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ORIGINAL ARTICLE

Metastability between the clicks of Muller's ratchet

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Abstract

We prove the existence and uniqueness of a quasi-stationary distribution for three stochastic processes derived from the model of Muller's ratchet. This model was invented with the aim of evaluating the limitations of an asexual reproduction mode in preventing the accumulation of deleterious mutations through natural selection alone. The main considered model is non-classical, as it is a stochastic diffusion evolving on an irregular set of infinite dimension with hard killing on a hyperplane. We are nonetheless able to prove exponential convergence in total variation to the quasi-stationary distribution even in this case. The parameters in this last convergence result are directly related to the core parameters of Muller's ratchet. The speed of convergence to the quasi-stationary distribution is deduced both for the infinite dimensional model and for approximations with a large yet finite number of potential mutations. Likewise, we give uniform moment estimates of the empirical distribution of mutations in the population under quasi-stationarity.

Keywords Quasi-stationary distribution · Convergence rate · Muller's ratchet · Diffusion on finite- and infinite-dimensional space · Purge of deleterious mutations · Natural selection

Mathematics Subject Classification Primary 37A25 · 60J70 · 92D15; Secondary 37A30 · 60J10 · 92D25

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1 Introduction

1.1 General presentation

Since deleterious mutations occur much more frequently than beneficial ones, it is crucial to understand how the fixation of these deleterious mutations is regulated. Notably, it is very exceptional that a subsequent mutation reverts a deleterious one, so that only natural selection can maintain some purity in the population. In this respect, there is a major distinction to be made between sexual and asexual reproduction. In a purely asexually reproducing population, a deleterious mutation can only be purged when the lineages carrying it go extinct. In a sexually reproducing population, such a deleterious mutation can be avoided through recombination, without getting rid of the whole set of other mutations carried by the lineages. There is actually no strong evidence that deleterious mutations are specifically targeted during this process of recombination. For natural selection to effectively reduce the mutational load, it appears sufficient that this random process of recombination prevents some lineages from carrying the mutation after a given time. This ability to better keep the population purified from deleterious mutations is one of the main explanations of the success of the sexual reproduction mode (see [1] for more details). Such an advantage for sexual reproduction is to be confronted with the cost (in terms of reproduction efficacy) of requiring two parents. The above scheme for purging deleterious mutations in asexual populations is the main object of study of the current paper.

We plan to justify the existence and uniqueness of a metastable state in which selective effects are able to maintain a subpopulation free from any deleterious mutation. At the time where this subpopulation goes extinct, we say that a click occurs in the population. It has been shown in [2] that clicks happen in finite time a.s. even for the infinite dimensional diffusion model (with infinitely many types of individuals). Rigorous definitions of such a metastable state (characterized by the absence of click) can be obtained in a broad generality by a conditioning of stochastic processes. We refer to Sect. 2.1 for the definition of several crucial characteristic of metastability, especially the notion of quasi-stationary distribution (abbreviated as QSD).

We treat in this paper three models of Muller's ratchet: the first one is discrete both in time and space, the second is a finite dimensional diffusion and the third one is an infinite dimensional diffusion, see Sect. 1.2 below. We prove the existence and uniqueness of a QSD for those three stochastic representations, see respectively Theorems 2.2.2, 2.3.2 and 2.4.1. To our knowledge, the existence and uniqueness of a QSD has not been rigorously proved until now except in the case of a finite state space. This result was nonetheless implicitly exploited for the approximations provided in [3].

We shall see that these QSD are concentrated on distributions with light tails, meaning that the proportion of the population carrying a large number of mutations remains negligible under the QSD. This claim is supported by our Proposition 2.3.4.

We also address the classical issue of specifying the conditions under which metastability is observed in practice. A generally accepted answer is to compare the so-called relaxation time t_R, which quantifies the rate at which the dependence in the past conditions vanishes, and the average clicking time t_C of the system. Metastability between clicks would be the most common observation provided $t_R \ll t_C$, so that a sequence of i.i.d. exponential law provides an accurate description of the sequence of intervals between clicks. This is where the comparison with the clicks of a ratchet comes from. If t_R is of the same order as t_C or larger, we a priori can not exclude that trains of short interdependent intervals could alter this observed distribution of interval length. But already if t_R is of the same order as t_C, there shall still be long realizations of inter-click intervals after which we can say that the dependence in the past is forgotten. This discussion is pursued in more details in Sect. 3.

The above mentioned theorems provide a proper definition of these two main quantities. The typical clicking time t_C is defined as the inverse of the extinction rate of the QSD, or equivalently as the expected waiting time of the next click with the QSD as the initial condition. On the other hand, the QSD is approached at an exponential rate by the marginal law of the process conditioned upon the fact that the click has not occurred. We describe the inverse of this exponential rate (which should be independent of the initial condition besides the QSD) as the typical relaxation time t_R.

As compared to the other models that we have treated by similar techniques as in the current paper, the proof of Theorem 2.4.1 is particularly difficult. It specifically exploits the effect of selection to obtain practical bounds on the maximal number of accumulated mutations. The argument is technical because at any time an infinitesimal proportion of heavily counter-selected mutants cannot be completely neglected.

A simplified version of such bounds is already needed for the proof of Theorem 2.2.2. This concerns the process defined in Sect. 1.2.1. The fact that the process describes a discrete population greatly simplifies the argument. We then extend the justification of the relaxation time and the clicking rate for large population limiting models. Note that the results of [3] or [4] already largely exploit the fact that the population is large. In the diffusive limit defined in Sect. 1.2.2, an additional difficulty arises in that the diffusion is degenerate on a non-smooth boundary that is partly absorbing and partly repulsive. In order to present a simplified analysis, we introduce in a first step a limitation in the number of carried mutations for the statement of Theorem 2.3.2 given in Sect. 2.3. In the last step given in Sect. 2.4 with Theorem 2.4.1, we establish the existence and uniqueness of a QSD for the more natural infinite dimensional model.

The paper is organized as follows. In the next Sect. 1.2, we specify the stochastic processes under consideration, first the individual-based model in Sect. 1.2.1 and then its diffusive limits in Sect. 1.2.2. Our results of quasi-stationarity are presented in Sect. 2. Starting in Sect. 2.1 with the general notion of exponential quasi-stationarity that we aim to establish, we treat respectively in Sects. 2.2, 2.3 and 2.4 each of the three stochastic processes mentioned above. The generic assumptions and theorems on which these proofs rely are stated in Sect. 2.5. Next, we discuss more precisely the interpretation of these results in Sect. 3. We justify in Sect. 3.1 to 3.3 under which conditions quasi-stationarity can be observed. Finally, we motivate our choice for not introducing a bound on the number of mutations in Sect. 3.4. The rest of the paper is dedicated to the proofs. Sections 4, 5 and 7 are devoted to the proofs of quasi-stationarity for each of the three processes, while Sect. 6 is devoted to uniform moment estimates of the QSDs. The proofs in Sect. 7 take advantage of the lemmas and propositions derived in the previous Sects. 5 and 6.

1.2 The mathematical model of Muller's ratchet

1.2.1 The individual-based model as a guideline

For the origin of the models which we study, we refer to the simplified mathematical model which has been proposed by Guess as the multiplicative fitness model [5] to quantify the regulation of deleterious mutations in an asexual population. The interest for this type of simplified models stems from general considerations on the evolutive advantage of recombination, as notably advanced by Muller in 1964 [6]. Since in any finite population, the ultimate fixation of deleterious mutations cannot be avoided (unless by the extinction of the population), yet a form of metastability can be observed. This "mechanism" of regulation has been called Muller's ratchet, notably by Haigh in [7].

Assuming a constant deleterious effect of mutations, at each time that the fittest individuals disappear, the ratchet clicks in the sense that the new fittest individuals carry an additional deleterious effect. The whole population is doomed after this time to carry at least this additional effect. Since the population size is constant, natural selection then acts as if the whole profile of mutations were translated by this value, so that the fittest individuals at that clicking time now become the new reference (at mutational burden 0). If the mutation rate is slow enough to allow these fittest individuals to maintain the stability of the system for a while, the dynamics shall rapidly follow the same behavior as before the click (taking into account that the empirical distribution of the number of carried mutations is translated).

This first model with discrete generations and fixed population size N evolves as follows. Mutations that occur are only deleterious and they occur at constant rate $\lambda > 0$. The cost in fitness of each mutation is quantified by $\alpha \in (0, 1)$. Assume that the current population is distributed with N_i individuals carrying i mutations and consider an individual from the next generation. Each one chooses its parent independently of the others according to the same probability distribution, which is specified by the fact that the chosen parent carries i mutations with probability:

$$\frac{N_i(1-\alpha)^i}{\sum_{k\geq 0} N_k \cdot (1-\alpha)^k}.$$

Remark that $\alpha = 0$ corresponds to neutral mutations, which are not under purifying selection. If $\alpha = 1$ on the other hand, no individual could survive the burden of a single mutation.

In addition to the mutations of its parent, each newborn gains ξ deleterious mutations, where ξ is a Poisson random variable with mean λ, specific to the newborn. ξ is drawn independently for each newborn and of the choice of the parent.

Existence and uniqueness for such a discrete-time Markov chain on a countable state-space is a classical result, which we shall take for granted.

Remark 1.2.1 Of course, the situation is more intricate in reality. Mutations certainly do not have constant effect, and combination effects are frequent (i.e. epistasis). In many asexual populations, there is evidence of the role of horizontal gene transfers, for instance with plasmids [8–11], which can be seen as a weak form of recombination. Moreover, the fact that mutations are deleterious is due to a change in the physiology that may be compensated by other means. It might even happen that after subsequent mutations, the carriers of an initially deleterious mutation become more adapted than the wild types [12]. Neglecting these effects enables however to gain insight on the main regulatory factor.

1.2.2 The stochastic diffusion under consideration

In the following, we also consider a description of the model that corresponds to a limit of large population size, accelerated time-scale (for which time is continuous), thus also small selective effect and small mutation rate. In the following statements, $d \in [\![1, \infty]\!]$ (i.e. $d \in \mathbb{N} \cup \{\infty\}$) defines an upper-bound on the number of deleterious mutations that can be carried by an individual. If $d := \infty$ in the following expression, $i \in [\![0, d]\!]$ has to be understood as $i \in \mathbb{Z}_+$.

We are interested in the following Fleming-Viot system of Stochastic Differential Equations (SDEs) for the $X_i^{(d)}(t)$'s, $i \in [\![0, d]\!]$, where $X_i^{(d)}(t)$ denotes the proportion of individuals in the population who carry exactly i deleterious mutations at time t (with $X_{-1}^{(d)} \equiv 0$):

$$\mathrm{d}X_i^{(d)}(t) = \alpha \cdot (M_1^{(d)}(t) - i) \cdot X_i^{(d)}(t)\,\mathrm{d}t + \lambda \cdot (X_{i-1}^{(d)}(t) - \mathbf{1}_{\{i<d\}} X_i^{(d)}(t))\,\mathrm{d}t \\ + \sqrt{X_i^{(d)}(t)}\,\mathrm{d}W_i(t) - X_i^{(d)}(t)\,\mathrm{d}W_{(d)}(s). \tag{1.1}$$

In (1.1), $(W_i)_{i \geq 0}$ denotes a family of mutually independent standard Brownian motions, where W_i specifies the demographic fluctuations that are specific to the subpopulations i. Secondly, the martingale process $W_{(d)}$ is defined as follows:

$$W_{(d)}(s) := \sum_{j=0}^{d} \int_0^t \sqrt{X_j^{(d)}(s)}\mathrm{d}W_j(s), \tag{1.2}$$

namely as an aggregated component according to which the fluctuations in the total population sizes are corrected. Finally, the process $M_1^{(d)}$ is defined as follows:

$$M_1^{(d)}(t) := \sum_{i=0}^{d} i \cdot X_i^{(d)}(t),$$

namely as the aggregated component according to which the variations in the total population sizes due to the selective effects are corrected. Unless otherwise specified, the Brownian motions that we introduce are all standard and this precision will possibly be omitted.

Remark 1.2.2 The martingale term is described by the following representation:

$$\mathrm{d}\mathcal{N}_i^{(d)}(t) := \sqrt{X_i^{(d)}(t)}\,\mathrm{d}W_i(t) - X_i^{(d)}(t)\,\mathrm{d}W_{(d)}(t)$$

which is actually equivalent to another commonly considered representation, namely:

$$\mathrm{d}\mathcal{N}_i^{(d)}(t) = \sum_{\{j\neq i\}} \sqrt{X_i^{(d)}(t)X_j^{(d)}(t)}\,\mathrm{d}W_{i,j}(t)$$

for a sequence $(W_{i,j})_{i<j}$ of independent Brownian motions, extended to any $i \neq j$ by the symmetry property $W_{i,j}(t) = W_{j,i}(t)$. As proved in Proposition A.0.1 of the appendix, weak existence and uniqueness hold for both systems of SDEs and the two representations have actually the same law. Since $\sum_{i=0}^{d} X^{(d)} \equiv 1$, and thanks to Lévy's caracterisation, the martingale $W_{(d)}$ has the law of a Brownian motion. Since

$$\mathrm{d}\langle\mathcal{N}_i^{(d)}\rangle_t = X_i^{(d)}(t)\cdot(1 - X_i^{(d)}(t))\,\mathrm{d}t,$$

there exists for any $i \in [\![1, d]\!]$ a Brownian motion B_i (which depends upon d) such that:

$$\mathrm{d}\mathcal{N}_i^{(d)}(t) := \sqrt{X_i^{(d)}(t)\cdot(1 - X_i^{(d)}(t))}\,\mathrm{d}B_i(t). \tag{1.3}$$

However the (B_i) are not mutually independent.

In [13], a closely related process with compensatory mutations is considered. We refer to this article for a detailed presentation of the connection to related individual-based models and only sketch next the interpretation of the parameters.

The selective effect of the deleterious mutations is the term proportional to α in the drift term. As we assume that all deleterious mutations carry the same burden and that the total population size is fixed, the growth rate of individuals carrying i mutations is proportional to the difference between i and the average number of mutations, i.e. $M_1^{(d)}(t)$. The appearance of new mutations is modeled by the term proportional to λ in the drift term. λ corresponds to the rate at which individuals carrying i mutations give birth to individuals carrying $i+1$ mutations. Finally, the neutral choice of the individuals replaced at each birth events gives rise to the martingale term. Our time-scale corresponds to the rescaling of time $t \mapsto t'/N$, where N is the population size.

1.2.3 Notations

For $\mathcal{X}$ a generic (Polish) space, hereafter $B(\mathcal{X})$ (respectively $B_+(\mathcal{X})$) denotes the space of bounded measurable (respectively nonnegative bounded measurable) functions from $\mathcal{X}$ into $\mathbb{R}$ and $\mathcal{M}_1(\mathcal{X})$ (respectively $\mathscr{M}(\mathcal{X})$) the space of Borel probability (respectively signed) measures. For any $f \in B(\mathcal{X})$ and $\mu \in \mathcal{M}(\mathcal{X})$, we use the abbreviation: $\langle \mu \,|\, f\rangle := \int_{\mathcal{X}} f(x)\mu(\mathrm{d}x)$. $\mathbb{R}_+$ (respectively $\mathbb{R}_+^*$) denotes the set of non-negative (respectively positive) reals. For any integers m, n such that $m \leq n$, $[\![m, n]\!]$

denotes the set of integers from m to n (included). We recall that $d = \infty$ is included in the expression $d \in [\![1, \infty]\!]$. $\mathbb{Z}_+$ (respectively $\mathbb{N}$) denote the set of non-negative (respectively positive) integers.

2 Exponential quasi-stationarity results

2.1 Exponential quasi-stationarity

The conclusions of the following theorems are expressed in terms of the notion of exponential quasi-stationarity that we borrow from [14].

Definition 2.1.1 For any linear, nonnegative, bounded and sub-conservative semi-group $(P_t)_{t\geq 0}$ acting on $\mathcal{M}(\mathcal{X})$, we say that P displays a uniform exponential quasi-stationary convergence (abbreviated as QSC) with characteristics $(\nu, h, \rho_0) \in \mathcal{M}_1(\mathcal{X})\times B_+(\mathcal{X})\times\mathbb{R}_+$ if $\langle \nu \,\big|\, h\rangle = 1$ and there exists $C, \gamma > 0$ such that the following inequality holds for any $t > 0$ and for any measure $\mu \in \mathcal{M}(\mathcal{X})$ with $\|\mu\|_{TV} \leq 1$:

$$\left\| e^{\rho_0 t} \mu P_t - \langle \mu \,\big|\, h\rangle\, \nu \right\|_{TV} \leq C e^{-\gamma t}. \tag{2.1}$$

By the term of characteristics, we mean that the triple (ν, h, ρ_0) is uniquely defined, as stated in [14, Remark 2.7]. Thanks to [14, Corollary 2.9], this definition of convergence for (P_t) implies the following convergence result to ν.

Corollary 2.1.1 *Assume (2.1). Then for any* $t \geq 0$ *and* $\mu \in \mathcal{M}_1(\mathcal{X})$ *such that* $\langle \mu \,\big|\, h\rangle > 0$:

$$\| \langle \mu P_t, \mathbf{1}\rangle^{-1} \cdot \mu P_t(\mathrm{d}x) - \nu(\mathrm{d}x) \|_{TV} \leq C \frac{\|\mu - \nu\|_{TV}}{\langle \mu \,\big|\, h\rangle}\, e^{-\gamma\, t}. \tag{2.2}$$

Remark 2.1.2 Choosing $\mu = \nu$ in (2.1), it is not hard to deduce the following relation:

$$\forall\, t \geq 0, \quad \nu P_t = e^{-\rho_0\, t}\, \nu, \tag{2.3}$$

and in particular $\langle \nu, \mathbf{1}\rangle = e^{-\rho_0\, t}$, cf [14, Fact 2.7]. This relation is what characterizes ν as a QSD since it implies that for any $t \geq 0$, $\langle \nu P_t, \mathbf{1}\rangle^{-1} \cdot \nu P_t(\mathrm{d}x) = \nu(\mathrm{d}x)$. By restricting the convergence stated in (2.1) on the evaluation of the measure on $\mathcal{X}$, we obtain a similar characterization of h. This latter convergence is what makes us call h the survival capacity.

There is an additional related notion that will be useful to describe the behavior of the process with the requirement of a long inter-click interval. This process is generically defined through the survival capacity h, on the state space: $\mathcal{H} := \{x \in \mathcal{X}\,;\; h(x) > 0\}$. In the following, we assume that $(P_t)_{t\geq 0}$ is the sub-conservative semi-group of a strong Markov process $(\Omega; (\mathcal{F}_t)_{t\geq 0}; (X_t)_{t\geq 0}; (\mathbb{P}_x)_{x\in\mathcal{X}})$ defined up to its extinction time τ_∂, which is expresssed as follows for any initial distribution $\mu \in \mathscr{M}_1(\mathcal{X})$:

$$\mu P_t(\mathrm{d}x) = \mathbb{E}_\mu\big(X_t \in \mathrm{d}x;\; t < \tau_\partial\big).$$

Note that for such a semi-group (P_t) displaying QSD with characteristics (ν, h, ρ_0) (in $\mathcal{M}_1(\mathcal{X}) \times B_+(\mathcal{X}) \times \mathbb{R}_+$), $\rho_0 > 0$ is equivalent to $\mathbb{P}_\nu(\tau_\partial < \infty) > 0$, thanks to Remark 2.1.2. Furthermore, $\mathbb{E}_\mu\big(X_t \in \mathrm{d}x \,|\, t < \tau_\partial\big) = \langle \mu P_t, \mathbf{1}\rangle^{-1} \cdot \mu P_t(\mathrm{d}x)$ in this framework, which converges to $\nu(\mathrm{d}x)$ according to (2.2).

Definition 2.1.2 We say that the Q-process exists if there exists a family $(\mathbb{Q}_x)_{x\in\mathcal{H}}$ of probability measures on Ω defined by:

$$\lim_{t\to\infty} \mathbb{P}_x(\Lambda_s \,\big|\, t < \tau_\partial) = \mathbb{Q}_x(\Lambda_s) \tag{2.4}$$

for any $\mathcal{F}_s$-measurable set Λ_s. We also require the process $(\Omega; (\mathcal{F}_t)_{t\geq 0}; (X_t)_{t\geq 0}; (\mathbb{Q}_x)_{x\in\mathcal{H}})$ to be a homogeneous strong Markov process.

With a slight adaptation of the proof of [15, Theorem 2.3], it was deduced in [14] that such a property of existence of the Q-process is a consequence of QSC.

Remark 2.1.3 The transition kernel of the Q-process is given by:

$$q(x; t; dy) = e^{\rho_0 t} \frac{h(y)}{h(x)}\, p(x; t; dy),$$

where $p(x; t; dy)$ is the transition kernel of the Markov process (X) under $(\mathbb{P}_x)$. Note that $\mathcal{X} \setminus \mathcal{H}$ is generally avoided by the process X under $\mathbb{Q}_x$. For the examples considered in this paper, h is positive, and the QSD ν is unique. No distinction has then to be made between $\mathcal{X}$ and $\mathcal{H}$ regarding the Q-process.

Thanks to [14, Corollary 2.12], our justification for the proof of QSC actually implies related results of convergence for the Q-process. Notably $\beta(\mathrm{d}x) := h(x)\,\nu(\mathrm{d}x)$ is the unique invariant probability measure of this process.

Remark 2.1.4 The probability space Ω is usually not made explicit. For the purpose of a following condition (namely Assumption $(A3_F)$ as stated in Sect. 2.5.1), we will require for the analysis of the diffusion processes $X^{(d)}$ that Ω is of path type. Following [16] (to exploit Proposition 8.8 within), it means that Ω is the canonical space of a strong Markov process $(\widehat{X}_t)_{t\in\mathbb{R}_+}$ (with values on a Polish space $\widehat{\mathcal{X}}$), while the filtration $(\mathcal{F}_t)$ is the one generated by the process $\widehat{X}$. In other words, $\widehat{X}$ is exactly the identity mapping, so that $\widehat{X}_t$ agrees with the evaluation map : $\omega \mapsto \omega_t$ for ω on the set of measurable functions from $\mathbb{R}_+$ to $\widehat{\mathcal{X}}$, and $(\mathcal{F}_t)$ is the smallest filtration that make this application $\mathcal{F}_t$-measurable for any $t \geq 0$. For our purposes, for any $d \in [\![1, \infty]\!]$, it will be sufficient to consider for $\Omega^{(d)}$ and for the associated filtration $(\mathcal{F}_t^{(d)})$ the canonical choice generated by the process $X^{(d)}$.

Remark 2.1.5 Originally in the quantitative work of Champagnat and Villemonais, the focus is more on estimates of convergence for the conditioned semi-group, as in (2.2). There was notably an efficient and full characterization of the case where the upper-bound can be taken as $Ce^{-\gamma t}$ uniformly over the initial condition [17]. In a more recent work [18], they also relate to the convergence of the semi-group scaled by $e^{\rho_0 t}$ [18, Corollary 2.4 and 2.6] and describe it as naturally adapted as well [18, Remark 5].

As in [14], we found the formulation (2.2) more generally adapted to the dependency over the initial condition due to the linearity of the semi-group P_t. A more general form of convergence than (2.1) with a weighted norm is the object of [19], with a full characterization as well.

2.2 The discrete population case

Let $N \geq 1$ be the number of individuals in the population. For $n \leq N$ and $t \in \mathbb{Z}_+$, let $D_n(t)$ be the number of mutations carried by the n-th individual. We consider the empirical measure at time $t > 0$ defined as follow:

$$\mathcal{Z}_t^N := \frac{1}{N} \sum\nolimits_{n=0}^{N} \delta_{D_n(t)}, \tag{2.5}$$

so that $\mathcal{Z}_t^N(i) \in N^{-1} \cdot [\![0, N]\!]$ specifies the proportion of individuals with exactly i mutations (since everything is discrete, we identify $\mathcal{Z}_t^N$ as a function from $\mathbb{Z}_+$ to $\mathbb{R}_+$). From the rules describing the next generation from the previous one, see Sect. 1.2.1, $\mathcal{Z}^N$ is a strong Markov process evolving on $\mathcal{M}_1^N(\mathbb{Z}_+)$, where:

$$\begin{aligned} \mathcal{M}_1^N(\mathbb{Z}_+) &:= \left\{ \frac{1}{N} \cdot \sum_{i \leq N} \delta_{d_i};\ d_i \in \mathbb{Z}_+,\ \sum_{i \in \mathbb{Z}_+} d_i = N \right\} \\ &\equiv \left\{ z : \mathbb{Z}_+ \mapsto N^{-1} \times [\![0, N]\!];\ \sum\nolimits_{\{i \in \mathbb{Z}_+\}} z(i) = 1 \right\}. \end{aligned} \tag{2.6}$$

This set $\mathcal{M}_1^N(\mathbb{Z}_+)$ is equipped with the Lévy-Prokhorov metric, which makes it a Polish space, that is a separable and complete metric space. The clicking time under consideration comes from the extinction of the fittest individuals, i.e.:

$$\tau_\partial^N := \inf \left\{ t \geq 0;\ \mathcal{Z}_t^N(0) = 0 \right\} = \inf \left\{ t \geq 0;\ \mathcal{Z}_t^N \notin \mathcal{M}_1^{(0),N}(\mathbb{Z}_+) \right\}, \tag{2.7}$$

where $\mathcal{M}_1^{(0),N}(\mathbb{Z}_+) = \left\{ z \in \mathcal{M}_1^N(\mathbb{Z}_+);\ z(0) \geq \frac{1}{N} \right\}$.

Remark 2.2.1 Classical theory on quasi-stationarity can be exploited by interpreting the clicking time τ_∂^N as an extinction time. We implicitly rely on the process $\bar{\mathcal{Z}}_t^N := \mathbf{1}_{\{t < \tau_\partial^N\}} \mathcal{Z}_t^N + \mathbf{1}_{\{\tau_\partial^N \leq t\}} \partial$ which is Markov and lives on $\mathcal{M}_1^{(0),N}(\mathbb{Z}_+) \cup \{\partial\}$. For the process $\bar{\mathcal{Z}}^N$, τ_∂^N is the hitting time of the absorbing state ∂ (the cemetery).

Our main conclusion for this process is the following theorem:

Theorem 2.2.2 *Consider for any $\alpha \in (0, 1)$, for any $\lambda > 0$ and for any $N \geq 1$ the Markov process Z^N whose transitions are prescribed as in Sect. 1.2.1, with clicking time τ_∂^N. Then, its semigroup P^N displays QSC with characteristics $(\nu^N, h^N, \rho_0^N) \in \mathcal{M}_1(\mathcal{M}_1^{(0),N}(\mathbb{Z}_+)) \times B_+(\mathcal{M}_1^{(0),N}(\mathbb{Z}_+)) \times \mathbb{R}_+^*$. Moreover, h^N is uniformly bounded away from 0.*

It implies that the convergence to ν^N given in (2.2) is uniform with respect to the initial condition and that the Q-process exists on the whole state space $\mathcal{M}_1^{(0),N}(\mathbb{Z}_+)$.

Remark 2.2.3 The discreteness of the process is strongly involved in the proof of Theorem 2.2.2, so that the derived convergence rate strongly depends on N.

2.3 The finite dimensional case

The system of SDEs for finite d evolves on the state space $\bar{\mathcal{X}}_d$, where:

$$\bar{\mathcal{X}}_d := \left\{(x_k)_{k\in[\![0,d]\!]} \in [0,1]^{d+1};\ \sum\nolimits_{k=0}^{d} x_k = 1\right\}.$$

With the L^1 distance, this set is also a Polish space. The solution to the system (1.1) is unique as stated in the next proposition, as a corrolary of [20, Theorem 2.1], whose proof is deferred for completeness to the appendix, Section A.

Proposition 2.3.1 *For any $d \in \mathbb{N}$, existence and weak uniqueness of solutions hold on the state space $\bar{\mathcal{X}}_d$ for the system (1.1) of SDEs.*

We denote by $\tau_\partial^{(d)}$ the clicking time of this process $X^{(d)}$, that is:

$$\tau_\partial^{(d)} := \inf\left\{t \geq 0;\ X_0^{(d)}(t) = 0\right\}. \tag{2.8}$$

The most natural state space for the process with extinction at time $\tau_\partial^{(d)}$ thus does not include this boundary:

$$\mathcal{X}_d := \left\{x \in \bar{\mathcal{X}}_d; x_0 \in (0,1]\right\}. \tag{2.9}$$

Our main conclusion for this process is the following theorem:

Theorem 2.3.2 *Consider the system of SDEs (1.1) for any $\alpha \in (0,1)$, for any $\lambda > 0$ and for any $d \in \mathbb{N}$, with clicking time $\tau_\partial^{(d)}$. Then, its semigroup $P^{(d)}$ displays a QSC with characteristics $(\nu^{(d)}, h^{(d)}, \rho_0^{(d)}) \in \mathcal{M}_1(\mathcal{X}_d)\times B_+(\mathcal{X}_d)\times\mathbb{R}_+^*$. In addition, for any $y_0 \in (0,1)$, $h^{(d)}$ is bounded away from 0 on $\{x \in \mathcal{X}_d;\ x_0 \geq y_0\}$. In particular, the associated Q-process exists on the whole state space $\mathcal{X}_d$.*

Remark 2.3.3 The fact that α is non-zero is actually not exploited in this proof. As in [21], we rely mainly on Harnack's inequality. The dependency in the dimension d is roughly considered. Nonetheless, we have here to be cautious in the way we handle jointly the absorbing and repulsive boundary conditions.

Moreover, we prove the following controls on the moments of the QSDs $\nu^{(d)}$, for $d \in \mathbb{N}$:

Proposition 2.3.4 *For any $\alpha \in (0, 1)$, for any $\lambda > 0$ and for any $k \geq 1$, we have uniform tightness in d over the moments of order k of the unique QSDs $\nu^{(d)}$ associated with the solution to (1.1), which means that the following supremum tends to 0 as m tends to infinity:*

$$\sup \Big\{ \int_{\mathcal{X}_d} \mathbf{1}_{\{M_k(x) \geq m\}} \nu^{(d)}(\mathrm{d}x);\ d \in \mathbb{N} \Big\},$$

where $M_k(x) := \sum_{i \in [\![0,d]\!]} i^k\, x_i$. In particular, the sequence $\hat{\nu}_k^{(d)}$, where the values for the coordinates larger than $k+1$ are put to 0, is tight in $\mathcal{M}_1(\mathbb{R}_+^{\mathbb{Z}})$.

Remark 2.3.5 This control on the moments is actually crucial for the proof of uniqueness when $d = \infty$. Given the above theorem, we expect the sequence $(\nu^{(d)})$ to converge as $d \to \infty$ to the unique QSD $\nu^{(\infty)}$ for the infinite system (for which the control extends), though it is beyond the scope of the current paper.

2.4 The infinite dimensional case

We consider now the infinite dimensional case, for which we require the existence of moments. Let us consider the following definition for any $\eta \in (2, \infty)$:

$$\bar{\mathcal{X}}^{\eta} := \left\{ (x_k)_{k \in \mathbb{Z}_+} \in [0, 1]^{\infty};\ \sum_{k=0}^{\infty} x_k = 1,\ \sum_{k=0}^{\infty} k^{\eta} x_k < \infty \right\}.$$

As in [2], we consider for $\bar{\mathcal{X}}^{\eta}$ the topology under which a probability $x^n = (x_k^n, k \geq 0)$ on $\mathbb{Z}_+$ converges to $x = (x_k, k \geq 0)$ if both it converges weakly, and $\sup_n \sum_{k \geq 0} k^{\eta} x_k^n < \infty$. This topology is actually generated by the following metric, defined for any $x, y \in \mathcal{X}^{\eta}$, which makes $\bar{\mathcal{X}}^{\eta}$ Polish, that is separable and complete:

$$d_{\eta}(x, y) = |x_0 - y_0| + \sum_{\{k \geq 1\}} k^{\eta} \cdot |x_k - y_k|.$$

Thanks to [2, Theorem 3] (and to Proposition A.0.1 in the appendix to show that the process under consideration is the same as ours), we know that for any $\eta > 2$ and for any initial condition x that belongs to $\bar{\mathcal{X}}^{\eta}$, (1.1) has a unique weak solution $X^{(\infty)}$ which is a.s. continuous with values in $\bar{\mathcal{X}}^{\eta}$. This process has been introduced in [4] and it has also been shown in [2] that clicks occur a.s. in finite time, with the same definition of $\tau_{\partial}^{(\infty)}$ as in (2.8). We consider the state space without the boundary $\{x_0 = 0\}$ as the state space with $\eta = 6$ for the process with extinction at time $\tau_{\partial}^{(\infty)}$:

$$\mathcal{X}_{\infty} := \{x \in \bar{\mathcal{X}}^{6};\ x_0 \in (0, 1]\}. \tag{2.10}$$

We now state the main theorem of the current paper:

Theorem 2.4.1 *Consider for any $\alpha \in (0, 1)$ and for any $\lambda > 0$ the system of SDEs (1.1) with $d = \infty$, defined on $\mathcal{X}_{\infty}$ with extinction at time $\tau_{\partial}^{(\infty)}$. Then, its semigroup $P^{(\infty)}$*

displays a QSC with characteristics $(\nu^{(\infty)}, h^{(\infty)}, \rho_0^{(\infty)}) \in \mathcal{M}_1(\mathcal{X}_\infty) \times B_+(\mathcal{X}_\infty) \times \mathbb{R}_+^*$. *In addition, for any* $y_0 \in (0,1)$, $h^{(\infty)}$ *is lower-bounded by a positive constant on* $\{x \in \mathcal{X}_\infty\,;\ x_0 \geq y_0\}$. *In particular, the Q-process exists on the whole state space* $\mathcal{X}_\infty$.

Besides, there exist $C, \gamma, d_\vee > 0$ *such that the convergences stated in (2.1) and (2.2) hold true with the constants* C, γ *for the processes given by (1.1) on* $\mathcal{X}_d$ *for any* $d \in [\![d_\vee, \infty]\!]$.

By the notation, $[\![d_\vee, \infty]\!]$, remark that we include ∞ besides all integers larger than $d_\vee$.

Remark 2.4.2 The core of our proof is based on the intuition that the slower the decay of the tail the more rapidly it gets erased and renewed. So we do not expect large tails to play a significant role. In practice, we exploit the finiteness of moments of order $2\eta'$ to control moments of order η', and that η' is strictly larger than 2 plays a significant role. For simplicity, we restrict ourselves to initial condition within $\mathcal{X}^6$ although our proof could likely be generalized to $\mathcal{X}^\eta$ provided at least that $\eta > 4$.

2.5 Some crucial sets of conditions ensuring exponential quasi-stationarity

The proof of QSC are exploiting the criteria given in [14, Subsection 2.3.1], with a trajectorial approach. The methods and statements taken from [15] have actually been adjusted in [14] with the current paper in mind. Thanks to Theorem 2.5.3 in Sect. 2.5.2 that summarizes these conclusions, it will remain to prove one of the following two sets of assumptions, (**A**) or (**AF**), to complete the proofs of Theorems 2.2.2, 2.3.2 and 2.4.1. Assumptions (**A**) and (**AF**) are made up of four basic assumptions, three being common to both. So first, we present these five basic assumptions in Sect. 2.5.1 in the general context of a strong Markov process X with extinction at time τ_∂ that makes it exit the state space $\mathcal{X}$.

$\mathcal{X}$ can here simply be thought to be a Polish space. The state spaces we consider for our theorems, namely $\mathcal{M}_1^{(0),N}(\mathbb{Z}_+)$ for any $N \geq 1$ (see (2.7)), $\mathcal{X}_d$ for any $d \geq 1$ (see (2.9)), and $\mathcal{X}_\infty$ (see (2.10)), are all Polish spaces.

2.5.1 Basic assumptions

The first assumption is on a sequence $(\mathcal{D}_\ell)_{\ell \geq 1}$ that shall be exploited for the following assumptions. $int(\mathcal{D})$ denotes the interior of any set $\mathcal{D}$.

$(A0)[(\mathcal{D}_\ell)]$: **Specification on the state space** with the sequence $(\mathcal{D}_\ell)_{\ell \geq 1}$:

For any ℓ, *it holds both that* $\mathcal{D}_\ell$ *is a closed subset of* $\mathcal{X}$ *and that* $\mathcal{D}_\ell \subset int(\mathcal{D}_{\ell+1})$.

Remark 2.5.1 Originally in [15], this assumption is strengthened with $\cup_{\ell \geq 1} \mathcal{D}_\ell = \mathcal{X}$. It was shown in [14] how to adapt the conclusions without this additional restriction. The state spaces $\mathcal{X}$ that we consider include boundaries in the vicinity of which the process $X^{(d)}$ is a degenerate diffusion (for instance the boundary corresponding to $x_1 = 0$). It is thus more convenient for some of our proofs not to assume $\cup_{\ell \geq 1} \mathcal{D}_\ell = \mathcal{X}$ and if possible to keep the diffusion process non-degenerate on any $\mathcal{D}_\ell$.

The sequence $\mathcal{D}_\ell$ will serve as a reference in the following, and we also denote:

$$\mathbf{D} := \left\{\mathcal{D} \subset \mathcal{X}\,;\ \overline{\mathcal{D}} = \mathcal{D}\,,\ \exists\, \ell \geq 1,\ \mathcal{D} \subset \mathcal{D}_\ell\right\}, \tag{2.11}$$

where $\overline{\mathcal{D}}$ denotes the closure of the subset $\mathcal{D}$ of $\mathcal{X}$, so that the elements of $\mathbf{D}$ are closed subsets of $\mathcal{X}$ that are countained in $\mathcal{D}_\ell$, for ℓ sufficiently large.

For this trajectorial approach, we strongly rely on the representation of the semi-group (P_t) in terms of a strong Markov process $(X_t)_{t\in[0,\tau_\partial)}$ defined up to some extinction time τ_∂. Recall $P_t f(x) = \mathbb{E}_x[f(X_t);\ t < \tau_\partial]$ for any $x \in \mathcal{X}$ and $f \in B(\mathcal{X})$. For the next statements, we will exploit the following notations for the exit and entry times of any subset $\mathcal{D}$ of $\mathcal{X}$:

$$T_{\mathcal{D}} := \inf\{t \geq 0\,;\ X_t \notin \mathcal{D}\}\ , \qquad \tau_{\mathcal{D}} := \inf\{t \geq 0\,;\ X_t \in \mathcal{D}\}\,.$$

Besides this dependency on the sequence $(\mathcal{D}_\ell)$ the next assumptions share as commom parameters the probability measure ζ on $\mathcal{X}$, the real number $\rho > 0$ and the set $E \in \mathbf{D}$. They are recalled in square brackets in the notations of the basic assumptions.
$(A1)[\zeta, (\mathcal{D}_\ell)]$: **Mixing property** for the reference probability measure $\zeta \in \mathcal{M}_1(\mathcal{X})$ according to the sequence $(\mathcal{D}_\ell)_{\ell\geq 1}$:

For any integer $\ell \geq 1$, there exist both an integer $L > \ell$ and $c, t > 0$ such that the following holds for any initial condition $y \in \mathcal{D}_\ell$:

$$\mathbb{P}_y\Big(X_t \in \mathrm{d}x\,;\ t < \tau_\partial \wedge T_{\mathcal{D}_L}\Big) \geq c\ \zeta(\mathrm{d}x).$$

$(A2)[\rho, E]$: **Escape from the Transitory domain** with penalty-rate $\rho > 0$, where the complementaty of this transitory domain is $E \in \mathbf{D}$:

$$\sup_{\{x\in\mathcal{X}\}}\ \mathbb{E}_x\big(\exp\big[\rho\cdot(\tau_\partial \wedge \tau_E)\big]\big) < \infty.$$

ρ in the previous exponential moment is required to be strictly larger than the following **"survival estimate"** ρ_S, which a priori depends on ζ:

$$\rho_S[\zeta] := \sup\Big\{\gamma \geq 0;\ \sup_{\{L\geq 1\}} \liminf_{\{t>0\}}\ e^{\gamma t}\, \mathbb{P}_\zeta\big(t < \tau_\partial \wedge T_{\mathcal{D}_L}\big) = 0\Big\} \vee 0.$$

Remark 2.5.2 It is proved in [14, Theorem 2.16] that $\rho_S[\zeta]$ coincides with the extinction rate ρ_0 (and is thus independent of ζ), provided the semi-group displays QSC.

The next two assumptions are proposed as alternatives and each alternative will be exploited in the current paper. The former is the assumption first introduced in [15, Section 2.1]. The latter provides a way to ensure the former given $(A0)$, $(A1)$ and $(A2)$ as proved in [14, Theorem 2.3].

$(A3)[\zeta, E]$: **Asymptotic comparison of survival** on the set $E \in \mathbf{D}$ of initial conditions, with reference probability measure ζ:

$$\limsup_{t\to\infty} \sup_{x\in E} \frac{\mathbb{P}_x(t < \tau_\partial)}{\mathbb{P}_\zeta(t < \tau_\partial)} < \infty.$$

$(A3_F)[\zeta, \rho, E]$: **Almost perfect harvest** on the set $E \in \mathbf{D}$ of initial conditions, with reference probability measure ζ and penalty-rate ρ:

For any $\epsilon \in (0, 1)$, *there exist* $t_F, c > 0$ *such that for any* $y \in E$ *there exist two stopping times* U_H *and* V *with the following properties:*

$$\mathbb{P}_y\big(X(U_H) \in \mathrm{d}x\,;\; U_H < \tau_\partial\big) \le c\,\mathbb{P}_\zeta\big(X(V) \in \mathrm{d}x\,;\; V < \tau_\partial\big).$$

including the next conditions on U_H:

$$\{\tau_\partial \wedge t_F < U_H\} = \{U_H = \infty\} \quad \text{and} \quad \mathbb{P}_x(U_H = \infty,\, t < \tau_\partial) \le \epsilon\, \exp(-\rho\, t_F).$$

Furthermore, Ω *is of path type.*

As stated in [14, Proposition 2.2], the assumption that Ω is of path type ensures sufficient regularity properties of stopping times with respect to iterative procedures exploiting the strong Markov property.

2.5.2 General theorems of convergence

We exploit the following definitions of Assumptions (**A**) and (**AF**) from [14].

Assumption (**A**): "There exists $\zeta \in \mathcal{M}_1(\mathcal{X})$ such that $(A1)[\zeta, (\mathcal{D}_\ell)]$ holds for a specific sequence $(\mathcal{D}_\ell)$ satisfying $(A0)$. Moreover, there exists $\rho > \rho_S[\zeta]$ and $E \in \mathbf{D}$ such that Assumptions $(A2)[\rho, E]$ and $(A3)[\zeta, E]$ hold."

Assumption (**AF**) is exactly Assumption (**A**) with $(A3)[\zeta, E]$ replaced by $(A3_F)[\zeta, \rho, E]$.

Theorem 2.8 in [14] can be restated for our purpose as follows:

Theorem 2.5.3 *Assume that either* (**A**) *or* (**AF**) *holds. Then, the semigroup* P *displays QSC with characteristics* $(\nu, h, \rho_0) \in \mathcal{M}_1(\mathcal{X}) \times B_+(\mathcal{X}) \times \mathbb{R}_+$, h *is bounded away from* 0 *on* $\mathcal{D}_\ell$ *for any* $\ell \ge 1$ *and the Q-process exists on* $\mathcal{H} := \{x \in \mathcal{X}\,;\; h(x) > 0\}$.

Remark 2.5.4 The proof of Theorem 2.5.3 is originally stated in continuous time. A careful check of the arguments given in [14, Section 3] and in [15, Section 3] shows that their extension to the discrete-time setting (exploited in the following Theorem 2.2.2) does not raise any issue.

Since the exploited sequence $(\mathcal{D}_\ell)_{\ell\ge1}$ usually does not cover the whole state space, we shall exploit [14, Proposition 2.10] to deduce some lower-bounds of h. The next proposition recalls its statement.

Proposition 2.5.5 *Assume that* (**A**) *or* (**AF**) *holds. Then, the survival capacity h is uniformly lower-bounded on any set $H \subset \mathcal{X}$ that satisfies the following condition:* (H_0) : *there exists $t > 0$, $\ell \geq 1$ such that* $\inf_{\{x \in H\}} \mathbb{P}_x(\tau_{\mathcal{D}_\ell} \leq t \wedge \tau_\partial) > 0$.
It implies the following identification of $\mathcal{H} := \{x \in \mathcal{X}\,;\; h(x) > 0\}$:

$$\mathcal{H} = \Big\{x \in \mathcal{X}\,;\; \exists\, \ell \geq 1,\; \mathbb{P}_x(\tau_{\mathcal{D}_\ell} < \tau_\partial) > 0\Big\}.$$

Thanks to Proposition 2.5.5, we shall prove in our models that h is actually positive. Note that this property entails the uniqueness of the QSD thanks to Corollary 2.1.1.

3 Outlook

3.1 Interpretation of crucial parameters

For the infinite-dimensional process, no parameter other than α and λ is introduced. We deduce from Theorem 2.4.1 that the QSD and the survival capacity depend only on α and λ, as well as the real numbers $C, \gamma > 0$ in (2.1) and (2.2).

As already noted by Haigh in [7], α/λ is the average number of deleterious mutations that are established in the deterministic limit (neglecting neutral fluctuations). The deterministic distribution of mutations is a function of α/λ, and actually follows a Poisson distribution with this mean, as shown in [4]. To infer the level of fluctuations around this deterministic equilibrium, we shall look at the coefficient in front of the martingale term in a new time-scale such that the mutation rate is set to 1. This gives $1/\lambda$, which we recall to scale as $\sqrt{1/N_e}$, where N_e is the population size. A large population size thus corresponds to letting λ go to infinity, making the deviations away from the deterministic distribution more unlikely.

A natural scale for the time between clicks can be directly derived from the notion of QSC, with the definition $t_C := \rho_0^{-1}$. On the other hand, we can propose the following definition for the relaxation time:

$$\begin{aligned} t_R := \inf \Big\{ t_r > 0\,;\; \exists\, C > 0,\; \forall \mu \in \mathcal{M}_1(\mathcal{X}),\; \forall t > 0, \\ \|\mu A_t - \nu\|_{TV} \leq (C/\langle \mu \,|\, h\rangle) \cdot e^{-t/t_r} \Big\}. \end{aligned} \tag{3.1}$$

Our results justify that this definition, which involves no ad-hoc parameters, leads to a finite quantity (upper-bounded by $1/\gamma$, where γ is the rate deduced in the proof of Theorem 2.4.1). Remark also that the convergence to h and β also occurs at a rate that is quicker than $1/t_R$, as one can check by adjusting the justification in [15].

By relying on the arguments of Theorem 2.4.1 and Proposition 2.3.4, we expect that truncating the number of accumulated mutations is not likely to alter much this value of t_R provided the threshold is sufficiently large. Since we cannot evaluate t_R precisely and are only able to provide an upper-bound, this is still conjectural. But substantial increase of these last components are proved to be rare thanks to Proposition 2.3.4 and not so significant when we look at Sect. 7.5.

Remark 3.1.1 The dependence on the initial condition in (3.1) is expected from the linearity of the semi-group (P_t), as observed in [14]. More general dependencies could nonetheless be imagined, relying for instance on Lyapunov functions as in [18] or in [19]. We simply do not think it would change the value of t_R because the confinement is mainly due to extinction and immediate repulsion from the boundaries.

3.2 Previous estimations

The study of this quasi-stationary regime arises naturally when one wishes to estimate the rate at which the ratchet clicks. To obtain quantitative estimates, several authors have justified their approach by assuming that the typical clicking time t_C is much larger than the typical relaxation time t_R of the system, usually with an empirical reference for the latter [3, 4].

In [3], an estimation of t_C in the context where $t_R \ll t_C$ is obtained through the characteristic equation of a certain QSD $\nu_\star$, of the form $\mathcal{L}\nu_\star = -\lambda \cdot \nu_\star$, with $\mathcal{L}$ is a certain infinitesimal generator and λ its main eigenvalue. This QSD $\nu_\star$ that they study is not the general QSD $\nu^{(\infty)}$ that we describe. The detailed description of the latter is reasonably argued to be too intricate. The former is in fact derived from a one-dimensional approximation of the process under metastability. It is argued that in the context of large populations, and given the number of fittest individuals, one can approximate the rest of the distribution as an almost deterministic profile. The dependence in this number of fittest individuals only occurs in the normalizing factor of this distribution. This latter argument of concentration could probably be made rigorous by using Large Deviation theory. Such results are beyond the scope of the present paper.

Note that the validity of this approximation relies upon the fact that $t_R \ll t_C$, where t_R is to be related to the QSD $\nu^{(\infty)}$. The relaxation rate of $\nu_\star$ is only a partial indication, although presumably carrying most of the information.

3.3 The quasi-stationary regime is generally observed for $t_R \ll t_C$

Provided $t_R \ll t_C$, we expect to generally observe the quasi-stationary regime between clicks. It is classical that with the QSD as an initial condition, the extinction time and extinction state are independent, the former being exponentially distributed, as it is stated in [22, Theorem 2.6]. Assuming that we start the analysis at a new click after a long time-interval without click, it implies that the profile of mutations just after the click is distributed as the restriction of the QSD to the hyperplane $\{X_0 = 0\}$.

Since having large values of M_1 makes it actually harder for the process to reach the hyperplane, we expect that, under the QSD restricted to $\{X_0 = 0\}$, M_1 tends to be smaller than the prediction $1 + \lambda/\alpha$ derived from the deterministic limit (under the constraint that $x_0 = 0$). Besides, the fittest individuals are altered by first changing into the type with only one mutation. So we expect also that under the QSD restricted to $\{X_0 = 0\}$, there is an over-representation of the proportion of individuals carrying a single mutation (the new optimal trait). Thus, we expect the distribution just after the click to be less prone to a future click than would be the QSD itself. Since $t_R \ll t_C$, the quasi-stationary regime is then rapidly reached.

Let us also imagine a dramatic situation where some clicks would rapidly follow each others. Then, it would imply that these fittest classes of individuals are rapidly wiped off, while not letting much time for the others to change much. Since we have seen that we have very strong controls of moments under the QSD, cf notably Proposition 2.3.4, such succession of clicks cannot hold for long. A class that is not prone to a quick extinction should be reached quite early and generate a new quasi-stationary regime. Such dramatic situations, which are very rare, are thus expected to be very isolated and of limited impact.

Expecting an exponential law for the inter-click intervals and the independence between them should be in conclusion a good approximation provided $t_R \ll t_C$.

As we discuss in [23, Subsection 2.3], one can also conclude whether or not the QSD profile is likely to be observed without conditioning by comparing ν to the survival capacity h. If quasi-stationarity is stable, we do not expect that the conditioning on having a click in the far future shall substantially alter the dynamics. In most trajectories, the Q-process shall thus behave as the original process. So h should be mostly constant on the support of $\beta(\mathrm{d}x) = h(x)\,\nu(\mathrm{d}x)$, implying $h \approx 1$ where the density of ν is large.

In practice, the QSD and the survival capacity are certainly quite difficult to specify with simulations because they live on a large dimensional space. Likewise, the convergence in total variation exploited in (3.1) is probably not very practical for numerical estimation.

3.4 Motivation for an unbounded number of deleterious mutations

In order to prove quasi-stationarity results, the case where $d < \infty$ can be treated more easily and provides an introduction to the case $d = \infty$. Nonetheless, the arguments for having a convergence at a given rate becomes more and more artificial as d tends to infinity. The constant involved in the Harnack inequalities goes to zero as the dimension increases. By considering the case $d = \infty$, we actually handle as a whole the case where d is sufficiently large. By these means, we are able to prove that the rate of convergence can be upper-bounded by a quantity that does not depend on the specific value of d. This is to be expected since, even when a large number of deleterious mutations is permitted, we expect individuals carrying a large number of mutations to remain negligible.

Referring for instance to [4], it is not difficult to prove that in the deterministic limit of a large population, the empirical measure of the number of mutations in the population tends to a Poisson distribution. The tail of the distribution is quickly disappearing. This deterministic limit corresponds to a limiting time-change of Eq. (1.1) of the form $t' = t/\epsilon$ with $\alpha = \alpha'/\epsilon$, $\lambda = \lambda'/\epsilon$ as ϵ tends to 0. The Poisson distribution has a mean of $\lambda'/\alpha' = \lambda/\alpha$ so that it may be possible to quantify much more precisely than we do the threshold in the number of deleterious mutations after which differentiating individuals is not so crucial. This could make it possible to obtain quantitative bounds from our arguments in the context of very large populations (in the vicinity of the deterministic limit).

Proofs

All the following properties strongly depend on the values of $\alpha > 0$ ($\alpha \in (0, 1)$ for the discrete state-space) and $\lambda > 0$. For brevity since the line of proof holds for any such values, this expected dependency is not recalled in the following statements. The dependencies in N for the discrete state-space and in d for the finite dimensional SDE are generally recalled (in the state spaces and the random times notably) because of the interest of having statements that are uniform over these parameters. They may still be omitted to avoid too heavy notations.

4 Proof of Theorem 2.2.2

The proof of Theorem 2.2.2 relies on the proof of Assumption (**A**) as stated in Sect. 2.5.2. For any $z \in \mathcal{M}_1^{(0),N}(\mathbb{Z}_+)$, we denote by $\mathbb{P}_z^N$ the law of the process Z^N with initial condition z, as defined in Sect. 1.2.1. This process is associated to the extinction time τ_∂^N in (2.7). The first step deals with the mixing estimate, while we focus on the persistence of large mutational burdens in the second step (for the estimate on the escape from the transitory domain).

Step 1: access to any focal state
We first prove that any focal state of the population can be reached uniformly in the initial condition with a non-neglible probability, as stated in the upcoming Proposition 4.0.1:

Proposition 4.0.1 *For any integer $N \geq 1$ and any element $z \in \mathcal{M}_1^{(0),N}(\mathbb{Z}_+)$:*

$$\inf \left\{ \mathbb{P}_{z_0}^N(Z^N(1) = z) \,\middle|\, z_0 \in \mathcal{M}_1^{(0),N}(\mathbb{Z}_+) \right\} > 0.$$

Proof : We impose that all the individuals of the next generation are the offspring of an individual without any mutation, and prescribe the number of mutations that they get from the profile of z. For any initial condition $z_0 \in \mathcal{M}_1^{(0),N}(\mathbb{Z}_+)$, the probability of choosing a fittest individual as a parent is: $z_0(0)/(\sum_{\{i \geq 0\}} z_0(i) \cdot (1-\alpha)^i)$. This probability is uniformly lower-bounded by $1/N$. The number of mutations is then chosen independently of z_0, and there is indeed a positive probability that the sequence of independent Poisson distributed random variables has an empirical law distributed as z. This concludes the proof of Proposition 4.0.1. □

Step 2: disappearance of any large mutational burden
With a probability close to 1, the sub-population of individuals carrying a large number of mutations leave no progeny, as stated in the upcoming Proposition 4.0.2:

Proposition 4.0.2 *For any $N \geq 1$ and $\epsilon > 0$, there exists $K \geq 1$ such that the following inequality holds with $E^N := \{z \in \mathcal{M}_1^{(0),N}(\mathbb{Z}_+)\,;\; z([\![K, \infty[\![) = 0\}$ for any $z \in \mathcal{M}_1^{(0),N}(\mathbb{Z}_+)$:*

$$\mathbb{P}_z^N(Z^N(1) \notin E^N) \leq \epsilon.$$

Proof : Let $k_1 \geq 1$ for the threshold in the number of mutations. The probability that an individual chooses a parent with more than k_1 mutations is upper-bounded by $N \cdot (1-\alpha)^{k_1}$, since $z(0) \geq 1$. For any $\epsilon > 0$, we thus choose $k_1 \geq 1$ such that, with a probability greater than $1 - \epsilon/2$, no individual in the next generation descends from an individual with more than k_1 mutations. Likewise, there exists $k_2 \geq 1$ such that, with a probability greater than $1-\epsilon/2$, the number of additional mutations is less than k_2 (for any individual, independently of the initial condition z). We can then conclude the proof of Proposition 4.0.2 in that:

$$\forall z \in \mathcal{M}_1^{(0),N}(\mathbb{Z}_+), \quad \mathbb{P}_z^N(Z^N(1) \notin E^N) \leq \epsilon,$$

where E^N is defined as in the Proposition 4.0.2 with $K = k_1 + k_2$. □

***Concluding the proof of Theorem* 2.2.2**

We simply set $\mathcal{D}_\ell^N$ to be the whole space $\mathcal{M}_1^{(0),N}(\mathbb{Z}_+)$ for any ℓ. Note that $(A0)$ is satisfied even for this degenerate case. Actually, the exit time are just infinite and the entry times in $\mathcal{D}_\ell^N$ always equal zero. Secondly, Proposition 4.0.1 implies Assumption $(A1)$. Concerning $(A2)$, we exploit Proposition 4.0.2 inductively over $k \geq 1$ to deduce an upper-bound of the following form, thanks to the Markov property and with τ_E^N the hitting time of E^N by the process Z^N:

$$\forall z \in \mathcal{M}_1^{(0),N}(\mathbb{Z}_+), \quad \mathbb{P}_z^N(k < \tau_\partial^N \wedge \tau_E^N) \leq 2^{-k} \cdot \exp(-\rho k).$$

It implies the upper-bound on the exponential moment for any z by splitting the expectation depending on the interval of the form $[k, k+1)$ that contains $\tau_\partial^N \wedge \tau_E^N$, i.e.:

$$\begin{aligned}\mathbb{E}_z^N(\exp[\rho \cdot (\tau_\partial^N \wedge \tau_E^N)]) &\leq \sum_{k\geq 0} \exp[\rho \cdot (k+1)] \cdot \mathbb{P}_z^N\big(\tau_\partial^N \wedge \tau_E^N \in [k, k+1)\big)\\ &\leq e^\rho \textstyle\sum_{k\geq 0} 2^{-k} = 2e^\rho < \infty.\end{aligned}$$

For the last criterion $(A3)$, we remark that E^N as defined in Proposition 4.0.2 is finite. Thanks to Proposition 4.0.1 and to the Markov property, we can thus choose $c = c(N) > 0$ such that the following comparisons of survival hold for any $t \geq 1$:

$$\mathbb{P}_{\delta_0}^N(t < \tau_\partial^N) \geq c \sup_{z\in E^N} \mathbb{P}_z^N(t-1 < \tau_\partial^N) \geq c \sup_{z\in E^N} \mathbb{P}_z^N(t < \tau_\partial^N). \qquad (4.1)$$

This concludes $(A3)$ and that Assumption $(\mathbf{A})$ is satisfied.

Thanks to Theorem 2.5.3, the semigroup P^N displays QSC with characteristics $(\nu^N, h^N, \rho_0^N) \in \mathcal{M}_1(\mathcal{M}_1^{(0),N}(\mathbb{Z}_+)) \times B_+(\mathcal{M}_1^{(0),N}(\mathbb{Z}_+)) \times \mathbb{R}_+$. Moreover, h^N is uniformly bounded away from 0. Besides, $\rho_0^N = -\log\left[\mathbb{P}_{\nu^N}^N(1 < \tau_\partial)\right] > 0$ because $\mathbb{P}_z^N(\tau_\partial = 1) > 0$ holds for any $z \in \mathcal{M}_1^{(0),N}(\mathbb{Z}_+)$. This concludes the proof of Theorem 2.2.2. $\square$

Remark 4.0.3 If we were to replace the law of ξ by a Bernoulli distribution (mutations occurring one by one), Proposition 4.0.1 would still hold with the restriction of $z = \delta_0$, which is the only case we need. It would extend to any z provided we change the time 1 by the maximal number of mutations in z. The proof would not be much more difficult with overlapping generations, except that individuals should then be removed one by one. The proof of the equivalent of Proposition 4.0.2 would merely be slightly more difficult. The details are left to the interested reader.

5 Proof of Theorem 2.3.2

The proof of Theorem 2.3.2 also relies on the set (**A**) of criteria, as stated in Sect. 2.5.2. The proofs of two of these criteria (especially the mixing estimate and the asymptotic comparison of survival) are very much inspired by those of [15, Subsection 4.2.2]. Likewise, they exploit Harnack's inequality –the following Property (H)– classically deduced for elliptic diffusions, see Sect. 5.1.1. The estimate of escape on the other exploits several comparisons with one-dimensional diffusions to deal with the behavior of the process near the boundary of the domain, by exploiting the classical results presented in Sect. 5.1.2.

The mixing estimate is then proved as the first step (in Sect. 5.2) and the asymptotic comparison of survival as the second step (in Sect. 5.3). We turn next to the estimate of the escape from the transitory domain (in Sect. 5.4), then to estimations of lower-bound on the survival capacity (in Sect. 5.5) before we can conclude the proof of Theorem 2.3.2 in Sect. 5.6.

We consider the following increasing closed subsets of $\mathcal{X}_d$ as a reference:

$$\mathcal{D}_\ell^{(d)} := \left\{ x = (x_i)_{i \in [\![0,d]\!]} \in \left[\frac{1}{2\ell d}\,,\ 1 - \frac{1}{2\ell d}\right]^d \ ;\ \sum_{i=0}^{d} x_i = 1 \right\}. \tag{5.1}$$

Remark that $T_{\mathcal{D}_\ell}^{(d)} \wedge \tau_\partial^{(d)} = T_{\mathcal{D}_\ell}^{(d)}$ holds for any ℓ, where $T_{\mathcal{D}_\ell}^{(d)}$ denotes the exit time of the process $X^{(d)}$ out of $\mathcal{D}_\ell^{(d)}$. Similarly and for simplicity, the superscript (d) for the set $\mathcal{D}_\ell^{(d)}$ is dropped in the corresponding entry time, namely $\tau_{\mathcal{D}_\ell}^{(d)}$.

For any $d \in [\![1, \infty]\!]$ and any $x \in \mathcal{X}_d$, the process $X^{(d)}$ is solution under $\mathbb{P}_x$ of the system (1.1) with initial condition $X_i^{(d)}(0) = x_i$.

There is no real ambiguity in the dependency in d, especially since this dependency shall be made visible for the process $X^{(d)}$ and the associated processes and stopping times, By extension, for any probability measure ζ on $\mathcal{X}_d$, $\mathbb{P}_\zeta(d\omega) = \int_{\mathcal{X}_d} \mathbb{P}_x(d\omega)\zeta(dx)$. Note that $\mathbb{P}$ can be seen as specifying the law of the sequence

$(W_i)_{i\in\mathbb{Z}_+}$ while $x \in \mathcal{X}_d$ can be seen as an element of $\mathcal{X}_\infty$ (with $x_i = 0$ for any $i \geq d+1$), so that a common notation makes sense as well.

5.1 First crucial properties

5.1.1 Harnack's inequality

Property (H) is defined as follows for any process $(Y(t))_{t\geq 0}$ on $\mathcal{Y}$ with generator $\mathcal{L}$ (including possibly an extinction rate ρ_c):

For any two connected open relatively compact sets $\mathfrak{K}^\wedge, \mathfrak{K}^\vee \subset \mathcal{Y}$ with $\mathcal{C}^\infty$ boundaries such that $\overline{\mathfrak{K}^\wedge} \subset \mathfrak{K}^\vee$ (where $\overline{\mathfrak{K}}$ denotes the closure of the set $\mathfrak{K}$), and any $0 < t_1 < t_2$, there exists $C = C(\mathcal{L}, t_1, t_2, \mathfrak{K}^\wedge, \mathfrak{K}^\vee) > 0$ such that the following properties hold for any positive $\mathcal{C}^\infty$ constraints: $u_{\partial\mathfrak{K}^\vee} : (\{0\} \times \mathfrak{K}^\vee) \cup ([0, t_2] \times \partial\mathfrak{K}^\vee) \to \mathbb{R}_+$. There exists a unique positive strong solution $u(t, x)$ to the following Cauchy problem:

$$\begin{aligned} \partial_t u(t,x) &= \mathcal{L}u(t,x) && \text{on } [0,t_2] \times \mathfrak{K}^\vee, \\ u(t,x) &= u_{\partial\mathfrak{K}^\vee}(x) && \text{on } (\{0\} \times \mathfrak{K}^\vee) \cup ([0,t_2] \times \partial\mathfrak{K}^\vee), \end{aligned}$$

and u satisfies the following inequality:

$$\inf_{x\in\mathfrak{K}^\wedge} u(t_2, x) \geq C \sup_{x\in\mathfrak{K}^\wedge} u(t_1, x).$$

For any $d \in \mathbb{N}$ and $\ell \in \mathbb{N}$, thanks to Proposition A.0.1 in the appendix, we identify the generator $\mathcal{L}^{(d)}$ of the finite dimensional process $X^{(d)}$ in restriction to the set $\mathcal{D}_\ell^{(d)}$ to the following nondivergence form, for $u \in C^2(\mathcal{D}_\ell^{(d)})$ and $x \in \mathcal{D}_\ell^{(d)}$:

$$\mathcal{L}^{(d)}u(x) = \frac{1}{2}\sum_{i,j=1}^{d} \sigma_{i,j}^{(d)}(x)\partial_{i,j}^2 u(x) + \sum_{i=1}^{d} b_i^{(d)}(x)\partial_i u(x).$$

Lemma 5.1.1 *Property (H) holds for any $d \in \mathbb{N}$ and any $\ell \in \mathbb{N}$ for the finite dimensional process $X^{(d)}$ with generator $\mathcal{L}^{(d)}$ in restriction to the set $\mathcal{D}_\ell^{(d)}$ as defined in (5.1).*

Proof : For any $d \in \mathbb{N}$ and any $\ell \in \mathbb{N}$, the diffusion matrix $\sigma^{(d)}$ is uniformly elliptic on $\mathcal{D}_\ell^{(d)}$ while $\sigma^{(d)}$ and the drift term $b^{(d)}$ are $\mathcal{C}^\infty$ on $\mathcal{D}_\ell^{(d)}$. As noted in [15, Section 4.2.2], this entails Property (H). We sketch the argument for completeness, rather referring to [24] for the clarity of its presentation. The existence and uniqueness of the solution u, with the fact that $u \in \mathcal{C}^\infty$, is a consequence of [24, Theorem 7]. Thanks to [24, Theorem 8], this solution is positive. We can then apply [24, Theorem 10] on u to deduce Harnack's inequality and complete the proof of Property (H). □

5.1.2 Boundary classification for one-dimensional diffusions

The following proofs rely on comparison principles with one-dimensional diffusions, for which the boundary classification is well-described. We first present briefly in the upcoming Lemma 5.1.2 the conclusions from [25, Section 6, Chapter 15] for the specific cases of solutions Y on the state-space $(0, 1)$ to SDEs of the following form:

$$dY(t) = b \cdot Y(t)dt + \sqrt{Y(t) \cdot (1 - Y(t))}dB(t), \tag{5.2}$$

where B is a standard Brownian motion and $b \in \mathbb{R}$.

To motivate this form of the diffusion coefficient, note that the martingale term of each coordinate X_i taken separately takes actually this form, as one can check from Proposition A.0.1 in the appendix. The vicinity of 0 will inform on the behavior of the boundary for a linear drift and the vicinity of 1 for a nearly constant drift.

Lemma 5.1.2 (i) *For any $b \in \mathbb{R}$, the left-boundary 0 is accessible to the solution Y of (5.2), which entails the following convergence for any $t > 0$:*

$$\lim_{y \to 0} \mathbb{P}_y(\tau_0^Y < t) = 1.$$

0 is actually an exit boundary, so that Y gets absorbed at 0.

(ii) *For any $b > -\frac{1}{2}$, the right-boundary 1 is accessible, which entails the following convergence for any $t > 0$:*

$$\lim_{y \to 1} \mathbb{P}_y(\tau_1^Y < t) = 1.$$

1 is actually an exit boundary iff $b \geq 0$. If $b \in (-\frac{1}{2}, 0)$, 1 is a regular reflecting boundary, which implies that $Y(t) < 1$ holds a.s. for any $t > 0$.

(iii) *For any $b \leq -\frac{1}{2}$, the right-boundary 1 is inaccessible and an entrance boundary, in the following sense: for any $y \in (0, 1)$,*

$$\lim_{t \to \infty} \inf_{z \in (y,1)} \mathbb{P}_z(\tau_y^Y < t) = 1.$$

Remark 5.1.3 In the context of one-dimensional diffusions, there is a well-established reformulation of the process that exploits the notions of scale function and speed measure. This framework is particularly beneficial for studying the behavior of the process close to its boundaries. This reformulation has also been exploited for the study of quasi-stationarity of one-dimensional diffusions, be it with spectral techniques that are specific and more classical for such diffusions (as in [26]) or recent extensions more closely related to our approach (notably in [27]). Since we do not exploit further the specific properties of an exit boundary and a regular reflecting boundary, these definitions are not specified and the interested reader is deferred to [25, Section 6, Chapter 15].

As a particular case of [28, Proposition 3.12], we have the upcoming Lemma 5.1.4, that will often be exploited for our comparison estimates on the boundaries:

Lemma 5.1.4 *Let $\hat{b}, \check{b} : \Omega \times \mathbb{R}_+ \times [0, 1] \mapsto \mathbb{R}$ and $L > 0$ be such that $\hat{b}(., ., y), \check{b}(., ., y)$ are measurable for any $y \in [0, 1]$ and that $\hat{b}(\omega, t, .), \check{b}(\omega, t, .)$ are L-Lispchitz continuous for all $\omega, t \in \Omega \times \mathbb{R}_+$. Assume that $\hat{b}(\omega, t, y) \geq \check{b}(\omega, t, y)$*

holds for any $(\omega, t, y) \in \Omega \times \mathbb{R}_+ \times [0,1]$ *and that* $\hat{y}, \check{y} \in [0,1]$ *are such that* $\hat{y} \geq \check{y}$. *Then the inequality* $\hat{Y}(t) \geq \check{Y}(t)$ *holds for any* $t \in \mathbb{R}_+$ *for the solution* $\hat{Y}$ *and* $\check{Y}$ *to the following SDEs:*

$$\mathrm{d}\hat{Y}(t) := \hat{b}_t(\hat{Y}(t))\mathrm{d}t + \sqrt{\hat{Y}(t)\cdot(1-\hat{Y}(t))}\mathrm{d}B(t) \quad \hat{Y}(0) = \hat{y},$$
$$\mathrm{d}\check{Y}(t) := \check{b}_t(\check{Y}(t))\mathrm{d}t + \sqrt{\check{Y}(t)\cdot(1-\check{Y}(t))}\mathrm{d}B(t) \quad \check{Y}(0) = \check{y},$$

where B *is a standard Brownian motion.*

In practice, we will exploit the following corollary instead of Lemma 5.1.2(ii) several times (the corollary being implied by this lemma together with Lemma 5.1.4).

Corollary 5.1.5 *Consider for any* $\varphi, \psi \in \mathbb{R}$ *the solution* Z *to the following SDE:*

$$\mathrm{d}Z(t) = \big[\varphi + \psi \cdot Z(t)\big]\mathrm{d}t + \sqrt{Z(t)\cdot(1-Z(t))}\mathrm{d}B(t),$$

where B *is a standard Brownian motion. For any* $\varphi \in (0, 1/2)$ *and* $\psi \in \mathbb{R}$*, 0 is a regular reflecting boundary for* Z.

5.2 Step 1: mixing estimate

The aim of this subsection is exactly $(A1)$, as stated in the upcoming Proposition 5.2.1:

Proposition 5.2.1 *For any* $d \in \mathbb{N}$, *there exists* $\zeta^{(d)} \in \mathcal{M}_1(\mathcal{X}_d)$ *with support in* $\mathcal{D}_2^{(d)}$ *such that the following property holds for any* $\ell \geq 1$. *There exists* $c > 0$ *such that:*

$$\forall x \in \mathcal{D}_\ell^{(d)}, \quad \mathbb{P}_x\left(X^{(d)}(1) \in \mathrm{d}y\,;\ 1 < T^{(d)}_{\mathcal{D}_{\ell+1}}\right) \geq c\,\zeta^{(d)}(\mathrm{d}y).$$

For the comparison with $(A1)$, recall that $T^{(d)}_{\mathcal{D}_\ell} = \tau^{(d)}_\partial \wedge T^{(d)}_{\mathcal{D}_\ell}$ holds for any ℓ.

Proof : Let $\ell \geq 1$. We choose two connected bounded open sets $\mathfrak{K}^\wedge, \mathfrak{K}^\vee \subset \mathcal{D}^{(d)}_{\ell+1}$ with $\mathcal{C}^\infty$ boundaries such that $\mathcal{D}^{(d)}_\ell \subset \mathfrak{K}^\wedge$ and $\overline{\mathfrak{K}^\wedge} \subset \mathfrak{K}^\vee$. Thanks to Lemma 5.1.1, we apply Property (H) (see Sect. 5.1.1) to $u(t,x) := \mathbb{E}_x\left(f(X^{(d)}(t))\,;\ t < T^{(d)}_{\mathfrak{K}^\vee}\right)$, where f is any non-negative $\mathcal{C}^\infty$ function with support in $\mathcal{D}^{(d)}_2 \subset \mathfrak{K}^\wedge$, and $T^{(d)}_{\mathfrak{K}^\vee} := \inf\{t \geq 0\,;\ X^{(d)}(t) \notin \mathfrak{K}^\vee\}$. Since in addition $T^{(d)}_{\mathcal{D}_2} \leq T^{(d)}_{\mathfrak{K}^\vee} \leq T^{(d)}_{\mathcal{D}_{\ell+1}}$, there exists a real $C^{(d)}_\ell > 0$ such that the following inequality holds for any $x \in \mathcal{D}^{(d)}_\ell$, $z \in \mathcal{D}^{(d)}_1$:

$$\mathbb{E}_x\left(f(X^{(d)}(1))\,;\ 1 < T^{(d)}_{\mathcal{D}_{\ell+1}}\right) \geq C^{(d)}_\ell\ \mathbb{E}_z\left(f(X^{(d)}(\tfrac{1}{2}))\,;\ \tfrac{1}{2} < T^{(d)}_{\mathcal{D}_2}\right).$$

With the arbitrary choices of $z^{(d)}$ as the barycenter of $\mathcal{D}^{(d)}_1$, we then define the probability measure $\zeta^{(d)}$ as follows:

$$\zeta^{(d)}(\mathrm{d}x) := \mathbb{P}_{z^{(d)}}\left(X^{(d)}(\tfrac{1}{2}) \in \mathrm{d}x \,\middle|\, \tfrac{1}{2} < T^{(d)}_{\mathcal{D}_2}\right),$$

which has support on $\mathcal{D}_2^{(d)}$ and is independent of ℓ. Since the constant $C_\ell^{(d)}$ does not depend on f, we deduce with $c_\ell = C_\ell^{(d)} \cdot \mathbb{P}_{z^{(d)}}\left(\frac{1}{2} < T_{\mathcal{D}_2}^{(d)}\right) > 0$ the following inequality between measures for any initial condition $x \in \mathcal{D}_\ell^{(d)}$:

$$\mathbb{E}_x\left(X^{(d)}(1) \in \mathrm{d}y\,;\; 1 < T_{\mathcal{D}_{\ell+1}}^{(d)}\right) \geq c_\ell\, \zeta^{(d)}(\mathrm{d}y),$$

which concludes the proof of Proposition 5.2.1. □

5.3 Step 2: asymptotic comparison of survival

The aim of this subsection is to prove the upcoming Proposition 5.3.1, thanks to which we will deduce $(A3)$:

Proposition 5.3.1 *The following boundedness property holds for any $d \in \mathbb{N}$ and any $\ell \geq 1$:*

$$\limsup_{t\to\infty} \sup_{x,x' \in \mathcal{D}_\ell^{(d)}} \frac{\mathbb{P}_x\left(t < \tau_\partial^{(d)}\right)}{\mathbb{P}_{x'}\left(t < \tau_\partial^{(d)}\right)} < \infty.$$

The proof of Proposition 5.3.1 is similar to the one of Proposition 5.2.1. It is nonetheless more technical because we can no longer neglect trajectories exiting $\mathcal{D}_{\ell+1}^{(d)}$.

Proof : We can find two connected open relatively compact sets $\mathfrak{K}^\wedge, \mathfrak{K}^\vee \subset \mathcal{X}_d$ with $\mathcal{C}^\infty$ boundaries such that $\mathcal{D}_\ell^{(d)} \subset \mathfrak{K}^\wedge$ and $\overline{\mathfrak{K}^\wedge} \subset \mathfrak{K}^\vee \subset int(\mathcal{D}_{\ell+1}^{(d)})$. We want to approximate the function:

$$u(t,x) := \mathbb{E}_x\left(f(X^{(d)}(t))\,;\; t < \tau_\partial^{(d)}\right), \quad \text{with } t \geq 1,\; x \in \mathfrak{K}^\vee$$

defined for some non-negative $f \in \mathcal{C}^\infty(\mathcal{X}_d)$. Thanks to [29, Theorem 5.1.15], u is continuous. It is clearly non-negative. However, it is a priori not regular enough to apply Harnack's inequality directly. Thus, we approximate it on the parabolic boundary $[1,\,\infty) \times \partial\mathfrak{K}^\vee \bigcup \{1\} \times \mathfrak{K}^\vee$ by some family $(U_k)_{k\geq 1}$ of non-negative smooth functions. We then deduce approximations of u in $[0,\,\infty) \times \mathfrak{K}^\vee$ by solutions u_k to the following Cauchy Problem:

$$\begin{aligned} &\partial_t u_k(t,x) - \mathcal{L}u_k(t,x) = 0, && \text{for } t \geq 0,\; x \in int(\mathfrak{K}^\vee) \\ &u_k(t,x) = U_k(t+1,x), && \text{for } t \geq 0,\; x \in \partial\mathfrak{K}^\vee, \quad \text{or } t = 0,\; x \in \mathfrak{K}^\vee. \end{aligned}$$

Thanks to Property (H) (in Sect. 5.1.1), the constant involved in Harnack's inequality does not depend on the values of U_k on the boundary. Thus, it applies with the same constant for the whole family of approximations u_k. With $t_1 := 1$ and $t_2 := 2$, we can thus choose $C_\ell^{(d)} > 0$ such that for any k and any $x, x' \in \mathcal{D}_\ell^{(d)}$:

$$u_k(1,x) \leq C_\ell^{(d)} u_k(2,x'),$$

where the constant $C_\ell^{(d)}$ does not depend on f either. Thanks to the proof in [21, Section 4, step 4], the Harnack inequality extends to the approximated function u, with the convergence of U_k on the parabolic boundary (and the time-shift of 1 taken into account). It means that the following inequality holds for any non-negative $f \in \mathcal{C}^\infty(\mathcal{Y})$ and any $x, x' \in \mathcal{D}_\ell^{(d)}$:

$$\mathbb{E}_x \left(f(X^{(d)}(2))\, ;\; 2 < \tau_\partial^{(d)} \right) \leq C_\ell^{(d)}\, \mathbb{E}_{x'} \left(f(X^{(d)}(3))\, ;\; 3 < \tau_\partial^{(d)} \right).$$

The inequality then extends to any measurable and bounded f. We now fix $t \geq 2$ and apply this result to the function $f_t(x) := \mathbb{P}_x(t-2 < \tau_\partial^{(d)})$, so that, thanks to the Markov property, the following comparison of survival holds for any $x, x' \in \mathcal{D}_\ell^{(d)}$ and any $t \geq 2$:

$$\begin{aligned}\mathbb{P}_x \left(t < \tau_\partial^{(d)} \right) &\leq C_\ell^{(d)}\, \mathbb{P}_{x'} \left(t+1 < \tau_\partial^{(d)} \right) \\ &\leq C_\ell^{(d)}\, \mathbb{P}_{x'} \left(t < \tau_\partial^{(d)} \right).\end{aligned}$$

Note that, since $C_\ell^{(d)}$ does not depend upon f, it does not depend upon t. This concludes the proof of Proposition 5.3.1. □

5.4 Step 3: escape from the transitory domain

The aim of this subsection is to prove the upcoming Proposition 5.4.1, that entails $(A2)$:

Proposition 5.4.1 *For any $d \in \mathbb{N}$ and $\rho > 0$, there exists $\ell \geq 1$ such that:*

$$\sup_{x \in \mathcal{X}_d} \mathbb{E}_x \exp\left[\rho \cdot \left(\tau_{\mathcal{D}_\ell}^{(d)} \wedge \tau_\partial^{(d)}\right)\right] \leq 16.$$

Two elementary steps

The proof is achieved with two forthcoming lemmas as intermediate steps. We first prove that the click is very likely to happen when the size of the optimal subpopulation is small, as stated in the upcoming Lemma 5.4.2:

Lemma 5.4.2 *For any $d \in \mathbb{N}$ and any time $t > 0$, the following supremum tends to 0 as y_0 tends to 0:*

$$\sup \left\{ \mathbb{P}_x \left(t < \tau_\partial^{(d)} \right) \middle| x \in \mathcal{X}_d, x_0 \leq y_0 \right\}.$$

Proof : Provided the initial condition x is such that $x_0 \leq y_0$, the process $X_0^{(d)}$, namely the initial component of the solution to (1.1)-(1.3), is upper-bounded by the solution Y to the following SDE, thanks to Lemma 5.1.4:

$$dY(t) = (\alpha d) \cdot Y(t)\, dt + \sqrt{Y(t) \cdot (1 - Y(t))}\, dB_0(t)\,, \quad Y(0) = y_0.$$

Thanks to Lemma 5.1.2(i), 0 is an exit boundary of Y, which concludes the proof of Lemma 5.4.2. □

We then deal iteratively with each subclass size to prove that these sizes escape the vicinity of 0, as stated in the upcoming Lemma 5.4.3.

Lemma 5.4.3 *For any integer $J \in [\![1, d]\!]$, any $y \in (0, 1)$, and any time $t > 0$, the next infimum tends to 1 as the constant $y' \in (0, y)$ tends to 0:*

$$\inf \left\{ \mathbb{P}_x \left(\tau_{y'}^{J,(d)} < t \wedge \tau_\partial^{(d)} \right) \middle| x \in \mathcal{X}_d, \forall j \leq J-1,\ x_j \geq y \right\},$$

where $\tau_{y'}^{J,(d)} := \inf\{s \geq 0\,;\ \forall j \leq J,\ X_j^{(d)}(s) \geq y'\}$.

Proof : Let $J \in [\![1, d]\!]$, $y \in (0, 1)$, $\epsilon, t > 0$. The process $X_j^{(d)}$, namely the j-th component of the solution to (1.1)-(1.3) for any $j \in [\![0, J-1]\!]$, is lower-bounded by solutions Y_j to SDEs of the following form, thanks to Lemma 5.1.4:

$$\mathrm{d}Y(t) = -(\lambda + \alpha J)\, \mathrm{d}t + \sqrt{Y(t) \cdot (1 - Y(t))}\, \mathrm{d}B(t)\,, \quad Y(0) = y,$$

where B is a standard Brownian motion. We then choose $t' \in (0, t)$ such that Y stays above $y/2$ on the time-interval $[0, t']$ with probability greater than $1 - \epsilon/(2J)$. We then exploit this property on Y_j for any $j \in [\![0, J-1]\!]$, so as to deduce the following inequality for any $d \in \mathbb{N}$ and any $x \in \mathcal{X}_d$ such that $x_j \geq y$ holds for any $j \leq J-1$:

$$\mathbb{P}_x \left(t < T_{y/2}^{J-1,(d)} \right) \geq 1 - \epsilon/2, \tag{5.3}$$

where $T_{y/2}^{J-1,(d)} := \inf\{s \geq 0\,;\ \exists\, j \leq J-1,\ X_j^{(d)}(s) \leq y/2\}$.

Let $y_1 := \lambda\, y/(4\lambda + 4\alpha J)$ and $\tau_{y_1}^{J,(d)}$ defined as in Lemma 5.4.3. The following inequalities thus hold for any $s \leq T_{y/2}^{J-1,(d)} \wedge \tau_{y_1}^{J,(d)}$:

$$\begin{aligned}(\alpha \cdot M_1^{(d)}(s) - \alpha J - \lambda) \cdot X_J^{(d)}(s) + \lambda \cdot X_{J-1}^{(d)}(s) &\geq -(\alpha J + \lambda) \cdot y_1 + \lambda\, y/2 \\ &\geq \lambda\, y/4.\end{aligned}$$

Thanks to Lemma 5.1.4, $X_J^{(d)}$ is thus a.s. lower-bounded on the time interval $[0, T_{y/2}^{J-1,(d)} \wedge \tau_{y_1}^{J,(d)}]$ by the solution Y_J to the following SDE:

$$\mathrm{d}Y_J(s) = \frac{(\lambda\, y) \wedge 1}{4}\, \mathrm{d}t + \sqrt{Y_J(t) \cdot (1 - Y_J(t))}\, \mathrm{d}B_J(t)\,, \quad Y_J(0) = 0,$$

where B_J is a standard Brownian Motion. Thanks to Corollary 5.1.5, the left-boundary 0 is regular reflecting for Y_J. Therefore, there exists $0 < y' \leq y_1 \wedge (y/2)$ such that:

$$\mathbb{P}(\sup_{\{s \leq t'\}} Y_J(s) < y') \leq \epsilon/2. \tag{5.4}$$

Provided that the initial condition satisfies that $x_j \geq y$ for any $j \in [\![0, J-1]\!]$, the property $\tau^{J,(d)}_{y'} < t \wedge \tau^{(d)}_\partial$ holds a.s. on the event $\left\{\sup_{s \leq t'} Y_J(s) \geq y'\right\} \cap \left\{t' < T^{J-1,(d)}_{y/2}\right\}$, which occurs with probability greater than $1 - \epsilon$ thanks to (5.3) and (5.4). This ends the proof of Lemma 5.4.3. □

***Concluding the proof of Proposition* 5.4.1**

Given $\rho > 0$, let $t_0 := \log(2)/\rho$. We can choose $y_0 \in (0, 1)$ thanks to Lemma 5.4.2 such that the probability of survival up to time t_0 is small as follows for any initial condition $x = (x_i)$ such that $x_0 < y_0$:

$$\mathbb{P}_x\left(t_0 < \tau^{(d)}_\partial\right) \leq \exp(-\rho\, t_0)/2 = \tfrac{1}{4}. \tag{5.5}$$

Thanks to Lemma 5.4.3, we iteratively obtain lower-bounds for the different components of $X^{(d)}$, so that for any $1 \leq J \leq d$ and any $y_{J-1} \in (0, 1)$, there exists $y_J \in (0, y_{J-1})$ such that the following inequality holds:

$$\inf\left\{\mathbb{P}_x\left(\tau^{J,(d)}_{y_J} < (t_0/d) \wedge \tau^{(d)}_\partial\right) \middle|\, x \in \mathcal{X}_d, \forall\, j \leq J-1,\ x_j \geq y_{J-1}\right\} \geq 1 - 1/(4d).$$

This property defines iteratively the sequence $(y_J)_{J \in [\![1,d]\!]}$ in terms of y_0.

Thanks to the strong Markov property at times $\tau^{J,(d)}_{y_J}$ and by induction on $0 \leq J \leq d$, we deduce that the following inequality holds for any $x \in \mathcal{X}_d$ such that $x_0 \geq y_0$:

$$\mathbb{P}_x\left(\tau^{J,(d)}_{y_J} \leq (J \cdot t_0/d) \wedge \tau^{(d)}_\partial\right) \geq 1 - J/(4d).$$

Let $E^{(d)} = \mathcal{D}^{(d)}_\ell$ for some $\ell \geq y_d/(2d)$, so that $E^{(d)} \in \mathbf{D}^{(d)}$ and $\tau^{(d)}_E \leq \tau^{d,(d)}_{y_d}$. Thanks to the previous inequality, the following one holds for any $x \in \mathcal{X}_d$ such that $x_0 \geq y_0$:

$$\begin{aligned}\mathbb{P}_x\left(t_0 < \tau^{(d)}_E \wedge \tau^{(d)}_\partial\right) &\leq 1 - \mathbb{P}_x\left(\tau^{d,(d)}_{y_d} \leq t_0 \wedge \tau^{(d)}_\partial\right)\\ &\leq \tfrac{1}{4} = \exp(-\rho\, t_0)/2,\end{aligned}$$

which extends the inequality stated in (5.5) for any x such that $x_0 < y_0$.

By induction over $k \geq 1$ thanks to the Markov property at times $k\, t_0$, the following inequality holds for any $x \in \mathcal{X}_d$ and any $k \geq 1$:

$$\mathbb{P}_x(k\, t_0 \leq \tau^{(d)}_\partial \wedge \tau^{(d)}_E) \leq 2^{-k} \exp(-\rho\, k\, t_0).$$

It entails the following upper-bound on the exponential moment:

$$\begin{aligned}&\sup_{\{x \in \mathcal{X}_d\}} \mathbb{E}_x\left[\exp\left(\rho \cdot (\tau^{(d)}_\partial \wedge \tau^{(d)}_E)\right)\right]\\ &\quad\leq \sum_{k \geq 0} \exp(\rho \cdot (k+1) \cdot t_0) \sup_{\{x \in \mathcal{X}_d\}} \mathbb{P}_x(k\, t_0 \leq \tau^{(d)}_\partial \wedge \tau^{(d)}_E)\\ &\quad\leq e^{\rho t_0} \textstyle\sum_{k \geq 0} 2^{-k} = 4 < \infty.\end{aligned}$$

This concludes the proof of Proposition 5.4.1. □

5.5 Step 4: Lower-bound of the survival capacity

The aim of this subsection is to prove the upcoming Lemma 5.5.1, which implies some lower-bound of the survival capacity:

Lemma 5.5.1 *For any $y_0 > 0$, there exists $t > 0$ and $\ell \geq 1$ such that the following holds:*

$$\inf \left\{ \mathbb{P}_x\big(\tau^{(d)}_{\mathcal{D}_\ell} \leq t \wedge \tau_\partial\big) \,\big|\, x \in \mathcal{X}_d\,;\; x_0 \geq y_0 \right\} > 0.$$

In other words, the sets $H^{(d)}_{y_0} := \{x \in \mathcal{X}_d\,;\; x_0 \geq y_0\}$ satisfy (H_0) as stated in Proposition 2.5.5, so that, for any $y_0 > 0$, $h^{(d)}$ is uniformly bounded away from 0 on $H^{(d)}_{y_0}$.

Proof : Let any $y_0 > 0$. The process $X^{(d)}_0$, namely the initial component of the solution to (1.1)-(1.3), is lower-bounded for any time a.s. under $\mathbb{P}_x$ by the solution Y to the following SDE, thanks to Lemma 5.1.4:

$$\mathrm{d}Y(s) = -\lambda\, \mathrm{d}s + \sqrt{Y(s)\cdot(1 - Y(s))}\, \mathrm{d}B_0(s)\,, \qquad Y(0) = y_0.$$

We consider the extinction time for Y as follows: $\tau^Y_\partial := \inf\{s\,;\; Y(s) = 0\}$. We deduce that the probability of survival up to time 1 is uniformly lower-bounded as follows for any $d \geq 1$ and any $x \in H^{(d)}_{y_0}$:

$$\mathbb{P}_x(1 < \tau^{(d)}_\partial) \geq c_S := \mathbb{P}_{y_0}(1 < \tau^Y_\partial) > 0. \tag{5.6}$$

The choice of 1 for the time is arbitrary. Thanks to Markov's inequality and to the exponential moment of $\tau^{(d)}_{\mathcal{D}_\ell} \wedge \tau^{(d)}_\partial$ derived in Proposition 5.4.1, there exists ℓ such that for any $d \in \mathbb{N}$ and $x \in \mathcal{X}_d$:

$$\mathbb{P}_x(1 < \tau^{(d)}_{\mathcal{D}_\ell} \wedge \tau^{(d)}_\partial) \leq c_S/2.$$

This implies (H_0) for $H^{(d)}_{y_0}$ in the sense that the following holds for any $x \in H^{(d)}_{y_0}$:

$$\mathbb{P}_x(\tau^{(d)}_{\mathcal{D}_\ell} \leq 1 \wedge \tau^{(d)}_\partial) \geq \mathbb{P}_x(1 < \tau^{(d)}_\partial) - \mathbb{P}_x(1 < \tau^{(d)}_{\mathcal{D}_\ell} \wedge \tau^{(d)}_\partial) \geq c_S/2.$$ □

Remark 5.5.2 Though this property (H_0) holds uniformly in d, it remains unchecked that the sequence $(h^{(d)})$ of functions is uniformly bounded away from zero on the sequence $(H^{(d)}_{y_0})$.

5.6 Concluding the proof of Theorem 2.3.2

For this proof, we plan to exploit Theorem 2.5.3 and first ensure Assumption (**A**) (see Sect. 2.5.2). Let $d \in \mathbb{N}$. The sets $\mathcal{D}_\ell^{(d)}$, defined in (5.1), satisfy $(A0)$.

Propositions 5.2.1, 5.3.1 and 5.4.1 ensure respectively $(A1)$, $(A3)$ and $(A2)$. Notably, for $(A3)$, since $\zeta^{(d)}$ has support in $\mathcal{D}_2^{(d)}$, for any $\ell \geq 2$, thanks to Proposition 5.3.1:

$$\limsup_{t\to\infty} \sup_{x\in\mathcal{D}_\ell^{(d)}} \frac{\mathbb{P}_x\left(t < \tau_\partial^{(d)}\right)}{\mathbb{P}_{\zeta^{(d)}}\left(t < \tau_\partial^{(d)}\right)} < \infty.$$

Thanks to Theorem 2.5.3, the semi-group therefore displays QSC with some characteristics $(\nu^{(d)}, h^{(d)}, \rho_0^{(d)})$. Thanks to Lemma 5.5.1 and to Proposition 2.5.5, the survival capacity $h^{(d)}$ is actually positive. Any QSD $\nu' \in \mathcal{M}_1(\mathcal{X}_d)$ must then satisfy both $\langle \nu' \,\big|\, h^{(d)} \rangle > 0$. Since ν' is a QSD, $\langle \nu' P_t^{(d)}, \mathbf{1}\rangle^{-1} \cdot \nu' P_t^{(d)} = \nu'$ holds for any t. Thanks to Corollary 2.1.1, this implies that $\nu' = \nu^{(d)}$, so that $\nu^{(d)}$ is in fact the unique QSD. With the upper-bound considered in Lemma 5.4.2, we deduce that $\mathbb{P}_y(\tau_\partial^{(d)} \leq 1) > 0$ holds for any $y \in \mathcal{X}_d$. Thus, $\rho_0^{(d)} = -\log[\mathbb{P}_{\nu^{(d)}}^{(d)}(1 < \tau_\partial^{(d)})] > 0$. This concludes the proof of Theorem 2.3.2. □

6 Proof of Proposition 2.3.4

The proof of Proposition 2.3.4, concluded in Sect. 6.3, relies on two main steps, handled uniformly over d. The first step in Sect. 6.2 is to ensure that descent from large values of the moment quickly occurs with probability close to one; the second step in Sect. 6.1 prove that a too large increase of the moment is unlikely to occur.

6.1 Step 1: descent of the moment

The aim of this subsection is the upcoming Proposition 6.1.1. The descent of the k-th moment $M_k^{(d)} = M_k(X^{(d)})$ of the process $X^{(d)}$ is stated in terms of the hitting time $\tau_m^{(k|d)}$:

$$\tau_m^{(k|d)} := \inf\left\{t \geq 0\,;\; M_k^{(d)}(t) \leq m\right\}. \tag{6.1}$$

Proposition 6.1.1 *For any time $t > 0$ and any $k \geq 1$, the following supremum tends to 0 as m tends to infinity:*

$$\sup\left\{\mathbb{P}_x\left(t < \tau_m^{(k|d)} \wedge \tau_\partial^{(d)}\right) \,\Big|\, d \in \mathbb{N}, x \in \mathcal{X}_d\right\}.$$

The proof of Proposition 6.1.1, as achieved in Sect. 6.1.4, relies on three steps detailed in the three forthcoming subsections. The first step is focused on the vicinity of $\{X_0 = 0\}$, the second on the vicinity of $\{M_1 = \infty\}$, the last one being iterated for each moment between 2 and k.

6.1.1 Step 1.1: rare survival for small optimal subpopulation

The aim of this subsection is the upcoming Lemma 6.1.2, which states, provided that the first moment is initially lower-bounded, that a click is very close to occurring when the initial size x_0 of the optimal subpopulation is very small:

Lemma 6.1.2 *For any time $t > 0$, the following supremum tends to 0 as δ tends to 0:*

$$\sup \left\{ \mathbb{P}_x \left(t < \tau_\partial^{(d)} \right) \middle| \, d \in \mathbb{N},\ x \in \mathcal{X}_d,\ M_1(x) \geq 1,\ x_0 \cdot M_1(x) \leq \delta \right\}.$$

Proof : This proof is an extension of the one of [2, Proposition 3.8].

Let $t > 0$ be fixed and define $\delta_\wedge$ as follows:

$$\delta_\wedge := \frac{1}{16\,\alpha} \wedge \frac{1}{4}.$$

We exploit two parameters $m_1 \geq 1$ and $\delta \in (0, \delta_\wedge)$: the upper-bound shall hold for initial conditions x such that both $m_1 \geq M_1(x)$ and $x_0\, m_1 \leq \delta$. m_1 is freely chosen and $\delta \leq \delta_\wedge$ is to be fixed below, according to (6.5). We will see nonetheless that the choice of δ and the upper-bound can be stated independently of m_1 provided $m_1 \geq 1$ (the larger is m_1, the better is the estimate).

Step 1: We introduce a crucial minoration of the process $X_0^{(d)}$ on the following event $\mathcal{E}_0^{(d)}$:

$$\mathcal{E}_0^{(d)} := \left\{ \sup_{\{s \leq t\}} X_0^{(d)}(s) \cdot M_1^{(d)}(s) \leq 2\,\delta_\wedge \right\} \cap \left\{ \sup_{\{s \leq t\}} X_0^{(d)}(s) \leq \tfrac{1}{2} \right\}.$$

On the event $\mathcal{E}_0^{(d)}$, the next inequalities hold for any $s \leq t$:

$$(\alpha \cdot M_1^{(d)}(s) - \lambda) \cdot X_0^{(d)}(s) \leq 2\,\alpha\,\delta_\wedge \leq \frac{1 - X_0^{(d)}(s)}{4}.$$

A.s. on the event $\mathcal{E}_0^{(d)}$, the process $X_0^{(d)}$ is thus upper-bounded on the time-interval $[0, t]$ by the solution Y to the following SDE, with $y_0 := \delta/m_1$ and thanks to Lemma 5.1.4:

$$\mathrm{d}Y(s) = \frac{1 - Y(s)}{4}\,\mathrm{d}s + \sqrt{Y(s) \cdot (1 - Y(s))}\,\mathrm{d}B_0(s)\,, \qquad Y(0) = y_0.$$

The main interest of this upper-bound is that it is explicitly given. For any $y > 0$, we denote $\tau_y^Y := \inf\{t \geq 0\,;\ Y(t) = y\}$, i.e. the hitting time of $y > 0$ by the process Y.

We consider the following martingale process, defined for any $s \in [0, \tau_0^Y)$:

$$d\mathcal{N}(s) := \sqrt{\frac{1 - Y(s)}{Y(s)}}\, dB_0(s), \quad \mathcal{N}(0) = 0.$$

Thanks to Itô's lemma, $Y(s)$ is expressed for any $s \in [0, \tau_0^Y)$ as follows in terms of $\mathcal{N}$:

$$Y(s) := y_0 \exp\Big(\mathcal{N}(s) - \frac{\langle \mathcal{N} \rangle_s}{4}\Big).$$

Let us consider the random time change $(\xi_u)_{u>0}$ as follows:

$$\xi(u) := \inf\big\{s \in [0, \tau_0^Y);\ \langle \mathcal{N} \rangle_s > u\big\}, \quad \mathcal{G}_u := \mathcal{F}_{\xi_u},$$

with value ∞ if the above set is empty. Thanks to [16, Theorem 18.4], there exists a Brownian motion W with respect to a standard extension of $\mathcal{G}$ such that a.s. $W = \mathcal{N} \circ \xi$ on $[0, \langle \mathcal{N} \rangle_{\tau_0^Y})$ and $\mathcal{N}(s) = W(\langle \mathcal{N} \rangle_s)$ for any $s \in [0, \tau_0^Y)$. Note that $\langle \mathcal{N} \rangle_{\tau_0^Y}$ is here defined as the left limit at τ_0^Y of the non-decreasing process $\langle \mathcal{N} \rangle$. Since the process $\langle \mathcal{N} \rangle$ is actually increasing in the time-interval $[0, \tau_0^Y)$, $\langle \mathcal{N} \rangle_{\xi(u)} = u$ holds for any $u \in [0, \langle \mathcal{N} \rangle_{\tau_0^Y})$. The following identity thus holds for any $u \in [0, \langle \mathcal{N} \rangle_{\tau_0^Y})$:

$$Y \circ \xi(u) = y_0 \exp\big(W_u - u/4\big).$$

Step 2: We prove that $\langle \mathcal{N} \rangle_{\tau_0^Y} = \infty$ holds a.s. on the event $\{\tau_0^Y < \tau_1^Y\}$.

Assume by absurdum that there exists $A > 0$ such that $\mathbb{P}(\langle \mathcal{N} \rangle_{\tau_0^Y} \leq A;\ \tau_0^Y < \tau_1^Y) > 0$ and let us denote this quantity $\epsilon \in (0, 1)$. Note that 0 is a regular reflecting boundary for Y, while 1 is an exit boundary, thanks to Lemma 5.1.2(ii) and Corollary 5.1.5. There exists $r > 0$ such that $\mathbb{P}(r \leq \tau_0^Y < \tau_1^Y) \leq \epsilon/2$. We consider the following sequence of stopping times, in terms of an integer ℓ:

$$T_\ell := \inf\{s \geq 0;\ \langle \mathcal{N} \rangle_s \geq A \text{ or } |\mathcal{N}(s)| \geq \ell\}.$$

By continuity of $\langle \mathcal{N} \rangle$ that starts from 0, we know that the following property holds for any $s > 0$ and any $\ell \geq 1$:

$$\mathbb{E}[\langle \mathcal{N} \rangle_{s \wedge T_\ell}] \leq A.$$

On the other hand, the local martingale $(\mathcal{N}(s))_s$ induces for any ℓ a continuous martingale $(\mathcal{N}(s \wedge T_\ell))_s$ whose predictible quadratic variation is exactly $(\langle \mathcal{N} \rangle_{s \wedge T_\ell})_s$. Thanks to Doob's inequality, the following inequalities thus hold for any $s > 0$, $\ell \geq 1$ and $B > 0$:

$$\mathbb{P}(\sup_{\{u \leq s\}} |\mathcal{N}(u \wedge T_\ell)| \geq B) \leq 4 \frac{\mathbb{E}[\langle \mathcal{N} \rangle_{s \wedge T_\ell}]}{B^2} \leq \frac{4A}{B^2}.$$

With $B = 4\sqrt{A/\epsilon}$ and any $\ell \geq B$, it yields the following upper-bound:

$$\mathbb{P}\Big(\sup_{\{u \in \xi([0,A])\}} |\mathcal{N}(u)| \geq B \Big) \leq \epsilon/4.$$

By virtue of the properties of ϵ and r, it entails that the following event has a positive probability, namely larger than $\epsilon/4$:

$$\mathcal{E} := \{\langle \mathcal{N} \rangle_{\tau_0^Y} \leq A\} \cap \{\tau_0^Y < \tau_1^Y \wedge r\} \cap \{\sup_{\{u \in \xi([0,A])\}} |\mathcal{N}(u)| < B\}.$$

Yet, on this event $\mathcal{E}$, it holds for any $s < \tau_0^Y$ that $\langle \mathcal{N} \rangle s \leq A$ and $|\mathcal{N}(s)| < B$, so that $Y(s) \geq y_0 \exp(-A/4 - B)$. Yet, since $\tau_0^Y < \infty$, the continuity of the process Y implies that $Y(\tau_0^Y -) = 0$, which is in contradiction with the previous statement. The event $\mathcal{E}$ is therefore empty, which contradicts that $\mathcal{E}$ has a positive probability. It concludes that $\langle \mathcal{N} \rangle_{\tau_0^Y} = \infty$ holds a.s. on the event $\{\tau_0^Y < \tau_1^Y\}$.

By the way, since 1 is an absorbing boundary for Y and by definition of $\langle \mathcal{N} \rangle$, $\langle \mathcal{N} \rangle_{\tau_0^Y} = \langle \mathcal{N} \rangle_{\tau_1^Y} < \infty$ holds a.s. on the complementary event $\{\tau_1^Y < \tau_0^Y\}$.

Step 3: We justify the choice of the constants involved in the definition of an event $\mathcal{A}^{(d)}$ on which survival of $X^{(d)}$ holds a.s. up to time t.

Let us consider some real number $\mu > 0$, to be fixed below according to (6.4). On the event $\{\tau_0^Y < \tau_{y_0+\mu}^Y\}$, $\langle \mathcal{N} \rangle_{\tau_0^Y} = \infty$ as a consequence of step 2, so that the following inequality holds for any $u \geq 0$:

$$y_0 \cdot \exp(W(u) - u/4) = Y(\xi(u)) < y_0 + \mu.$$

It implies that the following inequality holds a.s. on the event $\{\tau_0^Y < \tau_{y_0+\mu}^Y\}$:

$$\int_0^\infty \frac{y_0 \exp[W(r) - r/4]}{1 - y_0 \exp[W(r) - r/4]} \mathrm{d}r \leq \frac{y_0}{1 - y_0 - \mu} \int_0^\infty \exp[W(r) - r/4] \, \mathrm{d}r.$$

Note also that $\tau_0^Y = \lim_{u \to \infty} \xi(u)$ and the following expression for the derivative of ξ:

$$\xi'(r) = \frac{1}{\langle \mathcal{N} \rangle' \circ \xi(r)} = \frac{y_0 \exp[W(r) - r/4]}{1 - y_0 \exp[W(r) - r/4]}.$$

Therefore, the next inequality holds for any $t \geq 0$:

$$\mathbb{P}_{y_0}(t < \tau_0^Y < \tau_{y_0+\mu}^Y) \leq \mathbb{P}\left(\frac{t \cdot (1 - y_0 - \mu)}{y_0} < \int_0^\infty \exp[W(r) - r/4] \, \mathrm{d}r \right). \tag{6.2}$$

On the event $\{\tau^Y_{y_0+\mu} < \tau^Y_0\}$, $y_0+\mu = y_0 \cdot \exp(W(r)-r/4)$ for $r = \langle \mathcal{N} \rangle_{\tau^Y_{y_0+\mu}} \in (0, \infty)$. We thus deduce the following upper-bound:

$$\mathbb{P}_{y_0}\left(\tau^Y_{y_0+\mu} < \tau^Y_0\right) = \mathbb{P}\left((y_0+\mu)/y_0 \leq \sup_{r \geq 0} \exp\left[W(r) - r/4\right]\right). \tag{6.3}$$

Let $\epsilon > 0$. Since the law of W is the one of a Brownian motion, $W(r)/r \to 0$ as $t \to \infty$ holds a.s. We can thus choose $c_1, c_2 > 1$ such that:

$$\mathbb{P}\left(c_1 < \int_0^\infty \exp\left(W(r) - r/4\right), \mathrm{d}r\right) \leq \epsilon\,, \quad \mathbb{P}\left(c_2 < \sup_{r \geq 0} \exp\left[W(r) - r/4\right]\right) \leq \epsilon.$$

Thanks to [2, Lemma 3.2], we can choose $c_3 > 0$ such that the following inequality holds for any $x \in \mathcal{X}_d$:

$$\mathbb{P}_x\left(\sup_{\{s \leq t\}} M_1^{(d)}(s) - M_1^{(d)}(0) \geq \lambda\, t + c_3\right) \leq \epsilon.$$

This motivates the following choices for m_1' and μ:

$$m_1' := m_1 + \lambda\, t + c_3 > 1, \qquad \mu := \frac{\delta_\wedge}{m_1'}. \tag{6.4}$$

We then choose $\delta \leq \delta_\wedge$ sufficiently small to ensure the following two inequalities:

$$\frac{t \cdot (1 - \delta_\wedge)}{\delta} - t \geq c_1, \quad \frac{\delta_\wedge}{\delta} \cdot (1 + \lambda\, t + c_3)^{-1} \geq c_2. \tag{6.5}$$

These conditions are prescribed in order to ensure the following two inequalities, recalling that $y_0 = \delta/m_1$ and $m_1' \geq m_1 \geq 1$:

$$\frac{t \cdot (1 - y_0 - \mu)}{y_0} = \frac{t m_1}{\delta} \cdot (1 - \frac{\delta_\wedge}{m_1'}) - t \geq c_1, \quad \frac{y_0 + \mu}{y_0} \geq \frac{\delta_\wedge}{\delta} \cdot \frac{m_1}{m_1'} \geq c_2.$$

Therefore, thanks to (6.2), to (6.3) and to the above choices of the constants, $\mathbb{P}_x(\mathcal{A}^{(d)}) \geq 1 - 3\,\epsilon$, where:

$$\mathcal{A}^{(d)} := \left\{\sup_{\{s \leq t\}} M_1^{(d)}(s) \leq m_1'\right\} \cap \left\{\tau^Y_0 < t \wedge \tau^Y_{y_0+\mu}\right\}.$$

Step 4: We check that $t < \tau^{(d)}_\partial$ holds a.s. on the event $\mathcal{A}^{(d)}$.

To check the upper-bound by Y, let $T^{(d)}_B := \inf\{s \geq 0\,;\; X_0^{(d)}(s) \cdot M_1^{(d)}(s) \geq 2\,\delta_\wedge\}$. Then, a.s. on the event $\mathcal{A}^{(d)}$, both $M_1^{(d)}(s) \leq m_1'$ and $X_0^{(d)}(s) \leq Y(s) \leq y_0 + \mu$ hold for any $s \leq t \wedge T_B^{(d)}$. Since $\mu/y_0 \geq c_2 > 1$ and $\mu := \delta_\wedge/m'_1$, the process $X_0^{(d)} \cdot M_1^{(d)}$ is upper-bounded as follows, with the constant $c_4 := (y_0 + \mu) \cdot m'_1$ and for any $s \leq T_B^{(d)}$:

$$X_0^{(d)}(s) \cdot M_1^{(d)}(s) \leq c_4 < 2\delta_\wedge.$$

By continuity of $X_0^{(d)} \cdot M_1^{(d)}$, the event $\{T_B^{(d)} < t\}$ has an empty intersection with $\mathcal{A}^{(d)}$. Therefore $X_0^{(d)}(s) \leq Y(s)$ holds for any $s \leq t$, and in particular $\tau_\partial^{(d)} \leq \tau_0^Y \leq t$. So the following inequality holds for any $x \in \mathcal{X}_d$ such that $m_1 \geq M_1(x)$ and $x_0 \cdot m_1 \leq \delta$:

$$\mathbb{P}_x(\tau_\partial^{(d)} \leq t) \geq \mathbb{P}_x(\mathcal{A}^{(d)}) \geq 1 - 3\,\epsilon.$$

Though the event $\mathcal{A}^{(d)}$ shall be adjusted depending on m_1, the choice of δ as a function of ϵ (in (6.5)) is made independently of m_1, which concludes the proof of Lemma 6.1.2. □

6.1.2 Step 1.2: quick descent of the first moment

Provided that the optimal subpopulation is still significant, the first moment is unlikely to stay high for a significant time, as stated in the upcoming Lemma 6.1.3, whose proof is the purpose of this Sect. 6.1.2:

Lemma 6.1.3 *Given any $t, y_0 > 0$, the following supremum tends to 0 as $m_1 > 0$ tends to infinity:*

$$\sup \left\{ \mathbb{P}_x \left(t \leq \tau_{m_1}^{(1|d)} \wedge \tau_\partial^{(d)} \right) \middle| d \in \mathbb{N}, x \in \mathcal{X}_d, x_0 \geq y_0 \right\}.$$

Proof : A.s. on the event $\{\inf_{\{s \leq t\}} M_1^{(d)}(s) \geq m_1\}$, the process $X_0^{(d)}$, namely the initial component of the solution to (1.1)-(1.3), is lower-bounded on $[0, t]$ by the solution Y to the following SDE, thanks to Lemma 5.1.4:

$$\mathrm{d}Y(s) = r(m_1)\, Y(s)\, \mathrm{d}s + \sqrt{Y(s) \cdot (1 - Y(s))}\, \mathrm{d}B_0(s)\,, \quad Y(0) = y_0,$$

where $r(m_1) := \alpha\, m_1 - \lambda \to \infty$ as $m_1 \to \infty$.

Since $M_1^{(d)}(s) = 0$ whenever $X_0^{(d)}(s) = 1$, this lower-bound of $X_0^{(d)}$ by Y must have stopped before $T_1^Y := \inf\{t \geq 0;\; Y(t) \geq 1\}$, so $\tau_{m_1}^{(1|d)} \wedge \tau_\partial^{(d)} \leq T_1^Y$. We thus only have to prove that $\mathbb{P}(t < T_1^Y)$ tends to 0 as m_1 tends to infinity.

Let $\epsilon, t_1 > 0$. Let us denote by $\mathcal{N}^Y$ the following martingale process naturally associated to Y:

$$\mathcal{N}^Y(t) := \int_0^t \sqrt{Y(s) \cdot (1 - Y(s))}\, \mathrm{d}B_0(s).$$

The quadratic variation of $\mathcal{N}^Y$ until time $t_1 \le t$ is upper-bounded by $t_1/4$, so that Doob's inequality implies the following upper-bound:

$$\mathbb{P}_{y_0}\Big(\sup_{s\le t_1} |\mathcal{N}^Y(s)| > \frac{y_0}{2} \Big) \le \frac{16 t_1}{y_0^2}. \tag{6.6}$$

By choosing t_1 sufficiently small, we assume $16t_1/y_0^2 \le \epsilon$. On the complementary event $\{\sup_{\{s\le t_1\}} |\mathcal{N}^Y(s)| \le y_0/2\}$, Y stays above $y_0/2$ on the time-interval $[0, t_1]$. The drift term can thus be lower-bounded by $s \cdot r(m_1) \cdot y_0/2$ for any $s \le t_1 \wedge T_1^Y$. Since it cannot exceed $1 - y_0/2$ before T_1^Y, it necessarily implies that for $r(m_1)$ sufficiently large (that is m_1 sufficiently large), we must have $T_1^Y < t_1$ on the event $\{\sup_{s\le t_1} |\mathcal{N}^Y(s)| \le y_0/2\}$. Thanks to (6.6) and since $t_1 \le t$, this implies that $\mathbb{P}(t < T_1^Y)$ tends to 0 as m_1 tends to ∞ and concludes the proof of Lemma 6.1.3. □

6.1.3 Step 1.3: quick descent of the next moments

Provided that one of the moment is initially upper-bounded, it is unlikely for the next moment to stay high on a significant time-interval afterwards, as stated in the upcoming Lemma 6.1.4 (recall the notation (6.1)), whose proof is the purpose of this Sect. 6.1.3:

Lemma 6.1.4 *Given any integer $k \ge 1$ and any $t, m > 0$, the following supremum tends to 0 as $m' > 0$ tends to infinity:*

$$\sup \Big\{ \mathbb{P}_x \Big(t \le \tau_{m'}^{(k+1|d)} \wedge \tau_\partial^{(d)} \Big) \Big| d \in \mathbb{N},\ x \in \mathcal{X}_d,\ M_k(x) \le m \Big\}.$$

For the proof of Lemma 6.1.4, we exploit the following properties on the semi-martingale decomposition of the process $M_k^{(d)}$, summarised in the upcoming Lemma 6.1.5.

Lemma 6.1.5 *For any $d \in [\![1, \infty]\!]$, $k \in \mathbb{Z}$, and any x which belongs either to $\mathcal{X}_d$ if $d \in \mathbb{N}$ or to $\mathcal{X}^{2k}$ if $d = \infty$, the process $M_k^{(d)}$ can be decomposed as follows under $\mathbb{P}_x$:*

$$dM_k^{(d)}(t) = V_k^{(d)}(t)\, dt + d\mathcal{M}_k^{(d)}(t), \quad M_k^{(d)}(0) = M_k(x),$$

where $\mathcal{M}_k^{(d)}$ is a continuous local-martingale starting from 0, whose quadratic variation is

$$\langle \mathcal{M}_k^{(d)} \rangle_t = \int_0^t (M_{2k}^{(d)}(s) - M_k^{(d)}(s)^2)\, ds,$$

and $V_k^{(d)}$ is a bounded variation process. $(M_{2k}^{(d)}(t))_{t\ge 0}$ is a.s. locally upper-bounded. In addition, there exists a universal constant $C_k > 0$ such that the following upper-bounds *hold a.s. for any time-interval:*

$$V_k^{(d)} \leq \alpha \cdot (M_1^{(d)} \cdot M_k^{(d)} - M_{k+1}^{(d)}) + \lambda \cdot (C_k M_k^{(d)} + 1)$$
$$\leq \lambda \cdot (C_k M_k^{(d)} + 1).$$

Note that given the quadratic variation of $\mathcal{M}_k^{(d)}$, it is a martingale provided $d \in \mathbb{N}$.

Proof of Lemma 6.1.5 Given the definition of $M_k^{(d)}$ in terms of the solution $(X^{(d)})$ to the system of SDEs (1.1), we obtain the following expression:

$$V_k^{(d)} := \alpha \cdot (M_1^{(d)} \cdot M_k^{(d)} - M_{k+1}^{(d)}) + \lambda \sum_{\ell=0}^{d-1} (\ell+1)^k X_\ell^{(d)} - \lambda \cdot (M_k^{(d)} - \mathbf{1}_{\{d<\infty\}} d^k X_d^{(d)}). \tag{6.7}$$

Thanks to Hölder's inequality, the next inequalities hold for any $x \in \mathcal{X}_d$:

$$M_1(x) \leq (M_{k+1}(x))^{1/(k+1)}, \qquad M_k(x) \leq (M_{k+1}(x))^{k/(k+1)},$$

thus $M_1(x) \cdot M_k(x) \leq M_{k+1}(x)$. It thus implies the inequality $M_1^{(d)} \cdot M_k^{(d)} \leq M_{k+1}^{(d)}$ between the stochastic processes involved in (6.7).

Exploiting that $(\ell+1)^k \leq 2^k \cdot \ell^k$ for $\ell \geq 1$, and that $X_0^{(d)} \leq 1$ for $\ell = 0$, it yields the following inequality, with $C_k = 2^k$:

$$V_k^{(d)} \leq \lambda \cdot (C_k \cdot M_k^{(d)} + 1).$$

On the other hand, the local martingale term is defined as follows:

$$\mathrm{d}\mathcal{M}_k^{(d)}(t) := \sum_{i=1}^{d} i^k \sqrt{X_i^{(d)}(t)} \mathrm{d}W_i(t) - M_k^{(d)}(t) \mathrm{d}W_{(d)}(t), \quad \mathcal{M}_k^{(d)}(0) = 0.$$

Thanks to the independence between the Brownian motions $(W_i)_i$ and to (1.2), its quadratic variation satisfies the following identities:

$$\mathrm{d}\langle \mathcal{M}_k^{(d)} \rangle_t = (M_{2k}^{(d)}(t) + [M_k^{(d)}(t)]^2)\, \mathrm{d}t - 2M_k^{(d)}(t) \sum_{i=1}^{d} i^k \sqrt{X_i^{(d)}(t)}\, \mathrm{d}\langle W_i, W_{(d)} \rangle_t$$
$$= (M_{2k}^{(d)}(t) - [M_k^{(d)}(t)]^2)\, \mathrm{d}t.$$

In the case where $d = \infty$, since $M_{2k}^{(d)}(x) < \infty$, we know thanks to [2, Theorem 3] that $(M_{2k}^{(d)}(t))_{t \geq 0}$ is a.s. locally upper-bounded, i.e. that $\sup_{s \leq t} M_{2k}^{(d)}(s) < \infty$ holds a.s. To be precise, this is not stated directly in [2, Theorem 3], yet visibly obtained in the proof, first in the case where $\alpha = 0$, then exended for any $\alpha > 0$ because the

Girsanov change of density is uniformly upper-bounded (see Proposition B.0.1 in the appendix for completeness).

This expression for the quadratic variation is thus well-defined. This concludes the proof of Lemma 6.1.5. □

Proof of Lemma 6.1.4 Let $k \geq 1$, $t > 0$, $m > 0$, and $\epsilon > 0$. Note that $M_1(x) \leq M_k(x)$ holds for any $d \in \mathbb{N}$ and any $x \in \mathcal{X}_d$. Thanks to [2, Lemma 3.2], we can thus choose $m_1 > 0$ such that the following inequality holds for any $d \in \mathbb{N}$ and any $x \in \mathcal{X}_d$ such that $M_k(x) \leq m$:

$$\mathbb{P}_x(T^{(1|d)}_{m_1} < t) \leq \epsilon, \tag{6.8}$$

where $T^{(1|d)}_{m_1} := \inf\{t \geq 0\,;\ M^{(d)}_1(t) \geq m_1\}$, namely $T^{(1|d)}_{m_1}$ is the hitting time of m_1 by $M^{(d)}_1$.

Thanks to the analysis of the process $M^{(d)}_k$ conducted in Lemma 6.1.5, the following inequality holds then with $\check{C}_k = \alpha \cdot m_1 + \lambda C_k$ for any $d \in \mathbb{N}$ and any $x \in \mathcal{X}_d$:

$$\begin{aligned}\mathbb{E}_x[M^{(d)}_k(t \wedge T^{(1|d)}_{m_1})] \leq M_k(x) - \alpha \cdot \mathbb{E}_x\left(\int_0^t \mathbf{1}_{\left\{s \leq T^{(1|d)}_{m_1}\right\}} M^{(d)}_{k+1}(s)\,\mathrm{d}s\right)\\ + \check{C}_k \cdot \mathbb{E}_x\left(\int_0^t \mathbf{1}_{\left\{s \leq T^{(1|d)}_{m_1}\right\}} M^{(d)}_{k}(s)\,\mathrm{d}s\right).\end{aligned}$$

Since $M^{(d)}_k$ is a non-negative process, with $\bar{C}_k = \check{C}_k/\alpha$, the following upper-bound on the $k+1$-th moment thus holds for any $d \in \mathbb{N}$ and any $x \in \mathcal{X}_d$ such that $M_k(x) \leq m$:

$$\mathbb{E}_x\left(\int_0^t \mathbf{1}_{\left\{s \leq T^{(1|d)}_{m_1}\right\}} M^{(d)}_{k+1}(s)\,\mathrm{d}s\right) \leq \frac{m}{\alpha} + \bar{C}_k \cdot \mathbb{E}_x\left(\int_0^t \mathbf{1}_{\left\{s \leq T^{(1|d)}_{m_1}\right\}} M^{(d)}_{k}(s)\,\mathrm{d}s\right).$$

By immediate induction over k, there exists $\hat{C}_k > 0$ such that the following inequality holds for any $d \in \mathbb{N}$ and any $x \in \mathcal{X}_d$ such that $M_k(x) \leq m$:

$$\begin{aligned}\mathbb{E}_x\left(\int_0^t \mathbf{1}_{\left\{s \leq T^{(1|d)}_{m_1}\right\}} M^{(d)}_{k+1}(s)\,\mathrm{d}s\right) &\leq \frac{(k-1)\cdot m}{\alpha} + \hat{C}_k \cdot \mathbb{E}_x\left(\int_0^t \mathbf{1}_{\left\{s \leq T^{(1|d)}_{m_1}\right\}} M^{(d)}_{1}(s)\,\mathrm{d}s\right)\\ &\leq \frac{(k-1)\cdot m}{\alpha} + \hat{C}_k \cdot t \cdot m_1.\end{aligned}$$

Thanks to Markov's inequality, we can thus choose $m' > 0$ such that the following inequality holds for any $d \in \mathbb{N}$ and any $x \in \mathcal{X}_d$ such that $M_k(x) \leq m$:

$$\mathbb{P}_x\left(\int_0^t \mathbf{1}_{\left\{s \leq T^{(1|d)}_{m_1}\right\}} M^{(d)}_{k+1}(s)\,\mathrm{d}s \geq t \cdot m'\right) \leq \epsilon. \tag{6.9}$$

Recalling (6.8), this concludes the proof of Lemma 6.1.4, since:

$$\left\{t > \tau_{m'}^{(k+1|d)}\right\} \cap \left\{t \leq T_{m_1}^{(1|d)}\right\} \subset \left\{\int_0^t \mathbf{1}_{\left\{s \leq T_{m_1}^{(1|d)}\right\}} M_{k+1}^{(d)}(s)\, \mathrm{d}s \geq t \cdot m'\right\}.$$

□

6.1.4 Concluding the proof of Proposition 6.1.1

Let $t, \epsilon > 0$. Thanks to Lemma 6.1.2, we choose an upper-bound $\delta \in (0, \frac{1}{(16\alpha)})$ for the initial condition of the process $X_0^{(d)} \cdot M_1^{(d)}$, so that the following inequality holds, for any $d \in \mathbb{N}$ and any $x \in \mathcal{X}_d$ such that both $M_1(x) \geq 1$ and $x_0 \cdot M_1(x) \leq \delta$:

$$\mathbb{P}_x\left(t < \tau_\partial^{(d)}\right) \leq \epsilon. \tag{6.10}$$

We consider the following exit time:

$$T_\delta^{B,(d)} := \inf\{t \geq 0;\ X_0^{(d)}(t) \cdot M_1^{(d)}(t) \leq \delta\}. \tag{6.11}$$

Let $m_1^\vee = ({}^{2\lambda}/_{\alpha}) \vee 1$. We consider $\tau_{m_1}^{(1|d)} := \inf\{t \geq 0;\ M_1^{(d)}(X_t) \leq m_1\}$ for any $m_1 \geq m_1^\vee$ (so that $\tau_{m_1}^{(1|d)} \leq \tau_{m_1^\vee}^{(1|d)}$). Thanks to (6.10) and to the strong Markov property at time $T_\delta^{B,(d)}$, the following inequality holds for any $x \in \mathcal{X}_d$ and $d \geq 1$:

$$\mathbb{P}_x\left(T_\delta^{B,(d)} \leq t \leq \tau_{m_1^\vee}^{(1|d)},\ 2t < \tau_\partial^{(d)}\right) \leq \epsilon. \tag{6.12}$$

On the event $\{t \leq T_\delta^{B,(d)} \wedge \tau_{m_1^\vee}^{(1|d)} \wedge \tau_\partial^{(d)}\}$, recalling that $m_1^\vee \geq 2\lambda/\alpha$, the following inequality holds for any $s \leq t$:

$$(\alpha \cdot M_1^{(d)}(s) - \lambda) \cdot X_0^{(d)}(s) \geq \frac{\alpha \cdot \delta}{2}.$$

Thus, thanks to Lemma 5.1.4, $X_0^{(d)}$ is lower-bounded by the solution Y to the following SDE, a.s. on $[0, t]$ on the event $\{t \leq T_\delta^{B,(d)} \wedge \tau_{m_1^\vee}^{(1|d)} \wedge \tau_\partial^{(d)}\}$:

$$\mathrm{d}Y(s) = \frac{\alpha \cdot \delta}{2} \cdot \mathrm{d}s + \sqrt{Y(s) \cdot (1 - Y(s))}\, \mathrm{d}B_0(s), \qquad Y(0) = 0. \tag{6.13}$$

Thanks to Corollary 5.1.5 the left-boundary 0 is regular reflecting for Y. Thus we can choose $y_0 > 0$ such that: $\mathbb{P}(Y(t) \leq y_0) \leq \epsilon$. Recalling (6.12), this implies the following inequalities for any $d \in \mathbb{N}$ and $x \in \mathcal{X}_d$:

$$\mathbb{P}_x\left(X_0^{(d)}(t) \leq y_0,\ t \leq \tau_{m_1^\vee}^{(1|d)},\ 2t < \tau_\partial^{(d)}\right)$$

$$\leq \mathbb{P}_x\Big(T_\delta^{B,(d)} \leq t \leq \tau_{m_1^\vee}^{(1|d)}\,,\ 2\,t < \tau_\partial^{(d)}\Big) + \mathbb{P}_x\Big(Y(t) \leq y_0\,,\ t \leq T_\delta^{B,(d)} \wedge \tau_{m_1^\vee}^{(1|d)} \wedge \tau_\partial^{(d)}\Big)$$
$$\leq 2\,\epsilon. \tag{6.14}$$

Thanks to Lemma 6.1.3, we can then choose $m_1 \geq m_1^\vee$ associated with y_0. Thanks to (6.14) and to the Markov property at time t, we deduce the following inequalities:

$$\begin{aligned}\mathbb{P}_x\Big(2\,t < \tau_{m_1}^{(1|d)} \wedge \tau_\partial^{(d)}\Big) &\leq \mathbb{P}_x\Big(X_0^{(d)}(t) \leq y_0\,,\ t \leq \tau_{m_1^\vee}^{(1|d)}\,,\ 2\,t < \tau_\partial^{(d)}\Big)\\ &\quad + \mathbb{E}_x\Big[\mathbb{P}_{X^{(d)}(t)}\big(t \leq \tau_{m_1}^{(1|d)} \wedge \tau_\partial^{(d)}\big)\,;\ X_0^{(d)}(t) \geq y_0\Big]\\ &\leq 3\,\epsilon.\end{aligned} \tag{6.15}$$

Thanks to Lemma 6.1.4, we can choose $m_2 > 0$ associated with m_1 and $m_3 > 0$ associated with m_2 such that the following inequalities hold:

$$\mathbb{P}_x\Big(3\,t < \tau_{m_2}^{(2|d)} \wedge \tau_\partial^{(d)}\Big) \leq 4\,\epsilon\,, \quad \mathbb{P}_x\Big(4\,t < \tau_{m_3}^{(3|d)} \wedge \tau_\partial^{(d)}\Big) \leq 5\,\epsilon.$$

Applying the same argument inductively over $k \geq 3$, we can choose $m_k > 0$ such that the following inequality holds for any $d \in \mathbb{N}$ and any $x \in \mathcal{X}_d$:

$$\mathbb{P}_x\Big((k+1)\cdot t < \tau_{m_k}^{(k|d)} \wedge \tau_\partial^{(d)}\Big) \leq (k+2)\cdot \epsilon,$$

so as to treat any moment. This concludes the proof of Proposition 6.1.1. □

6.2 Step 2: rare large increase of the moment

With a probability close to 1, the increase of the moments after their descent can be upper-bounded uniformly over a given time-interval, as stated in the next proposition, whose proof is the purpose of this Sect. 6.2. For any $k \geq 1$ and $m \geq 1$, let:

$$T_m^{(k|d)} := \inf\Big\{t \geq 0;\ M_k^{(d)}(t) \geq m\Big\}. \tag{6.16}$$

Proposition 6.2.1 *For any $k > 1$, $t, m > 0$, the following supremum tends to 0 as $m' > 0$ tends to infinity:*

$$\sup\Big\{\mathbb{P}_x\Big(T_{m'}^{(k|d)} \leq t\Big)\ \Big|\ d \in \mathbb{N}, x \in \mathcal{X}_d, M_k(x) \leq m\Big\}.$$

Proof : Let $k > 1$, $t > 0$, $m' \geq m$, $d \in \mathbb{N}$ and $x \in \mathcal{X}_d$ such that $M_k(x) \leq m$.

We aim at exploiting Doob's inequality on a non-negative sub-martingale that is an upper-bound of $M_k^{(d)}$. Given the semi-martingale decomposition of M_k as stated in

Lemma 6.1.5, we consider the solution $\widehat{M}_k^{(d)}$ to the following SDE:

$$\widehat{M}_k^{(d)}(t) := m + \lambda t + \lambda C_k \int_0^t \widehat{M}_k^{(d)}(s)\, \mathrm{d}s + \mathcal{M}_k^{(d)}(t). \tag{6.17}$$

Thanks to [28, Proposition 3.12] (as for Lemma 5.1.4), $\widehat{M}_k^{(d)}(t) \geq M_k^{(d)}(t)$ holds for any $t \geq 0$. Since $M_k^{(d)}$ is non-negative, $\widehat{M}_k^{(d)}$ is also non-negative. As the solution to Eq. (6.17), $\widehat{M}_k^{(d)}$ is a sub-martingale. Since $\mathbb{E}_x\left[\widehat{M}_k^{(d)}(s)\right]$ is upper-bounded by d^k for any s, Grönwall's lemma implies the following inequality for any $x \in \mathcal{X}_d$ such that $M_k(x) \leq m$:

$$\sup_{\{s \leq t\}} \mathbb{E}_x\left[\widehat{M}_k^{(d)}(s)\right] \leq (m + \lambda t)\, e^{Ct}. \tag{6.18}$$

Exploiting Doob's inequality on $\widehat{M}_k^{(d)}$, then (6.18) with $C_M := (1 + \lambda t)\, e^{Ct}$, we obtain the following inequality for any $x \in \mathcal{X}_d$ such that $M_k^{(d)}(x) \leq m$:

$$\begin{aligned}\mathbb{P}_x\left(\sup_{\{s \leq t\}} M_k^{(d)}(s) > m'\right) &\leq \mathbb{P}_x\left(\sup_{\{s \leq t\}} \widehat{M}_k^{(d)}(s) > m'\right)\\ &\leq \frac{\mathbb{E}_x[\widehat{M}_k^{(d)}(t)]}{m'} \leq \frac{C_M\, m}{m'}.\end{aligned}$$

This concludes the proof of Proposition 6.2.1. □

6.3 Concluding the proof of Proposition 2.3.4

First of all, we show that we have a uniform upper-bound on the extinction rate $\rho_0^{(d)}$ associated with the system (1.1): $\sup_{\{d \in \mathbb{N}\}} \rho_0^{(d)} < \infty$. Indeed, whatever $d \in \mathbb{N}$, we can find $x^{(d)} \in \mathcal{X}_d$ such that $x_0^{(d)} \geq \frac{1}{2}$ so that, thanks to Lemma 5.1.4, $X_0^{(d)}$ is a.s. lower-bounded under $\mathbb{P}_x$ for any time by the solution Y to the following SDE:

$$\mathrm{d}Y(s) = -\lambda \cdot Y(s)\mathrm{d}s + \sqrt{Y(s)\,(1 - Y(s))}\,\mathrm{d}B_0(s)\,, \quad Y(0) = \tfrac{1}{2}.$$

Thanks to Lemma 5.1.2(i), the left-boundary 0 is an exit boundary. The semi-group governing Y, with extinction at τ_0^Y, corresponds exactly to the system (1.1) with $d = 1$, $\alpha = 0$, $X_0' = Y$ and $X_1' = 1 - Y$. Thanks to Theorem 2.3.2, the semigroup thus displays QSC with extinction rate $\rho_\vee$. Denoting $\mathbb{P}^Y_{\frac{1}{2}}$ the law of Y, it entails the following inequality from the convergences of the survival capacities:

$$\rho_0^{(d)} = \lim_{t \to \infty} \tfrac{-1}{t} \log \mathbb{P}_{x^{(d)}}(t < \tau_\partial^{(d)}) \leq \lim_{t \to \infty} \tfrac{-1}{t} \log \mathbb{P}^Y_{\frac{1}{2}}(t < \tau_0^Y) := \rho_\vee.$$

Thanks to Proposition 6.1.1 (similarly to the way Proposition 5.4.1 was deduced), we can choose $m > 0$ such that $\tau_m^{(1|d)}$ satisfies the following inequality for any $d \geq 1$ and any $x \in \mathcal{X}_d$:

$$\mathbb{E}_x \exp\Big[(\rho_\vee + 1) \cdot (\tau_m^{(1|d)} \wedge \tau_\partial^{(d)})\Big] \leq C < \infty.$$

In particular, it implies the following inequality for any $t > 0$ and any $d \in \mathbb{N}$:

$$\mathbb{P}_{\nu^{(d)}}\Big(t < \tau_m^{(1|d)} \wedge \tau_\partial^{(d)}\Big) \leq C \exp\Big[-(\rho_\vee + 1) \cdot t\Big]. \tag{6.19}$$

Then, for any $\epsilon > 0$, consider $t := -\log(\epsilon/(2\,C))$. Thanks to Proposition 6.2.1, the probability of large increase of the process $M_k^{(d)}$ can be made negligible. So we can choose a constant $m' > 0$ such that the following inequality holds for any initial condition $x \in \mathcal{X}_d$ such that $M_k(x) \leq m$:

$$\mathbb{P}_x\left(\sup_{\{s \leq t\}} M_k^{(d)}(s) \geq m'\right) \leq \epsilon/2 \exp[-\rho_\vee\, t]. \tag{6.20}$$

By virtue of the definition of t, recalling (6.19) and (6.20), the following inequalities thus hold for any $d \in \mathbb{N}$:

$$\begin{aligned}
\nu^{(d)}\Big(\Big\{M_k \geq m'\Big\}\Big) &= \exp[\rho_0^{(d)}\, t] \cdot \mathbb{P}_{\nu^{(d)}}\Big(M_k^{(d)}(t) \geq m'\,;\ t \leq \tau_\partial^{(d)}\Big) \\
&\leq \exp[\rho_\vee\, t] \cdot \Big(\mathbb{P}_{\nu^{(d)}}\Big(\tau_m^{(1|d)} > t\,;\ t < \tau_\partial^{(d)}\Big) \\
&+ \mathbb{E}_{\nu^{(d)}}\Big[\mathbb{P}_{X^{(d)}(\tau_m^{(1|d)})}\Big(\sup_{\{s \leq t\}} M_k^{(d)}(s) \geq m'\Big)\,;\ \tau_m^{(1|d)} < t \wedge \tau_\partial^{(d)}\Big]\Big) \\
&\leq C \cdot \epsilon/(2\,C) + \epsilon/2 \leq \epsilon.
\end{aligned}$$

This concludes the proof of Proposition 2.3.4. □

Remark 6.3.1 In fact, thanks to Lemma 6.1.4, we are thus able to upper-bound any moment, with probability close to 1 and under the QSD $\nu^{(d)}$ uniformly on $d \in \mathbb{N}$. These upper-bounds will extend to the limiting QSD on $\mathcal{X}_\infty$.

7 Proof of Theorem 2.4.1: the infinite dimensional case

The proof of Theorem 2.4.1 is achieved in Sect. 7.6 by ensuring Assumption (**AF**) as stated in Sect. 2.5.2. We will treat both the case of large yet finite values of d and $d = \infty$, for which we recall that any $x \in \mathcal{X}_\infty$ has a finite sixth moment (see (2.10)).

As one can imagine, the proof of Theorem 2.4.1 is much more technical than the ones of Theorem 2.2.2, Theorem 2.3.2 and Proposition 2.3.4. For instance, there is no explicit reference measure that seems to be exploitable as $\zeta^{(\infty)}$: the Lebesgue measure cannot be extended on an infinite dimensional space! Many elements of these previous proofs are however to be exploited with only slight adaptations, so that the reader is really encouraged to read them before the next proof.

The core idea behind the proof is still that the individuals carrying many mutations are actually wiped out very rapidly, implying rapid shuffle of the last coordinates. Quite unexpectedly, the criteria we developed to deal with jump events has proved to be very effective in this context. Notably, we could exploit the Girsanov transform to relate to the finite dimensional problem and deal with moments increasing too largely as exceptional events.

7.1 Outline of the proof

We now consider $d \in [\![1, \infty]\!]$, i.e. including the case $d = \infty$. For the purpose of Theorem 2.4.1 in this Sect. 7, we replace the notation given in (5.1) by the following one:

$$\mathcal{D}_\ell^{(d)} := \left\{ x \in \mathcal{X}_d \,;\; x_0 \geq \frac{1}{2\ell} \right\}. \tag{7.1}$$

In this approach, the family $\mathcal{D}_\ell^{(d)}$ covers the whole state space:

$$\cup_{\ell \geq 1} \mathcal{D}_\ell^{(d)} = \mathcal{X}_d.$$

Because it is close to the previous proof of Proposition 2.3.4, we will first focus on the result of Theorem 7.2.1, which can be interpreted as the statement of escape from the transitory domain (though the issue of the dependency in the parameter ϵ led us to integrate the result into the following estimate of almost perfect harvest). The sets $E^{(d)}$ that we will consider are defined through three parameters $m, y, \eta > 0$ as follow:

$$E^{(d)} := \left\{ x \in \mathcal{X}_d \,;\; M_3(x) \leq m \,,\; \forall\, j \leq \lfloor m/\eta \rfloor + 1,\; x_j \geq y \right\}. \tag{7.2}$$

The reference probability measure $\zeta^{(d)}$ on $\mathcal{X}_d$ is chosen to be especially adapted for our arguments, in a way that makes it actually complex to express. Its specific definition, stated in (7.59), is given in Sect. 7.4 that is dedicated to the mixing property and the accessibility of the subsets of $\mathcal{X}_d$. It relies on the notations and properties introduced in Sect. 7.3, where we justify a close connection with the finite dimensional SDEs. The estimate of almost perfect harvest is then conducted in Sect. 7.5, before we finally conclude the proof of Theorem 2.4.1 in Sect. 7.6.

As stated at the beginning of Sect. 5, for any $d \in [\![1, \infty]\!]$ and probability measure ζ on $\mathcal{X}_d$, the process $X^{(d)}$ is solution under $\mathbb{P}_\zeta$ of the system (1.1) with initial condition $X^{(d)}(0)$ distributed as ζ.

7.2 Escape from the Transitory domain

We prove in this subsection that the process rapidly escape the sets $E^{(d)}$ as proposed in (7.2), provided that the upper-bound on the moment is sufficiently large and the lower-bound on the optimal subpopulation size is sufficiently small, as stated in the next theorem. We recall the notation $\tau_E^{(d)}$ as the entry time of $E^{(d)}$.

Theorem 7.2.1 *For any $t, \eta, \epsilon > 0$, there exist a couple $m, y > 0$ such that the following inequality holds with $E^{(d)} = E^{(d)}(m, y, \eta)$ as in (7.2) for any $d \in [\![1, \infty]\!]$ and $x \in \mathcal{X}_d$:*

$$\mathbb{P}_x \left(t < \tau_\partial^{(d)} \wedge \tau_E^{(d)} \right) \leq \epsilon. \tag{7.3}$$

Remark 7.2.2 With the same reasoning as in the proof of Proposition 5.4.1, (7.3) has the following implication in terms of exponential moments. For any $\rho, \eta > 0$, there exist a couple $m, y > 0$ such that the following inequality holds for any $d \in [\![1, \infty]\!]$ and $x \in \mathcal{X}_d$:

$$\mathbb{E}_x \left(\exp[\rho \cdot (\tau_\partial^{(d)} \wedge \tau_E^{(d)})] \right) \leq 4.$$

The proof of Theorem 7.2.1 relies on the six forthcoming lemmas, mostly adapted from the uniform escape and the uniform descent of the moments in the finite dimensional systems. They are given in the order at which they will be exploited to conclude the proof in Sect. 7.2.2.

We first show in the upcoming Lemma 7.2.3 that the click is very likely when the growth of the optimal subpopulation size is initially very small, in the situation where the first moment is large so the initial size itself is small.

Lemma 7.2.3 *For any $t > 0$, the following supremum tends to 0 as δ tends to 0:*

$$\sup \left\{ \mathbb{P}_x \left(t < \tau_\partial^{(d)} \right) \middle| d \in [\![1, \infty]\!] \,,\; x \in \mathcal{X}_d \,,\; M_1(x) \in (1, \infty) \,,\; x_0 \cdot M_1(x) \leq \delta \right\}.$$

Then, provided the optimal subpopulation size is non-negligible, we show that the first moment is unlikely to stay very high, as stated in the upcoming Lemma 7.2.4.

Lemma 7.2.4 *For any two real numbers $t, y_0 > 0$, the following supremum tends to 0 as m_1 tends to ∞:*

$$\sup \left\{ \mathbb{P}_x \left(t \leq \tau_{m_1}^{(1|d)} \wedge \tau_\partial^{(d)} \right) \middle| d \in [\![1, \infty]\!] \,,\; x \in \mathcal{X}_d \,,\; x_0 \geq y_0 \right\},$$

where we recall $\tau_{m_1}^{(1|d)} := \inf\{t \geq 0 ;\; M_1^{(d)}(t) \leq m_1\}$.

The proofs of Lemmas 7.2.3 and 7.2.4, can be adapted mutatis mutandis from the ones of respectively Lemmas 6.1.2 (in Sect. 6.1.1) and 6.1.3 (in Sect. 6.1.2).

The next step is to handle the situation where both the first moment and the optimal subpopulation size are initially small, as stated in the upcoming Lemma 7.2.5.

Lemma 7.2.5 *For any two real numbers $t, m_1 > 0$, the following supremum tends to 0 as δ tends to 0:*

$$\sup \left\{ \mathbb{P}_x \left(t < \tau_\partial^{(d)} \right) \middle| d \in [\![1, \infty]\!] \,,\; x \in \mathcal{X}_d \,,\; M_1(x) \leq m_1 \,,\; x_0 \leq \delta \right\}.$$

Proof : As a generalization of Lemma 5.4.2, Lemma 7.2.5 is a consequence of the fact that $X_0^{(d)}$ is upper-bounded on the event $\{\sup_{s\leq t} M_1^{(d)}(s) \leq m_1'\}$, thanks to Lemma 5.1.4, by the solution Y to the following SDE:

$$\mathrm{d}Y(t) = \alpha\, m_1' \cdot Y(t)\,\mathrm{d}t + \sqrt{Y(t)\cdot(1-Y(t))}\,\mathrm{d}B_0(t)\,, \qquad Y(0) = \delta.$$

Thanks to Lemma 5.1.2(i), 0 is an exit boundary for Y. Thanks to [2, Lemma 3.2], we know an upper-bound of $\mathbb{P}_x(\sup_{s\leq t} M_1^{(d)}(s) \geq m_1')$ that tends to 0 as m_1' goes to ∞, uniformly in the $x \in \mathcal{X}_d$ such that $M_1(x) \leq m_1$. The combination of these two facts concludes the proof. □

As the next step, we justify with the upcoming Lemma 7.2.6 that, once a moment has descended, it is unlikely for the next moment to stay high on a significant time-interval afterwards:

Lemma 7.2.6 *Given any integer $k \in \{1, 2, 3\}$ and any two real numbers $t,\ m > 0$, the following supremum tends to 0 as m' tends to ∞:*

$$\sup\left\{\mathbb{P}_x\left(t \leq \tau_{m'}^{(k+1|d)} \wedge \tau_\partial^{(d)}\right) \,\middle|\, d \in [\![k,\infty]\!]\,,\ x \in \mathcal{X}_d\,,\ M_{2k}(x) < \infty\,,\ M_k(x) \leq m\right\}.$$

Proof : The proof of Lemma 7.2.6 generalizes the one of Lemma 6.1.4 in Sect. 6.1.3. We just sketch the localization argument for the case $d = \infty$, that is similar yet simpler than the one presented for the proof of the forthcoming Lemma 7.2.8.

Thanks to Lemma 6.1.5, recall that $M_{2k}^{(\infty)}$ is a.s. locally upper-bounded. We can thus introduce a sequence T_ℓ, $\ell \geq 1$, of stopping times such that $M_{2k}^{(\infty)}$ is upper-bounded by ℓ on $[0, t \wedge T_\ell]$ and that goes to infinity as ℓ tends to infinity. With the same arguments as for Lemma 6.1.4:

$$\mathbb{E}_x\left(\int_0^t \mathbf{1}_{\left\{s\leq T_{m_1}^{(1|d)}\wedge T_\ell\right\}} M_{k+1}^{(d)}(s)\,\mathrm{d}s\right) \leq \frac{(k-1)\cdot m}{\alpha} + \hat{C}_k \cdot t \cdot m_1.$$

Taking ℓ to infinity by monotone convergence, we can then proceed as previously and conclude the proof of Lemma 7.2.6. □

With the upcoming Lemma 7.2.7, we state that, on the event of its survival, the process is bound to reach non-negligible subpopulation sizes for the $(X_j^{(d)})_{j\in[\![0,J]\!]}$, for any $J \leq d$ (that are the J-optimal classes).

Lemma 7.2.7 *Given any integer $J \in \mathbb{N}$, and any three real numbers $t,\ m_1,\ y_0 > 0$, the following supremum tends to 0 as y tends to 0:*

$$\sup\left\{\mathbb{P}_x\left(t \wedge \tau_\partial^{(d)} < \tau_y^{J,(d)}\right) \,\middle|\, d \in [\![J,\infty]\!]\,,\ x \in \mathcal{X}_d\,,\ x_0 \geq y_0\,,\ M_1(x) \leq m_1\right\},$$

where we recall the notation $\tau_y^{J,(d)} := \inf\left\{s \geq 0;\ \forall\, j \leq J,\ X_j^{(d)}(s) \geq y\right\}$ *for any $y > 0$ and any integer $J \geq 0$.*

The above notation is to be understood as $\tau_y^{J,(d)} = \tau_y^{d,(d)}$ for any $J \geq d$ (for the case $d \in \mathbb{N}$). The proof of Lemma 7.2.7 is an adaptation mutatis mutandis of the one of Lemma 5.4.3, as stated in Sect. 5.4.

7.2.1 Upper-bound on the probability of moment increase

Section 7.2.1 is devoted to the upcoming Lemma 7.2.8. In the time-interval between the descent of the moment and the increase of the J-optimal population sizes $(X_j^{(d)})_{j \leq J}$, we show that the control on the corresponding moment stays tight. Recall the definition of $T_m^{(k|d)}$ from (6.16).

Lemma 7.2.8 *For any two real numbers $k > 1$ and $t > 0$, there exists $C \geq 1$ such that the following inequality holds for any $m, m' > 0$, any $d \in [\![1, \infty]\!]$ and any $x \in \mathcal{X}^{2k}$ ($x \in \mathcal{X}_d$ for $d \in \mathbb{N}$) such that $M_k(x) \leq m$,*

$$\mathbb{P}_x\left(T_{m'}^{(k|d)} \leq t\right) \leq \frac{Cm}{m'}.$$

Proof : The proof of Proposition 6.2.1 already implies the result provided $d < \infty$. We justify in the following that it extends to the case where $d = \infty$ in which the martingale part $\mathcal{M}_k^{(\infty)}$ in Lemma 6.1.5 is a priori only local.

The initial condition $x \in \mathcal{X}^{2k}$ is such that $M_k(x) \leq m$. The expression of $V_k^{(\infty)}$ in (6.7) implies the following inequalities:

$$-\alpha M_{k+1}^{(\infty)} \leq V_k^{(\infty)} \leq \lambda \cdot (C_k M_k^{(\infty)} + 1).$$

Thanks to Lemma 6.1.5 with the fact that $x \in \mathcal{X}^{2k}$, the process $M_{k+1}^{(\infty)}$ is a.s. locally upper-bounded (thus also $M_k^{(\infty)}$). Since $\mathcal{M}_k^{(\infty)}(t) = M_k^{(\infty)}(t) - M_k(x) - \int_0^t V_k^{(\infty)}(s)\mathrm{d}s$, $\mathcal{M}_k^{(\infty)}$ is thus also a.s. locally upper-bounded. Thanks to Duhamel's formula, this entails that the process $\widehat{M}_k^{(\infty)}$ is well-defined as a solution to the following SDE, similar to (6.17):

$$\widehat{M}_k^{(\infty)}(t) := m + \lambda t + \lambda C_k \int_0^t \widehat{M}_k^{(\infty)}(s)\,\mathrm{d}s + \mathcal{M}_k^{(\infty)}(t).$$

We localize this process thanks to the following sequence of stopping times T_ℓ, for any $\ell \geq 1$:

$$T_\ell := \inf\left\{s \geq 0\,;\ \langle \mathcal{M}_k^{(\infty)} \rangle_s \geq \ell\,,\ M_k^{(\infty)}(s) \geq \ell\right\}. \tag{7.4}$$

The process $(\widehat{M}_k^{(\infty)}(t \wedge T_\ell))_{t \geq 0}$ defines a non-negative submartingale such that the following inequality holds for any $t \geq 0$:

$$\mathbb{E}_x[\widehat{M}_k^{(\infty)}(t \wedge T_\ell)] := m + \lambda t + \lambda C_k \int_0^t \mathbb{E}_x[\widehat{M}_k^{(\infty)}(s \wedge T_\ell)]\,\mathrm{d}s.$$

Thanks to Grönwall's lemma (see for instance [28, Proposition 6.59]), since our localization procedure entails an upper-bound on $(\widehat{M}_k^{(\infty)}(t \wedge T_\ell))_{t \geq 0}$, the following inequality holds for any $t \geq 0$:

$$\mathbb{E}_x \left[\widehat{M}_k^{(\infty)}(t \wedge T_\ell) \right] \leq (m + \lambda t) \cdot e^{\lambda C_k t} \leq C \cdot m, \tag{7.5}$$

with $C := (1 + \lambda t) \cdot e^{\lambda C_k t}$ (recall that $m \geq 1$). Thanks to [28, Proposition 3.12], $\widehat{M}_k^{(\infty)} \geq M_k^{(\infty)}$. Thanks to Doob's inequality, the following inequalities thus hold for any $t \geq 0$ and $m' > 0$:

$$\begin{aligned} \mathbb{P}_x \Big(\sup_{\{s \leq t \wedge T_\ell\}} M_k^{(\infty)}(s) \geq m' \Big) &\leq \mathbb{P}_x \Big(\sup_{\{s \leq t \wedge T_\ell\}} \widehat{M}_k^{(\infty)}(s) \geq m' \Big) \\ &\leq \frac{\mathbb{E}_x [\widehat{M}_k^{(\infty)}(t \wedge T_\ell)]}{m'}. \end{aligned}$$

Recalling (7.5) and thanks to the monotone convergence theorem letting ℓ tend to infinity, it entails the following inequality for any $m, m' > 0$ and any $x \in \mathcal{X}^{2k}$ such that $M_k(x) \leq m$:

$$\mathbb{P}_x \left(T_{m'}^{(k|\infty)} \leq t \right) \leq \frac{Cm}{m'},$$

which concludes the proof of Lemma 7.2.8. □

7.2.2 Concluding the proof of Theorem 7.2.1

Let us consider any three real numbers $t, \eta, \epsilon > 0$. We consider the following event as a function of $m_1 > 0$ that describes a failure in the descent of the first moment:

$$\mathcal{E}_1^{(d)} := \left\{ \tau_{m_1}^{(1|d)} > 2t \right\} \cap \left\{ 2\,t < \tau_\partial^{(d)} \right\}.$$

With exactly the same reasoning as for Proposition 6.1.1, exploting Lemmas 7.2.3 and 7.2.4 instead of Lemmas 6.1.2 and 6.1.3, we can choose $m_1 > 0$ such that $\mathbb{P}_x(\mathcal{E}_1^{(d)}) \leq 3\,\epsilon$ holds for any $x \in \mathcal{X}_d$.

The following event is stated as a function of $m_1' > 0$ and describes a failure in having the first moment contained on a significant time-interval:

$$\mathcal{E}_2^{(d)} := \left\{ \tau_{m_1}^{(1|d)} \leq 2t \right\} \cap \left\{ 2t < \tau_\partial^{(d)} \right\} \cap \left\{ \widetilde{T}_{m_1'}^{(1|d)} \leq 5t \right\},$$

where $\widetilde{T}_{m_1'}^{(1|d)} := \inf \left\{ s \geq \tau_{m_1}^{(1|d)} \,;\; M_1^{(d)}(s) \leq m_1' \right\}$. Thanks to [2, Lemma 3.2] and the strong Markov property at time $\tau_{m_1}^{(1|d)}$, we can choose $m_1' > 0$ such that $\mathbb{P}_x(\mathcal{E}_2^{(d)}) \leq \epsilon$ holds for any $x \in \mathcal{X}_d$.

We then consider the following event as a function of $y_0 \in (0, 1)$, that describes a failure in having $X_0^{(d)}$ bounded away from 0:

$$\mathcal{E}_3^{(d)} := \left\{\tau_{m_1}^{(1|d)} \leq 2t\right\} \cap \left\{\widetilde{T}_{m_1'}^{(1|d)} > 5t\right\} \cap \left\{\widetilde{T}_{y_0}^{0,(d)} \leq 5t\right\} \cap \left\{6\,t < \tau_\partial^{(d)}\right\},$$

where $\widetilde{T}_{y_0}^{0,(d)} := \inf\left\{s \geq \tau_{m_1}^{(1|d)};\, X_0^{(d)}(s) < y_0\right\}$. Thank to Lemma 7.2.5 and the strong Markov property at time $\widetilde{T}_{y_0}^{0,(d)}$ (within the time-interval $[\tau_{m_1}^{(1|d)}, \widetilde{T}_{m_1'}^{(1|d)}]$), we can choose $y_0 > 0$ such that $\mathbb{P}_x(\mathcal{E}_3^{(d)}) \leq \epsilon$ holds for any $x \in \mathcal{X}_d$.

The following event is stated as a function of $m_3 > 0$ and describes a failure in the descent of the third moment:

$$\mathcal{E}_4^{(d)} := \left\{\tau_{m_1}^{(1|d)} \leq 2t\right\} \cap \left\{2t < \tau_\partial^{(d)}\right\} \cap \left\{\widetilde{\tau}_{m_3}^{(3|d)} > 4\,t\right\},$$

where $\widetilde{\tau}_{m_3}^{(3|d)} := \inf\left\{s \geq \tau_{m_1}^{(1|d)} + t\,;\ M_3^{(d)}(s) \leq m_3\right\}$. Thanks to Lemma 7.2.6 and the strong Markov property at time $\tau_{m_1}^{(1|d)}$, we can choose $m_3 > 0$ such that $\mathbb{P}_x\left(\mathcal{E}_4^{(d)}\right) \leq \epsilon$ holds for any $x \in \mathcal{X}_d$ (with an implicit step for the second moment).

The failure in the containment of the third moment is stated in terms of the following event, as a function of $m_3' > 0$:

$$\mathcal{E}_5^{(d)} := \left\{\tau_{m_1}^{(1|d)} \leq 2t\right\} \cap \left\{\widetilde{\tau}_{m_3}^{(3|d)} \leq 4\,t\right\} \cap \left\{\widetilde{T}_{m_3'}^{(3|d)} \leq 5\,t\right\} \cap \left\{5\,t < \tau_\partial^{(d)}\right\},$$

where $\widetilde{T}_{m_3'}^{(3|d)} := \inf\left\{s \geq \tau_{m_3}^{(3|d)};\ M_3^{(d)}(s) \leq m_3'\right\}$. Thanks to Lemma 7.2.8, we can choose $m_3' > 0$ such that $\mathbb{P}_x\left(\mathcal{E}_5^{(d)}\right) \leq \epsilon$ holds for any $x \in \mathcal{X}_d$.

Now, we can define $J := \lfloor m_3'/\eta \rfloor + 1$ (η being an imposed parameter in the statement of Theorem 7.2.1). The failure in having the J-optimal subpopulation sizes bounded away from 0 is stated in terms of the following event, as a function of $y \in (0, 1)$:

$$\mathcal{E}_6^{(d)} := \left\{\tau_{m_1}^{(1|d)} \leq 2t\right\} \cap \left\{\widetilde{\tau}_{m_3}^{(3|d)} \leq 4\,t\right\} \cap \left\{\widetilde{T}_{m_1'}^{(1|d)} > 5t\right\} \cap \left\{\widetilde{T}_{y_0}^{0,(d)} > 5t\right\} \\ \cap \left\{5t < \tau_\partial^{(d)}\right\} \cap \left\{\widetilde{\tau}_y^{J,(d)} > 5t\right\},$$

where $\widetilde{\tau}_y^{J,(d)} := \inf\left\{s \geq \widetilde{\tau}_{m_3}^{(3|d)};\ \forall\, j \leq J,\ X_j^{(d)}(s) \geq y\right\}$. Thanks to Lemma 7.2.7, we can choose $y \in (0, 1)$ such that $\mathbb{P}_x\left(\mathcal{E}_6^{(d)}\right) \leq \epsilon$ holds for any $x \in \mathcal{X}_d$.

Let $E^{(d)}$ take the following form, which agrees with (7.2):

$$E^{(d)} := \left\{x \in \mathcal{X}_d\,;\ M_3(x) \leq m_3'\,,\ \forall\, j \leq \lfloor m_3'/\eta \rfloor + 1,\ x_j \geq y\right\}. \tag{7.6}$$

Firstly, $\tau_{m_1}^{(1|d)} \leq 2\,t$ holds a.s. on the event $\{6t < \tau_\partial^{(d)}\} \setminus \mathcal{E}_1^{(d)}$. $\widetilde{T}_{m_1'}^{(1|d)} \geq 5t$ thus holds a.s. on the event $\{6t < \tau_\partial^{(d)}\} \setminus \cup_{i=1}^{2} \mathcal{E}_i^{(d)}$. Consequently, $\widetilde{T}_{y_0}^{0,(d)} \geq 5t$ holds a.s. on the event $\{6t < \tau_\partial^{(d)}\} \setminus \cup_{i=1}^{3} \mathcal{E}_i^{(d)}$. On the other hand, $\widetilde{\tau}_{m_3}^{(3|d)} \leq 4\,t$ holds a.s. on the event $\{6t < \tau_\partial^{(d)}\} \setminus \cup_{i=1}^{4} \mathcal{E}_i^{(d)}$. $\widetilde{T}_{m_3'}^{(3|d)} \geq 5\,t$ then holds a.s. on the event $\{6t < \tau_\partial^{(d)}\} \setminus \cup_{i=1}^{5} \mathcal{E}_i^{(d)}$. Finally, $\widetilde{\tau}_y^{J,(d)} \leq 5t$ holds a.s. on the event $\{6t < \tau_\partial^{(d)}\} \setminus \cup_{i=1}^{6} \mathcal{E}_i^{(d)}$. By definition of $\widetilde{\tau}_y^{J,(d)}$ and $\widetilde{T}_{m_3'}^{(3|d)}$, and since $\widetilde{\tau}_y^{J,(d)} \in [\widetilde{\tau}_{m_3}^{(3|d)}, \widetilde{T}_{m_3'}^{(3|d)}]$, $X^{(d)}(\widetilde{\tau}_y^{J,(d)})$ belongs to $E^{(d)}$ a.s. on the event $\{6t < \tau_\partial^{(d)}\} \setminus \cup_{i=1}^{6} \mathcal{E}_i^{(d)}$. This concludes the following inclusion $\{6t < \tau_\partial^{(d)} \wedge \tau_E^{(d)}\} \subset \cup_{i=1}^{6} \mathcal{E}_i^{(d)}$, which entails the following upper-bound in probability for any $d \in [\![1, \infty]\!]$ and any $x \in \mathcal{X}_d$:

$$\mathbb{P}_x\Big(6\,t < \tau_\partial^{(d)} \wedge \tau_E^{(d)}\Big) \leq 8\,\epsilon.$$

This concludes the proof of Theorem 7.2.1 (by adjusting the initial choices of t and ϵ). □

7.3 Aggregation of the last coordinates

The changes in the description of the system specified in this subsection will be crucial for the proofs of both the estimates of mixing (in Sect. 7.4) and of almost perfect harvest (in Sect. 7.5). Up to a multiplicative constant in the probabilities, they make it possible to gather the last coordinates in one specific block while keeping a Markovian description. Our aim is then to prove that the dependency in the initial values of these last coordinates vanishes very quickly.

More precisely, the current subsection is dedicated to the study of the law $\mathbb{P}_x^{(J:d)}$, for any $d \in [\![1, \infty]\!]$, any integer $J \in [\![1, d]\!]$ and any $x \in \mathcal{X}_d$, of the solution to the following set of equations, stated for any $i \in [\![0, d]\!]$:

$$\begin{aligned} \mathrm{d}X_i^{(d)}(t) &= \alpha \cdot (M_1^{(J:d)}(t) - i \wedge J) \cdot X_i^{(d)}(t)\,\mathrm{d}t + \lambda \cdot (X_{i-1}^{(d)}(t) - \mathbf{1}_{\{i<d\}}\, X_i^{(d)}(t))\,\mathrm{d}t \\ &\quad + \sqrt{X_i^{(d)}(t)}\,\mathrm{d}W_i(t) - X_i^{(d)}(t)\,\mathrm{d}W_{(d)}(t), \qquad X_i^{(d)}(0) = x, \end{aligned} \tag{7.7}$$

with $(W_i)_{i\geq 0}$ is still a family of mutually independent Brownian motions, $W_{(d)}$ is expressed as in (1.1):

$$\mathrm{d}W_{(d)}(t) := \textstyle\sum_{j\in[\![0,d]\!]} \sqrt{X_j^{(d)}(t)}\mathrm{d}W_j(t), \quad W_{(d)}(0) = 0, \tag{7.8}$$

and the process $M_1^{(J:d)}$ is the first moment saturated at value J:

$$M_1^{(J:d)} := \textstyle\sum_{i\in[\![0,d]\!]}(i \wedge J)\, X_i^{(d)} = \sum_{i\leq J-1} i\, X_i^{(d)} + J\, \sum_{i\geq J} X_i^{(d)}. \tag{7.9}$$

The reason we do not include the obvious dependency in J in the solution $X^{(d)}$ to the system (7.7) is that we want to connect this solution to the one to the system (1.1) under $\mathbb{P}_x$ with a change of probability density given by the Girsanov transform.

The main interest of this law lies in that it can be efficiently projected on a finite-dimensional system, as stated and proved in Sect. 7.3.1 (see Proposition 7.3.1). In the next Sect. 7.3.2, we will obtain comparison estimates relating $\mathbb{P}^{(J:d)}$ and $\mathbb{P}$ thanks to the Girsanov transform (see Proposition 7.3.2). Finally, in Sect. 7.3.3, we handle the probability of large increase of the moments (see Lemma 7.3.5).

7.3.1 Connexion between $\mathbb{P}^{(J:d)}$ and $\mathbb{P}$

Since the selection effect is identical on all the classes larger than J under $\mathbb{P}^{(J:d)}$, this law is naturally associated with the following projection π_J from $\mathcal{X}_d$ to $\mathcal{X}_J$:

$$\pi_J(x)_i = \begin{cases} x_i, & \text{if } i \leq J-1, \\ \sum_{j=J}^{d} x_j = 1 - \sum_{j=0}^{J-1} x_j, & \text{if } i = J. \end{cases} \tag{7.10}$$

The renormalized sequence of the tail classes under the projection will be described in terms of the law of the solution $X^{(F|d)}$ to the following set of equations, where F stands for "Final". We first denote its corresponding total size as the process $X^{(d)}_{(J)}$, with initial condition $x_{(J)} := [\pi_J(x)]_J$:

$$X^{(d)}_{(J)} := 1 - \sum_{\{i \leq J-1\}} X^{(d)}_i. \tag{7.11}$$

This process affects both the entrance flux on the class J, the associated correction term due to the renormalisation and the level of demographic fluctuations. For $i \in [\![J, d]\!]$, for any $t \geq 0$:

$$\mathrm{d}X^{(F|d)}_i(t) = V^{(F|d)}_i(t)\mathrm{d}t + \mathrm{d}\mathcal{N}^{(F|d)}_i(t), \quad X^{(F|d)}_i(0) = \frac{x_i}{x_{(J)}}, \tag{7.12}$$

with the following definitions of the process $V^{(F|d)}_i$:

$$V^{(F|d)}_i = \lambda \cdot \left[\frac{X^{(d)}_{J-1}}{X^{(d)}_{(J)}} \cdot (\mathbf{1}_{\{i=J\}} - X^{(F|d)}_i) + \mathbf{1}_{\{i \geq J+1\}} X^{(F|d)}_{i-1} - \mathbf{1}_{\{i<d\}} X^{(F|d)}_i \right], \tag{7.13}$$

and of the martingale $\mathcal{N}^{(d)}_{(J)}$:

$$\mathrm{d}\mathcal{N}^{(F|d)}_i(t) = \sqrt{\frac{X^{(F|d)}_i(t)}{X^{(d)}_{(J)}(t)}}\mathrm{d}W^{(F|d)}_i(t) - \frac{X^{(F|d)}_i(t)}{\sqrt{X^{(d)}_{(J)}(t)}}\mathrm{d}W^{(F|d)}_{(d)}(t), \quad \mathcal{N}^{(F|d)}_i(0) = 0, \tag{7.14}$$

in terms of the sequence $(W_i^{(F|d)})_{i\in[\![J,d]\!]}$, which defines a mutually independent family of Brownian motions that are mutually independent of the family $(W_i)_{i\in[\![0,d]\!]}$ and in terms of the martingale $W_{(d)}^{(F|d)}$:

$$\mathrm{d}W_{(d)}^{(F|d)}(t) := \sum\nolimits_{i=J}^{d} \sqrt{X_i^{(F|d)}(t)}\mathrm{d}W_i^{(F|d)}(t), \quad W_{(d)}^{(F|d)}(0) = 0.$$

We can then define the process $\bar{X}_i^{(d)}$ as $X_i^{(d)}$ for any $i \in [\![0, J-1]\!]$ and as $X_{(J)}^{(d)} \cdot \bar{X}_i^{(F|d)}$ for any $i \in [\![J, d]\!]$.

Proposition 7.3.1 *For any $J \geq 1$, $\pi_J(X^{(d)})$ is by itself a Markov process under any $\mathbb{P}_x^{(J:d)}$, whose law is independent of $d \in [\![J, \infty]\!]$ and depends on $x \in \mathcal{X}_d$ only through $\pi_J(x)$. Under $\mathbb{P}_x^{(J:d)}$, the process $\bar{X}^{(d)}$ on $\mathcal{X}_d$ has the same law as the process $X^{(d)}$.*

Proof : By virtue of (7.9) and of (7.11):

$$M_1^{(J:d)} := \sum\nolimits_{i=0}^{J-1} i\, X_i^{(d)} + J\, X_{(J)}^{(d)}.$$

Under $\mathbb{P}_x^{(J:d)}$, for any $x \in \mathcal{X}_d$, Itô's lemma then entails that the process $X_{(J)}^{(d)}$ is solution to the following SDE, for any $t \geq 0$:

$$\mathrm{d}X_{(J)}^{(d)}(t) = V_{(J)}^{(d)}(t)\mathrm{d}t + \mathrm{d}\mathcal{N}_{(J)}^{(d)}(t) - X_{(J)}^{(d)}(t)\,\mathrm{d}W_{(d)}(t), \tag{7.15}$$

with the definition of $W_{(d)}$ from (7.8) and the following definitions of the process $V_{(J)}^{(d)}$:

$$V_{(J)}^{(d)} = \alpha \cdot (M_1^{(J:d)} - J) \cdot X_{(J)}^{(d)} + \lambda \cdot X_{J-1}^{(d)}, \tag{7.16}$$

and of the martingale $\mathcal{N}_{(J)}^{(d)}$:

$$\mathrm{d}\mathcal{N}_{(J)}^{(d)}(t) = \sum\nolimits_{j=J}^{d} \sqrt{X_j^{(d)}(t)}\,\mathrm{d}W_j(t), \quad \widetilde{\mathcal{N}}_{(J)}^{(d)}(0) = 0. \tag{7.17}$$

Since the sequence $(W_i)_{i\in[\![0,d]\!]}$ defines a mutually independent family of Brownian motions, the following identity holds for any $t \geq 0$:

$$\mathrm{d}\langle\mathcal{N}_{(J)}^{(d)}\rangle_t = \sum\nolimits_{j=J}^{d} X_j^{(d)}(t)\mathrm{d}t = X_{(J)}^{(d)}(t)\mathrm{d}t.$$

Thanks to [16, Theorem 18.12], we can define a Brownian motion $W_{(J)}^{(d)}$ such that the following SDE hold: $\mathrm{d}\mathcal{N}_{(J)}^{(d)}(t) = \sqrt{X_{(J)}^{(d)}(t)}\mathrm{d}W_{(J)}^{(d)}(t)$. Recalling (7.8), it yields the following alternative identity for the Brownian motion $W_{(d)}$:

$$\mathrm{d}W_{(d)}(t) = \sum\nolimits_{i=0}^{J-1} \sqrt{X_i^{(d)}(t)}\mathrm{d}W_i(t) + \sqrt{X_{(J)}^{(d)}(t)}\mathrm{d}W_{(J)}^{(d)}(t). \tag{7.18}$$

The correlation between $W^{(d)}_{(J)}$ and the $(W_i)_{i\leq J-1}$ remains zero, while they constitute a system of Brownian motions under the same filtration. $W^{(d)}_{(J)}$ is thus independent of $\sigma(W_i;\, i\leq J-1)$, so that the system of equations satisfied by $\pi_J(X)$ is equivalent for any $\mathbb{P}^{(J:d)}_x$.

Thanks to Itô's lemma, the following identity holds for any $t\geq 0$ and any $i\in[\![J,d]\!]$:

$$d\bar{X}^{(d)}_i(t) = \bar{V}^{(d)}_i(t)\mathrm{d}t + \mathrm{d}\bar{\mathcal{N}}^{(d)}_i(t) + \tfrac{1}{2}\,\mathrm{d}\langle \mathcal{N}^{(d)}_{(J)}, \mathcal{N}^{(F|d)}_i\rangle_t, \tag{7.19}$$

with $\mathcal{N}^{(d)}_{(J)}(t)$ and $\mathcal{N}^{(F|d)}_i(t)$ defined respectively in (7.15) and (7.14), while the martingale component $\bar{\mathcal{N}}^{(d)}_i$ is expressed as follows after simplifications:

$$\begin{aligned}\mathrm{d}\bar{\mathcal{N}}^{(d)}_i(t) &= X^{(d)}_{(J)}(t)\mathrm{d}\bar{\mathcal{N}}^{(F|d)}_i(t) + \bar{X}^{(F|d)}_i(t)\mathrm{d}\mathcal{N}^{(d)}_{(J)}(t)\\ &= \sqrt{\bar{X}^{(d)}_i(t)}\mathrm{d}W_i(t) - \bar{X}^{(d)}_i(t)\mathrm{d}W_{(d)}(t),\quad \bar{\mathcal{N}}^{(d)}_i(0)=0,\end{aligned}$$

and the process $\bar{V}^{(F|d)}_i(t)$ as follows, thanks to (7.16) and to (7.13) after simplifications:

$$\begin{aligned}\bar{V}^{(F|d)}_i &= X^{(d)}_{(J)}\cdot \bar{V}^{(F|d)}_i + \bar{X}^{(F|d)}_i\cdot V^{(d)}_{(J)}\\ &= \alpha\cdot(M^{(J:d)}_1 - J)\cdot \bar{X}^{(d)}_i + \lambda\cdot(\bar{X}^{(d)}_{i-1} - \mathbf{1}_{\{i<d\}}\bar{X}^{(d)}_i).\end{aligned}$$

$\mathcal{N}^{(F|d)}_i$ is defined in terms of the sequence $(W^{(F|d)}_i)_{i\in[\![0,d]\!]}$ of Brownian motions, which is independent of the sequence $(W_i)_{i\in[\![0,d]\!]}$ in terms of which the martingale $\mathcal{N}^{(d)}_{(J)}$ is stated. Therefore, $\mathrm{d}\langle\mathcal{N}^{(d)}_{(J)}, \mathcal{N}^{(F|d)}_i\rangle\equiv 0$. The system of SDEs (7.19) satisfied by $(\bar{X}^{(d)}_i)_{i\in[\![0,d]\!]}$ thus coincides with the system (1.1) satisfied by $(X^{(d)}_i)_{i\in[\![0,d]\!]}$. By the uniqueness of the whole system, cf Proposition A.0.1 in the appendix, $X^{(d)}$ coincides with $\bar{X}^{(d)}$. This ends the proof of Proposition 7.3.1. □

7.3.2 The connexion formula between $\mathbb{P}^{(J:d)}$ and $\mathbb{P}$

Thanks to the Girsanov transform, we will establish a relevant quantification for the transfer between the original law of the process $\mathbb{P}$ and the law of $\mathbb{P}^{(J:d)}$. For the upcoming Proposition 7.3.2, we shall exploit a control on moments of order k. We recall the definition of $T^{(k|d)}_m$ from (6.16) for any $m>0$:

$$T^{(k|d)}_m := \inf\left\{s\geq 0;\ M^{(d)}_k(s)\geq m\right\}.$$

Proposition 7.3.2 *Given any $t,\epsilon>0$, $k\geq 2$, there exists $C_M, C_G>0$ for which the following holds. For any $m\geq 1$, with $m' := C_M\cdot m$, for any $d\in[\![1,\infty]\!]$, any $J\leq d$, and any $x\in\mathcal{X}_d\cap\mathcal{X}^{2k}$ such that $M_k(x)\leq m$, there exists a coupling between $\mathbb{P}^{(J:d)}$*

and $\mathbb{P}$ such that the following upper-bound holds a.s. on the event $\{t < T_{m'}^{(k|d)}\}$:

$$\Big| \log\left(\frac{\mathrm{d}\mathbb{P}_x^{(J:d)}}{\mathrm{d}\mathbb{P}_x}\Big|_{[0,t]} \right) \Big| \le C_G \frac{m}{J^{k-2}},$$

while the event $\{t < T_{m'}^{(k|d)}\}$ happens with a probability close to 1 in the following sense:

$$\mathbb{P}_x\left(T_{m'}^{(k|d)} \le t\right) \le \epsilon.$$

Remark 7.3.3 In this article, we exploit Proposition 7.3.2 only for $k = 3$. The proof is extended to any real number $k \ge 2$ to explicit our motivation for choosing $k > 2$ (see (7.76) below).

The explicit connexion formula

We express in this subsection the Girsanov transform that makes it possible to relate $\mathbb{P}^{(J:d)}$ and $\mathbb{P}$. It is expressed in the upcoming Lemma 7.3.4 in terms of the processes $R_1^{(J:d)}$ and $R_2^{(J:d)}$ with the following definitions:

$$R_1^{(J:d)} := \sum\nolimits_{i=J+1}^{d} (i-J)\cdot X_i^{(d)}, \quad R_2^{(J:d)} := \sum\nolimits_{i=J+1}^{d} (i-J)^2\cdot X_i^{(d)}.$$

One can notice that they correspond to the expectation and variance of the vector $(Y_i)_{i\in\mathbb{Z}_+}$ such that $Y_0 = \sum_{j=0}^{J} X_j^{(d)}$ and for any $i \in \mathbb{N}$, $Y_i = X_{J+i}^{(d)}$.

Lemma 7.3.4 *For any $d \in [\![1,\infty]\!]$ and $J \in [\![1,d]\!]$, there exists a coupling between $\mathbb{P}^{(J:d)}$ and $\mathbb{P}$ such that:*

$$\log \frac{\mathrm{d}\mathbb{P}_x^{(J:d)}}{\mathrm{d}\mathbb{P}_x}\Big|_{[0,t]} = \alpha\cdot R_1^{(J:d)}(0) - \alpha\cdot R_1^{(J:d)}(t) + \int_0^t G^{(J:d)}(s)\mathrm{d}s,$$

with the following definition of the process $G^{(J:d)}$:

$$\begin{aligned} G^{(J:d)} &:= \alpha^2\,(M_1^{(d)} - J)\,R_1^{(J:d)} - \alpha^2\,R_2^{(J:d)} + \alpha\,\lambda\,(X_{(J)}^{(d)} - \mathbf{1}_{\{d<\infty\}}X_d^{(d)}) \\ &\quad - \frac{\alpha^2}{2}\left[R_2^{(J:d)} - (R_1^{(J:d)})^2\right]. \end{aligned}$$

Proof : We define as follows the martingale $\mathcal{L}^{(J:d)}$, starting at 0:

$$\begin{aligned} \mathrm{d}\mathcal{L}^{(J:d)}(t) &:= -\alpha \sum_{i\ge J+1} (i-J)\sqrt{X_i^{(d)}(t)}\,\mathrm{d}W_i(t) + \alpha\cdot R_1^{(J:d)}(t)\,\mathrm{d}W_{(d)}(t) \\ &= -\alpha \sum_{i\ge J+1} (i-J)\cdot\left[\sqrt{X_i^{(d)}(t)}\,\mathrm{d}W_i(t) - X_i^{(d)}(t)\,\mathrm{d}W_{(d)}(t)\right]. \end{aligned}$$

By this choice, we obtain the following identities, for any $i \in [\![0, d]\!]$:

$$\begin{aligned} \mathrm{d}\langle \mathcal{L}^{(J:d)}, W_i \rangle_s &= \alpha \cdot \left[R_1^{(d)}(s) - (i-J)_+ \right] \cdot \sqrt{X_i^{(d)}(s)}\, \mathrm{d}s, \\ \mathrm{d}\langle \mathcal{L}^{(J:d)}, W_{(d)} \rangle_s &= 0. \end{aligned} \tag{7.20}$$

Recalling the systems of SDEs (1.1) and (7.7), it entails that the Girsanov transform of the law $\mathbb{P}$ with respect to the exponential martingale of $\mathcal{L}^{(J:d)}$ generates $\mathbb{P}^{(J:d)}$, in the sense that the following property holds for any $t > 0$ and $x \in \mathcal{X}_d$:

$$\log \frac{\mathrm{d}\mathbb{P}_x^{(J:d)}}{\mathrm{d}\mathbb{P}_x}\bigg|_{[0,t]} = \mathcal{L}^{(J:d)}(t) - \frac{1}{2}\, \mathrm{d}\langle \mathcal{L}^{(J:d)} \rangle_t. \tag{7.21}$$

Thanks to (7.20), the quadratic variation $\mathrm{d}\langle \mathcal{L}^{(J:d)} \rangle_t$ satisfies the following identity:

$$\begin{aligned} \mathrm{d}\langle \mathcal{L}^{(J:d)} \rangle_t &= -\alpha \sum_{i \geq J+1} (i-J) \sqrt{X_i^{(d)}(t)} \mathrm{d}\langle \mathcal{L}^{(J:d)}, W_i \rangle_t \\ &= \alpha^2 \cdot \left[R_2^{(J:d)}(t) - R_1^{(J:d)}(t)^2 \right]. \end{aligned} \tag{7.22}$$

On the other hand, we note the following identity for any $s \geq 0$:

$$\mathrm{d}R_1^{(J:d)}(s) = V_1^{(J:d)}(s)\mathrm{d}s - \frac{1}{\alpha} \mathrm{d}\mathcal{L}^{(J:d)}(s), \quad R_1^{(J:d)}(0) = \sum_{i \geq J+1} (i-J) \cdot x_i,$$

where the process $V_1^{(J:d)}$ is defined as follows:

$$V_1^{(J:d)} = \alpha \cdot \left[(M_1^{(d)} - J) \cdot R_1^{(J:d)} - R_2^{(J:d)} \right] \mathrm{d}s + \lambda \cdot (X_{(J)}^{(d)} - \mathbf{1}_{\{d<\infty\}} X_d^{(d)}).$$

With this alternative expression for $\mathcal{L}^{(J:d)}$, recalling (7.21) and (7.22), we conclude the proof of Lemma 7.3.4. □

***Concluding the proof of Proposition* 7.3.2**

The aim is now to get uniform upper-bound on the expression given in Lemma 7.3.4. We consider any $k \geq 2$. Firstly, the following inequalities hold by definitions of $R_1^{(J:d)} \geq 0$ and $M_k^{(d)}$:

$$R_1^{(J:d)} \leq J^{-(k-1)} \sum\nolimits_{\{i \geq J+1\}} i^k\, X_i^{(d)} \leq J^{-(k-1)} M_k^{(d)}.$$

Similarly for $R_2^{(J:d)} \geq 0$ and $X_{(J)}^{(d)} - \mathbf{1}_{\{d<\infty\}} X_d^{(d)}(s) \geq 0$:

$$R_2^{(J:d)} \leq J^{-(k-2)} M_k^{(d)}, \qquad X_{(J)}^{(d)} - \mathbf{1}_{\{d<\infty\}} X_d^{(d)}(s) \leq J^{-k} M_k^{(d)}.$$

Thanks to the Cauchy–Schwarz inequality, $(R_1^{(J:d)})^2 \leq R_2^{(J:d)}$. Thanks to Hölder's inequality, both $M_1^{(d)} \leq (M_k^{(d)})^{1/k}$ and $M_{k-1}^{(d)} \leq (M_k^{(d)})^{(k-1)/k}$ hold, which finally yields the following inequalities about $M_1^{(d)} \cdot R_1^{(J:d)} \geq 0$:

$$M_1^{(d)} \cdot R_1^{(J:d)} \leq J^{-(k-2)} M_1^{(d)} \cdot M_{k-1}^{(d)} \leq J^{-(k-2)} M_k^{(d)},$$

Thanks to Lemma 7.3.4, we can thus choose a constant $C_1 > 0$ such that the following inequality holds a.s. on the event $\{t < T_{m'}^{(k|d)}\}$ for any $m, m' \geq 1$ such that $m < m'$, any $d \in [\![1, \infty]\!]$, any $J \in [\![1, d]\!]$, and any $x \in \mathcal{X}_d$ such that $M_k(x) \leq m$:

$$\Big| \log \Big(\frac{\mathrm{d}\mathbb{P}_x^{(J:d)}}{\mathrm{d}\mathbb{P}_x} \Big|_{[0,t]} \Big) \Big| \leq C_1 \, \frac{m'}{J^{k-2}}. \tag{7.23}$$

Thanks to Lemma 7.2.8, we can choose $C_2 > 0$ such that the following inequality holds for any $m, m' \geq 1$ such that $m < m'$, any $d \in [\![1, \infty]\!]$ and any $x \in \mathcal{X}_d$ such that $M_k(x) \leq m$:

$$\mathbb{P}_x \left(T_{m'}^{(k|d)} \leq t \right) \leq \frac{C_2 m}{m'}.$$

Let $\epsilon > 0$. We thus define $m' = C_M \cdot m$, where $C_M := C_2/\epsilon$, so that the above upper-bound is exactly ϵ. Thanks to (7.23) with $C_G = C_1 \cdot C_2/\epsilon$, the following inequality holds a.s. on the event $\{t < T_{m'}^{(k|d)}\}$, for any $m \geq 1$, any $d \in [\![1, \infty]\!]$, any $J \in [\![1, d]\!]$, and any $x \in \mathcal{X}_d$ such that $M_k(x) \leq m$:

$$\Big| \log \Big(\frac{\mathrm{d}\mathbb{P}_x^{(J:d)}}{\mathrm{d}\mathbb{P}_x} \Big|_{[0,t]} \Big) \Big| \leq C_G \, \frac{m}{J^{k-2}}.$$

This concludes the proof of Proposition 7.3.2. □

7.3.3 Upper-bound on the probability of moment increase for $X^{(F|d)}$

Similarly as for Lemma 7.2.8, exploiting the decomposition in Proposition 7.3.1, we define the third moment $M_3^{(F|d)}$ of $X^{(F|d)}$:

$$M_3^{(F|d)} := \sum\nolimits_{i=J}^{d} i^3 \, X_i^{(F|d)} \in [J^3, \infty),$$

and the corresponding hitting time $T_m^{(F.3|d)}$ of the value $m > 0$:

$$T_m^{(F.3|d)} := \inf \Big\{ s \geq 0 \, ; \; M_3^{(F|d)}(s) \geq m \Big\}. \tag{7.24}$$

The upper-bound is to be obtained up to the following hitting time $\tau_0^{(J:d)}$:

$$\tau_0^{(J:d)} := \inf \Big\{ t \geq 0 \, ; \; X_{(J)}^{(d)}(t) = 0 \Big\}. \tag{7.25}$$

For clarity, we define $\mathcal{F}^{(J)} = \sigma\Big(W_i : i \le J-1;\ W^{(d)}_{(J)}\Big)$. Recall that the process $X^{(F|d)}_i$ is driven by Brownian motions $(W^{(F|d)}_i : i \ge J)$ that are independent of $\mathcal{F}^{(J)}$. The inclusion $\sigma(\pi_J(X)) \subset \mathcal{F}^{(J)}$ is directly obtained through the autonomous set of equation verified by $\pi_J(X)$. The following control on $M^{(F|d)}_3$ exploits the filtration $\mathcal{F}^{(J)}_t := \mathcal{F}^{(J)} \vee \mathcal{F}_t$.

Lemma 7.3.5 *For any $t > 0$, there exists $C \ge 1$ such that the following inequality holds a.s. for any $m, m' > 0$, any $d \in [\![1, \infty]\!]$ and any $x \in \mathcal{X}_d$ such that $M^{(F|d)}_3(x) \le m$:*

$$\mathbb{P}^{(J:d)}_x \left(T^{(F.3|d)}_{m'} \le t \wedge \tau^{(J:d)}_0 \,\big|\, \mathcal{F}^{(J)}\right) \le \frac{Cm}{m'}.$$

Proof : Under $\mathbb{P}^{(J:d)}$, we exploit the Itô formula to express $M^{(F|d)}_3$ as the solution to the following SDE:

$$\mathrm{d}M^{(F|d)}_3(t) := V^{(F|d)}_3(t)\mathrm{d}t + \mathrm{d}\mathcal{M}^{(F|d)}_3(t), \tag{7.26}$$

where $V^{(F|d)}_3$ is a bounded variation process defined as:

$$V^{(F|d)}_3 := \lambda \cdot \left[\frac{X^{(d)}_{J-1}}{X^{(d)}_{(J)}} \cdot (J^3 - M^{(F|d)}_3) + \sum_{\ell \ge J} (\ell+1)^3 \cdot X^{(F|d)}_\ell - M^{(F|d)}_3 + \mathbf{1}_{\{d<\infty\}} d^3 \cdot X^{(F|d)}_d\right].$$

Note that whatever the values of $(X^{(d)}_{J-1}/X^{(d)}_{(J)})$, with the rough estimate $(\ell+1)^3 \le 8\ell^3$ for $\ell \ge 1$, the inequality $V^{(F|d)}_3 \le 8\lambda\, M^{(F|d)}_3$ holds. On the other hand, the local martingale process $\mathcal{M}^{(F|d)}_3$ is expressed as follows in terms of the martingales $\mathcal{N}^{(F|d)}_i$ defined in (7.14):

$$\mathrm{d}\mathcal{M}^{(F|d)}_3(t) := \sum_{i \ge J} i^3 \cdot \mathcal{N}^{(F|d)}_i(t), \quad \mathcal{M}^{(F|d)}_3(0) = 0. \tag{7.27}$$

Relying on the same calculations as for M_3, $\mathcal{M}^{(F|d)}_3$ is a continuous local martingale starting from 0 for the filtration $\mathcal{F}^{(J)}_t$ whose quadratic variation satisfies the following identity:

$$\mathrm{d}\langle \mathcal{M}^{(F|d)}_3 \rangle_t = \frac{M^{(F|d)}_6(t) - (M^{(F|d)}_3(t))^2}{X^{(d)}_{(J)}(t)},$$

where $M_6^{(F|d)}(s) := \sum_{i\geq J} i^6\, X_i^{(F|d)}(t)$. The definition of the localization time can be adapted from (7.4) as follows for any positive integer ℓ:

$$T_\ell := \inf\{s \geq 0\,;\ \langle \mathcal{M}_3^{(F|d)}\rangle_s \geq \ell\,,\ M_3^{(F|d)}(s) \geq \ell\}. \tag{7.28}$$

Thanks to Proposition 7.3.1, the following inequalities hold for any t such that $X_{(J)}^{(d)}(t) > 0$:

$$M_3^{(F|d)}(t) \leq \frac{M_3(t)}{X_{(J)}^{(d)}(t)}, \qquad M_6^{(F|d)}(t) \leq \frac{M_6(t)}{X_{(J)}^{(d)}(t)}.$$

Thanks to the proof of [2, Theorem 3] (see also Propositon B.0.1 in the appendix), $M_6^{(\infty)}$ is locally upper-bounded, a.s. under $\mathbb{P}_x^{(J:d)}$ for any $x \in \mathcal{X}_\infty = \mathcal{X}^6$. It entails that both $M_3^{(F|d)}$ and $\langle \mathcal{M}_3^{(F|d)}\rangle$ are also a.s. upper-bounded for any $n \geq 1$ on the time-interval $[0, t \wedge \tau_{1/n}^{(J:d)}]$, where $\tau_{1/n}^{(J:d)} = \inf\{t \geq 0\,;\ X_{(J)}^{(d)}(t) = 1/n\}$. Taking the limit with n tending to infinity, $\lim_\ell T_\ell \geq t \wedge \tau_0^{(J:d)}$ holds a.s. The rest of the proof of Lemma 7.3.5 can be taken mutatis mutandis from the one of Lemma 7.2.8 (with $C = \exp[8\lambda t] \vee 1$). □

7.4 Mixing property and accessibility

Theorem 7.4.7, stated and proved in Sect. 7.4.4, is the main result of the current Sect. 7.4. It establishes the mixing estimate $(A2)$.

We consider three intermediate steps: first in Sect. 7.4.1, we justify that an interior subset of $\mathcal{X}_d$ with convenient properties can be accessed, cf Lemma 7.4.3; secondly in Sect. 7.4.2, we show a mixing estimate on the two optimal subpopulation sizes $X_0^{(d)}$ and $X_1^{(d)}$, cf Lemma 7.4.4; thirdly in Sect. 7.4.3, we prove the existence of a uniform lower-bound on the probability of the event on which we will condition to produce $\zeta^{(d)}$ as a probability measure, cf Lemma 7.4.3.

We recall the definition of $\mathcal{D}_\ell^{(d)}$ from (7.1) for any integer ℓ:

$$\mathcal{D}_\ell^{(d)} := \left\{x \in \mathcal{X}_d\,;\ x_0 \geq \frac{1}{2\ell}\right\},$$

and the fact that $T_{\mathcal{D}_\ell}^{(d)}$ denotes the exit time of $X^{(d)}$ out of $\mathcal{D}_\ell^{(d)}$.

Remark 7.4.1 It can be noted that for any $x \in \mathcal{D}_\ell^{(d)}$, $T_{\mathcal{D}_\ell}^{(d)} < \tau_\partial^{(d)}$, so that $T_{\mathcal{D}_\ell}^{(d)} = \tau_\partial^{(d)} \wedge T_{\mathcal{D}_\ell}^{(d)}$.

Remark 7.4.2 In the current Sect. 7.4, we apply the decomposition introduced in Sect. 7.3 for $J = 2$. The definitions (7.10) for π_J and (7.11) for $X_{(J)}^{(d)}$ will be used below in case $J = 2$. The proofs given in the next Sect. 7.5 will exploit a generalization of the argument for large values of J.

7.4.1 Access to an interior point

Section 7.4.1 is devoted to the proof of the upcoming Lemma 7.4.3, in which we justify the accessibility of $\mathcal{H}_\ell^{(d)}$, which is an interior subset of $\mathcal{X}_d$ with convenient properties. These properties entail that any state $x \in \mathcal{H}_\ell^{(d)}$ will constitute a suitable initial condition in order to exploit Property (H).

Lemma 7.4.3 *For any integer $\ell \geq 1$, there exist four real numbers $t, m, y > 0$, such that the following inequality holds for any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{D}_\ell^{(d)}$:*

$$\mathbb{P}_x\Big(M_3^{(d)}(t) \leq m\,,\ \ X_0^{(d)}(t) \wedge X_1^{(d)}(t) \wedge X_{(2)}^{(d)}(t) \geq y\,,\ \ t < T_{\mathcal{D}_{2\ell}}^{(d)}\Big) \geq c.$$

This lemma leads to the introduction of the following subset of $\mathcal{X}_d$ for any $d \in [\![2, \infty]\!]$ and any $\ell \in \mathbb{N}$:

$$\mathcal{H}_\ell^{(d)} = \mathcal{H}^{(d)}(m_\ell, y_\ell) := \Big\{x' \in \mathcal{X}_d\,;\ \ M_3(x') \leq m_\ell\,,\ \ x_0' \wedge x_1' \wedge \sum\nolimits_{\{k \geq 2\}} x_k' \geq y_\ell\},$$

where $m_\ell, y_\ell > 0$ are the values of m and y associated to ℓ through Lemma 7.4.3, so that the lower-bounded probability can be expressed as $\mathbb{P}_x\big(X^{(d)}(t) \in \mathcal{H}_\ell^{(d)}\,;\ t < T_{\mathcal{D}_{2\ell}}^{(d)}\big)$. Without restriction, $y_\ell \leq 1/4\ell$ can be assumed.

Proof : Let $\ell \geq 1$. We define $y_\wedge = 1/4\ell$. The process $X_0^{(d)}$, which is the initial component of the solution to (1.1)-(1.3), is lower-bounded under $\mathbb{P}_x$ for any d and $x \in \mathcal{D}_\ell^{(d)}$ by the solution Y_0 to the following SDE, thanks to Lemma 5.1.4:

$$\mathrm{d}Y_0(s) = -\lambda\,\mathrm{d}s + \sqrt{Y_0(s)\cdot(1 - Y_0(s))}\,\mathrm{d}B_0(s)\,, \qquad Y_0(0) = 2y_\wedge,$$

where B_0 is a Brownian motion. With $c_0 := \mathbb{P}\big(\inf_{t\in[0,1]} Y_0(t) > y_\wedge \,\big|\, Y_0(0) = 2y_\wedge\big) > 0$, we thus deduce the following inequality for any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{D}_\ell^{(d)}$:

$$\mathbb{P}_x\Big(1 < T_{\mathcal{D}_{2\ell}}^{(d)}\Big) \geq c_0. \tag{7.29}$$

Thanks to Lemmas 7.2.4 and 7.2.6 (similarly as in Sect. 7.2.2), there exists $m_D > 0$ such that the following inequality holds for any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{D}_\ell^{(d)}$:

$$\mathbb{P}_x\Big(\tau_{m_D}^{(3|d)} \leq \tfrac{1}{3}\Big) \geq \sqrt{1 - \frac{c_0}{2}}. \tag{7.30}$$

Thanks to Lemma 7.2.8, there exists $m \geq m_D$ such that the following inequality holds for any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{X}_d$ such that $M_3(x) \leq m_D$:

$$g_m[x] := \mathbb{P}_x\Big(1 < T_m^{(3|d)}\Big) \geq \sqrt{1 - \frac{c_0}{2}}. \tag{7.31}$$

Thanks to the strong Markov property at time $\tau_m^{(3|d)}$, recalling (7.29), (7.30) and (7.31), the following inequalities hold for any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{D}_\ell^{(d)}$:

$$\begin{aligned} &\mathbb{P}_x\Big(\tau_{m_D}^{(3|d)} \leq \tfrac{1}{3}\,,\; 1 < T_m^{(3|d)} \wedge T_{\mathcal{D}_{2\ell}}^{(d)}\Big) \\ &\geq \mathbb{P}_x\Big(1 < T_{\mathcal{D}_{2\ell}}^{(d)}\Big) - \Big[1 - \mathbb{E}_x\Big(g_m\big[X^{(d)}(\tau_{m_D}^{(3|d)})\big];\; \tau_{m_D}^{(3|d)} \leq \tfrac{1}{3}\Big)\Big] \\ &\geq \frac{c_0}{2}. \end{aligned} \tag{7.32}$$

Let $b_1 := (\lambda y_\wedge) \wedge (\frac{1}{4})$. Thanks to Lemma 5.1.4, the process $X_1^{(d)}$ is lower-bounded a.s. on the event $\{1 < T_{\mathcal{D}_{2\ell}}^{(d)}\}$ under $\mathbb{P}_x$ on the time-interval $[\frac{1}{3}, 1]$ by the solution Y_1 to the following SDE:

$$\mathrm{d}Y_1(s) = \varphi_1\, \mathrm{d}s - (\alpha + \lambda) \cdot Y_1(s)\, \mathrm{d}s + \sqrt{Y_1(s) \cdot (1 - Y_1(s))}\, \mathrm{d}B_1(s)\,, \qquad Y_1(\tfrac{1}{3}) = 0. \tag{7.33}$$

Thanks to Corollary 5.1.5, since $\varphi_1 \in (0, \frac{1}{4})$, 0 is a regular reflecting boundary for this process Y_1. Therefore, there exists $y_1 \in (0, \frac{1}{2\lambda})$ such that $\mathbb{P}(Y_1(2/3) < 2y_1) \leq c_0/4$. Recalling (7.32), the following inequality thus holds for any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{D}_\ell^{(d)}$:

$$\mathbb{P}_x\Big(\tau_{m_D}^{(3|d)} \leq \tfrac{1}{3}\,,\; X_1^{(d)}(2/3) \geq 2y_1\,,\; 1 < T_m^{(3|d)} \wedge T_{\mathcal{D}_{2\ell}}^{(d)}\Big) \geq \frac{c_0}{4}. \tag{7.34}$$

Thanks to Lemma 5.1.4, the process $X_1^{(d)}$ is lower-bounded a.s. on the event $\{1 < T_{\mathcal{D}_{2\ell}}^{(d)}\} \cap \{X_1^{(d)}(2/3) \geq 2y_1\}$ under $\mathbb{P}_x$ on the time-interval $[2/3, 1]$ by the solution Y_1^I to the following SDE, where the difference with (7.33) lies in the initial condition at time $2/3$:

$$\mathrm{d}Y_1^I(s) = \varphi_1\, \mathrm{d}s - (\alpha + \lambda) \cdot Y_1^I(s)\, \mathrm{d}s + \sqrt{Y_1^I(s) \cdot (1 - Y_1^I(s))}\, \mathrm{d}B_1(s)\,, \qquad Y_1^I(\frac{2}{3}) = 2y_1.$$

We consider the two following stopping times $T_{y_1}^1$ and $T_{y_1}^{1,(d)}$:

$$T_{y_1}^1 := \inf\{s \geq 2/3\,;\; Y_1^I(s) \leq y_1\}, \qquad T_{y_1}^{1,(d)} := \inf\{s \geq 2/3\,;\; X_1^{(d)}(s) \leq y_1\},$$

namely the hitting time of y_1 after time $2/3$ by the processes respectively Y_1^I and $X_1^{(d)}$. There exists $t_1 \in (0, \frac{1}{3})$ such that:

$$\mathbb{P}\Big(T_{y_1}^1 \leq 2/3 + t_1\Big) \leq \frac{c_0}{8}.$$

Let $t = 2/3 + t_1$. The following inequality thus holds for any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{D}_\ell^{(d)}$:

$$\mathbb{P}_x\Big(\tau_{m_D}^{(3|d)} \leq \tfrac{1}{3}\,,\ X_1^{(d)}(2/3) \geq 2y_1\,,\ t < T_{y_1}^{1,(d)} \wedge T_m^{(3|d)} \wedge T_{\mathcal{D}_{2\ell}}^{(d)}\Big) \geq \frac{c_0}{8}. \quad (7.35)$$

Let $\varphi_2 := \lambda\, y_1$. Thanks to Lemma 5.1.4, the process $X_2^{(d)}$ is lower-bounded a.s. on the event $\{X_1^{(d)}(2/3) \geq 2y_1\} \cap \{t < T_{y_1}^{1,(d)}\}$ under $\mathbb{P}_x$ on the time-interval $[2/3, t]$ by the solution Y_2 to the following SDE:

$$\mathrm{d}Y_2(s) = \varphi_2\, \mathrm{d}s - (2\alpha + \lambda)\cdot Y_2(s)\,\mathrm{d}s + \sqrt{Y_2(s)\cdot(1 - Y_2(s))}\,\mathrm{d}B_2(s)\,, \quad Y_2(\tfrac{2}{3}) = 0. \quad (7.36)$$

Thanks to Corrolary 5.1.5 since $\varphi_2 \in (0, \frac{1}{2})$, 0 is a regular reflecting boundary for this process Y_2. Therefore, there exists $y_2 \in (0, y_1)$ such that $\mathbb{P}(Y_2(t) < y_2) \leq c_0/16$. Recalling (7.35), the following inequality thus holds for any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{D}_\ell^{(d)}$:

$$\begin{aligned}
&\mathbb{P}_x\Big(M_3^{(d)}(t) \leq m\,,\ X_0^{(d)}(t) \wedge X_1^{(d)}(t) \wedge X_{(2)}^{(d)}(t) \geq y_2\,,\ t < T_{\mathcal{D}_{2\ell}}^{(d)}\Big)\\
&\quad\geq \mathbb{P}_x\Big(\tau_{m_D}^{(3|d)} \leq \tfrac{1}{3}\,,\ X_1^{(d)}(\tfrac{2}{3}) \geq 2y_1\,,\ X_2^{(d)}(t) \geq y_2\,,\ t < T_{y_1}^{1,(d)} \wedge T_m^{(3|d)} \wedge T_{\mathcal{D}_{2\ell}}^{(d)}\Big)\\
&\quad\geq \frac{c_0}{16}. \qquad (7.37)
\end{aligned}$$

This concludes the proof of Lemma 7.4.3. □

7.4.2 Mixing estimate on the two optimal subpopulation sizes

Section 7.4.2 is devoted to the proof of the upcoming Lemma 7.4.4, in which we both establish a mixing property for the process $(X_0^{(d)}, X_1^{(d)})$ and obtain a control on $X_1^{(d)}$ and on $X_{(2)}^{(d)}$. As a reference initial condition, we consider $z^{(d)} \in \mathcal{X}_d$ to be such that $z_k^{(d)} = 2^{-k-1}$ for any $k \in [\![0, d-1]\!]$ (that is $k \in \mathbb{Z}_+$ for $d = \infty$) and $z_d^{(d)} = 2^{-d}$ (in the case $d < \infty$). Note that $z^{(d)} \in \mathcal{D}_1^{(d)}$.

We introduce as follows some subsets $\mathcal{Y}_2(y)$ of $\mathcal{X}_2$ in terms of some parameter $y \in (0, y_\ell)$ that describes its gap to the boundary of $\mathcal{X}_2$:

$$\mathcal{Y}_2(y) := \Big\{z \in \mathcal{X}_2\,;\ z_0 \wedge z_1 \wedge z_2 \geq y\Big\}.$$

There exist two connected open relatively compact sets $\mathfrak{K}_2^{\wedge}(y), \mathfrak{K}_2^{\vee}(y)$ with C^∞-boundaries with the following properties:

$$\mathcal{Y}_2(2y) \subset \mathfrak{K}_2^{\wedge}(y), \quad \overline{\mathfrak{K}_2^{\wedge}(y)} \subset \mathfrak{K}_2^{\vee}(y) \subset \mathcal{Y}_2(y).$$

The stopping time $T_y^{2,(d)}$ is defined for any $y \in (0, y_\ell)$ as follows:

$$T_y^{2,(d)} := \inf \Big\{ t \geq 0 \, ; \; \pi_2(X^{(d)}(t)) \notin \mathfrak{K}_2^{\vee}(y) \Big\}, \tag{7.38}$$

namely the exit time of $\pi_2(X^{(d)})$ outside of $\mathfrak{K}_2^{\vee}(y)$. Note that for any $y > 0$, there exists an integer L such that $T_y^{2,(d)} < T_{\mathcal{D}_L}^{(d)}$.

Lemma 7.4.4 *For any $\ell \geq 1$ and $y_C \in (0, 1)$, there exists $y \in (0, y_C]$ and four real numbers $c_D, c_Z, m_U, m_D > 0$ such that the two following inequalities hold for any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{H}_\ell^{(d)}$:*

$$\begin{aligned} &\mathbf{1}_{\{z \in \mathcal{Y}_2(2y)\}} \mathbb{P}_x \Big(\pi_2(X^{(d)}(2)) \in \mathrm{d}z \, ; \; 2 < T_y^{2,(d)} \wedge T_{m_U}^{(3|d)} \Big) \\ &\quad \geq c_D \, \mathbf{1}_{\{z \in \mathcal{Y}_2(2y)\}} \mathbb{P}_{z^{(d)}} \Big(\pi_2(X^{(d)}(1)) \in \mathrm{d}z \, ; \; 1 < T_y^{2,(d)} \wedge T_{m_D}^{(3|d)} \Big), \end{aligned}$$

and:

$$\mathcal{Z}^{(d)} := \mathbb{P}_{z^{(d)}} \Big(X^{(d)}(1) \in \mathcal{R}_y \, , \; 1 < T_y^{2,(d)} \wedge T_{m_D}^{(3|d)} \Big) \geq c_Z,$$

where the subset $\mathcal{R}_y$ of $\mathcal{Y}_2(2y)$ is defined as follows:

$$\mathcal{R}_y := \Big\{ z \in \mathcal{Y}_2(2y) \, ; \; z_0 \in (1 - 5y, 1 - 4y) \, , \; z_1 \in (2y, 3y) \Big\}. \tag{7.39}$$

Proof : Let $\ell \geq 1$, m_H and y_H be the constants associated to $\mathcal{H}_\ell^{(d)}$ and y_C be given. Let $y = y_C \wedge (\frac{y_H}{2}) \wedge (\frac{1}{7})$. Recall from Proposition 7.3.1 that the system $(X_0^{(d)}, X_1^{(d)}, X_{(2)}^{(d)}) = \pi_2(X^{(d)})$ is autonomous under $\mathbb{P}_x^{(2:d)}$, whatever $d \in [\![2, \infty]\!]$ and $x \in \mathcal{H}_\ell^{(d)}$, with a common infinitesimal generator $\mathcal{L}^{(2)}$ on $\mathcal{X}_2$. As stated in Lemma 5.1.1, the process $\pi_2(X^{(d)})$ satisfies Property (H) on $\mathcal{Y}_2(y)$. Since $\pi_2(\mathcal{H}_\ell^{(d)}) \subset \mathcal{Y}_2(2y) \subset \mathfrak{K}_2^{\wedge}(y)$, we deduce as in the proof of Proposition 5.2.1 that there exists $c_D^1 > 0$ such that the following inequality holds for any $d \in [\![2, \infty]\!]$ and $x \in \mathcal{H}_\ell^{(d)}$:

$$\begin{aligned} &\mathbf{1}_{\{z \in \mathcal{Y}_2(2y)\}} \mathbb{P}_x^{(2:d)} \big(\pi_2(X^{(d)}(2)) \in \mathrm{d}z \, ; \; 2 < T_y^{2,(d)} \big) \\ &\quad \geq c_D^1 \, \mathbf{1}_{\{z \in \mathcal{Y}_2(2y)\}} \mathbb{P}_{z^{(d)}}^{(2:d)} \big(\pi_2(X^{(d)}(2)) \in \mathrm{d}z \, ; \; 1 < T_y^{2,(d)} \big). \end{aligned} \tag{7.40}$$

To prepare for the second inequality, recall (7.39). The set $\mathcal{R}_y$ has a non empty interior, so that we can find a smooth function $f : \mathfrak{K}_2^{\vee}(y) \mapsto [0, 1]$ with support on $\mathcal{R}_y$ that is non-zero. We are led to consider the corresponding Cauchy problem on $\mathbb{R}_+ \times \mathfrak{K}_2^{\vee}(y)$ with the value at the boundary given by the function $u_{\partial \mathfrak{K}_2^{\vee}(y)}$ defined as $u_{\partial \mathfrak{K}_2^{\vee}(y)}(0, z) = f(z)$ for any $z \in \mathfrak{K}_2^{\vee}(y)$ and as $u_{\partial \mathfrak{K}_2^{\vee}(y)}(t, z) = 0$ for any $t \geq 0$ and $z \in \partial \mathfrak{K}_2^{\vee}(y)$. Thanks to Property (H), there exists a unique positive strong solution u

to this Cauchy problem. Note that by construction $\pi_2(z^{(d)}) = z^{(2)}$ is independent of d. As a consequence:

$$\begin{aligned}\mathbb{P}^{(2:d)}_{z^{(d)}}\big(\pi_2(X^{(d)}(1)) \in \mathcal{R}_y\,;\ 1 < T^{2,(d)}_y\big) &\geq \mathbb{E}^{(2:d)}_{z^{(d)}}\Big(f\Big[\pi_2(X^{(d)}(1))\Big];\ 1 < T^{2,(d)}_y\Big)\\ &= u(1, z^{(2)}) > 0.\end{aligned} \tag{7.41}$$

So as to relate to the original dynamics prescribed by $\mathbb{P}_x$, we need upper-bounds of the third moment that are given independently of $\pi_2(X^{(d)})$, by referring to Sect. 7.3.3. Note that $M_3^{(F|d)}(x) \leq m_\ell$ holds for any $x \in \mathcal{H}^{(d)}_\ell$ and that $T^{2,(d)}_y < \tau^{(2:d)}_0$ hold also a.s. Thanks to Lemma 7.3.5, there exists $m'_U > 0$ such that the following inequality holds for any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{H}^{(d)}_\ell$:

$$\mathbb{P}^{(2:d)}_x\left(T^{(F.3|d)}_{m'_U} \leq 2 \wedge T^{2,(d)}_y \,\middle|\, \mathcal{F}^{(2)}\right) \leq \tfrac{1}{2}. \tag{7.42}$$

Similarly, there exists $m'_D > 0$ such that the following inequality holds for any $d \in [\![2, \infty]\!]$:

$$\mathbb{P}^{(2:d)}_{z^{(d)}}\left(T^{(F.3|d)}_{m'_D} \leq 1 \wedge T^{2,(d)}_y \,\middle|\, \mathcal{F}^{(2)}\right) \leq \tfrac{1}{2}. \tag{7.43}$$

Thanks to (7.40) and to (7.42), the following inequality holds for any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{H}^{(d)}_\ell$:

$$\begin{aligned}&\mathbf{1}_{\{z\in\mathcal{Y}_2(2y)\}}\mathbb{P}^{(2:d)}_x\Big(\pi_2(X^{(d)}(2)) \in \mathrm{d}z\,;\ 2 < T^{2,(d)}_y \wedge T^{(F.3|d)}_{m'_U}\Big)\\ &\quad\geq 2c^1_D\,\mathbf{1}_{\{z\in\mathcal{Y}_2(2y)\}}\mathbb{P}^{(2:d)}_{z^{(d)}}\Big(\pi_2(X^{(d)}(1)) \in \mathrm{d}z\,;\ 1 < T^{2,(d)}_y\Big).\end{aligned}$$

Thanks to Proposition 7.3.2, noting that $M_3(x) \leq M_3^{(F|d)}(x) + 1$ (it holds for any $d \geq 2$ and $x \in \mathcal{X}_d$), there exists $c_D > 0$ such that the following inequality holds for any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{H}^{(d)}_\ell$, with $m_U = m'_U + 1$ and $m_D = m'_D + 1$:

$$\begin{aligned}&\mathbf{1}_{\{z\in\mathcal{Y}_2(2y)\}}\mathbb{P}_x\Big(\pi_2(X^{(d)}(2)) \in \mathrm{d}z\,;\ 2 < T^{2,(d)}_y \wedge T^{(3|d)}_{m_U}\Big)\\ &\quad\geq c_D\,\mathbf{1}_{\{z\in\mathcal{Y}_2(2y)\}}\mathbb{P}_{z^{(d)}}\Big(\pi_2(X^{(d)}(1)) \in \mathrm{d}z\,;\ 1 < T^{2,(d)}_y \wedge T^{(3|d)}_{m_D}\Big),\end{aligned}$$

which is the first inequality in Lemma 7.4.4.

Similarly, thanks to Proposition 7.3.2, recalling (7.41) and (7.43), there exists $c_Z > 0$ such that the following inequality holds for any $d \in [\![2, \infty]\!]$:

$$\mathbb{P}_{z^{(d)}}\Big(\pi_2(X^{(d)}(1)) \in \mathcal{R}_y\,,\ 1 < T^{2,(d)}_y \wedge T^{(3|d)}_{m_D}\Big) \geq c_Z,$$

which concludes the proof of Lemma 7.4.4. □

7.4.3 The event of a regular behavior happen with a lower-bounded probability

Section 7.4.3 is devoted to the proof of the upcoming Lemma 7.4.5, in which we justify for well-prepared initial conditions that the dependency in the different components of $X^{(d)}_{(2)}$ can be forgotten in an event of non-negligible probability.

Lemma 7.4.5 *For any $\ell \geq 1$, there exists $y_C > 0$ such that the following property holds for any $y \in (0, y_C)$ and any $m > 0$. There exists $c > 0$ such that the following inequality holds for any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{X}_d$ such that $\pi_2(x) \in \mathcal{R}_y$ and $M_3(x) \leq m$:*

$$\mathbb{P}_x\left(\tau_0^{(2:d)} < 1 < T^{(d)}_{\mathcal{D}_3}\right) \geq c.$$

Proof : We wish to control the extinction of a uniform upper-bound of $X^{(d)}_{(2)}$, that is to be solution to the following SDE:

$$\mathrm{d}Z(t) := \frac{\mathrm{d}t}{4} + \sqrt{Z(t) \cdot (1 - Z(t))}\, \mathrm{d}B_{(2)}(t)\,, \quad Z(0) = y, \tag{7.44}$$

where W is a standard Brownian motion. Note that 0 is an accessible boundary for this process Z (it is actually regular refecting), thanks to Corollary 5.1.5.

To ensure that the flux of population into $X^{(d)}_{(2)}$ due to mutations remains lower than $\frac{1}{4}$ in the time-interval $[0, t_B]$ for some $t_B > 0$, we wish to impose that $X^{(d)}_1$ remains upper-bounded by $\frac{1}{4\lambda}$ on this time-interval. For practical reasons, the considered upper-bound is actually slightly adjusted:

$$y_M := \frac{1}{4\lambda} \wedge \frac{1}{2}. \tag{7.45}$$

Similarly as for $X^{(d)}_{(2)}$, we thus consider an upper-bound for the process $X^{(d)}_1$, as the solution to the following SDE:

$$\mathrm{d}Y(t) := (\lambda + \alpha)\, \mathrm{d}t + \sqrt{Y(t) \cdot (1 - Y(t))}\, \mathrm{d}B_1(t)\,, \quad Y(0) = y_M/4, \tag{7.46}$$

where B_1 is a standard Brownian motion (recall that $M^{(2:d)}_1 \leq 2$ under $\mathbb{P}^{(2:d)}_x$). We consider the following stopping time $T^Y_{y_M}$ (as a function of y_M):

$$T^Y_{y_M} := \inf\left\{t \geq 0\,;\ Y(t) \geq y_M\right\}, \tag{7.47}$$

namely the hitting time of y_M by the process Y. We consider also the martingale process $\mathcal{N}_1(t)$, defined as follows for any $t > 0$:

$$\mathcal{N}_1(t) := \int_0^t \left[\sqrt{Y(t) \cdot (1 - Y(t))} \wedge \sqrt{y_M \cdot (1 - y_M)}\right] \mathrm{d}B_1(t). \tag{7.48}$$

Its quadratic variation is upper-bounded as follows at time t_B:

$$\langle \mathcal{N}_1 \rangle_{t_B} \leq t_B \cdot y_M \cdot (1 - y_M).$$

Doob's inequality thus entails the following upper-bound:

$$\mathbb{P}\Big(\sup_{t \leq t_B} |\mathcal{N}_1(t)| \geq \frac{y_M}{2} \Big) \leq \frac{16 t_B}{y_M}. \tag{7.49}$$

Thanks to (7.46) and (7.48) since $y_M \leq \frac{1}{2}$, the following identity holds a.s. on the event $\left\{ t \leq T^Y_{y_M} \right\}$, for any $t \in [0, t_B]$:

$$Y(t) = \frac{y_M}{4} + (\lambda + \alpha) \cdot t + \mathcal{N}_1(t).$$

On the event $\{T^Y_{y_M} \leq t_B\}$, the evaluation of this identity at time $T^Y_{y_M}$ entails the following inequality a.s.:

$$\sup_{t \leq t_B} |\mathcal{N}_1(t)| \geq \frac{3 y_M}{4} - (\lambda + \alpha) \cdot t_B. \tag{7.50}$$

On the other hand, thanks to Property (H) (see Sect. 5.1.1) and to Proposition 7.3.1 since the event $\{1 < T^{(d)}_{\mathcal{D}_2}\}$ only depends on the process $X^{(d)}_0$, there exists $c_0 > 0$ such that the following inequality holds for any $d \in [\![2, \infty]\!]$ and for any $x \in \mathcal{X}_d$ such that $x_0 \in [\frac{1}{2}, 1]$:

$$\mathbb{P}^{(2:d)}_x \Big(1 < T^{(d)}_{\mathcal{D}_2} \Big) \geq c_0. \tag{7.51}$$

We then choose $t_B > 0$ as follows:

$$t_B := \Big(\frac{c_0 \cdot y_M}{2^5} \Big) \wedge \Big(\frac{y_M}{4(\lambda + \alpha)} \Big) \wedge 1, \tag{7.52}$$

which ensures thanks to (7.49) and to (7.50) the following inequality:

$$\mathbb{P}\Big(T^Y_{y_M} \leq t_B \Big) \leq \frac{c_0}{2}. \tag{7.53}$$

Recalling that 0 is an accessible boundary for the process Z defined in (7.44), we then choose $z \in (0, y_M/4)$ sufficiently small for the following inequality to hold:

$$\mathbb{P}\Big(t_B \leq \tau^Z_\partial \Big) \leq \frac{c_0}{4}, \tag{7.54}$$

where $\tau^Z_\partial := \inf\{t \geq 0 ;\ Z_t = 0\}$.

The definition of $y_C > 0$ is as follows:

$$y_C = \frac{z}{3}.$$

For any $y \in (0, y_C)$, $z \in \mathcal{R}_y$ thus implies that $z_0 > 1 - 5y_C > 1 - (\frac{5}{12}) \cdot y_M > \frac{1}{2}$ and $z_1 \wedge z_2 < 3y_C = z < y_M/4$. Thanks to Lemma 5.1.4, for any $y \in (0, y_C)$, any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{X}_d$ such that $\pi_2(x) \in \mathcal{R}_y$, $X_1^{(d)}$ under the law $\mathbb{P}_x^{(2:d)}$ is upper-bounded by the process Y, which entails with (7.53) the following inequality:

$$\mathbb{P}_x^{(2:d)}\left(T_{y_M}^{1,(d)} \le t_B\right) \le \frac{c_0}{2}, \tag{7.55}$$

where the stopping time $T_{y_M}^{1,(d)}$ is defined as follows:

$$T_{y_M}^{1,(d)} := \inf\left\{t \ge 0\,;\ X_1^{(d)} \ge y_M\right\},$$

namely the hitting time of y_M by the process $X_1^{(d)}$.

Thanks similarly to Lemma 5.1.4, $X_{(2)}^{(d)}$ under the law $\mathbb{P}_x^{(2:d)}$ is upper-bounded by the process Z. Recalling (7.54) and (7.55), the following inequalities thus hold for any $y \in (0, y_C)$, any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{X}_d$ such that $\pi_2(x) \in \mathcal{R}_y$:

$$\begin{aligned}\mathbb{P}_x^{(2:d)}\left(\tau_0^{(2:d)} < t_B\right) &\ge \mathbb{P}_x^{(2:d)}\left(\tau_0^{(2:d)} < t_B < T_{y_M}^{1,(d)}\right)\\ &\ge 1 - \frac{3c_0}{4}.\end{aligned} \tag{7.56}$$

Recalling (7.51), (7.55) and (7.56), the following inequality thus holds for any $y \in (0, y_C)$, any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{X}_d$ such that $\pi_2(x) \in \mathcal{R}_y$,

$$\mathbb{P}_x^{(2:d)}\left(\tau_0^{(2:d)} < t_B \wedge T_{\mathcal{D}_2}^{(d)}\right) \ge \frac{c_0}{4}. \tag{7.57}$$

Since the event $\{\tau_0^{(2:d)} < t_B \wedge T_{\mathcal{D}_2}^{(d)}\}$ belongs to the sigma-field $\mathcal{F}_{t_B}^{(2)}$, we can follow the same reasoning regarding the third moment as in Sect. 7.4.2 and find a constant $c_1 > 0$ such that the following inequality holds for any $y \in (0, y_C)$, any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{X}_d$ such that both $\pi_2(x) \in \mathcal{R}_y$ and $M_3(x) \le m$:

$$\mathbb{P}_x\left(\tau_0^{(2:d)} < t_B \wedge T_{\mathcal{D}_2}^{(d)}\right) \ge c_1. \tag{7.58}$$

On the other hand, there exists $c_2 > 0$ such that the following inequality holds for any $d \in [\![2, \infty]\!]$ and $x \in \mathcal{D}_2^{(d)}$:

$$\mathbb{P}_x\left(1 < T_{\mathcal{D}_3}^{(d)}\right) \ge c_2.$$

It is exactly (5.6) from Sect. 5.5, which extends to the case $d = \infty$. Thanks to the strong Markov property at time $\tau_0^{(2:d)}$, it concludes the proof of Lemma 7.4.5 with $c = c_1 \cdot c_2 > 0$. □

7.4.4 Main mixing estimate

After the proofs of the three Lemmas 7.4.3, 7.4.4 and 7.4.5, we are in conditions to prove the mixing estimate, as stated in the next theorem that is the main result of this Sect. 7.4. It exploits the following definition of $\zeta^{(d)}$. In this formula, the two real numbers y and m_D are associated by Lemma 7.4.5 with the (arbitrary) choice $t := 1$.

$$\zeta^{(d)}(\mathrm{d}x) := \int_{\mathcal{X}_d} \mathbb{P}_z\left(X^{(d)}(1) \in \mathrm{d}x \,\middle|\, \tau_0^{(2:d)} < 1 < T^{(d)}_{\mathcal{D}_3}\right) \nu^{(d)}(\mathrm{d}z)\,, \tag{7.59}$$

where the measure $\nu^{(d)}$ is defined as follows:

$$\nu^{(d)}(\mathrm{d}z) := \frac{\mathbf{1}_{\{z\in\mathcal{R}_y\}}}{\mathcal{Z}^{(d)}} \mathbb{P}_{z^{(d)}}\left(\pi_2(X^{(d)}(2)) \in \mathrm{d}z\,;\; 1 < T_y^{2,(d)} \wedge T_{m_D}^{(3|d)}\right),$$

while $z \in \mathcal{R}_y$ is seen as an element of $\mathcal{X}_d$ by defining $z_k = 0$ for any $k \in [\![3, d]\!]$.

Note that $\nu^{(d)}$ is related to the measure that appears in the lower-bound in Lemma 7.4.4, with a restriction to the set $\mathcal{R}_y \subset \mathcal{Y}_2(2y)$ followed by its renormalisation. The second inequality in Lemma 7.4.4 ensures that the normalizing term $\mathcal{Z}^{(d)}$ is lower-bounded away from 0 uniformly in $d \in [\![2, \infty]\!]$.

Remark 7.4.6 As exploited in the final Sect. 7.6, the constraint $1 < T^{(d)}_{\mathcal{D}_3}$ ensures that $\zeta^{(d)}$ is supported on $\mathcal{D}_3^{(d)}$.

Theorem 7.4.7 *For any $\ell \geq 1$, there exist an integer $L > \ell$ and two real number $t, c > 0$ such that the following inequality holds for any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{D}_\ell^{(d)}$, where $\zeta^{(d)}$ is defined in (7.59):*

$$\mathbb{P}_x\left(X^{(d)}(t) \in \mathrm{d}x'\,;\; t < T^{(d)}_{\mathcal{D}_L}\right) \geq c\,\zeta^{(d)}(\mathrm{d}x').$$

Proof : Let $\ell \geq 1$. Thanks to Lemma 7.4.3, the set $\mathcal{H}_\ell^{(d)}$, the two real numbers $c_H, t_H > 0$ and the integer $L_H \geq 1$ are such that the following inquality holds for any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{D}_\ell^{(d)}$:

$$\mathbb{P}_x\left(X^{(d)}(t_H) \in \mathcal{H}_\ell^{(d)}\,;\; t_H < T^{(d)}_{\mathcal{D}_{L_H}}\right) \geq c_H. \tag{7.60}$$

We then define $y_C > 0$ according to Lemma 7.4.5. Thanks to Lemma 7.4.4, there exists $y \in (0, y_C]$ and four real numbers $c_D, c_Z, m_U, m_D > 0$ such that the two following inequalities hold for any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{H}_\ell^{(d)}$:

$$\mathbf{1}_{\{z\in\mathcal{Y}_2(2y)\}}\mathbb{P}_x\left(\pi_2(X^{(d)}(2)) \in \mathrm{d}z\,;\; 2 < T_y^{2,(d)} \wedge T_{m_U}^{(3|d)}\right)$$

$$\geq c_D \, \mathbf{1}_{\{z \in \mathcal{Y}_2(2y)\}} \mathbb{P}_{z^{(d)}}\Big(\pi_2(X^{(d)}(1)) \in \mathrm{d}z \,;\; 1 < T_y^{2,(d)} \wedge T_{m_D}^{(3|d)}\Big), \qquad (7.61)$$

and:

$$\mathcal{Z}^{(d)} := \mathbb{P}_{z^{(d)}}\Big(\pi_2(X^{(d)}(1)) \in \mathcal{R}_y \,,\; 1 < T_y^{2,(d)} \wedge T_{m_D}^{(3|d)}\Big) \geq c_Z. \qquad (7.62)$$

Note that there exists an integer $L \geq 3 \vee L_H$ such that $T_y^{2,(d)} < T_{\mathcal{D}_L}^{(d)}$ (recall (7.38)). Thanks to Lemma 7.4.5 with this value of y and $m_C = m_D \vee m_U$, there exists $c_C > 0$ such that the following inequality holds for any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{X}_d$ such that $\pi_2(x) \in \mathcal{R}_y$ and $M_3(x) \leq m_C$:

$$\mathbb{P}_x\Big(\tau_0^{(2:d)} < 1 < T_{\mathcal{D}_3}^{(d)}\Big) \geq c_C. \qquad (7.63)$$

Thanks to Proposition 7.3.1 since $X_j^{(d)}(\tau_0^{(2:d)}) = 0$ for any $j \geq 2$, the following identity holds for any d and any $x, x' \in \mathcal{X}_d$ such that $\pi_2(x) = \pi_2(x')$:

$$\begin{aligned}&\mathbb{P}_x^{(2:d)}\Big(X^{(d)}(\tau_0^{(2:d)}) \in \mathrm{d}z \,,\; \tau_0^{(2:d)} \in \mathrm{d}t \,;\; \tau_0^{(2:d)} < 1 \wedge T_{\mathcal{D}_3}^{(d)}\Big)\\ &\quad = \mathbb{P}_{x'}^{(2:d)}\Big(X^{(d)}(\tau_0^{(2:d)}) \in \mathrm{d}z \,,\; \tau_0^{(2:d)} \in \mathrm{d}t \,;\; \tau_0^{(2:d)} < 1 \wedge T_{\mathcal{D}_3}^{(d)}\Big).\end{aligned}$$

As in the proof given in Sect. 7.4.2, this entails that there exists $c_G > 0$ such that the following inequality holds for any d and any $x, x' \in \mathcal{X}_d$ such that $\pi_2(x) = \pi_2(x')$, $M_3(x) \leq m_U$ and $M_3(x) \leq m_D$:

$$\begin{aligned}&\mathbb{P}_x\Big(X^{(d)}(\tau_0^{(2:d)}) \in \mathrm{d}z \,,\; \tau_0^{(2:d)} \in \mathrm{d}t \,;\; \tau_0^{(2:d)} < 1 \wedge T_{\mathcal{D}_3}^{(d)}\Big)\\ &\quad \geq c_G \mathbb{P}_{x'}\Big(X^{(d)}(\tau_0^{(2:d)}) \in \mathrm{d}z \,,\; \tau_0^{(2:d)} \in \mathrm{d}t \,;\; \tau_0^{(2:d)} < 1 \wedge T_{\mathcal{D}_3}^{(d)}\Big).\end{aligned}$$

Thanks to the strong Markov property at time $\tau_0^{(2:d)}$, the following inequality thus holds for any d and any $x, x' \in \mathcal{X}_d$ such that $\pi_2(x) = \pi_2(x')$, $M_3(x) \leq m_U$ and $M_3(x) \leq m_D$:

$$\mathbb{P}_x\Big(X^{(d)}(1) \in \mathrm{d}z \,;\; \tau_0^{(2:d)} < 1 < T_{\mathcal{D}_3}^{(d)}\Big) \geq c_G \mathbb{P}_{x'}\Big(X^{(d)}(1) \in \mathrm{d}z \,;\; \tau_0^{(2:d)} < 1 < T_{\mathcal{D}_3}^{(d)}\Big).$$

Thanks to the Markov property at time 1, recalling (7.59), (7.61), (7.62) and (7.63), we deduce the following inequality for any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{H}_\ell^{(d)}$:

$$\mathbb{P}_x\Big(X^{(d)}(3) \in \mathrm{d}x' \,;\; 3 < T_{\mathcal{D}_{L_D}}^{(d)}\Big) \geq (c_D \cdot c_Z \cdot c_C \cdot c_G) \cdot \zeta^{(d)}(\mathrm{d}x').$$

Thanks to the Markov property at time t_H, recalling (7.60) and that $L \geq L_H$, with $c = c_H \cdot c_D \cdot c_Z \cdot c_C \cdot c_G > 0$ and $t = t_H + 3$, the following inequality thus holds for any $d \in [\![2, \infty]\!]$ and any $x \in \mathcal{D}_\ell^{(d)}$:

$$\mathbb{P}_x\Big(X^{(d)}(t) \in \mathrm{d}x' \,;\ t < T^{(d)}_{\mathcal{D}_L}\Big) \geq c \cdot \zeta^{(d)}(\mathrm{d}x').$$

The proof of Theorem 7.4.7 is complete. □

Remark 7.4.8 In the above proof of Theorem 7.4.7, Lemma 7.3.5 was exploited to deduce upper-bounds of the third moments. Instead of third moments, we could have concluded to a similar result by considering second moments instead. However, the proof of Theorem 7.5.1 below strongly exploits a control of a moment with exponent strictly greater than 2.

7.5 Almost perfect harvest

The crucial result of this subsection is the upcoming Theorem 7.5.1, whose conclusion is very close to the property $(A3_F)$ of "almost perfect harvest", see Sect. 2.5.

Theorem 7.5.1 *Given any $\rho, m, \eta, y > 0$ and any $\epsilon \in (0,\ 1)$, there exists $c_V > 0$ and an integer $J \geq 1$ such that the following property holds for any $d \in [\![J, \infty]\!]$ and any $x \in \mathcal{X}_d$. There exists two stopping times $U_H^{(d)}$ and $V^{(d)}$ such that the following three conditions hold for any $x_\zeta \in \mathcal{D}_3^{(d)}$:*

$$\Big\{\tau_\partial^{(d)} \wedge 1 < U_H^{(d)}\Big\} = \Big\{U_H^{(d)} = \infty\Big\}\,, \qquad \mathbb{P}_x(U_H^{(d)} = \infty,\ 1 < \tau_\partial^{(d)}) \leq \epsilon\, e^{-\rho}$$
$$\mathbb{P}_x\,\Big(X^{(d)}(U_H^{(d)}) \in \mathrm{d}y\,;\ U_H^{(d)} < \tau_\partial^{(d)}\Big) \leq c_V\, \mathbb{P}_{x_\zeta}\,\Big(X^{(d)}(V^{(d)}) \in \mathrm{d}y\,;\ V^{(d)} < \tau_\partial^{(d)}\Big)\,.$$

In addition, the probability space Ω and the filtration $\mathcal{F}_t$ according to which $U_H^{(d)}$ and $V^{(d)}$ are stopping times can be chosen to be the canonical representation of the process $X^{(d)}$, see Remark 2.1.4.

Remark 7.5.2 The definition of $\zeta^{(d)}$ in (7.59) makes it supported on $\mathcal{D}_3$. To emphasize that this property is sufficient for our purposes, we consider for the upper-bound in the last inequality Dirac initial conditions of the form $x_\zeta \in \mathcal{D}_3$.

The proof of Theorem 7.5.1 is split into three parts: we start in Sect. 7.5.1 with the choice of the parameters and the statement of the corresponding properties, then introduce the definition of $U_H^{(d)}$ and $V^{(d)}$ with their intrinsic properties in Sect. 7.5.2 before we conclude with the comparison of densities at time $U_H^{(d)}$ versus $V^{(d)}$ in Sect. 7.5.3.

7.5.1 Choice of the parameters

The choice of $t_F = 1$ is made for simplicity. In the first time-interval of length $\frac{1}{3}$, we justify the access to a suitable set $E^{(d)}(m, y, \eta)$ for η well-chosen as a function of ϵ, according to (7.73), then m, $1/y$ sufficiently large according to Theorem 7.2.1. In the next time-interval of length $t_H \leq \frac{1}{3}$, we couple the first J coordinates between two processes with different initial conditions. The duration t_H is to be fixed below (in (7.80)), sufficiently small to restrict on trajectories away from the boundaries. Then, we impose that $X^{(d)}_{(J)}$ gets extinct in the next time-interval of length t_B, with a similar approach as in Sect. 7.4.3 for Lemma 7.4.5. Remark that we rely for the two last time-intervals on the notations and results of Sect. 7.3. Notably, we recall that the process $X^{(d)}$ is solution under $\mathbb{P}^{(J:d)}$ to (7.7), that $X^{(d)}_{(J)}$ is defined in (7.11) and that the process $M_1^{(J:d)}$ is upper-bounded by J by virtue of its definition in (7.9).

We wish to control the extinction of a uniform upper-bound $(Z(t))_{t\geq 0}$ of the process $\big(X^{(d)}_{(J)}(\tau_E^{(d)} + t_H + t)\big)_{t\geq 0}$, that is to be solution to the following SDE, analogous to (7.44), with $t \in [0, t_B]$:

$$\mathrm{d}Z(t) := \frac{\mathrm{d}t}{4} + \sqrt{Z(t)\cdot(1 - Z(t))}\,\mathrm{d}B(t)\,, \qquad Z(0) = z, \tag{7.64}$$

where B is a standard Brownian motion. Note that 0 is an accessible boundary for this process Z, thanks to Corollary 5.1.5.

Upper-bound on the incomming flux of population:
To ensure that the flux of population into $X^{(d)}_{(J)}$ due to mutations remains lower than $\frac{1}{4}$ in the time-interval of length t_B that follows $\tau_E^{(d)} + t_H$, we wish to impose that the process $\big(X^{(d)}_{J-1}(\tau_E^{(d)} + t_H + t)\big)_{t\in[0,t_B]}$ remains upper-bounded by $\frac{1}{4\lambda}$. As in (7.45), we define:

$$y_M := \frac{1}{4\lambda} \wedge \frac{1}{2}. \tag{7.65}$$

Similarly as in (7.46), we thus consider an upper-bound $(Y(t))_{t\geq 0}$ for the process $\big(X^{(d)}_{J-1}(\tau_E^{(d)} + t_H + t)\big)_{t\in[0,t_B]}$, as the solution to the following SDE with $t \in [0, t_B]$:

$$\mathrm{d}Y(t) := (\lambda + \alpha)\,\mathrm{d}t + \sqrt{Y(t)\cdot(1 - Y(t))}\,\mathrm{d}B_{(-1)}(t)\,, \qquad Y(0) = y_M/4, \tag{7.66}$$

where $B_{(-1)}$ is a standard Brownian motion (recall that $M_1^{(J:d)} \leq J$ and that $X^{(d)}_{J-1}$ is solution to (7.7) under $\mathbb{P}_x^{(J:d)}$). Let $T^Y_{y_M}$ denote the hitting time of y_M by this process Y (posterior to t_H). The definition of the martingale $\mathcal{N}_{(-1)}(t)$ is then slightly adapted from (7.48) with a start at time t_H with value 0, so that the two following inequalities hold, generally for the first one and a.s. on the event $\{T^Y_{y_M} \leq t_B\}$ for the second one:

$$\mathbb{P}\Big(\sup_{t\leq t_B} |\mathcal{N}_{(-1)}(t)| \geq y_M/2\Big) \leq \frac{16 t_B}{y_M}, \tag{7.67}$$

$$\sup_{t\leq t_B} |\mathcal{N}_{(-1)}(t)| \geq \frac{3y_M}{4} - (\lambda + \alpha)\cdot t_B. \tag{7.68}$$

Lower-bound on the survival with an initial condition in $\mathcal{D}_3$
Thanks to Lemma 5.1.4, for any initial condition $x_\zeta \in \mathcal{D}_3$, $X_0^{(d)}$ is lower-bounded by the solution Y_0 to the following SDE:

$$\mathrm{d}Y_0(s) = -\lambda\,\mathrm{d}s + \sqrt{Y_0(s)\cdot(1-Y_0(s))}\,\mathrm{d}B_0(s)\,, \quad Y_0(0) = \tfrac{1}{6}.$$

Thus, denoting $c_\zeta := (\frac{1}{2})\cdot\mathbb{P}_{\frac{1}{6}}(\inf_{t\in[0,\frac{1}{3}]} Y_0(t) > 0) > 0$, we deduce that the following inequality holds for any $d \in [\![2,\infty]\!]$ and any $x_\zeta \in \mathcal{D}_3^{(d)}$:

$$\mathbb{P}_{x_\zeta}(\tfrac{1}{3} < \tau_\partial^{(d)}) \geq 2c_\zeta. \tag{7.69}$$

Choice of t_B ***and*** z:
Let $\epsilon > 0$, that we assume without loss of generality to be smaller than c_ζ. We then choose $t_B > 0$ as follows:

$$t_B := \Big(\frac{\epsilon\cdot y_M}{16}\Big)\wedge\Big(\frac{y_M}{4(\lambda+\alpha)}\Big)\wedge\Big(\frac{1}{3}\Big), \tag{7.70}$$

which ensures thanks to (7.67) and (7.68) the following upper-bound in probability:

$$\mathbb{P}\Big(T^Y_{y_M} \leq t_B \,\Big|\, Y(0) = y_M/4\Big) \leq \epsilon. \tag{7.71}$$

Recalling that 0 is an accessible boundary for the process Z defined in (7.64), we then choose $z \in (0, y_M/4)$ sufficiently small for the following inequality to hold:

$$\mathbb{P}\Big(t_B \leq \tau_\partial^Z \,\Big|\, Z(0) = z\Big) \leq \epsilon, \tag{7.72}$$

where $\tau_\partial^Z := \inf\{t \geq 0\,;\ Z_t = 0\}$.

Choice of η, m ***and*** y:

Now, with the constants C_G, C_M associated by Proposition 7.3.2 with the choices of $k = 3$, $\epsilon > 0$ and $t = 1$, we can choose $\eta > 0$ as follows:

$$\eta = \Big(\frac{z}{C_M}\Big)\wedge\Big(\frac{\log(2)}{C_G}\Big), \tag{7.73}$$

where we recall that z is an implicit function of ϵ. Thanks to Theorem 7.2.1, given $\rho > 0$, we can choose the two real numbers $m, y > 0$ such that the following inequality holds for any $d \in [\![2,\infty]\!]$ and any $x \in \mathcal{X}_d$:

$$\mathbb{P}_x\big(\tfrac{1}{3} < \tau_E^{(d)} \wedge \tau_\partial^{(d)}\big) \leq \epsilon, \tag{7.74}$$

where we recall the definition of $E^{(d)}$ given in (7.2):

$$E^{(d)} = E^{(d)}(m,y,\eta) := \Big\{x \in \mathcal{X}_d\,;\ M_3(x) \leq m\,,\ \forall j \leq \Big\lfloor\frac{m}{\eta}\Big\rfloor + 1,\ x_j \geq y\Big\}.$$

Recalling (7.69) and that $c_\zeta \geq \epsilon$, (7.74) has the following implication for any $d \in [\![2,\infty]\!]$ and any initial condition $x_\zeta \in \mathcal{D}_3$:

$$\mathbb{P}_{x_\zeta}(\tau_E^{(d)} \leq \tfrac{1}{3} < \tau_\partial^{(d)}) \geq c_\zeta. \tag{7.75}$$

With $m_M = C_M \cdot m$, recalling the definition of C_M in relation to Proposition 7.3.2, we deduce that the following inequalities hold for any $d \in [\![2,\infty]\!]$ and any $x \in E^{(d)}$:

$$\Big|\log\Big(\frac{\mathrm{d}\mathbb{P}_x^{(J:d)}}{\mathrm{d}\mathbb{P}_x}\Big|_{[0,1]}\Big)\Big| \leq \frac{C_G \cdot m}{J} \leq \log(2), \qquad \mathbb{P}_x\Big(T_{m_M}^{(3|d)} \leq 1\Big) \leq \epsilon, \tag{7.76}$$

with the following definition of the integer J:

$$J := \Big\lfloor \frac{m}{\eta} \Big\rfloor + 2. \tag{7.77}$$

Since $\eta \leq \frac{z}{C_M}$, recalling the definition of $T_m^{(3|d)}$ from (6.16), the following inequalities hold a.s. on the event $\{1 < T_{m_M}^{(3|d)}\}$, for any $t_H \leq \frac{1}{3}$ and any $d \in [\![J,\infty]\!]$:

$$X_{(J-1)}^{(d)}(t_H) \leq \frac{C_M \cdot m}{(J-1)^3} \leq z. \tag{7.78}$$

Recalling that $z < y_M/4$, we deduce that $X_{J-1}^{(d)}(t_H) \leq y_M/4$ and that $X_{(J)}^{(d)}(t_H) \leq z$, as intended for Y (see (7.66)) and for Z (see (7.64)) to be appropriate upper-bounds.

Choice of t_H:
Note that $\pi_J(x) \in \mathcal{Y}_J(y)$ holds for any $d \in [\![J,\infty]\!]$ and any $x \in E^{(d)}$, with the following definition of $\mathcal{Y}_J(y)$:

$$\mathcal{Y}_J(y) := \Big\{x \in \mathcal{X}_J\,;\ \Big(\bigwedge\nolimits_{\{i \in [\![0,J]\!]\}} x_i\Big) > y\Big\},$$

and the projection π_J defined in (7.10). On $\mathcal{Y}_J(y)$, the diffusion term in the system $(S^{(J)})$ of SDEs, i.e. the system (1.1) with d replaced by J, is uniformly elliptic. In practice, we need a bit more space for Property (H) to hold (see Lemma 5.1.1), so that we consider the following exit time $T_{y'}^{J,(d)}$ generally for any $y' \in (0,y)$:

$$T_{y'}^{J,(d)} := \inf\Big\{t \geq 0\,;\ \pi_J(X^{(d)}(t)) \notin \mathcal{Y}_J(y')\Big\} < \tau_\partial^{(d)}. \tag{7.79}$$

The probability of such an escape is required to be very small, uniformly in $d \in [\![J,\infty]\!]$ and in $x \in E^{(d)}$, as stated in the upcoming Lemma 7.5.3, where $\mathbb{P}^{(J)}$ denotes the law of the system given by $(S^{(J)})$:

Lemma 7.5.3 *The following supremum tends to 0 as t_H tends to 0:*

$$\sup\Big\{\mathbb{P}_{\pi_J(x)}^{(J)}\Big(T_{y/2}^{J,(d)} \leq t_H\Big)\,\Big|\, d \in [\![J,\infty]\!],\ x \in E^{(d)}\Big\}.$$

Proof : Since the system $(S^{(J)})$ is uniformly elliptic on some connected open relatively compact subset $\mathfrak{K}_J$ of $\mathcal{Y}_J(y/2)$ with $\mathcal{C}^\infty$-boundary that contains $\mathcal{Y}_J(y)$, and recalling Proposition 7.3.1, Lemma 7.5.3 is deduced thanks to Lemma 5.1.1 (also thanks e.g. to [30, Proposition V.2.5]). □

Thanks to Proposition 7.3.1, we can thus choose $t_H \leq \frac{1}{3}$ sufficiently small such that the following inequality holds for any $d \in [\![J, \infty]\!]$, and any $x \in E^{(d)}$:

$$\mathbb{P}_x^{(J:d)}\left(T_{y/2}^{J,(d)} \leq t_H\right) \leq \epsilon. \tag{7.80}$$

7.5.2 Definition of $U_H^{(d)}$ with a control of exceptional events

For the definition of $U_H^{(d)}$, the following extinction time $\widehat{\tau}_0^{(J:d)}$ is considered for the process $X_{(J)}^{(d)}$ after time $\tau_E^{(d)} + t_H$:

$$\widehat{\tau}_0^{(J:d)} := \inf\left\{t \geq \tau_E^{(d)} + t_H\,;\ X_{(J)}^{(d)}(t) = 0\right\}.$$

In view of Theorem 7.5.1, we define $U_H^{(d)} := \widehat{\tau}_0^{(J:d)}$ on the following event:

$$\left\{\tau_E^{(d)} < \tfrac{1}{3}\right\} \cap \left\{\tau_E^{(d)} + t_H < \widehat{T}_{y/2}^{J,(d)}\right\} \cap \left\{\widehat{\tau}_0^{(J:d)} < (\tau_E^{(d)} + t_H + t_B) \wedge \widehat{T}_{m_M}^{(3|d)} \wedge \tau_\partial^{(d)}\right\} \tag{7.81}$$

and otherwise $U_H^{(d)} := \infty$, where the definitions of $\widehat{T}_{y/2}^{J,(d)}$ and $\widehat{T}_{m_M}^{(3|d)}$ are as follows, adjusted from those of $T_{y/2}^{J,(d)}$ and $T_{m_M}^{(3|d)}$ with a specific time-shift:

$$\widehat{T}_{y/2}^{J,(d)} := \inf\left\{t \geq \tau_E^{(d)}\,;\ \pi_J(X^{(d)}(t)) \notin \mathcal{Y}_J(y/2)\right\},$$
$$\widehat{T}_{m_M}^{(3|d)} := \inf\left\{t \geq \tau_E^{(d)} + t_H\,;\ M_3^{(d)}(t) \geq m_M\right\}.$$

Since $t_H, t_B \leq \frac{1}{3}$, we deduce that $\left\{1 \wedge \tau_\partial^{(d)} < U_H^{(d)}\right\} = \left\{U_H^{(d)} = \infty\right\}$. On the other hand, the stopping time $V^{(d)}$ is defined as follows:

$$V^{(d)} := \inf\left\{t \geq \tau_E^{(d)} + 2t_H\,;\ X_{(J)}^{(d)}(t) = 0\right\}. \tag{7.82}$$

Remark that $U_H^{(d)}$ and $V^{(d)}$ are regularly expressed in terms of the process $X^{(d)}$, so that they can be expressed as stopping times for a path space representation of Ω and $(\mathcal{F}_t)$, namely the canonical representation of $X^{(d)}$, as stated in Theorem 7.5.1.

Thanks to the strong Markov property at time $\tau_E^{(d)}$, the following inequality holds for any $d \in [\![J, \infty]\!]$ and $x \in \mathcal{X}_d$:

$$\mathbb{P}_x\left(U_H^{(d)} = \infty, 1 < \tau_\partial^{(d)}\right)$$

$$\leq \mathbb{P}_x\Big(\tfrac{1}{3} < \tau_E^{(d)} \wedge \tau_\partial^{(d)}\Big) + \mathbb{E}_x\Big[g\big(X^{(d)}(\tau_E^{(d)})\big)\,;\ \tau_E^{(d)} < \tau_\partial^{(d)}\Big], \tag{7.83}$$

where the function g is expressed as follows for any $x \in E^{(d)}$:

$$\begin{aligned} g(x) = \mathbb{P}_x\Big(T_{m_M}^{(3|d)} \leq 1\Big) + \mathbb{P}_x\Big(T_{y/2}^{J,(d)} \leq t_H\ ,\ 1 < T_{m_M}^{(3|d)}\Big) \\ +\mathbb{P}_x\Big(t_H + t_B < \widetilde{\tau}_0^{(J:d)}\ ,\ 1 < T_{m_M}^{(3|d)}\Big), \end{aligned} \tag{7.84}$$

where $\widetilde{\tau}_0^{(J:d)} := \inf\Big\{t \geq t_H\,;\ X_{(J)}^{(d)}(t) = 0\Big\}$. Recalling (7.76), the function g is upper-bounded as follows in terms of the probability measure $\mathbb{P}_x^{(J:d)}$ for any $x \in E^{(d)}$:

$$g(x) \leq \mathbb{P}_x\Big(T_{m_M}^{(3|d)} \leq 1\Big) + 2\mathbb{P}_x^{(J:d)}\Big(T_{y/2}^{J,(d)} \leq t_H\Big) + 2\mathbb{P}_x^{(J:d)}\Big(t_H + t_B < \widetilde{\tau}_0^{(J:d)}\Big). \tag{7.85}$$

Thanks to the Markov Property at time t_H and to Lemma 5.1.4 with (7.78), the following upper-bound for the last term is expressed in terms of the processes Y (see (7.66)) and Z (see (7.64)) for any $x \in E^{(d)}$:

$$\mathbb{P}_x^{(J:d)}\Big(t_H + t_B < \widetilde{\tau}_0^{(J:d)}\Big) \leq \mathbb{P}\Big(T_{y_M}^Y \leq t_B \,\Big|\, Y(0) = y_M/4\Big) + \mathbb{P}\Big(t_B \leq \tau_\partial^Z \,\Big|\, Z(0) = z\Big).$$

Recalling (7.71) and (7.72), this term is thus upper-bounded by 2ϵ. Injecting this upper-bound in (7.85) and the similar ones deduced from (7.76) and (7.80), we deduce that $g(x) \leq 7\epsilon$ for any $x \in E^{(d)}$. Injecting this upper-bound and the one from (7.74) into (7.83), the inequality $\mathbb{P}_x\Big(U_H^{(d)} = \infty, 1 < \tau_\partial^{(d)}\Big) \leq 8\epsilon$ holds for any $d \in [\![J, \infty]\!]$ and $x \in \mathcal{X}_d$. To arrive at the upper-bound that $\mathbb{P}_x(U_H^{(d)} = \infty, 1 < \tau_\partial^{(d)}) \leq \epsilon' \cdot e^{-\rho}$, as stated in Theorem 7.5.1 for some $\epsilon' > 0$, we just have to choose $\epsilon = \epsilon' \cdot e^{-\rho}/8$. Since ϵ is freely chosen, so is ϵ'.

7.5.3 Comparison of densities

The core argument for this Sect. 7.5.3 is Harnack's inequality (from Sect. 5.1.1) applied on a reduction of the system to a finite dimensional projection. We need however to handle carefully both the distance from the boundary of the finite dimensional projection, since we need the diffusion to be elliptic, and the control of the third moment, to convert $\mathbb{P}$ into $\mathbb{P}^{(J:d)}$ and vice-versa.

For any $d \in [\![J, \infty]\!]$, let us first assume that x and x_ζ both belong to $E^{(d)}$, so that $\tau_E^{(d)} = 0$ for both initial conditions x and x_ζ. By virtue of the definition of $U_H^{(d)}$ (see (7.81)), then thanks to (7.76):

$$\mathbb{P}_x\Big(X^{(d)}(U_H^{(d)}) \in dx'\ ,\ U_H^{(d)} < \infty\Big) \leq 2\,\mathbb{P}_x^{(J:d)}\Big(X^{(d)}(\widehat{\tau}_0^{(J:d)}) \in dx'\ ,\ U_H^{(d)} < \infty\Big). \tag{7.86}$$

Recall the definition of $\tau_0^{(J:d)}$ from (7.25). Thanks to the Markov property at time t_H, the following inequality holds:

$$\mathbb{P}_x^{(J:d)}\Big(X^{(d)}(\widehat{\tau}_0^{(J:d)}) \in \mathrm{d}x'\,,\ U_H^{(d)} < \infty\Big) \leq \int_{\mathcal{X}_d} \nu_x^1(\mathrm{d}x_H)\nu_{x_H}^2(\mathrm{d}x'), \tag{7.87}$$

where:

$$\nu_x^1(\mathrm{d}x_H) := \mathbb{P}_x^{(J:d)}\Big(X^{(d)}(t_H) \in \mathrm{d}x_H\,;\ t_H < T_{y/2}^{J,(d)} \wedge T_{m_M}^{(J.3|d)}\Big),$$

so that $\nu_x^1(\mathrm{d}x_H)$ describes the transition of the process $X^{(d)}$ during the time-interval $[0, t_H]$ on the event $\{t_H < T_{y/2}^{J,(d)} \wedge T_{m_M}^{(J.3|d)}\}$, while:

$$\nu_{x_H}^2(\mathrm{d}x') := \mathbb{P}_{x_H}^{(J:d)}\Big(X^{(d)}(\tau_0^{(J:d)}) \in \mathrm{d}x'\,;\ \tau_0^{(J:d)} < t_B \wedge T_{m_M}^{(J.3|d)} \wedge \tau_\partial^{(d)}\Big), \tag{7.88}$$

so that $\nu_{x_H}^2(\mathrm{d}x')$ describes the transition of $X^{(d)}$ during the time-interval $[0, \tau_0^{(J:d)}]$ on the event $\{\tau_0^{(J:d)} < t_B \wedge T_{m_M}^{(J.3|d)} \wedge \tau_\partial^{(d)}\}$. Note on this event that $X^{(d)}(\tau_0^{(J:d)})_i = 0$ holds for any $i \geq J$, so that $X^{(d)}(\tau_0^{(J:d)})$ is directly expressed in terms of $\pi_J(X^{(d)}(\tau_0^{(J:d)}))$. The measure $\nu_{x_H}^2$ only depends on $\pi_J(x_H)$ thanks to Proposition 7.3.1, so that the formula $\bar{\nu}_{\pi_J(x_H)}^2 = \nu_{x_H}^2$ produces a well-defined quantity. Therefore:

$$\int_{\mathcal{X}_d} \nu_x^1(\mathrm{d}x_H)\nu_{x_H}^2(\mathrm{d}x') = \int_{\mathcal{X}_J} \bar{\nu}_x^1(\mathrm{d}x'_H)\bar{\nu}_{x'_H}^2(\mathrm{d}x'), \tag{7.89}$$

where $\bar{\nu}_x^1$ is the image of ν_x^1 by the projection $\pi_J : \mathcal{X}_d \mapsto \mathcal{X}_J$, i.e.:

$$\bar{\nu}_x^1(\mathrm{d}x'_H) := \mathbb{P}_x^{(J:d)}\Big(\pi_J(X^{(d)}(t_H)) \in \mathrm{d}x'_H\,;\ t_H < T_{y/2}^{J,(d)} \wedge T_{m_M}^{(J.3|d)}\Big).$$

Note that $T_{y/2}^{J,(d)} \wedge T_{m_M}^{(J.3|d)}$ corresponds to the exit time of $\pi_J(X^{(d)})$ out of some domain $\mathfrak{H}_J(y/2, m_M)$. There exist two connected open relatively compact sets $\mathfrak{K}_J^\wedge, \mathfrak{K}_J^\vee$ with $\mathcal{C}^\infty$-boundaries such that the following inclusions hold:

$$\mathfrak{H}_J(y/2, m_M) \subset \mathfrak{K}_J^\wedge, \quad \overline{\mathfrak{K}_J^\wedge} \subset \mathfrak{K}_J^\vee \subset \mathfrak{H}_J(y/4, 2m_M).$$

Considering Property (H) (see Sect. 5.1.1) with Dirichlet boundary conditions on $\mathfrak{K}_J^\vee$ (that is with $u_{\partial\mathfrak{K}_J^\vee}(z,t) \equiv 0$ for any $z \in \partial\mathfrak{K}_J^\vee$ and $t \in [0, t_H]$) amounts to the following inequality, which holds with a fixed constant $C_H > 0$ for any $d \in [\![J, \infty]\!]$ and any $x_1, x_2 \in \mathfrak{K}_J^\wedge$:

$$\begin{aligned}&\mathbf{1}_{\{x' \in \mathfrak{K}_J^\wedge\}}\mathbb{P}_{x_1}^{(J:d)}\Big(\pi_J(X^{(d)}(t_H)) \in \mathrm{d}x'\,,\ t_H < T_{\mathfrak{K}_J^\vee}\Big)\\ &\quad\leq C_H\,\mathbf{1}_{\{x' \in \mathfrak{K}_J^\wedge\}}\mathbb{P}_{x_2}^{(J:d)}\Big(\pi_J(X^{(d)}(2t_H)) \in \mathrm{d}x'\,,\ 2t_H < T_{\mathfrak{K}_J^\vee}\Big).\end{aligned}$$

It implies in particular that $\bar{\nu}^1_x \leq C_H \cdot \check{\nu}^1_{x_\zeta}$, where the time t_H and the constants $y/2$ and m_M are slightly relaxed:

$$\check{\nu}^1_{x_\zeta}(\mathrm{d}x'_H) := \mathbb{P}^{(J:d)}_{x_\zeta}\Big(\pi_J(X^{(d)}(2t_H)) \in \mathrm{d}x'_H\,;\ 2t_H < T^{J,(d)}_{y/4} \wedge T^{(J.3|d)}_{2m_M}\Big).$$

Thanks in addition to (7.86), (7.87), (7.89), we obtain as an intermediate step the following inequality, which holds for any $d \in [\![J, \infty]\!]$ and any $x, x_\zeta \in E^{(d)}$:

$$\mathbb{P}_x\Big(X^{(d)}(U^{(d)}_H) \in \mathrm{d}x',\ U^{(d)}_H < \infty\Big) \leq 2C_H \int_{\mathcal{X}_J} \check{\nu}^1_{x_\zeta}(\mathrm{d}x'_H)\bar{\nu}^2_{x'_H}(\mathrm{d}x'). \quad (7.90)$$

To go back to an upper-bound in terms of the original process, we want again to exploit Proposition 7.3.2. So we need to again ensure upper-bounds on the third moments for the last components for which we lost the information. We exploit the representation given in Proposition 7.3.1, firstly on the time-interval $[0, 2t_H]$. Since $x_\zeta \in E^{(d)}$, we have $M^{(F|d)}_3(0) \leq m/y$. Thanks to Lemma 7.3.5, we can thus define $m_H > 0$ such that the following inequality holds a.s. on the event $\{2t_H < T^{J,(d)}_{y/4}\}$, for any $d \in [\![J, \infty]\!]$ and any $x_\zeta \in E^{(d)}$:

$$\mathbb{P}^{(J:d)}_{x_\zeta}\left(T^{(F.3|d)}_{m_H} \leq 2t_H \,\middle|\, \mathcal{F}^{(J)}\right) \leq \tfrac{1}{2}.$$

With the fact that $\tau^{(d)}_\partial > T^{J,(d)}_{y/4}$, it implies that $\check{\nu}^1_{x_\zeta} \leq 2\tilde{\nu}^1_{x_\zeta}$, where:

$$\tilde{\nu}^1_{x_\zeta}(\mathrm{d}x_H) := \mathbb{P}^{(J:d)}_{x_\zeta}\Big(\pi_J\big(X^{(d)}(2t_H)\big) \in \mathrm{d}x'_H\,;\ 2t_H < \tau^{(d)}_\partial \wedge T^{(J.3|d)}_{2m_M} \wedge T^{(F.3|d)}_{m_H}\Big). \quad (7.91)$$

Recalling that the formula $\bar{\nu}^2_{\pi_J(x_H)} = \nu^2_{x_H}$ produces a well-defined quantity, it implies that $\int_{\mathcal{X}_J} \tilde{\nu}^1_{x_\zeta}(\mathrm{d}x'_H)\bar{\nu}^2_{x'_H}(\mathrm{d}x') = \int_{\mathcal{X}_d} \hat{\nu}^1_{x_\zeta}(\mathrm{d}x_H)\nu^2_{x_H}(\mathrm{d}x')$, where:

$$\hat{\nu}^1_{x_\zeta}(\mathrm{d}x_H) := \mathbb{P}^{(J:d)}_{x_\zeta}\Big(X^{(d)}(2t_H) \in \mathrm{d}x_H\,;\ 2t_H < \tau^{(d)}_\partial \wedge T^{(J.3|d)}_{2m_M} \wedge T^{(F.3|d)}_{m_H}\Big). \quad (7.92)$$

Note that this measure $\hat{\nu}^1_{x_\zeta}$ is supported on the set $\big\{x_H \in \mathcal{X}_d\,;\ M^{(F|d)}_3(x_H) \leq m_H\big\}$. Thanks again to Lemma 7.3.5, we can thus define $m_F \geq m_H$ such that the following inequality holds a.s. for any $d \in [\![J, \infty]\!]$ and any $x_H \in \mathcal{X}_d$ such that $M^{(F|d)}_3(x_H) \leq m_H$:

$$\mathbb{P}^{(J:d)}_{x_H}\left(T^{(F.3|d)}_{m_F} < t_B \wedge \tau^{(J:d)}_0 \,\middle|\, \mathcal{F}^{(J)}\right) \leq \tfrac{1}{2}.$$

It entails that $\nu^2_{x_H} \leq 2\tilde{\nu}^2_{x_H}$, where:

$$\tilde{\nu}^2_{x_H}(\mathrm{d}x') := \mathbb{P}^{(J:d)}_{x_H}\Big(X^{(d)}(\tau^{(J:d)}_0) \in \mathrm{d}x'\,;\ \tau^{(J:d)}_0 < t_B \wedge \tau^{(d)}_\partial \wedge T^{(J.3|d)}_{m_M} \wedge T^{(F.3|d)}_{m_F}\Big).$$

(7.93)

Since $M_3^{(d)}(x) \leq M_3^{(J:d)}(x) + M_3^{(F|d)}(x)$, and thanks to the Markov property at time t_H, recalling (7.90), (7.91), (7.92) and (7.93), the following inequality holds with $V^{(d)}$ as defined in (7.82) and $\hat{m} := [2m_M + m_H] \vee [m_M + m_F]$, for any $x, x_\zeta \in E^{(d)}$:

$$\begin{aligned} &\mathbb{P}_x\Big(X^{(d)}(U_H^{(d)}) \in \mathrm{d}x',\ U_H^{(d)} < \infty\Big) \\ &\leq 8C_H \cdot \mathbb{P}_{x_\zeta}^{(J:d)}\Big(X^{(d)}(V^{(d)}) \in \mathrm{d}x';\ V^{(d)} < \tau_\partial^{(d)} \wedge T_{\hat{m}}^{(3|d)}\Big). \end{aligned}$$

We can then relate to the original law $\mathbb{P}_{x_\zeta}$ thanks to Proposition 7.3.2 with an additional factor $C_G > 0$ such that the following inequality holds for any $x, x_\zeta \in E^{(d)}$:

$$\begin{aligned} &\mathbb{P}_x\Big(X^{(d)}(U_H^{(d)}) \in \mathrm{d}x',\ U_H^{(d)} < \infty\Big) \\ &\leq 8C_H \cdot C_G \cdot \mathbb{P}_{x_\zeta}\Big(X^{(d)}(V^{(d)}) \in \mathrm{d}x';\ V^{(d)} < \tau_\partial^{(d)}\Big). \end{aligned} \tag{7.94}$$

To conclude, we consider general initial conditions $\bar{x} \in \mathcal{X}_d$ and $\bar{x}_\zeta \in \mathcal{D}_3$ and define:

$$\nu_x^E(\mathrm{d}x') := \mathbb{P}_x\Big(X^{(d)}(\tau_E^{(d)}) \in \mathrm{d}x';\ \tau_E^{(d)} < \tfrac{1}{3} \wedge \tau_\partial^{(d)}\Big).$$

We deduce from (7.75) that $\nu_{\bar{x}_\zeta}^E(E^{(d)}) \geq c_\zeta$ while $\nu_{\bar{x}}^E(E^{(d)}) \leq 1$. Thanks (twice) to the strong Markov property at time $\tau_E^{(d)}$ and by virtue of the definitions of $U_H^{(d)}$ and $V^{(d)}$ in (7.81) and (7.82), (7.94) is extended as follows for any $d \in [\![J, \infty]\!]$ to any $\bar{x} \in \mathcal{X}_d$ and $\bar{x}_\zeta \in \mathcal{D}_3$:

$$\begin{aligned} &\mathbb{P}_{\bar{x}}\Big(X^{(d)}(U_H^{(d)}) \in \mathrm{d}x',\ U_H^{(d)} < \infty\Big) \\ &= \int_{E^{(d)}} \nu_{\bar{x}}^E(\mathrm{d}x)\mathbb{P}_x\Big(X^{(d)}(U_H^{(d)}) \in \mathrm{d}x',\ U_H^{(d)} < \infty\Big) \\ &\leq 8C_H \cdot C_G \cdot \frac{\nu_{\bar{x}}^E(E^{(d)})}{\nu_{\bar{x}_\zeta}^E(E^{(d)})} \int_{E^{(d)}} \nu_{\bar{x}_\zeta}^E(dx_\zeta)\mathbb{P}_{x_\zeta}\Big(X^{(d)}(V^{(d)}) \in \mathrm{d}x';\ V^{(d)} < \tau_\partial^{(d)}\Big) \\ &\leq c_V \cdot \mathbb{P}_{\bar{x}_\zeta}\Big(X^{(d)}(V^{(d)}) \in \mathrm{d}x';\ V^{(d)} < \tau_\partial^{(d)}\Big), \end{aligned}$$

where $c_V = 8C_H\, C_G / c_\zeta > 0$. This concludes the proof of Theorem 7.5.1. □

7.6 Concluding the proof of Theorem 2.4.1

We plan to exploit Theorem 2.5.3 and ensure Assumption (**AF**) (recall Sect. 2.5). Firstly, the sets $\mathcal{D}_\ell^{(d)}$ satisfy Assumption $(A0)$ (recall (7.1)). Thanks to Theorem 7.4.7, Assumption $(A1)$ holds true for the reference measure $\zeta^{(d)}$ defined by (7.59). Thanks to $(A1)$ and [15, Lemma 3.0.2], ρ_S is upper-bounded by some real number $\widetilde{\rho}_S$ that only depends on the constants involved in $(A1)$, and can thus be chosen independent of d. In order to satisfy $\rho > \rho_S$, we set $\rho := 2\widetilde{\rho}_S$. For this choice of ρ, we deduce Assumption $(A2)$ as a consequence of Theorem 7.2.1 (recall also Remark 7.2.2), where the complementary to the transitory domain can be taken as the proposed set $E^{(d)}(m, y, 1)$ for some $m, y > 0$ independent of $d \in [\![1, \infty]\!]$, or simply $\mathcal{D}_L^{(d)}$ with $L \geq \frac{1}{y}$ (we choose $\eta = 1$ for simplicity).

When $d = \infty$ we infer Assumption $(A3_F)$ thanks to Theorem 7.5.1 (regardless of the transitory domain that is chosen), for this choice of ρ and $\zeta^{(\infty)}$, noting that $\zeta^{(\infty)}$ is supported on $\mathcal{D}_3^{(\infty)}$. Assumption (**AF**) thus holds true for $d = \infty$. Thanks to Theorem 2.5.3, we then conclude the proof that the semi-group displays QSC and that the Q-process exists on $\mathcal{X}_\infty$.

In addition, recall that $\rho_0^{(\infty)} = -\log[\mathbb{P}_{\nu^{(\infty)}}(1 < \tau_\partial^{(\infty)})]$ (see Remark 2.1.2). Since $\cup_\ell \mathcal{D}_\ell^{(\infty)} = \mathcal{X}_\infty$ (recall (7.1)), there exists ℓ such that $\nu^{(\infty)}(\mathcal{D}_\ell^{(\infty)}) > 0$. With a slight adaptation of the proof of Lemma 7.2.5, we deduce that $\mathbb{P}_x^{(\infty)}(\tau_\partial^{(\infty)} \leq 1)$ is uniformly bounded away from 0 for any $x \in \mathcal{D}_\ell^{(\infty)}$. This concludes that $\rho_0^{(\infty)} > 0$.

As noted in [14, Subsection 6.1], the statement of Assumption $(A3_F)$ is actually required for a single value of ϵ defined in terms of the parameters involved in Assumptions $(A1)$ and $(A2)$. Since these parameters can be chosen uniformly in d, the corresponding value of ϵ can also be chosen uniformly so that the required properties in Assumption (**AF**) holds uniformly for any d larger than a given threshold (see the definition of J in (7.77)). In addition and for the same reason, all parameter involved in the convergences can be chosen independently of d sufficiently large. There are indeed intricate yet explicit relations between all these parameters introduced in [14] and the corresponding assumptions. This concludes the proof of Theorem 2.4.1. □

Appendix A Two representations for the same process

In this Section A, we justify more precisely (see Proposition A.0.1) than we did in Remark 1.2.2 that the process we consider is exactly the same as defined notably in [2], though the representation has been adapted for our purposes.

Let us recall from (1.1) for any $d \in [\![2, \infty]\!]$ the definition of our focal process $X^{(d)}$ as follows for any $i \in [\![0, d]\!]$:

$$\mathrm{d}X_i^{(d)}(t) = b_i^{(d)}(X^{(d)}(t))\,\mathrm{d}t + \mathrm{d}\mathcal{N}_i^{(d)}(t), \tag{A1}$$

where the vectorial function b encodes the drift term as follows for any $x \in \mathcal{X}_d$ and $i \in [\![0, d]\!]$:

$$b_i^{(d)}(x) := \alpha \cdot \left(\sum\nolimits_{\{j \in [\![0,d]\!]\}} j \cdot x_j - i\right) \cdot x_i + \lambda \cdot \left(x_{i-1} - \mathbf{1}_{\{i<d\}} x_i\right),$$

while the martingale process $\mathcal{N}_i^{(d)}$, which starts at 0 for $t = 0$, is expressed as follows:

$$\begin{aligned} \mathrm{d}\mathcal{N}_i^{(d)}(t) &:= \sum_{j=0}^{d} \left(\delta_{ij} - X_i^{(d)}(t)\right) \cdot \sqrt{X_j^{(d)}(t)}\,\mathrm{d}W_j(t) \\ &= \sqrt{X_i^{(d)}(t)}\,\mathrm{d}W_i(t) - X_i^{(d)}(t)\,\mathrm{d}W_{(d)}(t). \end{aligned} \tag{A2}$$

in terms of a family $(W_i)_{i\in\mathbb{Z}_+}$ of mutually independent Brownian motions and an aggregated martingale $W_{(d)}$ with the following definition:

$$\mathrm{d}W_{(d)}(t) := \textstyle\sum_{j=0}^{d} \sqrt{X_j^{(d)}(t)}\mathrm{d}W_j(t), \quad W_{(d)}(0) = 0.$$

On the other hand, we next consider the more classical way that Muller's ratchet diffusion $\widehat{X}$ is defined, notably in [2]:

$$\mathrm{d}\widehat{X}_i^{(d)}(t) = b_i^{(d)}\left(\widehat{X}^{(d)}(t)\right)\mathrm{d}t + \mathrm{d}\widehat{\mathcal{N}}_i^{(d)}(t),$$

where the martingale term $\widehat{\mathcal{N}}_i^{(d)}$ is now expressed as follows:

$$\mathrm{d}\widehat{\mathcal{N}}_i^{(d)}(t) = \textstyle\sum_{\{j\neq i\}} \sqrt{\widehat{X}_i^{(d)}(t) \cdot \widehat{X}_j^{(d)}(t)}\,\mathrm{d}W_{i,j}(t), \quad \widehat{\mathcal{N}}_i^{(d)}(0) = 0.$$

in terms of a family $(W^{i,j})_{i<j}$ of independent Brownian motions, extended to any $i \neq j$ by the symmetry property $W_{i,j}(t) = W_{j,i}(t)$.

For the infinite-dimensional case, we recall the definition of $\mathcal{X}^\eta$ in 2.10 as the set of probabilities on $\mathbb{N}$ with finite η-th moment ($\eta = 6$ being considered for simplicity in our proofs). The upcoming Proposition A.0.1 summarises the result of this Section A and notably implies Proposition 2.3.1.

Proposition A.0.1 *The processes $(X_i^{(d)})$ and $(\widehat{X}_i^{(d)})$ share the same law. Existence and uniqueness in law holds, for processes on $\mathcal{X}_d$ for any $d \in \mathbb{N}$, and, for the case $d = \infty$, on $\mathcal{X}^\eta$ for any $\eta > 2$. For any d, the associated infinitesimal generator takes the following nondivergence form, for any $u \in C^2(\mathcal{D}_\ell^{(d)})$ and $x \in \mathcal{D}_\ell^{(d)}$:*

$$\mathcal{L}^{(d)}u(x) = \frac{1}{2}\sum_{i,j=1}^{d} \sigma_{i,j}^{(d)}(x)\partial_{i,j}^2 u(x) + \sum_{i=1}^{d} b_i^{(d)}(x)\partial_i u(x),$$

where the state-dependent diffusion matrix $\sigma^{(d)}$ is expressed as follows:

$$\sigma_{i,j}^{(d)}(x) := \begin{cases} -x_i \cdot x_j & \textit{if } i \neq j, \\ x_i \cdot (1 - x_i) & \textit{if } i = j, \end{cases}$$

and the state-dependent drift term $b^{(d)}$ is expressed as follows:

$$b_i^{(d)}(x) := \alpha \cdot \left(M_1^{(d)}(x) - i\right) \cdot x_i + \lambda \cdot \left(x_{i-1} - \mathbf{1}_{\{i<d\}} \cdot x_i\right).$$

Remark A.0.2 Recall that our representation of the martingale term corresponds to the idea that internal demographic fluctuations W^i specific to each subpopulation level i get compensated thanks to the process W. On the other hand, the classical representation is associated to the principles of Moran models, where demographic fluctuations are generated by choosing one parent at random whose offspring replaces another inidividual also chosen at random. By distinguishing these contributions in terms of the two involved types (i, j) (be it the parental type or the killed type), we can derive the contribution $\sqrt{\widehat{X}_i^{(d)}(t) \cdot \widehat{X}_j^{(d)}(t)}\, \mathrm{d}W_{i,j}(t)$ to the fluctuations. It explains why $W_{i,j} = -W_{j,i}$ and how the previous term can be interpreted as a random flux, which does not need to be compensated.

Proof : According to [20, Theorem 2.1], the existence and the uniqueness in law of solutions hold for the system A1 of SDEs (for both finite and infinite d) provided that the drift term $b_i^{(d)}$ satisfies the following three conditions:

- $b_i(x) \geq 0$ uniformly in $x \in \mathcal{X}_d$,
- $\sum_{i \in [\![0,d]\!]} b_i(x) = 0$,
- there exists a matrix $(q_{ij})_{i,j \in [\![0,d]\!]}$ such that $q_{ij} \geq 0$ and $\sup_{k \in [\![0,d]\!]} \sum_{i=0}^{d} q_{ik} < +\infty$ for every i and j of $[\![0, d]\!]$, and such that the following inequality holds for any x and x' of $\mathcal{X}_d$ $(i \in [\![0, d]\!])$:

$$|b_i^{(d)}(x) - b_i^{(d)}(x')| \leq \sum_{\{j \in S\}} q_{ij} \cdot |x_j - x'_j|.$$

The first two properties are satisfied for any $d \in [\![2, \infty]\!]$. For finite $d \in \mathbb{N}$, the following estimate on the drift term holds for any $i \in [\![0, d]\!]$ and $x, x' \in \mathcal{X}_d$:

$$|b_i(x) - b_i(x')| \leq (2\alpha \cdot d + \lambda) \cdot \sum_{j=0}^{d} |x_j - x'_j|.$$

The corresponding choice of $q_{ij} \equiv 2\alpha \cdot d + \lambda$ ensures for any $d \in \mathbb{N}$ the existence and uniqueness in law for the system A1 of SDEs. Thanks to [2, Theorem 3] (which relies on the above result of [20]), existence and uniqueness in law also hold on the set $\mathcal{X}^\eta$ for any $\eta > 2$. The associated infinitesimal generator is expressed in [20, Theorem 1.1] as stated in Proposition A.0.1.

For completeness and to check that the laws of $\mathcal{N}_i^{(d)}(t)$ and $\widehat{\mathcal{N}}_i^{(d)}(t)$ actually coincide, we provide next the computations of the quadratic variations of these martingale terms. We first consider the quadratic variation of each component i for the process $\mathcal{N}_i^{(d)}$:

$$\begin{aligned}
\mathrm{d}\langle \mathcal{N}_i^{(d)} \rangle_t &= X_i^{(d)}(t)\, \mathrm{d}\langle W_i \rangle_t + (X_i^{(d)}(t))^2\, \mathrm{d}\langle W_{(d)} \rangle_t - 2\sqrt{X_i^{(d)}(t)} \cdot X_i^{(d)}(t)\, \mathrm{d}\langle W_i, W_{(d)} \rangle_t \\
&= X_i^{(d)}(t) \cdot (1 - X_i^{(d)}(t))\, \mathrm{d}t,
\end{aligned}$$

then for the process $\widehat{\mathcal{N}}_i^{(d)}$:

$$\begin{aligned} d\langle \widehat{\mathcal{N}}_i^{(d)} \rangle_t &= \sum_{\{j\neq i\}} \widehat{X}_i^{(d)}(t)\cdot \widehat{X}_j^{(d)}(t)\, dt \\ &= \widehat{X}_i^{(d)}(t)\cdot (1-\widehat{X}_i^{(d)}(t))\, dt. \end{aligned}$$

We secondly consider the cross-variations for $i \neq j$, which yields for the process $M^{(d)}$:

$$\begin{aligned} & d\langle \mathcal{N}_i^{(d)}, \mathcal{N}_j^{(d)} \rangle_t \\ &= \sqrt{X_i^{(d)}(t)\cdot X_j^{(d)}(t)}\, d\langle W_i, W_j\rangle_t - \sqrt{X_i^{(d)}(t)}\cdot X_j^{(d)}(t)\, d\langle W_i, W_{(d)}\rangle_t \\ &- X_i^{(d)}(t)\cdot \sqrt{X_j^{(d)}(t)}\, d\langle W_{(d)}, W_j\rangle_t + X_i^{(d)}(t)\cdot X_j^{(d)}(t)\, d\langle W_{(d)}\rangle_t \\ &= \Bigg[0 - \sqrt{X_i^{(d)}(t)}\cdot X_j^{(d)}(t)\cdot \sqrt{X_i^{(d)}(t)} - X_i^{(d)}(t) \\ &\quad \cdot \sqrt{X_j^{(d)}(t)}\cdot \sqrt{X_j^{(d)}(t)} + X_i^{(d)}(t)\cdot X_j^{(d)}(t)\Bigg]\, dt, \\ &= -X_i^{(d)}(t)\cdot X_j^{(d)}(t)\, dt, \end{aligned}$$

then yields for the process $\widehat{M}^{(d)}$:

$$\begin{aligned} d\langle \widehat{\mathcal{N}}_i^{(d)}, \widehat{\mathcal{N}}_j^{(d)} \rangle_t &= -\sum_{\{k\neq i\}} \sum_{\{\ell\neq j\}} \sqrt{\widehat{X}_i^{(d)}(t)\cdot \widehat{X}_k^{(d)}(t)\cdot \widehat{X}_j^{(d)}(t)\cdot \widehat{X}_\ell^{(d)}(t)}\, d\langle W_{i,k}, W_{\ell,j}\rangle_t \\ &= -\widehat{X}_i^{(d)}(t)\cdot \widehat{X}_j^{(d)}(t)\, dt, \end{aligned}$$

since $d\langle W_{i,k}, W_{\ell,j}\rangle_t = \mathbf{1}_{\{i=\ell, j=k\}} dt$.

Since the quadratic variations are expressed in the same ways for the solutions $X^{(d)}$ and $\widehat{X}^{(d)}$ to the systems respectively A1 and A2, we conclude thanks to [20, Theorem 1.1] that the laws of these two processes coincide. This concludes the proof of Proposition A.0.1. □

Appendix B Justification for the boundedness of the moments

In this Section B, we derive from the proof of [2, Theorem 3] the local boundedness of the moment processes, as stated in this last Proposition B.0.1.

Proposition B.0.1 *The process $M_k^{(\infty)}$ is a.s. locally upper-bounded for any $k \geq 1$, any $\alpha, \lambda > 0$ and any $x \in \mathcal{X}_\infty \cap \mathcal{X}^k$ under both $\mathbb{P}_x$ and $\mathbb{P}_x^{(J:d)}$, whatever $J \in \mathbb{N}$.*

Proof : Without loss of generality, we assume that $k \geq 3$ and consider any $x \in \mathcal{X}^k$, any $\lambda > 0$ and any $t > 0$. The conclusion of the proof of [2, Proposition 2.2] can be restated as the following upper-bound, which holds in the case where $\alpha = 0$ (as indicated with the notation $\mathbb{P}^{[0]}$):

$$\mathbb{P}_x^{[0]}\Big(\sup_{\{s\le t\}} M_k^{(\infty)}(s) < \infty\Big) = 1. \tag{B1}$$

We consider any $\alpha > 0$ and denote for clarity $\mathbb{P}^{[\alpha]}$ instead of $\mathbb{P}^{(\infty)}$ for the law of the process $X^{(\infty)}$ solution to (1.1). Thanks to [2, Subsection 2.1, proof of Theorem 3], $\mathbb{P}^{[\alpha]}$ is related to $\mathbb{P}^{[0]}$ with a Girsanov transform, namely $\mathbb{P}_x^{[\alpha]}|_{\mathcal{F}_t} = Z_\alpha(t)\cdot \mathbb{P}_x^{[0]}|_{\mathcal{F}_t}$ holds for any $x \in \mathcal{X}_\infty$ and any $t > 0$, which involves the following martingale process $(Z_\alpha(s))_{s\ge 0}$:

$$\begin{aligned} Z_\alpha(s) &:= \exp\left(-\alpha \mathcal{M}_1^{(\infty)}(s) - (\alpha^2/2)\cdot \int_0^s M_2^{(\infty)}(s)\mathrm{d}s\right),\\ &\le \exp\left(-\alpha\cdot[M_1^{(\infty)}(x) + \lambda s]\right). \end{aligned}$$

In the above expression, $\mathcal{M}_1^{(\infty)}(s)$ is the martingale component of $M_1^{(\infty)}(s)$, as stated in Lemma 6.1.5. Since $Z_\alpha(t)$ is uniformly upper-bounded, (B1) entails that $\sup_{\{s\le t\}} M_k^{(\infty)}(s) < \infty$ holds also a.s. under $\mathbb{P}_x^{[\alpha]}$, provided $x \in \mathcal{X}^k$.

We next consider $J \in \mathbb{N}$ and the law $\mathbb{P}_x^{[\alpha],J} = \mathbb{P}_x^{[\alpha],(J:\infty)}$ as stated in Sect. 7.3. Thanks to Proposition 7.3.1, and with the same arguments as above, $\mathbb{P}^{[\alpha],J}$ is related to $\mathbb{P}^{[0]}$ with another Girsanov transform, namely $\mathbb{P}_x^{[\alpha],J}|_{\mathcal{F}_t} = Z_\alpha^J(t)\cdot \mathbb{P}_x^{[0]}|_{\mathcal{F}_t}$ holds for any $x \in \mathcal{X}_\infty$ and any $t > 0$, which involves the following martingale process $(Z_\alpha^J(s))_{s\ge 0}$:

$$\begin{aligned} Z_\alpha^J(s) &:= \exp\left(-\alpha \mathcal{M}_1^{(J:\infty)}(s) - (\alpha^2/2)\cdot \int_0^s M_2^{(J:\infty)}(s)\mathrm{d}s\right),\\ &\le \exp\left(-\alpha\cdot\big[(M_1^{(J:\infty)}(x)\wedge J) + \lambda s\big]\right). \end{aligned}$$

Similarly, $\mathcal{M}_1^{(J:\infty)}(s)$ is the martingale component of $M_1^{(J:\infty)}(s)$. With the same arguments as for the proof of Lemma 6.1.5, its quadratic variation involves the process $M_2^{(J:\infty)}$, which is the second moment saturated at value J, i.e.:

$$\begin{aligned} M_2^{(J:\infty)}(t) &:= \sum_{i\le J-1} i^2\cdot X_i^{(\infty)}(t) + J^2\cdot X_{(J)}^{(\infty)}(t)\\ &= \sum_{i\ge 0} \big(i\wedge J\big)^2\cdot X_i^{(\infty)}(t). \end{aligned}$$

Since $Z_\alpha^J(t)$ is uniformly upper-bounded, (B1) entails that $\sup_{\{s\le t\}} M_k^{(\infty)}(s) < \infty$ holds a.s. under $\mathbb{P}_x^{[\alpha],J}$, provided $x \in \mathcal{X}^k$. This concludes the proof of Propostion B.0.1. □

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