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Functional law of large numbers and central limit theorem for Crump–Mode–Jagers branching processes

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ABSTRACT

We establish a Law of Large Numbers and a Central Limit Theorem for a class of Crump–Mode–Jagers continuous time branching processes, where the birth rate is age–dependent, and also random, in the limit of a large number of ancestors. The only difficulty concerns the tightness in the Skorohod space D for the central limit theorem. We exploit a criterion for the CLT in D due to Hahn (1978).

1. Introduction

Crump–Mode–Jagers processes (CMJ-processes), as general continuous time and discrete state space branching process models with age-dependent reproduction mechanism, were introduced in Crump and Mode (1968), Jagers (1969). A CMJ-process starts with a single individual at time 0 and this individual gives birth during its lifetime to a random number of offspring at a sequence of random times. Every child that is born evolves in the same way, independently of all other individuals in the population. In most applications, the lifetimes do not follow an exponential distribution, and/or the process of successive births is not Poissonian, hence the process is not Markovian.

Such processes are used as used in the study of epidemics, see e.g. Ball et al. (2014), Olofsson and Sindi (2014), in particular for the detailed analysis of the start (resp. the end) of an epidemic, see Britton and Pardoux (2019) (resp. Mougabe–Peurkor et al., 2024), where the epidemic model is in general non Markovian, thus leading to a CMJ process which is exactly of the type which will be considered in the present paper. Other applications concern the allelic distribution of a population which reproduces according to a CMJ process, and is subject to mutations, see Champagnat and Lambert (2012), queuing theory, see e.g. Grishechkin (1992), Lambert et al. (2013), and generalized Pólya urn schemes, see Athreya and Karlin (1968) and Janson (2006).

Since most CMJ-processes are neither Markov processes nor semi-martingales, the law of such a process cannot be easily computed, so that approximations of that law can be useful, whenever it is available. This is our motivation for establishing a functional law of large numbers and a functional central limit theorem, in the limit of the number of ancestors tending to infinity.

Let us first describe the precise class of CMJ processes which will be considered in this paper. The law of the lifetime of each individual will be very general, with a Hölder continuous distribution function. Each alive individual gives birth to offspring at a rate which is a random function of its age. Moreover, we will allow multiple births at each birth time.

As already explained, our goal is to approximate, via both the law of large numbers and the central limit theorem, the law of our CMJ branching process, in case of a large number of ancestors. It is fair to say that the law of large numbers and the finite dimensional convergence in the functional CLT are easy consequences of very classical results. The only difficulty in our work is

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to establish the functional convergence in the Skorohod space D for the central limit theorem. For that sake, our basic tool is the result of Hahn (1978), which gives a sufficient condition for the CLT in D. Our technique for verifying the conditions in the main theorem of Hahn (1978) uses formulas for some moments of products of integrals w.r.t. a compensated Poisson Random Measure.

Moreover, while most of this paper considers the number of individuals in the population alive at a given time t, we also explore the option of counting individuals by the value of a specific characteristic, which is chosen as an arbitrary function of the age of each individual, and we deduce from our first results similar results for the empirical measure of the ages of the individuals, see the *Supplementary Material*.

There does not exist a vast literature on limit theorems for CMJ processes in the asymptotic of a large number of ancestors. We have found four papers which treat that subject. Wang proves a LLN in Wang (1975) and a CLT in Wang (1977) for the total population process, Solomon (1987) and Oelschläger (1990) establish both the LLN and the CLT for the empirical measure of ages in the population. All four consider generalized CMJ processes, in the sense that the birth rates (and also the death rates in Oelschläger, 1990) depend on the empirical age distribution. However their birth rates either do not depend upon the individual's age, or else they need a regularity of that dependence which we do not assume, and moreover those rates are deterministic, unlike in the present paper.

Notations All random variables will be defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We shall use the following notations: $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$. For $x \in \mathbb{R}_+$, $[x]$ denotes the integer part of x. Let $L^2(\Omega)$ be the space of square integrable random variables, $C([0,\infty), \mathbb{R}_+)$ denote the space of continuous functions from $[0,\infty)$ into $\mathbb{R}_+$, $D = D([0,\infty), \mathbb{R})$ and $D_+ = D([0,\infty), \mathbb{R}_+)$ denote resp. the space of functions from $[0,\infty)$ into $\mathbb{R}$, resp. $\mathbb{R}_+$, which are right continuous and have left limits at any $t > 0$. We shall always equip the spaces D and D_+ with the Skorohod topology, as described in Chapter 3 of Billingsley (1999). For $x, y \in \mathbb{R}$, $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. A unique letter C will denote a constant which may differ from line to line. We shall establish convergence results in D. We recall, see Theorem 16.2 in Billingsley (1999), that convergence in D is implied by convergence in $D([0, T])$ for all $T > 0$.

2. Preliminaries

We first consider the evolution in time of a population started with a single ancestor, i.e. we consider the random function $Z(t)$ representing the number of individuals alive at time t, who are descendants of a single individual born at time $t = 0$. Each individual appearing in the population lives for a random length of time whose law is that of η, and at random times during its life gives birth to offspring that behave in a similar fashion, the behavior of each individual being independent of all others. Let $0 < t^{(1)} < t^{(2)} < \cdots < t^{(n)} < \cdots$ be the birth times of the offspring of the ancestor. Then the random function $A(t) = \sum_{j=1}^{\infty} \mathbb{1}_{t^{(j)} \le t}$ is the number of birth events of offspring of the ancestor up to time t. Clearly, we have

$$Z(t) = \mathbb{1}_{\eta > t} + \sum_{j=1}^{A(t)} \sum_{i=1}^{L_j} Z_{j,i}(t - t^{(j)}), \tag{2.1}$$

where $\{Z_{j,i}(t),\ t \ge 0\} \overset{(d)}{=} \{Z(t),\ t \ge 0\}$ and are independent and identically distributed (i.i.d) random variables, and L_j denotes the number of twins born at the birth time $t^{(j)}$. We assume that the random variables $\{L^j,\ Z_{j,i},\ j, i \ge 1\}$ are mutually independent, and globally independent of the sequence $\{t^{(j)},\ j \ge 1\}$, and $t^{(1)},\ t^{(2)}, \dots$ are the jump times of the process

$$A(t) = \int_0^t \int_0^{b(s)} Q(ds, du),$$

where $Q(ds, du)$ is a Poisson Random Measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with mean measure $ds du$, and b is an $\mathbb{R}_+$–valued random process independent of Q. Let π denote the common law of the random processes $\{Z_{j,i}(.),\ j, i \ge 1\}$. We can rewrite (2.1) as

$$Z(t) = \mathbb{1}_{\eta > t} + \int_0^t \int_0^{b(s)} \int_{\mathscr{D}} \sum_{i=1}^{\ell} z_i(t - s) \mathscr{M}(ds, du, d\mathbf{z}), \tag{2.2}$$

where $\mathscr{M}(ds, du, d\mathbf{z})$ is a Poisson Random Measure on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathscr{D}$ with mean measure $\mu(ds, du, d\mathbf{z}) = ds du \Pi(d\mathbf{z})$, again independent of b, and such that Q is the projection of $\mathscr{M}$ on the first two coordinates. Π is a probability distribution on $\mathscr{D} := \mathbb{N} \times D^\infty$. It is the law of $(L, Z_1, Z_2, \dots,)$, where $L, Z_1, Z_2, \dots$ are mutually independent, the law of L is that of the number of twins at each birth event and $Z_1, Z_2, \dots$ are i.i.d., all having the law π. We shall denote below by $\bar{L}$ the expectation of L. Above $\mathbf{z}$ stands for $(\ell, z_1, z_2, \dots)$.

The joint distribution of b and η is the same for each individual, but we do not require that b and η be independent. On the other hand, we assume that there is independence between the pair (b, η) and $\mathscr{M}$.

Let us define $\bar{b}(t) = \mathbb{E}(b(t))$ the expectation of the random function b, $F(t) = \mathbb{P}(\eta \le t)$ the cumulative distribution function of η, $F^c(t) = 1 - F(t)$ its survival function or reliability function and $M_1(t) = \mathbb{E}(Z(t))$ the expectation of the random process Z. It is easy to see that the function $M_1(t)$ satisfies the integral equation

$$M_1(t) = F^c(t) + \int_0^t \bar{L} M_1(t - s) \bar{b}(s) ds. \tag{2.3}$$

Let us rewrite (2.2) in the following form, with $\hat{\mathscr{M}} = \mathscr{M} - \mu$ and $f_{t,b}(s, u, \mathbf{z}) = \mathbf{1}_{s \le t, u \le b(s)} \sum_{i=1}^{\ell} z_i(t - s)$,

$$Z(t) = \mathbb{1}_{\eta > t} + \int_0^t \bar{L} M_1(t - s) b(s) ds + \hat{\mathscr{M}}(f_{t,b}). \tag{2.4}$$

Note that for any $c > 0$, if $f_{t,b,c} := \mathbf{1}_{s\le t, u\le b(s)} \sum_{i=1}^{\ell}(z_i(t-s)\wedge c)$, from (2.2), $Z(t)\wedge c \le \mathbb{1}_{\eta>t} + \mathscr{M}(f_{t,b,c})$, hence

$$Z(t)\wedge c \le \mathbb{1}_{\eta>t} + \int_0^t \bar{L} M_1(t-s)b(s)ds + \hat{\mathscr{M}}(f_{t,b,c})\,. \tag{2.5}$$

3. Functional law of large numbers and central limit theorem

Let $x > 0$ be arbitrary, and let $N \geqslant 1$ be an integer which will eventually go to infinity. Let $(Z^{N,x}(t))_{t\ge 0}$ denote the CMJ-process which describes the number of descendants alive at time t of $[Nx]$ ancestors. We attribute to each individual in the population a mass equal to $1/N$ and we want to study the behavior of the process $X^{N,x}(t) = N^{-1}Z^{N,x}(t)$. Since $Z^{N,x}$ satisfies the branching property, it can be written as $Z^{N,x}(t) = \sum_{k=1}^{[Nx]} Z_k(t)$, where for any $k \ge 1$, $Z_k(t)$ is a CMJ-process which describes the number of descendants alive at time t of a single ancestor. Thus, we obtain

$$X^{N,x}(t) = N^{-1}\sum_{k=1}^{[Nx]} Z_k(t). \tag{3.6}$$

3.1. Functional law of large numbers

In this subsection, we only need to assume that the random function b and L satisfy the following assumption

$$(\mathbf{H_1}):\quad \bar{L} < \infty,\quad \text{and for any } T > 0,\ \sup_{0<t<T} \bar{b}(t) < \infty.$$

It clearly follows from $(\mathbf{H_1})$ and Gronwall's Lemma that M_1 is the unique solution of (2.3), and it satisfies:

$$\sup_{0<t<T} M_1(t) < \infty,\quad \text{for all } T > 0\,. \tag{3.7}$$

Let us now establish our functional Law of Large Numbers.

Theorem 3.1. *Suppose that Assumption $(\mathbf{H_1})$ is satisfied. Then, as $N\to\infty$, $(X^{N,x}(t), t\ge 0)$ converges to $(xM_1(t), t\ge 0)$ almost surely in D_+, where M_1 is the solution of the integral Eq. (2.3).*

Proof. From (3.6), we deduce

$$X^{N,x}(t) - xM_1(t) = \frac{[Nx]}{N}\left(\frac{1}{[Nx]}\sum_{k=1}^{[Nx]} Z_k(t) - M_1(t)\right) + \left(\frac{[Nx]-Nx}{N}\right) M_1(t).$$

From Theorem 1 in Rango and Rao (1963), the first term on the right converges to 0 a.s. in D, while the convergence to 0 of the second term locally uniformly in t follows from (3.7). ■

3.2. Functional central limit theorem

In the rest of this section, we study the convergence in distribution of the process defined by

$$\mathscr{R}^{N,x}(t) = \sqrt{N}\left(X^{N,x}(t) - xM_1(t)\right).$$

From (2.3), (2.4) and (3.6), we can rewrite $\mathscr{R}^N$ in the form

$$\mathscr{R}^{N,x}(t) = \sqrt{\frac{[Nx]}{N}}\left(\mathscr{U}_1^{N,x}(t) + \mathscr{U}_2^{N,x}(t) + \mathscr{U}_3^{N,x}(t)\right) + \varepsilon^{N,x}(t), \tag{3.8}$$

with $\varepsilon^{N,x}(t) = \left(\frac{[Nx]-Nx}{\sqrt{N}}\right) M_1(t)$, and[1] $\mathscr{U}_1^{N,x}(t) = \sqrt{[Nx]}\left(\frac{1}{[Nx]}\sum_{k=1}^{[Nx]} \mathbb{1}_{\eta_k>t} - F^c(t)\right)$,

$$\mathscr{U}_2^{N,x}(t) = \frac{1}{\sqrt{[Nx]}}\sum_{k=1}^{[Nx]}\int_0^t \left(b_k(s)-\bar{b}(s)\right)\bar{L}\, M_1(t-s)ds,\quad \mathscr{U}_3^{N,x}(t) = \frac{1}{\sqrt{[Nx]}}\sum_{k=1}^{[Nx]} \hat{\mathscr{M}}_k(f_{t,b_k}),$$

where $(\eta_k)_{k\ge1}$, $\{b_k(t), t\ge 0\}_{k\ge1}$ and $(\mathscr{M}_k)_{k\ge1}$ are respectively identically distributed independent copies of η, $\{b(t), t\ge 0\}$ and $\mathscr{M}$, defined in (2.2). Moreover the pair (b_k, η_k) and $\mathscr{M}_k$ are independent, for all $k\ge 1$.

In the next result, we shall assume that L and the random function b satisfy the following hypothesis

$$(\mathbf{H_2}):\quad \mathbb{E}[L^2] < \infty,\quad \text{and for any } T>0,\ \sup_{0<t<T}\mathbb{E}(b^2(t)) < \infty.$$

The next statement follows readily from the usual central limit theorem and easy computations.

[1] The first term reflects the randomness of η, the second one the randomness of b, and the third one the randomness of $\mathscr{M}$.

Proposition 3.2. *If Assumption* $(\mathbf{H_2})$ *is satisfied, the final-dimensional distributions of* $(\mathscr{U}_1^{N,x}, \mathscr{U}_2^{N,x}, \mathscr{U}_3^{N,x})$ *converge as* $N \to +\infty$, *to those of the centered Gaussian process* $(\mathscr{U}_1, \mathscr{U}_2, \mathscr{U}_3)$, *whose covariances are*[2]*:*

$$\mathbb{C}ov(\mathscr{U}_1(t), \mathscr{U}_1(s)) = F^c(t \vee s) - F^c(t)F^c(s)$$

$$\mathbb{C}ov(\mathscr{U}_2(t), \mathscr{U}_2(s)) = \int_0^t \int_0^s \bar{L}^2 M_1(t-r) M_1(s-u) \mathbb{C}ov\,(b(r), b(u))\, du dr$$

$$\mathbb{C}ov(\mathscr{U}_3(t), \mathscr{U}_3(s)) = \int_0^{t \wedge s} \left\{ \bar{L}\bar{b}(r)\mathbb{E}[Z(t-r)Z(s-r)] + \mathbb{E}[L^2 - L] M_1(t-r) M_1(s-r) \right\} dr$$

$$\mathbb{C}ov(\mathscr{U}_1(t), \mathscr{U}_3(s)) = \mathbb{C}ov(\mathscr{U}_2(t), \mathscr{U}_3(s)) = 0,\ \mathbb{C}ov(\mathscr{U}_1(t), \mathscr{U}_2(s)) = \int_0^s \bar{L} M_1(s-r) \mathbb{C}ov\left(b(r), \mathbb{1}_{\eta > t}\right) dr.$$

For the rest of this subsection, we shall assume that the random function b satisfies the following hypothesis.

$$(\mathbf{H_3}): \quad \mathbb{E}[L^4] < \infty, \quad \text{and for any } T > 0,\ \sup_{0<t<T} \mathbb{E}(b^4(t)) < \infty.$$

Our assumption concerning the distribution function F of the random variable η will be

$(\mathbf{H_4})$: For any $T > 0$, there exists $\alpha \in [0, 1]$ and a constant $C > 0$ such that

$$F(t) - F(s) \le C|t - s|^\alpha, \quad \forall 0 \le s < t \le T.$$

We have the second main result of this section

Theorem 3.3. *Suppose that Assumptions* $(\mathbf{H_3})$ *and* $(\mathbf{H_4})$ *are satisfied. Then, as* $N \to \infty$, $(\mathscr{R}^{N,x}(t), t \ge 0)$ *converges to* $(\mathscr{R}^x(t), t \ge 0)$ *in distribution in* D*, where* $\mathscr{R}^x$ *is given by*

$$\mathscr{R}^x(t) = \sqrt{x}\left(\mathscr{U}_1(t) + \mathscr{U}_2(t) + \mathscr{U}_3(t)\right), \quad \textit{where}$$

$(\mathscr{U}_1, \mathscr{U}_2, \mathscr{U}_3)$ *is the Gaussian random element of* $C([0, \infty))^3$ *whose law is specified in Proposition 3.2.*

To prove the Theorem, we will need some preliminary results. The following is Theorem 14.3 in Billingsley (1999).

Lemma 3.4. *As* $N \to \infty$, $(\mathscr{U}_1^{N,x}(t), t \ge 0)$ *converges to the continuous* $(\mathscr{U}_1(t), t \ge 0)$ *in distribution in* D.[3]

We shall need the

Lemma 3.5. *Suppose that Assumption* $(\mathbf{H_2})$ *is satisfied. Then, as* $N \to \infty$, $(\mathscr{U}_2^{N,x}(t), t \ge 0)$ *converges to* $(\mathscr{U}_2(t), t \ge 0)$ *in distribution in* $C([0, +\infty))$.

Proof. Let us rewrite $\mathscr{U}_2^{N,x}(t)$ in the form

$$\mathscr{U}_2^{N,x}(t) = \int_0^t \bar{L}\, V^{N,x}(s) M_1(t-s) ds, \quad \text{where} \quad V^{N,x}(t) = \frac{1}{\sqrt{[Nx]}} \sum_{k=1}^{[Nx]} \left(b_k(t) - \bar{b}(t)\right).$$

From the theorem in Kandelaki and Sozanov (1964) (see also Lemma 1.6 of Prokhorov, 1956), we deduce that $(V^{N,x}(t), t \ge 0)$ converges to $(V(t), t \ge 0)$ in $L^2([0, \infty))$, where V is the centered Gaussian continuous process such that $\mathbb{E}(V(t)V(s)) = \mathbb{C}ov(b(t), b(s))$, for $s, t > 0$. Now, for $T > 0$, let us define the continuous linear map Φ from the Hilbert space $L^2([0, T])$ equipped with the scalar product $(f, g)_T = \int_0^T f(t)g(t)dt$ into the Banach space $C([0, T])$ equipped with the sup norm, by $\Phi(f) = \int_0^{\cdot} f(s)\bar{L}\, M_1(\cdot - s)ds$. Since $\mathscr{U}_2^{N,x} = \Phi(V^{N,x})$, the result follows. ■

We need the following result, whose proof is developed in the *supplementary material*.

Lemma 3.6. *The sequence* $(\mathscr{U}_3^{N,x}, N \ge 1)$ *converges weakly in* D *and its limit belongs a.s. to* $C([0, \infty))$.

We now have

Proposition 3.7. *As* $N \to +\infty$, $(\mathscr{U}_1^{N,x}(t) + \mathscr{U}_2^{N,x}(t) + \mathscr{U}_3^{N,x}(t), t \ge 0) \Longrightarrow (\mathscr{U}_1(t) + \mathscr{U}_2(t) + \mathscr{U}_3(t), t \ge 0)$ *in* D.

Proof. It follows from Lemmas 3.4–3.6 that the sequence $(\mathscr{U}_1^{N,x}, \mathscr{U}_2^{N,x}, \mathscr{U}_3^{N,x})_{N \ge 1}$ is tight in D^3, and its unique limit point is a.s. continuous. Hence the limit of the sum equals the sum of the limits. ■

We can now turn to the

Proof of Theorem 3.3. The result follows from (3.8), Proposition 3.7, together with (3.7). ■

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[2] The independence of $(\mathscr{U}_1, \mathscr{U}_2)$ and $\mathscr{U}_3$ is a consequence of that of (η, b) and $\mathscr{M}$.

[3] Indeed, the proof of Theorem 4.3 in Billingsley (1999) specifies that $\mathscr{U}_1$ has the law of $W^0(F(t))$, where W^0 is the Brownian bridge, and our F is continuous, see $(\mathbf{H_4})$.

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.spl.2026.110916.

Data availability

No data was used for the research described in the article.

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