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On the regularization effect of space-time white noise on quasi-linear parabolic partial differential equations

István Gyöngy[1,⋆] **and E. Pardoux**[2,⋆⋆]

[1] Dept. of Probability Theory & Statistics, Eötvös Loránd University Budapest, Múzeum krt. 6–8, H-1088 Budapest, Hungary
[2] Université de Provence, Mathématiques URA 225, F-13331 Marseille Cedex 3, France

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Summary. We show that an existence and uniqueness and a comparison theorem hold if we add a space time white noise to a quasi-linear parabolic equation in one space dimension, even if the nonlinearity is only measurable and not even locally bounded.

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1 Introduction

We consider the quasi-linear stochastic partial differential equation

$$\frac{\partial u}{\partial t}(t, x) = \frac{\partial^2}{\partial x^2} u(t, x) + f(u)(t, x) + \frac{\partial^2}{\partial t \partial x} W(t, x) \qquad \text{(Eq(f))}$$

with Neumann boundary condition

$$\frac{\partial}{\partial x} u(t, 0) = \frac{\partial}{\partial x} u(t, 1) = 0 \quad t \in [0, T]$$

and the initial condition

$$u(0, x) = u_0(x) \quad x \in [0, 1]$$

where $\frac{\partial^2}{\partial t \partial x} W$ is a space time white noise, and $f(u)(t, x)$ denotes $f(t, x, u(t, x))$ for a function

$$f: [0, T] \times [0, 1] \times \mathbf{R} \to \mathbf{R}.$$

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The solution is understood in the generalized sense defined in Walsh [10]. It is well-known from [10] that the above problem admits a unique solution if f satisfies a linear growth and a Lipschitz condition. In [2] we prove that there exists a unique solution even in the case when f is locally bounded and only measurable. Moreover a comparison theorem for the solutions of such equations is obtained, which generalizes the comparison theorem of [1]. In the present paper we extend these results to the case when f is not even locally bounded, but it satisfies only some integrability condition. We use the methods of our paper [2]. The important point is that we can extend our estimate from [2] on the distributions of the solutions of Eq(f) to the case when an appropriate power of f is locally integrable. Moreover we show also that the solution depends continuously on the initial value u_0, and we approximate it in $C([0, T] \times [0, 1])$. This implies, in particular, that the solution is a $C[0, 1]$-valued Markov process. We remark that using the exponential estimate of Theorem 3.2.1 of the present paper one can also show its germ-field Markov property by the method of [7].

In the last section we adapt the results obtained for (Eq(f)) with Neumann boundary condition to the case of the same equation with Dirichlet condition.

We note that for measurable locally unbounded f the equation (Eq(f)) without the space-time white noise may not have a solution, or it may not be unique (and the continuous dependence on the initial value may fail). Obviously our results are also valid for Eq(f) with $c \frac{\partial^2}{\partial t \partial x} W(t, x)$ instead of $\frac{\partial^2}{\partial t \partial x} W$ if $c \neq 0$. That means arbitrary small space-time white noise regularizes the quasi-linear PDEs with irregular non-linearity of above type.

In the proof we make use of the fact that the solution of Eq(0) is a Gaussian random field of which density has the estimate given by Proposition 3.2.3 below. Getting similar estimates one can extend our results to the case where $\frac{\partial^2}{\partial x^2}$ is replaced by the more general operator

$$a(t, x, \omega) \frac{\partial^2}{\partial x^2} + b(t, x, \omega) \frac{\partial}{\partial x} + c(t, x, \omega) ,$$

provided a, b, c are progressively measurable, bounded and $a(t, x, \omega) \geqq \lambda > 0$.

Finally we note that the solutions we deal with are strong solutions in the sense the distinction between strong and weak solutions is made in the theory of SDEs. Analogous results for weak solutions of finite dimensional SDEs are obtained in Portenko [8], in Stummer [9] proving Girsanov theorem with locally unbounded drifts, and in [4] using deep estimates from [5]. (See also the references of these papers.)

2 Formulation of the problem and the results

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geqq 0}, P)$ be a stochastic basis carrying a space time white noise W on $\mathbf{R}_+ \times [0, 1]$. That means we are given an application $W: \mathscr{B}(\mathbf{R}_+ \times [0, 1]) \to L_2(\Omega, \mathscr{F}, P)$ such that

(i) $\forall A, B \in \mathscr{B}(\mathbf{R}_+ \times [0, 1])$ with $A \cap B = \emptyset$, $W(A)$ and $W(B)$ are independent Gaussian random variables:

(ii) $\forall C \in \mathscr{B}[0, 1]$, $\{W([0, t] \times C): t \geqq 0\}$ is an $\mathscr{F}_t$-Brownian motion with covariance $t\lambda(C)$ where λ is Lebesgue measure.

Let also be given an $\mathscr{F}_0$-measurable $C([0, 1])$-valued random variable u_0 and a function

$$f: \mathbf{R}_+ \times [0, 1] \times \mathbf{R} \to \mathbf{R}$$

which is $\mathscr{B}(\mathbf{R}_+ \times [0, 1] \times \mathbf{R})$-measurable.

We use the notation $\mathscr{B}(V)$ for the Borel σ-algebra on V (for a topological space V) and $\mathscr{P}$ for the progressively measurable subsets of $\mathbf{R}_+ \times \Omega$.

We now give a rigorous meaning to Eq(f) from the Introduction.

We say that a $\mathscr{P} \otimes \mathscr{B}[0, 1]$-measurable and continuous random field $u = \{u(t, x): t \in [0, T], x \in [0, 1]\}$ is a solution (on the interval $[0, T]$) of Eq(u_0;f) if for any $\varphi \in C^2([0, 1])$ s.t. $\varphi'(0) = \varphi'(1) = 0$,

$$\int_0^1 u(t, x)\varphi(x)\,dx = \int_0^1 u_0(x)\varphi(x)\,dx + \int_0^t \int_0^1 \left[u(s, x)\frac{\partial^2}{\partial x^2}\varphi(x) + f(u)(s, x)\varphi(x) \right] dx\,ds$$

$$+ \int_0^t \int_0^1 \varphi(x)\, W(ds, dx) \quad t \in [0, T] \quad \text{(a.s.)}$$

where the last integral is a Wiener integral, and

$$f(u)(s, x) := f(s, x, u(s, x)) .$$

When f is bounded then u is a solution of Eq(u_0;f) iff u satisfies

$$\begin{aligned} u(t, x) = & \int_0^1 G_t(x, y) u_0(y)\,dy \\ & + \int_0^t \int_0^1 G_{t-s}(x, y) f(u)(s, y)\,dy\,ds + \int_0^t \int_0^1 G_{t-s}(x, y)\, W(ds, dy) \end{aligned} \tag{2.1}$$

$$t \geqq 0, \quad 0 \leqq x \leqq 1 \quad \text{(a.s.)}$$

where

$$G_t(x, y) = \frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{\infty} \left\{ \exp\left(-\frac{(y - x - 2n)^2}{4t} \right) + \exp\left(-\frac{(y + x - 2n)^2}{4t} \right) \right\}$$

is the fundamental solution of the heat equation on $\mathbf{R}_+ \times [0, 1]$ with Neumann boundary condition (see [10]). It is well known from [10] that Eq(u_0;f) admits a unique solution when f is Lipschitz in r and satisfies the linear growth condition. Moreover in this case one has the following comparison of the solutions: If u and v are solutions to Eq(u_0;f) and Eq(v_0; F) respectively where u_0, v_0 are $C([0, 1])$-valued $\mathscr{F}_0$-measurable random fields and f, F are Lipschitz functions in r s.t.

$$u_0(x) \leqq v_0(x) \quad \text{(a.s.)} \quad \text{for all } x \in [0, 1]$$

$$f(t, x, r) \leqq F(t, x, r) \quad dt \otimes dx \quad \text{a.e.} (t, x) \quad \text{for all } r,$$

then $u(t, x) \leqq v(t, x)$ a.s. for all t, x. (See [1].)

In [2] we extended these results to the case of locally bounded f, which are only Borel measurable. In the present paper we show that an existence and uniqueness theorem and a comparison theorem hold even without assuming that f is locally bounded. Instead of the local boundedness of f we assume a condition on local integrability of f.

To formulate the results let $L_{p,\beta,\infty}$ denote the space of Borel measurable functions $f: [0, T] \times [0, 1] \times \mathbf{R} \to \mathbf{R}$ s.t.

$$|f|_{p,\beta,\infty} := \sup_{x \in [0,1]} \left(\int_0^T \left(\int |f(t,x,r)|^p dr \right)^{\frac{\beta}{p}} dt \right)^{\frac{1}{\beta}} < \infty .$$

If f does not depend on $x \in [0, 1]$ then we write $f \in L_{p,\beta}$ instead of $f \in L_{p,\beta,\infty}$ and $|f|_{p,\beta}$ instead $|f|_{p,\beta,\infty}$.

The main result (Theorem 2.7 below) is an existence and uniqueness result for Eq(u_0;$f + g$), where f is in some $L_{p,\beta,\infty}$ and g is locally bounded and satisfies a growth condition with respect to r. We now state our results. The proof of Theorem 2.1 and Theorem 2.5 will be given in the last section, while the other results (except for Theorem 2.6, whose proof is very close to the uniqueness proof in [2], and which we therefore do not repeat) will be deduced below from Theorem 2.1.

Let $\{u_{0n}\}$ be a sequence of $C([0, 1])$-valued $\mathscr{F}_0$-measurable random variables and let $\{f_n\}$ be a sequence of Borel functions $f_n: [0, T] \times [0, 1] \times \mathbf{R} \to \mathbf{R}$, s.t. the following conditions hold:

(i) $u_{0n} \to u_0$ almost surely in $C([0, 1])$;
(ii) $f_n(t, x, r) \to f(t, x, r)$ $dt \otimes dx \otimes dr$ a.e.
(iii) there exist a constant C and a function $F \in L_{p,\beta,\infty}$, for some $p > 1$, $\beta > 4p/(4p-1)$, such that

$$|f_n(t,x,r)|^2 \leqq C + F(t,x,r) \quad dt \otimes dx \otimes dr \quad \text{a.e.}$$

Theorem 2.1 *Assume* (i)–(iii). *Suppose that f_n is Lipschitzian in $r \in \mathbf{R}$ for each n, uniformly in $(t, x, r) \in [0, T] \times [0, 1] \times [-R, R]$ for each R. Then the solution u_n of* Eq(u_{0n};f_n) *converges almost surely to a random field $\{u(t, x); (t, x) \in [0, T] \times [0, 1]\}$ uniformly in $(t, x) \in [0, T] \times [0, 1]$. Moreover u is uniquely determined by u_0 and f, and it is a solution to* Eq(u_0;f).

This theorem motivates the following temporary definition:

Definition 2.2 A solution u to Eq(u_0;f) constructed by Theorem 2.1 is called a constructable solution of Eq(u_0;f).

Thus Theorem 2.1 implies the following existence and uniqueness result:

Theorem 2.3 *Let u_0 be an $\mathscr{F}_0$-measurable $C([0, 1])$-valued random variable and let f be a Borel measurable function s.t.*

$$|f(t,x,r)|^2 \leqq C + F(t,x,r)\ dt \otimes dx \otimes dr \quad \text{a.e.}$$

where C is a constant and $F \in L_{p,\beta,\infty}$ for some $p > 1$, $\beta > 4p/(4p-1)$. Then Eq(u_0;f) *has a unique constructable solution u.*

Proof. For non-negative integers m, k we set

$$f_{m,k}(t,x,r) := m \int_{\mathbf{R}} \kappa(mz) f_{(k)}(t,x,r-z)\,dz, \quad f_{(k)} := k \wedge (f \vee (-k)) \tag{2.2}$$

where κ is a non-negative smooth C_0^∞ kernel. Then one can choose some sequences $\{m(k)\}_{k=1}^\infty$, $\{k(n)\}_{n=1}^\infty$ of non-negative integers, such that $|F - F_n|_{p,\beta,\infty} \leqq 2^{-n}$ for every n and $f_n \to f$ $dt \otimes dx \otimes dr$ a.e. as $n \to \infty$, where $f_n := f_{m(k(n)),k(n)}$ and F_n is obtained from F by smoothing it with the kernel $\kappa_n(z) := m(k(n))\kappa(m(k(n))z)$. Then

$$|f_n|^2 \leqq 2C + 2F_n \leqq 2C + 2F'$$

with the constant C and with $F' := F + \Sigma_n |F - F_n|$, which belongs to $L_{p,\beta,\infty}$. Thus we can apply Theorem 2.1 to the sequence of $\mathrm{Eq}(u_0; f_n)$.

Theorem 2.1 implies also the following result on comparison of constructable solutions:

Theorem 2.4 *Let $u^{(1)}$ and $u^{(2)}$ be constructable solutions of* $\mathrm{Eq}(u_0^{(1)}; f^{(1)})$ *and* $\mathrm{Eq}(u_0^{(2)}; f^{(2)})$ *respectively where $u_0^{(i)}$ is an $\mathscr{F}_0$-measurable $C([0,1])$-valued random field and $f^{(i)}$ is a Borel function s.t.*

$$|f^{(i)}(t,x,r)|^2 \leqq C + F^{(i)}(t,x,r)\ dt \otimes dx \otimes dr \quad a.e.$$

with a constant C and a function $F^{(i)} \in L_{p_i,\beta_i,\infty}$ for some $p_i > 1$, $\beta_i > 4p_i/(4p_i - 1)$ for $i = 1, 2$. Assume that

$$u_0^{(1)}(x) \leqq u_0^{(2)}(x) \quad \text{a.s. } for \text{ a.e. } x \in [0,1]$$

$$f^{(1)}(t,x,r) \leqq f^{(2)}(t,x,r)\ dt \otimes dx \otimes dr \quad \text{a.e}$$

Then almost surely $u^{(1)}(t,x) \leqq u^{(2)}(t,x)$ for all t, x.

Proof. Define $f_{mk}^{(1)}$ and $f_{mk}^{(2)}$ by (2.2). Then there are some sequences $\{m(k)\}_{k=1}^{\infty}$, $\{k(n)\}_{n=1}^{\infty}$, such that $f_n^{(i)} := f_{m(k(n)),k(n)}^{(i)}$ satisfy the conditions of Theorem 2.1. Since $f_n^{(i)}$ is Lipschitzian in r for every n, we have

$$u_n^{(1)}(t,x) \leqq u_n^{(2)}(t,x) \quad \text{a.s. for all } t, x$$

for the solutions $u_n^{(1)}$ and $u_n^{(2)}$ of $\mathrm{Eq}(u_0^{(1)}; f_n^{(1)})$ and $\mathrm{Eq}(u_0^{(2)}; f_n^{(2)})$ respectively. Hence, taking $n \to \infty$ we obtain Theorem 2.4 by Theorem 2.1.

Using Theorem 2.4 Theorem 2.1 can be generalized as follows:

Theorem 2.5 *Let u_{n_0} be a sequence of $\mathscr{F}_0$-measurable $C([0,1])$-valued random variables and let f_n be a sequence of Borel functions, satisfying* (i)–(iii). *Then the solution u_n of* $\mathrm{Eq}(u_{n_0}; f_n)$ *converges almost surely to a random field $\{u(t,x): t \in [0,T], x \in [0,1]\}$, uniformly in $(t,x) \in [0,T] \times [0,1]$. Moreover u is uniquely determined by u_0 and f, and it is the unique constructable solution of* $\mathrm{Eq}(u_0; f)$.

Next one may ask the following question: Is there a nonconstructable solution? For this we have the following answer.

Theorem 2.6 *Assume the conditions of Theorem 2.3. Then every solution is constructable.*

Proof. One can repeat our theorem on uniqueness from [2] (Th. 4.2) without essential changes.

Finally we note that one can weaken the conditions on the drift f as follows. Let

$$g: \mathbf{R}_+ \times [0,1] \times \mathbf{R} \to \mathbf{R}$$

be a Borel function which is locally bounded, and has one-sided linear growth, i.e., there exists a constant L such that

$$rg(t,x,r) \leqq L(1 + r^2); \quad t \geqq 0, \quad 0 \leqq x \leqq 1, \quad r \in \mathbf{R}.$$

Then our existence and uniqueness theorem can be formulated as follows:

Theorem 2.7 *Let g be as above. Assume the conditions of Theorem 2.3. Then there exists a unique solution to* $\mathrm{Eq}(u_0; f + g)$.

Proof. Repeating the truncation procedure for g from Sect. 5 of [2] we get this result from Theorems 2.3 and 2.6.

3 Preliminary results

In this section we establish the basic tools of the paper.

3.1 Girsanov transformation

Let $(\Omega, \mathscr{F}, \mathscr{F}_{t\geqq 0}, P)$ be a stochastic basis carrying a space-time white noise W on $[0, T]\times D$ where D is a Borel set of $\mathbf{R}^d$. That means the conditions (i) and (ii) from paragraph 2 are satisfied with D in place of $[0, 1]$.

Let $\{g(t, x): t \geqq 0, x \in D\}$ be a $\mathscr{B}(\mathbf{R}_+)\otimes\mathscr{F}\otimes\mathscr{B}(D)$-measurable random field, such that $g(t, x)$ is $\mathscr{F}_t$-measurable for every $t \geqq 0$ $x\in D$, and

$$\int_0^T \int_D g^2(t, x)\,dx\,dt < \infty \quad \text{a.s.}$$

Define the measure $\tilde{P}$ by

$$d\tilde{P} = Z dP, \quad Z = \exp\left(\int_0^T \int_D g(t, x)\,W(dt, dx) - \frac{1}{2}\int_0^T \int_D g^2(t, x)\,dx\,dt\right).$$

The following result is proved e.g. in [3].

Theorem 3.1.1 *Assume that $\tilde{P}$ is a probability measure. Then under $\tilde{P}$ the application $\tilde{W}: \mathscr{B}([0, T]\times D) \to L_2(\Omega, \mathscr{F}, P)$ defined by*

$$\tilde{W}(C) = W(C) - \int_0^T \int_D \mathbb{1}_C g(t, x)\,dt\,dx \quad \textit{for } C \in \mathscr{B}([0, T]\times D)$$

is a space-time white noise on $[0, T]\times D$.

Corollary 3.1.2 *Let $u = (u(t, x))$ be a $\mathscr{P}\otimes\mathscr{B}([0, 1])$ measurable random field on $[0, T]\times[0, 1]$ and let $f: \mathbf{R}_+\times\Omega\times[0, 1]\times\mathbf{R}\to\mathbf{R}$ be a $\mathscr{P}\otimes\mathscr{B}([0, 1]\times\mathbf{R})$ measurable function such that for every $\varphi\in C^2([0, 1])$ with $\varphi'(0) = \varphi'(1) = 0$, for $dt\otimes P$* a.e. $(t, \omega)\in[0, T]\times\Omega$

$$\int_0^1 u(t, x)\varphi(x)\,dx = \int_0^1 u(0, x)\varphi(x)\,dx + \int_0^1 \int_0^1 \left[u(s, x)\frac{\partial^2}{\partial x^2}\varphi(x) + f(u)(s, x)\varphi(x)\right] dx\,ds$$

$$+ \int_0^t \int_0^1 \varphi(x)\,W(ds, dx). \quad dt\otimes P \text{ a.e. } (t, \omega)\in[0, T]\times\Omega\,. \tag{3.1.1}$$

Assume that $u(0)\in C([0, 1])$ a.s., *and that*

$$E\exp\left(-\int_0^T \int_0^1 f(u)(s, x)\,W(ds, dx) - \frac{1}{2}\int_0^T \int_0^1 f^2(u)(s, x)\,dx\,ds\right) = 1\,. \tag{3.1.2}$$

Then u has an a.s. continuous modification which solves $\text{Eq}(u(0); f)$ *on* $[0, T]$.

Proof. Define the measure $\tilde{P}$ by

$$d\tilde{P} = \gamma_T dP, \quad \gamma_T = \exp\left(-\int_0^T \int_0^1 f(u)(s,x)\, W(ds,dx) - \frac{1}{2} \int_0^T \int_0^1 f^2(u)(s,x)\, dx\, ds \right).$$

Then $\tilde{P}$ is a probability measure by (3.1.2) and by virtue of Theorem 3.1.1 for every $\varphi \in C^2([0,1])$ with $\varphi'(0) = \varphi'(1) = 0$, for $dt \otimes \tilde{P}$ a.e. (t, ω)

$$\int_0^1 u(t,x)\varphi(x)dx = \int_0^1 u(0,x)\varphi(x)dx + \int_0^t \int_0^1 u(s,x) \frac{\partial^2}{\partial x^2} \varphi(x)\, dx\, ds$$
$$+ \int_0^t \int_0^1 \varphi(x)\, \tilde{W}(ds, dx).$$

with the $\tilde{P}$-white noise $\tilde{W}(ds, dx) := f(u)(s,x) ds dx + W(ds, dx)$. Hence

$$u(t,x) = \int_0^1 G_t(x,y) u(0,y) dy + \int_0^t \int_0^1 G_{t-s}(x,y)\, \tilde{W}(ds, dy) \tag{3.1.3}$$

for $dt \otimes P \otimes dx$ a.e. $(t, \omega, x) \in [0,T] \times \Omega \times [0,1]$. Define now $\tilde{u}(t,x)$ by the right side of (3.1.3). Then $\tilde{u}$ is $\tilde{P}$-almost surely in $C([0,T] \times [0,1])$. Hence $\tilde{P}$-almost every $\omega \in \Omega$ (3.1.1) holds for all $t \in [0,T]$ with $\tilde{u}$ in place of u. Then P-almost every $\omega \in \Omega$ (3.1.1) also holds for all $t \in [0,T]$ with $\tilde{u}$ in place of u. Consequently $\tilde{u}$ is a solution of Eq$(u(0); f)$.

3.2 *A priori estimates*

In the next theorem we formulate the important estimates we use for the solutions of Eq$(u_0; f)$. We recall that the Borel functions

$$F: [0,T] \times [0,1] \times \mathbf{R} \to \mathbf{R}, \quad h: [0,T] \times \mathbf{R} \to \mathbf{R}$$

are said to belong to $L_{p,\alpha,\infty}$ and to $L_{p,\alpha}$ respectively, if

$$|F| := |F|_{p,\alpha,\infty} := \sup_{x \in [0,1]} \left\{ \int_0^T \left(\int_{-\infty}^{\infty} |F(t,x,r)|^p dr \right)^{\frac{\alpha}{p}} dt \right\}^{1/\alpha} < \infty$$

$$|h|_{p,\alpha} := \left(\int_0^T \left(\int_{-\infty}^{\infty} |h(t,r)|^p dr \right)^{\frac{\alpha}{p}} dt \right)^{1/\alpha} < \infty .$$

For $v \in C([0,1])$ we use the notation $\|v\| := \sup_{x \in [0,1]} |v(x)|$. The basic assumption on the Borel function f is the following:

(E) For $dt \otimes dx \otimes dr$ a.e. $(t,x,r) \in [0,T] \times [0,1] \times \mathbf{R}$

$$|f(t,x,r)|^2 \leqq C + F(t,x,r)$$

where C is a constant and $F \in L_{p,\beta,\infty}$ for some $p > 1$ and $\beta > 4p/(4p-1)$.

Theorem 3.2.1 *Assume (E). Let u be the solution to* Eq(u_0;f). *Then the following estimates hold:*

(i) *for every* $\gamma > 1$, $\delta > 0$ *and* $T > 0$

$$E\{(\exp(-\delta\|u_0\|)|u(t,x)|^\gamma\} \leqq K$$

for all $t \in [0, T]$, $x \in [0, 1]$ *where K is a constant depending on* $C, p, \beta, |F|_{p,\beta,\infty}$ *from (E) and on* δ, γ;

(ii) *for every Borel function* $h: \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$

$$E\int_0^T h(t, u(t,x))\,dt \leqq K|h|_{q,\alpha}$$

for every $x \in [0, 1]$ *for every* $q > 1$ *and* $\alpha > 4q/(4q-1)$, *where K is a constant depending on* $C, p, \beta, |F|_{p,\beta,\infty}$ *and on* q, α;

(iii) *there exists a real analytic function A such that for every Borel function* h: $\mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$

$$E\exp\left(\int_0^T |h(s, u(s,x))|\,ds\right) \leqq A(|h|_{q,\alpha})$$

for every $x \in [0, 1]$, *and for every* $q > 1$ *and* $\alpha > 4q/(4q-1)$.

The proof will follow after some propositions.

Proposition 3.2.2 *For every solution u of* Eq(u_0; 0) *the statements* (i)–(ii) *of Theorem* 3.2.1 *hold.*

Proof. By (2.1) we have $u(t,x) = \eta(t,x) + \xi(t,x)$ for every $t \in [0, T]$, $x \in [0, 1]$ where

$$\xi(t,x) := \int_0^t G_{t-s}(x,y)\,W(ds,dy)$$

is a Gaussian random variable with $E\xi(t,x) = 0$, variance

$$\sigma^2(t,x) = \int_0^t \int_0^1 G^2_{t-s}(x,y)\,dy\,ds,$$

and

$$\eta(t,x) := \int_0^1 G_t(x,y)u_0(y)\,dy\,.$$

Note that $\sigma^2(t,x) \leqq K$ and $|\eta(t,x)| \leqq K\|u_0\|$ for all $t \in [0, T]$, $x \in [0, 1]$ where $K = K(T)$ is a constant for each $T \in \mathbf{R}_+$. Hence (i) and (ii) are obvious.

Proposition 3.2.3 *If u is a solution of* Eq(u_0; 0) *and* $h: \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ *is a Borel function then*

$$E\left(\int_{t_0}^T h(s, u(s,x))\,ds \,\middle|\, \mathscr{F}_{t_0}\right) \leqq C \int_{t_0}^T \left(\frac{1}{(t-t_0)^{1/4}} \int_{\mathbf{R}} |h(t,r)|^q\,dr\right)^{\frac{1}{q}} dt$$

for every $x \in [0, 1]$, $t_0 \in [0, T]$ *and* $q > 1$ *where C is a constant depending only on* q *and* $T \in \mathbf{R}_+$.

Proof. We have the decomposition $u(t, x) = \eta(t, x) + \xi(t, x)$ for every $t \in [t_0, T]$, $x \in [0, 1]$ where

$$\xi(t, x) := \int_{t_0}^{t} G_{t-s}(x, y) W(ds, dy)$$

is a Gaussian random variable which is independent of $\mathscr{F}_{t_0}$. In particular, $\xi(t, x)$ is independent of the variable

$$\eta(t, x) := \int_0^1 G_{t-t_0}(x, y) u(t_0, y) \, dy .$$

Note that $E\xi(t, x) = 0$ and

$$\sigma^2(t, x) := E\xi^2(t, x) = \int_{t_0}^{t} \int_0^1 G_{t-s}^2(x, y) \, dy \, ds$$

$$\geqq \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{t-t_0} \frac{1}{s} \int_0^1 \left\{ \exp\left(-\frac{(y-x-2n)^2}{2s} \right) + \exp\left(-\frac{(y+x-2n)^2}{2s} \right) \right\} dy \, ds$$

$$\geqq \frac{1}{2\pi} \int_0^{t-t_0} \frac{1}{s} \int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2s} \right) dz \, ds = \sqrt{\frac{2}{\pi}} \sqrt{t - t_0}$$

for every $x \in [0, 1]$ and $t \in [t_0, T]$. Note also that $\eta(t, x)$ is $\mathscr{F}_{t_0}$-measurable for every $t \in [t_0, T]$, $x \in [0, 1]$. Hence

$$E\left(\int_{t_0}^{T} h(s, u(s, x)) \, ds \,\middle|\, \mathscr{F}_{t_0} \right) = \int_{t_0}^{T} E(h(s, \eta(s, x) + \xi(s, x)) \,|\, \mathscr{F}_{t_0}) \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{t_0}^{T} \frac{1}{\sigma(s, x)} \int_{-\infty}^{\infty} h(s, \eta(s, x) + r) \exp\left(-\frac{1}{2} \frac{r^2}{\sigma^2(s, x)} \right) dr \, ds$$

$$\leqq \frac{1}{\sqrt{2\pi}} \int_{t_0}^{T} \left\{ \frac{1}{\sigma(s, x)} \left(\int_{\mathbf{R}} |h(s, r)|^q \, dr \right)^{\frac{1}{q}} \left(\int_{\mathbf{R}} \exp\left(-\frac{1}{2} \frac{r^2 p}{\sigma^2(s, x)} \right) dr \right)^{\frac{1}{p}} \right\} ds$$

$$\leqq C \int_{t_0}^{T} \left(\frac{1}{\sqrt[4]{t - t_0}} \int_{-\infty}^{\infty} |h(t, r)|^q \, dr \right)^{\frac{1}{q}} dt$$

by Hölder's inequality for every $q > 1$ where $C := \left(\frac{q-1}{q} \right)^{\frac{1-q}{2q}}$, and $\frac{1}{p} + \frac{1}{q} = 1$.

Corollary 3.2.4 *If u is a solution of* Eq$(u_0; 0)$ *on* $[0, T]$ *and* $h : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ *is a Borel function, then for every* $q > 1$ *and* $\alpha > 4q/4q - 1$

$$E\left(\int_{t_0}^{T} h(s, u(s, x)) \, ds \,|\, \mathscr{F}_{t_0} \right) \leqq K \left(\int_{t_0}^{T} \left(\int_{\mathbf{R}} |h(s, r)|^q \, dr \right)^{\frac{\alpha}{q}} ds \right)^{\frac{1}{\alpha}}$$

for every $x \in [0, 1]$, $t_0 \in [0, T]$ *where* K *is a constant depending on* q, α, T.

Proof. By Hölder's inequality for every $q > 1$

$$\int_{t_0}^{T} \left(\frac{1}{\sqrt[4]{t - t_0}} \int_{\mathbf{R}} |h(t, r)|^q \, dr \right)^{\frac{1}{q}} dt \leqq K \left(\int_{t_0}^{T} \left(\int_{\mathbf{R}} |h(s, r)|^q \, dr \right)^{\frac{\alpha}{q}} ds \right)^{\frac{1}{\alpha}}$$

where

$$K := \left(\int_{t_0}^{T} (t - t_0)^{-\frac{\beta}{4q}} dt \right)^{\frac{1}{\beta}}$$

is finite if $\alpha > 4q/(4q-1)$.

Proposition 3.2.5 *If u is a solution of* Eq$(u_0; 0)$ *and h is a Borel function then for every $T \geqq 0$, $q > 1$, $\alpha > 4q/(4q-1)$ there is a real analytic function A s.t.*

$$E \exp\left(\int_0^T h(s, u(s,x))\, ds \right) \leqq A(|h|_{q,\alpha}) \tag{3.2.1}$$

for every $x \in [0,1]$ where

$$|h|_{q,\alpha} := \left(\int_0^T \left(\int_{-\infty}^{\infty} |h(s,r)|^q dr \right)^{\frac{\alpha}{q}} ds \right)^{\frac{1}{\alpha}}.$$

Proof. (Similar method of getting exponential estimates is used may be first by Khasminskii.) We fix $x \in [0,1]$ and for the sake of notational convenience we omit the arguments of u below. Clearly

$$\begin{aligned} E \exp\left(\int_0^T |h(s,u(s))|\, ds \right) &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} E\left(\int_0^T |h(s,u(s))|\, ds \right)^n \\ &= 1 + \sum_{n=1}^{\infty} E\left(\frac{1}{n!} \prod_{i=1}^{n} \int_0^T |h(s_i, u(s_i))|\, ds_i \right) = 1 + \sum_{n=1}^{\infty} I_n \end{aligned} \tag{3.2.2}$$

where

$$I_n := E \int_0^T \int_{s_1}^T \dots \int_{s_{n-1}}^T h(s_1, u(s_1)) \dots h(s_n, u(s_n))\, ds_n \dots ds_1.$$

By Corollary 3.2.4

$$E\left(\int_{s_{n-1}}^T |h(s_n, u(s_n))|\, ds_n \,\Big|\, \mathscr{F}_{s_{n-1}} \right) \leqq C \left(\int_{s_{n-1}}^T |h(s_n)|_q^\alpha\, ds_n \right)^{\frac{1}{\alpha}}$$

where

$$|h(s)|_q := \left(\int_{-\infty}^{\infty} |h(s,r)|^q dr \right)^{\frac{1}{q}}.$$

Hence

$$I_n \leqq CE \int_0^T \int_{s_1}^T \dots \int_{s_{n-1}}^T h(s_1, u(s_1)) \dots h(s_{n-1}, u(s_{n-1})) \left(\int_{s_{n-1}}^T |h(s_n)|_q^\alpha \right)^{\frac{1}{\alpha}} ds_1 \dots ds_{n-1}.$$

Applying this argument again, conditioning by $\mathscr{F}_{s_{n-2}}$, then by $\mathscr{F}_{s_{n-3}}$ and so on, finally by $\mathscr{F}_{s_1}$, we get

$$\begin{aligned} I_n &\leqq C^n \left(\int_0^T \int_{s_1}^T \dots \int_{s_{n-1}}^T |h(s_1)|_q^\alpha \dots |h(s_n)|_q^\alpha\, ds_n \dots ds_1 \right)^{\frac{1}{\alpha}} \\ &= C^n (n!)^{-\frac{1}{\alpha}} \left(\int_0^T |h(s)|_q^\alpha \right)^{\frac{n}{\alpha}} \end{aligned} \tag{3.2.3}$$

from every n, where C is the constant from Corollary 3.2.4. Hence, defining the function A by

$$A(z) := 1 + \sum_{n=1}^{\infty} C^n (n!)^{-\frac{1}{\alpha}} z^n \quad z \in \mathbf{C}$$

we get (3.2.1) from (3.2.2) and (3.2.3).

Proposition 3.2.6 *Assume* (*E*) *and let* u *be a solution of* $\mathrm{Eq}(u_0; f)$. *Then*

$$Z := \exp\left(- \int_0^T \int_0^1 f(u)(s,x)\, W(ds,dx) - \frac{1}{2} \int_0^T \int_0^1 f^2(u)(s,x)\, dx\, ds \right)$$

defines a random variable such that the measure $\tilde{P}$ *defined by* $d\tilde{P} = Z dP$ *is a probability measure which is equivalent to* P. *Moreover for every* $\gamma \in \mathbf{R}$ *there is a constant* K_γ *depending on* γ, T, p, β *and on* $|f|_{p,\beta,\infty}$ s.t.

$$\tilde{E}|Z|^\gamma \leqq K_\gamma \tag{3.2.4}$$

$$E|Z|^\gamma \leqq K_\gamma \tag{3.2.5}$$

where $\tilde{E}$ *denotes the expectation w.r. to* $\tilde{P}$.

Proof. Assume first that f is bounded. Then Z and $\tilde{P}$ are defined, under $\tilde{P}$ the random measure $\tilde{W}(dt,dx) = f(u)(t,x)\, dt\, dx + W(dt,dx)$ is a white noise, and u is a solution of $\mathrm{Eq}(u_0; 0)$ with $\tilde{W}$ in place of W. Hence

$$\tilde{E} \int_0^T \int_0^1 |f(u)(s,x)|^2\, dx\, dt \leqq CT + K|F|_{p,\beta,\infty}$$

by Corollary 3.2.4 where C is the constant from (*E*) and K is the constant from Corollary 3.2.4. Moreover

$$\begin{aligned}
\tilde{E}|Z|^\gamma &= \tilde{E} \exp\left(-\gamma \int_0^T \int_0^1 f(u)(s,x)\, W(ds,dx) - \frac{\gamma}{2} \int_0^T \int_0^1 f^2(u)(s,x)\, dx\, ds \right) \\
&= \tilde{E} \exp\left(-\gamma \int_0^T \int_0^1 f(u)(s,x)\, \tilde{W}(ds,dx) - \gamma^2 \int_0^T \int_0^1 f^2(u)(s,x)\, dx\, ds \right) \\
&\quad \times \exp\left(\left| \gamma^2 + \frac{\gamma}{2} \right| \int_0^T f^2(u)(s,x)\, dx\, ds \right) \\
&\leqq \left(\tilde{E} \exp\left(2 \left| \gamma^2 + \frac{\gamma}{2} \right| \int_0^T \int_0^1 f^2(u)(s,x)\, dx\, ds \right) \right)^{\frac{1}{2}} \\
&\leqq \exp\left(CT \left| \gamma^2 + \frac{\gamma}{2} \right| \right) \left(A\left(2 \left| \gamma^2 + \frac{\gamma}{2} \right| |F|_{p,\beta,\infty} \right) \right)^{\frac{1}{2}}
\end{aligned} \tag{3.2.6}$$

by Hölder's inequality and by Proposition 3.2.5. Consequently (3.2.4) and hence (3.2.5) are valid for bounded f. Hence

$$\begin{aligned}
E \int_0^T \int_0^1 |f(u)(s,x)|^2\, dx\, ds &\leqq (\tilde{E}|Z^{-1}|^\gamma)^{\frac{1}{\gamma}} \left(\tilde{E} \int_0^T \int_0^1 |f(u)(s,x)|^{2\delta}\, dx\, ds \right)^{\frac{1}{\delta}} \\
&\leqq (K_{-\gamma})^{\frac{1}{\gamma}} (CT + K|F|_{p\delta,\beta\delta,\infty})^{\frac{1}{\delta}} \quad \forall \delta > 1, \frac{1}{\delta} + \frac{1}{\gamma} = 1
\end{aligned} \tag{3.2.7}$$

Now we can finish the proof of this proposition by taking δ sufficiently close to 1 and applying (3.2.6) with $f_n := f\mathbb{1}_{|f| \leqq n}$ in place of f and letting $n \to \infty$.

Proof of Theorem 3.2.1 We define Z and $\tilde{P}$ as in Proposition 3.2.6. Then $\tilde{P}$ is a probability measure and by Hölder's inequality for every $\varrho > 1$

$$E(\exp(-\delta\|u_0\|\,|u(t,x)|^\gamma) \leqq C(\varrho)(\tilde{E}(\exp(-\varrho\delta\|u_0\|)|u(t,x)|^{\varrho\gamma}))^{\frac{1}{\varrho}}$$

$$E\left(\exp(-\delta\|u_0\|)\int_0^T\left(\int_0^1|u(t,x)|\,dx\right)^\gamma dt\right) \leqq C(\varrho)\left(\tilde{E}\left(\int_0^T\left(\int_0^1|u(t,x)|\,dx\right)^\gamma dt\right)^\varrho\right)^{\frac{1}{\varrho}}$$

$$E\int_0^T h(t,u(t,x))\,dt \leqq C(\varrho)\,T^{\varrho-1}\left(\tilde{E}\int_0^T|h(t,u(t,x))^\varrho|\,dt\right)^{\frac{1}{\varrho}}$$

$$E\exp\left(\int_0^T|h(s,u(s,x)|\,ds\right) \leqq C(\varrho)\left(\tilde{E}\exp\left(\int_0^T\varrho|h(s,u(s,x)|\,ds\right)\right)^{\frac{1}{\varrho}}$$

where $C(\varrho) := \left(\tilde{E}|Z|^{-\frac{\varrho}{\varrho-1}}\right)^{\frac{\varrho-1}{\varrho}}$ and $\tilde{E}$ denotes expectation w.r. to $\tilde{P}$. Note that under $\tilde{P}$ the random field u solves Eq$(u_0; 0)$ with the $\tilde{P}$-white noise

$$\tilde{W}(dt,dx) = f(t,x)\,dt\,dx + W(dt,dx)$$

in place of W. Hence we can finish the proof, taking ϱ close to 1 and applying Proposition 3.2.2, Corollary 3.2.4 and Proposition 3.2.5.

3.3 Passage to the limit in the equations

Here we describe the technique of taking the limit of functions of converging sequences of random variables, when the functions are only measurable. Such a technique was used first by N.V. Krylov [6] in constructing the weak solutions of finite dimensional SDEs with measurable coefficients.

Let $\{f_n\}_{n=1}^\infty$ and $\{h_n\}_{n=1}^\infty$ be sequences of Borel functions $f_n: \mathbf{R}_+ \times [0,1] \times \mathbf{R} \to \mathbf{R}$ and $h_n: \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ satisfying the following conditions:

(A) There exist some constants C_1, C_2 and functions $F^{(1)} \in L_{p,\beta,\infty}$, $F^{(2)} \in L_{q,\alpha}$ for some $p > 1$, $\beta > 4p/(4p-1)$, $q > 1$, $\alpha > 4q/(4q-1)$ such that for all n

$$|f_n(t,x,r)|^2 \leqq C_1 + F^{(1)}(t,x,r) \quad dt \otimes dx \otimes dr \quad \text{a.e.}$$

$$|h_n(t,r)| \leqq C_2 + F^{(2)}(t,r) \quad dt \otimes dr \quad \text{a.e.}$$

(B) Suppose there exists a solution u_n of Eq$(u_{0n}; f_n)$ s.t. for every $t \in [0,T]$, $x \in [0,1]$

$$\lim_{n\to\infty} u_n(t,x) = u(t,x)$$

for almost every $\omega \in \Omega$, where u is some random field.

(C) For each $R > 0$ the set $\{h_n\}$ is relatively compact in $L_{q,\alpha}([0,T] \times [-R,R])$. First we extend the estimates of Theorem 3.2.1:

Proposition 3.3.1 *Assume (A) and (B). Then Theorem 3.2.1 and Proposition 3.2.6 hold with* $u := \lim_{n\to\infty} u_n$.

Proof. By Theorem 3.2.1 the estimates (i)–(iv) hold with a constant K independent of n. Therefore (i)–(ii) hold also for $u := \lim u_n$. The estimates (iii) and (iv) hold also for u for continuous functions h and hence for every measurable h by the monotone class theorem. We can similarly show that u satisfies Proposition 3.2.6.

The following Proposition and Corollary are very similar to Proposition 3.3 and Corollary 3.4 of [2]. Although the proofs follow with minor changes those from [2], for convenience of the reader we present them below.

Proposition 3.3.2 *Assume (A)–(C). Then*

$$\lim_{n\to\infty} \sup_k E \int_0^T |h_k(t, u_n(t,x)) - h_k(t, u(t,x))|\, dt = 0$$

for every $x \in [0, 1]$.

Proof. Let $\kappa : \mathbf{R} \to \mathbf{R}$ be a smooth function such that $0 \leqq \kappa(r) \leqq 1$ for every r, $\kappa(r) = 0$ for $|r| \geqq 1$ and $\kappa(0) = 1$. Let us fix $x \in \mathbf{R}$. For a given $\varepsilon > 0$ and $\delta > 1$ let $R > 0$ be such that

$$\left(E \int_0^T |1 - \kappa(u(t,x)/R)|^\delta\, dt \right)^{\frac{1}{\delta}} < \varepsilon .$$

We can find finitely many bounded smooth functions $H_1, \dots, H_N$ such that for each k

$$\left(\int_0^T \left(\int_{-R}^R |h_k(t,r) - H_i(t,r)|^p\, dr \right)^{\frac{\beta}{p}} dt \right)^{\frac{1}{\beta}} < \varepsilon$$

for some H_i. Clearly

$$I(n,k) := E \int_0^T |h_k(u_n) - h_k(u)|\, dt \leqq I_1(n,k) + I_3(n) + I_2(k)$$

where

$$I_1(n,k) := E \int_0^T |h_k(u_n) - H_i(u_n)|\, dt$$

$$I_3(n) := \sum_{j=1}^N E \int_0^T |H_i(u_n) - H_i(u)|\, dt$$

$$I_2(k) := E \int_0^T |h_k(u) - H_i(u)|\, dt .$$

(For notational convenience we omit the variables t, x of the integrands.) By Theorem 3.2.1, using Hölder's inequality

$$I_1(n,k) = E \int_0^T \kappa(u_n/R) |h_k(u_n) - H_i(u_n)|\, dt + E \int_0^T |1 - \kappa(u_n/R)| |h_k(u_n) - H_i(u_n)|\, dt$$

$$\leqq K_1 \left(\int_0^T \left(\int_{-R}^R |h_k(t,r) - H_i(t,r)|^q\, dr \right)^{\frac{\alpha}{q}} dt \right)^{\frac{1}{\alpha}} + L_1 E \int_0^T |1 - \kappa(u_n/R)|\, dt$$

$$+ \left(E \int_0^T |1 - \kappa(u_n/R)|^\delta\, dt \right)^{\frac{1}{\delta}} \left(E \int_0^T |F^{(2)}(t, u_n)|^\gamma\, dt \right)^{\frac{1}{\gamma}}$$

where L_1, K_1 are constants, $F^{(2)}$ is from (A), δ, $\gamma \in (0, 1)$ s.t. $\frac{1}{\delta} + \frac{1}{\gamma} = 1$. Hence, choosing γ sufficiently close to 1,

$$I_1(n, k) \leqq K_1 \varepsilon + L_1 E \int_0^T |1 - \kappa(u_n/R)| \, dt + M_1 \left(E \int_0^T |1 - \kappa(u_n/R)|^\delta \, dt \right)^{\frac{1}{\delta}}$$

where the constants K_1, L_1, M_1 are independent of n and k. Consequently

$$\begin{aligned} \limsup_{n \to \infty} \sup_k I_1(n, k) &\leqq K_1 \varepsilon + L_1 E \int_0^T |1 - \kappa(u/R)| \, dt + M_1 \left(E \int_0^T |1 - \kappa(u/R)|^\delta \, dt \right)^{\frac{1}{\delta}} \\ &\leqq (K + L_1' + M_1)\varepsilon . \end{aligned}$$

Similarly,

$$\sup_k I_2(k) \leqq (K_2 + L_2' + M_2)\varepsilon$$

with constants K_2, C_2, L_2. It is clear that

$$\lim_{n \to \infty} I_3(n) = 0 .$$

Consequently

$$\limsup_{n \to \infty} \sup_k I(n, k) \leq \sum_{i=1}^{2} (K_i + L_i' + M_i)\varepsilon ,$$

and the proof is complete, since we can take $\varepsilon > 0$ arbitrary small.

Corollary 3.3.3 *Assume (A) and (B) and suppose that for* $n \to \infty$

$$h_n \to h \quad in \quad L_{p,\beta}([0, T] \times [-R, R])$$

for every $R \geqq 0$. *Then*

$$\lim_{n \to \infty} E \int_0^T |h_n(t, u_n(t, x)) - h(t, u(t, x))| \, dt = 0$$

for every $x \in [0, 1]$.

Proof. Clearly

$$E \int_0^T |h_n(u_n) - h(u)| \, dt \leqq \mathscr{T}_1(n) + \mathscr{T}_2(n) ,$$

where

$$\mathscr{T}_1(n) := \sup_k E \int_0^T |h_k(u_n) - h_k(u)| \, dt$$

$$\mathscr{T}_2(n) := E \int_0^T |h_n(u) - h(u)| \, dt .$$

Note that $\lim_{n\to\infty} \mathcal{T}_1(n) = 0$ by Proposition 3.3.2 and

$$\mathcal{T}_2(n) = E\int_0^T \kappa(u/R)|h_n(u) - h(u)|\,dt + 2C_2 E\int_0^T |1-\kappa(u/R)|\,dt$$

$$+\left(E\int_0^T |1-\kappa(u/R)|^\delta\,dt\right)^{\frac{1}{\delta}}\left(E\int_0^T |F^{(2)}(t,u)|^\gamma\,dt\right)^{\frac{1}{\gamma}}$$

$$\leqq K\left(\int_0^T\left(\int_{-R}^R |h_n(t,r)-h(t,r)|^p dr\right)^{\frac{\beta}{p}} dt\right)^{\frac{1}{p}} + 2C_2 E\int_0^T |1-\kappa(u/R)|\,dt$$

$$+ 2L\left(E\int_0^T |1-\kappa(u/R)|^\delta dt\right)^{\frac{1}{\delta}}$$

by Theorem 3.2.1 if $\gamma > 1$ is sufficiently close to 1, where K, L are constants, C_2 and $F^{(2)}$ are from (A). Letting here first $n\to\infty$ then $R\to\infty$ we finish the proof of the corollary.

Corollary 3.3.4 *Let $\{u_{0n}\}$ be a sequence of $\mathcal{F}_0$-measurable $C([0,1])$-valued random variables. Let $\{f_n\}_{n=1}^\infty$ be a sequence of measurable functions $f_n: [0,T]\times[0,1]\times\mathbf{R}\to\mathbf{R}$ satisfying (A). Assume that*

(i) *u_{0n} converges in $C([0,1])$ to a random field u_0 almost surely, and*

$$\lim_{n\to\infty} f_n(t,x,r) = f(t,x,r) \quad dt\otimes dx\otimes dr \quad \text{a.e.}$$

where f is a Borel function;

(ii) Eq$(u_{0n};f_n)$ *admits a solution u_n* s.t. *almost surely* $\lim_{n\to\infty} u_n(t,x) = u(t,x)$ *for every $t\in[0,T]$, $x\in[0,1]$, where u is some random field. Then u solves* Eq$(u_0;f)$.

Proof. By (ii) of Theorem 3.2.1 the sequence of u_n is uniformly integrable on $\Omega\times[0,T]\times[0,1]$ with respect to $\exp(-\|u_0\|)\,dP\,dt\,dx$. By Corollary 3.3.3, if $\varphi\in C^\infty([0,1])$

$$E\left|\int_0^t\int_0^1 f_n(u_n)(s,x) - f(u)(s,x)\varphi(x)\,dx\,ds\right|$$

$$\leqq \int_0^1 |\varphi(x)|\,E\int_0^T |f_n(u_n)(s,x) - f(u)(s,x)|\,ds\,dx \to 0$$

for $n\to\infty$. Thus letting $n\to\infty$ in the equation

$$\int_0^1 u_n(t,x)\varphi(x)\,dx = \int_0^1 u_{0n}(x)\varphi(x)\,dx + \int_0^t\int_0^1\left[u_n(s,x)\frac{\partial^2}{\partial x^2}\varphi(x) + f_n(u_n)(s,x)\varphi(x)\right]dx\,ds + \int_0^t\int_0^1 \varphi(x)\,W(ds,dx)$$

one easily gets

$$\int_0^1 u(t,x)\varphi(x)\,dx = \int_0^1 u_0(x)\varphi(x)\,dx + \int_0^t\int_0^1\left[u(s,x)\frac{\partial^2}{\partial x^2}\varphi(x) + f(u)(s,x)\varphi(x)\right]dx\,ds + \int_0^t\int_0^1 \varphi(x)\,W(ds,dx)$$

for every $\varphi \in C^\infty([0, 1])$. By Proposition 3.3.1 the condition (3.1.2) of Corollary 3.1.2 holds. Hence by Corollary 3.1.2 we have a continuous modification of u which solves Eq$(u_0; f)$.

4 The proof of Theorem 2.1 and Theorem 2.5

Set for non-negative integers $n < k$

$$f_{nk} := \bigwedge_{i=n}^{k} f_i := \min(f_n, f_{n+1}, \ldots, f_k) \quad \hat{u}_{0nk} := \bigwedge_{i=n}^{k} u_{0i}$$

$$F_{nk} := \bigvee_{i=n}^{k} f_i := \max(f_n, f_{n+1}, \ldots, f_k) \quad \check{u}_{0nk} := \bigvee_{i=n}^{k} u_{0i}$$

$$f_{(n)} := \bigwedge_{i=n}^{\infty} f_i, \qquad F_{(n)} := \bigvee_{i=n}^{\infty} f_i .$$

Then f_{nk}, F_{nk} are Lipschitz in r. Moreover, F_{nk} is increasing (f_{nk} is decreasing) for $k \uparrow \infty$, and

$$F_{nk} \geqq F_{mk} \geqq f_{mk} \geqq f_{nk} \quad \text{for} \quad n \leqq m \leqq k$$

$$\check{u}_{0nk} \geqq \check{u}_{mk} \geqq \hat{u}_{0mk} \geqq \hat{u}_{nk} .$$

Therefore Eq$(\hat{u}_{0nk}; f_{nk})$ and Eq$(\check{u}_{0nk}; F_{nk})$ admit the unique solutions u_{nk} and w_{nk} respectively for every $n < k$. Moreover by the well-known theorem on comparison u_{nk} is decreasing (w_{nk} is increasing) for $k \uparrow \infty$ and

$$w_{nk} \geqq w_{mk} \geqq u_{mk} \geqq u_{nk} \quad \text{for} \quad n \leqq m \leqq k . \tag{4.1}$$

Note also that by Theorem 3.2.1 we have constant K such that for all $n < k$ and $(t, x) \in [0, T] \times [0, 1]$

$$E\{\exp(-\|\hat{u}_{0nk}\|)|u_{nk}(t, x)|^2\} < K$$

$$E\{\exp(-\|\check{u}_{0nk}\|)|w_{nk}(t, x)|^2\} < K .$$

Therefore we can construct the random fields

$$\begin{aligned} u_{(n)}(t, x) &:= \lim_{k \to \infty} u_{nk}(t, x), \quad & w_{(n)}(t, x) &:= \lim_{k \to \infty} w_{nk}(t, x) \\ u(t, x) &:= \lim_{n \to \infty} u_{(n)}(t, x) \quad & w(t, x) &:= \lim_{n \to \infty} w_{(n)}(t, x) \\ \hat{u}_{0n}(x) &:= \lim_{n \to \infty} \hat{u}_{0nk}(x) \quad & \check{w}_{0n}(x) &:= \lim_{k \to \infty} \check{w}_{0nk}(x) \end{aligned} \tag{4.2}$$

By Corollary 3.3.4 we can see first that $u_{(n)}$ is a solution of Eq$(\hat{u}_{0n}; f_{(n)})/w_{(n)}$ is a solution of Eq$(\check{u}_{0n}; F_{(n)})/$, and afterwards that u and w are solutions of Eq$(u_0; f)$. Hence by Dini's theorem the above convergences are uniform in $(t, x) \in [0, T] \times [0, 1]$. From (4.1) we get $w \geqq u$ almost surely for all t, x. Hence $u = w$, because Girsanov transformation yields that the solutions of Eq$(u_0; f)$ have the same law. Using the well-known comparison theorem again we have

$$w_{nk} \geqq \check{u}_{nk} \geqq \hat{u}_{nk} \geqq u_{nk}$$

where

$$\check{u}_{nk} := \bigvee_{i=n}^{k} u_i, \quad \hat{u}_{nk} := \bigwedge_{i=n}^{k} u_i ,$$

and u_i is the solution of Eq($u_{0i};f_i$). Letting here first $k \to \infty$ and then $n \to \infty$ we get

$$w \geqq \limsup_{n\to\infty} u_n \geqq \liminf_{n\to\infty} u_n \geqq u = w .$$

That means almost surely u_n converges to $u = w$ for every t, x. Moreover since the convergences in (4.2) are uniform in $(t, x) \in [0, T] \times [0, 1]$, the convergence

$$u_n(t, x) \to u(t, x) \quad \text{for } n \to \infty$$

is also uniform in $(t, x) \in [0, T] \times [0, 1]$. Let now v_{0n}, v_0 be $C([0, 1])$-valued $\mathscr{F}_0$-measurable random fields and g_n, g be Borelian functions. We assume that they satisfy the same conditions as u_{0n}, u_0 and f_n, f in Theorem 2.1. Let $S(w_0; h)$ denote the solution of Eq($w_0; h$) for an $\mathscr{F}_0$-measurable $C([0, 1])$ valued random variable w_0 and a Borel measurable bounded function h satisfying the Lipschitz condition in $r \in \mathbf{R}$. We are going to show that $v_n := S(v_{0n}; g_n)$ converges also to the same random field u constructed above. By the well-known comparison theorem

$$S(u_{0n} \vee v_{0n}; f_n \vee g_n) \geqq u_n \vee v_n \geqq u_n \wedge v_n \geqq S(u_{0n} \wedge v_{0n}; f_n \wedge g_n) .$$

Hence v_n converges to u because $f_n \vee g_n \to f$, $f_n \wedge g_n \to f$ imply (as we have seen above) the convergence of $S(u_{0n} \vee v_{0n}; f_n \vee g_n)$ and of $S(u_{0n} \wedge v_{0n}; f_n \wedge g_n)$ to some solutions of Eq($u_0; f$), which have the same law and are comparable, i.e. are equal to one another. The proof of Theorem 2.1 is complete.

To obtain Theorem 2.5 we repeat the above proof with obvious changes. Namely, instead of saying solution we say constructable solution and instead of using the "well-known comparison theorem" we use Theorem 2.4.

5 The case of Dirichlet boundary conditions

With a slight modification the above results remain true for Eq($u_0; f$) with the Neumann boundary condition replaced by the Dirichlet condition

$$u(t, 0) = u(t, 1) = 0, \quad t \in [0, T] ,$$

where we assume that $u_0(0) = u_0(1) = 0$. Namely, one needs only to substitute the space $L_{p,\beta,\infty}$ (considered for $p > 1$, $\beta > 4p/(4p - 1)$) in the assumptions by the spaces $L^{p,\gamma,\beta}$ with $p > 1$, $\gamma > \dfrac{p}{p-1}$, $\beta > \dfrac{4p}{4p-1}$, where $L^{p,\gamma,\beta}$ is the Banach space of functions

$$f: [0, T] \times [0, 1] \times \mathbf{R} \to \mathbf{R}$$

with the norm

$$\|f\|_{p,\gamma,\beta} := \left(\int_0^T \left(\int_0^1 \left(\int_{\mathbf{R}} |f(t, x, r)|^p \, dr \right)^{\frac{\gamma}{p}} dx \right)^{\frac{\beta}{\gamma}} dt \right)^{\frac{1}{\beta}} < \infty .$$

We now explain the minor changes which need to be made in Sect. 3. The difference is that the lower bound for $\sigma^2(t, x)$ which is derived in the proof of Proposition 3.2.3 is no longer valid. Following Lemma 6.1 in [2] we have that

$$\sigma^2(t, x) \geqq c^2(t, x)\sqrt{t - t_0}$$

and there exists a constant $c > 0$ such that

$$c(t, x) \geqq c \inf_{\substack{s \leqq t - t_0 \\ \frac{1}{3} \leqq y \leqq \frac{2}{3}}} P_x(\tau > s | \sqrt{2} B_s = y)$$

where under P_x the process $\{B_s : s \geqq 0\}$ is a standard Brownian motion starting from x and τ is the first exit time from (0, 1) for the process $\{\sqrt{2} B_s : s \geqq t_0\}$. For x close to 0 the above quantity is bounded from below, up to a multiplicative constant by

$$P_0\left(\sup_{s \leqq t - t_0} B_s < x\right) = P_0(|B_{t-t_0}| \leqq x) .$$

Hence there exist three constants a, b, ε such that $0 < a < b < 1$, $\varepsilon > 0$ and

$$\sigma^2(t, x) \geqq \varepsilon x^2 \sqrt{t - t_0} \quad \text{if } 0 \leqq x \leqq a ,$$

$$\sigma^2(t, x) \geqq \varepsilon \sqrt{t - t_0} \quad \text{if } a \leqq x \leqq b ,$$

$$\sigma^2(t, x) \geqq \varepsilon (1 - x)^2 \sqrt{t - t_0} \quad \text{if } b \leqq x \leqq 1 .$$

Consequently,

$$\sigma(t, x) \geqq C_T x(1 - x)\sqrt{t - t_0} \quad \text{for } x \in [0, 1], \quad 0 \leqq t_0 \leqq t \leqq T$$

where $C_T > 0$ is a constant depending on T. Thus it follows from the computations in the proof of Proposition 3.2.3 that

$$E\left(\int_{t_0}^{T} h(s, x, u(s, x))\, ds \middle| \mathscr{F}_{t_0}\right)$$

$$\leqq C \int_{t_0}^{T} \left(\frac{(t - t_0)^{-\frac{1}{4}}}{x(1 - x)} \int_{\mathbf{R}} |h(t, x, r)|^q\, dr\right)^{\frac{1}{q}} dt$$

for every $q > 1$ and Borel function $h : [0, T] \times [0, 1] \times \mathbf{R} \to \mathbf{R}$ where C is a constant depending on q and T. Hence

$$E\left(\int_{t_0}^{T} \int_0^1 h(s, x, u(s, x))\, dx\, ds \middle| \mathscr{F}_{t_0}\right) \leqq C C_1 C_2 \left(\int_{t_0}^{T} \|h(t)\|_{q,\gamma}^{\alpha}\, dt\right)^{\frac{1}{\alpha}}$$

by Hölder's inequality for every $q > 1$, $\gamma > \dfrac{q}{q - 1}$ and $\alpha > \dfrac{4q}{4q - 1}$, where

$$\|h(t)\|_{q,\gamma} := \left(\int_0^1 \left(\int_{\mathbf{R}} |h(t, x, r)|^q\, dr\right)^{\frac{\gamma}{q}} dx\right)^{\frac{1}{\gamma}}$$

and

$$C_1 := \int_0^1 (x(1-x))^{\frac{\gamma}{q(1-\gamma)}}\, dx < \infty$$

$$C_2 := \int_0^T t^{\frac{\alpha}{4q(1-\alpha)}}\, dt < \infty \,.$$

Next (3.2.1) in Proposition 3.2.5 should be replaced by

$$E\left(\exp \int_0^T \int_0^1 h(s, x, u(s, x))\, dx ds\right) \leqq A(\|h\|_{q,\gamma,\alpha})$$

and the proof of that follows exactly the same argument as the one there. The changes in the rest of the proofs are obvious and are left to the reader.

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