\sum_{k=1}^N \mathbf{1}_{\mathfrak{a}_k(t)< \eta_{k,B_k(t)}}.$$

It remains to verify the formulas

$$\overline{I}(t) = \mathbb{E}\big[\mathbf{1}_{\mathfrak{a}_k(t)<\eta_{k,B_k(t)}}\big] \qquad \text{and} \qquad \overline{U}(t) = \mathbb{E}\big[\mathbf{1}_{\mathfrak{a}_k(t)\geq\eta_{k,B_k(t)}}\big]$$

are coherent with (7)-(8).

If $t-\tau_i < \eta_{k,i}$, then $\gamma_{k,i}(t-\tau_i) = 0$, from Assumption 2.4. Hence

$$\int_{\tau_i}^t \gamma_{k,i}(r-\tau_i)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr = 0,$$

and $B_k(t) = B_k(\tau_i)$. Since $\mathbf{1}_{t<\eta_{k,0}} = \mathbf{1}_{t<\eta_{k,0}}\mathbf{1}_{B_k(t)=0}$ a.s., we have

$$\begin{aligned}\mathbb{E}\big[\mathbf{1}_{\mathfrak{a}_k(t)<\eta_{k,B_k(t)}}\big] &= \mathbb{E}\left[\mathbf{1}_{t<\eta_{k,0}}\mathbf{1}_{B_k(t)=0} + \sum_{i\geq 1}\mathbf{1}_{t-\tau_i<\eta_{k,i}}\mathbf{1}_{B_k(t)=i}\right]\\ &= \overline{I}_H(0)\,F_0^c(t) + \sum_{i\geq 1}\mathbb{E}\big[\mathbb{P}\big(t-\tau_i<\eta_{k,i}\mid \tau_i\big)\,\mathbf{1}_{\tau_i\leq t}\big].\end{aligned}$$

Since $\eta_{k,i}$ and τ_i are independent, it follows that

$$\begin{aligned}\mathbb{E}\big[\mathbf{1}_{\mathfrak{a}_k(t)<\eta_{k,B_k(t)}}\big] &= \overline{I}_H(0)\,F_0^c(t) + \sum_{i\geq 1}\mathbb{E}[F^c(t-\tau_i)\,\mathbf{1}_{\tau_i\leq t}]\\ &= \overline{I}_H(0)\,F_0^c(t) + \mathbb{E}\left[\int_0^t F^c(t-s)\,B_k(ds)\right]\\ &= \overline{I}_H(0)\,F_0^c(t) + \kappa\int_0^t F^c(t-s)\,\overline{\mathfrak{S}}(s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(s)\,ds. \qquad (5.27)\end{aligned}$$

This identifies $\overline{I}(t)$ as given by (8). If $\eta_{k,0} > t$ (resp. $\eta_{k,i} > t$), then by Assumption 2.4

$$\gamma_{k,i}(s) = 0 \quad (\text{resp. } \gamma_{k,0}(s) = 0), \qquad 0\leq s\leq t.$$

We write γ (resp. γ_0) for a random function with the same law as $\gamma_{k,i}$ for $i\geq 1$ (resp. as $\gamma_{k,0}$).

Exploiting the fact that $\gamma_{k,0}(t) = 0$ whenever $\eta_{k,0} > t$, a similar statement concerning $\gamma_{k,i}(t)$, and the argument leading to (5.27), we also have

$$\begin{aligned}\mathbb{E}\big[\mathbf{1}_{\mathfrak{a}_k(t)<\eta_{k,B_k(t)}}\big] &= \mathbb{E}\left[\mathbf{1}_{\eta_{k,0}>t}\exp\Big(-\kappa\int_0^t \gamma_0(r)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\right]\\ &\quad + \kappa\int_0^t \mathbb{E}\left[\mathbf{1}_{\eta_{k,i}>t-s}\exp\Big(-\kappa\int_s^t \gamma(r-s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\right]\overline{\mathfrak{S}}(s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(s)\,ds.\end{aligned}$$

Combining the last identity with (9) yields

$$\begin{aligned}\mathbb{E}\big[\mathbf{1}_{\mathfrak{a}_k(t)\geq\eta_{k,B_k(t)}}\big] &= 1 - \mathbb{E}\big[\mathbf{1}_{\mathfrak{a}_k(t)<\eta_{k,B_k(t)}}\big]\\ &= \mathbb{E}\left[\mathbf{1}_{t\geq\eta_{k,0}}\exp\Big(-\kappa\int_0^t \gamma_0(r)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\right]\end{aligned}$$

$$+\kappa\int_0^t \mathbb{E}\Big[\mathbf{1}_{t-s\geq\eta_{k,i}}\exp\Big(-\kappa\int_s^t\gamma(r-s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\Big]\,\overline{\mathfrak{S}}(s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(s)\,ds,$$

which identifies $\overline{U}(t)$ as given by (7). $\square$

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