PDE model for multi-patch epidemic models with migration and infection-age dependent infectivity

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Abstract. We study a stochastic epidemic model with multiple patches (locations), where individuals in each patch are categorized into three compartments, Susceptible, Infected and Recovered/Removed, and may migrate from one patch to another in any of the compartments. Each individual is associated with a random infectivity function which dictates the force of infectivity while the interactive infection process depends on the age of infection (elapsed time since infection). We prove a functional law of large number for the epidemic evolution dynamics including the aggregate infectivity process, the numbers of susceptible and recovered individuals as well as the number of infected individuals at each time that have been infected for a certain amount of time. From the limits, we derive a PDE model for the density of the number of infected individuals with respect to the infection age, which is a systems of linear PDE equations with a boundary condition that is determined by a set of integral equations.

1. Introduction

Multi-patch epidemic models have been used to study infectious disease dynamics in different geographic areas [21, 1, 23, 2, 20, 9]. Most of the literature concerns Markovian models and the associated ODEs. In [20], the authors study a non-Markovian multi-patch model with general exposed and infectious distributions as well as Markovian migration among the patches. That work extends the study of the homogeneous stochastic epidemic models in [18]. However, both works assumed a constant infection rate. In [8], a stochastic epidemic model is studied to take into account varying infectivity, capturing the varying viral load phenomenon during infection as observed in [11]. In fact, Kermack and McKendrick [14] already proposed deterministic epidemic models to study varying infectivity, and the FLLN limit in [8] coincides with the integral equations in [14]. By tracking the age of infection (elapsed time since infection) in that model with varying infectivity, in [19], the authors have studied the process counting the number of individuals at each time that have been infected for less than a certain amount time, and derived a PDE model for the density of that process with respect to the infection age. The PDE model is comparable with the well known PDE models introduced by Kermack and McKendrick [15]. This homogeneous model with varying infectivity in [8] is extended to a multi-patch multi-type model in [9], however, the processes do not take into account the infection ages.

In this paper, we extend the study of epidemic models with infection-age dependent infectivity in [19] to multi-patch models, and derive the associated PDE models. Specifically, we consider an individual-based stochastic epidemic model with multiple patches, where each individual is associated with a random infectivity function of the same law, and can migrate from one patch to another in each of the infection stages (susceptible, infected or recovered). The evolution dynamics at each time is described by the total force of infectivity, the number of susceptible individuals,

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the number of infected individuals that have been infected for less than a certain amount of time, and the number of recovered individuals. We prove a functional law of large numbers (FLLN) for these processes (Theorem 2.1), where the limits are a set of Volterra-type integral equations. We then derive a PDE model (Theorem 3.1) from the limit of the proportions of infected individuals tracking the infection age distributions, together with the other limits. The PDE model concerns the density of the limit of the number of infected individuals tracking the infection age distributions with respect to the infection age. We show that the PDE model is characterized by a system of linear equations, with a boundary condition also given by a set of Volterra-type integral equations.

Since the seminar work in [15], a few articles have used PDE models to describe epidemic dynamics with infection-age dependent infectivity. See, for example, [12, 13, 24, 17, 4] and references therein. They all use the hazard rate function of the infection durations as a way to model the dependence upon the infection ages. For many scenarios, constructing the PDE models directly using the hazard rate functions is feasible, and sometimes, it is a very convenient method. However, for the multi-patch model with migration as we consider in this paper, it does not seem possible to directly construct the PDE model using hazard rate functions to describe the dependence on the infection ages together with the migration dynamics. See the PDE equation in (3.1). It is then important that we start with an individual-based stochastic model and then derive the PDE models as the scaling limits of the stochastic models.

The authors in [7] and [10] also take this individual-based stochastic model approach but they use measure-valued stochastic processes to describe the epidemic dynamics, where the hazard rate functions of the infection durations are used for that purpose. However, for the homogeneous model, our previous work [19] differs in that the the infectious duration can be of any arbitrary distribution, and then the PDE model is also derived from the FLLN limit when the infectious distribution is absolutely continuous (the hazard rate function exists) and when it is more general (for which a hazard measure is used). In this paper, we extend the approach in [19] to the multi-patch model, and establish the FLLN limit without requiring any condition on the infectious distribution. The PDE model is then derived when the infectious distribution is absolutely continuous in Section 3 (see also Remark 3.3 for the more general case).

To prove the FLLN, we employ the weak convergence criterion for stochastic processes taking values in the $\mathbf{D}_\mathbf{D}$ space, see Theorem 5.1 (established in [19]). The proof for the multi-patch model relies on an important observation that the process tracking how long individuals have been infected has an integral representation (Lemma 2.1 and (4.10)). The convergence criterion is used for three components in the integral representation (Lemmas 4.3, 4.4 and 4.5), together with properties of stochastic integrals with respect to the associated Poisson random measures.

Organization of the paper. The rest of the paper is organized as follows. In Section 2, we describe the model in detail, and state the FLLN result. In Section 3, we state the PDE model and its derivation, and prove the existence and uniqueness of its solution. The proof for the FLLN is given in Section 4. In the Appendix, we state two results on the weak convergence of stochastic processes used in the proof.

2. The Model and FLLN

We consider a multi-patch epidemic model with infection-age dependent infectivity described as follows. Individuals in each patch belong to either of the susceptible, infected or recovered/removed compartments, and may migrate from one patch to another in any of the three compartments. Each individual is associated with a random infectivity function, and the infection process depends on the age of infection (elapsed time since infection).

Let N be the total population size and L be the number of patches. For each $\ell \in \mathcal{L} := \{1, \dots, L\}$, let $S^N_\ell(t)$, $I^N_\ell(t)$ and $R^N_\ell(t)$ denote the numbers of individuals in patch ℓ that are susceptible, infectious and recovered at time t, respectively. Then we have the balance equation:

$$N \;=\; \sum_{\ell=1}^{L} \left(S_\ell^N(t) + I_\ell^N(t) + R_\ell^N(t) \right), \quad t \geq 0\,.$$

Assume that $S_\ell^N(0) > 0$ and $\sum_{\ell=1}^L I_\ell^N(0) > 0$, $\ell \in \mathcal{L}$. Let $B_\ell^N(t) = S_\ell^N(t) + I_\ell^N(t) + R_\ell^N(t)$ for $t \geq 0$ and each ℓ. In addition, let $\mathfrak{I}_\ell^N(t, \mathfrak{a})$ be the number of infected individuals in patch ℓ at time t that have been infected for less than or equal to $\mathfrak{a}$. Also, let $A_\ell^N(t)$ be the number of newly infected individuals in patch ℓ by time t after time 0.

For each individual i that becomes infected in patch ℓ, let $\lambda_i^\ell : \mathbb{R}_+ \to \mathbb{R}_+$ be the associated random infectivity function. Similarly, for each individual j that is infected in patch ℓ at time zero, let $\lambda_j^{0,\ell}(t)$ be the associated infectivity function. Assume that the random functions $\{\lambda_i^\ell(\cdot)\}_{i,\ell}$ and $\{\lambda_j^{0,\ell}(\cdot)\}$ are independent and have the same law. This is reasonable since we model the same disease. We write $\lambda(t)$ and $\lambda^0(t)$ as the generic random functions for these sequences.

Let $\{\tau_i^{\ell,N}, i \in \mathbb{N}\}$ be the event times associated with the infection process $A_\ell^N(t)$. For the initially infected individuals in patch ℓ, we let $\{\tau_{j,0}^{\ell,N}, j = 1, \dots, I_\ell^N(0)\}$ be the times at which the initially infected individuals at time 0 became infected. Note that we label the initially infected individuals by the patch where they are at time 0, irrespective of where they have been infected. We do not follow the movements of the individuals before time 0. Then $\tilde{\tau}_{j,0}^{\ell,N} = -\tau_{j,0}^{\ell,N}$, $j = 1, \dots, I_\ell^N(0)$, represent the amount of time that an initially infected individual has been infected by time 0, that is, the age of infection at time 0. WLOG, assume that $0 > \tau_{1,0}^{\ell,N} > \tau_{2,0}^{\ell,N} > \cdots > \tau_{I_\ell^N(0),0}^{\ell,N}$ (or equivalently $0 < \tilde{\tau}_{1,0}^{\ell,N} < \tilde{\tau}_{2,0}^{\ell,N} < \cdots < \tilde{\tau}_{I_\ell^N(0),0}^{\ell,N}$). Set $\tilde{\tau}_{0,0}^{\ell,N} = 0$. Define $\mathfrak{I}_\ell^N(0, \mathfrak{a}) = \max\{j \geq 0 : \tilde{\tau}_{j,0}^{\ell,N} \leq \mathfrak{a}\}$, which represents the number of initially infected individuals in patch ℓ that have been infected for less or equal to $\mathfrak{a}$ at time 0. Assume that there exists $0 \leq \bar{\mathfrak{a}} < \infty$ such that $I_\ell^N(0) = \mathfrak{I}_\ell^N(0, \bar{\mathfrak{a}})$ a.s. for all $\ell \in \mathcal{L}$.

We define

$$\eta_i^\ell = \inf\{t > 0 : \lambda_i^\ell(t + r) = 0,\, \forall r \geq 0\},$$
$$\eta_j^{0,\ell} = \inf\{t > 0 : \lambda_j^{0,\ell}(\tilde{\tau}_{j,0}^{\ell,N} + r) = 0,\, \forall r \geq t\}.$$

By the i.i.d. assumption on $\{\lambda_i^\ell(\cdot)\}_{i,\ell}$, the variables $\{\eta_i^\ell\}_i$ are also i.i.d. and let F be the associated c.d.f. The variables $\eta_j^{0,\ell}$ represent the remaining infected period for the initially infected individual. They are independent for all the initially infected individuals but their distributions depend on the time being infected. That is, the conditional distribution of $\eta_j^{0,\ell}$ given that $\tilde{\tau}_{j,0}^{\ell,N} = s > 0$ is given by

$$\mathbb{P}(\eta_j^{0,\ell} > t | \tilde{\tau}_{j,0}^{\ell,N} = s) = \frac{F^c(t+s)}{F^c(s)}, \quad \text{for} \quad t, s > 0.$$

For notational convenience, we let $F_0(t|s) := 1 - \frac{F^c(t+s)}{F^c(s)}$.

Individuals may migrate from one patch to another in any of the S-I-R stages. Assume that the migration rates depend on the patch and the stage of the epidemic (S-I-R), that is, an individual in patch ℓ in stage S (resp. stages I and R), migrates from patch ℓ to ℓ' with rates $\nu_{\ell,\ell'}^S$ (resp. $\nu_{\ell,\ell'}^I$ and $\nu_{\ell,\ell'}^R$). Note that for each $\ell \in \mathcal{L}$, $\nu_{\ell,\ell}^S = -\sum_{\ell':\ell' \neq \ell} \nu_{\ell,\ell'}^S$, and similarly, for $\nu_{\ell,\ell'}^I$ and $\nu_{\ell,\ell'}^R$. In order to follow the movement of the infected individuals, let $X_i^\ell(t)$ be the patch at time t of the i-th newly infected individual that becomes infected in patch ℓ. Then $X_i^\ell(t)$ is a Markov process associated with rates $\nu_{\ell,\ell'}^I$, $\ell' \in \mathcal{L}$, and let $p_{\ell,\ell'}(t) = \mathbb{P}(X_i^\ell(t) = \ell' | X_i^\ell(0) = \ell)$ for $\ell, \ell' \in \mathcal{L}$ and $t \geq 0$. Let $X_j^{0,\ell}(t)$ be the patch at time t of the j-th initially infected individual that was in patch ℓ at time 0. Assume that $\{X_j^{0,\ell}(t)\}_{j,\ell}$ has the same transition probability functions $p_{\ell,\ell'}(\cdot)$ as $\{X_i^\ell(t)\}_{i,\ell}$ for each $\ell \in \mathcal{L}$.

The aggregate infectivity in patch ℓ at time t is given by

$$\mathfrak{F}_\ell^N(t) = \sum_{\ell'=1}^{L} \sum_{j=1}^{I_{\ell'}^N(0)} \lambda_j^{0,\ell'}(\tilde{\tau}_{j,0}^{\ell',N}+t)\mathbf{1}_{X_j^{0,\ell'}(t)=\ell} + \sum_{\ell'=1}^{L} \sum_{i=1}^{A_{\ell'}^N(t)} \lambda_i^{\ell'}(t-\tau_i^{\ell',N})\mathbf{1}_{X_i^{\ell'}(t-\tau_i^{\ell',N})=\ell}\,. \tag{2.1}$$

We consider the following instantaneous infection rate function:

$$\Upsilon_\ell^N(t) = \frac{S_\ell^N(t)\sum_{\ell'=1}^L \beta_{\ell\ell'}\mathfrak{F}_{\ell'}^N(t)}{N^{1-\gamma}(B_\ell^N(t))^\gamma} = \left(\frac{B_\ell^N(t)}{N}\right)^{1-\gamma} \frac{S_\ell^N(t)}{B_\ell^N(t)} \sum_{\ell'=1}^{L} \beta_{\ell\ell'}\mathfrak{F}_{\ell'}^N(t)\,, \tag{2.2}$$

where the constants $\beta_{\ell\ell'} > 0$ represent the impact from patch ℓ' upon patch ℓ, and $\gamma \in [0,1]$. Assume that $\beta_{\ell\ell'} \le \beta^* < \infty$. When $\gamma = 0$, an infected individual in patch ℓ' encounters a susceptible individual from patch ℓ with a rate $S_\ell^N(t)/N$, and when $\gamma = 1$, that rate is equal to $\frac{S_\ell^N(t)}{B_\ell^N(t)}$. When $\gamma \in (0,1)$, the infection rate is a mix of the two extreme scenarios. We also refer the readers to [20, 9] for further discussions on such infection rate functions. Then we can write the counting process of newly infected individuals at patch ℓ as

$$A_\ell^N(t) = \int_0^t \int_0^\infty \mathbf{1}_{u \le \Upsilon_\ell^N(s^-)} Q_\ell(ds, du), \tag{2.3}$$

where Q_ℓ's are mutually independent standard Poisson random measures on $\mathbb{R}_+^2$.

The number of susceptible individuals in patch ℓ at each time t can be represented by

$$S_\ell^N(t) = S_\ell^N(0) - A_\ell^N(t) - \sum_{\ell'=1}^{L} P_{\ell,\ell'}^S\left(\nu_{\ell,\ell'}^S \int_0^t S_\ell^N(s)ds\right) + \sum_{\ell'=1}^{L} P_{\ell',\ell}^S\left(\nu_{\ell',\ell}^S \int_0^t S_{\ell'}^N(s)ds\right)\,, \tag{2.4}$$

where $P_{\ell,\ell'}^S$'s are mutually independent rate-1 Poisson processes.

The number of infected individuals in patch ℓ at each t that have been infected for less than or equal to $\mathfrak{a}$ can be represented by

$$\begin{aligned} \mathfrak{I}_\ell^N(t,\mathfrak{a}) &= \sum_{\ell'=1}^{L} \sum_{j=1}^{I_{\ell'}^N(0)} \mathbf{1}_{\eta_j^{0,\ell'}>t}\mathbf{1}_{\tilde{\tau}_{j,0}^{\ell',N} \le (\mathfrak{a}-t)^+}\mathbf{1}_{X_j^{0,\ell'}(t)=\ell} \\ &\quad + \sum_{\ell'=1}^{L} \sum_{i=A_{\ell'}^N((t-\mathfrak{a})^+)+1}^{A_{\ell'}^N(t)} \mathbf{1}_{\tau_i^{\ell',N}+\eta_i^{\ell'}>t}\mathbf{1}_{X_i^{\ell'}(t-\tau_i^{\ell',N})=\ell}\,. \end{aligned} \tag{2.5}$$

The number of infected individuals in patch ℓ at each t is then equal to

$$I_\ell^N(t) = \mathfrak{I}_\ell^N(t,\infty).$$

The number of recovered/removed individuals in patch ℓ at each time t can be represented by

$$\begin{aligned} R_\ell^N(t) &= R_\ell^N(0) + \sum_{\ell'=1}^{L} \sum_{j=1}^{I_{\ell'}^N(0)} \mathbf{1}_{\eta_j^{0,\ell'} \le t}\mathbf{1}_{X_j^{0,\ell'}(\eta_j^{0,\ell'})=\ell} + \sum_{\ell'=1}^{L} \sum_{i=1}^{A_{\ell'}^N(t)} \mathbf{1}_{\tau_i^{\ell',N}+\eta_i^{\ell'} \le t}\mathbf{1}_{X_i^{\ell'}(\eta_i^{\ell'})=\ell} \\ &\quad - \sum_{\ell'=1}^{L} P_{\ell,\ell'}^R\left(\nu_{\ell,\ell'}^R \int_0^t R_\ell^N(s)ds\right) + \sum_{\ell'=1}^{L} P_{\ell',\ell}^R\left(\nu_{\ell',\ell}^R \int_0^t R_{\ell'}^N(s)ds\right)\,, \end{aligned} \tag{2.6}$$

where $P_{\ell,\ell'}^R$'s are mutually independent rate-1 Poisson processes, independent of $P_{\ell,\ell'}^S$'s.

It is clear that the four processes $S_\ell^N, \mathfrak{F}_\ell^N, \mathfrak{I}_\ell^N, R_\ell^N$ describe the epidemic evolution dynamics of our model. We provide an alternative representation of $\mathfrak{I}_\ell^N(t,\mathfrak{a})$ in the following lemma.

Lemma 2.1. *We have*

$$\begin{aligned}\mathfrak{I}_\ell^N(t,\mathfrak{a}) &= \mathfrak{I}_\ell^N(0,(\mathfrak{a}-t)^+) - \sum_{\ell'=1}^L \sum_{j=1}^{\mathfrak{I}_{\ell'}^N(0,(\mathfrak{a}-t)^+)} \mathbf{1}_{\eta_j^{0,\ell'}\le t}\mathbf{1}_{X_j^{0,\ell'}(\eta_j^{0,\ell'})=\ell} \\ &\quad + A_\ell^N(t) - A_\ell^N((t-\mathfrak{a})^+) - \sum_{\ell'=1}^L \sum_{i=A_{\ell'}^N((t-\mathfrak{a})^+)+1}^{A_{\ell'}^N(t)} \mathbf{1}_{\tau_i^{\ell',N}+\eta_i^{\ell'}\le t}\mathbf{1}_{X_i^{\ell'}(\eta_i^{\ell'})=\ell} \\ &\quad - \sum_{\ell=1}^L \int_{(t-\mathfrak{a})^+}^t \int_0^\infty \mathbf{1}_{u\le \nu_{\ell,\ell'}^I \mathfrak{I}_\ell^N(s,\mathfrak{a}-(t-s))} Q_{\ell,\ell'}^I(ds,du) \\ &\quad + \sum_{\ell'=1}^L \int_{(t-\mathfrak{a})^+}^t \int_0^\infty \mathbf{1}_{u\le \nu_{\ell',\ell}^I \mathfrak{I}_{\ell'}^N(s,\mathfrak{a}-(t-s))} Q_{\ell',\ell}^I(ds,du)\,, \end{aligned} \tag{2.7}$$

where $Q_{\ell,\ell'}^I(ds,du)$, $\ell,\ell'\in\mathcal{L}$, are mutually independent standard PRMs on $\mathbb{R}_+^2$, independent of $P_{\ell,\ell'}^S$ and $P_{\ell,\ell'}^I$, $\ell,\ell'\in\mathcal{L}$.

Proof. Recall the expression of $\mathfrak{I}_\ell^N(t,\mathfrak{a})$ in (2.5). For the first term, we have in the summation over ℓ', if $\ell'=\ell$,

$$\sum_{j=1}^{\mathfrak{I}_\ell^N(0,(\mathfrak{a}-t)^+)} \mathbf{1}_{\eta_j^{0,\ell}>t}\mathbf{1}_{X_j^{0,\ell}(t)=\ell} = \mathfrak{I}_\ell^N(0,(\mathfrak{a}-t)^+) - \sum_{j=1}^{\mathfrak{I}_\ell^N(0,(\mathfrak{a}-t)^+)} \mathbf{1}_{\eta_j^{0,\ell}\le t}\mathbf{1}_{X_j^{0,\ell}(\eta_j^{0,\ell})=\ell} - \sum_{\ell':\ell'\ne\ell} Y_{\ell,\ell'}^{N,0}(t,\mathfrak{a})\,,$$

and if $\ell'\neq\ell$,

$$\sum_{j=1}^{\mathfrak{I}_{\ell'}^N(0,(\mathfrak{a}-t)^+)} \mathbf{1}_{\eta_j^{0,\ell'}>t}\mathbf{1}_{X_j^{0,\ell'}(t)=\ell} = Y_{\ell',\ell}^{N,0}(t,\mathfrak{a}) - \sum_{j=1}^{\mathfrak{I}_{\ell'}^N(0,(\mathfrak{a}-t)^+)} \mathbf{1}_{\eta_j^{0,\ell'}\le t}\mathbf{1}_{X_j^{0,\ell'}(\eta_j^{0,\ell'})=\ell}\,,$$

where $Y_{\ell',\ell}^{N,0}(t,\mathfrak{a})$ is the number of the initially infected individuals in patch ℓ' that are in patch ℓ at time $t\wedge\eta_j^{0,\ell'}$, for $j=1,\dots,\mathfrak{I}_{\ell'}^N(0,(\mathfrak{a}-t)^+)$.

Next, for the second term, we have in the summation, if $\ell'=\ell$,

$$\begin{aligned}&\sum_{i=A_\ell^N((t-\mathfrak{a})^+)+1}^{A_\ell^N(t)} \mathbf{1}_{\tau_i^{\ell,N}+\eta_i^\ell>t}\mathbf{1}_{X_i^\ell(t-\tau_i^{\ell,N})=\ell} \\ &= A_\ell^N(t) - A_\ell^N((t-\mathfrak{a})^+) - \sum_{i=A_\ell^N((t-\mathfrak{a})^+)+1}^{A_\ell^N(t)} \mathbf{1}_{\tau_i^{\ell,N}+\eta_i^\ell\le t}\mathbf{1}_{X_i^\ell(\eta_i^\ell)=\ell} - \sum_{\ell':\ell'\ne\ell} Y_{\ell,\ell'}^N(t,\mathfrak{a})\,,\end{aligned}$$

and if $\ell'\neq\ell$,

$$\sum_{i=A_{\ell'}^N((t-\mathfrak{a})^+)+1}^{A_{\ell'}^N(t)} \mathbf{1}_{\tau_i^{\ell',N}+\eta_i^{\ell'}>t}\mathbf{1}_{X_i^{\ell'}(t-\tau_i^{\ell',N})=\ell} = Y_{\ell',\ell}^N(t,\mathfrak{a}) - \sum_{i=A_{\ell'}^N((t-\mathfrak{a})^+)+1}^{A_{\ell'}^N(t)} \mathbf{1}_{\tau_i^{\ell',N}+\eta_i^{\ell'}\le t}\mathbf{1}_{X_i^{\ell'}(\eta_i^{\ell'})=\ell}\,,$$

where $Y_{\ell',\ell}^N(t,\mathfrak{a})$ is the number of the newly infected individuals in patch ℓ' that are in patch ℓ at time $t\wedge(\tau_i^{\ell',N}+\eta_i^{\ell'})$, for $i=A_{\ell'}^N((t-\mathfrak{a})^+)+1,\dots,A_{\ell'}^N(t)$.

We then observe that

$$\sum_{\ell'=1}^L \Big(Y_{\ell',\ell}^{N,0}(t,\mathfrak{a}) + Y_{\ell',\ell}^N(t,\mathfrak{a}) - Y_{\ell,\ell'}^{N,0}(t,\mathfrak{a}) - Y_{\ell,\ell'}^N(t,\mathfrak{a})\Big)$$

$$= \sum_{\ell'=1}^{L} \left(\int_{(t-\mathfrak{a})^+}^{t} \int_0^\infty \mathbf{1}_{u \le \nu^I_{\ell,\ell'} \mathfrak{I}_\ell^N(s,\mathfrak{a}-(t-s))} Q^I_{\ell,\ell'}(ds,du) - \int_{(t-\mathfrak{a})^+}^{t} \int_0^\infty \mathbf{1}_{u \le \nu^I_{\ell',\ell} \mathfrak{I}_{\ell'}^N(s,\mathfrak{a}-(t-s))} Q^I_{\ell',\ell}(ds,du) \right).$$

The interpretation of the identity is as follows. The left hand side counts the total number of individuals (initially and newly infected) that have migrated from all patches ℓ' into patch ℓ, minus those out of patch ℓ, but only the individuals with an infection age less than or equal to $(t-\mathfrak{a})^+$ at time t, or recovered by time t. The right hand side represents the same counts by using the processes $\mathfrak{I}_\ell^N(t,\mathfrak{a})$, but noting the $\mathfrak{I}_\ell^N(s,\mathfrak{a}-(t-s))$ inside the integral as the infection age evolves with s changes from $(t-\mathfrak{a})^+$ to t.

Combining the above, we obtain the representation of $\mathfrak{I}_\ell^N(t,\mathfrak{a})$ in the lemma. ☐

Throughout the paper, let $\mathbf{D} = \mathbf{D}(\mathbb{R}_+;\mathbb{R})$ denote the space of $\mathbb{R}$–valued càdlàg functions defined on $\mathbb{R}_+$. Convergence in $\mathbf{D}$ means convergence in the Skorohod J_1 topology, see Chapter 3 of [3]. Also, $\mathbf{D}^k$ stands for the k-fold product equipped with the product topology. Let $\mathbf{C}$ be the subset of $\mathbf{D}$ consisting of continuous functions. Let $\mathbf{D_D} = \mathbf{D}(\mathbb{R}_+;\mathbf{D}(\mathbb{R}_+;\mathbb{R}))$ be the $\mathbf{D}$-valued $\mathbf{D}$ space. In particular, the processes $\mathfrak{I}_\ell^N(t,\mathfrak{a})$ have sample paths in $\mathbf{D_D}$.

We define the LLN scaled processes $\bar{Z}^N = N^{-1}Z^N$ for any process Z. We first impose the following conditions on the initial quantities.

Assumption 2.1. *There exist deterministic continuous nondecreasing functions $\bar{\mathfrak{I}}_\ell(0,\cdot)$ on $\mathbb{R}_+$ with $\bar{\mathfrak{I}}_\ell(0,0)=0$ and constants $\bar{S}_\ell(0), \bar{R}_\ell(0) \in [0,1]$, $\ell \in \mathcal{L}$, such that*

$$(\bar{S}_\ell^N(0), \bar{\mathfrak{I}}_\ell^N(0,\cdot), \bar{R}_\ell^N(0))_{\ell\in\mathcal{L}} \to (\bar{S}_\ell(0), \bar{\mathfrak{I}}_\ell(0,\cdot), \bar{R}_\ell^N(0))_{\ell\in\mathcal{L}} \quad \text{in} \quad \mathbb{R}_+^L \times \mathbf{D}^L \times \mathbb{R}_+^L$$

in probability as $N\to\infty$. Let $\bar{I}_\ell(0) = \bar{\mathfrak{I}}_\ell(0,\bar{\mathfrak{a}})$ for each $\ell \in \mathcal{L}$. Then the convergence implies that $(\bar{I}_\ell^N(0), \ell\in\mathcal{L}) \to (\bar{I}_\ell^N(0), \ell\in\mathcal{L})$ in $\mathbb{R}_+^L$ in probability as $N\to\infty$. In addition, assume that $\sum_{\ell\in\mathcal{L}} \bar{I}_\ell(0) > 0$, $\sum_{\ell\in\mathcal{L}}(\bar{S}_\ell(0)+\bar{I}_\ell(0)+\bar{R}_\ell(0)) = 1$, and that the functions $\mathfrak{a} \mapsto \mathfrak{I}_\ell(0,\mathfrak{a})$ satisfy the following assumptions: $\mathfrak{I}_\ell(0,\bar{\mathfrak{a}}) = \mathfrak{I}(0,+\infty)$, for some $\bar{\mathfrak{a}}>0$, all $1\le\ell\le L$, and there exist constants $C,\alpha>0$ such that $\mathfrak{I}_\ell(0,\mathfrak{a}) - \mathfrak{I}(0,\mathfrak{a}-\delta) \le C\delta^\alpha$ for all $1\le\ell\le L$, $\mathfrak{a}\in[0,\bar{\mathfrak{a}}]$, $\delta>0$.

We then impose the following conditions on the random infectivity functions. Recall that $\{\lambda_j^{0,\ell}(\cdot)\}_{j,\ell}$ and $\{\lambda_i^\ell(\cdot)\}_{i,\ell}$ have the same law.

Assumption 2.2. *Let $\lambda(\cdot)$ be a process representing $\{\lambda_j^{0,\ell}(\cdot)\}_{j,\ell}$ and $\{\lambda_i^\ell(\cdot)\}_{i,\ell}$ with the same law. Assume that $\lambda(\cdot)\in\mathbf{D}$, and there exists a constant λ^* such that for each $T\in(0,\infty)$, $\sup_{t\in[0,T]}\lambda(t) \le \lambda^*$ a.s. Let $\bar{\lambda}(t) = \mathbb{E}[\lambda(t)]$ for $t\ge0$.*

Theorem 2.1. *Under Assumptions 2.1 and 2.2,*

$$(\bar{S}_\ell^N, \bar{\mathfrak{F}}_\ell^N, \bar{\mathfrak{I}}_\ell^N, \bar{R}_\ell^N)_{\ell\in\mathcal{L}} \to (\bar{S}_\ell, \bar{\mathfrak{F}}_\ell, \bar{\mathfrak{I}}_\ell, \bar{R}_\ell)_{\ell\in\mathcal{L}} \tag{2.8}$$

in probability, locally uniformly in t and $\mathfrak{a}$ as $N\to\infty$, where the limits are the unique continuous solution to the following set of integral equations, for $t,\mathfrak{a}\ge0$,

$$\bar{S}_\ell(t) = \bar{S}_\ell(0) - \int_0^t \bar{\Upsilon}_\ell(s)ds + \sum_{\ell'=1}^{L}\int_0^t \Big(\nu^S_{\ell',\ell}\bar{S}_{\ell'}(t) - \nu^S_{\ell,\ell'}\bar{S}_\ell(t)\Big)ds\,, \tag{2.9}$$

$$\bar{\mathfrak{F}}_\ell(t) = \sum_{\ell'=1}^{L} p_{\ell',\ell}(t)\int_0^{\bar{\mathfrak{a}}} \bar{\lambda}(\mathfrak{a}+t)\bar{\mathfrak{I}}_{\ell'}(0,d\mathfrak{a}) + \sum_{\ell'=1}^{L}\int_0^t \bar{\lambda}(t-s)p_{\ell',\ell}(t-s)\bar{\Upsilon}_{\ell'}(s)ds\,, \tag{2.10}$$

$$\begin{aligned}\bar{\mathfrak{I}}_\ell(t,\mathfrak{a}) &= \bar{\mathfrak{I}}_\ell(0,(\mathfrak{a}-t)^+) - \sum_{\ell'=1}^{L}\int_0^{(\mathfrak{a}-t)^+}\Big(\int_0^t p_{\ell',\ell}(u)F_0(du|y)\Big)\bar{\mathfrak{I}}_{\ell'}(0,dy)\\ &\quad + \int_{(t-\mathfrak{a})^+}^t \bar{\Upsilon}_\ell(s)ds - \sum_{\ell'=1}^{L}\int_{(t-\mathfrak{a})^+}^t\int_0^{t-s} p_{\ell',\ell}(u)F(du)\bar{\Upsilon}_{\ell'}(s)ds \end{aligned} \tag{2.11}$$

$$+\sum_{\ell'=1}^{L}\int_{(t-\mathfrak{a})^+}^{t}\Big(\nu^I_{\ell',\ell}\bar{\mathfrak{I}}_{\ell'}(s,\mathfrak{a}-(t-s))-\nu^I_{\ell,\ell'}\bar{\mathfrak{I}}_{\ell}(s,\mathfrak{a}-(t-s))\Big)ds\,,$$

$$\bar{R}_\ell(t)=\bar{R}_\ell(0)+\sum_{\ell'=1}^{L}\int_0^{\bar{\mathfrak{a}}}\Big(\int_0^t p_{\ell',\ell}(u)F_0(du|\mathfrak{a})\Big)\bar{\mathfrak{I}}_{\ell'}(0,d\mathfrak{a})$$
$$+\sum_{\ell'=1}^{L}\int_0^t\int_0^{t-s}p_{\ell',\ell}(u)F(du)\bar{\Upsilon}_{\ell'}(s)ds+\sum_{\ell'=1}^{L}\int_0^t\Big(\nu^R_{\ell',\ell}\bar{R}_{\ell'}(t)-\nu^R_{\ell,\ell'}\bar{R}_\ell(t)\Big)ds\,, \tag{2.12}$$

and

$$\bar{\Upsilon}_\ell(t)=\frac{\bar{S}_\ell(t)}{(\bar{B}_\ell(t))^\gamma}\sum_{\ell'=1}^{L}\beta_{\ell\ell'}\bar{\mathfrak{F}}_{\ell'}(t), \tag{2.13}$$

where $\bar{B}_\ell=\bar{S}_\ell+\bar{I}_\ell+\bar{R}_\ell$ *and* $\bar{I}_\ell(t)=\bar{\mathfrak{I}}_\ell(t,\infty)$*. In addition,* $(\bar{I}^N_\ell)_\ell\to(\bar{I}_\ell)_\ell$ *in* $\mathbf{D}^L$ *in probability where*

$$\bar{I}_\ell(t)=\bar{I}_\ell(0)-\sum_{\ell'=1}^{L}\int_0^{\bar{\mathfrak{a}}}\Big(\int_0^t p_{\ell',\ell}(u)F_0(du|\mathfrak{a})\Big)\bar{\mathfrak{I}}_{\ell'}(0,d\mathfrak{a})+\int_0^t\bar{\Upsilon}_\ell(s)ds$$
$$-\sum_{\ell'=1}^{L}\int_0^t\int_0^{t-s}p_{\ell',\ell}(u)F(du)\bar{\Upsilon}_{\ell'}(s)ds+\sum_{\ell'=1}^{L}\int_0^t\Big(\nu^I_{\ell',\ell}\bar{I}_{\ell'}(t)-\nu^I_{\ell,\ell'}\bar{I}_\ell(t)\Big)ds\,. \tag{2.14}$$

3. The PDE model

In this section we present the PDE model that is derived from the limiting integral equations. We assume that the distribution function F is absolutely continuous, with a density f. For the extension of the results of this section to general F, see Remark 3.3 below. Recall that $\{X^\ell_i(t)\}_{i,\ell}$ and $\{X^{0,\ell}_j(t)\}_{j,\ell}$ are the Markov processes representing the migrations of newly and initially infected individuals, and have the same law, with transition probability functions $p_{\ell,\ell'}(\cdot)$ and transition rates $\nu^I_{\ell,\ell'}$. For notational convenience, we use the Markov process $\{X(t):t\ge 0\}$ to represent a typical migration process in the infected compartment. Let $Q=(Q_{\ell,\ell'})$ denote the infinitesimal generator of $X(t)$, that is,

$$Q_{\ell,\ell'}=\begin{cases}\nu^I_{\ell,\ell'}, & \text{if } \ell'\neq\ell,\\ -\sum_{\ell'\neq\ell}\nu^I_{\ell,\ell'}, & \text{if } \ell'=\ell.\end{cases}$$

Then the transition probability function satisfies $p_{\ell,\ell'}(t)=\big(e^{Qt}\big)_{\ell,\ell'}$. We also define

$$P(t,\mathfrak{a})=\int_0^{t\wedge\mathfrak{a}}e^{Qs}\frac{f((\mathfrak{a}-t)^++s)}{F^c((\mathfrak{a}-t)^+)}ds$$
$$=\begin{cases}\int_0^t e^{Qs}\frac{f(\mathfrak{a}-t+s)}{F^c(\mathfrak{a}-t)}ds & \text{in case } t\le\mathfrak{a}\,,\\ \int_0^{\mathfrak{a}}e^{Qs}f(s)ds & \text{in case } \mathfrak{a}<t\,.\end{cases}$$

Note that the matrix $P(t,\mathfrak{a})$ commutes with the matrix Q, hence also with e^{Qs} for all $s\ge 0$.

Lemma 3.1. *If* $F^c(\mathfrak{a})>0$*, then the matrix* $I-P(t,\mathfrak{a})$ *is invertible.*

Proof. Let us first consider the case $\mathfrak{a}<t$. Let $A:=I-P(t,\mathfrak{a})=I-\int_0^{\mathfrak{a}}e^{Qs}f(s)ds$. It is clear that the diagonal entries of the matrix A are all >0, while the off-diagonal entries are all ≤ 0. Since $\sum_{\ell'}p_{\ell,\ell'}(s)=1$ for all s, denoting by $\mathbf{1}$ the L–dimensional vector whose entries are all 1, we have that all entries of the vector $A\mathbf{1}$ equal $F^c(\mathfrak{a})>0$. Let now $x\in\mathbb{R}^L$ be non zero, and let $1\le\ell\le L$

be such that $|x_\ell| = \max_{1\le i\le L}|x_i|$. We claim that $(Ax)_\ell \neq 0$, hence $Ax\neq 0$, and this is true for any $x\neq 0$, hence the result follows from the claim.

Assume w.l.o.g. that $x_\ell > 0$ (otherwise we consider $y=-x$).

$$\begin{aligned}(Ax)_\ell &= A_{\ell,\ell}x_\ell + \sum_{i\neq\ell} A_{\ell,i}x_i\\ &\ge A_{\ell,\ell}x_\ell + \sum_{i\neq\ell} A_{\ell,i}x_i^+\\ &\ge x_\ell(A\mathbf{1})_\ell > 0\,,\end{aligned}$$

where the first and second inequalities follow from the fact that the off-diagonal entries of A are non positive.

Now consider the case $t\le\mathfrak{a}$. We have for any $1\le\ell\le L$,

$$\begin{aligned}(A\mathbf{1})_\ell &= 1-\frac{\int_0^t f(\mathfrak{a}-s)ds}{F^c(\mathfrak{a}-t)}\\ &= \frac{F^c(\mathfrak{a})}{F^c(\mathfrak{a}-t)}\,,\end{aligned}$$

and the same argument as in the first case applies. □

Let $\bar{\mathfrak{i}}_\ell(t,\mathfrak{a}) = \frac{\partial}{\partial\mathfrak{a}}\bar{\mathfrak{I}}_\ell(t,\mathfrak{a})$ and consider $\bar{\mathfrak{i}}(t,\mathfrak{a}) = (\bar{\mathfrak{i}}_\ell(t,\mathfrak{a}))_{\ell\in\mathcal{L}}$ as a row vector.

Theorem 3.1. *Suppose that $\bar{\mathfrak{I}}_\ell(0,\cdot)$ is absolutely continuous with density $\bar{\mathfrak{i}}_\ell(0,\cdot)$ for each $\ell\in\mathcal{L}$, and that F is absolutely continuous with density f. Then $\bar{\mathfrak{i}}(t,\mathfrak{a})$ is the unique solution to the following PDE:*

$$\frac{\partial}{\partial t}\bar{\mathfrak{i}}(t,\mathfrak{a}) + \frac{\partial}{\partial\mathfrak{a}}\bar{\mathfrak{i}}(t,\mathfrak{a}) = -\frac{f(\mathfrak{a})}{F^c((\mathfrak{a}-t)^+)}\,\bar{\mathfrak{i}}(t,\mathfrak{a})(I-P(t,\mathfrak{a}))^{-1} + \bar{\mathfrak{i}}(t,\mathfrak{a})Q\,, \tag{3.1}$$

for $(t,\mathfrak{a})$ in $(0,\infty)^2$, where the initial condition is given by $\bar{\mathfrak{i}}(0,\mathfrak{a})$, and the boundary condition reads

$$\bar{\mathfrak{i}}_\ell(t,0) = \frac{\bar{S}_\ell(t)}{(\bar{B}_\ell(t))^\gamma}\sum_{\ell'=1}^L \beta_{\ell,\ell'}\left[\int_0^{t+\bar{\mathfrak{a}}}\bar{\lambda}(\mathfrak{a})\,\bar{\mathfrak{i}}(t,\mathfrak{a})(I-P(t,\mathfrak{a}))^{-1}du\right]_{\ell'}\,, \tag{3.2}$$

with $\bar{B}_\ell(t) = \bar{S}_\ell(t)+\bar{I}_\ell(t)+\bar{R}_\ell(t)$. Moreover, the PDE has a unique solution: for $t\le\mathfrak{a}$,

$$\bar{\mathfrak{i}}(t,\mathfrak{a}) = \bar{\mathfrak{i}}(0,\mathfrak{a}-t)(I-P(t,\mathfrak{a}))e^{Qt}\,, \tag{3.3}$$

and for $t>\mathfrak{a}$,

$$\bar{\mathfrak{i}}(t,\mathfrak{a}) = \bar{\mathfrak{i}}(t-\mathfrak{a},0)(I-P(t,\mathfrak{a}))e^{Q\mathfrak{a}}\,, \tag{3.4}$$

where the boundary condition $\bar{\mathfrak{i}}(t,0)$ is the first component of the unique solution to the following set of integral equations:

$$\begin{aligned}\bar{\mathfrak{i}}_\ell(t,0) &= \frac{\bar{S}_\ell(t)}{(\bar{B}_\ell(t))^\gamma}\sum_{\ell'=1}^L\beta_{\ell,\ell'}\bigg(\sum_{\ell''=1}^L p_{\ell'',\ell'}(t)\int_0^{\bar{\mathfrak{a}}}\bar{\lambda}(\mathfrak{a}+t)\,\bar{\mathfrak{i}}_{\ell''}(0,\mathfrak{a})d\mathfrak{a} \\ &\qquad + \sum_{\ell''=1}^L\int_0^t\bar{\lambda}(t-s)p_{\ell'',\ell'}(t-s)\,\bar{\mathfrak{i}}_{\ell''}(s,0)ds\bigg)\,,\end{aligned} \tag{3.5}$$

$$\bar{S}_\ell(t) = \bar{S}_\ell(0) - \int_0^t\bar{\mathfrak{i}}_\ell(s,0)ds + \sum_{\ell'=1}^L\int_0^t\Big(\nu^S_{\ell',\ell}\bar{S}_{\ell'}(t) - \nu^S_{\ell,\ell'}\bar{S}_\ell(t)\Big)ds\,, \tag{3.6}$$

$$\bar{I}_\ell(t) = \bar{I}_\ell(0) - \sum_{\ell'=1}^L\int_0^{\bar{\mathfrak{a}}}\bigg(\int_0^t p_{\ell',\ell}(u)\frac{f(u+\mathfrak{a})}{F^c(\mathfrak{a})}du\bigg)\bar{\mathfrak{i}}_{\ell'}(0,\mathfrak{a})d\mathfrak{a} + \int_0^t\bar{\mathfrak{i}}_\ell(s,0)ds$$

$$- \sum_{\ell'=1}^{L} \int_0^t \int_0^{t-s} p_{\ell',\ell}(u) f(u) du \, \bar{\mathfrak{i}}_\ell(s,0) ds + \sum_{\ell'=1}^{L} \int_0^t \Big(\nu^I_{\ell',\ell} \bar{I}_{\ell'}(t) - \nu^I_{\ell,\ell'} \bar{I}_\ell(t) \Big) ds \,, \tag{3.7}$$

$$\bar{R}_\ell(t) = \bar{R}_\ell(0) + \sum_{\ell'=1}^{L} \int_0^{\bar{\mathfrak{a}}} \left(\int_0^t p_{\ell',\ell}(u) \frac{f(u+\mathfrak{a})}{F^c(\mathfrak{a})} du \right) \bar{\mathfrak{i}}_{\ell'}(0,\mathfrak{a}) d\mathfrak{a}$$

$$+ \sum_{\ell'=1}^{L} \int_0^t \int_0^{t-s} p_{\ell',\ell}(u) f(u) du \, \bar{\mathfrak{i}}_\ell(s,0) ds + \sum_{\ell'=1}^{L} \int_0^t \Big(\nu^R_{\ell',\ell} \bar{R}_{\ell'}(t) - \nu^R_{\ell,\ell'} \bar{R}_\ell(t) \Big) ds \,. \tag{3.8}$$

Remark 3.1. *For $\mathfrak{a}$ such that $F^c(\mathfrak{a}) = 0$, we do not know whether the matrix $I - P(t,\mathfrak{a})$ is invertible. However, in that case we can assume w.l.o.g. that $f(\mathfrak{a}) = 0$, hence we replace $f(\mathfrak{a})(I - P(t,\mathfrak{a}))^{-1}$ by zero, and the PDE (3.1) makes sense, without any assumption on the support of the law of η. Similarly, in that case we can also assume $\bar{\lambda}(\mathfrak{a}) = 0$ and replace $\bar{\lambda}(\mathfrak{a})(I - P(t,\mathfrak{a}))^{-1}$ by zero, so that the boundary condition in the expression (3.2) makes sense.*

Remark 3.2. *Consider the particular case where $Q = 0$, i.e., $\nu^I_{\ell,\ell'} = 0$ for all ℓ, ℓ', where individuals (at least the infected individuals) do not move. In that case, the matrix $P(t,\mathfrak{a})$ is equal to a scalar times the identity matrix. The PDE in (3.1) simplifies to*

$$\frac{\partial}{\partial t} \bar{\mathfrak{i}}_\ell(t,\mathfrak{a}) + \frac{\partial}{\partial \mathfrak{a}} \bar{\mathfrak{i}}_\ell(t,\mathfrak{a}) = -\mu(\mathfrak{a}) \bar{\mathfrak{i}}_\ell(t,\mathfrak{a}), \tag{3.9}$$

for all $1 \le \ell \le L$, where $\mu(\mathfrak{a}) = f(\mathfrak{a})/F^c(\mathfrak{a})$ is the hazard rate function of the law of η. The formulas (3.3) and (3.4) reduce to

$$\bar{\mathfrak{i}}_\ell(t,\mathfrak{a}) = \frac{F^c(\mathfrak{a})}{F^c(\mathfrak{a}-t)} \bar{\mathfrak{i}}_\ell(0,\mathfrak{a}-t) \ \text{ if } t \le \mathfrak{a}, \quad \text{and} \quad \bar{\mathfrak{i}}_\ell(t,\mathfrak{a}) = F^c(\mathfrak{a}) \bar{\mathfrak{i}}_\ell(t-\mathfrak{a},0), \ \text{ if } \mathfrak{a} < t \,,$$

for all $1 \le \ell \le L$, which are exactly the formulas for the homogeneous model in our previous work [19]. We remark that although it is natural to use hazard rate function of the infectious duration in the formulation of the PDE models of epidemics with infection-age dependent infection as first proposed in [15], for the multi–patch model with migration, it does not seem possible to represent the PDE model in (3.1) via the hazard rate function.

Proof. We first derive the PDE model. Recall the expression of $\bar{\mathfrak{I}}_\ell(t,\mathfrak{a})$ in (2.11). Since both $\mathfrak{I}_\ell(0,\cdot)$ (for each ℓ) and F are absolutely continuous, then $\bar{\mathfrak{I}}_\ell(t,\mathfrak{a})$ is differentiable in t and $\mathfrak{a}$, and we have

$$\frac{\partial}{\partial t} \bar{\mathfrak{I}}_\ell(t,\mathfrak{a}) + \frac{\partial}{\partial \mathfrak{a}} \bar{\mathfrak{I}}_\ell(t,\mathfrak{a}) = - \sum_{\ell'=1}^{L} \int_0^{(\mathfrak{a}-t)^+} p_{\ell',\ell}(t) \frac{f(t+y)}{F^c(y)} \bar{\mathfrak{I}}_{\ell'}(0,dy) + \bar{\Upsilon}_\ell(t)$$
$$- \sum_{\ell'=1}^{L} \int_{(t-\mathfrak{a})^+}^t p_{\ell',\ell}(t-s) f(t-s) \bar{\Upsilon}_{\ell'}(s) ds + \sum_{\ell'=1}^{L} \left[\nu^I_{\ell',\ell} \bar{\mathfrak{I}}_{\ell'}(t,\mathfrak{a}) - \nu^I_{\ell,\ell'} \bar{\mathfrak{I}}_\ell(t,\mathfrak{a}) \right] . \tag{3.10}$$

We also note that for $0 < \mathfrak{a} < t$ and $\mathfrak{a}$ small, $\frac{\partial}{\partial \mathfrak{a}} \bar{\mathfrak{I}}_\ell(t,\mathfrak{a}) = \bar{\Upsilon}_\ell(t-\mathfrak{a}) + O(\mathfrak{a})$, and consequently, letting $\mathfrak{a} \to 0$, we deduce that

$$\bar{\Upsilon}_\ell(t) = \bar{\mathfrak{i}}_\ell(t,0) \,. \tag{3.11}$$

We differentiate (3.10) with respect to $\mathfrak{a}$, at least in the distributional sense, and deduce the following identify from the fact that $\frac{\partial}{\partial \mathfrak{a}} \frac{\partial}{\partial t} \bar{\mathfrak{I}}_\ell(t,\mathfrak{a}) = \frac{\partial}{\partial t} \frac{\partial}{\partial \mathfrak{a}} \bar{\mathfrak{I}}_\ell(t,\mathfrak{a})$:

$$\frac{\partial}{\partial t} \bar{\mathfrak{i}}_\ell(t,\mathfrak{a}) + \frac{\partial}{\partial \mathfrak{a}} \bar{\mathfrak{i}}_\ell(t,\mathfrak{a}) = -\mathbf{1}_{t \le \mathfrak{a}} \sum_{\ell'=1}^{L} p_{\ell',\ell}(t) \frac{f(\mathfrak{a})}{F^c(\mathfrak{a}-t)} \bar{\mathfrak{i}}_{\ell'}(0,\mathfrak{a}-t)$$

$$- \mathbf{1}_{\mathfrak{a}<t} \sum_{\ell'=1}^{L} p_{\ell',\ell}(\mathfrak{a}) f(\mathfrak{a}) \bar{\mathfrak{i}}_{\ell'}(t-\mathfrak{a},0) + \sum_{\ell'=1}^{L} \left[\nu^I_{\ell',\ell} \bar{\mathfrak{i}}_{\ell'}(t,\mathfrak{a}) - \nu^I_{\ell,\ell'} \bar{\mathfrak{i}}_{\ell}(t,\mathfrak{a})\right] . \quad (3.12)$$

We next obtain a relation between $\mathfrak{i}(t,\mathfrak{a})$ and $\mathfrak{i}(0,\mathfrak{a}-t)$ or $\mathfrak{i}(t-\mathfrak{a},0)$, depending upon whether $t \le \mathfrak{a}$ or $\mathfrak{a} < t$, which will lead to the expressions of $\bar{\mathfrak{i}}(t,\mathfrak{a})$ in (3.3) and (3.4). We start with the first case $t \le \mathfrak{a}$. For $0 \le s \le t$, by (3.12),

$$\begin{aligned} \frac{d}{ds}\bar{\mathfrak{i}}_\ell(s,\mathfrak{a}-t+s) &= \Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathfrak{a}}\Big)\bar{\mathfrak{i}}_\ell(s,\mathfrak{a}-t+s) \\ &= -\sum_{\ell'=1}^{L} p_{\ell',\ell}(s) \frac{f(\mathfrak{a}-t+s)}{F^c(\mathfrak{a}-t)} \bar{\mathfrak{i}}_{\ell'}(0,\mathfrak{a}-t) + (\bar{\mathfrak{i}}(s,\mathfrak{a}-t+s)Q)_\ell \,. \end{aligned}$$

The value at time $s=t$ of the solution of this linear system of ODEs is given by (3.3), that is

$$\bar{\mathfrak{i}}(t,\mathfrak{a}) = \bar{\mathfrak{i}}(0,\mathfrak{a}-t)(I-P(t,\mathfrak{a}))e^{Qt} \,.$$

We then consider the case $\mathfrak{a} < t$. For $0 \le s \le \mathfrak{a}$, by (3.12),

$$\begin{aligned} \frac{d}{ds}\bar{\mathfrak{i}}_\ell(t-\mathfrak{a}+s,s) &= \Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathfrak{a}}\Big)\bar{\mathfrak{i}}_\ell(t-\mathfrak{a}+s,s) \\ &= -\sum_{\ell'=1}^{L} p_{\ell',\ell}(s) f(s) \bar{\mathfrak{i}}_{\ell'}(t-\mathfrak{a},0) + (\bar{\mathfrak{i}}(s,\mathfrak{a}-t+s)Q)_\ell \,. \end{aligned}$$

The value at time $s=\mathfrak{a}$ of the solution of this linear system of ODEs is given by (3.4), that is,

$$\bar{\mathfrak{i}}(t,\mathfrak{a}) = \bar{\mathfrak{i}}(t-\mathfrak{a},0)(I-P(t,\mathfrak{a}))e^{Q\mathfrak{a}} \,.$$

Thus, by (3.12) and these two identities together with the invertibility of $I-P(t,\mathfrak{a})$ in Lemma 3.1 (see Remark 3.1 for the case where $F(\mathfrak{a})=1$), we obtain the PDE in (3.1).

We then derive the boundary condition. By (3.11) and (2.13), using (2.10), we obtain the boundary condition expression (3.5) for $\bar{\mathfrak{i}}_\ell(t,0)$:

$$\begin{aligned} \bar{\mathfrak{i}}_\ell(t,0) = \frac{\bar{S}_\ell(t)}{(\bar{B}_\ell(t))^\gamma} \sum_{\ell'=1}^{L} \beta_{\ell,\ell'} \Bigg(&\sum_{\ell''=1}^{L} p_{\ell'',\ell'}(t) \int_0^{\bar{\mathfrak{a}}} \bar{\lambda}(\mathfrak{a}+t)\,\bar{\mathfrak{i}}_{\ell''}(0,\mathfrak{a})d\mathfrak{a} \\ &+ \sum_{\ell''=1}^{L} \int_0^t \bar{\lambda}(t-s) p_{\ell'',\ell'}(t-s)\,\bar{\mathfrak{i}}_{\ell''}(s,0)ds \Bigg) . \end{aligned}$$

We rewrite the first integral on the right (in vector form) as follows

$$\begin{aligned} \int_0^{\bar{\mathfrak{a}}} \bar{\lambda}(\mathfrak{a}+t)\,\bar{\mathfrak{i}}(0,\mathfrak{a})e^{Qt}d\mathfrak{a} &= \int_t^{t+\bar{\mathfrak{a}}} \bar{\lambda}(u)\bar{\mathfrak{i}}(0,u-t)e^{Qt}du \\ &= \int_t^{t+\bar{\mathfrak{a}}} \bar{\lambda}(u)\bar{\mathfrak{i}}(t,u)(I-P(t,u))^{-1}du \,, \end{aligned}$$

where in the second equality we have used (3.3) and Lemma 3.1. We rewrite the second integral as follows

$$\begin{aligned} \int_0^t \bar{\lambda}(t-s)\bar{\mathfrak{i}}(s,0)e^{Q(t-s)}ds &= \int_0^t \bar{\lambda}(u)\bar{\mathfrak{i}}(t-u,0)e^{Qu}du \\ &= \int_0^t \bar{\lambda}(u)\bar{\mathfrak{i}}(t,u)(I-P(t,u))^{-1}du \,, \end{aligned}$$

where in the second equality we have used (3.4) and Lemma 3.1. From these we obtain the boundary condition expression of $\bar{\mathfrak{i}}_\ell(t,0)$ in (3.2).

In addition, the expressions of $\bar{S}_\ell(t)$ in (3.6), $\bar{I}_\ell(t)$ in (3.7) and $\bar{R}_\ell(t)$ in (3.8), are obtained from the equations in (2.9), (2.14) and (2.12) by replacing $\bar{\Upsilon}_\ell(t) = \bar{\mathfrak{i}}_\ell(t,0)$ and using the density $\bar{\mathfrak{i}}_\ell(0,\mathfrak{a})$.

It is then clear that existence and uniqueness of the solution to (3.1) follows from the existence and uniqueness of the solution to the system of equations satisfied by the boundary condition, $\bar{S}$, $\bar{I}$ and $\bar{R}$ (Lemma 3.2 below), as well as the explicit expressions of the PDEs (3.1) in both cases $\mathfrak{a} \geq t$ and $\mathfrak{a} < t$ in terms of the initial conditions and boundary conditions in (3.3) and (3.4). ☐

We next show that there exists a unique solution to the boundary conditions determined by the set of equations in (3.5)–(3.8).

Lemma 3.2. *The system of integral equations* (3.5), (3.6), (3.7) *and* (3.8) *has a unique solution in* $\mathbf{C}(\mathbb{R}_+;\mathbb{R}_+^{4L})$.

Proof. We consider the cases of $\gamma = 0$ and $\gamma \in (0,1]$ separately. When $\gamma = 0$, the set of equations reduces to the systems of linear Volterra equations of $\bar{\mathfrak{i}}_\ell(t,0)$ and $\bar{S}_\ell(t)$, that is,

$$\bar{\mathfrak{i}}_\ell(t,0) = \bar{S}_\ell(t)\sum_{\ell'=1}^{L}\beta_{\ell,\ell'}\Bigg(\sum_{\ell''=1}^{L}p_{\ell'',\ell'}(t)\int_0^{\bar{\mathfrak{a}}}\bar{\lambda}(\mathfrak{a}+t)\,\bar{\mathfrak{i}}_{\ell''}(0,\mathfrak{a})d\mathfrak{a} + \sum_{\ell''=1}^{L}\int_0^t\bar{\lambda}(t-s)p_{\ell'',\ell'}(t-s)\,\bar{\mathfrak{i}}_{\ell''}(s,0)ds\Bigg),$$

and

$$\bar{S}_\ell(t) = \bar{S}_\ell(0) - \int_0^t\bar{\mathfrak{i}}_\ell(s,0)ds + \sum_{\ell'=1}^{L}\int_0^t\Big(\nu^S_{\ell',\ell}\bar{S}_{\ell'}(t) - \nu^S_{\ell,\ell'}\bar{S}_\ell(t)\Big)ds\,.$$

Thus, the existence and uniqueness of a solution follow from the well known theory of linear Volterra integral equations (see, e.g., [5]).

We next consider the case $\gamma \in (0,1]$. Define

$$\bar{V}_\ell(t) = \bar{\mathfrak{i}}(t,0)\frac{(\bar{B}_\ell(t))^\gamma}{\bar{S}_\ell(t)}, \quad 1 \leq \ell \leq L,\ t \geq 0.$$

Let moreover

$$\begin{aligned}
f_\ell(t) &= \sum_{\ell',\ell''=1}^{L}\beta_{\ell,\ell'}p_{\ell'',\ell'}(t)\int_0^{\bar{\mathfrak{a}}}\bar{\lambda}(\mathfrak{a}+t)\,\bar{\mathfrak{i}}_{\ell''}(0,\mathfrak{a})d\mathfrak{a}\,,\\
g_\ell(t) &= \bar{I}_\ell(0) - \sum_{\ell'=1}^{L}\int_0^{\bar{\mathfrak{a}}}\left(\int_0^t p_{\ell',\ell}(u)\frac{f(u+\mathfrak{a})}{F^c(\mathfrak{a})}du\right)\bar{\mathfrak{i}}_{\ell'}(0,\mathfrak{a})d\mathfrak{a},\\
h_\ell(t) &= \bar{R}_\ell(0) + \sum_{\ell'=1}^{L}\int_0^{\bar{\mathfrak{a}}}\left(\int_0^t p_{\ell',\ell}(u)\frac{f(u+\mathfrak{a})}{F^c(\mathfrak{a})}du\right)\bar{\mathfrak{i}}_{\ell'}(0,\mathfrak{a})d\mathfrak{a},\\
H_{\ell,\ell'}(t) &= \sum_{\ell''=1}^{L}\beta_{\ell,\ell''}\bar{\lambda}(t)p_{\ell',\ell''}(t),\\
G_{\ell,\ell'}(t) &= \sum_{\ell'=1}^{L}\int_0^t p_{\ell',\ell}(u)f(u)du\,.
\end{aligned}$$

With these notations, the system of equations (3.5), (3.6), (3.7) and (3.8) can be rewritten as

$$\bar{V}_\ell(t) = f_\ell(t) + \sum_{\ell'=1}^{L}\int_0^t H_{\ell,\ell'}(t-s)\frac{\bar{S}_{\ell'}(s)}{(\bar{B}_{\ell'}(s))^\gamma}\bar{V}_{\ell'}(s)ds,$$

$$\bar{S}_\ell(t) = \bar{S}_\ell(0) - \int_0^t \frac{\bar{S}_\ell(s)}{(\bar{B}_\ell(s))^\gamma}\bar{V}_\ell(s)ds + \sum_{\ell'=1}^L \int_0^t \Big(\nu^S_{\ell',\ell}\bar{S}_{\ell'}(s) - \nu^S_{\ell,\ell'}\bar{S}_\ell(s)\Big)ds,$$

$$\bar{I}_\ell(t) = g_\ell(t) - \sum_{\ell'=1}^L \int_0^t G_{\ell,\ell'}(t-s)\frac{\bar{S}_{\ell'}(s)}{(\bar{B}_{\ell'}(s))^\gamma}\bar{V}_{\ell'}(s)ds + \sum_{\ell'=1}^L \int_0^t \Big(\nu^I_{\ell',\ell}\bar{I}_{\ell'}(s) - \nu^I_{\ell,\ell'}\bar{I}_\ell(s)\Big)ds,$$

$$\bar{R}_\ell(t) = h_\ell(t) + \sum_{\ell'=1}^L \int_0^t G_{\ell,\ell'}(t-s)\frac{\bar{S}_{\ell'}(s)}{(\bar{B}_{\ell'}(s))^\gamma}\bar{V}_{\ell'}(s)ds + \sum_{\ell'=1}^L \int_0^t \Big(\nu^R_{\ell',\ell}\bar{R}_{\ell'}(s) - \nu^R_{\ell,\ell'}\bar{R}_\ell(s)\Big)ds\,.$$

In order to deduce existence and uniqueness of a unique solution of this system of $4L$ equations from standard results on integral equations (see, e.g., [5]), it suffices to show that $\frac{\bar{S}_\ell(s)}{(\bar{S}_\ell(s)+\bar{I}_\ell(s)+\bar{R}_\ell(s))^\gamma}$ is a bounded and uniformly Lipschitz function of its three arguments.

By (3.6), (3.7) and (3.8), we have

$$\begin{aligned}
\bar{B}_\ell(t) &= \bar{B}_\ell(0) + \sum_{\ell'=1}^L \int_0^t \Big(\nu^S_{\ell',\ell}\bar{S}_{\ell'}(t) - \nu^S_{\ell,\ell'}\bar{S}_\ell(t)\Big)ds + \sum_{\ell'=1}^L \int_0^t \Big(\nu^I_{\ell',\ell}\bar{I}_{\ell'}(t) - \nu^I_{\ell,\ell'}\bar{I}_\ell(t)\Big)ds \\
&\qquad + \sum_{\ell'=1}^L \int_0^t \Big(\nu^R_{\ell',\ell}\bar{R}_{\ell'}(t) - \nu^R_{\ell,\ell'}\bar{R}_\ell(t)\Big)ds \\
&\geq \bar{B}_\ell(0) + \sum_{\ell'=1}^L \int_0^t \Big(\underline{\nu}_{\ell',\ell}\bar{B}_{\ell'}(t) - \bar{\nu}_{\ell,\ell'}\bar{B}_\ell(t)\Big)ds \\
&= \bar{B}_\ell(0) - \int_0^t \Big(\sum_{\ell'=1}^L \bar{\nu}_{\ell,\ell'}\Big)\bar{B}_\ell(t)ds + \sum_{\ell'=1}^L \int_0^t \underline{\nu}_{\ell',\ell}\bar{B}_{\ell'}(t)ds
\end{aligned}$$

where $\underline{\nu}_{\ell',\ell} = \min\{\nu^S_{\ell',\ell}, \nu^I_{\ell',\ell}, \nu^R_{\ell',\ell}\}$ and $\bar{\nu}_{\ell',\ell} = \max\{\nu^S_{\ell',\ell}, \nu^I_{\ell',\ell}, \nu^R_{\ell',\ell}\}$. From this we can deduce that there exists a constant $c_T > 0$ such that for each $0 \leq t \leq T$, $1 \leq \ell \leq L$, $\bar{B}_\ell(t) > c_T$. Then the boundedness and uniform Lipschitz properties follow easily. This completes the proof of the lemma. □

Remark 3.3. *We can follow a similar argument as in* [19] *for the homogeneous model to derive the PDE model when the distribution function F is not necessarily absolutely continuous. We replace $f(x)dx$ by $F(dx)$. Then the PDE in* (3.1) *becomes*

$$\frac{\partial}{\partial t}\bar{\mathfrak{i}}(t,\mathfrak{a}) + \frac{\partial}{\partial \mathfrak{a}}\bar{\mathfrak{i}}(t,\mathfrak{a}) = -\frac{F(d\mathfrak{a})}{F^c((\mathfrak{a}-t)^-)}\,\bar{\mathfrak{i}}(t,\mathfrak{a})(I-P(t,\mathfrak{a}))^{-1} + \bar{\mathfrak{i}}(t,\mathfrak{a})Q\,. \tag{3.13}$$

The boundary condition can be modified accordingly. We omit the details here.

4. Proof of the FLLN

4.1. **Convergence of $\bar{\mathfrak{F}}^N_\ell$.** Recall the expression of A^N_ℓ in (2.3) and the instantaneous infectivity rate function Υ^N_ℓ in (2.2). The process A^N_ℓ has the semimartingale decomposition

$$A^N_\ell(t) = M^N_{A,\ell}(t) + \int_0^t \Upsilon^N_\ell(s)ds, \tag{4.1}$$

where

$$M^N_{A,\ell}(t) = \int_0^t \int_0^\infty \mathbf{1}_{u \leq \Upsilon^N_\ell(s^-)}\overline{Q}_\ell(ds,du) \tag{4.2}$$

and $\overline{Q}_\ell(ds,du) := Q_\ell(ds,du) - dsdu$ is the compensated PRM associated with $Q_\ell(ds,du)$. It can be shown that $\{M^N_{A,\ell}(t) : t \geq 0\}$ is a square-integrable martingale with respect to the filtration $\{\mathcal{F}^N_t : t \geq 0\}$ where

$$\mathcal{F}^N_t := \sigma\Big\{S^N_\ell(0), I^N_\ell(0), R^N_\ell(0), \mathfrak{I}^N_\ell(0,\cdot), \{\lambda^{0,\ell}_j(\cdot)\}_{j\geq 1}, \{\lambda^\ell_i(\cdot)\}_{i\geq 1}, \ell \in \mathcal{L}\Big\}$$
$$\vee\, \sigma\Big\{\int_0^{t'}\int_0^\infty \mathbf{1}_{u\leq \Upsilon^N_\ell(s^-)} Q_\ell(ds,du) : t' \leq t, \ell \in \mathcal{L}\Big\}.$$

See, e.g., [6, Chapter VI]. The martingale $M^N_{A,\ell}(t)$ has a finite quadratic variation

$$\langle M^N_{A,\ell}\rangle(t) = \int_0^t \Upsilon^N_\ell(s)ds, \quad t \geq 0,$$

which satisfies

$$0 \leq \int_s^t \bar{\Upsilon}^N_\ell(u)du \leq \lambda^*\beta^*(t-s), \quad \text{w.p.1} \quad \text{for} \quad 0 \leq s \leq t. \tag{4.3}$$

Since $\langle \bar{M}^A_{N,\ell}\rangle(t) = N^{-1}\int_0^t \bar{\Upsilon}^N_\ell(s)ds \leq N^{-1}\lambda^*\beta^* t$, from Doob's inequality we deduce that locally uniformly in t,

$$\bar{M}^N_{A,\ell}(t) \to 0 \tag{4.4}$$

in probability as $N \to \infty$, and as a consequence, the following lemma holds (analogous as the proof of Lemma 4.1 in [8]).

Lemma 4.1. *For each $\ell \in \mathcal{L}$, the sequence $\{\bar{A}^N_\ell : N \in \mathbb{N}\}$ is tight in $\mathbf{D}$, and the limit of each convergent subsequence of $\{\bar{A}^N_\ell\}$, denoted by $\bar{A}_\ell$, satisfies*

$$\bar{A}_\ell = \lim_{N\to\infty} \bar{A}^N_\ell = \lim_{N\to\infty}\int_0^\cdot \bar{\Upsilon}^N_\ell(s)ds,$$

and

$$0 \leq \bar{A}_\ell(t) - \bar{A}_\ell(s) \leq \lambda^*\beta^*(t-s), \quad \textit{w.p.1} \quad \textit{for} \quad 0 \leq s \leq t.$$

It clearly follows from the last inequality that for each $\ell \in \mathcal{L}$, the measure whose distribution function is the increasing function $\bar{A}_\ell(t)$ is absolutely continuous with respect to Lebesgue's measure. In fact, since the sequence $\bar{\Upsilon}^N_\ell$ is bounded in $L^2(0,T)$ for any $T > 0$, the above converging subsequence can be chosen such that $\bar{\Upsilon}^N_\ell$ converges in law in $L^2(0,T)$ equipped with its weak topology. But we do not know yet that its limit is the function $\bar{\Upsilon}_\ell$ given by (2.13).

Recall $\mathfrak{F}^N_\ell(t)$ in (2.1). Let

$$\bar{\mathfrak{F}}^{N,0}_\ell(t) := N^{-1}\sum_{\ell'=1}^{L}\sum_{j=1}^{I^N_{\ell'}(0)} \lambda^{0,\ell'}_j(\tilde{\tau}^{\ell',N}_{j,0} + t)\mathbf{1}_{X^{0,\ell'}_j(t)=\ell}\,, \tag{4.5}$$

$$\bar{\mathfrak{F}}^{N,1}_\ell(t) := N^{-1}\sum_{\ell'=1}^{L}\sum_{i=1}^{A^N_{\ell'}(t)} \lambda^{\ell'}_i(t-\tau^{\ell',N}_i)\mathbf{1}_{X^{\ell'}_i(t-\tau^{\ell',N}_i)=\ell}\,. \tag{4.6}$$

Lemma 4.2. *Under Assumptions 2.1 and 2.2, along any convergent subsequence of $\{\bar{A}^N_\ell\}$ with the limit $\{\bar{A}_\ell\}$ for each $\ell \in \mathcal{L}$,*

$$\big(\bar{\mathfrak{F}}^N_\ell\big)_{\ell\in\mathcal{L}} \Rightarrow \big(\bar{\mathfrak{F}}_\ell\big)_{\ell\in\mathcal{L}} \quad \textit{in} \quad \mathbf{D}^L$$

as $N\to\infty$, where $\bar{\mathfrak{F}}_\ell = \bar{\mathfrak{F}}_\ell^0 + \bar{\mathfrak{F}}_\ell^1$ with

$$\bar{\mathfrak{F}}_\ell^0(t) := \sum_{\ell'=1}^L p_{\ell',\ell}(t)\int_0^{\bar{\mathfrak{a}}} \bar\lambda(\mathfrak{a}+t)\bar{\mathfrak{I}}_{\ell'}(0,d\mathfrak{a}), \quad t\ge 0, \tag{4.7}$$

and

$$\bar{\mathfrak{F}}_\ell^1(t) := \sum_{\ell'=1}^L \int_0^t \bar\lambda(t-s)p_{\ell',\ell}(t-s)d\bar A_\ell(s)\,, \quad t\ge 0. \tag{4.8}$$

Note that the limit $\bar{\mathfrak{F}}_\ell$ is not yet the same as that given in (2.10) since $\bar A_{\ell'}(ds)$ in (4.8) remains to be identified as $\bar\Upsilon_{\ell'}(s)ds$. So we are abusing the notation to use $\bar{\mathfrak{F}}_\ell$ in this lemma. The proof of this lemma follows from a slight modification of the proof approach in [9] to take into account the difference in the initial condition, which is omitted for brevity. We remark that the approach in [9] uses an argument adopted from the "propagation of chaos" for interacting particle systems [22] which requires only the conditions $\lambda(\cdot)\in\mathbf{D}$ a.s. and $\sup_{t\in[0,T]}\lambda(t)\le\lambda^*$ for any $T>0$, instead of the regularity conditions as stated in Assumption 2.1 in [8].

4.2. **Convergence of $(\bar S_\ell^N, \bar{\mathfrak{I}}_\ell^N, \bar R_\ell^N)_{\ell\in\mathcal{L}}$.** We first have the following representations of the LLN-scaled processes, from (2.1)–(2.7):

$$\begin{aligned}\bar S_\ell^N(t) &= \bar S_\ell^N(0) - \bar A_\ell^N(t) + \sum_{\ell'=1}^L\big(\bar M^N_{S,\ell',\ell}(t) - \bar M^N_{S,\ell,\ell'}(t)\big)\\ &\quad + \sum_{\ell'=1}^L\bigg(\nu^S_{\ell',\ell}\int_0^t \bar S^N_{\ell'}(s)ds - \nu^S_{\ell,\ell'}\int_0^t \bar S^N_\ell(s)ds\bigg),\end{aligned} \tag{4.9}$$

$$\begin{aligned}\bar{\mathfrak{I}}_\ell^N(t,\mathfrak{a}) &= \bar{\mathfrak{I}}_\ell^N(0,(\mathfrak{a}-t)^+) - \bar{\mathfrak{I}}_\ell^{N,0}(t,\mathfrak{a}) + \bar A_\ell^N(t) - \bar A_\ell^N(t-\mathfrak{a})^+) - \bar{\mathfrak{I}}_\ell^{N,1}(t,\mathfrak{a})\\ &\quad + \sum_{\ell'=1}^L\big(\bar M^N_{\mathfrak{I},\ell',\ell}(t,\mathfrak{a}) - \bar M^N_{\mathfrak{I},\ell,\ell'}(t,\mathfrak{a})\big)\\ &\quad + \sum_{\ell'=1}^L\bigg(\nu^I_{\ell',\ell}\int_{(t-\mathfrak{a})^+}^t \bar{\mathfrak{I}}^N_{\ell'}(s,\mathfrak{a}-(t-s))ds - \nu^I_{\ell,\ell'}\int_{(t-\mathfrak{a})^+}^t \bar{\mathfrak{I}}^N_{\ell}(s,\mathfrak{a}-(t-s))ds\bigg),\end{aligned} \tag{4.10}$$

$$\begin{aligned}\bar R_\ell^N(t) &= \bar R_\ell^N(0) + \bar R_\ell^{N,0}(t) + \bar R_\ell^{N,1}(t) + \sum_{\ell'=1}^L\big(\bar M^N_{R,\ell',\ell}(t) - \bar M^N_{R,\ell,\ell'}(t)\big)\\ &\quad + \sum_{\ell'=1}^L\bigg(\nu^R_{\ell',\ell}\int_0^t \bar R^N_{\ell'}(s)ds - \nu^R_{\ell,\ell'}\int_0^t \bar R^N_\ell(s)ds\bigg),\end{aligned} \tag{4.11}$$

and

$$\bar\Upsilon_\ell^N(t) = \frac{\bar S_\ell^N(t)}{(\bar B_\ell^N(t))^\gamma}\sum_{\ell'=1}^L \beta_{\ell\ell'}\bar{\mathfrak{F}}^N_{\ell'}(t)\,, \tag{4.12}$$

where $\bar B_\ell^N(t) = \bar S_\ell^N(t) + \bar I_\ell^N(t) + \bar S_\ell^N(t)$ with $\bar I_\ell^N(t) = \bar{\mathfrak{I}}_\ell^N(t,\infty)$, $\bar M^N_{A,\ell}(t)$ is given in (4.2),

$$\bar M^N_{Z,\ell,\ell'}(t) = \frac{1}{N}\left(P^Z_{\ell,\ell'}\left(\nu^Z_{\ell,\ell'}\int_0^t Z^N_\ell(s)ds\right) - \nu^Z_{\ell,\ell'}\int_0^t Z^N_\ell(s)ds\right)\,, \quad Z=S,R,$$

$$\bar M^N_{\mathfrak{I},\ell,\ell'}(t,\mathfrak{a}) = \frac{1}{N}\left(\int_{(t-\mathfrak{a})^+}^t\int_0^\infty \mathbf{1}_{u\le\nu^I_{\ell,\ell'}\mathfrak{I}^N_\ell(s,\mathfrak{a}-(t-s))}Q^I_{\ell,\ell'}(ds,du) - \nu^I_{\ell,\ell'}\int_{(t-\mathfrak{a})^+}^t \mathfrak{I}^N_{\ell'}(s,\mathfrak{a}-(t-s))ds\right)\,,$$

$$\bar{\mathfrak{I}}_\ell^{N,0}(t,\mathfrak{a}) = \frac{1}{N}\sum_{\ell'=1}^{L}\sum_{j=1}^{\mathfrak{I}_{\ell'}^N(0,(\mathfrak{a}-t)^+)} \mathbf{1}_{\eta_j^{0,\ell'}\le t}\mathbf{1}_{X_j^{0,\ell'}(\eta_j^{0,\ell'})=\ell}\,,$$

$$\bar{\mathfrak{I}}_\ell^{N,1}(t,\mathfrak{a}) = \frac{1}{N}\sum_{\ell'=1}^{L}\sum_{i=A_{\ell'}^N((t-\mathfrak{a})^+)+1}^{A_{\ell'}^N(t)} \mathbf{1}_{\tau_i^{\ell',N}+\eta_i^{\ell'}\le t}\mathbf{1}_{X_i^{\ell'}(\eta_i^{\ell'})=\ell}\,.$$

$$\bar{R}_\ell^{N,0}(t) = \frac{1}{N}\sum_{\ell'=1}^{L}\sum_{j=1}^{I_{\ell'}^N(0)} \mathbf{1}_{\eta_j^{0,\ell'}\le t}\mathbf{1}_{X_j^{0,\ell'}(\eta_j^{0,\ell'})=\ell} \;=\; \bar{\mathfrak{I}}_\ell^{N,0}(t,\infty)\,,$$

$$\bar{R}_\ell^{N,1}(t) = \frac{1}{N}\sum_{\ell'=1}^{L}\sum_{i=1}^{A_{\ell'}^N(t)} \mathbf{1}_{\tau_i^{\ell',N}+\eta_i^{\ell'}\le t}\mathbf{1}_{X_i^{\ell'}(\eta_i^{\ell'})=\ell} \;=\; \bar{\mathfrak{I}}_\ell^{N,1}(t,\infty)\,.$$

Lemma 4.3. *Under Assumptions 2.1 and 2.2, for each $\ell,\ell'\in\mathcal{L}$,*

$$\big(\bar{M}_{S,\ell,\ell'}^N(t), \bar{M}_{\mathfrak{I},\ell,\ell'}^N(t,\mathfrak{a}), \bar{M}_{R,\ell,\ell'}^N(t)\big) \to 0 \tag{4.13}$$

in probability, uniformly in t and $\mathfrak{a}$, as $N\to\infty$.

Proof. The process $\bar{M}_{S,\ell,\ell'}^N(t)$ is a square-integrable martingale with respect to the filtration:

$$\mathcal{F}_S^N(t) = \mathcal{F}_t^N \vee \sigma\bigg\{P_{\ell,\ell'}^S\left(\nu_{\ell,\ell'}^S\int_0^{t'} S_\ell^N(s)ds\right) : t'\le t,\ \ell,\ell'\in\mathcal{L}\bigg\},$$

with quadratic variation

$$\langle \bar{M}_{S,\ell,\ell'}^N\rangle(t) = \frac{1}{N}\nu_{\ell,\ell'}^S\int_0^t \bar{S}_\ell^N(s)ds \le \frac{1}{N}\nu_{\ell,\ell'}^S\sum_{\ell\in\mathcal{L}}\bar{S}_\ell^N(0)t\,,$$

which converges to zero in probability as $N\to\infty$. This implies that $\bar{M}_{S,\ell,\ell'}^N(t)\to 0$ locally uniformly in t in probability as $N\to\infty$. Similarly for $\bar{M}_{S,\ell,\ell'}^N\to 0$.

We next prove the convergence of $\bar{M}_{\mathfrak{I},\ell,\ell'}^N(t,\mathfrak{a})$. We apply Theorem 5.1. First, for each $t,\mathfrak{a}\ge 0$,

$$\mathbb{E}\left[\big(\bar{M}_{\mathfrak{I},\ell,\ell'}^N(t,\mathfrak{a})\big)^2\right] = \frac{1}{N}\nu_{\ell,\ell'}^I\mathbb{E}\int_{(t-\mathfrak{a})^+}^t \bar{\mathfrak{I}}_{\ell'}^N(s,\mathfrak{a}-(t-s))ds\,.$$

Observe that by (2.7), for each ℓ, and for each $t,\mathfrak{a}\ge 0$,

$$\bar{\mathfrak{I}}_\ell^N(t,\mathfrak{a}) \le \sum_{\ell'\in\mathcal{L}}\Big(\bar{\mathfrak{I}}_{\ell'}^N(0,(\mathfrak{a}-t)^+) + \bar{A}_{\ell'}^N(t) - \bar{A}_{\ell'}^N((t-\mathfrak{a})^+)\Big)\,. \tag{4.14}$$

So for $(t-\mathfrak{a})^+ < s < t$, we have

$$\bar{\mathfrak{I}}_{\ell'}^N(s,\mathfrak{a}-(t-s)) \le \sum_{\ell\in\mathcal{L}}\Big(\bar{\mathfrak{I}}_{\ell}^N(0,(\mathfrak{a}-t)^+) + \bar{A}_{\ell}^N(s) - \bar{A}_{\ell}^N((t-\mathfrak{a})^+)\Big)\,.$$

Hence

$$\begin{aligned}
\mathbb{E}\left[\big(\bar{M}_{\mathfrak{I},\ell,\ell'}^N(t,\mathfrak{a})\big)^2\right] &\le \frac{1}{N}\nu_{\ell,\ell'}^I\mathbb{E}\int_{(t-\mathfrak{a})^+}^t\sum_{\ell\in\mathcal{L}}\Big(\bar{\mathfrak{I}}_{\ell}^N(0,(\mathfrak{a}-t)^+) + \bar{A}_{\ell}^N(s) - \bar{A}_{\ell}^N((t-\mathfrak{a})^+)\Big)ds\\
&\le \frac{1}{N}\nu_{\ell,\ell'}^I\mathfrak{a}\,\mathbb{E}\sum_{\ell\in\mathcal{L}}\Big(\bar{\mathfrak{I}}_{\ell}^N(0,(\mathfrak{a}-t)^+) + \bar{A}_{\ell}^N(t) - \bar{A}_{\ell}^N((t-\mathfrak{a})^+)\Big)\\
&\to 0 \quad \text{as} \quad N\to\infty,
\end{aligned}$$

where the convergence follows by the conditions on $\bar{\mathfrak{I}}_\ell^N(0,\cdot)$ in Assumption 2.1 and the tightness of $\bar{A}_\ell^N$ in Lemma 4.1. Thus by Markov's inequality, for any $\epsilon>0$,

$$\sup_{t\in[0,T]}\sup_{\mathfrak{a}\in[0,T']}\mathbb{P}\big(\big|\bar{M}^N_{\mathfrak{I},\ell,\ell'}(t,\mathfrak{a})\big|>\epsilon\big)\to 0$$

as $N\to\infty$. (In fact, we can set $T'=T+\bar{\mathfrak{a}}$. Similarly in the following proofs.)

Then, we check the two requirements of condition (ii) in Theorem 5.1. For the first one, we show that for $\epsilon>0$, as $\delta\to 0$,

$$\limsup_N \sup_{t\in[0,T]}\frac{1}{\delta}\mathbb{P}\left(\sup_{w\in[0,\delta]}\sup_{\mathfrak{a}\in[0,T']}\big|\bar{M}^N_{\mathfrak{I},\ell,\ell'}(t+w,\mathfrak{a})-\bar{M}^N_{\mathfrak{I},\ell,\ell'}(t,\mathfrak{a})\big|>\epsilon\right)\to 0. \tag{4.15}$$

We have

$$\begin{aligned}
&\Big|\bar{M}^N_{\mathfrak{I},\ell,\ell'}(t+w,\mathfrak{a})-\bar{M}^N_{\mathfrak{I},\ell,\ell'}(t,\mathfrak{a})\Big|\\
&\le \frac{1}{N}\bigg|\int_t^{t+w}\int_0^\infty \mathbf{1}_{u\le \nu^I_{\ell,\ell'}\mathfrak{I}_\ell^N(s,\mathfrak{a}-(t+w-s))}\bar{Q}^I_{\ell,\ell'}(ds,du)\bigg|\\
&\quad+\frac{1}{N}\bigg|\int_{(t-\mathfrak{a})^+}^{(t+w-\mathfrak{a})^+}\int_0^\infty \mathbf{1}_{u\le \nu^I_{\ell,\ell'}\mathfrak{I}_\ell^N(s,\mathfrak{a}-(t+w-s))}\bar{Q}^I_{\ell,\ell'}(ds,du)\bigg|\\
&\quad+\frac{1}{N}\bigg|\int_{(t-\mathfrak{a})^+}^{t}\int_0^\infty \mathbf{1}_{\nu^I_{\ell,\ell'}\mathfrak{I}_\ell^N(s,\mathfrak{a}-(t+w-s))<u\le \nu^I_{\ell,\ell'}\mathfrak{I}_\ell^N(s,\mathfrak{a}-(t-s))}\bar{Q}^I_{\ell,\ell'}(ds,du)\bigg|.
\end{aligned}\tag{4.16}$$

The first term on the right of (4.16) satisfies

$$\begin{aligned}
\frac{1}{N}\bigg|\int_t^{t+w}\int_0^\infty \mathbf{1}_{u\le \nu^I_{\ell,\ell'}\mathfrak{I}_\ell^N(s,\mathfrak{a}-(t+w-s))}\bar{Q}^I_{\ell,\ell'}(ds,du)\bigg| &\le \frac{1}{N}\int_t^{t+w}\int_0^\infty \mathbf{1}_{u\le \nu^I_{\ell,\ell'}\mathfrak{I}_\ell^N(s,\mathfrak{a}-(t+w-s))}Q^I_{\ell,\ell'}(ds,du)\\
&\quad+\nu^I_{\ell,\ell'}\int_t^{t+w}\bar{\mathfrak{I}}_\ell^N(s,\mathfrak{a}-(t+w-s))ds,
\end{aligned}$$

hence

$$\begin{aligned}
&\sup_{w\in[0,\delta],\mathfrak{a}\in[0,T']}\frac{1}{N}\bigg|\int_t^{t+w}\int_0^\infty \mathbf{1}_{u\le \nu^I_{\ell,\ell'}\mathfrak{I}_\ell^N(s,\mathfrak{a}-(t+w-s))}\bar{Q}^I_{\ell,\ell'}(ds,du)\bigg|\\
&\qquad\le \sup_{\mathfrak{a}\in[0,T']}\frac{1}{N}\int_t^{t+\delta}\int_0^\infty \mathbf{1}_{u\le \nu^I_{\ell,\ell'}\mathfrak{I}_\ell^N(s,\mathfrak{a}-(t+\delta-s))}Q^I_{\ell,\ell'}(ds,du)+\nu^I_{\ell,\ell'}\delta\\
&\qquad\le \sup_{\mathfrak{a}\in[0,T']}\frac{1}{N}\bigg|\int_t^{t+\delta}\int_0^\infty \mathbf{1}_{u\le \nu^I_{\ell,\ell'}\mathfrak{I}_\ell^N(s,\mathfrak{a}-(t+\delta-s))}\bar{Q}^I_{\ell,\ell'}(ds,du)\bigg|+2\nu^I_{\ell,\ell'}\delta\,.
\end{aligned}$$

We note that for each fixed t, δ, N and ℓ,

$$Y^{N,\ell}_{\mathfrak{a}}:=\frac{1}{N}\int_t^{t+\delta}\int_0^{\nu^I_{\ell,\ell'}\mathfrak{I}_\ell^N(s,\mathfrak{a}-(t-s))}\overline{Q}^I_{\ell,\ell'}(ds,du)$$

is a $\mathcal{G}_a$–martingale, where

$$\mathcal{G}_a=\sigma\left\{\mathfrak{I}_\ell^N(s,\mathfrak{a}-t+s),t\le s\le t+\delta,\ Q^I_{\ell,\ell'}\Big|_{\mathcal{Q}^{N,\ell}_{t,t+\delta}}\right\},$$

with

$$\mathcal{Q}^{N,\ell}_{t,t+\delta}=\big\{(s,u),\ t\le s\le t+\delta,\ 0\le u\le \mathfrak{I}_\ell^N(s,\mathfrak{a}-t+s)\big\}\ .$$

Consequently from Doob's inequality,

$$\mathbb{E}\left[\sup_{\mathfrak{a}\in[0,T']}|Y^{N,\ell}_{\mathfrak{a}}|^2\right]\le 4\frac{\nu^I_{\ell,\ell'}}{N}\delta\,.$$

Finally

$$\begin{aligned}\limsup_N \frac{1}{\delta}\mathbb{P}&\left(\sup_{w\in[0,\delta],\mathfrak{a}\in[0,T']}\frac{1}{N}\left|\int_t^{t+w}\int_0^\infty \mathbf{1}_{u\le \nu^I_{\ell,\ell'}\mathfrak{I}^N_\ell(s,\mathfrak{a}-(t+w-s))}\bar{Q}^I_{\ell,\ell'}(ds,du)\right|>\epsilon\right)\\ &\le \epsilon^{-2}\limsup_N\frac{1}{\delta}\mathbb{E}\left(\sup_{w\in[0,\delta],\mathfrak{a}\in[0,T']}\frac{1}{N^2}\left|\int_t^{t+w}\int_0^\infty \mathbf{1}_{u\le \nu^I_{\ell,\ell'}\mathfrak{I}^N_\ell(s,\mathfrak{a}-(t+w-s))}\bar{Q}^I_{\ell,\ell'}(ds,du)\right|^2\right)\\ &\le 8(\nu^I_{\ell,\ell'})^2\delta\,,\end{aligned}$$

which tends to 0 as $\delta\to 0$, as required by (4.17).

The second term on the right of (4.16) is treated as the first term. We finally consider the third term. We have

$$\begin{aligned}\sup_{w\in[0,\delta],\mathfrak{a}\in[0,T']}&\frac{1}{N}\left|\int_{(t-\mathfrak{a})^+}^t\int_0^\infty \mathbf{1}_{\nu^I_{\ell,\ell'}\mathfrak{I}^N_\ell(s,\mathfrak{a}-(t+w-s))<u\le \nu^I_{\ell,\ell'}\mathfrak{I}^N_\ell(s,\mathfrak{a}-(t-s))}\bar{Q}^I_{\ell,\ell'}(ds,du)\right|\\ &\le \sup_{\mathfrak{a}\in[0,T']}\frac{1}{N}\int_0^t\int_0^\infty \mathbf{1}_{\nu^I_{\ell,\ell'}\mathfrak{I}^N_\ell(s,\mathfrak{a}-(t+\delta-s))<u\le \nu^I_{\ell,\ell'}\mathfrak{I}^N_\ell(s,\mathfrak{a}-(t-s))}Q^I_{\ell,\ell'}(ds,du)\\ &\quad+\nu^I_{\ell,\ell'}\sup_{\mathfrak{a}\in[0,T']}\int_0^t\left(\bar{\mathfrak{I}}^N_\ell(s,\mathfrak{a}-(t-s))-\bar{\mathfrak{I}}^N_\ell(s,\mathfrak{a}-(t+\delta-s))\right)ds\\ &\le \sup_{\mathfrak{a}\in[0,T']}\frac{1}{N}\left|\int_0^t\int_0^\infty \mathbf{1}_{\nu^I_{\ell,\ell'}\mathfrak{I}^N_\ell(s,\mathfrak{a}-(t+\delta-s))<u\le \nu^I_{\ell,\ell'}\mathfrak{I}^N_\ell(s,\mathfrak{a}-(t-s))}\bar{Q}^I_{\ell,\ell'}(ds,du)\right|\\ &\quad+2\nu^I_{\ell,\ell'}\sup_{\mathfrak{a}\in[0,T']}\int_0^t\left(\bar{\mathfrak{I}}^N_\ell(s,\mathfrak{a}-(t-s))-\bar{\mathfrak{I}}^N_\ell(s,\mathfrak{a}-(t+\delta-s))\right)ds\\ &\le \sup_{\mathfrak{a}\in[0,T']}\frac{1}{N}\left|\int_0^t\int_0^\infty \mathbf{1}_{u\le \nu^I_{\ell,\ell'}\mathfrak{I}^N_\ell(s,\mathfrak{a}-(t-s))}\bar{Q}^I_{\ell,\ell'}(ds,du)\right|\\ &\quad+\sup_{\mathfrak{a}\in[0,T']}\frac{1}{N}\left|\int_0^t\int_0^\infty \mathbf{1}_{u\le \nu^I_{\ell,\ell'}\mathfrak{I}^N_\ell(s,\mathfrak{a}-(t+\delta-s))}\bar{Q}^I_{\ell,\ell'}(ds,du)\right|\\ &\quad+2\nu^I_{\ell,\ell'}\sup_{\mathfrak{a}\in[0,T']}\int_0^t\left(\bar{\mathfrak{I}}^N_\ell(s,\mathfrak{a}-(t-s))-\bar{\mathfrak{I}}^N_\ell(s,\mathfrak{a}-(t+\delta-s))\right)ds\,.\end{aligned}$$

The two first terms of the last right hand side are resp. $\mathcal{G}'_a$ and $\mathcal{G}''_a$ martingales, for appropriate filtrations, by an argument similar to the one used above. And we conclude that the second moment of their sup over $a\in[0,T']$ tends to 0 as $N\to\infty$. It remains to treat the last term in the last right hand side. We have, for any $p\ge 1$, thanks to Assumption 2.1,

$$\begin{aligned}\limsup_N\frac{1}{\delta}\mathbb{P}&\left(\sup_{\mathfrak{a}\in[0,T']}\int_0^t\left(\bar{\mathfrak{I}}^N_\ell(s,\mathfrak{a}-(t-s))-\bar{\mathfrak{I}}^N_\ell(s,\mathfrak{a}-(t+\delta-s))\right)ds>\epsilon\right)\\ &\le\frac{1}{\delta}\left(\frac{Ct}{\epsilon}\right)^p\delta^{\alpha p},\end{aligned}$$

which tends to 0 as $\delta\to 0$, provided $p>\alpha^{-1}$.

For the second requirement of condition (ii) in Theorem 5.1, we show that for $\epsilon > 0$, as $\delta \to 0$,

$$\limsup_N \sup_{t\in[0,T]} \frac{1}{\delta}\mathbb{P}\left(\sup_{v\in[0,\delta]}\sup_{t\in[0,T]}\left|\bar{M}^N_{\mathfrak{I},\ell,\ell'}(t,\mathfrak{a}+v)-\bar{M}^N_{\mathfrak{I},\ell,\ell'}(t,\mathfrak{a})\right|>\varepsilon\right)\to 0. \tag{4.17}$$

We have

$$\begin{aligned}
&\left|\bar{M}^N_{\mathfrak{I},\ell,\ell'}(t,\mathfrak{a}+v)-\bar{M}^N_{\mathfrak{I},\ell,\ell'}(t,\mathfrak{a})\right|\\
&=\Bigg|\frac{1}{N}\left(\int_{(t-\mathfrak{a}-v)^+}^t\int_0^\infty \mathbf{1}_{u\le \nu^I_{\ell,\ell'}\mathfrak{I}^N_\ell(s,\mathfrak{a}+v-(t-s))}Q^I_{\ell,\ell'}(ds,du)-\nu^I_{\ell,\ell'}\int_{(t-\mathfrak{a}-v)^+}^t \mathfrak{I}^N_{\ell'}(s,\mathfrak{a}+v-(t-s))ds\right)\\
&\quad-\frac{1}{N}\left(\int_{(t-\mathfrak{a})^+}^t\int_0^\infty \mathbf{1}_{u\le \nu^I_{\ell,\ell'}\mathfrak{I}^N_\ell(s,\mathfrak{a}-(t-s))}Q^I_{\ell,\ell'}(ds,du)-\nu^I_{\ell,\ell'}\int_{(t-\mathfrak{a})^+}^t \mathfrak{I}^N_{\ell'}(s,\mathfrak{a}-(t-s))ds\right)\Bigg|\\
&\le \frac{1}{N}\int_{(t-\mathfrak{a}-v)^+}^t\int_0^\infty \mathbf{1}_{\nu^I_{\ell,\ell'}\mathfrak{I}^N_\ell(s,\mathfrak{a}-(t-s))<u\le \nu^I_{\ell,\ell'}\mathfrak{I}^N_\ell(s,\mathfrak{a}+v-(t-s))}Q^I_{\ell,\ell'}(ds,du)\\
&\quad+\frac{1}{N}\int_{(t-\mathfrak{a}-v)^+}^{(t-\mathfrak{a})^+}\int_0^\infty \mathbf{1}_{u\le \nu^I_{\ell,\ell'}\mathfrak{I}^N_\ell(s,\mathfrak{a}-(t-s))}Q^I_{\ell,\ell'}(ds,du)\\
&\quad+\nu^I_{\ell,\ell'}\int_{(t-\mathfrak{a}-v)^+}^{(t-\mathfrak{a})^+}\bar{\mathfrak{I}}^N_{\ell'}(s,\mathfrak{a}+v-(t-s))ds\,.
\end{aligned}$$

Clearly, the same arguments used to verify condition (i) allow us to conclude condition (ii) of Theorem 5.1. ☐

We next prove the convergence of the processes $\bar{\mathfrak{I}}^{N,0}_\ell(t,\mathfrak{a})$ and $\bar{\mathfrak{I}}^{N,1}_\ell(t,\mathfrak{a})$. We will only provide the detailed proof for the convergence of $\bar{\mathfrak{I}}^{N,1}_\ell(t,\mathfrak{a})$ since the proof of that of $\bar{\mathfrak{I}}^{N,0}_\ell(t,\mathfrak{a})$ follows the same steps with some modifications.

Lemma 4.4. *Under Assumptions 2.1 and 2.2, for each $\ell\in\mathcal{L}$,*

$$\bar{\mathfrak{I}}^{N,0}_\ell(t,\mathfrak{a})\to\bar{\mathfrak{I}}^0_\ell(t,\mathfrak{a}) \tag{4.18}$$

in probability, uniformly in t and $\mathfrak{a}$, as $N\to\infty$, where

$$\bar{\mathfrak{I}}^0_\ell(t,\mathfrak{a})=\sum_{\ell'=1}^L\int_0^{(\mathfrak{a}-t)^+}\left(\int_0^t p_{\ell',\ell}(u)F_0(du|y)\right)\bar{\mathfrak{I}}_{\ell'}(0,dy)\,. \tag{4.19}$$

Lemma 4.5. *Under Assumptions 2.1 and 2.2, for each $\ell\in\mathcal{L}$, along a convergent subsequence of $\bar{A}^N_\ell$ with limit $\bar{A}_\ell$, as $N\to\infty$,*

$$\bar{\mathfrak{I}}^{N,1}_\ell(t,\mathfrak{a})\Rightarrow\bar{\mathfrak{I}}^1_\ell(t,\mathfrak{a}) \tag{4.20}$$

for the topology of locally uniform convergence in t and $\mathfrak{a}$, where

$$\bar{\mathfrak{I}}^1_\ell(t,\mathfrak{a})=\sum_{\ell'=1}^L\int_{(t-\mathfrak{a})^+}^t\int_0^{t-s}p_{\ell',\ell}(u)F(du)d\bar{A}_{\ell'}(s)\,. \tag{4.21}$$

In fact we have the joint convergence $(\bar{A}^N_\ell(t),\bar{\mathfrak{I}}^{N,1}_\ell(t,\mathfrak{a}))\Rightarrow(\bar{A}_\ell(t),\bar{\mathfrak{I}}^1_\ell(t,\mathfrak{a}))$, for the topology of locally uniform convergence in t and $\mathfrak{a}$.

Proof. Define

$$\breve{\mathfrak{I}}^{N,1}_\ell(t,\mathfrak{a}):=\frac{1}{N}\sum_{\ell'=1}^L\sum_{i=A^N_{\ell'}((t-\mathfrak{a})^+)+1}^{A^N_{\ell'}(t)}\int_0^{t-\tau_i^{\ell',N}}p_{\ell',\ell}(u)F(du)$$

$$= \frac{1}{N}\sum_{\ell'=1}^{L}\int_{(t-\mathfrak{a})^+}^{t}\int_0^{t-s} p_{\ell',\ell}(u)F(du)d\bar{A}_{\ell'}^N(s)\,. \tag{4.22}$$

(Here the integral $\int_a^b$ stands for $\int_{(a,b]}$.) By Lemma 5.1, for each $t,\mathfrak{a}\geq 0$,

$$\breve{\mathfrak{I}}_\ell^{N,1}(t,\mathfrak{a}) \Rightarrow \bar{\mathfrak{I}}_\ell^1(t,\mathfrak{a}) \quad \text{as} \quad N\to\infty. \tag{4.23}$$

Then to show that the convergence $\breve{\mathfrak{I}}_\ell^{N,1}(t,\mathfrak{a}) \Rightarrow \bar{\mathfrak{I}}_\ell^1(t,\mathfrak{a})$ holds locally uniformly in t and $\mathfrak{a}$, it suffices to show that for any $\varepsilon>0$, there exists $\delta>0$ such that for any $t,\mathfrak{a}\geq 0$,

$$\limsup_N \mathbb{P}\left(\sup_{t\leq t'\leq t+\delta,\,\mathfrak{a}\leq\mathfrak{a}'\leq\mathfrak{a}+\delta}\left|\breve{\mathfrak{I}}_\ell^{N,1}(t,\mathfrak{a})-\breve{\mathfrak{I}}_\ell^{N,1}(t',\mathfrak{a}')\right|>\varepsilon\right)=0. \tag{4.24}$$

This follows from the second representation in (4.22), and the convergence of $\bar{A}_\ell^N$ in Lemma 4.1.

Next we consider the difference

$$\begin{aligned}
V^N(t,\mathfrak{a}) &:= \bar{\mathfrak{I}}_\ell^{N,1}(t,\mathfrak{a})-\breve{\mathfrak{I}}_\ell^{N,1}(t,\mathfrak{a})\\
&= \frac{1}{N}\sum_{\ell'=1}^{L}\sum_{i=A_{\ell'}^N((t-\mathfrak{a})^+)+1}^{A_{\ell'}^N(t)}\left(\mathbf{1}_{\tau_i^{\ell',N}+\eta_i^{\ell'}\leq t}\mathbf{1}_{X_i^{\ell'}(\eta_i^{\ell'})=\ell}-\int_0^{t-\tau_i^{\ell',N}}p_{\ell',\ell}(u)F(du)\right).
\end{aligned}$$

We apply Theorem 5.1 to show that

$$V^N(t,\mathfrak{a})\to 0$$

in probability in the topology of locally uniform convergence in t and $\mathfrak{a}$ as $N\to\infty$. For condition (i) in Theorem 5.1, we have

$$\begin{aligned}
\mathbb{P}(V^N(t,\mathfrak{a})>\epsilon) &\leq \frac{1}{\epsilon^2}\mathbb{E}\big[V^N(t,\mathfrak{a})^2\big]\\
&\leq \frac{2}{\epsilon^2 N}\sum_{\ell'=1}^{L}\mathbb{E}\left[\int_{(t-\mathfrak{a})^+}^{t}\int_0^{t-s}p_{\ell',\ell}(u)F(du)\left(1-\int_0^{t-s}p_{\ell',\ell}(u)F(du)\right)d\bar{A}_{\ell'}^N(s)\right]\\
&\leq \frac{2}{\epsilon^2 N}\sum_{\ell'=1}^{L}\mathbb{E}\left[\int_{(t-\mathfrak{a})^+}^{t}\int_0^{t-s}p_{\ell',\ell}(u)F(du)\bar{\Upsilon}_{\ell'}^N(s)ds\right]\\
&\leq \frac{2}{\epsilon^2 N}\lambda^*\beta^*\sum_{\ell'=1}^{L}\int_{(t-\mathfrak{a})^+}^{t}\int_0^{t-s}p_{\ell',\ell}(u)F(du)ds\,,
\end{aligned}$$

and thus,

$$\sup_{t\in[0,T],\,\mathfrak{a}\in[0,T']}\mathbb{P}(V^N(t,\mathfrak{a})>\epsilon)\to 0 \quad \text{as} \quad N\to\infty.$$

We then check the tightness requirements in condition (ii) of Theorem 5.1. For the first, we show that for $\epsilon>0$, as $\delta\to 0$,

$$\limsup_N \sup_{t\in[0,T]}\frac{1}{\delta}\mathbb{P}\left(\sup_{u\in[0,\delta]}\sup_{\mathfrak{a}\in[0,T']}\left|V^N(t+u,\mathfrak{a})-V^N(t,\mathfrak{a})\right|\right)\to 0. \tag{4.25}$$

We have

$$\begin{aligned}
&\left|V^N(t+u,\mathfrak{a})-V^N(t,\mathfrak{a})\right|\\
&=\left|\frac{1}{N}\sum_{\ell'=1}^{L}\sum_{i=A_{\ell'}^N((t+u-\mathfrak{a})^+)+1}^{A_{\ell'}^N(t+u)}\left(\mathbf{1}_{\tau_i^{\ell',N}+\eta_i^{\ell'}\leq t+u}\mathbf{1}_{X_i^{\ell'}(\eta_i^{\ell'})=\ell}-\int_0^{t+u-\tau_i^{\ell',N}}p_{\ell',\ell}(r)F(dr)\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{1}{N}\sum_{\ell'=1}^{L}\sum_{i=A^N_{\ell'}((t-\mathfrak{a})^+)+1}^{A^N_{\ell'}(t)} \bigg(\mathbf{1}_{\tau_i^{\ell',N}+\eta_i^{\ell'}\le t}\mathbf{1}_{X_i^{\ell'}(\eta_i^{\ell'})=\ell} - \int_0^{t-\tau_i^{\ell',N}} p_{\ell',\ell}(r)F(dr)\bigg)\bigg| \\
&= \bigg|\frac{1}{N}\sum_{\ell'=1}^{L}\sum_{i=A^N_{\ell'}((t-\mathfrak{a})^+)+1}^{A^N_{\ell'}(t+u)} \bigg[\bigg(\mathbf{1}_{\tau_i^{\ell',N}+\eta_i^{\ell'}\le t+u}\mathbf{1}_{X_i^{\ell'}(\eta_i^{\ell'})=\ell} - \int_0^{t+u-\tau_i^{\ell',N}} p_{\ell',\ell}(r)F(dr)\bigg) \\
&\qquad\qquad - \bigg(\mathbf{1}_{\tau_i^{\ell',N}+\eta_i^{\ell'}\le t}\mathbf{1}_{X_i^{\ell'}(\eta_i^{\ell'})=\ell} - \int_0^{t-\tau_i^{\ell',N}} p_{\ell',\ell}(r)F(dr)\bigg)\bigg] \\
&\quad - \frac{1}{N}\sum_{\ell'=1}^{L}\sum_{i=A^N_{\ell'}((t-\mathfrak{a})^+)+1}^{A^N_{\ell'}((t+u-\mathfrak{a})^+)} \bigg(\mathbf{1}_{\tau_i^{\ell',N}+\eta_i^{\ell'}\le t+u}\mathbf{1}_{X_i^{\ell'}(\eta_i^{\ell'})=\ell} - \int_0^{t+u-\tau_i^{\ell',N}} p_{\ell',\ell}(r)F(dr)\bigg) \\
&\quad + \frac{1}{N}\sum_{\ell'=1}^{L}\sum_{i=A^N_{\ell'}(t)+1}^{A^N_{\ell'}(t+u)} \bigg(\mathbf{1}_{\tau_i^{\ell',N}+\eta_i^{\ell'}\le t}\mathbf{1}_{X_i^{\ell'}(\eta_i^{\ell'})=\ell} - \int_0^{t-\tau_i^{\ell',N}} p_{\ell',\ell}(r)F(dr)\bigg)\bigg| \\
&\le \frac{1}{N}\sum_{\ell'=1}^{L}\sum_{i=A^N_{\ell'}((t-\mathfrak{a})^+)+1}^{A^N_{\ell'}(t+u)} \bigg|\mathbf{1}_{t<\tau_i^{\ell',N}+\eta_i^{\ell'}\le t+u}\mathbf{1}_{X_i^{\ell'}(\eta_i^{\ell'})=\ell} - \int_{t-\tau_i^{\ell',N}}^{t+u-\tau_i^{\ell',N}} p_{\ell',\ell}(r)F(dr)\bigg| \\
&\quad + \frac{1}{N}\sum_{\ell'=1}^{L}\sum_{i=A^N_{\ell'}((t-\mathfrak{a})^+)+1}^{A^N_{\ell'}((t+u-\mathfrak{a})^+)} \bigg|\mathbf{1}_{\tau_i^{\ell',N}+\eta_i^{\ell'}\le t+u}\mathbf{1}_{X_i^{\ell'}(\eta_i^{\ell'})=\ell} - \int_0^{t+u-\tau_i^{\ell',N}} p_{\ell',\ell}(r)F(dr)\bigg| \\
&\quad + \frac{1}{N}\sum_{\ell'=1}^{L}\sum_{i=A^N_{\ell'}(t)+1}^{A^N_{\ell'}(t+u)} \bigg|\mathbf{1}_{\tau_i^{\ell',N}+\eta_i^{\ell'}\le t}\mathbf{1}_{X_i^{\ell'}(\eta_i^{\ell'})=\ell} - \int_0^{t-\tau_i^{\ell',N}} p_{\ell',\ell}(r)F(dr)\bigg| \\
&\le \frac{1}{N}\sum_{\ell'=1}^{L}\sum_{i=A^N_{\ell'}((t-\mathfrak{a})^+)+1}^{A^N_{\ell'}(t+u)} \mathbf{1}_{t<\tau_i^{\ell',N}+\eta_i^{\ell'}\le t+u}\mathbf{1}_{X_i^{\ell'}(\eta_i^{\ell'})=\ell} \\
&\quad + \frac{1}{N}\sum_{\ell'=1}^{L}\sum_{i=A^N_{\ell'}((t-\mathfrak{a})^+)+1}^{A^N_{\ell'}(t+u)} \int_{t-\tau_i^{\ell',N}}^{t+u-\tau_i^{\ell',N}} p_{\ell',\ell}(r)F(dr) \\
&\quad + \sum_{\ell'=1}^{L}\Big(\bar{A}^N_{\ell'}(t+u-\mathfrak{a}) - \bar{A}^N_{\ell'}((t-\mathfrak{a})^+)\Big) + \sum_{\ell'=1}^{L}\Big(\bar{A}^N_{\ell'}(t+u) - \bar{A}^N_{\ell'}(t)\Big).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\mathbb{P}\bigg(\sup_{u\in[0,\delta]}\sup_{\mathfrak{a}\in[0,T']}\big|V^N(t+u,\mathfrak{a}) - V^N(t,\mathfrak{a})\big| > \epsilon\bigg) \\
&\quad \le \mathbb{P}\Bigg(\frac{1}{N}\sum_{\ell'=1}^{L}\sum_{i=A^N_{\ell'}((t-T')^+)+1}^{A^N_{\ell'}(t+\delta)} \mathbf{1}_{t<\tau_i^{\ell',N}+\eta_i^{\ell'}\le t+\delta}\mathbf{1}_{X_i^{\ell'}(\eta_i^{\ell'})=\ell} > \epsilon/3\Bigg)
\end{aligned}
$$

$$+\mathbb{P}\left(\frac{1}{N}\sum_{\ell'=1}^{L}\sum_{i=A_{\ell'}^N((t-T')^+)+1}^{A_{\ell'}^N(t+\delta)}\int_{t-\tau_i^{\ell',N}}^{t+\delta-\tau_i^{\ell',N}} p_{\ell',\ell}(r)F(dr)>\epsilon/3\right)$$

$$+2\mathbb{P}\left(\sup_{0\le t\le T}\sum_{\ell'=1}^{L}\left|\bar{A}_{\ell'}^N(t+\delta)-\bar{A}_{\ell'}^N(t)\right|>\epsilon/6\right). \tag{4.26}$$

For the first term, let $\{\widetilde{Q}_\ell(ds,du,dr,d\theta),\ 1\le\ell\le L\}$ denote a collection of i.i.d. PRM on $\mathbb{R}_+^3\times\mathcal{L}$ with mean measure $ds\times du\times F(dr)\times\mu_\ell(r,d\theta)$, where for each $r>0$, $\mu_\ell(r,\{\ell'\})=p_{\ell,\ell'}(r)$. We denote by $\overline{Q}_\ell(ds,du,dr,d\theta)$ be the compensated PRM associated to Q_ℓ, $1\le\ell\le L$. We have

$$\sum_{i=A_{\ell'}^N((t-T')^+)+1}^{A_{\ell'}^N(t+\delta)}\mathbf{1}_{t<\tau_i^{\ell',N}+\eta_i^{\ell'}\le t+\delta}\mathbf{1}_{X_i^{\ell'}(\eta_i^{\ell'})=\ell}=\int_{(t-T')^+}^{t+\delta}\int_0^\infty\int_{t-s}^{t+\delta-s}\int_{\{\ell\}}\mathbf{1}_{u\le\Upsilon_\ell^N(s^-)}\widetilde{Q}_{\ell'}(ds,du,dr,d\theta).$$

Thus, we have the first term

$$\begin{aligned}
&\mathbb{P}\left(\frac{1}{N}\sum_{\ell'=1}^{L}\sum_{i=A_{\ell'}^N((t-T')^+)+1}^{A_{\ell'}^N(t+\delta)}\mathbf{1}_{t<\tau_i^{\ell',N}+\eta_i^{\ell'}\le t+\delta}\mathbf{1}_{X_i^{\ell'}(\eta_i^{\ell'})=\ell}>\epsilon/3\right)\\
&\le 9\epsilon^{-2}\mathbb{E}\left[\left(\frac{1}{N}\sum_{\ell'=1}^{L}\int_{(t-T')^+}^{t+\delta}\int_0^\infty\int_{t-s}^{t+\delta-s}\int_{\{\ell\}}\mathbf{1}_{u\le\Upsilon_\ell^N(s^-)}\widetilde{Q}_{\ell'}(ds,du,dr,d\theta)\right)^2\right]\\
&\le 18\epsilon^{-2}\mathbb{E}\left[\frac{1}{N^2}\sum_{\ell'=1}^{L}\left(\int_{(t-T')^+}^{t+\delta}\int_0^\infty\int_{t-s}^{t+\delta-s}\int_{\{\ell\}}\mathbf{1}_{u\le\Upsilon_\ell^N(s^-)}\overline{Q}_{\ell'}(ds,du,dr,d\theta)\right)^2\right]\\
&\quad+18L\epsilon^{-2}\mathbb{E}\left[\frac{1}{N^2}\sum_{\ell'=1}^{L}\left(\int_{(t-T')^+}^{t+\delta}\int_{t-s}^{t+\delta-s}p_{\ell',\ell}(r)F(dr)\Upsilon_{\ell'}^N(s)ds\right)^2\right]\\
&=18\epsilon^{-2}\mathbb{E}\left[\frac{1}{N}\sum_{\ell'=1}^{L}\int_{(t-T')^+}^{t+\delta}\int_{t-s}^{t+\delta-s}p_{\ell',\ell}(r)F(dr)\bar{\Upsilon}_\ell^N(s)ds\right]\\
&\quad+18L\epsilon^{-2}\sum_{\ell'=1}^{L}\left(\int_{(t-T')^+}^{t+\delta}\int_{t-s}^{t+\delta-s}p_{\ell',\ell}(r)F(dr)\bar{\Upsilon}_{\ell'}^N(s)ds\right)^2\\
&\le 18\epsilon^{-2}\lambda^*\beta^*\frac{1}{N}\sum_{\ell'=1}^{L}\int_{(t-T')^+}^{t+\delta}\int_{t-s}^{t+\delta-s}p_{\ell',\ell}(r)F(dr)ds\\
&\quad+18L\epsilon^{-2}(\lambda^*\beta^*)^2\sum_{\ell'=1}^{L}\left(\int_{(t-T')^+}^{t+\delta}\int_{t-s}^{t+\delta-s}p_{\ell',\ell}(r)F(dr)ds\right)^2
\end{aligned}\tag{4.27}$$

where the first term on the right hand converges to zero as $N\to\infty$. It remains to consider the second term divided by δ. Each summand in the sum over ℓ' is bounded from above by (with $F(s)=0$ for $s<0$)

$$\begin{aligned}
\left(\int_0^{t+1}[F(t-s+\delta)-F(t-s)]ds\right)^2&=\left(\int_{-1}^{t}[F(s+\delta)-F(s)]ds\right)^2\\
&=\left(\int_0^{t+\delta}F(r)dr-\int_0^tF(s)ds\right)^2
\end{aligned}$$

$$\le \delta^2 .$$

We have shown that this term satisfies (4.25). Now for the second term on the right hand side of (4.26), we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{N}\sum_{\ell'=1}^{L}\sum_{i=A_{\ell'}^N((t-T')^+)+1}^{A_{\ell'}^N(t+\delta)}\int_{t-\tau_i^{\ell',N}}^{t+\delta-\tau_i^{\ell',N}} p_{\ell',\ell}(r)F(dr)\right)^2\right]\\
&\le L\mathbb{E}\left[\sum_{\ell'=1}^{L}\left(\int_{(t-T')^+}^{t+\delta}\int_{t-s}^{t+\delta-s} p_{\ell',\ell}(r)F(dr)dsd\bar{A}_{\ell'}^N(s)\right)^2\right]\\
&\le 2L\mathbb{E}\left[\sum_{\ell'=1}^{L}\left(\int_{(t-T')^+}^{t+\delta}\int_{t-s}^{t+\delta-s} p_{\ell',\ell}(r)F(dr)dsd\bar{M}_{A,\ell'}^N(s)\right)^2\right]\\
&\quad+2L\mathbb{E}\left[\sum_{\ell'=1}^{L}\left(\int_{(t-T')^+}^{t+\delta}\int_{t-s}^{t+\delta-s} p_{\ell',\ell}(r)F(dr)ds\bar{\Upsilon}_{\ell'}^N(s)ds\right)^2\right],
\end{aligned}$$

where the first term converges to zero as $N\to\infty$ by the convergence $\bar{M}_{A,\ell'}^N(s)\to 0$ in mean square, locally uniformly in t, and the second is estimated as the second term in (4.27). The third term on the right hand side of (4.26) satisfies (forgetting the sum over ℓ' for notational simplicity)

$$\mathbb{P}\left(\sup_{0\le t\le T}|\bar{A}_{\ell'}^N(t+\delta)-\bar{A}_{\ell'}^N(t)|>\epsilon'\right)\le\frac{1}{(\epsilon')^2}\mathbb{E}\left[\left(\sup_{0\le t\le T}\int_t^{t+\delta}\bar{\Upsilon}_{\ell'}^N(s)ds+2\sup_{0\le t\le T+\delta}|\bar{M}_{N,\ell'}^A(t)|\right)^2\right]$$

It follows readily from the bound on $\bar{\Upsilon}_{\ell'}^N$ and the properties of the sequence of martingales $\bar{M}_{N,\ell'}^A$ that

$$\limsup_N\frac{1}{\delta}\mathbb{P}\left(\sup_{0\le t\le T}|\bar{A}_{\ell'}^N(t+\delta)-\bar{A}_{\ell'}^N(t)|>\epsilon'\right)\le\frac{C}{\epsilon'^2}\delta .$$

Combining these results gives us the property in (4.25).

For the second condition in (ii) of Theorem 5.1, we show that for $\epsilon>0$, as $\delta\to 0$,

$$\limsup_N\sup_{\mathfrak{a}\in[0,T']}\frac{1}{\delta}\mathbb{P}\left(\sup_{v\in[0,\delta]}\sup_{t\in[0,T]}\left|V^N(t,\mathfrak{a}+v)-V^N(t,\mathfrak{a})\right|\right)\to 0. \tag{4.28}$$

We have

$$\begin{aligned}
&\left|V^N(t,\mathfrak{a}+v)-V^N(t,\mathfrak{a})\right|\\
&=\left|\frac{1}{N}\sum_{\ell'=1}^{L}\sum_{i=A_{\ell'}^N((t-\mathfrak{a}-v)^+)+1}^{A_{\ell'}^N((t-\mathfrak{a})^+)}\left(\mathbf{1}_{\tau_i^{\ell',N}+\eta_i^{\ell'}\le t}\mathbf{1}_{X_i^{\ell'}(\eta_i^{\ell'})=\ell}-\int_0^{t-\tau_i^{\ell',N}} p_{\ell',\ell}(r)F(dr)\right)\right|\\
&\le\sum_{\ell'=1}^{L}(\bar{A}_{\ell'}^N((t-\mathfrak{a})^+)-\bar{A}_{\ell'}^N((t-\mathfrak{a}-v)^+)).
\end{aligned}$$

Thus

$$\mathbb{P}\left(\sup_{v\in[0,\delta]}\sup_{t\in[0,T]}\left|V^N(t,\mathfrak{a}+v)-V^N(t,\mathfrak{a})\right|>\epsilon\right)$$

$$\leq \mathbb{P}\left(\sup_{v\in[0,\delta]}\sup_{t\in[0,T]}\sum_{\ell'=1}^{L}(\bar{A}_{\ell'}^N((t-\mathfrak{a})^+)-\bar{A}_{\ell'}^N((t-\mathfrak{a}-v)^+))>\varepsilon\right)$$

$$\leq \mathbb{P}\left(\sup_{t\in[0,T]}\sum_{\ell'=1}^{L}(\bar{A}_{\ell'}^N((t-\mathfrak{a})^+)-\bar{A}_{\ell'}^N((t-\mathfrak{a}-\delta)^+))>\varepsilon\right).$$

Then the claim in (4.28) follows the same argument as in the third term on the right hand side of (4.26). ☐

As an immediate consequence of Lemma 4.5, we obtain the following convergence results for $(\bar{R}_\ell^{N,0}(t),\bar{R}_\ell^{N,1}(t))$.

Corollary 4.1. *Under Assumptions 2.1 and 2.2, along a convergent subsequence of $\bar{A}_\ell^N$ with limit $\bar{A}_\ell$, for each $\ell\in\mathcal{L}$, $\bar{R}_\ell^{N,0}(t)\to\bar{R}_\ell^0(t)$ in probability, uniformly in t, and $\bar{R}_\ell^{N,1}(t)\Rightarrow\bar{R}_\ell^1(t)$ in $\mathbf{D}$, as $N\to\infty$, where*

$$\bar{R}_\ell^0(t)=\bar{\mathfrak{I}}_\ell^0(t,\infty)=\sum_{\ell'=1}^{L}p_{\ell',\ell}(t)\int_0^{\bar{\mathfrak{a}}}\left(1-\frac{F^c(t+\mathfrak{a})}{F^c(\mathfrak{a})}\right)\bar{\mathfrak{I}}_{\ell'}(0,dy)\,, \tag{4.29}$$

$$\bar{R}_\ell^1(t)=\bar{\mathfrak{I}}_\ell^1(t,\infty)=\sum_{\ell'=1}^{L}\int_0^t\int_0^{t-s}p_{\ell',\ell}(u)F(du)d\bar{A}_{\ell'}(s)\,. \tag{4.30}$$

Proof of the convergence of $(\bar{S}_\ell^N,\bar{\mathfrak{I}}_\ell^N,\bar{R}_\ell^N)_{\ell\in\mathcal{L}}$. We first consider the convergence provided with the convergent subsequence of $(\bar{A}_\ell^N)_{\ell\in\mathcal{L}}$ with the limit $(\bar{A}_\ell)_{\ell\in\mathcal{L}}$ in Lemma 4.1.

By (4.9), and by the convergence of $(\bar{M}_{S,\ell,\ell'}^N,\ell,\ell\in\mathcal{L})\to 0$ in Lemma 4.3 and that of $(\bar{S}_\ell^N(0),\ell\in\mathcal{L})\to(\bar{S}_\ell(0),\ell\in\mathcal{L})$ under Assumption 2.1, applying the continuous mapping theorem, we obtain the convergence of $(\bar{S}_\ell^N,\ell\in\mathcal{L})$ to $(\bar{S}_\ell^N,\ell\in\mathcal{L})$ in $\mathbf{D}^L$ as $N\to\infty$, where

$$\bar{S}_\ell(t)=\bar{S}_\ell(0)-\bar{A}_\ell(t)+\sum_{\ell'=1}^{L}\left(\nu_{\ell',\ell}^S\int_0^t\bar{S}_{\ell'}(s)ds-\nu_{\ell,\ell'}^S\int_0^t\bar{S}_\ell(s)ds\right). \tag{4.31}$$

We want to show the convergence of $(\bar{\mathfrak{I}}_\ell^N(t,\mathfrak{a}),\ell\in\mathcal{L})$ to $(\bar{\mathfrak{I}}_\ell(t,\mathfrak{a}),\ell\in\mathcal{L})$ locally uniformly in t and $\mathfrak{a}$ as $N\to\infty$, where

$$\begin{aligned}\bar{\mathfrak{I}}_\ell(t,\mathfrak{a})&=\bar{\mathfrak{I}}_\ell(0,(\mathfrak{a}-t)^+)-\bar{\mathfrak{I}}_\ell^0(t,\mathfrak{a})+\bar{A}_\ell(t)-\bar{A}_\ell((t-\mathfrak{a})^+)-\bar{\mathfrak{I}}_\ell^1(t,\mathfrak{a})\\&\quad+\sum_{\ell'=1}^{L}\left(\nu_{\ell',\ell}^I\int_{(t-\mathfrak{a})^+}^t\bar{\mathfrak{I}}_{\ell'}(s,\mathfrak{a}-(t-s))ds-\nu_{\ell,\ell'}^I\int_{(t-\mathfrak{a})^+}^t\bar{\mathfrak{I}}_\ell(s,\mathfrak{a}-(t-s))ds\right).\end{aligned} \tag{4.32}$$

We first deduce from (4.32) an explicit formula for $\bar{\mathfrak{I}}_\ell(t,\mathfrak{a})$ in terms of $\mathfrak{I}_\ell(0,\cdot)$, $\bar{\mathfrak{I}}_\ell^0(t,\mathfrak{a})$, $\bar{A}_\ell$ and $\bar{\mathfrak{I}}_\ell^1(t,\mathfrak{a})$. For that sake, we use again the matrix Q defined at the start of section 3.

Lemma 4.6. *The row vector $\{\bar{\mathfrak{I}}_\ell(t,\mathfrak{a}),\ 1\leq\ell\leq L,\ t\geq 0,\mathfrak{a}>0\}$ is given by the formula*

$$\begin{aligned}\bar{\mathfrak{I}}(t,\mathfrak{a})&=\bar{\mathfrak{I}}(0,(\mathfrak{a}-t)^+)-\bar{\mathfrak{I}}^0(t,\mathfrak{a})+\bar{A}(t)-\bar{A}((t-\mathfrak{a})^+)-\bar{\mathfrak{I}}^1(t,\mathfrak{a})\\&\quad+\int_{(t-\mathfrak{a})^+}^t\Big\{\bar{\mathfrak{I}}(0,(\mathfrak{a}-t)^+)-\bar{\mathfrak{I}}^0(s,\mathfrak{a}-t+s)\\&\qquad+\bar{A}(s)-\bar{A}((t-\mathfrak{a})^+)-\bar{\mathfrak{I}}^1(s,\mathfrak{a}-t+s)\Big\}e^{Q(t-s)}Qds\,.\end{aligned} \tag{4.33}$$

Proof. Equation (4.32) for all $t\geq 0$, $\mathfrak{a}\geq 0$ implies that for all $(t-\mathfrak{a})^+\leq s\leq t$, we have the following identity between row vectors

$$\bar{\mathfrak{I}}(s,\mathfrak{a}-t+s)=\bar{\mathfrak{I}}(0,(\mathfrak{a}-t)^+)-\bar{\mathfrak{I}}^0(s,\mathfrak{a}-t+s)+\bar{A}(s)-\bar{A}((t-\mathfrak{a})^+)-\bar{\mathfrak{I}}^1(s,\mathfrak{a}-t+s)$$

$$+\int_{(t-\mathfrak{a})^+}^{s}\bar{\mathfrak{I}}(r,\mathfrak{a}-t+r)Qdr\,.$$

It follows that $\bar{\mathfrak{I}}(t,\mathfrak{a})$ is the value at time $s=t$ of the solution to the system of linear ODEs:

$$x(s)=f(s)+\int_{(t-\mathfrak{a})^+}^{s}x(r)Qdr, \tag{4.34}$$

where, for $1\le \ell\le L$,

$$f_\ell(s)=\bar{\mathfrak{I}}_\ell(0,(\mathfrak{a}-t)^+)-\bar{\mathfrak{I}}^0_\ell(s,\mathfrak{a}-t+s)+\bar{A}_\ell(s)-\bar{A}_\ell((t-\mathfrak{a})^+)-\bar{\mathfrak{I}}^1_\ell(s,\mathfrak{a}-t+s)\,.$$

Formula (4.33) now follows readily from the explicit formula for the solution of the linear ODE (4.34). □

Comparing (4.10) and (4.32), we deduce that the row vector $\bar{\mathfrak{I}}^N(t,\mathfrak{a})$ is given by an analog of formula (4.33), namely

$$\begin{aligned}\bar{\mathfrak{I}}^N(t,\mathfrak{a})&=\bar{\mathfrak{I}}^N(0,(\mathfrak{a}-t)^+)-\bar{\mathfrak{I}}^{N,0}(t,\mathfrak{a})+\bar{A}^N(t)-\bar{A}^N((t-\mathfrak{a})^+)-\bar{\mathfrak{I}}^{N,1}(t,\mathfrak{a})+\bar{\mathcal{M}}^N(t,\mathfrak{a})\\&\quad+\int_{(t-\mathfrak{a})^+}^{t}\Big\{\bar{\mathfrak{I}}^N(0,(\mathfrak{a}-t)^+)-\bar{\mathfrak{I}}^{N,0}(s,\mathfrak{a}-t+s)+\bar{A}^N(s)-\bar{A}^N((t-\mathfrak{a})^+)\\&\qquad-\bar{\mathfrak{I}}^{N,1}(s,\mathfrak{a}-t+s)+\bar{\mathcal{M}}^N(s,\mathfrak{a}-t+s)\Big\}Qe^{Q(t-s)}ds,\end{aligned} \tag{4.35}$$

where

$$\bar{\mathcal{M}}^N_\ell(t,\mathfrak{a})=\sum_{\ell'=1}^{L}\Big(\bar{M}^N_{\mathfrak{I},\ell',\ell}(t,\mathfrak{a})-\bar{M}^N_{\mathfrak{I},\ell,\ell'}(t,\mathfrak{a})\Big)\,.$$

Comparing (4.35) and (4.33), it now follows from Assumption 2.1, Lemma 4.1, Lemma 4.3, Lemma 4.4 and Lemma 4.5 that $\bar{\mathfrak{I}}^N(t,\mathfrak{a})\Rightarrow\bar{\mathfrak{I}}(t,\mathfrak{a})$ for the topology of locally uniform convergence in t and $\mathfrak{a}$.

As a consequence, letting $\bar{I}_\ell(t)=\bar{\mathfrak{I}}_\ell(t,\infty)$, we also get the weak convergence of $(\bar{I}^N_\ell,\ell\in\mathcal{L})$ to $(\bar{I}_\ell,\ell\in\mathcal{L})$ locally uniformly in t as $N\to\infty$, where

$$\bar{I}_\ell(t)=\bar{I}_\ell(0)+\bar{A}_\ell(t)-\bar{R}^0_\ell(t)-\bar{R}^1_\ell(t)+\sum_{\ell'=1}^{L}\int_0^t\Big(\nu^I_{\ell',\ell}\bar{I}_{\ell'}(t)-\nu^I_{\ell,\ell'}\bar{I}_\ell(t)\Big)ds\,. \tag{4.36}$$

Then by (4.11), and by the convergence of $(\bar{M}^N_{R,\ell,\ell'},\ell,\ell'\in\mathcal{L})\to 0$ in Lemma 4.3, of $(\bar{R}^{N,0}_\ell,\bar{R}^{N,1}_\ell,\ell\in\mathcal{L})\to(\bar{R}^0_\ell,\bar{R}^1_\ell,\ell\in\mathcal{L})$ in Corollary 4.1, and that of $(\bar{R}^N_\ell(0),\ell\in\mathcal{L})\to(\bar{R}_\ell(0),\ell\in\mathcal{L})$ under Assumption 2.1, applying the continuous mapping theorem, we obtain the convergence of $(\bar{R}^N_\ell,\ell\in\mathcal{L})$ to $(\bar{R}^N_\ell,\ell\in\mathcal{L})$ in $\mathbf{D}^L$ as $N\to\infty$, where

$$\bar{R}_\ell(t)=\bar{R}_\ell(0)+\bar{R}^0_\ell(t)+\bar{R}^1_\ell(t)+\sum_{\ell'=1}^{L}\left(\nu^R_{\ell',\ell}\int_0^t\bar{R}_{\ell'}(s)ds-\nu^R_{\ell,\ell'}\int_0^t\bar{R}_\ell(s)ds\right). \tag{4.37}$$

Next we identify the limit $(\bar{A}_\ell,\ell\in\mathcal{L})$ in terms of the limits $(\bar{\mathfrak{F}}_\ell,\bar{S}_\ell,\bar{I}_\ell,\bar{R}_\ell,\ell\in\mathcal{L})$ and let $\bar{B}_\ell=\bar{S}_\ell+\bar{I}_\ell+\bar{R}_\ell$. By (4.31), (4.36) and (4.37), it can be easily shown that there exists a constant $C^T_\ell>0$ such that for each $0\le t\le T$, $\bar{S}_\ell(t)+\bar{I}_\ell(t)+\bar{R}_\ell(t)\ge c^T_\ell$. (Similar to Lemma 4.10 in [20]). The mapping from $(\bar{S}_\ell(t),\bar{I}_\ell(t),\bar{R}_\ell(t),\sum_{\ell'=1}^{L}\beta_{\ell',\ell}\bar{\mathfrak{F}}_{\ell'})$ to $\bar{\Upsilon}_\ell(t)$ is continuous in the Skorohod topology whenever $\bar{S}_\ell(t)+\bar{I}_\ell(t)+\bar{R}_\ell(t)>0$. Then we obtain the convergence

$$\bar{\Upsilon}^N_\ell(t)=\frac{\bar{S}^N_\ell(t)\sum_{\ell'=1}^{L}\beta_{\ell',\ell}\bar{\mathfrak{F}}^N_{\ell'}}{(\bar{S}^N_\ell(t)+\bar{I}^N_\ell(t)+\bar{R}^N_\ell(t))^\gamma}\Rightarrow\bar{\Upsilon}_\ell(t):=\frac{\bar{S}_\ell(t)\sum_{\ell'=1}^{L}\beta_{\ell',\ell}\bar{\mathfrak{F}}_{\ell'}}{(\bar{S}_\ell(t)+\bar{I}_\ell(t)+\bar{R}_\ell(t))^\gamma}\,,$$

in $\mathbf{D}$ as $N \to \infty$. Then by Lemma 4.1, we obtain the convergence of $(\bar{A}_\ell^N, \ell \in \mathcal{L})$ to $(\bar{A}_\ell, \ell \in \mathcal{L})$ in $\mathbf{D}^L$, where

$$\bar{A}_\ell(t) = \int_0^t \bar{\Upsilon}_\ell(s) ds \,,$$

with $\bar{\Upsilon}_\ell(s)$ given above. Since all converging sub-sequences have the same limit, which is deterministic, we have the convergence in probability of the whole sequence. This completes the proof. □

5. Appendix

The following theorem was stated in Theorem 5.1 in [19]. It extends the Corollary on page 83 of [3], and also Theorem 3.5.1 in Chapter 6 of [16] in the space $\mathbf{C}([0,1]^k, \mathbb{R})$.

Theorem 5.1. *Let $\{X^N : N \geq 1\}$ be a sequence of random elements in $\mathbf{D}_\mathbf{D}$. If the following two conditions are satisfied: for any $T, S > 0$,*

(i) *for any $\epsilon > 0$, $\sup_{t\in[0,T]} \sup_{s\in[0,S]} \mathbb{P}\big(|X^N(t,s)| > \epsilon\big) \to 0$ as $N \to \infty$, and*
(ii) *for any $\epsilon > 0$, as $\delta \to 0$,*

$$\limsup_{N\to\infty} \sup_{t\in[0,T]} \frac{1}{\delta} \mathbb{P}\left(\sup_{u\in[0,\delta]} \sup_{s\in[0,S]} |X^N(t+u,s) - X^N(t,s)| > \epsilon \right) \to 0,$$
$$\limsup_{N\to\infty} \sup_{s\in[0,S]} \frac{1}{\delta} \mathbb{P}\left(\sup_{v\in[0,\delta]} \sup_{t\in[0,T]} |X^N(t,s+v) - X^N(t,s)| > \epsilon \right) \to 0,$$

then $X^N(t,s) \to 0$ in probability, locally uniformly in t and s, as $N \to \infty$.

The following lemma was stated in Lemma 5.1 in [19]. The spaces $\mathbf{D}_\uparrow$ and $\mathbf{C}_\uparrow$ are the subspaces of $\mathbf{D}$ and $\mathbf{C}$ of increasing functions.

Lemma 5.1. *Let $f \in \mathbf{D}$ and $\{g_N\}_{N\geq 1}$ be a sequence of elements of $\mathbf{D}_\uparrow$ which is such that $g_N \to g$ locally uniformly, where $g \in \mathbf{C}_\uparrow$. Then for any $T > 0$,*

$$\int_{[0,T]} f(t) g_N(dt) \to \int_{[0,T]} f(t) g(dt) \,.$$

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