Stochastics and Stochastics Reports, Vol. 60, pp. 57–83
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BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS AND INTEGRAL-PARTIAL DIFFERENTIAL EQUATIONS

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(Received 5 June 1995; In final form 1 August 1996)

We consider a backward stochastic differential equation, whose data (the final condition and the coefficient) are given functions of a jump-diffusion process. We prove that under mild conditions the solution of the BSDE provides a viscosity solution of a system of parabolic integral-partial differential equations. Under an additional assumption, that system of equations is proved to have a unique solution, in a given class of continuous functions.

Keywords: Backward stochastic differential equations; jump-diffusion processes; integral-partial differential operators; viscosity solutions of systems of partial differential equations

AMS Classification: 60H10, 60G44, 35K55

INTRODUCTION

Backward stochastic differential equations (in short BSDE's) are new types of stochastic differential equations, whose value is prescribed at the final time T, see Pardoux, Peng [8]. It has been noted in Pardoux, Peng [9] that

* The research of this author has been done during a visit at the Université de Provence, and was supported by a grant of the German Deutsche Forschungsgesellschaft.
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BSDE's provide a probabilistic formula for the solutions of certain classes of systems of quasi-linear parabolic PDE's of second order, and in particular that BSDE's are naturally associated with viscosity solutions of PDE's.

The aim of this paper is to generalize the above results to the case of BSDE's with respect to both Brownian motion and a Poisson random measure. The associated system of parabolic PDE's is then a system of integro-partial differential equations. We prove, under appropriate assumptions, that our BSDE has a unique solution. We then put us in a Markovian framework, in which case a certain function defined through the solution of the BSDE is the unique viscosity solution of a system of parabolic integro-partial differential equations.

1. A JUMP-DIFFUSION PROCESS

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis such that $\mathcal{F}_0$ contains all P-null elements of $\mathcal{F}$, and $\mathcal{F}_{t^+} \stackrel{\triangle}{=} \cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon} = \mathcal{F}_t, t \geq 0$, and suppose that the filtration is generated by the following two mutually independent processes:

—a d-dimensional standard Brownian motion $\{W_t\}_{t\geq 0}$, and

—a Poisson random measure μ on $\mathbb{R}_+ \times E$, where $E \stackrel{\triangle}{=} \mathbb{R}^\ell \setminus \{0\}$ is equipped with its Borel field $\mathcal{E}$, with compensator $\nu(dt, de) = dt\lambda(de)$, such that $\{\tilde{\mu}([0,t] \times A) = (\mu - \nu)([0,t] \times A)\}_{t\geq 0}$ is a martingale for all $A \in \mathcal{E}$ satisfying $\lambda(A) < \infty$. λ is assumed to be a σ-finite measure on $(E, \mathcal{E})$ satisfying

$$\int_E (1 \wedge |e|^2)\lambda(de) < +\infty.$$

Let $b : \mathbb{R}^d \to \mathbb{R}^d, \sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be globally Lipschitz and $\beta : \mathbb{R}^d \times E \to \mathbb{R}^d$ be measurable and such that for some real K, and for all $e \in E$,

$$|\beta(x,e)| \leq K(1 \wedge |e|), \quad x \in \mathbb{R}^d,$$
$$|\beta(x,e) - \beta(x',e)| \leq K|x - x'|(1 \wedge |e|), \quad x, x' \in \mathbb{R}^d.$$

We now consider the following SDE:

$$dX_s = b(X_s)ds + \sigma(X_s)dW_s + \int_E \beta(X_{s^-}, e)\tilde{\mu}(ds, de). \tag{1.1}$$

We denote by $\{X_s^t(x)\}_{s\geq t}$ the unique solution of equation (1.1) starting from x at time $s = t$, and define $X_t = X_t^0(x_0), t \geq 0$, for some $x_0 \in \mathbb{R}^d$.

Existence and uniqueness, as well as the properties of the solution of (1.1) are stated in the following result, which follows from Theorems 2.2 and 2.3 in Fujiwara, Kunita [5]:

PROPOSITION 1.1 *For each* $t \geq 0$, *there exists a version of* $\{X_s^t(x); x \in \mathbb{R}^d, s \geq t\}$ *such that* $s \to X_s^t$ *is a* $C^2(\mathbb{R}^d)$*- valued càdlàg process. Moreover*

(i) X_s^t *and* X_{s-t}^0 *have the same distribution,* $0 \leq t \leq s$;

(ii) $X_{t_1}^{t_0}, X_{t_2}^{t_1}, \ldots, X_{t_n}^{t_{n-1}}$ *are independent, for all* $n \in \mathbb{N}, 0 \leq t_0 < t_1 < \cdots < t_n$;

(iii) $X_r^t = X_r^s \circ X_s^t, 0 \leq t < s < r$.
Furthermore, for all $p \geq 2$, *there exists a real* M_p *such that for all* $0 \leq t < s, x, x' \in \mathbb{R}^d$,

(iv)

$$E\left(\sup_{t \leq r \leq s} |X_r^t(x) - x|^p\right) \leq M_p(s-t)(1+|x|^p)$$

$$E\left(\sup_{t \leq r \leq s} |X_r^t(x) - X_r^t(x') - (x-x')|^p\right) \leq M_p(s-t)|x-x'|^p$$

2. BSDEs WITH RESPECT TO BROWNIAN MOTION AND LÉVY PROCESS. EXISTENCE AND UNIQUENESS OF A SOLUTION

From now on, we fix a terminal time $T > 0$. We define some spaces of processes. Let $\mathcal{S}^2$ denote the set of $\mathcal{F}_t$-adapted càdlàg k-dimensional processes $\{Y_t, 0 \leq t \leq T\}$ which are such that

$$||Y||_{\mathcal{S}^2} \triangleq \left\|\sup_{0 \leq t \leq T} |Y_t|\right\|_{L^2(\Omega)} < \infty.$$

Let $L^2(W)$ be the set of $\mathcal{F}_t$-progressively measurable $k \times d$ dimensional processes $\{Z_t, 0 \leq t \leq T\}$ which are such that

$$||Z||_{L^2(W)} \triangleq \left(E\int_0^T |Z_t|^2 dt\right)^{1/2} < \infty.$$

By $L^2(\tilde{\mu})$ we denote the set of mappings $U: \Omega \times [0,T] \times E \to \mathbb{R}^k$ which are $\mathcal{P} \otimes \mathcal{E}$ measurable[1] and such that

$$||U||_{L^2(\tilde{\mu})} \triangleq \left(E\int_0^T \int_E U_t(e)^2 \lambda(de)dt\right)^{1/2} < \infty.$$

[1] $\mathcal{P}$ denotes the σ-algebra of $\mathcal{F}_t$-predictable subsets of $\Omega \times [0,T]$.

Finally, we define $\mathcal{B}^2 = \mathcal{S}^2 \times L^2(W) \times L^2(\tilde{\mu})$. A proof of the next result can be found in Tang, Li [12], Lemma 2.4.

THEOREM 2.1 *Let* $Q \in (L^2(\Omega, \mathcal{F}_T, P))^k$ *and* $f : \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \times L^2 (E, \mathcal{E}, \lambda; \mathbb{R}^k) \to \mathbb{R}^k$ *be* $\mathcal{P} \otimes \mathcal{B}_k \otimes \mathcal{B}_{k\times d} \times \mathcal{B}(L^2(E, \mathcal{E}, \lambda; \mathbb{R}^k))$ *measurable and satisfy:*

$$(A.1.i) \qquad E\int_0^T |f_t(0,0,0)|^2 dt < \infty;$$

(A.1.ii) there exists $K > 0$ *such that*

$$|f_t(y,z,u) - f_t(y',z',u')| \le K(|y-y'| + |z-z'| + \|u-u'\|)$$

for all $0 \le t \le T, y, y' \in \mathbb{R}^k, z, z' \in \mathbb{R}^{k\times d}, u, u' \in L^2(E, \mathcal{E}, \lambda; \mathbb{R}^k)$. *Then there exists a unique triple* $(Y, Z, U) \in \mathcal{B}^2$ *which solves the B.S.D.E.*

$$Y_t = Q + \int_t^T f_s(Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T\int_E U_s(e)\tilde{\mu}(ds,de), 0 \le t \le T. \tag{2.1}$$

We now establish a continuity result, which will be useful in the sequel.

PROPOSITION 2.2 *Let* (Q,f) *and* (Q',f') *be two data satisfying the assumptions of theorem* 2.1 *and let* (Y,Z,U) *denote the solution of the BSDE with data* (Q,f) *and* (Y',Z',U') *that of the BSDE with data* (Q',f').

Define $(\Delta Q, \Delta f) = (Q - Q', f - f')$ *and* $(\Delta Y, \Delta Z, \Delta U) = (Y - Y', Z - Z', U - U')$. *Then there exists a constant c such that*

$$E\left[\sup_{0\le t\le T}|\Delta Y_t|^2 + \int_0^T |\Delta Z_t|^2 dt + \int_0^T\int_E |\Delta U_t(e)|^2\lambda(de)dt\right]$$
$$\le cE\left[|\Delta Q|^2 + \int_0^T |\Delta f(Y_t, Z_t, U_t)|^2 dt\right]$$

Proof It follows from Itô's formula that

$$E\left[|\Delta Y_t|^2 + \int_t^T |\Delta Z_s|^2 ds + \int_t^T\int_E |\Delta U_s(e)|^2\lambda(de)ds\right]$$
$$= E\left[|\Delta Q|^2 + 2\int_t^T < \Delta Y_s, \Delta f_s(Y_s,Z_s,U_s) + f'_s(Y_s,Z_s,U_s) - f'_s(Y'_s,Z'_s,U'_s) > ds\right]$$
$$\le E\left[|\Delta Q|^2 + cE\int_t^T [|\Delta Y_s|^2 + |\Delta f_s(Y_s,Z_s,U_s)|^2]ds\right]$$
$$+ \frac{1}{2}E\int_t^T |\Delta Z_s|^2 ds + \frac{1}{2}E\int_t^T\int_E |\Delta U(s,e)|^2\lambda(de)ds$$

The inequality which we want to prove, but with "$\sup_{0\le t\le T}$" outside the expectation, follows from Gronwall's lemma. We apply again Itô's formula, yielding

$$|\Delta Y_t|^2+\int_t^T|\Delta Z_s|^2ds+\int_t^T\int_E|\Delta U_s(e)|^2\lambda(de)ds$$
$$=|\Delta Q|^2+2\int_t^T\langle\Delta Y_s,f_s(Y_s,Z_s,U_s)-f'_s(Y'_s,Z'_s,U'_s)\rangle ds$$
$$-2\int_t^T\langle\Delta Y_s,\Delta Z_s dW_s\rangle-\int_t^T\int_E(|\Delta Y_{s^-}+\Delta U_s(e)|^2-|\Delta Y_{s^-}|^2)\tilde\mu(ds,de).$$

The result now follows from the last identity, Doob's inequality and the previous estimate. □

In the sequel, we are concerned by a specific class of BSDEs where both Q and for each t,y,z,u, the process $\{f_s(y,z,u),t\le s\le T\}$ are given functions of the process $X_s^t(x)$ constructed in section 1.

More precisely, we are given two continuous functions

$$g:\mathbb{R}^d\to\mathbb{R}^k,\quad f:[0,T]\times\mathbb{R}^d\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\times\mathbb{R}^k\to\mathbb{R}^k$$

and a measurable function $\gamma:\mathbb{R}^d\times E\to\mathbb{R}^k$ such that, for each $1\le i\le k$, $f_i(t,x,y,z,q)$ depends on the matrix z only through its i-th column z_i, and on the vector q only through its i-th coordinate q_i. The first one of these assumptions is essential for the notion of viscosity solution of the system of integro-partial differential equations to be considered below to make sense. The second restriction is less essential, but will be useful for proving the uniqueness result. We assume specifically that

(A.2i) $|f(t,x,0,0,0)|\le C(1+|x|^p)$, $|g(x)|\le C(1+|x|^p)$, for some $C,p>0$;
(A.2ii) $f=f(t,x,y,z,q)$ is globally Lipschitz in (y,z,q), uniformly in (t,x);
(A.2iii) for each $(t,x,y,z)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^k\times\mathbb{R}^{k\times d}$, $1\le i\le k$, the function

$$p\mapsto f_i(t,x,y,z,p)\text{ is non-decreasing;}$$

(A.2iv) there is some real $C>0$ such that, for all $1\le i\le k$,

$$0\le\gamma_i(x,e)\le C(1\wedge|e|),\quad x\in\mathbb{R}^d,e\in E$$
$$|\gamma_i(x,e)-\gamma_i(x',e)|<C|x-x'|(1\wedge|e|),x,x'\in\mathbb{R}^d,e\in E.$$

Under the assumptions (A.2i)–(A.2iv), for each $t \in [0, T]$ and $x \in \mathbb{R}^d$, we consider the BSDE

$$\begin{aligned} Y^t_{s,i}(x) &= g_i(X^t_T(x)) \\ &+ \int_s^T f_i\Big(r, X^t_r(x), Y^t_r(x), Z^t_{r,i}(x), \int_E U^t_{r,i}(x,e)\gamma_i(X^t_r(x),e)\lambda(de)\Big)dr \\ &- \int_s^T Z^t_{r,i}(x)dW_r - \int_s^T \int_E U^t_{r,i}(x,e)\tilde{\mu}(drde), \quad t \le s \le T, 1 \le i \le k \end{aligned} \tag{2.2}$$

Note that in (2.2) and in the sequel, f depends on U in a very specific way. The main reason for this restriction is that we shall need the comparison theorem (proposition 2.6 below), which would not be true in general, as it is explained in remark 2.7.

It follows readily from Theorem 2.1

COROLLARY 2.3 *For each* $t \in [0, T], x \in \mathbb{R}^d$, *the BSDE* (2.2) *has a unique solution*

$$(Y^t(x), Z^t(x), U^t(x,\cdot)) \in \mathcal{B}^2,$$

and $(t,x) \mapsto Y^t_t(x)$ *defines a deterministic mapping from* $[0,T] \times \mathbb{R}^d$ *into* $\mathbb{R}^k$.

Remark 2.4 From Proposition 1.1 (ii) and (iii), and from the uniqueness of the solution of the BSDE (2.2) we obtain that, for any $0 \le t \le t' \le T$, $x \in \mathbb{R}^d$,

$$\begin{aligned} &(Y^{t'}_s \circ X^t_{t'}(x)), Z^{t'}_s \circ X^t_{t'}(x), U^{t'}_s(X^t_{t'}(x),\cdot))_{t' \le s \le T} \\ &\quad = (Y^t_s(x), Z^t_s(x), U^t_s(x,\cdot))_{t' \le s \le T} \text{ in } \mathcal{B}^2 \end{aligned}.$$

Let us denote the deterministic function $Y^t_t(x)$ by $u(t,x)$ and indicate some of its basic properties.

PROPOSITION 2.5 *Under the assumptions (A.2), the function* $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}^k$ *is continuous in* (t,x) *and for some real C and p,*

$$|u(t,x)| \le C(1+|x|^p), \quad (t,x) \in [0,T] \times \mathbb{R}^d.$$

Moreover, if g and $f(t,\cdot,y,z,q)$ *are uniformly continuous, uniformly with respect to* (t,y,z,q) *and bounded, then u is uniformly continuous and bounded.*

Proof From Corollary 3.2 in [3], we know that, for any $q \ge 2$, there exists $C_q > 0$ such that, for all $0 \le t \le s \le T, x \in \mathbb{R}^d$,

$$E[|Y^t_s(x)|^q] + E\left[\left(\int_s^T |Z^t_r(x)|^2 dr + \int_s^T \int_E |U^t_r(x,e)|^2 \lambda(de)dr\right)^{q/2}\right] \le C_q(1+|x|^q).$$

Choosing $s = t$, we obtain the wished estimate for the growth of $u(t, x)$. We now define $Y_s^{t,x}$ for all $s \in [0, T]$ by choosing $Y_s^{t,x} = Y_t^{t,x}$ for $0 \leq s \leq t$. In order to prove the continuity of u, we use proposition 2.2 to estimate

$$\begin{aligned} |Y_t^t(x) - Y_{t'}^{t'}(x')|^2 &= |Y_0^t(x) - Y_0^{t'}(x')|^2 \\ &\leq E\Big(\sup_{0\leq s\leq T} |Y_s^t(x) - Y_s^{t'}(x')|^2\Big) \\ &\leq cE\Big[|g(X_T^t(x)) - g(X_T^{t'}(x'))|^2 \\ &\quad + \int_0^T \big|\mathbf{1}_{[t,T]}(s)f(s,, X_s^t(x), Y_s^t(x), Z_s^t(x), U_s^t(x)) \\ &\quad - \mathbf{1}_{[t',T]}(s)f(s, X_s^{t'}(x'), Y_s^{t'}(x'), Z_s^{t'}(x'), U_s^{t'}(x'))\big|^2 ds\Big] \end{aligned}$$

and the continuity of u follows from the assumption (A.2v) and the polynomial growth of g and f in all their variables (except s), which results from (A.2i),(A.2ii) and (A.2iv).

If moreover g and $f(t, \cdot, y, z, q)$ are uniformly continuous, then the uniform continuity of u follows from the same estimate and the boundedness is immediate. □

Finally, in preparation to the next section, we provide a comparison theorem for one-dimensional BSDE's.

PROPOSITION 2.6 *Let* $h : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ *be* $\mathcal{P} \otimes \mathcal{B} \otimes \mathcal{B}_d \otimes \mathcal{B}$-*measurable and satisfy*

(i) $E[\int_0^T |h(t, 0, 0, 0)|^2 dt] < +\infty$,

(ii) $|h(t, y, z, q) - h(t, y', z', q')| \leq K(|y - y'| + |z - z'| + |q - q'|)$, *for any* $y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d, q, q' \in \mathbb{R}, \quad t \in [0, T]$ *and for some real* $K > 0$;

(iii) $q \mapsto h(t, y, z, q)$ *is non-decreasing, for all* $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$.

Furthermore, let $\gamma : \Omega \times [0, T] \times E \to \mathbb{R}$ *be* $\mathcal{P} \otimes \mathcal{E}$-*measurable and satisfy*

$$0 \leq \gamma_t(e) \leq C(1 \wedge |e|), \quad e \in E.$$

We set

$$f(t, \omega, y, z, \varphi) = h(t, \omega, y, z, \int_E \varphi(e)\gamma_t(\omega, e)\lambda(de)),$$

for $(t, \omega, y, z, \varphi) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{E}, \lambda)$.

Let $Q, Q' \in L^2(\Omega, \mathcal{F}_T, P)$ *and let denote by* $(Y, Z, U) \in \mathcal{B}^2$ *(resp.* $(Y', Z', U') \in \mathcal{B}^2$*) the unique solution of eq.* (2.1) *with final condition* Q

(resp. Q'). Then, if $Q \geq Q'$, it follows

$$Y_t \geq Y'_t, \quad \textit{for} \quad 0 \leq t \leq T. \quad \square$$

The proof using Itô's formula follows the argument of the proof of the comparison theorem [9] for BSDE's without jumps.

Remark 2.7 If we impose on f only the assumptions (A.1i) and (A.1ii), then, in general, the comparison theorem does not hold. We give now a counter-example.

Let $E = \mathbb{R} - \{0\}, \lambda(de) = \delta_1(de)$ and $f(t, \omega, y, z, \varphi) = -2\varphi(1)$.

Then

$$N_t = \int_0^t \int_E \mu(dsde), 0 \leq t \leq T,$$

is standard Poisson process, and if we choose

$$Q = N_T \textit{ and } Q' = 0,$$

then

$$(Y_t, Z_t, U_t) = (N_t - (T - t), 0, I_{\{e=1\}}),$$
$$(Y'_t, Z'_t, U'_t) = (0, 0, 0).$$

Clearly, $Q \geq Q'$, but $P\{Y_t < Y'_t\} > 0$, for all $0 \leq t < T$.

3. ASSOCIATED INTEGRAL-PARTIAL DIFFERENTIAL EQUATIONS. EXISTENCE AND UNIQUENESS

We consider the system of integral-partial differential equations of parabolic type

$$\begin{cases} -\frac{\partial}{\partial t} u_i(t,x) - Lu_i(t,x) - f_i(t,x,u(t,x), (\nabla u_i \sigma)(t,x), B_i u_i(t,x)) = 0, \\ (t,x) \in [0,T] \times \mathbb{R}^d, \qquad 1 \leq i \leq k \\ u_i(T,x) = g_i(x), \quad x \in \mathbb{R}^d, \quad 1 \leq i \leq k, \end{cases} \tag{3.1}$$

where the second-order integral-differential operator L is of the form

$$L = A + K,$$

with

$$A\varphi(x) = \frac{1}{2}Tr\left(a(x)\frac{\partial^2\varphi}{\partial x^2}(x)\right) + \langle b(x), \nabla\varphi(x)\rangle, \quad \varphi \in C^2(\mathbb{R}^d),$$
$$a_{ij}(x) = (\sigma(x)\sigma(x)^*)_{i,j},$$
$$K\varphi(x) = \int_E (\varphi(x+\beta(x,e)) - \varphi(x) - \langle\nabla\varphi(x), \beta(x,e)\rangle)\lambda(de), \quad \varphi \in C^2(\mathbb{R}^d),$$

and B_i is an integral operator defined as

$$B_i\varphi(x) = \int_E (\varphi(x+\beta(x,e)) - \varphi(x))\gamma_i(x,e)\lambda(de), \quad \varphi \in C^1(\mathbb{R}^d).$$

The functions σ, b and β are supposed to satisfy the assumptions made in section 1, f, g and γ shall satisfy (A.2i)–(A.2iv).

For such a system (3.1), we introduce the notion of viscosity solution.

DEFINITION 3.1 We say that $u \in C([0,T] \times \mathbb{R}^d; \mathbb{R}^k)$ is

(i) *a viscosity subsolution of* (3.1) if

$$u_i(T,x) \le g_i(x), \quad x \in \mathbb{R}^d, 1 \le i \le k$$

and if, for any $1 \le i \le k, \varphi \in C^2([0,T] \times \mathbb{R}^d)$, wherever $(t,x) \in [0,T[\times\mathbb{R}^d$ is a global maximum point of $u_i - \varphi$,

$$-\frac{\partial\varphi}{\partial t}(t,x) - A\varphi(t,x) - K^\delta(u_i,\varphi)(t,x) - f_i(t,x,u(t,x), (\nabla\varphi\sigma)(t,x), B_i^\delta(u_i,\varphi)(t,x)) \le 0.$$

for any $\delta > 0$ where

$$K^\delta(u_i,\varphi)(t,x) = \int_{E_\delta} (\varphi(t,x+\beta(x,e)) - \varphi(t,x) - \langle\nabla\varphi(t,x), \beta(x,e)\rangle)\lambda(de) + \int_{E_\delta^c} (u_i(t,x+\beta(x,e)) - u_i(t,x) - \langle\nabla\varphi(t,x), \beta(x,e)\rangle)\lambda(de)$$

and

$$B_i^\delta(u_i,\varphi)(t,x) = \int_{E_\delta} (\varphi(t,x+\beta(x,e)) - \varphi(t,x))\gamma_i(x,e)\lambda(de) + \int_{E_\delta^c} (u_i(t,x+\beta(x,e)) - u_i(t,x))\gamma_i(x,e)\lambda(de)$$

with $E_\delta = \{e \in E; |e| < \delta\}$.

(ii) *a viscosity supersolution of* (3.1) if

$$u_i(T,x) \geq g_i(T,x),, x \in \mathbb{R}^d, 1 \leq i \leq k,$$

and, for any $1 \leq i \leq k, \varphi \in C^2([0,T] \times \mathbb{R}^d)$, whenever $(t,x) \in [0,T[\times R^d$ is a global minimum point of $u_i - \varphi$,

$$-\frac{\partial \varphi}{\partial t}(t,x) - A\varphi(t,x) - K^\delta(u_i,\varphi)(t,x) - f_i(t,x,u(t,x),$$
$$(\nabla\varphi\sigma)(t,x), B_i^\delta(u_i,\varphi)(t,x)) \geq 0.$$

(iii) *a viscosity solution of* (3.1) if it is both a sub and a supersolution of (3.1).

Remark 3.2 The introduction of the operators K^δ and B_i^δ in Definition 3.1 is necessary: indeed, since we assume only u to be continuous in x, Ku_i and B_iu_i are not well-defined because of the singularity of $\lambda(de)$ at 0. On the contrary, since φ is a C^2-function

$$|\varphi(t,x+\beta(x,e)) - \varphi(t,x) - \langle D\varphi(t,x), \beta(x,e)\rangle| \leq C|\beta(x,e)|^2$$

and

$$|\varphi(t,x+\beta(x,e)) - \varphi(t,x)|\gamma_i(x,e) \leq C|\beta(x,e)|\gamma_i(x,e)$$

for some constant C and therefore the two first terms in the definition of K^δ and B_i^δ have a sense.

Non linear elliptic and parabolic equations with integro-differential terms have been studied using viscosity solutions theory by A. Sayah [10] and H.M. Soner [11] (see also O. Alvarez and A. Tourin [1]). They consider either a different class of solutions or a different type of integro-differential terms. We borrow here some of the arguments of these works but it is worth mentionning that the particular form of the system (3.1) – linear terms + Lipschitz continuous nonlinearities–allows us to provide a simpler proof of uniqueness.

We first give an equivalent definition of viscosity solutions which will be useful later on.

LEMMA 3.3 *In the definition of u being a viscosity sub- (resp. super-) solution of* (3.1), *we can replace*

$$K^\delta(u_i,\varphi)(t,x) \quad by \quad K\varphi(t,x),$$
$$B_i^\delta(u_i,\varphi)(t,x) \quad by \quad B_i\varphi(t,x). \quad \square$$

In the same way, we can replace "global maximum point" or "global minimum point" by "strict global maximum point" or "strict global minimum point". The proof of this claim is very simple and we leave it as an exercise for the reader.

Proof We treat only the subsolution case. If (t, x) is a global maximum point of $u_i - \varphi$, we have

$$u_i(s, y) - \varphi(s, y) \leq u_i(t, x) - \varphi(t, x),$$

for all $(s, y) \in [0, T] \times \mathbb{R}^d$. Therefore

$$u_i(t, y) - u_i(t, x) \leq \varphi(t, y) - \varphi(t, x),$$

for any $y \in \mathbb{R}^d$ and this yields, for any $\delta > 0$,

$$K^\delta(u_i, \varphi)(t, x) \leq K\varphi(t, x),$$
$$B_i^\delta(u_i, \varphi)(t, x) \leq B_i\varphi(t, x).$$

From the inequality given by Definition 3.1, using assumption (A.2iii), we deduce easily that

$$-\frac{\partial \varphi}{\partial t}(t, x) - A\varphi(t, x) - K\varphi(t, x) - f_i(t, x, u(x, t), (\nabla\varphi\sigma)(t, x), B_i\varphi(t, x)) \leq 0. \tag{3.2}$$

It remains to show that this last condition implies that of the definition.

Changing φ into $\varphi - (\varphi(t, x) - u_i(t, x))$, we may assume that $u_i(t, x) = \varphi(t, x), u_i \leq \varphi$. Moreover we may assume w.l.o.g. that, for all $\alpha > 0$, there is some $\eta_\alpha > 0, \eta_\alpha \to 0$ as $\alpha \to 0$, such that

$$\varphi(s, y) - u_i(s, y) \geq \eta_\alpha, \text{for all}(s, y) \in [0, T] \times \mathbb{R}^d \text{ with} |(s, y) - (t, x)| > \alpha.$$

But we will show below that, under these circumstances, there exists a sequence $(\varphi_\varepsilon)_\varepsilon$ of elements of $C^2([0, T] \times \mathbb{R}^d)$ such that

(i) $\varphi_\varepsilon(s, y) = \varphi(s, y)$, if $|(s, y) - (t, x)| \notin (\frac{\delta}{2}, \frac{1}{\varepsilon})$;

(ii) $u_i(s, y) < \varphi_\varepsilon(s, y) \leq \varphi(s, y)$, if $\frac{\delta}{2} < |(s, y) - (t, x)| < \frac{1}{\varepsilon}$;

(iii) $\varphi_\varepsilon(s, y) \to u_i(s, y)$, as $\varepsilon \to 0$, for all $(s, y) \in [0, T] \times \mathbb{R}^d$ such that $|(s, y) - (t, x)| \geq \delta$.

In particular, we have

$$\nabla\varphi_\varepsilon(t, x) = \nabla\varphi(t, x), \quad \frac{\partial \varphi_\varepsilon}{\partial t}(t, x) = \frac{\partial \varphi}{\partial t}(t, x), \quad D^2\varphi_\varepsilon(t, x) = D^2\varphi(t, x).$$

Since moreover $\varphi_\varepsilon(t,x) = u_i(t,x)$ and $\varphi_\varepsilon \geq u_i$, it follows from (3.2) that

$$-\frac{\partial}{\partial t}\varphi_\varepsilon(t,x) - A\varphi_\varepsilon(t,x) - K\varphi_\varepsilon(t,x) - f_i(t,x,u(t,x),\\ (\nabla\varphi_\varepsilon\sigma)(t,x), B_i\varphi_\varepsilon(t,x)) \leq 0.$$

Then the property (ii) above together with (A.2iii) yields

$$-\frac{\partial}{\partial t}\varphi(t,x) - A\varphi(t,x) - K^\delta(\varphi_\varepsilon,\varphi)(t,x) - f_i(t,x,u(t,x),\\ (\nabla\varphi\sigma)(t,x), B_i^\delta(\varphi_\varepsilon,\varphi)(t,x)) \leq 0.$$

Now, using (ii) and (iii), we deduce from the Lebesgue's dominated convergence theorem that

$$\lim_{\varepsilon\to 0} K^\delta(\varphi_\varepsilon,\varphi)(t,x) = K^\delta(u_i,\varphi)(t,x),$$
$$\lim_{\varepsilon\to 0} B_i^\delta(\varphi_\varepsilon,\varphi)(t,x) = B_i^\delta(u_i,\varphi)(t,x),$$

and letting ε tend to 0 in the above relation yields the desired result.

We now prove the existence of a sequence $(\varphi_\varepsilon)_\varepsilon$ having the required properties. For convenience but without loss of generality of the method we forget about the variable t and suppose that we have a $\varphi \in C^2(\mathbb{R}^d)$ and a $u_i \in C(\mathbb{R}^d)$ such that

(a) $\varphi(x) = u_i(x)$

(b) for all $\alpha > 0$ there is a $\eta_\alpha > 0$ such that $\eta_\alpha \to 0$ as $\alpha \to 0$ and

$$\varphi(y) - u_i(y) \geq \eta_\alpha, \text{ for all } y \in \mathbb{R}^d \text{ with } |y-x| \geq \alpha.$$

Fix now $\varepsilon \in (0, \frac{4}{3\delta})$ and introduce the non-negative function

$$\psi_\varepsilon(y) = \left(\varphi(y) - u_i(y) - \frac{\eta}{2}\right)\mathbf{1}_{\{\frac{3\delta}{4}\leq|x-y|\leq\frac{1}{\varepsilon}\}}, \quad y \in \mathbb{R}^d,$$
where $\eta = \eta_\alpha$ given by (b) with $\alpha = 3\delta/4$.

Let X be a non-negative element of $C^\infty(\mathbb{R}^d)$ with support in the unit ball of $\mathbb{R}^d$ such that

$$\int X(y)dy = 1,$$

and we set

$$X_\mu(y) = \mu^{-d}X(\mu^{-1}y), \quad \text{for } \mu > 0.$$

Since $\varphi - u_i$ is continuous, we can find some $\mu_\varepsilon \in (0, \delta/4)$ such that

$$|(\varphi - u_i)(y) - (\varphi - u_i)(y - z)| \leq \frac{\eta}{4},$$

for all $y, z \in \mathbb{R}^d$ with $|x - y| \leq \frac{2}{\varepsilon}$ and $|z| \leq \mu_\varepsilon$.

Finally we define the function

$$\varphi_\varepsilon(y) = \varphi(y) - \int_{\mathbb{R}^d} X_{\mu_\varepsilon}(y - z)\psi_\varepsilon(z)dz, \quad y \in \mathbb{R}^d.$$

One checks easily that the following properties hold

(i) $\varphi_\varepsilon(y) = \varphi(y)$, for all $y \in \mathbb{R}^d$ with $|y - x| \notin \left(\frac{3\delta}{4} - \mu_\varepsilon, \frac{1}{\varepsilon} + \mu_\varepsilon\right)$;

(ii) $u_i(y) + \frac{\eta}{4} \leq \varphi_\varepsilon(y) \leq \varphi(y)$, for $|y - x| \in \left(\frac{\delta}{2} - \mu_\varepsilon, +\infty\right)$;

(iii) $\varphi_\varepsilon(y) \to u_i(y)$, as $\varepsilon \to 0$, for all $y \in \mathbb{R}^d$.

This completes the proof. □

We now prove that our BSDE provides a viscosity solution of (3.1).

THEOREM 3.4 *The function $u(t, x) = Y_t^t(x), (t, x) \in [0, T] \times \mathbb{R}^d$, introduced in Section 2, is a viscosity solution of eq.* (3.1).

Proof Due to Proposition 2.5, the function u belongs to $C([0, T] \times \mathbb{R}^d; \mathbb{R}^k)$. It clearly satisfies the boundary condition at time $t = T$. We now show that it is a viscosity subsolution of eq. (3.1). A similar argument would show that it is a viscosity supersolution of eq. (3.1).

Let $1 \leq i \leq k, \varphi \in C^2([0, T] \times \mathbb{R}^d), (t, x) \in [0, T] \times \mathbb{R}^d$ such that $\varphi \geq u_i$, $\varphi(t, x) = u_i(t, x)$. Taking into account the corresponding properties of u_i, we can assume additionally that φ and its derivatives have at most polynomial growth as $|y| \to \infty$, uniformly in $s \in [0, T]$.

Next we note that from uniqueness of the solution of our BSDE, for any $t \leq s \leq T, x \in \mathbb{R}^d$,

$$Y_s^t(x) = Y_s^s(X_s^t(x)) = u(s, X_s^t(x)).$$

Choose $h > 0$ such that $t + h \leq T$. It follows from the last remark that for $t \leq s \leq t + h$,

$$Y_{s,i}^t(x) = u_i(t+h, X_{t+h}^t(x)) + \int_s^{t+h} f_i(r, X_r^t(x), Y_r^t(x), Z_{r,i}^t(x),$$
$$\int_E U_{r,i}^t(x,e)\gamma_i(X_r^t(x),e)\lambda(de))dr - \int_s^{t+h} Z_{r,i}^t(x)dB_r$$
$$- \int_s^{t+h}\int_E U_{r,i}^t(x,e)\tilde{\mu}(drde).$$

If $y \in \mathbb{R}, z \in \mathbb{R}^k$, we denote by $(y, \tilde{z}_i)$ the k-dimensional vector whose i-th component equals y, and all the other components equal the corresponding ones of z.

Consider the one-dimensional BSDE

$$\overline{Y}_s^h(x) = \varphi(t+h, X_{t+h}^t(x))$$
$$+ \int_s^{t+h} f_i(r, X_r^t(x), (\overline{Y}_r^h, \tilde{Y}_{r,i}^t(x)), \overline{Z}_r^h, \int_E \overline{U}_r^h(e)\gamma_i(X_r^t(x),e)\lambda(de))dr$$
$$- \int_s^{t+h} \overline{Z}_r^h dB_r - \int_s^{t+h}\int_E \overline{U}_r^h(e)\tilde{\mu}(drde), \quad t \leq s \leq t+h.$$

Taking into account that $\varphi \geq u_i$, it follows from Proposition 2.6 that

$$\overline{Y}_s^h \geq Y_{s,i}^t(x), \quad t \leq s \leq t+h, \quad \text{a.e.},$$

and, in particular,

$$\overline{Y}_t^h \geq u_i(t,x) = \varphi(t,x).$$

Furthermore, putting

$$\psi(s,y) = \frac{\partial}{\partial s}\varphi(s,y) + L\varphi(s,y),$$
$$\Phi(s,y,e) = \varphi(s, y + \beta(y,e)) - \varphi(s,y),$$

we have by Itô's formula,

$$\varphi(s, X_s^t(x)) = \varphi(t+h, X_{t+h}^t(x)) - \int_s^{t+h}\psi(r, X_r^t(x))dr - \int_s^{t+h}(\nabla\varphi\sigma)(r, X_r^t(x))dB_r$$
$$- \int_s^{t+h}\int_E \Phi(r, X_{r^-}^t(x), e)\tilde{\mu}(drde).$$

Define, for $t \le s \le t+h, e \in E$,

$$\hat{Y}_s^h = \overline{Y}_s^h - \varphi(s, X_s^t(x)),$$
$$\hat{Z}_s^h = \overline{Z}_s^h - (\nabla\varphi\sigma)(s, X_s^t(x)),$$
$$\hat{U}_s^h(e) = \overline{U}_s^h(e) - \Phi(s, X_{s^-}^t(x), e)$$

It follows from the above identifies that $(\hat{Y}^h, \hat{Z}^h, \hat{U}^h)$ is the unique solution of the following one-dimensional BSDE:

$$\hat{Y}_s^h = \int_s^{t+h} [\psi(r, X_r^t(x)) + f_i(r, X_r^t(x), \tilde{Y}_r^h, \tilde{Z}_r^h, \tilde{U}_r^h)]dr - \int_s^{t+h} \hat{Z}_r^h dB_r - \int_s^{t+h}\int_E \hat{U}_r^h(e)\tilde{\mu}(drde),$$

where

$$\tilde{Y}_r^h = (\varphi(r, X_r^t(x)) + \hat{Y}_r^h, \tilde{Y}_{r,i}^t(x))$$
$$\tilde{Z}_r^h = (\nabla\varphi\sigma)(r, X_r^t(x)) + \hat{Z}_r^h$$
$$\tilde{U}_r^h = \int_E (\Phi(r, X_{r^-}^t(x), e) + \hat{U}_r^h(e))\gamma_i(X_r^t(x), e)\lambda(de)$$

Hence, by using standard techniques to estimate the squared norm of the solution $(\hat{Y}^h, \hat{Z}^h, \hat{U}^h)$, we obtain for $t \le s \le t+h$,

$$E[|\hat{Y}_s^h|^2] + E\left[\int_s^{t+h} |\hat{Z}_r^h|^2 dr\right] + E\left[\int_s^{t+h}\int_E |\hat{U}_r^h(e)|^2\lambda(de)dr\right] \le cE\left[\int_s^{t+h} |\hat{Y}_r^h|(1 + |\hat{Y}_r^h| + |\hat{Z}_r^h| + \left(\int_E |\hat{U}_r^h(e)|^2\lambda(de)\right)^{1/2}\right)dr\right],$$

i.e., for some $c > 0$,

$$E[|\hat{Y}_s^h|^2] + \frac{1}{2}\left(E\left[\int_s^{t+h} |\hat{Z}_r^h|^2 dr\right] + E\left[\int_s^{t+h}\int_E |\hat{U}_r^h(e)|^2\lambda(de)dr\right]\right) \le cE\left[\int_s^{t+h} ([\hat{Y}_r^h| + |\hat{Y}_r^h|^2)dr\right]$$

Hence, in particular

$$E[|\hat{Y}_s^h|^2] \le 2c\left(h + E\left[\int_s^{t+h} |\hat{Y}_r^h|^2 dr\right]\right),$$

and consequently for some $C > 0$

$$E[|\hat{Y}_s^h|^2] \le Ch.$$

On the other hand,

$$\frac{1}{2}\left(E\left[\int_s^{t+h}|\hat{Z}_r^h|^2dr\right]+E\left[\int_s^{t+h}\int_E|\hat{U}_r^h(e)|^2\lambda(de)dr\right]\right) \leq cE\left[\int_s^{t+h}(|\hat{Y}_r^h|+|\hat{Y}_r^h|^2)dr\right],$$

and therefore, for some $C>0$

$$\frac{1}{h}\left(E\left[\int_t^{t+h}|\hat{Z}_r^h|^2dr\right]+E\left[\int_t^{t+h}\int_E|\hat{U}_r^h(e)|^2\lambda(de)dr\right]\right)\leq C\sqrt{h},\quad h>0.$$

We suppose now that

$$\frac{\partial}{\partial t}\varphi(t,x)+L\varphi(t,x)+f_i(t,x,u(t,x),(\nabla\varphi\sigma)(t,x),B_i\varphi(t,x))<0,$$

and find a contradiction by using the above estimate. Indeed, under the above assumptions, there exist some $\delta>0$ and some $h_0>0$ such that, for all $0<h\leq h_0$,

$$\begin{aligned}\xi_h \quad &:=\frac{1}{h}E\int_t^{t+h}[\psi(r,X_r^t(x))\\&+f_i(r,X_r^t(x),(\varphi(r,X_r^t(x)),\tilde{u}_i(r,X_r^t(x))),(\nabla\varphi\sigma)(r,X_r^t(x)),\\&B_i\varphi(r,X_r^t(x))]dr\\&\leq -\delta.\end{aligned}$$

Now we have that $\hat{Y}_t^h\geq 0$, hence

$$\begin{aligned}0&\leq h^{-1}\hat{Y}_t^h\\&=\frac{1}{h}E\int_t^{t+h}[\psi(r,X_r^t(x))+f_i(r,X_r^t(x),(\varphi(r,X_r^t(x))+\hat{Y}_r^h,\tilde{u}_i(r,X_r^t(x))),\\&(\nabla\varphi\sigma)(r,X_r^t(x))+\hat{Z}_r^h,B_i\varphi(r,X_r^t(x))+\int_E\hat{U}_r^h(e)\gamma_i(X_r^t(x),e)\lambda(de))]dr.\end{aligned}$$

Therefore, for all $0<h\leq h_0$,

$$\begin{aligned}\delta&\leq|h^{-1}\hat{Y}_t^h-\xi_h|\\&\leq cE\left[\frac{1}{h}\int_t^{t+h}\left(|\hat{Y}_s^h|+|\hat{Z}_s^h|+\left(\int_E|\hat{U}_s^h(e)|^2\lambda(de)\right)^{1/2}\right)ds\right]\\&\leq c'\left(\sup_{t\leq s\leq t+h}E[|\hat{Y}_s^h|^2]^{1/2}+\left(\frac{1}{h}E\left[\int_t^{t+h}|\hat{Z}_r^h|^2dr\right]\right)^{1/2}\right.\\&\left.+\left(\frac{1}{h}E\left[\int_t^{t+h}\int_E|\hat{U}_r^h(e)|^2\lambda(de)dr\right]\right)^{1/2}\right)\\&\leq ch^{1/4}.\end{aligned}$$

This is impossible, consequently

$$-\frac{\partial}{\partial t}\varphi(t,x) - L\varphi(t,x) - f_i(t,x,u(t,x),(\nabla\varphi\sigma)(t,x),B_i\varphi(t,x)) \leq 0.$$

In view of Lemma 3.3, this completes the proof. □

Now we give a uniqueness result for (3.1). This result is obtained under more restrictive assumptions that the existence one : namely we need the two following additional assumptions

(A.2 v) $\quad |f_i(t,x,r,p,q) - f_i(t,y,r,p,q)| \leq m_R^i(|x-y|(1+|p|))$

for $1 \leq i \leq k$, where $m_R^i(s) \to 0$ when $s \to 0^+$, for all $t \in [0,T], |x|, |y| \leq R$, $|r| \leq R,\ p \in \mathbb{R}^d,\ q \in \mathbb{R}\ (\forall R < \infty)$.

For γ, we assume in addition

(A.2 vi) $\quad |\gamma_i(x,e) - \gamma_i(y,e)| \leq C_1|x-y|(1 \wedge |e|^2)$ for $1 \leq i \leq k$

for some constant $C_1 > 0$ and for any $x, y \in \mathbb{R}^d, e \in E$.

Our result is the

THEOREM 3.5 *Assume that f, g and γ satisfy $(A2)$. Then there exists at most one viscosity solution u of* (3.1) *such that*

$$\lim_{|x|\to+\infty} |u(t,x)|e^{-\tilde{A}[\log(|x|)]^2} = 0, \tag{3.3}$$

uniformly for $t \in [0,T]$, for some $\tilde{A} > 0$.

In particular, the function $u(t,x) = Y_t^t(x)$ is the unique viscosity solution of (3.1) *in the class of solutions which satisfy* (3.3) *for some $\tilde{A} > 0$.*

Remark 3.6 Notice that, by Proposition 2.5, $u(t,x) = Y_t^t(x)$ has at most a polynomial growth at infinity and therefore it satisfies (3.3).

The growth condition (3.3) is optimal to get such a uniqueness result for (3.1). Indeed, consider the equation

$$\frac{\partial u}{\partial t} - \frac{x^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{x}{2}\frac{\partial u}{\partial x} = 0 \quad \text{in}(0,T) \times (0,+\infty); \tag{3.4}$$

then u is a solution of (3.4) if and only if the function $v(t,y) = u(t,e^y)$ is a solution of the Heat Equation

$$\frac{\partial v}{\partial t} - \frac{1}{2}\frac{\partial^2 v}{\partial x^2} = 0 \quad \text{in}(0,T) \times \mathbb{R}. \tag{3.5}$$

But it is well-known that, for the Heat Equation, the uniqueness holds in the class of solutions v satisfying

$$\lim_{|y|\to+\infty} |v(t,y)|e^{-\tilde{A}|y|^2} = 0, \tag{3.6}$$

uniformly for $t \in [0, T]$, for some $\tilde{A} > 0$. And (3.6) gives back (3.3) for (3.4) since $y = \log(x)$.

Let us finally mention that, in our case, the growth condition (3.3) is mainly a consequence of the assumptions on the coefficients of the differential operator and in particular on $a = (a_{i,j})_{i,j}$; under the assumptions of Theorem 3.5, the matrix a has a priori a quadratic growth at infinity. If a is assumed to have a linear growth at infinity, an easy adaptation of the proof of Theorem 3.5 shows that the uniqueness holds in the class of solutions satisfying

$$\lim_{|x|\to+\infty} |u(t,x)|e^{-\tilde{A}|x|} = 0,$$

uniformly for $t \in [0, T]$, for some $\tilde{A} > 0$.

Proof of Theorem 3.5 Let u and v be two viscosity solutions of (3.1). The proof consists in two steps. We first show that $u - v$ and $v - u$ are viscosity subsolutions of an integral partial differential system; then we build a suitable sequence of smooth supersolutions of this system to show that $|u - v| = 0$ in $[0, T] \times \mathbb{R}^d$. Here and below, we denote by $|\cdot|$ the sup norm in $\mathbb{R}^k$.

LEMMA 3.7 *Let u be a subsolution and v a supersolution of* (3.1). *Then the function $\omega := u - v$ is a viscosity subsolution of the system*

$$-\frac{\partial \omega_i}{\partial t} - L\omega_i - \widetilde{K}\big[|\omega| + |\nabla\omega_i\sigma| + (B_i\omega_i)^+\big] = 0 \text{ in } [0,T] \times \mathbb{R}^d \tag{3.7}$$

for $1 \le i \le k$, where $\widetilde{K}$ is the Lipschitz constant of f in (r, p, q).

Proof Let $\varphi \in C^2([0,T] \times \mathbb{R}^d)$ and let $(t_0, x_0) \in (0,T) \times \mathbb{R}^d$ be a strict global maximum point of $\omega_i - \varphi$ for some $1 \le i \le k$.

We introduce the function

$$\psi_{\varepsilon,\alpha}(t,x,s,y) = u_i(t,x) - v_i(s,y) - \frac{|x-y|^2}{\varepsilon^2} - \frac{(t-s)^2}{\alpha^2} - \varphi(t,x)$$

where ε, α are positive parameters which are devoted to tend to zero.

Since (t_0, x_0) is a strict global maximum point of $u_i - v_i - \varphi$, by a classical argument in the theory of viscosity solutions, there exists a sequence $(\bar{t}, \bar{x}, \bar{s}, \bar{y})$ such that

(i) $(\bar{t}, \bar{x}, \bar{s}, \bar{y})$ is a global maximum point of $\psi_{\varepsilon,\alpha}$ in $([0, T] \times \overline{B}_R)^2$ where B_R is a ball with a large radius R.

(ii) $(\bar{t}, \bar{x}), (\bar{s}, \bar{y}) \to (t_0, x_0)$ as $(\varepsilon, \alpha) \to 0$.

(iii) $\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2}, \frac{(\bar{t}-\bar{s})^2}{\alpha^2}$ are bounded and tend to zero when $(\varepsilon, \alpha) \to 0$.

We have dropped above the dependence of $\bar{t}, \bar{x}, \bar{s}$ and $\bar{y}$ in ε and α for the sake of simplicity of notations.

It follows from Theorem 8.3 in [4] that there exists $X, Y \in \mathcal{S}^d$ such that

$$\left(\bar{a} + \frac{\partial \varphi}{\partial t}(\bar{t}, \bar{x}), \bar{p} + D\varphi(\bar{t}, \bar{x}), X\right) \in \overline{D}^{2,+} u_i(\bar{t}, \bar{x})$$

$$(\bar{a}, \bar{p}, Y) \in \overline{D}^{2,-} v_i(\bar{s}, \bar{y})$$

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{4}{\varepsilon^2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} D^2\varphi(\bar{t}, \bar{x}) & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$\bar{a} = \frac{2(\bar{t} - \bar{s})}{\alpha^2} \text{ and } \bar{p} = \frac{2(\bar{x} - \bar{y})}{\varepsilon^2}.$$

Modifying if necessary $\psi_{\varepsilon,\alpha}$ by adding terms of the form $\chi(x)$ and $\chi(y)$ with supports in $B^c_{R/2}$, we may assume that $(\bar{t}, \bar{x}, \bar{s}, \bar{y})$ is a global maximum point of $\psi_{\varepsilon,\alpha}$ in $([0, T] \times \mathbb{R}^d)^2$. Since u and v are respectively sub and supersolution of (3.1), we have for δ small enough

$$\begin{aligned}
&\bar{a} - \frac{\partial \varphi}{\partial t}(\bar{t}, \bar{x}) - \frac{1}{2} Tr(a(\bar{x})X) - \langle b(\bar{x}), \bar{p} + D\varphi(\bar{t}, \bar{x})\rangle \\
&- \int_{E_\delta} \frac{|\beta(\bar{x}, e)|^2}{\varepsilon^2} \lambda(de) - \int_{E_\delta} (\varphi(\bar{t}, \bar{x} + \beta(\bar{x}, e)) - \varphi(\bar{t}, \bar{x}) \\
&- \langle D\varphi(\bar{t}, \bar{x}), \beta(\bar{x}, e)\rangle)\lambda(de) - \int_{E^c_\delta} (u_i(\bar{t}, \bar{x} + \beta(\bar{x}, e)) - u_i(\bar{t}, \bar{x}) \\
&- \langle \bar{p} + D\varphi(\bar{t}, \bar{x}), \beta(\bar{x}, e)\rangle)\lambda(de) \\
&- f_i(\bar{t}, \bar{x}, u(\bar{t}, \bar{x}), (\bar{p} + D\varphi(\bar{t}, \bar{x}))\sigma(\bar{x}), \widehat{B}^\delta_i) \leq 0
\end{aligned}$$

where

$$\widehat{B}_i^\delta = \int_{E^\delta} \left(\langle \overline{p}, \beta(\overline{x}, e) \rangle + \frac{|\beta(\overline{x}, e)|^2}{\varepsilon^2} \right) \gamma_i(\overline{x}, e) \lambda(de)$$
$$\int_{E_\delta} (\varphi(\overline{t}, \overline{x} + \beta(\overline{x}, e)) - \varphi(\overline{t}, \overline{x})) \gamma_i(\overline{x}, e) \lambda(de)$$
$$+ \int_{E_\delta^c} (u_i(\overline{t}, \overline{x} + \beta(\overline{x}, e)) - u_i(\overline{t}, \overline{x})) \gamma_i(\overline{x}, e) \lambda(de)$$

and

$$\overline{a} - \frac{1}{2} Tr(a(\overline{y}) Y - \langle b(\overline{y}), \overline{p} \rangle + \int_{E_\delta} \frac{|\beta(\overline{y}, e)|^2}{\varepsilon^2} \lambda(de)$$
$$- \int_{E_\delta^c} (v_i(\overline{s}, \overline{y} + \beta(\overline{y}, e)) - v_i(\overline{s}, \overline{y}) - \langle \overline{p}, \beta(\overline{y}, e) \rangle) \lambda(de)$$
$$- f_i(\overline{s}, \overline{y}, v(\overline{s}, \overline{y}), \overline{p}\sigma(\overline{y}), \widehat{\widehat{B}}_i^\delta) \geq 0$$

where

$$\widehat{\widehat{B}}_i^\delta = \int_{E_\delta} \left(- \langle \overline{p}, \beta(\overline{y}, e) \rangle - \frac{|\beta(\overline{y}, e)|^2}{\varepsilon^2} \right) \gamma_i(\overline{y}, e) \lambda(de)$$
$$+ \int_{E_\delta^c} (v_i(\overline{s}, \overline{y} + \beta(\overline{y}, e)) - v_i(\overline{s}, \overline{y})) \gamma_i(\overline{y}, e) \lambda(de).$$

It is worth noticing that the χ terms we have to add to $\psi_{\varepsilon,\alpha}$ to have a global maximum point do not appear in the two inequalities above because β is bounded and they have a support which is included in $\overline{B}_{R/2}^c$ for R large.

Of course, we are going to substract these inequalities and we need to estimate differences between terms of the same type.

First, if $(e_1 \ldots e_d)$ is an orthonormal basis of $\mathbb{R}^d$,

$$Tr(a(\overline{x})X) - Tr(a(\overline{y})Y) = Tr(\sigma^*(\overline{x}) X \sigma(\overline{x})) - Tr(\sigma^*(\overline{y}) Y \sigma(\overline{y}))$$
$$= \sum_{i=1}^{d} [\langle X\sigma(\overline{x})e_i, \sigma(\overline{x})e_i \rangle - \langle Y\sigma(\overline{y})e_i, \sigma(\overline{y})e_i \rangle]$$

To estimate this sum, we use the matrix inequality above together with the Lipschitz continuity of σ. We get

$$Tr(a(\overline{x})X) - Tr(a(\overline{y})Y) \leq C \frac{|\overline{x} - \overline{y}|^2}{\varepsilon^2} + Tr(a(\overline{x}) D^2 \varphi(\overline{t}, \overline{x}))$$

for some constant C. Then

$$|\langle b(\overline{x}),\overline{p}\rangle - \langle b(\overline{y}),\overline{p}\rangle| \leq C_1|\overline{x}-\overline{y}||\overline{p}| \leq C\frac{|\overline{x}-\overline{y}|^2}{\varepsilon^2}$$

because of the Lipschitz continuity of b.

To estimate the differences of the integro-differential term, we strongly use the fact that $(\overline{t},\overline{x},\overline{s},\overline{y})$ is a global maximum point of $\psi_{\varepsilon,\alpha}$ in $\overline{B}_{R/2}$. From the inequality

$$\psi_{\varepsilon,\alpha}(\overline{t},\overline{x}+\beta(\overline{x},e),\overline{s},\overline{y}+\beta(\overline{y},e)) \leq \psi_{\varepsilon,\alpha}(\overline{t},\overline{x},\overline{s},\overline{y})$$

we deduce

$$\begin{aligned}
&[u_i(\overline{t},\overline{x}+\beta(\overline{x},e)) - u_i(\overline{t},\overline{x})] - [v_i(\overline{s},\overline{y}+\beta(\overline{y},e)) - v_i(\overline{s},\overline{y})] \\
&- \langle\overline{p},\beta(\overline{x},e)-\beta(\overline{y},e)\rangle - \frac{1}{\varepsilon^2}|\beta(\overline{x},e)-\beta(\overline{y},e)|^2 \\
&\leq \varphi(\overline{t},\overline{x}+\beta(\overline{x},e)) - \varphi(\overline{t},\overline{x}).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{E_\delta^c}(u_i(\overline{t},\overline{x}+\beta(\overline{x},e)) - u_i(\overline{t},\overline{x}) - \langle\overline{p}+D\varphi(\overline{t},\overline{x}),\beta(\overline{x},e)\rangle)\lambda(de) \\
&- \int_{E_\delta^c}(v_i(\overline{s},\overline{y}+\beta(\overline{y},e)) - v_i(\overline{t},\overline{y}) - \langle\overline{p},\beta(\overline{y},e)\rangle)\lambda(de) \\
&\leq \int_{E_\delta^c}[\varphi(\overline{t},\overline{x}+\beta(\overline{x},e)) - \varphi(\overline{t},\overline{x}) - \langle D\varphi(\overline{t},\overline{x}),\beta(\overline{x},e)\rangle]\lambda(de) \\
&+ \int_{E_\delta^c}\frac{|\beta(\overline{x},e)-\beta(\overline{y},e)|^2}{\varepsilon^2}\lambda(de)
\end{aligned}$$

Notice that the last term of the right-hand side is estimated by $\frac{C|\overline{x}-\overline{y}|^2}{\varepsilon^2}$ with C independent of δ, because of the assumptions on β. In the same way,

$$\begin{aligned}
\widehat{B}_i^\delta - \widehat{\widehat{B}}_i^\delta \leq &\int_{E_\delta}\left(\langle\overline{p},\beta(\overline{x},e)\rangle + \frac{|\beta(\overline{x},e)|^2}{\varepsilon^2}\right)\gamma_i(\overline{x},e)\lambda(de) \\
&+ \int_{E_\delta}\left(\langle\overline{p},\beta(\overline{y},e)\rangle + \frac{|\beta(\overline{y},e)|^2}{\varepsilon^2}\right)\gamma_i(\overline{y},e)\lambda(de) \\
&+ \int_E[\varphi(\overline{t},\overline{x}+\beta(\overline{x},e)) - \varphi(\overline{t},\overline{x})]\gamma_i(\overline{x},e)\lambda(de) \\
&+ \int_{E_\delta^c}[v_i(\overline{s},\overline{y}+\beta(\overline{y},e)) - v_i(\overline{s},\overline{y})](\gamma_i(\overline{x},e)-\gamma_i(\overline{y},e))\lambda(de) \\
&+ \int_{E_\delta^c}[\langle\overline{p},\beta(\overline{x},e)-\beta(\overline{y},e)\rangle + \frac{1}{\varepsilon^2}|\beta(\overline{x},e)-\beta(\overline{y},e)|^2]\gamma_i(\overline{x},e)\lambda(de)
\end{aligned}$$

Because of the assumption on β and γ, the last integral is estimated by $\frac{C|\bar{x}-\bar{y}|^2}{\varepsilon^2}$ where C is independent of δ and the preceeding one is estimated by $C|\bar{x}-\bar{y}|$ since v is continuous (and therefore locally bounded) and because of the additional assumption (A2.vi) made on γ in the statement of Theorem 3.5.

Finally, we consider the difference between the nonlinear terms

$$\begin{aligned} &f_i(\bar{t},\bar{x},u(\bar{t},\bar{x}),(\bar{p}+D\varphi(\bar{t},\bar{x}))\sigma(\bar{x}),\widehat{B}_i^\delta) - f_i(\bar{s},\bar{y},v(\bar{s},\bar{y}),\bar{p}\sigma(\bar{y}),\widehat{\widehat{B}}{}_i^\delta) \\ &\quad \le \rho_{\varepsilon,\delta}(|\bar{t}-\bar{s}|) + m^i(|\bar{x}-\bar{y}|(1+|\bar{p}\sigma(\bar{y})|)) + \widetilde{K}|u(\bar{t},\bar{x})-v(\bar{s},\bar{y})| \\ &\qquad + \widetilde{K}|\bar{p}(\sigma(\bar{x})-\sigma(\bar{y})) + D\varphi(\bar{t},\bar{x})\sigma(\bar{x})| + \widetilde{K}(\widehat{B}_i^\delta - \widehat{\widehat{B}}{}_i^\delta)^+. \end{aligned}$$

The first term in the right-hand side comes from the continuity of f_i in t : $\rho_{\varepsilon,\delta}(s) \to 0$ when $s \to 0^+$ for fixed ε and δ. The second term comes from (A.2 v): we have denoted by m^i the modulus m_R^i which appears in this assumption for R large enough. The three last terms come from the Lipschitz continuity of f_i w.r.t. the three last variables and the fact that it is non-decreasing with respect to the last one.

We notice that

$$|\bar{p}(\sigma(\bar{x})-\sigma(\bar{y}))| \le C\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2}$$

because of the Lipschitz continuity of σ and that

$$|\bar{x}-\bar{y}|\cdot|\bar{p}\sigma(\bar{y})| \le C\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2}.$$

Now we substract the viscosity inequalities for u and v : thanks to the above estimates, we can write the obtained inequality in the following way

$$\begin{aligned} &-\frac{\partial\varphi}{\partial t}(\bar{t},\bar{x}) - A\varphi(\bar{t},\bar{x}) - K\varphi(\bar{t},\bar{x}) - \widetilde{K}|u(\bar{t},\bar{x})-v(\bar{s},\bar{y})| \\ &-\widetilde{K}|D\varphi(\bar{t},\bar{x})\sigma(\bar{x})| - \widetilde{K}(B_i\varphi(\bar{t},\bar{x}))^+ \le \rho_{\varepsilon,\delta}(|\bar{t}-\bar{s}|) + \omega_1(\varepsilon,\alpha) + \omega_2^\varepsilon(\delta) \end{aligned}$$

where we have gathered in the $\omega_1(\varepsilon,\alpha)$ term, all the term of the form $\frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2}$ and $|\bar{x}-\bar{y}|$; $\omega_1(\varepsilon,\alpha) \to 0$ when (ε,α) tends to 0. The $\omega_2^\varepsilon(\delta)$ term contains all the remaining integrals on E_δ. To conclude we first let α go to zero : since $\frac{|\bar{t}-\bar{s}|^2}{\alpha^2}$ is bounded, $|\bar{t}-\bar{s}| \to 0$ and we get rid of the first term of the right-hand side above. Then we let δ go to zero keeping ε fixed and finally we let $\varepsilon \to 0$. Since $(\bar{t},\bar{x}),(\bar{s},\bar{y}) \to (t_0,x_0)$, we obtain :

$$\begin{aligned} -\frac{\partial\varphi}{\partial t}(t_0,x_0) - L\varphi(t_0,x_0) - \widetilde{K}|\omega(t_0,x_0)| - \widetilde{K}|D\varphi(t_0,x_0)\sigma(x_0)| \\ - \widetilde{K}(B_i\varphi)^+(t_0,x_0) \le 0 \end{aligned}$$

and therefore ω is a subsolution of the desired equation by Lemma 3.3.

Now we are going to build suitable smooth supersolutions for the equation (3.7).

LEMMA 3.8 *For any $\tilde{A} > 0$, there exists $C_1 > 0$ such that the function*

$$\chi(t,x) = \exp[(C_1(T-t) + \tilde{A})\psi(x)]$$

where

$$\psi(x) = [\log((|x|^2+1)^{1/2}) + 1]^2,$$

satisfies

$$-\frac{\partial \chi}{\partial t} - L\chi - \widetilde{K}\chi - \widetilde{K}|D\chi\sigma| - \widetilde{K}(B_i\chi)^+ > 0 \text{ in } [t_1, T] \times \mathbb{R}^d$$

for $1 \le i \le k$ where $t_1 = T - \tilde{A}/C_1$.

Proof We first estimate the terms $K\chi$ and $B_i\chi$, the main point being their dependence in x. For the sake of simplicity of notations, we denote below by C all the positive constants which enter in these estimates. These constants depend only on $\tilde{A}$ and on the bounds on the coefficients of the equations.

We first give estimates on the first and second derivatives of ψ : easy computations yield

$$|D\psi(x)| \le \frac{2[\psi(x)]^{1/2}}{(|x|^2+1)^{1/2}} \le 4 \quad \text{in } \mathbb{R}^d,$$

and

$$|D^2\psi(x)| \le \frac{C(1+[\psi(x)]^{1/2})}{|x|^2+1} \quad \text{in } \mathbb{R}^d.$$

These estimates imply that, if $t \in [t_1, T]$

$$\begin{aligned}|D\chi(t,x)| &\le (C_1(T-t)+\tilde{A})\chi(t,x)|D\psi(x)| \\ &\le C\chi(t,x)\frac{[\psi(x)]^{1/2}}{(|x|^2+1)^{1/2}},\end{aligned}$$

and, in the same way

$$|D^2\chi(t,x)| \le C\chi(t,x)\frac{\psi(x)}{|x|^2+1}.$$

It is worth noticing that, because of our choice of t_1, the above estimates do not depend on C_1.

Then, since β is bounded and since ψ is lipschitz continuous in $\mathbb{R}^d$, tedious but straight-forward computations imply

$$\chi(t,x+\beta(x,e))-\chi(t,x)-\langle D\chi(x,t),\beta(x,e)\rangle \leq C\chi(t,x)\frac{\psi(x)}{|x|^2+1}|\beta(x,e)|^2,$$

and

$$\chi(t,x+\beta(x,e))-\chi(t,x) \leq C\chi(t,x)\frac{[\psi(x)]^{1/2}}{(|x|^2+1)^{1/2}}|\beta(x,e)|.$$

Since σ and b grow at most linearly at infinity, we have

$$-\frac{\partial\chi}{\partial t}(t,x)-L\chi(t,x)-\widetilde{K}\chi(t,x)-\widetilde{K}|D\chi(t,x)\sigma(x)|-\widetilde{K}(B_i\chi)^+$$
$$\geq \chi\left[C_1\psi(x)-C\psi(x)+C[\psi(x)]^{1/2}-C\frac{\psi(x)}{|x|^2+1}-\widetilde{K}-C\widetilde{K}[\psi(x)]^{1/2}\right.$$
$$\left.-C\widetilde{K}\frac{[\psi(x)]^{1/2}}{(|x|^2+1)^{1/2}}\right]$$

Since $\psi(x)\geq 1$ in $\mathbb{R}^d$, by using Cauchy-Schwartz inequality, it is clear enough that for C_1 large enough the quantity in the brackets is positive and the proof is complete. □

To conclude the proof, we are going to show that $\omega = u - v$ satisfies

$$|\omega(t,x)| \leq \alpha\chi(t,x) \quad \text{in } [0,T]\times\mathbb{R}^d$$

for any $\alpha > 0$. Then we will let α tend to zero.

To prove this inequality, we first remark that because of (3.3)

$$\lim_{|x|\to+\infty}|\omega(t,x)|e^{-\tilde{A}[\log((|x|^2+1)^{1/2})]^2}=0$$

uniformly for $t\in[0,T]$, for some $\tilde{A}>0$. This implies, in particular, that $|\omega_i|-\alpha\chi$ is bounded from above in $[t_1,T]\times\mathbb{R}^d$ for any $1\leq i\leq k$ and that

$$M=\max_{1\leq i\leq k}\max_{[t_1,T]\times\mathbb{R}^d}(|\omega_i|-\alpha\chi)(t,x)e^{\widetilde{K}(T-t)}$$

is achieved at some point (t_0,x_0) and for some i_0.

We first remark that, since $|\cdot|$ is the sup norm in $\mathbb{R}^k$, we have

$$M=\max_{[t_1,T]\times\mathbb{R}^d}(|\omega|-\alpha\chi)(t,x)e^{\widetilde{K}(T-t)}$$

and $|\omega_{i_0}(t_0,x_0)| = |\omega(t_0,x_0)|$. We may assume w.l.o.g. that $|\omega_{i_0}(t_0,x_0)| > 0$, otherwise we are done.

Then two cases : either $\omega_{i_0}(t_0,x_0) > 0$ or $\omega_{i_0}(t_0,x_0) < 0$. We treat the first case, the second one is treated in a similar way since the roles of u and v are symmetric.

From the maximum point property, we deduce that

$$\omega_{i_0}(t,x) - \alpha\chi(t,x) \leq (\omega_{i_0} - \alpha\chi)(t_0,x_0)e^{\widetilde{K}(t-t_0)}$$

and this inequality can be interpreted as the property for the function $\omega_{i_0} - \phi$ to have a global maximum point at (t_0,x_0) where

$$\phi(t,x) = \alpha\chi(t,x) + (\omega_{i_0} - \alpha\chi)(t_0,x_0)e^{\widetilde{K}(t-t_0)}$$

Since ω is a viscosity subsolution of (3.7), if $t_0 \in [t_1,T]$, we have

$$-\frac{\partial\phi}{\partial t}(t_0,x_0) - L\phi(t_0,x_0) - \widetilde{K}|\omega(t_0,x_0)| - \widetilde{K}|D\phi(t_0,x_0)\sigma(x_0)| - \widetilde{K}(B_i\phi)^+ \leq 0.$$

But the left-hand side of this inequality is nothing but

$$\alpha\Bigg[-\frac{\partial\chi}{\partial t}(t_0,x_0) - L\chi(t_0,x_0) - \widetilde{K}\chi(t_0,x_0) - \widetilde{K}|D\chi(t_0,x_0)\sigma(x_0)| \\ -\widetilde{K}(B_i\chi)^+(t_0,x_0)\Bigg]$$

since $\omega_{i_0}(t_0,x_0) = |\omega(t_0,x_0)|$; so, by Lemma 3.8, we have a contradiction. Therefore $t_0 = T$ and since $|\omega(T,x)| = 0$, we have

$$|\omega(t,x)| - \alpha\chi(t,x) \leq 0 \text{ in } [t_1,T] \times \mathbb{R}^d.$$

Letting α tend to zero, we obtain

$$|\omega(t,x)| = 0 \text{ in } [t_1,T] \times \mathbb{R}^d.$$

Applying successively the same argument on the intervals $[t_2,t_1]$ where $t_2 = (t_1 - \tilde{A}/C_1)^+$ and then, if $t_2 > 0$ on $[t_3,t_2]$ where $t_3 = (t_2 - \tilde{A}/C_1)^+ \ldots$ etc. We finally obtain that

$$|\omega(t,x)| = 0 \text{ in } [0,T] \times \mathbb{R}^d$$

and the proof is complete.

Remark 3.9 The assumption (A.2 vi) on γ is used in the proof to estimate the difference $(\widehat{B}_i^\delta - \widehat{\widehat{B}}_i^\delta)^+$: if u or v is assumed to be locally Lipschitz continuous this additional assumption is not necessary anymore to obtain the result of Theorem 3.5.

Moreover if the functions f_i satisfy $r_\ell \to f_i(t,x,r,p,q)$ is non-decreasing for $\ell \neq i$ for any $t \in [0,T], x \in \mathbb{R}^d, p \in \mathbb{R}^d, q \in \mathbb{R}$ and $(r_1,\ldots,r_{\ell-1},r_{\ell+1},\ldots,r_k)$ in $\mathbb{R}^{k-1}$ then easy modifications in the proof show that a comparison result is true for (3.1). More precisely, if u, v are respectively viscosity sub- and supersolutions of (3.1) satisfying (3.3) and

$$u(T,x) \leq v(T,x) \ in \ \mathbb{R}^d$$

then

$$u(t,x) \leq v(t,x) \ in \ [0,T] \times \mathbb{R}^d.$$

Our proof, which avoids this assumption, is inspired from H. Ishii and S. Koike [7].

Finally, we want to mention that, under the above monotonicity assumptions on the f_i, the case of semicontinuous solutions can also be treated. We refer to O. Alvarez and A. Tourin [1] for a complete description of the properties of semicontinuous solutions for such integral partial differential equations.

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