Stochastic Epidemic Model for Malaria : the Law of Large Numbers.

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Abstract

We study an individual-based stochastic host–vector epidemic model for malaria, in which humans can experience repeated infections over their lifetime. In contrast to classical Ross–Macdonald or compartmental SIR/SEIR models, each infection episode is characterised by a random time-dependent infectivity profile: after infection, a human host transmits parasites to susceptible mosquitoes according to a random infectivity function of the time since infection, while recovered hosts gradually regain susceptibility according to a random susceptibility function. On the vector side, susceptible mosquitoes become infected through contact with infectious humans and then contribute to transmission until death, under a demographic regime that combines birth and mortality processes.

We analyse the large-population asymptotic behaviour of this coupled host–vector system and prove a functional law of large numbers (FLLN) by constructing a sequence of i.i.d. auxiliary processes . The limiting dynamics are described by a nonlinear deterministic system of renewal-type integral equations that generalises both the classical Kermack–McKendrick age-of-infection framework and standard malaria models. In this limit, the solution of the limiting deterministic system depends on the expectation of a complicated functional of the random susceptibility functions, but only on the mean infectivity functions of humans and mosquitoes.

Keywords: Epidemic modeling, vector-borne disease, infection age dependent infectivity, waning immunity

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1 Introduction

Vector-borne diseases remain a major global health burden. They arise when an infection is transmitted indirectly between hosts through a biological vector—most often blood-feeding insects such as mosquitoes, but also ticks, flies, fleas, or freshwater snails , depending on the pathogen. Beyond their wide ecological diversity, vector-borne diseases collectively account for a substantial share of infectious morbidity and mortality worldwide and continue to spread under the combined effects of environmental change and globalization.

From a modelling perspective, vector transmission requires coupling two interacting populations (hosts and vectors), typically with distinct epidemiological structures and time scales. In many mosquito-borne infections, vectors do not recover and remain infectious until death, so that an SI/SEI description is natural on the vector side, while hosts may follow SIRS/SEIRS -type dynamics. Because vector lifespans are often much shorter than host lifespans, demographic turnover in the vector population (births and deaths) is an essential modelling component, and it is common to incorporate explicit vector demography and, when appropriate, time-scale separation arguments. Historically, the foundations of mathematical epidemiology for vector-borne diseases trace back to Ross [10]' pioneering work on malaria, which led to the celebrated insight that reducing vector density below a threshold can prevent sustained transmission (often referred to as Ross' " mosquito theorem"). In parallel, Kermack and McKendrick introduced a general framework for infectious disease dynamics in which infectivity depends on the infection age (time since infection) [6] , and later works in their series also accounted for progressive loss of immunity [7] ,[8]. These formulations naturally lead to integral or partial differential equation models rather than ordinary differential equations, but they capture more realistic within-host and between-host heterogeneities in infectiousness and immunity. A complementary and now classical viewpoint is that deterministic compartmental systems can be obtained as law-of-large-numbers limits of finite-population stochastic epidemic models. Recent developments have shown that Kermack–McKendrick-type infection-age models can arise as limits of *non-Markovian* stochastic systems, thereby providing a probabilistic foundation for varying infectivity and waning immunity beyond the Markovian setting. This probabilistic perspective is also crucial to describe early epidemic stages and extinction phases, which are inherently driven by stochasticity and cannot be captured by purely deterministic models [5, 4, 3, 2] .

The goal of this work is to extend this non-Markovian, varying-infectivity and waning-immunity framework to a Ross-type vector-borne setting. In contrast with classical malaria models where human immunity loss is often assumed instantaneous , we incorporate a *progressive* waning of host immunity, and we allow both host and vector infectivity to depend on infection age and to vary across individuals. On the vector side, we explicitly include demography through births and deaths. Our main contributions are (i) the formulation of a finite-population stochastic host–vector model with infection-age-dependent transmission and progressive immunity decay, and (ii) the derivation of the corresponding deterministic limit as the population size tends to infinity.

Notation. Throughout the paper, all the random variables and processes are defined on a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We use $\xrightarrow[N\to+\infty]{a.s}$ to denote almost sure convergence as the parameter $N \to \infty$. We use $\mathbf{1}_{\{.\}}$ for the indicator function. Let $D = D(\mathbb{R}_+; \mathbb{R})$ be the space of $\mathbb{R}$-valued càdlàg functions defined on $\mathbb{R}_+$, with convergence in D meaning convergence in the Skorohod J_1 topology (see, e.g., [1, Chapter 3]). We also define $D_+ = D(\mathbb{R}_+, \mathbb{R}_+)$.

2 Model Description

We model malaria transmission between a host population of size N and a vector population of size N^V and we shall let $N, N^V \to \infty$ while $N^V/N = \kappa \in (0, \infty)$. Transmission occurs in both directions:

$$\text{host} \longrightarrow \text{vector}, \qquad \text{vector} \longrightarrow \text{host}.$$

2.1 Host Infectivity and Susceptibility Profiles

Let

$$(\lambda_0, \gamma_0) \qquad \text{and} \qquad (\lambda, \gamma)$$

be random elements of $D^2 := D(\mathbb{R}_+; \mathbb{R}_+)^2$.
For each host $k \in \{1, \dots, N\}$, we let

$$(\lambda_{k,0}, \gamma_{k,0})$$

be i.i.d. copies of (λ_0, γ_0), and for $i \geq 1$,

$$(\lambda_{k,i}, \gamma_{k,i})$$

are i.i.d. copies of (λ, γ), independent of the previous family.

$(\lambda_{k,0}, \gamma_{k,0})$ characterises the status of individual k at time 0, while for $i \geq 1$,

- $\lambda_{k,i}(t)$ is the infectivity of host k towards vectors at infection-age t (host $\to$ vector) after its i-th infection.
- $\gamma_{k,i}(t)$ is the susceptibility of host k to vector-borne infection at infection-age t (vector $\to$ host) after its i-th infection.

If the individual k is initially infected, then $\gamma_{k,0}(0) = 0$ and $\eta_{k,0} = \sup\{t > 0 : \lambda_{k,0} > 0\} > 0$.
If the individual k is initially susceptible, then $\gamma_{k,0}(0) = 1$, and $\gamma_{k,0}(t)$ remains equal to 1, until individual k gets infected. In that case, $\lambda_{k,0} \equiv 0$ and $\eta_{k,0} = 0$.
Let $\overline{I}_H(0) = \mathbb{P}\big(\eta_{k,0} > 0\big)$. This is the probability that any of the individuals in the population is initially infected.

2.2 Host Infection Counting Process and Age Since Infection

For each host k, the counting process of infection events is

$$B_k^N(t) := \text{number of infection events experienced by host } k \text{ in } (0,t].$$

The time since its most recent infection is

$$\mathfrak{a}_k^N(t) := t - \sup\{s \in (0,t] : B_k^N(s) = B_k^N(s^-) + 1\},$$

with $\sup \varnothing = 0$.

The current infectivity and susceptibility of host k at time t are

$$\lambda_k(t) := \lambda_{k,B_k^N(t)}\big(\mathfrak{a}_k^N(t)\big), \qquad \gamma_k(t) := \gamma_{k,B_k^N(t)}\big(\mathfrak{a}_k^N(t)\big).$$

2.3 Vector Infectivity Profiles (Vector $\to$ Host)

Let

$$\lambda_0^V \qquad \text{and} \qquad \lambda^V$$

be random elements of $D(\mathbb{R}_+;\mathbb{R}_+)$, representing vector infectivity towards hosts.

For each vector $\ell \in \{1,\dots,N_V\}$,

$$\lambda_{\ell,0}^V \sim \lambda_0^V, \qquad \lambda_{\ell,1}^V \sim \lambda^V,$$

independent of host processes. For each vector ℓ, the counting process of infection events is

$$C_\ell^{N_V}(t) := 0 \text{ or } 1 \text{ depending upon wether the individual has been infected or not }.$$

Vectors do not lose immunity, so each vector is infected at most once:

$$C_\ell^{N_V}(t) \in \{0,1\}, \qquad \forall t \geq 0, t \to C_\ell^{N_V}(t) \quad \text{is non-decreasing.}$$

At time 0, the vector population may contain both susceptible and infected individuals. We encode the initial status of vector ℓ by $C_\ell^{N_V}(0) \in \{0,1\}$. If ℓ is initially infected, i.e. $C_\ell^{N_V}(0) = 1$, then (since $t \mapsto C_\ell^{N_V}(t)$ is non-decreasing and $\{0,1\}$-valued)

$$C_\ell^{N_V}(t) = 1, \qquad \forall\, t \geq 0.$$

Moreover, its infectiousness toward hosts is described by the vector infectivity profile λ_0^V.

The time of infection of the initially susceptible vector ℓ is

$$\tau_\ell^{N_V}(t) := \inf\{t : C_\ell^{N_V}(t) = 1\}.$$

Its current infectivity toward hosts is $\lambda^V_{\ell,C_\ell^{N_V}(t)}\big(t - \tau_\ell^{N_V}(t)\big)$

Assumption 2.1. *We assume that, almost surely, the host infectivity and susceptibility profiles satisfy*

$$0 \leq \lambda_0(t),\, \lambda(t) \leq \lambda^*, \qquad 0 \leq \gamma_0(t),\, \gamma(t) \leq 1,$$

and that the host susceptibility becomes positive only when its infectivity vanishes:

$$\sup\{t \geq 0 : \lambda_0(t) > 0\} \leq \inf\{t \geq 0 : \gamma_0(t) > 0\},$$
$$\sup\{t \geq 0 : \lambda(t) > 0\} \leq \inf\{t \geq 0 : \gamma(t) > 0\}.$$

In addition, we impose analogous assumptions for the vector infectivity profile:

$$0 \leq \lambda_0^V(t),\, \lambda^V(t) \leq \lambda^*,$$

We also assume that, for all $t < 0$,

$$\lambda_0(t) = \lambda(t) = \lambda_0^V(t) = \lambda^V(t) = 0.$$

We need to take into account the demography of vectors . For each vector i which is not present in the population at time 0 , we denote by α_i its birth time and by σ_i its death time.

Assumption 2.2. *We assume that $\mu_V(t)$, which is both the birth and death rate of the vectors is locally integrable, i.e.,*

$$\mu_V \in L^1_{\text{loc}}(\mathbb{R}_+;\mathbb{R}_+).$$

At time t , susceptible vectors are borns at rate $N_V\mu_V(t)$ and each vector, whether infected or susceptible, dies at rate $\mu_V(t)$

Remark 2.1. *The following two probabilistic conditions hold:*

1. *There exists $t_0 > 0$ such that, for every $t \in [0, t_0]$,*
$$\mathbb{P}\big(\gamma(t) < 1\big) > 0.$$
2. *For every $t > 0$,*
$$\mathbb{P}\big(\lambda_0(t) < \lambda^*\big) > 0.$$

These conditions are natural and non-restrictive within our framework.

Justification of (1). There exists $t_0 > 0$ such that ,

$$\mathbb{P}\big(\gamma(t) = 0, 0 \le t \le t_0\big) > 0,$$

and therefore

$$\mathbb{P}\big(\gamma(t) < 1\big) > 0, \qquad \forall\, t \in [0, t_0],$$

which establishes condition (1).

Justification of (2). Assume by contradiction that there exists $t > 0$ such that

$$\mathbb{P}\big(\lambda_0(t) = \lambda^*\big) = 1.$$

This would imply that the baseline infection intensity almost surely attains its maximal value at time t, meaning that the proportion of initially infected individuals equals one at that time. By monotonicity of the infection process, this would in turn imply that the proportion of infected individuals was already equal to one at time $t = 0$. Consequently, the initial proportion of susceptible individuals would be zero, which contradicts the standing assumptions of the model. Hence,

$$\mathbb{P}\big(\lambda_0(t) < \lambda^*\big) > 0$$

must hold for every $t > 0$, hence condition (2).

2.4 Average infectivity and susceptibility

Let $\big(\lambda_{k,i}, \gamma_{k,i}\big)_{i\ge 0,\, 1\le k\le N}$, $\big(\lambda^V_{\ell,i}\big)_{i\in\{0,1\},\, \ell\ge 1}$, be a collection of independent random variables (i.i.d.) and $\big(Q_k\big)_{1\le k\le N_V}$ (resp. $\big(Q^V_k\big)_{1\le k\le N}$) be a collection of independent and identically distributed (i.i.d.) standard Poisson random measures (PRMs) on $\mathbb{R}^2_+$.

For host the quantities

$$\overline{\mathfrak{F}}^N_1(t) := \frac{1}{N}\sum_{k=1}^{N} \lambda_{k,B^N_k(t)}\big(\mathfrak{a}^N_k(t)\big).$$

$$\overline{\mathfrak{S}}^N(t) := \frac{1}{N}\sum_{k=1}^{N} \gamma_{k,B^N_k(t)}\big(\mathfrak{a}^N_k(t)\big).$$

are respectively the average infectivity and the average susceptibility in the human population.

For vectors the quantities

$$\overline{\mathfrak{F}}^{N_V}_2(t) := \frac{1}{N_V}\left(\sum_{i=1}^{I^{N_V}(0)} \lambda^0_i(t)\,\mathbf{1}_{\{t<\sigma_i\}} + \sum_{j=1}^{S^{N_V}(0)} \lambda_j\big(t-\tau^{N_V}_j\big)\,\mathbf{1}_{\{t<\sigma_j\}} + \sum_{\alpha_k\in(0,t]} \lambda^V_{\alpha_k}\big(t-\tau^{N_V}_{\alpha_k}\big)\,\mathbf{1}_{\{\alpha_k\le t<\sigma_k\}}\right).$$

$$\bar{S}^{N_V}_V(t) := \frac{1}{N_V}\, S^{N_V}_V(t).$$

are respectively the average infectivity in the vector population and the proportion of susceptible vectors , where $S_V^{N_V}(t)$ denotes the number of susceptible vectors in the vector population at time t.
Note that the vector population at time $t=0$ is

$$N_V = I^{N_V}(0) + S^{N_V}(0),$$

partitioned into $I^{N_V}(0)$ initially infected and $S^{N_V}(0)$ initially susceptible.

Assumption 2.3. *As $N_V \to \infty$,*

$$\frac{I_V^{N_V}(0)}{N_V} \longrightarrow \overline{I}_V(0), \qquad \frac{S_V^{N_V}(0)}{N_V} \longrightarrow \overline{S}_V(0).$$

Moreover, vectors are born susceptible at rate $N_V \mu_V(t)$, and each vector dies at rate $\mu_V(t)$. Thus, a host k becomes infected by vectors at the instantaneous rate

$$\Upsilon_k^N(t) := \kappa.\gamma_{k,B_k^N(t)}(\mathfrak{a}_k^N(t))\,\overline{\mathfrak{F}}_2^{N_V}(t).$$

and a susceptible vector ℓ becomes infected by biting infectious hosts with intensity

$$\Gamma_\ell^{N_V}(t) := \frac{1}{\kappa}\overline{\mathfrak{F}}_1^N(t).$$

2.5 Construction of the Counting Processes B_k^N and $C_\ell^{N_V}$

For each host $k \in \{1, \dots, N\}$, the counting process of its successive infection events is defined as the solution of

$$B_k^N(t) = \int_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{\{u \le \Upsilon_k^N(r-)\}}\, Q_k(\mathrm{d}r, \mathrm{d}u), \qquad t \ge 0, \tag{2.1}$$

By construction, $(B_k^N(t))_{t\ge 0}$ is a càdlàg, integer-valued, nondecreasing process with jumps of size 1.

Similarly, let $\left(Q_\ell^V\right)_{1\le \ell \le S^{N_V}(0)}$ and $\left(Q_k^V\right)_{k > S^{N_V}}$ be an i.i.d. family of standard Poisson random measures on $\mathbb{R}_+^2$, independent of everything else. For each $\ell \in \{1, \dots, N_V\}$, the counting process of infection events is defined by

$$C_\ell^{N_V}(t) = \int_0^t \int_0^\infty \mathbf{1}_{\{C_\ell^{N_V}(s^-)=0\}}\, \mathbf{1}_{\{u \le \Gamma_\ell^{N_V}(s^-)\}}\, Q_\ell^V(\mathrm{d}s, \mathrm{d}u), \qquad t \ge 0, \tag{2.2}$$

and, for a vector born at time α_k,

$$C_{\alpha_k}^{N_V}(t) = \int_{\alpha_k}^t \int_0^\infty \mathbf{1}_{\{C_{\alpha_k}^{N_V}(s^-)=0\}}\, \mathbf{1}_{\left\{u \le \frac{\overline{\mathfrak{F}}_1^N(s^-)}{\kappa}\right\}}\, Q_{S^{N_V}(0)+k}^V(\mathrm{d}s, \mathrm{d}u).$$

Each vector can be infected at most once.

2.6 Infectious Periods for Hosts

We now define the random infectious periods associated with the infectivity profiles of hosts and vectors.

Host infectious periods

The infectious periods for hosts are defined by

$$\eta_0 := \sup\{t \ge 0 : \lambda_0(t) > 0\}, \qquad \eta := \sup\{t \ge 0 : \lambda(t) > 0\},$$

with the convention $\sup \varnothing = 0$. Thus η_0 (resp. η) represents the duration during which an initially infected host (resp. a newly infected host) can infect vectors.

For each host $k \in \{1, \dots, N\}$ and infection index $i \ge 0$, we define the individual infectious period

$$\eta_{k,i} := \sup\{t \ge 0 : \lambda_{k,i}(t) > 0\}.$$

By construction, $\eta_{k,0}$ has the same distribution as η_0, and $\eta_{k,i}$ (for $i \ge 1$) has the same distribution as η.

2.7 Numbers of Infected and Non-Infected Hosts

We now define the indicators of current infection status using the infection counting processes and infectious periods previously introduced.

A host k is *currently infected (infectious)* at time t if it has experienced at least one infection before t and its time since the last infection is strictly smaller than the corresponding infectious period. We thus define the indicator

$$I_k^N(t) := \mathbf{1}_{\left\{ \mathfrak{a}_k^N(t) < \eta_{k, B_k^N(t)} \right\}},$$

where $B_k^N(t)$ is the infection counting process of host k, $\mathfrak{a}_k^N(t)$ is its age since last infection, and $\eta_{k,B_k^N(t)}$ is the infectious period associated with its most recent infection. In the case $B_k^N(t) = 0$, we have $\mathfrak{a}_k^N(t) = t$, and individual k is infected at time t if it was initially infected and $\eta_{k,0} > t$.

The total number of (currently) infected hosts at time t is

$$I_H^N(t) := \sum_{k=1}^{N} I_k^N(t),$$

and the number of non-infected (non-infectious) hosts is

$$U_H^N(t) := N - I_H^N(t).$$

Here, $U_H^N(t)$ groups together susceptible and immune hosts that are not currently infectious.

Vector demography (births and deaths). Let $(\alpha_i)_{i\geq 1}$ and $(\sigma_i)_{i\geq 1}$ denote respectively the birth times and the death times of vectors. We model the birth of vectors through the birth point process

$$\mathcal{A}^{N_V}(\mathrm{d}s) := \sum_{i > S^{N_V}(0)} \delta_{\alpha_i}(\mathrm{d}s),$$

and vector removal through the death point process

$$\mathcal{M}^{N_V}(\mathrm{d}s) := \sum_{i\geq 1} \delta_{\sigma_i}(\mathrm{d}s).$$

The associated counting processes are

$$A^{N_V}(t) = \mathcal{A}^{N_V}([0,t]) = \sum_{i > S^{N_V}(0)} \mathbf{1}_{\{\alpha_i \leq t\}}, \qquad M^{N_V}(t) = \mathcal{M}^{N_V}([0,t]) = \sum_{i\geq 1} \mathbf{1}_{\{\sigma_i \leq t\}}.$$

Hence the vector population size evolves as

$$N_V(t) = N_V + A^{N_V}(t) - M^{N_V}(t).$$

Remark 2.2 (Deaths due to the disease)**.** *We have not explicitly modeled the fact that the disease is likely to kill both vectors and humans. Concerning the vectors, we could easily increase the rate of death of infected mosquitos (compared to the rate of death of susceptible mosquitos).*

Concerning the humans, we can decide that with a certain probability, $\gamma \equiv 0$. This means that in that case the concerned human will no longer contribute to the epidemic, and we could decide that those humans die at the end of the infected period.

2.8 Markovian SIRS–SI host–vector model as a special case

In this subsection we show how the classical SIRS–SI host–vector model can be recovered as a particular case of the general stochastic framework introduced above, by choosing specific infectivity and susceptibility profiles for hosts and vectors.

Host side: SIRS structure

We assume that each host, once infected, goes through three successive phases:

- an *infectious* phase which duration follows an exponential distribution with parameter $\eta_H > 0$,

- an *immune* phase which duration follows an exponential distribution with parameter $\theta_H > 0$,
- followed by a fully susceptible phase.

We denote by $\beta_{HV} > 0$ the transmission rate from an infectious host to vectors, and we neglect host heterogeneity by taking identical profiles for all infection episodes. At the level of the prototype functions, this corresponds to choosing

$$\lambda(t) := \beta_{HV}\,\mathbf{1}_{\{0\le t<\xi_H\}}, \qquad t \ge 0, \quad \text{where } \xi_H \sim \mathrm{Exp}(\eta_H).$$

and

$$\gamma(t) := \mathbf{1}_{\{t\ge \xi_H+\zeta_H\}}, \qquad t \ge 0, \quad \text{where } \zeta_H \sim \mathrm{Exp}(\theta_H), \text{ and } \zeta_H \text{ and } \xi_H \text{ are independent.}$$

Note that β_{HV}, η_H, and θ_H are identical for all hosts, whereas the random variables ξ_H and ζ_H are i.i.d. across hosts.

Vector side: SI structure with demography

For the vectors, we consider a simple SI structure: after infection, vectors remain infectious until they die and never return to a susceptible state. We denote by $\beta_{VH} > 0$ the transmission rate from an infectious host to susceptible vectors.

In the general framework, this corresponds to choosing a vector infectivity profile of the form

$$\lambda^V(t) := \beta_{VH}\,\mathbf{1}_{\{\alpha\le t<\sigma\}}, \qquad t \ge 0,$$

α and σ denote respectively the birth and death time. Vectors who are alive at time 0 are supposed be born at time 0. Susceptible vectors are born at rate $\mu_V N_V > 0$, and die naturally at rate $\mu_V > 0$. Equivalently, each vector has an i.i.d. lifetime $\sigma - \alpha \sim \mathrm{Exp}(\mu_V)$. Denoting by $S_V(t)$ and $I_V(t)$ the numbers of susceptible and infected vectors, infection occurs at rate $\beta_{VH}\, I_H(t)/N_H$ and infected vectors remain infectious until death. The following deterministic model can be justified in this Markovian context :

$$\frac{dS_H}{dt} = -\beta_{HV}\frac{I_V}{N_V}S_H + \theta_H R_H, \qquad \frac{dI_H}{dt} = \beta_{HV}\frac{I_V}{N_V}S_H - \eta_H I_H, \qquad \frac{dR_H}{dt} = \eta_H I_H - \theta_H R_H$$

$$\frac{dS_V}{dt} = \mu_V N_V - \beta_{VH}\frac{I_H}{N}S_V - \mu_V S_V, \qquad \frac{dI_V}{dt} = \beta_{VH}\frac{I_H}{N}S_V - \mu_V I_V.$$

3 Functional Law of Large Numbers and Limit Integral System

In this section we state the functional law of large numbers (FLLN) for the scaled host–vector processes and introduce the deterministic limit through a *two-dimensional* system of integral equations. The key point is that, in the large-population limit, the vector layer can be expressed explicitly as a functional of the host-to-vector renormalised force of infection noted by y in this section , so that the core limit dynamics closes on two unknowns.

3.1 Mean infectivities and deterministic inputs

We denote by $\bar{I}_H(0)$ and $\bar{I}_V(0)$ the initial fractions of infected hosts and vectors, and by $\bar{S}_V(0)$ the initial fraction of susceptible vectors. We set the mean infectivities and the (conditional) survival functions

$$\begin{aligned} \bar{\lambda}_0(t) &:= \mathbb{E}\big[\lambda_0(t)\big], & \bar{\lambda}(t) &:= \mathbb{E}\big[\lambda(t)\big], \\ \bar{\lambda}_0^V(t) &:= \mathbb{E}\big[\lambda_0^V(t)\big], & \bar{\lambda}^V(t) &:= \mathbb{E}\big[\lambda^V(t)\big], \\ F_0^c(t) &:= \mathbb{P}\big(\eta_0 > t \mid \eta_0 > 0\big), & F^c(t) &:= \mathbb{P}(\eta > t). \end{aligned}$$

3.2 Vector layer as a functional of y

Given a nonnegative function y on $\mathbb{R}_+$ representing the renormalised host force of infection, define the induced fraction of susceptible vector

$$\mathcal{S}(y)(t) = \bar{S}_V(0)\,\exp\left(-\int_0^t \big[\frac{y(r)}{\kappa} + \mu_V(r)\big]\,\mathrm{d}r\right) + \int_0^t \mu_V(s)\,\exp\left(-\int_s^t \big[\frac{y(r)}{\kappa} + \mu_V(r)\big]\,\mathrm{d}r\right)\mathrm{d}s, \quad (3.1)$$

and the induced vector-to-host force of infection

$$\mathcal{V}(y)(t) = \bar{I}_V(0)\,\bar{\lambda}_0^V(t)\,\exp\Big(-\int_0^t \mu_V(u)\,\mathrm{d}u\Big) + \frac{1}{\kappa}\int_0^t \bar{\lambda}^V(t-s)\,\exp\Big(-\int_s^t \mu_V(u)\,\mathrm{d}u\Big)\,\mathcal{S}(y)(s)\,y(s)\,\mathrm{d}s. \tag{3.2}$$

$\mathcal{S}(y)(t)$ and $\mathcal{V}(y)(t)$ represent, respectively, the susceptible fraction and the renormalised vector-to-host force of infection in the vector population at time t as a functions of the renormalised host force of infection between time 0 and time t.

3.3 Two-dimensional limit system

To describe the deterministic limit of the FLLN, we introduce a two-dimensional system of integral equations. Observe that the host layer depends on the vector layer only through $\mathcal{V}(y)$ defined in (3.2). We thus look for a solution $(x, y) \in D_+^2$ of the closed system

$$\begin{cases} x(t) = \mathbb{E}\Big[\gamma_0(t)\,\exp\Big(-\kappa\int_0^t \gamma_0(r)\,\mathcal{V}(y)(r)\,\mathrm{d}r\Big)\Big] \\ \qquad + \kappa\int_0^t \mathbb{E}\Big[\gamma(t-s)\,\exp\Big(-\kappa\int_s^t \gamma(r-s)\,\mathcal{V}(y)(r)\,\mathrm{d}r\Big)\Big]\,x(s)\,\mathcal{V}(y)(s)\,\mathrm{d}s, \\ y(t) = \bar{\lambda}_0(t) + \kappa\int_0^t \bar{\lambda}(t-s)\,x(s)\,\mathcal{V}(y)(s)\,\mathrm{d}s. \end{cases} \tag{3.3}$$

In (3.3), the only random ingredients from the microscopic model enter through the expectations defining the mean infectivities $(\bar{\lambda}_0, \bar{\lambda}, \bar{\lambda}_0^V, \bar{\lambda}^V)$ and through two laws of γ_0 and γ.

3.4 Well-posedness and FLLN statement

Our first result establishes existence and uniqueness for the reduced limit system (3.3).

Theorem 3.1 (Existence and uniqueness)**.** *Assume that Assumption 2.1 holds . Then the system (3.3) admits a unique solution $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}}_1) \in D_+^2$. Moreover, if $t \mapsto \mathbb{E}[\lambda_0(t)]$ is continuous and $t \mapsto \gamma_0(t)$ is continuous in probability, then $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}}_1) \in C(\mathbb{R}_+)^2$.*

We now state the functional law of large numbers. Let

$$(\overline{\mathfrak{F}}_1^N, \overline{\mathfrak{S}}^N, \bar{S}_V^{N_V}, \overline{\mathfrak{F}}_2^{N_V})$$

be the scaled host–vector processes in the (N, N_V)-population model (defined in Subsection 2.4).

Theorem 3.2 (FLLN)**.** *Under Assumptions 2.1, 2.2, and 2.3 as $N, N_V \to \infty$ while $\frac{N_v}{N} = \kappa \in (0, \infty)$,*

$$(\overline{\mathfrak{S}}^N, \overline{\mathfrak{F}}_1^N) \longrightarrow (\overline{\mathfrak{S}}, \overline{\mathfrak{F}}_1) \qquad \text{in } D^2(\mathbb{R}_+),$$

where $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}}_1)$ is the unique solution of (3.3) and moreover,

$$(\bar{S}_V^{N_V}, \overline{\mathfrak{F}}_2^{N_V}) \longrightarrow (\mathcal{S}(\overline{\mathfrak{F}}_1), \mathcal{V}(\overline{\mathfrak{F}}_1)) \qquad \text{in } D^2(\mathbb{R}_+).$$

Corollary 3.1. *Given the solution $(\overline{\mathfrak{S}}, \overline{\mathfrak{F}}_1)$ of (3.3),*

$$(\overline{U}^N, \overline{I}^N) \xrightarrow[N\to+\infty]{\mathbb{P}} (\overline{U}, \overline{I}) \qquad \text{in } D^2,$$

With $(\overline{U}^N, \overline{I}^N) = (\frac{I_H^N(t)}{N}, \frac{U_H^N(t)}{N})$ and $(\overline{U}, \overline{I})$ is given by

$$\overline{U}(t) = \mathbb{E}\Big[\mathbf{1}_{t\geq\eta_0}\exp\Big(-\kappa\int_0^t \gamma_0(r)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\Big] + \kappa\int_0^t \mathbb{E}\Big[\mathbf{1}_{t-s\geq\eta}\exp\Big(-\kappa\int_s^t \gamma(r-s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\Big]\,\overline{\mathfrak{S}}(s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(s)\,ds, \tag{3.4}$$

$$\overline{I}(t) = \overline{I}_H(0)\,F_0^c(t) + \kappa\int_0^t F^c(t-s)\,\overline{\mathfrak{S}}(s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(s)\,ds. \tag{3.5}$$

Theorem 3.1 and Theorem 3.2 will be proved in the next two sections, and Corollary (3.1) will follow immediately thereafter.

4 Proof of Theorem 3.1

Before starting the proof of Theorem 3.1, we first establish the following lemma.

Lemma 4.1 (Mass conservation)**.** *Assume that the pair (x, y) is a solution of the system* (3.3)*. Then, for every $t \geq 0$, we have*

$$\mathbb{E}\left[\exp\left(-\kappa \int_0^t \gamma_0(r)\, \mathcal{V}(y)(r)\, \mathrm{d}r\right)\right] + \kappa \int_0^t \mathbb{E}\left[\exp\left(-\kappa \int_s^t \gamma(r-s)\, \mathcal{V}(y)(r)\, \mathrm{d}r\right)\right]\, x(s)\, \mathcal{V}(y)(s)\, \mathrm{d}s = 1. \tag{4.1}$$

Proof of Lemma 4.1. We note as soon as (x, y) solves (3.3) ,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbb{E}\left[\exp\Big(-\kappa \int_0^t \gamma_0(r)\, \mathcal{V}(y)(r)\, \mathrm{d}r\Big)\right] + \kappa \int_0^t \mathbb{E}\left[\exp\Big(-\kappa \int_s^t \gamma(r-s)\, \mathcal{V}(y)(r)\, \mathrm{d}r\Big)\right] x(s)\, \mathcal{V}(y)(s)\, \mathrm{d}s\right)$$
$$= \Bigg(\mathbb{E}\left[-\gamma_0(t)\, \exp\Big(-\kappa \int_0^t \gamma_0(r)\, \mathcal{V}(y)(r)\, \mathrm{d}r\Big)\right] + x(t)$$
$$-\kappa \int_0^t \mathbb{E}\left[\gamma(t-s)\, \exp\Big(-\kappa \int_s^t \gamma(r-s)\, \mathcal{V}(y)(r)\, \mathrm{d}r\Big)\right] x(s)\, \mathcal{V}(y)(s)\, \mathrm{d}s\Bigg)\kappa\, \mathcal{V}(y)(t) = 0.$$

Hence the left-hand side of (4.1) is equal to its value at $t = 0$, which is 1. □

A priori estimates of the solution

Let $(x, y) \in D$ be a solution of the system (3.3). From Assumption 2.1 we have that $0 \leq \gamma_0(t) \leq 1$ and $0 \leq \gamma(t-s) \leq 1$ for all $0 \leq s \leq t \leq T$. Hence, the combination of the first equation of (3.3) and Lemma 4.1 implies that

$$0 \leq x(t) \leq 1, \qquad t \in [0, T].$$

Moreover, we have

$$\mathcal{S}(y)(t) = \bar{S}_V(0)\, \exp\left(-\int_0^t \Big(\frac{y(r)}{\kappa} + \mu_V(r)\Big)\, \mathrm{d}r\right) + \int_0^t \mu_V(s)\, \exp\left(-\int_s^t \Big(\frac{y(r)}{\kappa} + \mu_V(r)\Big)\, \mathrm{d}r\right)\, \mathrm{d}s.$$

For all $t \geq 0$,

$$\bar{S}_V(0)\, \exp\left(-\int_0^t \Big(\frac{y(r)}{\kappa} + \mu_V(r)\Big)\, \mathrm{d}r\right) \leq \bar{S}_V(0)$$

and

$$\int_0^t \mu_V(s)\, \exp\left(-\int_s^t \Big(\frac{y(r)}{\kappa} + \mu_V(r)\Big)\, \mathrm{d}r\right)\, \mathrm{d}s \leq \int_0^t \mu_V(s)\, \exp\left(-\int_s^t \mu_V(r)\, \mathrm{d}r\right)\, \mathrm{d}s \leq 1.$$

Consequently, we obtain the useful estimate

$$\bar{\mathcal{S}}(y)(t) \leq 1 + \bar{S}_V(0) \qquad t \geq 0. \tag{4.2}$$

In addition, by Assumption 2.1, if $\lambda(t) > 0$ (resp. $\lambda_0(t) > 0$), then $\gamma(s) = 0$ (resp. $\gamma_0(s) = 0$) for all $0 \leq s \leq t$. From the second line of (3.3) we deduce that for any $t \in [0, T]$,

$$y(t) = \mathbb{E}\left[\lambda_0(t)\, \exp\left(-\kappa \int_0^t \gamma_0(r)\, \mathcal{V}(y)(r)\, \mathrm{d}r\right)\right]$$
$$+\kappa \int_0^t \mathbb{E}\left[\lambda(t-s)\, \exp\left(-\kappa \int_s^t \gamma(r-s)\, \mathcal{V}(y)(r)\, \mathrm{d}r\right)\right] x(s)\, \mathcal{V}(y)(s)\, \mathrm{d}s$$
$$\leq \lambda^*.$$

Finally we can deduce that for any $T > 0, 0 \leq t \leq T$:

$$\mathcal{V}(y)(t) = \bar{I}_V(0)\, \bar{\lambda}_0^V(t)\, \exp\left(-\int_0^t \mu_V(u)\, \mathrm{d}u\right)$$
$$+\frac{1}{\kappa} \int_0^t \bar{\lambda}^V(t-s)\, \exp\left(-\int_s^t \mu_V(u)\, \mathrm{d}u\right)\, \mathcal{S}(y)(s)\, y(s)\, \mathrm{d}s$$
$$\leq \lambda^* + \frac{(\lambda^*)^2}{\kappa}\, (S_V(0) + 1)\, T =: C_T. \tag{4.3}$$

Uniqueness of the solution to the integral system

We prove that the integral system admits at most one solution on any bounded interval $[0, T]$.

For a *càdlàg* function $f : \mathbb{R}_+ \to \mathbb{R}$, define

$$\|f\|_t := \sup_{0\le r\le t} |f(r)|, \qquad t > 0.$$

Let (x, y) and $(\tilde{x}, \tilde{y})$ be two elements of D_+^2 solutions of the system on $[0, T]$ with the same initial conditions. Set

$$\Delta x(t) := x(t) - \tilde{x}(t), \qquad \Delta y(t) := y(t) - \tilde{y}(t), \qquad t \in [0, T].$$

Moreover, we write

$$V(t) := \mathcal{V}(y)(t), \qquad \tilde{V}(t) := \mathcal{V}(\tilde{y})(t), \qquad \Delta V(t) := V(t) - \tilde{V}(t),$$
$$S(t) := \mathcal{S}(y)(t), \qquad \tilde{S}(t) := \mathcal{S}(\tilde{y})(t), \qquad \Delta S(t) := S(t) - \tilde{S}(t),$$

For all $t \in [0, T]$ we have,

$$\Delta y(t) = \kappa \int_0^t \bar{\lambda}(t-s) \Big(x(s)\, V(s) - \tilde{x}(s)\, \tilde{V}(s) \Big)\, \mathrm{d}s.$$

Using the bounds $0 \le x, \tilde{x} \le 1$ and (4.3), we obtain

$$|\Delta y(t)| \le \lambda^* \kappa \int_0^t \Big(C_T |\Delta x(s)| + |\Delta V(s)| \Big)\, \mathrm{d}s, \qquad t \in [0, T]. \tag{4.4}$$

we have for all $t \in [0, T]$,

$$\Delta V(t) = \frac{1}{\kappa} \int_0^t \bar{\lambda}^V(t-s)\, e^{-\int_s^t \mu_V(u)\,\mathrm{d}u} \Big(S(y)(s)y(s) - \tilde{S}(y)(s)\tilde{y}(s) \Big)\, \mathrm{d}s.$$

using $0 \le \tilde{x}_V \le 1$, the bound $\bar{\lambda}^V \le \lambda^*$, and (4.2) we obtain

$$|\Delta V(t)| \le \frac{\lambda^*}{\kappa} max(\lambda^*, \bar{S_V}(0) + 1) \int_0^t \Big(|\Delta S(s)| + |\Delta y(s)| \Big)\, \mathrm{d}s. \tag{4.5}$$

Since the exponential is 1- lipschitizian on $\mathbb{R}_-$, we get, for all $t \in [0, T]$,

$$\begin{aligned} |\Delta S(t)| &\le \frac{\bar{S}_V(0)}{\kappa} \int_0^t |\Delta y(r)|\, \mathrm{d}r + \frac{1}{\kappa} \int_0^t \mu_V(s) exp(-\int_s^t \mu_V(r) dr) \int_s^t |\Delta y(r)|\, \mathrm{d}r\, \mathrm{d}s \\ &\le \frac{1}{\kappa} \Big(\bar{S}_V(0) + \int_0^t \mu_V(s) exp(-\int_s^t \mu_V(r) dr)\, \mathrm{d}s \Big) \int_0^t |\Delta y(r)|\, \mathrm{d}r \\ &\le \frac{\bar{S}_V(0) + 1}{\kappa} \int_0^t |\Delta y(r)|\, \mathrm{d}r. \end{aligned}$$

Consequently, setting

$$C_2 := \frac{\bar{S}_V(0) + 1}{\kappa},$$

$$\|\Delta S\|_t \le C_2 \int_0^t \|\Delta y\|_s\, \mathrm{d}s, \qquad t \in [0, T]. \tag{4.6}$$

Combining (4.5) and (4.6), we obtain :

$$\|\Delta V\|_t \le C_3(T) \int_0^t \|\Delta y\|_s\, \mathrm{d}s, \qquad t \in [0, T], \tag{4.7}$$

with

$$C_3(T) = \frac{\lambda^*}{\kappa} \max(\lambda^*, S_V(0) + 1)\, (1 + C_2 T).$$

Now we decompose x as

$$x(t) = A(V)(t) + B(x,V)(t), \qquad t \in [0,T],$$

where,

$$A(V)(t) := \mathbb{E}\Big[\gamma_0(t)\, \exp\Big(-\kappa \int_0^t \gamma_0(r)\, V(r)\, \mathrm{d}r\Big)\Big]\,,$$

$$B(x,V)(t) := \kappa \int_0^t \mathbb{E}\Big[\gamma(t-s)\, \exp\Big(-\kappa \int_s^t \gamma(r-s)\, V(r)\, \mathrm{d}r\Big)\Big]\; x(s)\, V(s)\, \mathrm{d}s.$$

For every $t \in [0,T]$, we have

$$\big|A(V)(t) - A(\widetilde{V})(t)\big| \le \kappa \int_0^t \big|V(r) - \widetilde{V}(r)\big|\, \mathrm{d}r \le \kappa C_3(T) \int_0^t \|\Delta y\|_r\, \mathrm{d}r, \tag{4.8}$$

We next estimate $B(x,V) - B(\tilde{x},\tilde{V})$. We add and subtract the quantity

$$\kappa\mathbb{E}\Big[\gamma(t-s)\, \exp\Big(-\kappa \int_s^t \gamma(r-s)\, V(r)\, \mathrm{d}r\Big)\Big]\; x(s)\, \tilde{V}(s),$$

to obtain

$$\big|B(x,V)(t) - B(\tilde{x},\tilde{V})(t)\big| \le \kappa \int_0^t \Big(C_3(T)\, |\Delta x(s)| + |\Delta y(s)|\Big)\, \mathrm{d}s. \tag{4.9}$$

Combining (4.9) and (4.4), we conclude that :

$$\|\Delta x\|_t \le \kappa\, C_4(T) \int_0^t \left(\|\Delta x\|_s + \|\Delta V\|_s\right)\, \mathrm{d}s. \tag{4.10}$$

for some constant $C_4(T) > 0$ depending only on λ^* and T .

Combining (4.7), (4.8), (4.9) and (4.10), we deduce that there exists a constant $K > 0$ such that, for all $t \in [0,T]$,

$$D(t) := \|\Delta x\|_t + \|\Delta y\|_t \le K \int_0^t D(s)\, \mathrm{d}s.$$

Since the two solutions have the same initial conditions, we have $D(0) = 0$. By Grönwall's lemma it follows that $D(t) = 0$ for all $t \in [0,T]$, that is,

$$\Delta x(t) = \Delta y = 0 \quad \text{for all } t \in [0,T].$$

Consequently, the two solutions coincide, which proves the uniqueness of the solution to the integral system on $[0,T]$, for all $T > 0$, hence on $\mathbb{R}_+$.

Existence for the integral system via Picard iteration

1) Operators and truncated system

Host operators. For bounded measurable functions $x, y : [0,T] \to \mathbb{R}_+$ and $V : [0,T] \times D \to \mathbb{R}_+$ define

$$\begin{aligned} F(t;x,y) &:= \mathbb{E}\Big[\gamma_0(t)\exp\Big(-\kappa \int_0^t \gamma_0(r)\, V(r,y)\, dr\Big)\Big] \\ &\quad + \kappa \int_0^t \mathbb{E}\Big[\gamma(t-s)\exp\Big(-\kappa \int_s^t \gamma(r-s)\, V(r,y)\, dr\Big)\Big]\; x(s)\, V(s,y)\, ds, \end{aligned} \tag{4.11}$$

$$G(t;x,y) := \bar{\lambda}_0(t) + \kappa \int_0^t \bar{\lambda}(t-s)\, x(s)\, V(s,y)\, ds. \tag{4.12}$$

Truncations. Define the 1-Lipschitz truncation maps

$$\Pi_1(u) := u \wedge 1, \qquad \Pi_2(u) := u \wedge \lambda^*$$

Truncated system. We consider the truncated fixed-point problem on $[0,T]$:

$$\begin{cases} x(t) = \Pi_1\Big(F(t;x,y)\Big), \\ y(t) = \Pi_2\Big(F(t;x,y)\Big), \end{cases} \tag{4.13}$$

2) Picard iteration and uniform bounds

Picard scheme. We initialize

$$x^{(0)}(t) = 0, \qquad y^{(0)}(t) = 0, \qquad t \in [0,T],$$

and , for $n \geq 0$, set

$$\begin{cases} x^{(n+1)}(t) := \Pi_1\Big(F(t;x^{(n)},y^{(n)})\Big), \\ y^{(n+1)}(t) := \Pi_2\Big(F(t;x^{(n)},y^{(n)})\Big), \end{cases} \tag{4.14}$$

Uniform bounds. By construction for all $n \geq 0$ and $t \in [0,T]$,

$$0 \leq x^{(n)}(t) \leq 1, \qquad 0 \leq y^{(n)}(t) \leq \lambda^*, \tag{4.15}$$

3) Contraction-type estimate and convergence

Differences. For $n \geq 0$ define, for $t \in [0,T]$,

$$\delta_x^{(n)}(t) := |x^{(n+1)}(t) - x^{(n)}(t)|, \quad \delta_y^{(n)}(t) := |y^{(n+1)}(t) - y^{(n)}(t)|,$$

$$\delta_V^{(n)}(s) = V(t, y^{(n+1)}) - V(t, y^{(n)})$$

Set

$$\Delta_n(t) := \delta_x^{(n)}(t) + \delta_y^{(n)}(t). \tag{4.16}$$

Step 1: estimate for $\delta_y^{(n)}$. Using (4.14), (4.7), together with $0 \leq \bar{\lambda} \leq \lambda^*$ and (4.15),

$$\begin{aligned} \delta_y^{(n)}(t) &\leq \kappa \int_0^t \bar{\lambda}(t-s) \left| x^{(n)}(s) V\big(t, y^{(n)}\big) - x^{(n-1)}(s) V\big(t, y^{(n-1)}\big) \right| ds \\ &\leq \lambda^* \kappa \int_0^t \Big(C_1\, \delta_x^{(n-1)}(s) + \delta_V^{(n-1)}(s) \Big)\, ds \\ &\leq \lambda^* \kappa \int_0^t \Big(C_1\, \delta_x^{(n-1)}(s) + C_3\, \delta_y^{(n-1)}(s) \Big)\, ds \\ &\leq \lambda^* \kappa \, \max\{C_1, C_3\} \int_0^t \Delta_{n-1}(s)\, ds. \end{aligned} \tag{4.17}$$

Step 2: estimate for $\delta_x^{(n)}$.

$$\delta_x^{(n)}(t) \ \leq \big| F(t; x^{(n)}, y^{(n)})) - F(t; x^{(n-1)}, y^{(n-1)})) \big|.$$

From (4.9) and (4.7) , we deduce that :

$$\delta_x^{(n)}(t) \leq C_4(T) \int_0^t \Delta_{n-1}(s)\, ds. \tag{4.18}$$

Step 4: Grönwall–Picard estimate. Summing (4.17) and (4.18), we obtain

$$\Delta_n(t) \leq C_T \int_0^t \Delta_{n-1}(s)\, ds, \qquad t \in [0,T], \tag{4.19}$$

with

$$C_T := \lambda^* \kappa \, \max\{C_1, C_3\} + C_4(T). \tag{4.20}$$

Iterating (4.19) gives, for $n \geq 1$,

$$\sup_{0 \leq t \leq T} \Delta_n(t) \leq \frac{(C_T T)^n}{n!} \sup_{0 \leq t \leq T} \Delta_0(t),$$

hence $\sum_{n \geq 0} \sup_{t \leq T} \Delta_n(t) < \infty$. Therefore $(x^{(n)}, y^{(n)})$ is Cauchy in $(D([0,T]))^2$ for the sup norm and converges uniformly to some $(x,y) \in (D([0,T]))^2$.

4) Passage to the limit and solution of the truncated system

By the uniform bounds (4.15), all integrands in (4.11)–(4.12) with (x,y) remplaced by $(x^{(n)},y^{(n)})$ are dominated by constants (depending on T only). Hence, by dominated convergence, we may pass to the limit in (4.14) and obtain that (x,y) satisfies the truncated system (4.13) on $[0,T]$ for all $T>0$, hence on $\mathbb{R}_+$.

Assume that (x,y) is a solution of the truncated system. Then

$$F(0,x,y)=\mathbb{E}\big[\gamma_0(0)\big],\qquad G(0,x,y)=\bar{\lambda}_0(0).$$

Hence

$$x(0)=F(0,x,y)\wedge 1,\qquad y(0)=G(0,x,y)\wedge\lambda^*,$$

satisfy $x(0)<1$ and $y(0)<\lambda^*$ see Remark 2.1 . By the right continuous of the map $t\to(F(t;x,y),G(t;x,y))$ there exists $t_1>0$ such that, for any $0\le t\le t_1$,

$$x(t)=F(t,x,y)<1,\qquad \text{and}\quad y(t)=G(t,x,y)<\lambda^*.$$

Consequently, the solution of (4.13) is also a solution of (3.3) on $[0,t_1]$.

Define the (possible) first exit time from the truncated domain by

$$\tau:=\inf\Big\{t\ge 0:\ \big(F(t,x,y)-1\big)\ \vee\ \big(G(t,x,y)-\lambda_*\big)\ge 0\Big\}.$$

Assume that $\tau<+\infty$. Then, for every $t\in[0,\tau)$,

$$F(t,x,y)<1\quad\text{and}\quad G(t,x,y)<\lambda^*,$$

and hence (x,y) is a solution of (3.3) on $[0,\tau)$.

Moreover, the mapping

$$t\longmapsto\mathbb{E}\Big[\exp\Big(-\kappa\int_0^t\gamma_0(r)\,V(r,y)\,dr\Big)\Big]+\kappa\int_0^t\mathbb{E}\Big[\exp\Big(-\kappa\int_s^t\gamma(r-s)\,y_V(r)\,dr\Big)\Big]\,x(s)\,V(s,y)\,ds$$

is continuous and from Lemma 4.1 it is equals 1 on $[0,\tau)$. Hence

$$\mathbb{E}\Big[\exp\Big(-\kappa\int_0^\tau\gamma_0(r)\,V(r,y)\,dr\Big)\Big]+\kappa\int_0^\tau\mathbb{E}\Big[\exp\Big(-\kappa\int_s^\tau\gamma(r-s)\,V(r,y)\,dr\Big)\Big]\,x(s)\,V(s,y)\,ds=1.$$

So, we have $x(\tau)=1$ if and only if

$$\gamma_0(\tau)=1\text{ a.s., }\forall\,t\ge 0,\qquad\text{and}\qquad\gamma(r-s)=1\text{ a.s., }\forall\,r\in[s,\tau].$$

Likewise, $y(t)=\lambda^*$ if and only if

$$\lambda_0(\tau)=\lambda^*\text{ a.s., }\forall\,t\ge 0,\qquad\text{and}\qquad\lambda(\tau-s)=\lambda^*\text{ a.s., }\forall\,s\in[0,\tau].$$

Using Remark (2.1), we infer that

$$x(\tau)<1\qquad\text{and}\qquad y(\tau)<\lambda^*.$$

Hence

$$F(\tau,x,y)<1\quad\text{and}\quad G(\tau,x,y)<\lambda^*,$$

which contradicts the definition of τ.

So, necessarily $\tau=+\infty$. Consequently, for every $t\ge 0$, the solution of (4.13) is also a solution of (3.3), which proves the existence of a solution to (3.3) We now prove the continuity of the solutions $t\mapsto\big(\overline{\mathfrak{F}}_1(t),\,\overline{\mathfrak{S}}(t)\big)$ under the additional assumption.
To this end, we prove the continuity of $t\mapsto\big(F(t,x,y),\,G(t,x,y)\big)$.

1. **Continuity of** $t\mapsto F(t,x,y)$
   - **Continuity of the first expectation term in** $F(t,x,y)$**.**
     $\mathbb{E}\Big[\gamma_0(t)\exp\Big(-\kappa\int_0^t\gamma_0(r)\,\mathcal{V}(y)(r)\,dr\Big)\Big]$ is a.s. continuous and takes its values in $[0,1]$.

     Since we assume that $\gamma_0(t)$ is continuous in probability and takes also its values in $[0,t]$, it is clear that the first term in $F(t,x,y)$ is continuous.

- **Continuity of the integral term in $F(t,x,y)$.**
  For $t \in [0,T]$, define
$$I(t) := \int_0^t \Phi(t,s)\,ds, \qquad \Phi(t,s) := \Psi(t,s)\,x(s)\,\mathcal{V}(y)(s),$$
where
$$\Psi(t,s) := \mathbb{E}\Big[\gamma(t-s)\exp\Big(-\kappa\int_s^t \gamma(r-s)\,\mathcal{V}(y)(r)\,dr\Big)\Big], \qquad 0 \le s \le t \le T.$$
Fix $t \in [0,T]$ and let $(t_n)_{n\ge1} \subset [0,T]$ be such that $t_n \to t$. Choose $T^\star > t \vee \sup_n t_n$ We have, for all admissible (t,s),
$$|\Phi(t,s)| \le |\Psi(t,s)|\,|x(s)\mathcal{V}(y)(s)| \le \mathbb{E}[|\gamma(t-s)|]\,|x(s)\mathcal{V}(y)(s)| \le C_T.$$
For each n,
$$\begin{aligned}|I(t)-I(t_n)| &= \left|\int_0^t \Phi(t,s)\,ds - \int_0^{t_n}\Phi(t_n,s)\,ds\right| \\ &\le \left|\int_{t\wedge t_n}^{t\vee t_n}\Phi(t\vee t_n,s)\,ds\right| + \left|\int_0^{t\wedge t_n}\big(\Phi(t\vee t_n,s)-\Phi(t\wedge t_n,s)\big)\,ds\right| \\ &=: |A_n| + |B_n|.\end{aligned} \tag{4.21}$$
First
$$|A_n| \le \int_{t\wedge t_n}^{t\vee t_n} C_T\,ds = C_T\,|t-t_n| \xrightarrow[n\to\infty]{} 0.$$
Fix $s \in [0,t]$ and define
$$P(t,s) := \int_s^t \gamma(r-s)\,\mathcal{V}(y)(r)\,dr.$$
Then
$$|P(t\vee t_n,s)-P(t\wedge t_n,s)| \le \int_{t\wedge t_n}^{t\vee t_n}|\gamma(r-s)|\,|\mathcal{V}(y)(r)|\,dr \le C_T\,|t-t_n| \xrightarrow[n\to\infty]{} 0,$$
hence
$$\exp\big(-\kappa P(t\vee t_n,s)\big) - \exp\big(-\kappa P(t\wedge t_n,s)\big) \to 0.$$
γ is càdlàg on $[0,T^\star]$. Its set of discontinuity points is at most countable. Therefore, for ds-a.e. $s \in [0,t]$, the point $t-s$ is a continuity point of γ, and thus
$$\gamma(t_n-s) \to \gamma(t-s) \qquad \text{for } ds\text{-a.e. } s \in [0,t].$$
Combining with the continuity of the exponential factor in t, we obtain for ds-a.e. :
$$\gamma(t\vee t_n - s)e^{-\kappa P(t\vee t_n,s)} - \gamma(t\wedge t_n - s)e^{-\kappa P(t\wedge t_n,s)} \to 0.$$
By dominated convergence in $\mathbb{E}$, we get $\Psi(t\vee t_n,s)-\Psi(t\wedge t_n,s) \to 0$ for ds-a.e. s. Multiplying by $x(s)\mathcal{V}(y)(s)$ yields
$$\Phi(t\vee t_n,s)-\Phi(t\wedge t_n,s) \to 0 \qquad \text{for } ds\text{-a.e. } s \in [0,t].$$
Finally, for all such s,
$$|\Phi(t\vee t_n,s)-\Phi(t\wedge t_n,s)| \le |\Phi(t\vee t_n,s)| + |\Phi(t\wedge t_n,s)| \le 2C_T.$$
Moreover,
$$B_n = \int_0^t \mathbf{1}_{[0,t\wedge t_n]}(s)\,\big(\Phi(t\vee t_n,s)-\Phi(t\wedge t_n,s)\big)\,ds,$$
Hence , again by dominated convergence ,
$$B_n \xrightarrow[n\to\infty]{} 0.$$

2. **Continuity of $t \mapsto G(t,x,y)$**
   Fix $t \in [0,T]$ and let $t_n \to t$. Proceed as above:
$$G(t_n)-G(t) = \big(\bar\lambda_0(t_n)-\bar\lambda_0(t)\big) + \int_0^{t_n}\bar\lambda(t_n-s)x(s)\mathcal{V}(y)(s)\,ds - \int_0^t \bar\lambda(t-s)x(s)\mathcal{V}(y)(s)\,ds.$$
The first term tends to 0 by continuity of $\bar\lambda_0$. For the integral term, one repeats the decomposition (4.21) and uses the bound $|\bar\lambda(t-s)x(s)\mathcal{V}(y)(s)| \le C_T\lambda^*$. Hence G is continuous.

5 Proof of Theorem 3.2

This proof section is organized as follows. We start with Lemma 5.1, which provides an explicit representation of the unique solution to system (3.3) in terms of new counting processes; see (5.3) and (5.8). Next, Lemma 5.2 is used to characterize two key quantities associated to the vectors, namely the force of infection and the proportion of susceptible vectors. Finally, we show that the approximation errors arising from replacing the counting processes introduced in Subsection 2.5 by those constructed in (5.3) and (5.8) converge to 0 as N and N_V tend to infinity.

Let $(Q_k)_{1\le k\le N}$ and $(Q^V_\ell)_{1\le \ell\le N_V}$ be i.i.d. standard Poisson random measures on $\mathbb{R}^2_+$, which are mutually independent.
For the host population, the model can be written as follows:

$$\overline{\mathfrak{F}}_1^N(t) := \frac{1}{N}\sum_{k=1}^N \lambda_{k,B_k^N(t)}\big(\mathfrak{a}_k^N(t)\big), \tag{5.1}$$

$$\overline{\mathfrak{S}}^N(t) := \frac{1}{N}\sum_{k=1}^N \gamma_{k,B_k^N(t)}\big(\mathfrak{a}_k^N(t)\big), \tag{5.2}$$

where

$$B_k^N(t) := \int_0^t\int_0^\infty \mathbf{1}_{\{u\le \Upsilon_k^N(s^-)\}}\, Q_k(\mathrm{d}s,\mathrm{d}u). \tag{5.3}$$

Moreover we have,

$$\Upsilon_k^N(t) := \kappa\, \gamma_{k,B_k^N(t)}\big(\mathfrak{a}_k^N(t)\big)\, \overline{\mathfrak{F}}_2^{N_V}(t). \tag{5.4}$$

For vectors the model can be written as :

$$\bar{S}^{N_V}(t) = \frac{1}{N^V}\Big(\sum_{i=1}^{S^{N_V}(0)} \mathbf{1}_{\{t<\sigma_i\}}\mathbf{1}_{\{C_i^{N_V}(t)=0\}} + \sum_{i>S^{N_V}(0):\,\alpha_i\in(0,t]} \mathbf{1}_{\{\alpha_i\le t<\sigma_i\}}\,\mathbf{1}_{\{C_{\alpha_i}^{N_V}(t)=0\}}\Big).$$

$$\begin{aligned}\overline{\mathfrak{F}}_2^{N_V}(t) = \frac{1}{N^V}\Big(&\sum_{i=1}^{I^{N_V}(0)} \lambda_i^0(t)\,\mathbf{1}_{\{t<\sigma_i\}} + \sum_{i=1}^{S^{N_V}(0)} \lambda_i\big(t-\tau_i^{N_V}\big)\,\mathbf{1}_{\{t<\sigma_i\}}\\ &+ \sum_{\substack{k>S^{N_V}(0)\\ \alpha_k\in(0,t]}} \lambda_k^V\big(t-\tau_k^{N_V}\big)\,\mathbf{1}_{\{\alpha_k\le t<\sigma_k\}}\Big).\end{aligned} \tag{5.5}$$

where $\tau_i^{N_V}$ and $\tau_k^{N_V}$ are, respectively, the jump times of the processes

$$C_i^{N_V}(t) = \int_0^t\int_0^\infty \mathbf{1}_{\{C_i^{N_V}(s^-)=0\}}\,\mathbf{1}_{\Big\{u\le \frac{\overline{\mathfrak{F}}_1^N(s^-)}{\kappa}\Big\}}\, Q_i^V(ds,du),$$

and

$$C_{\alpha_k}^{N_V}(t) = \int_{\alpha_k}^t\int_0^\infty \mathbf{1}_{\{C_{\alpha_k}^{N_V}(s^-)=0\}}\,\mathbf{1}_{\Big\{u\le \frac{\overline{\mathfrak{F}}_1^N(s^-)}{\kappa}\Big\}}\, Q_k^V(ds,du).$$

We now define processes $\{(B,\varsigma)\}$ and $\{(C,\vartheta)\}$ using the *same* PRMs (Q_k) and (Q^V_ℓ).
For hosts,

$$B_k(t) = \int_0^t\int_0^\infty \mathbf{1}_{\{u\le \Upsilon_k(s^-)\}}\, Q_k(\mathrm{d}s,\mathrm{d}u), \qquad \Upsilon_k(s) = \kappa\,\gamma_{B_k(s)}\big(\mathfrak{a}_k(s)\big)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(s), \tag{5.6}$$

Where the *age* of infection $\mathfrak{a}_k$ (or elapsed time since the last jump of B) is defined by

$$\mathfrak{a}_k(t) := t - \sup\Big\{\, s\in(0,t] : B_k(s) = B_k(s^-)+1\Big\}. \tag{5.7}$$

Then $\{(B_k,\mathfrak{a}_k)\}_{k\ge 1}$ is an i.i.d. family.
For vectors,

$$\begin{cases} C_i(t) = \displaystyle\int_0^t\int_0^\infty \mathbf{1}_{C_i(s^-)=0}\,\mathbf{1}_{u\le \frac{\overline{\mathfrak{F}}_1(s)}{\kappa}}\, Q_i(ds,du),\\[2ex] C_{\alpha_k}(t) = \displaystyle\int_{\alpha_k}^t\int_0^\infty \mathbf{1}_{C_{\alpha_k}(s^-)=0}\,\mathbf{1}_{u\le \frac{\overline{\mathfrak{F}}_1(s)}{\kappa}}\, Q_{S^{N_V}(0)+k}(ds,du).\end{cases} \tag{5.8}$$

Lemma 5.1. *The equation* (5.6) *admits a unique càdlàg solution* $B_k(t)$. *Moreover, the associated pair :*

$$\Big(\mathbb{E}\big[\gamma_{B_k(t)}(\mathfrak{a}_k(t))\big],\, \mathbb{E}\big[\lambda_{B_k(t)}(\mathfrak{a}_k(t))\big]\Big).$$

is the unique solution to the limiting system (3.3).

Proof of Lemma 5.1 . Existence and uniqueness is quite elementary . We leave its verification to the reader.
Filtration and infection times. For each host k we work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ supporting the Poisson random measure Q_k and the i.i.d. marks $(\lambda_{k,i}, \gamma_{k,i})_{i\geq 0}$. We denote by $(\mathcal{F}_t)_{t\geq 0}$ the filtration generated by these sources of randomness, i.e.

$$\mathcal{F}_t := \sigma\Big((\lambda_{k,i}, \gamma_{k,i})_{i\leq B_k(t)},\, Q_k|[0,t]\times\mathbb{R}_+\Big).$$

Let τ_i denote the i-th jump time of the counting process B_k, namely

$$\tau_i := \inf\{t \geq 0 :\; B_k(t) \geq i\}, \qquad i \geq 1,$$

with the convention $\inf\varnothing = +\infty$. Then $(\tau_i)_{i\geq 1}$ is a sequence of $(\mathcal{F}_t)_{t\geq 0}$-stopping times, and the restriction $Q_k|[\tau_i, t]\times\mathbb{R}_+$ is independent of $\mathcal{F}_{\tau_i}$. Hence,

$$\mathbb{P}(B_k(t) = i \,|\, \mathcal{F}_{\tau_i}) = \exp\Big(-\int_{\tau_i}^t \gamma_i(r-\tau_i)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(s)\,\mathrm{d}r\Big)\,.$$

We denote by ϱ the law of γ,

Step 1: Proof of $\mathbb{E}\big[\gamma_{B_k(t)}(\zeta(t))\big] = \overline{\mathfrak{S}}_1(t)$.

$$\gamma_{B_k(t)}(\zeta(t)) = \gamma_0(t)\mathbf{1}_{\{B_k(t)=0\}} + \sum_{i\geq 1}\gamma_i(t-\tau_i)\mathbf{1}_{\{B_k(t)=i\}},$$

$$\mathbb{E}\big[\gamma_0(t)\,\mathbf{1}_{\{B_k(t)=0\}}\big] = \mathbb{E}[\gamma_0(t)\,\mathbb{P}(B_k(t)=0\,|\,\gamma_0)] = \mathbb{E}\Big[\gamma_0(t)\exp\Big(-\kappa\int_0^t \gamma_0(r)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\Big]\,.$$

$$\begin{aligned}
\mathbb{E}\Big[\sum_{i\geq 1}\gamma_i(t-\tau_i)\,\mathbf{1}_{\{B_k(t)=i\}}\Big] &= \sum_{i\geq 1}\mathbb{E}\Big[\gamma(t-\tau_i)\exp\Big(-\kappa\int_{\tau_i}^t \gamma(r-\tau_i)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\Big]\\
&= \sum_{i\geq 1}\mathbb{E}\Big[\int_D \gamma(t-\tau_i)\exp\Big(-\kappa\int_{\tau_i}^t \gamma(r-\tau_i)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\;\varrho(d\gamma)\Big]\\
&= \mathbb{E}\Big[\int_0^t \mathbb{E}\Big[\gamma(t-s)\exp\Big(-\kappa\int_s^t \gamma(r-s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\Big]\;dB_k(s)\Big]\\
&= \kappa\mathbb{E}\Big[\int_0^t \gamma(t-s)\exp\Big(-\int_s^t \gamma(r-s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\;\mathbb{E}\big[\gamma_{B_k(s)}(\mathfrak{a}(s))\big]\,\mathcal{V}(\overline{\mathfrak{F}}_1)(s)\,ds\Big]\,.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\big[\gamma_{B_k(t)}(\mathfrak{a}(t))\big] &= \mathbb{E}\Big[\gamma_0(t)\exp\Big(-\kappa\int_0^t \gamma_0(r)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\Big]\\
&\quad + \mathbb{E}\Big[\int_0^t \gamma(t-s)\exp\Big(-\kappa\int_s^t \gamma(r-s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\;\kappa\mathbb{E}\big[\gamma_{B_k(s)}(\mathfrak{a}(s))\big]\,\mathcal{V}(\overline{\mathfrak{F}}_1)(s)\,ds\Big]\,.
\end{aligned}$$

This proves that $\mathbb{E}\big[\gamma_{B_k(t)}(\mathfrak{a}(t))\big] = \overline{\mathfrak{S}}(t)$, as a consequence of the uniqueness of a solution to the above linear integral equation, given that $\overline{\mathfrak{F}}_2(t) \leq \lambda^*$ for all $t \geq 0$.

Step 2: Proof of $\mathbb{E}\big[\lambda_{B_k(t)}(\mathfrak{a}(t))\big] = \overline{\mathfrak{F}}_1(t)$.

$$\begin{aligned}\lambda_{B_k(t)}\big(\zeta(t)\big) &= \lambda_0(t)\,\mathbf{1}_{\{B_k(t)=0\}} + \sum_{i\geq 1}\lambda_i(t-\tau_i)\,\mathbf{1}_{\{B_k(t)=i\}}\\ &= \lambda_0(t) + \sum_{i\geq 1}\lambda_i(t-\tau_i)\,.\end{aligned}$$

We have

$$\begin{aligned}\mathbb{E}\big[\lambda_{B_k(t)}\big(\zeta(t)\big)\big] &= \mathbb{E}[\lambda_0(t)] + \mathbb{E}\left[\sum_{i\geq 1}\lambda_i\big(t-\tau_i\big)\right]\\ &= \bar{\lambda}_0(t) + \mathbb{E}\left[\sum_{i\geq 1}\bar{\lambda}\big(t-\tau_i\big)\right]\\ &= \bar{\lambda}_0(t) + \mathbb{E}\left[\int_0^t \bar{\lambda}(t-s)\,\mathrm{d}B_k(s)\right]\\ &= \bar{\lambda}_0(t) + \int_0^t \bar{\lambda}(t-s)\,\kappa\mathbb{E}\big[\gamma_{B_k(s)}\big(\zeta(s)\big)\big]\ \mathcal{V}(\overline{\mathfrak{F}}_1)(s)\,\mathrm{d}s\\ &= \bar{\lambda}_0(t) + \kappa\int_0^t \bar{\lambda}(t-s)\,\overline{\mathfrak{S}}(s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(s)\,\mathrm{d}s,\end{aligned}$$

where we have used the result of step 1 . Hence the result. □

We now define $\widetilde{S}^{N_V}(t)$ and $\widetilde{\mathfrak{F}}_2^{N_V}(t)$ by replacing $C_i^{N_V}$ with C_i in $\bar{S}^{N_V}(t)$ and $\overline{\mathfrak{F}}_2^{N_V}(t)$.

Lemma 5.2. *As $N, N_V \to \infty$, with $\frac{N_V}{N} = \kappa$,*

$$\widetilde{S}^{N_V}(t) \to \mathcal{S}\big(\overline{\mathfrak{F}}_1\big)(t) \quad \text{and} \quad \widetilde{\mathfrak{F}}_2^{N_V}(t) \to \mathcal{V}\big(\overline{\mathfrak{F}}_1\big)(t).$$

a.s. in D

Proof of Lemma 5.2. Step 1: $\widetilde{S}^{N_V}(t)$

We have :

$$\begin{aligned}\mathbb{E}[C_i(t)] &= \mathbb{E}\int_0^t\int_0^\infty \mathbf{1}_{C(s^-)=0}\mathbf{1}_{u\leq\frac{\overline{\mathfrak{F}}_1(s^-)}{\kappa}}Q_i^V(ds,du)\\ &= \frac{1}{\kappa}\int_0^t \exp\left(-\frac{1}{\kappa}\int_0^s \overline{\mathfrak{F}}_1(r)dr\right)\overline{\mathfrak{F}}_1(s)ds,\end{aligned}$$

We introduce the following two quantities:

$$\widetilde{S}_1(t) := \frac{1}{N^V}\sum_{i=1}^{S^{N_V}(0)}\mathbf{1}_{\{t<\sigma_i\}}\,\mathbf{1}_{\{C_i(t)=0\}},\qquad \widetilde{S}_2(t) := \frac{1}{N_V}\sum_{k\geq 1}\mathbf{1}_{\alpha_k\leq t<\sigma(\alpha_k)}\mathbf{1}_{C_{\alpha_k}(t)=0}$$

On the event $\{\alpha_i < t\}$

$$\mathbb{P}(C_{\alpha_i}(t)=0\,|\,\alpha_i) = \exp\bigg(-\frac{1}{\kappa}\int_{\alpha_i}^t \overline{\mathfrak{F}}_1(r)\,\mathrm{d}r\bigg)\,,$$

Similarly, the survival probability up to time t is given by

$$\mathbb{P}(t<\sigma_i|\alpha_i) = \exp\bigg(-\int_{\alpha_i}^t \mu_V(r)\,dr\bigg)\,.$$

Since the infection and death mechanisms are independent, we obtain for $i < S^{V_N}(0)$

$$\mathbb{P}(t<\sigma_i, C_{\alpha_i}(t)=0) = \exp\bigg(-\int_0^t\Big[\mu_V(r)+\frac{\overline{\mathfrak{F}}_1(r)}{\kappa}\Big]dr\bigg)\,.$$

By linearity of the expectation,

$$\mathbb{E}[\widetilde{S}_1(t)] = \bar{S}^{N_V}(0)\,\exp\Big(-\int_0^t \Big[\mu_V(r) + \frac{\overline{\mathfrak{F}}_1(r)}{\kappa}\Big]dr\Big)\,.$$

By the law of large numbers in D (see [9]), we have that, as $N_V \to \infty$,

$$\tilde{S}_1^{N_V}(t) \to \bar{S}^{N_V}(0)\exp\left(-\int_0^t \left[\mu_V(r) + \frac{\overline{\mathfrak{F}}_1(r)}{\kappa}\right]dr\right)$$

a.s. in D.

Next, $\{\alpha_k,\ k \geq 1\}$ denoting the points of a Poisson process with intensity $N_V\mu_V(t)$ on $\mathbb{R}_+$, we consider :

$$\begin{aligned}\widetilde{S}_2(t) : &= \frac{1}{N^V}\sum_{k\geq 1}\mathbf{1}_{\{\alpha_k\leq t<\sigma_k(\alpha_k)\}}\,\mathbf{1}_{\{C_{\alpha_k}(t)=0\}}.\\ &= \frac{1}{N_V}\int_{[0,t]\times\mathbb{R}_+^3}\mathbf{1}_{u\leq N_v\mu_V(s)}\mathbf{1}_{t-s<\sigma}\mathbf{1}_{t-s<\tau}Q(ds,du,d\sigma,d\tau),\end{aligned}$$

where Q is a PRM on $\mathbb{R}_+^4$ with mean measure

$$ds\,du\,\mathrm{Exp}_{\mu_V(s+\cdot)}(d\sigma)\,\mathrm{Exp}_{\kappa^{-1}\overline{\mathfrak{F}}_1(s+\cdot)}(d\tau).$$

$$\mathbb{E}\Big[\widetilde{S}_2(t)\Big] = \int_0^t \mu_V(s)\,\mathbb{E}\big[\mathbf{1}_{\{t<\sigma(s)\}}\mathbf{1}_{\{C_s(t)=0\}}\big]\,\mathrm{d}s.$$

It is clear that :

$$\mathbb{E}\Big[\widetilde{S}_2(t)\Big] = \int_0^t \mu_V(s)\,\exp\Big(-\int_s^t \Big(\mu_V(r) + \frac{\overline{\mathfrak{F}}_1(r)}{\kappa}\Big)\,\mathrm{d}r\Big)\,\mathrm{d}s.$$

Now the wished convergence a.s. in D of $\tilde{S}_2^{N_V}$ towards

$$\int_0^{\cdot} \mu_V(s)\exp\left(-\int_s^t \left[\mu_V(r) + \kappa^{-1}\overline{\mathfrak{F}}_1(r)\right]dr\right)$$

follows again from Rao's [9] LLN in D, if we note that

$$\begin{aligned}&\frac{1}{N_V}\int_{[0,t]\times\mathbb{R}_+^3}\mathbf{1}_{u\leq N_v\mu_V(s)}\mathbf{1}_{t-s<\sigma}\mathbf{1}_{t-s<\tau}Q(ds,du,d\sigma,d\tau)\\ &\qquad = \frac{1}{N_V}\sum_{k=1}^{N_V}\int_{[0,t]\times\mathbb{R}_+^3}\mathbf{1}_{(k-1)\mu_V(s)<u\leq k\mu_V(s)}\mathbf{1}_{t-s<\sigma}\mathbf{1}_{t-s<\tau}Q(ds,du,d\sigma,d\tau),\end{aligned}$$

and from the well-known properties of Poisson random measures, the above sum is a sum of i.i.d. r.v.'s.

Hence $\widetilde{S}^{N_V}(t) \to \mathcal{S}\big(\overline{\mathfrak{F}}_1\big)(t)$ a.s. in D

Step 2: $\widetilde{\mathfrak{F}}_2^{N_V}(t)$

We next consider the three terms $\tilde{\mathfrak{F}}_{2,0}^{N_V}(t)$, $\tilde{\mathfrak{F}}_{2,1}^{N_V}(t)$ and $\tilde{\mathfrak{F}}_{2,2}^{N_V}(t)$.

We have

$$\tilde{\mathfrak{F}}_{2,0}^{N_V}(t) := \frac{1}{N_V}\sum_{j=1}^{I^{N_V}(0)}\lambda_j^{V,0}(t)\mathbf{1}_{t<\sigma_{-j}}\,,$$

where $\sigma_{-1}, \sigma_{-2}, \ldots$ stand for the death times of the initially infected individuals.

We clearly have the following a.s. convergence in D, as $N_V \to \infty$,

$$\tilde{\mathfrak{F}}_{2,0}^{N_V}(t) \to \bar{I}_V(0)\bar{\lambda}_0^V(t)\exp\left(-\int_0^t \mu_V(r)dr\right)\,.$$

Concerning the second term, as $N_V \to \infty$,

$$\widetilde{\mathfrak{F}}^{N_V}_{2,1}(t) = \frac{1}{N^V} \sum_{i=1}^{S^{N_V}(0)} \lambda^V_i\big(t-\tau^V_i\big)\,\mathbf{1}_{\{t<\sigma_i\}} = \frac{1}{N^V} \sum_{i\ge 1}^{S^{N_V}(0)} \int_0^t \lambda^V_i(t-s)\mathbf{1}_{\{t<\sigma_i\}}\,C_i(ds)$$

$$\longrightarrow \bar{S}_V(0)\,\mathbb{E}\Big[\int_0^t \lambda^V_1(t-s)\mathbf{1}_{\{s<\sigma_i\}}\exp\Big(-\int_0^s \frac{\overline{\mathfrak{F}}_1(r)}{\kappa}\,dr\Big)\,\frac{\overline{\mathfrak{F}}_1(s)}{\kappa}\,ds\Big]$$

$$= \bar{S}_V(0)\int_0^t \bar{\lambda}^V(t-s)\,\exp\Big(-\int_s^t \mu_V(r)\,dr\Big)\exp\Big(-\int_0^s \mu_V(r)+\frac{\overline{\mathfrak{F}}_1(r)}{\kappa}\,dr\Big)\,\frac{\overline{\mathfrak{F}}_1(s)}{\kappa}\,ds,$$

For the last term in $\widetilde{\mathfrak{F}}^{N_V}_{2,2}(t)$, we will need to introduce the following formalism.

Let $Q'(\mathrm{d}s,\mathrm{d}u,\mathrm{d}\sigma,\mathrm{d}\tau,\mathrm{d}\lambda)$ be a Poisson random measure (PRM) on $\mathbb{R}^4_+\times D$, with mean (intensity) measure :

$$\bar{Q}'(\mathrm{d}s,\mathrm{d}u,\mathrm{d}\sigma,\mathrm{d}\tau,\mathrm{d}\lambda) = \mathrm{d}u\,\mathrm{d}s\,\mathrm{Exp}_{\mu_V(s+\cdot)}(\mathrm{d}\sigma)\,\mathrm{Exp}_{\frac{\overline{\mathfrak{F}}_1(t)}{\kappa}(s+\cdot)}(\mathrm{d}\tau)\,\nu_V(\mathrm{d}\lambda).$$

Note that the PRM Q introduced in Step 1 of the proof is the projection of Q' on $\mathbb{R}^4_+$.

Now,

$$\widetilde{\mathfrak{F}}^{N_V}_{2,2}(t) = \frac{1}{N_V}\sum_{\substack{i>S^{N_V}(0)\\ \alpha_i\in(0,t]}} \lambda^V_i\big(t-\tau^V_i\big)\,\mathbf{1}_{\{\alpha_i\le t<\sigma(\alpha_i)\}},$$

and

$$\frac{1}{N_V}\sum_{\substack{i>S^{N_V}(0)\\ \alpha_i\in(0,t]}} \lambda^V_i\big(t-\tau^V_i\big)\,\mathbf{1}_{\{\alpha_i\le t<\sigma_i\}} = \frac{1}{N^V}\int_{[0,t]\times\mathbb{R}^3_+\times D} \mathbf{1}_{\{u\le N_V\mu_V(s)\}}\,\lambda^V(t-\tau-s)\,\mathbf{1}_{\{\tau\le t-s<\sigma\}}\,Q'(\mathrm{d}s,\mathrm{d}u,\mathrm{d}\sigma,\mathrm{d}\tau,\mathrm{d}\lambda).$$

We have :

$$\mathbb{E}\Bigg[\frac{1}{N_V}\int_{[0,t]\times\mathbb{R}^3_+\times D} \mathbf{1}_{\{u\le N_V\mu_V(s)\}}\,\lambda^V(t-s-\tau)\,\mathbf{1}_{\{\tau\le t-s<\sigma\}}\,Q'(\mathrm{d}s,\mathrm{d}u,\mathrm{d}\sigma,\mathrm{d}\tau,\mathrm{d}\lambda)\Bigg]$$

$$= \int_0^t\int_s^t \mu_V(s)\,\bar{\lambda}^V(t-r)\,\exp\Big(-\int_s^t \mu_V(v)\,\mathrm{d}v\Big)\,\exp\Big(-\int_s^r \frac{\overline{\mathfrak{F}}_1(u)}{\kappa}\,\mathrm{d}u\Big)\,\frac{\overline{\mathfrak{F}}_1(r)}{\kappa}\,\mathrm{d}r\,\mathrm{d}s,\qquad \text{a.s.}$$

Again, we easily obtain that, as $N_V\to\infty$, a.s. in D,

$$\widetilde{\mathfrak{F}}^{N_V}_{2,2}(t) \to \int_0^t\int_s^t \mu_V(s)\bar{\lambda}^V(t-r)\exp\Big(-\int_s^t \mu_V(v)dv\Big)\,\frac{\overline{\mathfrak{F}}_1(r)}{\kappa}\exp\Big(-\int_s^r \frac{\overline{\mathfrak{F}}_1(u)}{\kappa}du\Big)\,drds\,.$$

When combinig (3.1) and (3.2) we see that $\mathcal{V}(y)$ is the sum of three terms.We have shown that $\mathbb{E}(\widetilde{\mathfrak{F}}^{N_V}_{2,0}(t)), \mathbb{E}(\widetilde{\mathfrak{F}}^{N_V}_{2,1}(t)$ and $\mathbb{E}(\widetilde{\mathfrak{F}}^{N_V}_{2,2}(t))$ are respectively equal to the first, the second and the third term in $\mathcal{V}(\overline{\mathfrak{F}}_1)$. Finally, we have shown that $\widetilde{\mathfrak{F}}^{N_V}_2(t) \to \mathcal{V}\big(\overline{\mathfrak{F}}_1\big)(t)$ a.s. in D. □

Completing the proof of Theorem 3.2.

Now, we introduce the *disagreement counting process*

$$\Delta^H_k(t) := \int_0^t\int_0^\infty \mathbf{1}_{\Big\{\min\big(\Upsilon^N_k(s^-),\Upsilon_k(s^-)\big)<u\le\max\big(\Upsilon^N_k(s^-),\Upsilon_k(s^-)\big)\Big\}}\,Q_k(ds,du).$$

By construction of the coupling, we have

$$\big|B^N_k(t)-B_k(t)\big| \le \Delta^H_k(t),\qquad t\ge 0.$$

Moreover,since t $\to \Delta^H_k$ is nondecreasing

$$\mathbb{E}\Bigg[\sup_{r\in[0,t]}\big|B^N_k(r)-B_k(r)\big|\Bigg] \le \mathbb{E}\big[\Delta^H_k(t)\big]$$

$$= \int_0^t \mathbb{E}\big[\big|\Upsilon^N_k(s)-\Upsilon_k(s)\big|\big]\,ds. \tag{5.9}$$

Recall that

$$\Upsilon_k^N(t) := \kappa\, \gamma_{k,B_k^N(t)}\big(\mathfrak{a}_k^N(t)\big)\, \overline{\mathfrak{F}}_2^{N_V}(t), \qquad \Upsilon_k(t) := \kappa\, \gamma_{k,B_k(t)}\big(\mathfrak{a}_k(t)\big)\, \overline{\mathfrak{F}}_2(t).$$

We have $\gamma_{k,i} \le 1$ for all i, and for all $t \ge 0, 0 \le \overline{\mathfrak{F}}_2^{N_V}(t)$, $\overline{\mathfrak{F}}_2(t) \le \lambda^*$. Then, for all $t \ge 0$,

$$\mathbb{E}\big[\big|\Upsilon_k^N(t) - \Upsilon_k(t)\big|\big] \le \mathbb{E}\Big[\big|\Upsilon_k^N(t) - \Upsilon_k(t)\big|\, \mathbf{1}_{\{B_k^N(t) = B_k(t),\, \mathfrak{a}_k^N(t) = \mathfrak{a}_k(t)\}}\Big] + \lambda^*\, \mathbb{P}\big(B_k^N(t) \neq B_k(t) \text{ or } \mathfrak{a}_k^N(t) \neq \mathfrak{a}_k(t)\big)\,. \tag{5.10}$$

On the event $\{B_k^N(t) = B_k(t),\, \mathfrak{a}_k^N(t) = \mathfrak{a}_k(t)\}$ we have $\gamma_{k,B_k^N(t)}(\mathfrak{a}_k^N(t)) = \gamma_{k,B_k(t)}(\mathfrak{a}_k(t)) \le 1$, hence

$$\mathbb{E}\Big[\big|\Upsilon_k^N(t) - \Upsilon_k(t)\big|\, \mathbf{1}_{\{B_k^N(t) = B_k(t),\, \mathfrak{a}_k^N(t) = \mathfrak{a}_k(t)\}}\Big] \le \kappa\, \mathbb{E}\Big[\big|\overline{\mathfrak{F}}_2^{N_V}(t) - \overline{\mathfrak{F}}_2(t)\big|\Big]. \tag{5.11}$$

$$\begin{aligned}
\mathbb{E}\Big[\Big|\overline{\mathfrak{F}}_2^{N_V}(t) - \overline{\mathfrak{F}}_2(t)\Big|\Big] \le{}& \mathbb{E}\left[\left|\frac{1}{N^V} \sum_{j=1}^{I^{N_V}(0)} \Big(\lambda_j^{V,0}(t)\mathbf{1}_{\{t<\sigma_j\}} - \mathbb{E}\big[\lambda_j^{V,0}(t)\mathbf{1}_{\{t<\sigma_j\}}\big]\Big)\right|\right] + \lambda^*\, \mathbb{E}\Big[\big|\overline{I}^{N_V}(0) - \overline{I}(0)\big|\Big] \\
&+ \mathbb{E}\left[\left|\frac{1}{N^V} \sum_{j=1}^{S^{N_V}(0)} \Big(\lambda_j^V\big(t - \tau_j^{N_V}\big)\mathbf{1}_{\{t<\sigma_j\}} - \mathbb{E}\big[\lambda_j^V\big(t - \tau_j^V\big)\mathbf{1}_{\{t<\sigma_j\}}\big]\Big)\right|\right] + \lambda^*\, \mathbb{E}\Big[\big|\overline{S}^{N_V}(0) - \overline{S}(0)\big|\Big] \\
&+ \mathbb{E}\left[\left|\frac{1}{N^V} \sum_{\substack{k > IN_V(0) + SN_V(0) \\ \alpha_k \in (0,t]}} \Big(\lambda_k^V\big(t - \tau_k^{N_V}\big)\mathbf{1}_{\{\alpha_k \le t < \sigma(\alpha_k)\}} - \lambda_k^V\big(t - \tau_k^V\big)\mathbf{1}_{\{\alpha_k \le t < \sigma_k\}}\Big)\right|\right] \\
&+ \mathbb{E}\left[\frac{1}{N_V}\left|\int_{[0,t]\times\mathbb{R}_+^3\times D} \mathbf{1}_{\{u \le N_V \mu_V(s)\}}\, \lambda(t - s - \tau)\, \mathbf{1}_{\{\tau \le t - s < \sigma\}}\, \big(Q' - \overline{Q'}\big)(\mathrm{d}s, \mathrm{d}u, \mathrm{d}\sigma, \mathrm{d}\tau, \mathrm{d}\lambda)\right|\right].
\end{aligned} \tag{5.12}$$

Lemma 5.3.

$$\mathbb{E}\left[\frac{1}{N_V}\left|\int_{[0,t]\times\mathbb{R}_+^3\times D} \mathbf{1}_{\{u \le N_V \mu_V(s)\}}\, \lambda(t - \tau - s)\, \mathbf{1}_{\{\tau \le t - s < \sigma\}}\, \big(Q' - \overline{Q'}\big)(\mathrm{d}s, \mathrm{d}u, \mathrm{d}\sigma, \mathrm{d}\tau, \mathrm{d}\lambda)\right|\right] \xrightarrow[N_V \to \infty]{} 0,$$

proof of Lemma 5.3. Define

$$f_{N_V}(s, u, \sigma, \tau, \lambda) := \mathbf{1}_{\{u \le N_V \mu_V(s)\}}\, \lambda(t - \tau)\, \mathbf{1}_{\{\tau \le t < \sigma\}},$$

and

$$M_{N_V}(t) := \int_{[0,t]\times\mathbb{R}_+^3\times D} f_{N_V}(s, u, \sigma, \tau, \lambda)\, \big(Q' - \overline{Q'}\big)(\mathrm{d}s, \mathrm{d}u, \mathrm{d}\sigma, \mathrm{d}\tau, \mathrm{d}\lambda).$$

So we have

$$\mathbb{E}\big[|M_{N_V}(t)|^2\big] = \int_{[0,t]\times\mathbb{R}_+^3\times D} f_{N_V}(s, u, \sigma, \tau, \lambda)^2\, \overline{Q'}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}\sigma, \mathrm{d}\tau, \mathrm{d}\lambda).$$

by Cauchy–Schwarz,

$$\mathbb{E}\left[\frac{1}{N_V}|M_{N_V}(t)|\right] \le \frac{1}{N_V}\Big(\mathbb{E}[|M_{N_V}(t)|^2]\Big)^{1/2}.$$

Therefore,

$$\begin{aligned}
\mathbb{E}\big[|M_{N_V}(t)|^2\big] &\le (\lambda^*)^2 \int_{[0,t]\times\mathbb{R}_+^3\times D} \mathbf{1}_{\{u \le N_V \mu_V(s)\}}\, \overline{Q'}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}\sigma, \mathrm{d}\tau, \mathrm{d}\lambda) \\
&= (\lambda^*)^2 \int_0^t \big(N_V \mu_V(s)\big)\, \mathrm{d}s = (\lambda^*)^2 N_V \int_0^t \mu_V(s)\, \mathrm{d}s,
\end{aligned}$$

Consequently,

$$\mathbb{E}\left[\frac{1}{N_V}|M_{N_V}(t)|\right] \le \frac{1}{N_V}\, \lambda^* \sqrt{N_V \int_0^t \mu_V(s)\, \mathrm{d}s} = \frac{\lambda^*}{\sqrt{N_V}} \sqrt{\int_0^t \mu_V(s)\, \mathrm{d}s} \xrightarrow[N_V \to \infty]{} 0,$$

□

It follows from our assumptions that as $N_V \to +\infty$,

$$\mathbb{E}\Big[\big|\overline{I}^{N_V}(0) - \overline{I}(0)\big|\Big] + \mathbb{E}\Big[\big|\overline{S}^{N_V}(0) - \overline{S}(0)\big|\Big]$$
$$\mathbb{E}\left[\frac{1}{N_V}\left|\int_{[0,t]\times\mathbb{R}_+^3\times D} \mathbf{1}_{\{u\le N_V\mu_V(s)\}}\,\lambda(t-\tau)\,\mathbf{1}_{\{\tau\le t<\sigma\}}\,\big(Q'-\overline{Q'}\big)(\mathrm{d}s,\mathrm{d}u,\mathrm{d}\sigma,\mathrm{d}\tau,\mathrm{d}\lambda)\right|\right] \xrightarrow[N_V\to\infty]{} 0. \tag{5.13}$$

For each $t \geq 0$, the $\lambda_j^0(t)$ are i.i.d. and bounded, and globally independent of $I^N(0)$, hence, from the classical strong law of large numbers ,

$$\mathbb{E}\left[\left|\frac{1}{N_V}\sum_{j=1}^{I^{N_V}(0)}\Big(\lambda_j^{V,0}(t)\mathbf{1}_{\{t<\sigma_j\}} - \mathbb{E}[\lambda_j^{V,0}(t)\mathbf{1}_{\{t<\sigma_j\}}]\Big)\right|\right] \xrightarrow[N_V\to\infty]{} 0. \tag{5.14}$$

Second

$$\begin{aligned}
&\mathbb{E}\left[\left\|\frac{1}{N_V}\sum_{j=1}^{S^{N_V}(0)}\Big(\lambda_j\big(t-\tau_j^{N_V}\big)\mathbf{1}_{\{t<\sigma_j\}} - \mathbb{E}\big[\lambda_j\big(t-\tau_j^{V}\big)\mathbf{1}_{\{t<\sigma_j\}}\big]\Big)\right\|\right] \\
&\le \mathbb{E}\left[\left\|\frac{1}{N}\sum_{j=1}^{S^{N_V}(0)}\Big(\lambda_j^V\big(t-\tau_j^{N_V}\big) - \lambda_j\big(t-\tau_j^{V}\big)\Big)\mathbf{1}_{\{t<\sigma_j\}}\right\|\right] \\
&\quad + \mathbb{E}\left[\left\|\frac{1}{N}\sum_{j=1}^{S^{N_V}(0)}\Big(\lambda_j^V\big(t-\tau_j^{V}\big)\mathbf{1}_{\{t<\sigma_j\}} - \mathbb{E}\big[\lambda_j^V\big(t-\tau_j^{V}\big)\mathbf{1}_{\{t<\sigma_j\}}\big]\Big)\right\|\right].
\end{aligned} \tag{5.15}$$

By the same argument as that used for establishing (5.14), we get that, as $N_V \to \infty$,

$$\mathbb{E}\left[\left\|\frac{1}{N_V}\sum_{j=1}^{S^{N_V}(0)}(\lambda_j^V(t-\tau_j^V)\mathbf{1}_{\{t<\sigma_j\}} - \mathbb{E}[\lambda^V(t-\tau_j^V)\mathbf{1}_{\{t<\sigma_j\}}])\right\|\right] \to 0. \tag{5.16}$$

In order to bound the first term on the right of (5.15), we first note that

$$|\lambda_j^V(t-\tau_j^{N_V}) - \lambda_j^V(t-\tau_j^V)| \le \lambda^*\mathbf{1}_{\tau_j^{V,N_V}\wedge t\neq\tau_j^V\wedge t}$$

$$\begin{aligned}
\mathbb{E}[|\lambda_j^V(t-\tau_j^{N_V}) - \lambda_j^V(t-\tau_j^V)|] &\le \lambda^*\mathbb{P}(\tau_j^{N_V}\wedge t \neq \tau_j^V\wedge t) \\
&= \lambda^*\mathbb{P}\left(\sup_{0\le r\le t}|C_j^N(r) - C_j(t)| \ge 1\right) \\
&\le \lambda^*\mathbb{E}\left[\sup_{0\le r\le t}|C_j^N(r) - C_j(t)|\right]
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
\varepsilon_{N_V}(t) := \mathbb{E}\Big[\big|\overline{I}^{N_V}(0) - \overline{I}(0)\big|\Big] &+ \mathbb{E}\Big[\big|\overline{S}^{N_V}(0) - \overline{S}(0)\big|\Big] \\
&+ \mathbb{E}\left[\frac{1}{N_V}\left|\int_{[0,t]\times\mathbb{R}_+^3\times D} \mathbf{1}_{\{u\le N_V\mu_V(s)\}}\,\lambda(t-\tau)\,\mathbf{1}_{\{\tau\le t<\sigma\}}\,\big(Q'-\overline{Q'}\big)(\mathrm{d}s,\mathrm{d}u,\mathrm{d}\sigma,\mathrm{d}\tau,\mathrm{d}\lambda)\right|\right]. \\
&+ \mathbb{E}\left[\left\|\frac{1}{N_V}\sum_{j=1}^{I^{N_V}(0)}\big(\lambda_j^0(t) - \overline{\lambda}^0(t)\big)\right\|\right] \\
&+ \mathbb{E}\left[\left\|\frac{1}{N_V}\sum_{j=1}^{S^{N_V}(0)}\Big(\lambda_j^V(t-\tau_j^V)\mathbf{1}_{\{t<\sigma_j\}} - \mathbb{E}\Big[\overline{\lambda}^V(t-\tau_j^V)\mathbf{1}_{\{t<\sigma_j\}}\Big]\Big)\right\|\right]
\end{aligned}$$

.

We have that for all $t \geq 0$, $\varepsilon_{N_V}(t) \to 0$ as $N_V \to \infty$, and moreover $\varepsilon_N(t) \leq 2 + 3\lambda^*$, hence also $\int_0^t \varepsilon_{N_V}(s)ds \to 0$ as $N_V \to \infty$. combining the above inequalities, we have that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|B_k^N(t) - B_k(t)\right|\right] \leq \int_0^T \varepsilon_{N_V}(r)\, dr + \int_0^t \frac{1}{N_V}\,\mathbb{E}\left[\sum_{j=1}^{S^{N_V}(0)} \sup_{0\leq s\leq r}\left|C_j^{N_V}(s) - C_j(s)\right|\right] dr$$
$$+ \int_0^t \frac{1}{N_V}\,\mathbb{E}\left[\sum_{j\geq S^{N_V}(0)+1} \sup_{0\leq s\leq r}\left|C_j^{N_V}(s) - C_j(s)\right|\right] dr + \int_0^t \mathbb{E}\left[\sup_{0\leq r'\leq u}\left|B_k^N(r') - B_k(r')\right|\right] du, \tag{5.18}$$

On the other hand, define

$$\Delta_k^V(t) := \int_0^t \int_{\overline{\mathfrak{F}}_1^N(s^-)\wedge\overline{\mathfrak{F}}_1(s^-)}^{\overline{\mathfrak{F}}_1^N(s^-)\vee\overline{\mathfrak{F}}_1(s^-)} Q_k^V(ds, du).$$

we have

$$\left|C_k^{N_V}(t) - C_k(t)\right| \leq \Delta_k^V(t), \qquad t \geq 0.$$

In particular since t $\to \Delta_k^V$ is nondecreasing,

$$\mathbb{E}\left[\sup_{r\in[0,t]}\left|C_k^{N_V}(r) - C_k(r)\right|\right] \leq \mathbb{E}\left[\Delta_k^V(t)\right]$$
$$= \int_0^t \mathbb{E}\left[\left|\overline{\mathfrak{F}}_1^N(s^-) - \overline{\mathfrak{F}}_1(s^-)\right|\right] ds. \tag{5.19}$$

$$\begin{aligned}\mathbb{E}\left[\left|\overline{\mathfrak{F}}_1^N(t) - \overline{\mathfrak{F}}_1(t)\right|\right] &= \kappa\,\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N\Big(\lambda_{j,B_j^N(t)}\big(\mathfrak{a}_j^N(t)\big) - \mathbb{E}\big[\lambda_{1,B_1(t)}\big(\mathfrak{a}_1(t)\big)\big]\Big)\right|\right]\\ &\leq \kappa\,\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N\Big(\lambda_{j,B_j^N(t)}\big(\mathfrak{a}_j^N(t)\big) - \lambda_{j,B_j(t)}\big(\mathfrak{a}_j(t)\big)\Big)\right|\right]\\ &\quad + \kappa\,\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N\big(\lambda_{j,B_j(t)}\big(\mathfrak{a}_j(t)\big) - \mathbb{E}\big[\lambda_{1,B_1(t)}\big(\mathfrak{a}_1(t)\big)\big]\big)\right|\right].\end{aligned} \tag{5.20}$$

Since $\big((\lambda_{k,i})_i,\, B_k,\, \mathfrak{a}_k\big)_k$ are i.i.d., the family $\big(\lambda_{k,B_k(t)}(\mathfrak{a}_k(t))\big)_k$ is i.i.d. Hence, by Cauchy–Schwarz,

$$\begin{aligned}\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N\Big(\lambda_{j,B_j(t)}\big(\mathfrak{a}_j(t)\big) - \mathbb{E}\big[\lambda_{1,B_1(t)}\big(\mathfrak{a}_1(t)\big)\big]\Big)\right|\right] &\leq \frac{1}{N}\left(\mathbb{E}\left[\left(\sum_{j=1}^N\Big(\lambda_{j,B_j(t)}\big(\mathfrak{a}_j(t)\big) - \mathbb{E}\big[\lambda_{j,B_j(t)}\big(\mathfrak{a}_j(t)\big)\big]\Big)\right)^2\right]\right)^{1/2}\\ &= \frac{1}{N}\left(\sum_{j=1}^N\mathbb{E}\left[\Big(\lambda_{j,B_j(t)}\big(\mathfrak{a}_j(t)\big) - \mathbb{E}\big[\lambda_{j,B_j(t)}\big(\mathfrak{a}_j(t)\big)\big]\Big)^2\right]\right)^{1/2}\\ &\leq \frac{\lambda^*}{\sqrt{N}}.\end{aligned} \tag{5.21}$$

. In addition, as $(B_j^N(t), \mathfrak{a}_j^N(t), B_j(t), \mathfrak{a}_j(t))_{1\leq j\leq N}$ are exchangeable, we have

$$\begin{aligned}\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N\Big(\lambda_{j,B_j^N(t)}\big(\mathfrak{a}_j^N(t)\big) - \lambda_{j,B_j(t)}\big(\mathfrak{a}_j(t)\big)\Big)\right|\right] &= \mathbb{E}\Bigg[\Bigg|\frac{1}{N}\sum_{j=1}^N\Big(\lambda_{j,B_j^N(t)}\big(\mathfrak{a}_j^N(t)\big) - \lambda_{j,B_j(t)}\big(\mathfrak{a}_j(t)\big)\Big)\\ &\qquad\times\mathbf{1}_{\left\{B_j(t)\neq B_j^N(t)\text{ or }\mathfrak{a}_j(t)\neq\mathfrak{a}_j^N(t)\right\}}\Bigg|\Bigg]\\ &\leq \frac{\lambda^*}{N}\sum_{j=1}^N\mathbb{P}\big(B_j(t)\neq B_j^N(t)\text{ or }\mathfrak{a}_j(t)\neq\mathfrak{a}_j^N(t)\big)\\ &= \lambda^*\,\mathbb{P}\big(B_k(t)\neq B_k^N(t)\text{ or }\mathfrak{a}_k(t)\neq\mathfrak{a}_k^N(t)\big).\end{aligned}$$

On the other hand, since

$$\{B_k^N(t) \neq B_k(t) \text{ or } \mathfrak{a}_k^N(t) \neq \mathfrak{a}_k(t)\} \subset \left\{ \sup_{r\in[0,t]} \left|B_k^N(r) - B_k(r)\right| \geq 1 \right\},$$

$$\mathbb{P}\big(B_k^N(t) \neq B_k(t) \text{ or } \mathfrak{a}_k^N(t) \neq \mathfrak{a}_k(t)\big) \leq \mathbb{E}\left[\sup_{r\in[0,t]} \left|B_k^N(r) - B_k(r)\right| \right]. \tag{5.22}$$

Therefore,

$$\mathbb{E}\left[\left|\overline{\mathfrak{F}}_1^N(t) - \overline{\mathfrak{F}}_1(t)\right|\right] \leq \kappa \frac{\lambda^*}{\sqrt{N}} + \lambda^* \mathbb{E}\left[\sup_{r\in[0,t]} \left|B_k^N(r) - B_k(r)\right| \right]. \tag{5.23}$$

we deduce that for any $t \geq 0$,

$$\mathbb{E}\left[\sup_{r\in[0,t]} \left|C_k^{N_V}(r) - C_k(r)\right| \right] \leq \frac{\lambda^*}{\sqrt{N}} t + 2\lambda^* \int_0^t \mathbb{E}\left[\sup_{r\in[0,t]} \left|B_k^N(r) - B_k(r)\right| \right] dt. \tag{5.24}$$

Lemma 5.4. *Fix $T > 0$ and $k \in \mathbb{N}$. For $t \in [0,T]$, define*

$$b_k^N(t) := \mathbb{E}\left[\sup_{0\leq r\leq t} \left|B_k^N(r) - B_k(r)\right| \right], \qquad c_k^{N_V}(t) := \mathbb{E}\left[\sup_{0\leq r\leq t} \left|C_k^{N_V}(r) - C_k(r)\right| \right].$$

.

Then, for all $t \in [0,T]$,

$$b_k^N(t) + c_k^{N_V}(t) \ \leq \ a_{N_V}(t) \ + \ K_{N_V} \int_0^t \big(b_k^N(u) + c_k^{N_V}(u)\big)\, du,$$

where

$$a_{N_V}(t) := \int_0^t \varepsilon_{N_V}(r)\, dr + \frac{\lambda^*}{\sqrt{N_V}} t, \qquad K_{N_V} := \big(1 + 2\lambda^*\big) + \frac{S^{N_V}(0)}{N_V}.$$

In particular, by Gronwall's inequality,

$$\sup_{t\in[0,T]} \big(b_k^N(t) + c_k^{N_V}(t)\big) \ \leq \ a_{N_V}(T) \exp\big(K_{N_V} T\big).$$

Proof. From (5.24) we have :

$$c_k^{N_V}(t) \leq \frac{\lambda^*}{\sqrt{N_V}} t + 2\lambda^* \int_0^t b_k^N(u)\, du. \tag{5.25}$$

From (5.18) we have

$$b_k^N(t) \leq \int_0^t \varepsilon_{N_V}(r)\, dr + \frac{S^{N_V}(0)}{N_V} \int_0^t c_k^{N_V}(r)\, dr + \int_0^t b_k^N(u)\, du. \tag{5.26}$$

Let $y_k^{N,N_V}(t) := b_k^N(t) + c_k^{N_V}(t)$. Adding (5.24) and (5.26) gives

$$y_k^{N,N_V}(t) \leq \int_0^t \varepsilon_{N_V}(r)\, dr + \frac{\lambda^*}{\sqrt{N_V}} t + \int_0^t b_k^N(u)\, du + 2\lambda^* \int_0^t b_k^N(u)\, du + \frac{S^{N_V}(0)}{N_V} \int_0^t c_k^{N_V}(r)\, dr.$$

Since $\int_0^t b_k^N \leq \int_0^t y_k^{N,N_V}$ and $\int_0^t c_k^{N_V} \leq \int_0^t y_k^{N,N_V}$, we obtain

$$y_k^{N,N_V}(t) \leq \left(\int_0^t \varepsilon_{N_V}(r)\, dr + \frac{\lambda^*}{\sqrt{N_V}} t\right) + \big((1 + 2\lambda^*) + \frac{S^{N_V}(0)}{N_V}\big) \int_0^t y_k^{N,N_V}(u)\, du.$$

This is precisely an integral Gronwall inequality, hence

$$y_k^{N,N_V}(t) \leq a_{N_V}(t) \exp(K_{N_V} t) \leq a_{N_V}(T) \exp(K_{N_V} T), \qquad t \in [0,T].$$

This concludes the proof of Theorem 3.2. $\square$

Proof of Corollary 3.1. We keep the same process $B_k(t)$ defined in (5.6) with its jumps $(\tau_i)_{i\geq 1}$. It follows from the proof of the theorem 3.2 that

$$\left(\frac{1}{N}\sum_{k=1}^{N}\Big(\mathbf{1}_{\mathfrak{a}_k^N(t)\geq \eta_{k,B_k^N(t)}}-\mathbf{1}_{\mathfrak{a}_k(t)\geq \eta_{k,B_k(t)}}\Big),\ \frac{1}{N}\sum_{k=1}^{N}\Big(\mathbf{1}_{\mathfrak{a}_k^N(t)< \eta_{k,B_k^N(t)}}-\mathbf{1}_{\mathfrak{a}_k(t)< \eta_{k,B_k(t)}}\Big)\right)\xrightarrow[N\to+\infty]{}(0,0),$$

locally uniformly in t.

Moreover, since $\big(\mathfrak{a}_k(t),\eta_{k,B_k(t)}\big)_{k\geq 1}$ is a collection of i.i.d. random variables in D^2, the law of large numbers in D^2 [[9], Theorem 1] yields

$$\left(\frac{1}{N}\sum_{k=1}^{N}\mathbf{1}_{\mathfrak{a}_k(t)\geq \eta_{k,B_k(t)}},\ \frac{1}{N}\sum_{k=1}^{N}\mathbf{1}_{\mathfrak{a}_k(t)< \eta_{k,B_k(t)}}\right)\xrightarrow[N\to+\infty]{a.s}\Big(\mathbb{E}\big[\mathbf{1}_{\mathfrak{a}_k(t)\geq \eta_{k,B_k(t)}}\big],\ \mathbb{E}\big[\mathbf{1}_{\mathfrak{a}_k(t)< \eta_{k,B_k(t)}}\big]\Big)\quad \text{in } D^2.$$

We decompose:

$$\overline{U}^N(t)=\frac{1}{N}\sum_{k=1}^{N}\Big(\mathbf{1}_{\mathfrak{a}_k^N(t)\geq \eta_{k,B_k^N(t)}}-\mathbf{1}_{\mathfrak{a}_k(t)\geq \eta_{k,B_k(t)}}\Big)+\frac{1}{N}\sum_{k=1}^{N}\mathbf{1}_{\mathfrak{a}_k(t)\geq \eta_{k,B_k(t)}},$$

$$\overline{I}^N(t)=\frac{1}{N}\sum_{k=1}^{N}\Big(\mathbf{1}_{\mathfrak{a}_k^N(t)< \eta_{k,B_k^N(t)}}-\mathbf{1}_{\mathfrak{a}_k(t)< \eta_{k,B_k(t)}}\Big)+\frac{1}{N}\sum_{k=1}^{N}\mathbf{1}_{\mathfrak{a}_k(t)< \eta_{k,B_k(t)}}.$$

It remains to verify the formulas

$$\overline{I}(t)=\mathbb{E}\big[\mathbf{1}_{\mathfrak{a}_k(t)<\eta_{k,B_k(t)}}\big]\qquad\text{and}\qquad \overline{U}(t)=\mathbb{E}\big[\mathbf{1}_{\mathfrak{a}_k(t)\geq\eta_{k,B_k(t)}}\big]$$

are coherent with (3.4)-(3.5).
If $t-\tau_i<\eta_{k,i}$, then $\gamma_{k,i}(t-\tau_i)=0$, from Assumption (2.1), Hence

$$\int_{\tau_i}^{t}\gamma_{k,i}(r-\tau_i)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr=0,$$

Hence $B_k(t)=B_k(\tau_i)$. Since $\mathbf{1}_{t<\eta_{k,0}}=\mathbf{1}_{t<\eta_{k,0}}\mathbf{1}_{B_k(t)=0}$ a.s., we have

$$\begin{aligned}\mathbb{E}\big[\mathbf{1}_{\mathfrak{a}_k(t)<\eta_{k,B_k(t)}}\big]&=\mathbb{E}\left[\mathbf{1}_{t<\eta_{k,0}}\mathbf{1}_{B_k(t)=0}+\sum_{i\geq 1}\mathbf{1}_{t-\tau_i<\eta_{k,i}}\mathbf{1}_{B_k(t)=i}\right]\\&=\overline{I}_H(0)\,F_0^c(t)+\sum_{i\geq1}\mathbb{E}\big[\mathbb{P}\big(t-\tau_i<\eta_{k,i}\mid\tau_i\big)\,\mathbf{1}_{\tau_i\leq t}\big]\,.\end{aligned}$$

Since $\eta_{k,i}$ and τ_i are independent, it follows that

$$\begin{aligned}\mathbb{E}\big[\mathbf{1}_{\mathfrak{a}_k(t)<\eta_{k,B_k(t)}}\big]&=\overline{I}_H(0)\,F_0^c(t)+\sum_{i\geq1}\mathbb{E}[F^c(t-\tau_i)\,\mathbf{1}_{\tau_i\leq t}]\\&=\overline{I}_H(0)\,F_0^c(t)+\mathbb{E}\left[\int_0^t F^c(t-s)\,B_k(ds)\right]\\&=\overline{I}_H(0)\,F_0^c(t)+\kappa\int_0^t F^c(t-s)\,\overline{\mathfrak{S}}(s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(s)\,ds.\end{aligned}\tag{5.27}$$

This identifies $\overline{I}(t)$ as in (3.5).
If $\eta_{k,0}>t$ (resp. $\eta_{k,i}>t$), then by Assumption (2.1)

$$\gamma_{k,i}(s)=0\quad(\text{resp. }\gamma_{k,0}(s)=0),\qquad 0\leq s\leq t.$$

We write γ (resp. γ_0) for a random variable with the same law as $\gamma_{k,i}$ for $i\geq1$ (resp. as $\gamma_{k,0}$).

By the argument leading to (5.27), we also have

$$\begin{aligned}\mathbb{E}\big[\mathbf{1}_{\mathfrak{a}_k(t)<\eta_{k,B_k(t)}}\big]=\mathbb{E}&\left[\mathbf{1}_{\eta_{k,0}>t}\exp\Big(-\kappa\int_0^t\gamma_0(r)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\right]\\&+\kappa\int_0^t\mathbb{E}\left[\mathbf{1}_{\eta_{k,i}>t-s}\exp\Big(-\kappa\int_s^t\gamma(r-s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\right]\overline{\mathfrak{S}}(s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(s)\,ds.\end{aligned}$$

Combining with (4.1) yields

$$
\begin{aligned}
\mathbb{E}\big[\mathbf{1}_{\mathfrak{a}_k(t)\geq\eta_{k,B_k(t)}}\big] &= 1-\mathbb{E}\big[\mathbf{1}_{\mathfrak{a}_k(t)<\eta_{k,B_k(t)}}\big] \\
&= \mathbb{E}\bigg[\mathbf{1}_{t\geq\eta_{k,0}}\exp\Big(-\kappa\int_0^t \gamma_0(r)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\bigg] \\
&\quad+\kappa\int_0^t \mathbb{E}\bigg[\mathbf{1}_{t-s\geq\eta_{k,i}}\exp\Big(-\kappa\int_s^t \gamma(r-s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(r)\,dr\Big)\bigg]\,\overline{\mathfrak{S}}(s)\,\mathcal{V}(\overline{\mathfrak{F}}_1)(s)\,ds,
\end{aligned}
$$

which identifies $\overline{U}(t)$ as in (3.4). □

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