Electron. Commun. Probab. **23** (2018), no. 73, 1–14.
https://doi.org/10.1214/18-ECP176
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Approximation of a generalized continuous-state branching process with interaction

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Abstract

In this work, we consider a continuous–time branching process with interaction where the birth and death rates are non linear functions of the population size. We prove that after a proper renormalization our model converges to a generalized continuous state branching process solution of the SDE

$$\begin{aligned} Z_t^x =& x + \int_0^t f(Z_r^x)dr + \sqrt{2c}\int_0^t \int_0^{Z_r^x} W(dr,du) + \int_0^t \int_0^1 \int_0^{Z_{r-}^x} z\,\overline{M}(ds,dz,du) \\ &+ \int_0^t \int_1^\infty \int_0^{Z_{r-}^x} z\,M(ds,dz,du), \end{aligned}$$

where W is a space-time white noise on $(0,\infty)^2$ and $\overline{M}(ds,dz,du) = M(ds,dz,du) - ds\mu(dz)du$, with M being a Poisson random measure on $(0,\infty)^3$ independent of W, with mean measure $ds\mu(dz)du$, where $(1\wedge z^2)\mu(dz)$ is a finite measure on $(0,\infty)$.

Keywords: continuous-state branching processes; interaction; Galton-Watson processes; tightness.
AMS MSC 2010: 60J80; 60F17; 92D25.
Submitted to ECP on May 25, 2018, final version accepted on October 2, 2018.

1 Introduction

Consider a population evolving in continuous time with m ancestors at time $t=0$, in which to each individual is attached a random vector describing her lifetime and her number of offsprings. We assume that those random vectors are independent and identically distributed. The rate of reproduction is governed by a finite measure ν on $\mathbb{Z}_+ = \{0,1,2,...\}$, satisfying $\nu(1)=0$. More precisely, each individual lives for an exponential time with parameter $\nu(\mathbb{Z}_+)$, and is replaced by a random number of children according to the probability $\nu(\mathbb{Z}_+)^{-1}\nu$. For each individual we superimpose additional birth and death rates due to interactions with others at a certain rate which depends upon the other individuals in the population. More precisely, given a function $f:\ \mathbb{R}^+ \to \mathbb{R}$, which satisfies assumption (**H2**) below, whenever the total size of the population is k, the total additional birth rate due to interactions is $\sum_{i=1}^k (f(i)-f(i-1))^+$, while the total additional death rate due to interactions is $\sum_{i=1}^k (f(i)-f(i-1))^-$.

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In this work, we prove that, when properly renormalized, the above continuous time branching process with interaction converges to the solution of the SDE

$$Z_t^x = x + \int_0^t f(Z_r^x)dr + \sqrt{2c}\int_0^t \int_0^{Z_r^x} W(dr,du) + \int_0^t \int_0^1 \int_0^{Z_{r-}^x} z\,\overline{M}(ds,dz,du) + \int_0^t \int_1^\infty \int_0^{Z_{r-}^x} z\,M(ds,dz,du), \tag{1.1}$$

where W is a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}_+$, $M(dr,dz,du)$ is a Poisson random measure with mean measure $ds\mu(dz)du$ independent of W, $c \geq 0$ and μ is a σ-finite measure on $(0,\infty)$ which satisfies

Assumption (H1) $\int_0^\infty (1 \wedge z^2)\mu(dz) < \infty$, and $\overline{M}$ is the compensated measure of M. Our assumption concerning the function f will be

Assumption (H2) $f \in \mathcal{C}(\mathbb{R}_+;\mathbb{R})$, $f(0)=0$, and there exists a constant $\beta > 0$ such that

$$f(x+y) - f(x) \leq \beta y \quad \forall x,y \geq 0.$$

Note that the assumption (**H2**) implies that $f(x) \leq \beta x$, for all $x \geq 0$.

Under the assumptions (**H1**) and (**H2**), the existence and uniqueness of a strong solution of (1.1) is proved in [5]. We thus generalize the convergence result in [3], see also [12], where the limit was a continuous process.

We will need to consider the CSBP Y^x solution of the SDE

$$Y_t^x = x + \beta\int_0^t Y_r^x dr + \sqrt{2c}\int_0^t \int_0^{Y_r^x} W(dr,du) + \int_0^t \int_0^1 \int_0^{Y_{r-}^x} z\,\overline{M}(ds,dz,du) + \int_0^t \int_1^\infty \int_0^{Y_{r-}^x} z\,M(ds,dz,du), \tag{1.2}$$

whose branching mechanism is given by

$$\psi(\lambda) = -\beta\lambda + c\lambda^2 + \int_0^\infty (e^{-\lambda z} - 1 + \lambda z \mathbf{1}_{\{z\leq 1\}})\mu(dr). \tag{1.3}$$

In this work, we assume that Y^x does not explode, which is equivalent to (see [8])

Assumption (H3) $\int_{0+} \frac{d\lambda}{|\psi(\lambda)|} = +\infty$.

The paper is organised as follows. We first define a discrete model jointly for all initial population sizes. This imposes a non symmetric competition rule between the individuals, which we will describe in section 2 below. We do a suitable renormalization of the parameters of the discrete model in section 3, and we prove the convergence of the renormalized model in the large population limit in section 4.

Note that due to our weak assumption (**H1**), Z^x does not have a finite moment of order 1. This induces difficulties for checking tightness of the approximation. We use comparison with two branching processes.

2 Discrete model of population with interaction

2.1 The model

We consider a continuous time $\mathbb{Z}_+$-valued population process $\{X_t^m, t \geq 0\}$, which starts at time zero from $X_0^m = m$ ancestors who are arranged from left to right, and evolves in continuous time. The left/right order is passed on to their offsprings. Moreover, at each death/birth event all newborn are arranged in an arbitrary left-right order. Those rules apply inside each genealogical tree, so that distinct branches of the tree never cross. This means that the forest of genealogical trees of the population is a planar forest of trees, where the ancestor of the population X_t^1 is placed on the far left, the ancestor of $X_t^2 - X_t^1$ immediately on her right, etc... This defines in a non-ambiguous way an order from left to right within the population alive at each time t. We decree that each individual feels the interaction with the others placed on her left but not with those on her right. In order to simplify our formulas, we suppose moreover that the first individual in the left/right order gives birth to ℓ offspring at rate $\nu(\ell) + f^+(1)\mathbf{1}_{\{\ell=2\}}$ and dies at rate $\nu(0) + f^-(1)$.

$\{X_t^m, t \geq 0\}$ is a $\mathbb{Z}_+$-valued Markov process, which starts from $X_0^m = m$. and evolves as follows. If $X_t^m = 0$, then $X_s^m = 0$ for all $s \geq t$. While at state $k \geq 1$, the process

$$X_t^m \text{ jumps to } \begin{cases} k+\ell-1, \text{ at rate } \nu(\ell)k \ + \ \mathbf{1}_{\{\ell=2\}} \sum_{j=1}^k (f(j) - f(j-1))^+, \text{ for all } \ell \geq 2; \\ k-1, \quad \text{at rate } \nu(0)k \ + \ \sum_{j=1}^k (f(j) - f(j-1))^-. \end{cases} \tag{2.1}$$

Hence the total interaction birth rates minus the total interaction death rates endured by the population X_t^m at time t is

$$\sum_{k=1}^{X_t^m} [(f(k) - f(k-1))^+ - (f(k) - f(k-1))^-] = \sum_{k=1}^{X_t^m} (f(k) - f(k-1)) = f(X_t^m).$$

2.2 Coupling over ancestral population size

The above description specifies the joint evolution of all $\{X_t^m, t \geq 0\}_{m\geq 1}$, or in other words of the two-parameter process $\{X_t^m, t \geq 0, \ m \geq 1\}$. In the case of a linear function f, for each fixed $t > 0$, $\{X_t^m, \ m \geq 1\}$ is an independent increments process. Here $\{X_\cdot^m, \ m \geq 1\}$ is a Markov chain with values in the space $\mathcal{D}([0,\infty); \mathbb{Z}_+)$ of càdlàg functions from $[0,\infty)$ into $\mathbb{Z}_+$, which starts from 0 at $m = 0$. Consequently, in order to describe the law of the two-parameter process $\{X_t^m, t \geq 0, \ m \geq 1\}$, it suffices to describe the conditional law of $X_\cdot^n$, given $X_\cdot^{n-1}$ for each $n \geq 1$. We now describe the conditional law of $X_\cdot^n$ given $X_\cdot^m$, for arbitrary $1 \leq m < n$. Let $V_t^{m,n} = X_t^n - X_t^m$, $t \geq 0$. Conditionally upon $\{X_\cdot^j, j \leq m\}$, and given that $X_t^m = x(t)$, $t \geq 0$, $\{V_t^{m,n}, t \geq 0\}$ is a $\mathbb{Z}_+$-valued time inhomogeneous Markov process starting from $V_0^{m,n} = n - m$, whose time-dependent infinitesimal generator $\{Q_{k,j}(t), \ k, j \in \mathbb{Z}_+\}$ is such that its non zero off-diagonal terms are given by

$$Q_{k,k+\ell-1}(t) = k\nu(\ell) \ + \ \mathbf{1}_{\{\ell=2\}} \sum_{i=1}^k (f(x(t)+i) - f(x(t)+i-1))^+, \text{ for all } \ell \geq 2,$$

$$Q_{k,k-1}(t) = k\nu(0) \ + \ \sum_{i=1}^k (f(x(t)+i) - f(x(t)+i-1))^-.$$

This description of the conditional law of $\{X_t^n - X_t^m, t \geq 0\}$, given $X_\cdot^m$, is prescribed by what we have just said, and $\{X_\cdot^m, \ m \geq 1\}$ is indeed a Markov chain. Note that $V_t^{0,n}$ evolves as X_t^n. Our nonlinear function f is very general. It can both model the Allee effect and competition, in case it is increasing for moderate values of x, and decreasing for large x.

3 A renormalized process

In this section, we first construct our continuous time branching process with interaction. We then proceed to its renormalisation. Let $N \geqslant 1$ be an integer which will eventually go to infinity.

3.1 Preliminaries

Let us start with a construction, which will allow us to separate small jumps and big jumps of the population process. Let us define ψ_1 and $\psi_2 \in \mathcal{C}([0,+\infty))$ by

$$\psi_1(u) = \int_0^1 (e^{-uz} - 1 + uz)\mu(dz) \quad \text{and} \quad \psi_2(u) = \int_1^\infty (e^{-uz} - 1)\mu(dz),$$

where μ satisfies (**H1**). In what follows, we set

$$h_{1,N}(s) = s + \frac{\psi_1(N(1-s))}{N\psi_1'(N)} \quad \text{and} \quad h_{2,N}(s) = s - \frac{\psi_2(N(1-s))}{N\psi_2'(N)}, \quad |s| \leqslant 1.$$

For $i = 1, 2$, $h_{i,N}$ is a probability generating function, and we have

$$h_{1,N}(s) = \sum_{\ell \geq 0} \pi^{-,N}(\ell) s^\ell \quad \text{and} \quad h_{2,N}(s) = \sum_{\ell \geq 0} \pi^{+,N}(\ell) s^\ell, \quad |s| \leqslant 1,$$

where $\pi^{-,N}$ and $\pi^{+,N}$ are two probability measures on $\mathbb{Z}_+$. Let us define

$$d_{-,N} = \psi_1'(N), \quad d_{+,N} = -\psi_2'(N) \quad \text{and} \quad d_N = d_{-,N} + d_{+,N}.$$

For any $\ell \geq 0$, we define

$$\pi_N(\ell) = \frac{1}{d_N}\left[d_{-,N}\pi^{-,N}(\ell) + d_{+,N}\pi^{+,N}(\ell)\right]. \tag{3.1}$$

It is easy to check that for all $N \geq 1$,

$$\sum_{\ell \geq 0} \pi_N(\ell) = \sum_{\ell \geq 0} \pi^{-,N}(\ell) = \sum_{\ell \geq 0} \ell \pi^{-,N}(\ell) = \sum_{\ell \geq 0} \pi^{+,N}(\ell) = 1. \tag{3.2}$$

We denote by h_N the probability generating function of π_N. We have

$$h_N(s) = \frac{1}{d_N}\left[d_{-,N}h_{1,N}(s) + d_{+,N}h_{2,N}(s)\right]. \tag{3.3}$$

In what follows, we will need the

Remark 3.1. For any $\lambda > 0$,

$$\psi_1(\lambda) + \psi_2(\lambda) = \int_0^\infty (e^{-\lambda z} - 1 + \lambda z \mathbf{1}_{\{z \leq 1\}})\mu(dz) = N d_N \left[h_N\Big(1 - \frac{\lambda}{N}\Big) - \Big(1 - \frac{\lambda}{N}\Big)\right].$$

Consider $g(s) = \frac{1}{2}(1 + s^2)$, which is the generating function of the probability $\pi = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$. We notice that

$$\sum_{\ell \geq 0} \pi(\ell) = \sum_{\ell \geq 0} \ell \pi(\ell) = 1. \tag{3.4}$$

We define one more probability on $\mathbb{Z}_+$. For any $\ell \geq 0$,

$$\nu_N(\ell) = \frac{1}{2cN + d_N}\left[d_N \pi_N(\ell) + 2cN\pi(\ell)\right]. \tag{3.5}$$

We denote by L_N the probability generating function of ν_N. We have

$$L_N(s) = \frac{1}{(2cN + d_N)}\left[2cNg(s) + d_N h_N(s)\right]. \tag{3.6}$$

From (3.1), we can rewrite (3.5) in the form

$$(2cN + d_N)\nu_N(\ell) = 2cN\pi(\ell) + d_{-,N}\pi^{-,N}(\ell) + d_{+,N}\pi^{+,N}(\ell). \tag{3.7}$$

3.2 Renormalized discrete model

Now we proceed with the renormalization of the model defined by (2.1). For $x \in \mathbb{R}_+$ and $N \in \mathbb{Z}_+$, we choose $m = [Nx]$, and $\nu(\ell) = (d_N + 2cN)\nu_N(\ell)$ for all $\ell \geq 2$, $\nu(0) = (2cN + d_N)\nu_N(0)$, we multiply f by N and divide its argument by N. We attach to each individual in the population a mass equal to $1/N$. Then the total mass process $Z^{N,x}$, which starts from $\frac{[Nx]}{N}$ at time $t = 0$, is a Markov process whose evolution can be described as follows. $Z^{N,x}$ jumps from k/N

$$\text{to} \begin{cases} \frac{k+\ell-1}{N} \text{ at rate } (2cN + d_N)\nu_N(\ell)k \; + \; N\mathbf{1}_{\{\ell=2\}} \sum_{i=1}^k (f(\frac{i}{N}) - f(\frac{i-1}{N}))^+, \text{ for all } \ell \geq 2; \\ \frac{k-1}{N} \;\; \text{ at rate } (2cN + d_N)\nu_N(0)k \; + \; N \sum_{i=1}^k (f(\frac{i}{N}) - f(\frac{i-1}{N}))^-. \end{cases} \tag{3.8}$$

Let $M^{1,N}(dr, dz, du)$, $M^{2,N}(dr, dz, du)$ and $Q^N(dr, dz, du)$ be three independent Poisson random measures on $(0, \infty) \times \mathbb{Z}_+ \times (0, \infty)$, with respective mean measures $2cNdr\pi(dz)du$, $d_{-,N}dr\pi^{-,N}(dz)du$ and $d_{+,N}dr\pi^{+,N}(dz)du$.

Let us define $M^N = M^{1,N} + M^{2,N} + Q^N$. It is clear from (3.7) that M^N is a Poisson random measure on $(0, \infty) \times \mathbb{Z}_+ \times (0, \infty)$, with mean measure $(2cN + d_N)ds\nu_N(dz)du$. Let P_1 and P_2 be two standard Poisson processes, such that M^N, P_1 and P_2 are independent. From (3.8), $Z^{N,x}$ can be expressed as

$$\begin{aligned} Z_t^{N,x} =& \frac{[Nx]}{N} + \frac{1}{N} \int_0^t \int_{\mathbb{Z}_+} \int_0^{NZ_{r^-}^{N,x}} (z-1) M^N(dr, dz, du) \\ &+ \frac{1}{N} P_1\Big(\int_0^t \Big\{ N \sum_{i=1}^{NZ_r^{N,x}} \Big(f(\frac{i}{N}) - f(\frac{i-1}{N}) \Big)^+ \Big\} dr \Big) \\ &- \frac{1}{N} P_2\Big(\int_0^t \Big\{ N \sum_{i=1}^{NZ_r^{N,x}} \Big(f(\frac{i}{N}) - f(\frac{i-1}{N}) \Big)^- \Big\} dr \Big). \end{aligned} \tag{3.9}$$

We introduce the notations

$$\begin{aligned} \bar{M}^{1,N}(dr, dz, du) &= M^{1,N}(dr, dz, du) - 2cNdr\pi(dz)du, \\ \bar{M}^{2,N}(dr, dz, du) &= M^{2,N}(dr, dz, du) - d_{-,N}dr\pi^{-,N}(dz)du, \\ M_1(t) = P_1(t) - t, &\quad M_2(t) = P_2(t) - t. \end{aligned}$$

For the rest of this subsection, we define the martingale

$$\begin{aligned} \mathcal{M}_t^{N,x} =& \frac{1}{N} \int_0^t \int_{\mathbb{Z}_+} \int_0^{NZ_{r^-}^{N,x}} (z-1) \bar{M}^{1,N}(dr, dz, du) \\ &+ \frac{1}{N} \int_0^t \int_{\mathbb{Z}_+} \int_0^{NZ_{r^-}^{N,x}} (z-1) \bar{M}^{2,N}(dr, dz, du) \\ &+ \frac{1}{N} M_1\Big(\int_0^t \Big\{ N \sum_{i=1}^{NZ_r^{N,x}} \Big(f(\frac{i}{N}) - f(\frac{i-1}{N}) \Big)^+ \Big\} dr \Big) \\ &- \frac{1}{N} M_2\Big(\int_0^t \Big\{ N \sum_{i=1}^{NZ_r^{N,x}} \Big(f(\frac{i}{N}) - f(\frac{i-1}{N}) \Big)^- \Big\} dr \Big) \end{aligned}$$

and

$$\mathbf{Q}_t^{N,x} = \frac{1}{N} \int_0^t \int_{\mathbb{Z}_+} \int_0^{NZ_{r^-}^{N,x}} (z-1) Q^N(dr, dz, du). \tag{3.10}$$

Since $\int_{\mathbb{Z}_+}(z-1)\pi(dz) = \int_{\mathbb{Z}_+}(z-1)\pi^{-,N}(dz) = 0$, (3.9) can be rewritten as

$$Z_t^{N,x} = \frac{[Nx]}{N} + \int_0^t f(Z_r^{N,x})dr + \mathcal{M}_t^{N,x} + \mathbf{Q}_t^{N,x}. \tag{3.11}$$

Since $\mathcal{M}^{N,x}$ is purely discontinuous, its quadratic variation $\left[\mathcal{M}^{N,x}\right]$ is the sum of the squares of its jumps:

$$\begin{aligned}\left[\mathcal{M}^{N,x}\right]_t = \frac{1}{N^2}\Bigg[&\int_0^t\int_{\mathbb{Z}_+}\int_0^\infty (z-1)^2\mathbf{1}_{\{u\le NZ_{r^-}^{N,x}\}}M^{1,N}(dr,dz,du)\\ &+\int_0^t\int_{\mathbb{Z}_+}\int_0^\infty (z-1)^2\mathbf{1}_{\{u\le NZ_{r^-}^{N,x}\}}M^{2,N}(dr,dz,du)\\ &+P_1\Big(\int_0^t\Big\{N\sum_{i=1}^{NZ_r^{N,x}}\Big(f(\frac{i}{N})-f(\frac{i-1}{N})\Big)^+\Big\}dr\Big)\\ &+P_2\Big(\int_0^t\Big\{N\sum_{i=1}^{NZ_r^{N,x}}\Big(f(\frac{i}{N})-f(\frac{i-1}{N})\Big)^-\Big\}dr\Big)\Bigg].\end{aligned}$$

From this, (3.2), (3.4) and the identities $\int_{\mathbb{Z}_+} z^2\pi^{-,N}(dz) = h''_{1,N}(1) + h'_{1,N}(1)$ and $\int_{\mathbb{Z}_+} z^2\pi(dz) = g''(1)+g'(1)$, we deduce that the predictable quadratic variation $\langle\mathcal{M}^{N,x}\rangle$ of $\mathcal{M}^{N,x}$ is given by

$$\langle\mathcal{M}^{N,x}\rangle_t = \int_0^t\left\{\left(2c+\int_0^1 z^2\mu(dz)\right)Z_r^{N,x} + \frac{1}{N}\|f\|_{N,0,Z_r^{N,x}}\right\}dr, \tag{3.12}$$

where for any $v=\frac{k}{N}$, $v'=\frac{k'}{N}$ with $k\le k'$, $k,k'\in\mathbb{Z}_+$, $\|f\|_{N,v,v'} = \sum_{i=k+1}^{k'}|f(\frac{i}{N})-f(\frac{i-1}{N})|$, hence

$$\|f\|_{N,v,v'} = \sum_{i=k+1}^{k'}\left\{2\left(f(\frac{i}{N})-f(\frac{i-1}{N})\right)^+ - \left(f(\frac{i}{N})-f(\frac{i-1}{N})\right)\right\}.$$

It now follows from Assumption (**H2**) that

$$\|f\|_{N,v,v'} \le 2\beta(v'-v) + f(v) - f(v'). \tag{3.13}$$

4 Convergence of $Z^{N,x}$

The aim of this section is to prove the convergence in law as $N\to\infty$ of the two parameter process $\{Z_t^{N,x},\ t\ge 0,\ x\ge 0\}$ defined in subsection 3.2 towards the process $\{Z_t^x,\ t\ge 0,\ x\ge 0\}$ solution of the SDE (1.1). We note that the processes $Z_t^{N,x}$ and Z_t^x are Markov processes indexed by x, with values in the space of càdlàg functions of t : $\mathcal{D}([0,\infty);\mathbb{R}_+)$.

We now state our main result (here and later in this paper, $\Rightarrow$ means convergence in law).

Theorem 4.1. *Suppose that Assumptions (**H1**), (**H2**) and (**H3**) are satisfied. Then for all* $n\ge 1$, $0<x_1<x_2<\cdots<x_n$,

$$(Z_\cdot^{N,x_1}, Z_\cdot^{N,x_2},\cdots,Z_\cdot^{N,x_n}) \Rightarrow (Z_\cdot^{x_1}, Z_\cdot^{x_2},\cdots,Z_\cdot^{x_n})$$

in $\mathcal{D}([0,\infty);\mathbb{R}^n)$, *as* $N\to\infty$, *where* $\{Z_t^x,\ t\ge 0,\ x\ge 0\}$ *is the unique solution of the SDE* (1.1).

In the direction x, we could only obtain the convergence in the sense of finite dimensional distributions. Our result is sufficient to declare that the coupling of the various initial conditions described by (1.1) is the natural one. For the proof of this theorem, we first consider $Z^{N,x}$ for a fixed $x>0$.

4.1 Tightness of $Z^{N,x}$

The main difficulty for proving tightness of the sequence $Z^{N,x}$ comes from the fact that, as a result of our very weak assumption on the Lévy measure μ, the limiting process Z^x does not have a first moment (since the large jumps may not be integrable). Hence we cannot hope for a uniform estimate of the first moment of $Z^{N,x}$ like in section 7.1 of [12], and another method is necessary for establishing tightness. We have chosen to use comparison of $Z^{N,x}$ (resp. Z^x) with a branching process $Y^{N,x}$ (resp. with a CSBP Y^x).

To prove the tightness criterion of $Z^{N,x}$, we will proceed in several steps. Let $Y^{N,x}$ be the Markov process which starts from $\frac{[Nx]}{N}$ at time $t = 0$, and evolves as follows

$$Y^{N,x} \text{ jumps from } \frac{k}{N} \text{ to } \begin{cases} \frac{k+\ell-1}{N} \text{ at rate } \{(2cN + d_N)\nu_N(\ell) + \beta \mathbf{1}_{\{\ell=2\}}\}k, \text{ for all } \ell \geq 2, \\ \frac{k-1}{N} \quad \text{at rate } (2cN + d_N)\nu_N(0)k. \end{cases} \tag{4.1}$$

$Y^{N,x}$ is obtained from $Z^{N,x}$ by replacing $f(z)$ by βz. Let $X^{N,x} = NY^{N,x}$ and μ_N be the finite measure on $\mathbb{Z}_+$:

$$\mu_N(0) = (2cN + d_N)\nu_N(0), \quad \mu_N(1) = 0 \quad \text{and} \quad \mu_N(\ell) = (2cN + d_N)\nu_N(\ell) + \beta \mathbf{1}_{\{\ell=2\}},$$

for every $\ell \geq 2$. Hence the dynamics of the continuous time Markov process $X^{N,x}$ is entirely characterized by the measure μ_N. We have the following Proposition, which can be found in Athreya-Ney [2] (4) page 106 and in Pardoux [12] Prop. 3 page 10.

Proposition 4.2. *The generating function of the process $X^{N,x}$ is given by*

$$\mathbb{E}_1\left(s^{X_t^{N,x}}\right) = w_t^N(s), \quad s \in [0,1],$$

with

$$w_t^N(s) = s + \int_0^t \Phi_N(w_r^N(s))dr,$$

and the function Φ_N is defined by

$$\begin{aligned} \Phi_N(s) &= \sum_{\ell=0}^{\infty}(s^\ell - s)\mu_N(\ell) \\ &= (2cN + d_N)(L_N(s) - s) + \beta(s^2 - s), \quad s \in [0,1], \end{aligned}$$

where L_N is the generating function given by (3.6).

The continuous time process $\{Y_t^{N,x},\ t \geqslant 0\}$ is a Markov process with values in the set $E_N = \{k/N,\ k \geqslant 1\}$. For $\lambda \geqslant 0$

$$\begin{aligned} \mathbb{E}\left(e^{-\lambda Y_t^{N,x}}\right) &= \mathbb{E}_{[Nx]}\left(e^{-\lambda X_t^{N,x}/N}\right) \\ &= \exp\left([Nx]\log w_t^N\left(e^{-\lambda/N}\right)\right), \end{aligned}$$

with w_t^N from Proposition 4.2. This suggests to define

$$u_t^N(\lambda) = N\left(1 - w_t^N\left(e^{-\lambda/N}\right)\right). \tag{4.2}$$

The function u_t^N solves the equation

$$u_t^N(\lambda) + \int_0^t \psi^N(u_r^N(\lambda))dr = N\left(1 - e^{-\lambda/N}\right), \tag{4.3}$$

where $\psi^N(u) = N\Phi_N(1 - \frac{u}{N})$. Combining the definition of Φ_N in Proposition 4.2 with (3.3), (3.6) and (1.3), we get

$$\psi^N(u) = \psi(u) + \beta \frac{u^2}{N}, \quad 0 \le u \le N. \tag{4.4}$$

It is clear that $Y^x = (Y_t^x,\ t \geqslant 0)$ defined by (1.2), is a Markov process taking values in $[0,\infty]$, where 0 and ∞ are two absorbing states, and satisfying the branching property. Its Laplace transform satisfies

$$\mathbb{E}\left[\exp(-\lambda Y_t^x)\right] = \exp\left\{-x u_t(\lambda)\right\}, \quad \text{for } \lambda \geqslant 0,$$

for some non negative function u_t which is the unique nonnegative solution of (see Silverstein [13])

$$u_t(\lambda) = \lambda - \int_0^t \psi(u_r(\lambda))dr. \tag{4.5}$$

We fix $\lambda > 0$. Since ψ and ψ_N are locally Lipschitz, $-\psi_N(u) \le -\psi(u)$ and $u_0^N(\lambda) \le u_0(\lambda)$, it follows from a well-known comparison theorem for one-dimensional ODEs that

$$u_t^N(\lambda) \le u_t(\lambda), \quad \forall\, t \ge 0.$$

We also notice from $(\mathbf{H3})$ that Y_t^x does not explode and the facts that $t \to (u_t(\lambda), u_t^N(\lambda))$ is continuous and $N \to u_t^N(\lambda)$ is monotone increasing imply that there exist $\bar{u}(\lambda)$, $\underline{u}^{N_0}(\lambda)$ such that for $0 \le t \le T$, $N \ge N_0$,

$$0 < \underline{u}^{N_0}(\lambda) \le u_t^N(\lambda) \le u_t(\lambda) \le \bar{u}(\lambda) < +\infty. \tag{4.6}$$

We have

Proposition 4.3. *Let $(t,\lambda) \to u_t(\lambda)$ be the unique locally bounded positive solution of* (4.5)*. For every $\lambda \geqslant 0$, as $N \to \infty$, $u_t^N(\lambda) \to u_t(\lambda)$ locally uniformly in t.*

Proof. We take the difference between (4.3) and (4.5), and use (4.4) to deduce that for $0 \le t \le T$,

$$\left|u_t(\lambda) - u_t^N(\lambda)\right| \leqslant K_\lambda \int_0^t \left|u_s(\lambda) - u_s^N(\lambda)\right| ds + k_N(\lambda) + \frac{\beta}{N} \int_0^t \left(u_s^N(\lambda)\right)^2 ds,$$

where $k_N(\lambda) = \lambda - N\left(1 - e^{-\lambda/N}\right) \to 0$, as $N \to \infty$, and K_λ is the Lipschitz constant for ψ on $[\underline{u}^{N_0}(\lambda), \bar{u}(\lambda)]$. From (4.6), the last term tends to 0, as N goes to infinity. We conclude thanks to Gronwall's lemma. □

Now some simple algebra yields (recall that Y^x is the CSBP given by (1.2))

Proposition 4.4. *For all $T > 0$, $x \ge 0$, for all $\lambda \ge 0$, $\mathbb{E}\left(e^{-\lambda Y_T^{N,x}}\right) \to \mathbb{E}\left(e^{-\lambda Y_T^x}\right)$, hence $Y_T^{N,x} \Rightarrow Y_T^x$, as $N \to \infty$.*

We next establish

Proposition 4.5. *For all $T, \epsilon > 0$, there exists $k_\epsilon > 0$ such that*

$$\limsup_{N\to\infty} \mathbb{P}\left(\sup_{0\le t\le T} Y_t^{N,x} > k_\epsilon\right) \le \epsilon.$$

For the proof of this proposition we need an intermediate result. We define the process $\bar{Y}^{N,x}$ in the same way as $Y^{N,x}$ with the measure μ replaced by $\bar{\mu} = \mu \mathbf{1}_{[0,1]}$.Thus, from (3.8), (3.11), (3.12) and (4.1), $\bar{Y}^{N,x}$ takes the form

$$\bar{Y}_t^{N,x} = \frac{[Nx]}{N} + \beta \int_0^t \bar{Y}_r^{N,x} dr + \bar{\mathcal{M}}_t^{N,x}, \tag{4.7}$$

where $\mathcal{M}^{\bar{N},x}$ is a purely discontinuous martingale, whose predictable quadratic variation reads

$$\langle \bar{\mathcal{M}}^{N,x} \rangle_t = \int_0^t \left\{ \left(2c + \int_0^1 z^2 \mu(dz) + \beta \right) \bar{Y}_r^{N,x} \right\} dr. \tag{4.8}$$

We have

Lemma 4.6. *For all $T > 0$, $x \geq 0$, there exists a constant $C_0 > 0$ such that for all $N \geq 1$,*

$$\sup_{0 \leq t \leq T} \mathbb{E}\left(\bar{Y}_t^{N,x} \right) \leq C_0.$$

Proof. Taking the expectation on both side of equation (4.7), we obtain

$$\mathbb{E}\left(\bar{Y}_t^{N,x} \right) = \frac{[Nx]}{N} + \beta \mathbb{E}\left(\int_0^t \bar{Y}_r^{N,x} dr \right).$$

It remains to use Gronwall's Lemma to conclude with $C_0 = xe^{\beta T}$. □

We next establish

Proposition 4.7. *For all $T > 0$, $x \geq 0$, there exists a constant $C_1 > 0$ such that for all $N \geq 1$,*

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} \bar{Y}_t^{N,x} \right) \leq C_1.$$

Proof. From (4.7),

$$\sup_{0 \leq t \leq T} \bar{Y}_t^{N,x} \leq \frac{[Nx]}{N} + \beta \int_0^T \bar{Y}_r^{N,x} dr + \sup_{0 \leq t \leq T} \left| \bar{\mathcal{M}}_t^{N,x} \right|.$$

From Cauchy-Schwartz, Doob's inequality for the L^2 norm, $|y| \leq 1 + y^2$ and (4.8),

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} \bar{Y}_t^{N,x} \right) \leq \frac{[Nx]}{N} + 1 + \left[4 \left(2c + \int_0^1 z^2 \mu(dz) + \beta \right) + \beta \right] \mathbb{E} \int_0^T \sup_{0 \leq r' \leq r} \bar{Y}_{r'}^{N,x} dr.$$

It remains to use Gronwall's Lemma to conclude. □

Proof of Proposition 4.5. Combining Proposition 4.4 and the Portmanteau theorem, we have

$$\forall\, M_\epsilon > 0, \quad \limsup_{N \to \infty} \mathbb{P}\left(Y_T^{N,x} \geq M_\epsilon \right) \leq \mathbb{P}\left(Y_T^x \geq M_\epsilon \right). \tag{4.9}$$

Since from $(\mathbf{H3})$ $Y_T^x < \infty$ $a.s$, we can choose M_ϵ such that

$$\mathbb{P}\left(Y_T^x \geq M_\epsilon \right) \leq \frac{\epsilon}{2}. \tag{4.10}$$

However, we have

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq t \leq T} Y_t^{N,x} > k_\epsilon \right) &\leq \mathbb{P}\left(\sup_{0 \leq t \leq T} Y_t^{N,x} > k_\epsilon,\ Y_T^{N,x} < M_\epsilon \right) + \mathbb{P}\left(Y_T^{N,x} \geq M_\epsilon \right) \\ &\leq \mathbb{P}\left(\sup_{0 \leq t \leq T} Y_t^{N,x} > k_\epsilon \middle| \ Y_T^{N,x} < M_\epsilon \right) + \mathbb{P}\left(Y_T^{N,x} \geq M_\epsilon \right) \end{aligned} \tag{4.11}$$

Now, switching from $Y^{N,x}$ to $\bar{Y}^{N,x}$ consists in removing some of the positive jumps of $Y^{N,x}$. So, the time reversed process $\bar{K}_t^{N,x} = \bar{Y}_{T-t}^{N,x}$ behaves as $K_t^{N,x} = Y_{T-t}^{N,x}$, with some negative jumps deleted. Consequently

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq t \leq T} Y_t^{N,x} > k_\epsilon \middle| \ Y_T^{N,x} < M_\epsilon \right) &\leq \mathbb{P}\left(\sup_{0 \leq t \leq T} \bar{Y}_t^{N,x} > k_\epsilon \middle| \ \bar{Y}_T^{N,x} < M_\epsilon \right) \\ &\leq \frac{\mathbb{E}\left(\sup_{0 \leq t \leq T} \bar{Y}_t^{N,x} \right)}{k_\epsilon \mathbb{P}\left(\bar{Y}_T^{N,x} < M_\epsilon \right)}. \end{aligned} \tag{4.12}$$

The arguments leading to Proposition 4.4 yield that $\bar{Y}_T^{N,x} \Rightarrow \bar{Y}_T^x$, where $\bar{Y}^x$ is the same as Y^x, but with μ replaced by $\bar{\mu}$. Using again the Portmanteau theorem and (4.10), we obtain

$$\begin{aligned}\liminf_{N\to\infty} \mathbb{P}\left(\bar{Y}_T^{N,x} < M_\epsilon\right) &\geq \mathbb{P}\left(\bar{Y}_T^x < M_\epsilon\right) \\ &\geq \mathbb{P}\left(Y_T^x < M_\epsilon\right) \\ &\geq 1 - \frac{\epsilon}{2}.\end{aligned}$$

Finally, combining this inequality with (4.11), (4.12) and Proposition 4.7, we deduce that

$$\limsup_{N\to\infty} \mathbb{P}\left(\sup_{0\leq t\leq T} Y_t^{N,x} > k_\epsilon\right) \leq \frac{C_1}{k_\epsilon(1-\frac{\epsilon}{2})} + \frac{\epsilon}{2}.$$

The result follows by choosing $k_\epsilon = 2C_1/\epsilon(1-\epsilon/2)$. □

Thanks to these results, we are in a position to establish the tightness of $Z^{N,x}$. Proposition 4.5 combined with the fact that $\sup_{0\leq t\leq T} Z_t^{N,x} \leq \sup_{0\leq t\leq T} Y_t^{N,x}$ stochastically [see the definitions (3.8), (4.1) and Assumption (**H2**)] leads to

Corollary 4.8. *For all $T, \epsilon > 0$, there exists $k_\epsilon > 0$ such that*

$$\limsup_{N\to\infty} \mathbb{P}\left(\sup_{0\leq t\leq T} Z_t^{N,x} > k_\epsilon\right) \leq \frac{\epsilon}{2}.$$

We want to check tightness of the sequence $\{Z^{N,x},\ N \geq 1\}$ using Aldous' criterion. Let $\{\tau_N,\ N\geq 1\}$ be an arbitrary sequence of $[0,T]$-valued stopping times. We deduce from the above Corollary

Lemma 4.9. *For any $T > 0$, and η, $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\sup_{N\geq 1}\sup_{0\leq\theta\leq\delta} \mathbb{P}\left(\left|\int_{\tau_N}^{(\tau_N+\theta)\wedge T} f(Z_r^{N,x})dr\right| \geq \eta\right) \leq \frac{\epsilon}{2}.$$

Proof. Let $J : \mathbb{R}_+ \to \mathbb{R}_+$ be the continuous increasing function defined by $J(z) = \sup_{0\leq r\leq z} |f(r)|$.

Provided $0 \leq \theta \leq \delta$, we have

$$\left|\int_{\tau_N}^{(\tau_N+\theta)\wedge T} f(Z_r^{N,x})dr\right| \leq \theta \sup_{0\leq t\leq T}\left|f(Z_t^{N,x})\right|, \quad \text{hence}$$

$$\sup_{0\leq\theta\leq\delta} \mathbb{P}\left(\left|\int_{\tau_N}^{(\tau_N+\theta)\wedge T} f(Z_r^{N,x})dr\right| \geq \eta\right) \leq \mathbb{P}\left(\sup_{0\leq t\leq T} Z_t^{N,x} \geq J^{-1}\left(\frac{\eta}{\delta}\right)\right).$$

The result follows by using Corollary 4.8 and choosing $\delta < \eta/J(k_\epsilon)$. □

We need to check tightness of the sequence of processes $\mathbf{Q}^{N,x}$. We have

Lemma 4.10. *For any $T > 0$, and η, $\epsilon > 0$, there exists $\theta_0 > 0$ such that*

$$\sup_{N\geq 1}\sup_{0\leq\theta\leq\theta_0} \mathbb{P}\left(\left|\mathbf{Q}_{\tau_N+\theta}^{N,x} - \mathbf{Q}_{\tau_N}^{N,x}\right| \geq \eta\right) \leq \epsilon.$$

Proof. From (3.10), we notice that

$$\left|\mathbf{Q}_{\tau_N+\theta}^{N,x} - \mathbf{Q}_{\tau_N}^{N,x}\right| \leq \frac{1}{N}\int_{\tau_N}^{\tau_N+\theta}\int_{\mathbb{Z}_+}\int_0^{NZ_{r^-}^{N,x}} (z+1)Q^N(dr,dz,du),$$

recall $Q^N(dr,dz,du)$ is a Poisson random measures on $(0,\infty)\times\mathbb{Z}_+\times(0,\infty)$, with mean measure $d_{+,N}dr\pi^{+,N}(dz)du$, where $d_{+,N}$ and $\pi^{+,N}$ were defined in subsection 3.1. It is easy to check that $Nd_{+,N}\le e^{-1}\mu[1,+\infty)$. We have

$$\begin{aligned}
&\mathbb{P}\left(\left|\mathbf{Q}^{N,x}_{\tau_N+\theta}-\mathbf{Q}^{N,x}_{\tau_N}\right|\ge\eta\right)\\
&\le\mathbb{P}\left(\sup_{0\le t\le T}Z^{N,x}_t>k_\epsilon\right)+\mathbb{P}\left(\left|\mathbf{Q}^{N,x}_{\tau_N+\theta}-\mathbf{Q}^{N,x}_{\tau_N}\right|\ge\eta,\ \sup_{0\le t\le T}Z^{N,x}_t\le k_\epsilon\right)\\
&\le\mathbb{P}\left(\sup_{0\le t\le T}Z^{N,x}_t>k_\epsilon\right)+\mathbb{P}\Big(Q^N\left([\tau_N,\tau_N+\theta]\times\mathbb{Z}_+\times[0,Nk_\epsilon]\right)>0\Big)\\
&\le\frac{\epsilon}{2}+1-\exp(-e^{-1}\mu[1,+\infty)\theta k_\epsilon).
\end{aligned}$$

The result follows by choosing $\theta_0\le e[\log(1/(1-\frac{\epsilon}{2}))]/[k_\epsilon\mu[1,+\infty)]$. □

From Lemma 4.9 and Lemma 4.10, we deduce that the second and fourth terms in the right-hand side of (3.11) satisfy Aldous' criterion in [1]. Corollary 4.8, (3.12) and (3.13) imply that $\langle\mathcal{M}^{N,x}\rangle$ is both tight and continuous, hence $\mathcal{C}$-tight in the terminology of [10], and from Theorem VI 4.13 in [10], $\mathcal{M}^{N,x}$ is tight. We have proved

Proposition 4.11. *For any fixed $x\ge 0$, the sequence of processes $\{Z^{N,x},\ N\ge 1\}$ is tight in $\mathcal{D}([0,\infty);\mathbb{R}_+)$.*

4.2 Convergence of $Z^{N,x}$ for fixed x

For the rest of this section we set

$$\mathcal{K}_+(NZ^{N,x}_s)=\sum_{i=1}^{NZ^{N,x}_s}\left(f(\frac{i}{N})-f(\frac{i-1}{N})\right)^+\quad\text{and}\quad\mathcal{K}_-(NZ^{N,x}_s)=\sum_{i=1}^{NZ^{N,x}_s}\left(f(\frac{i}{N})-f(\frac{i-1}{N})\right)^-.\tag{4.13}$$

The argument leading to (3.13) implies

Lemma 4.12. *For any $s>0$, we have*

$$\mathcal{K}_+(NZ^{N,x}_s)+\mathcal{K}_-(NZ^{N,x}_s)\le 2\beta Z^{N,x}_s-f(Z^{N,x}_s).$$

It follows from (1.1) and Itô's formula that, for $\lambda\ge 0$, the following is a martingale

$$\begin{aligned}
&e^{-\lambda Z^x_t}-e^{-\lambda Z^x_0}+\lambda\int_0^t e^{-\lambda Z^x_r}f(Z^x_r)dr-c\lambda^2\int_0^t Z^x_r e^{-\lambda Z^x_r}dr\\
&\quad-\int_0^t Z^x_r e^{-\lambda Z^x_r}\left\{\int_0^\infty(e^{-\lambda z}-1+\lambda z\mathbf{1}_{\{z\le 1\}})\mu(dz)\right\}dr.
\end{aligned}\tag{4.14}$$

Thanks to these results, we can now establish the convergence of $Z^{N,x}$.

Proposition 4.13. *For any fixed $x\ge 0$, $Z^{N,x}\Rightarrow Z^x$ in $\mathcal{D}([0,\infty);\mathbb{R}_+)$ as $N\to\infty$, where Z^x is the unique solution of the SDE (1.1).*

Proof. By Proposition 4.11, along a subsequence (denoted as the whole sequence) $\{Z^{N,x}_t,\ t\ge 0\}$ converges weakly to a process $\{Z^x_t,\ t\ge 0\}$ for the Skorohod topology of $\mathcal{D}([0,\infty);\mathbb{R}_+)$. Let for $\lambda\ge 0$, $F(u)=e^{-\lambda u}$, from (3.9), (4.13) and Itô's formula, we deduce that the following is a martingale

$$\begin{aligned}
&e^{-\lambda Z^{N,x}_t}-e^{-\lambda Z^{N,x}_0}-\int_0^t Z^{N,x}_r e^{-\lambda Z^{N,x}_r}\left\{Nd_N\int_{\mathbb{Z}_+}\left(e^{-\lambda\left(\frac{z-1}{N}\right)}-1\right)\pi_N(dz)\right\}dr\\
&\quad-\int_0^t Z^{N,x}_r e^{-\lambda Z^{N,x}_r}\left\{2cN^2\int_{\mathbb{Z}_+}\left(e^{-\lambda\left(\frac{z-1}{N}\right)}-1\right)\pi(dz)\right\}dr-\Gamma_x(t,N),
\end{aligned}\tag{4.15}$$

where we have used the decomposition $(2cN+d_N)\nu_N = d_N\pi_N + 2cN\pi$, and

$$\Gamma_x(t,N) = N\int_0^t \mathcal{K}_+(NZ_r^{N,x})e^{-\lambda Z_r^{N,x}}(e^{-\frac{\lambda}{N}}-1)dr + N\int_0^t \mathcal{K}_-(NZ_r^{N,x})e^{-\lambda Z_r^{N,x}}(e^{\frac{\lambda}{N}}-1)dr.$$

Using Taylor's formula and the fact that $\mathcal{K}_+(NZ_s^{N,x})-\mathcal{K}_-(NZ_s^{N,x}) = f(Z_s^{N,x})$, we deduce

$$\Gamma_x(t,N) = -\lambda\int_0^t e^{-\lambda Z_r^{N,x}}f(Z_r^{N,x})dr + 0\left(\frac{1}{N}\right)\int_0^t e^{-\lambda Z_r^{N,x}}[\mathcal{K}_+(NZ_r^{N,x})+\mathcal{K}_-(NZ_r^{N,x})]dr.$$

However, we have that

$$\begin{aligned} Nd_N\int_{\mathbb{Z}_+}\left(e^{-\lambda\left(\frac{z-1}{N}\right)}-1\right)\pi_N(dz) &= Nd_N e^{\frac{\lambda}{N}}\int_{\mathbb{Z}_+}\left(e^{-\frac{\lambda z}{N}}-e^{-\frac{\lambda}{N}}\right)\pi_N(dz) \\ &= Nd_N e^{\frac{\lambda}{N}}\left(h_N\left(e^{-\frac{\lambda}{N}}\right)-e^{-\frac{\lambda}{N}}\right). \end{aligned}$$

From an easy adaptation of the argument of the proof of Proposition 3.40 of Li [11], we have that

$$\lim_{N\to\infty} Nd_N\left[h_N\Big(1-\frac{\lambda}{N}\Big)-\Big(1-\frac{\lambda}{N}\Big)\right] = \lim_{N\to\infty} Nd_N\left[h_N\left(e^{-\frac{\lambda}{N}}\right)-e^{-\frac{\lambda}{N}}\right].$$

Combining this with Remark 3.1, we deduce that

$$\lim_{N\to\infty} Nd_N\int_{\mathbb{Z}_+}\left(e^{-\lambda\left(\frac{z-1}{N}\right)}-1\right)\pi_N(dz) = \psi_1(\lambda)+\psi_2(\lambda) = \int_0^\infty (e^{-\lambda z}-1+\lambda z\mathbf{1}_{\{z\le 1\}})\mu(dz).$$

However, we deduce from Taylor's formula that

$$2cN^2\int_{\mathbb{Z}_+}\left(e^{-\lambda\left(\frac{z-1}{N}\right)}-1\right)\pi(dz) = c\lambda^2 + o\left(\frac{1}{N}\right).$$

Now, by combining the above results with Lemma 4.12 and Proposition 4.11, we obtain (4.14) by letting $N\to\infty$ in (4.15). Let $g\in C_K^2(\mathbb{R}_+)$ (the space of C^2 functions from $\mathbb{R}_+$ into $\mathbb{R}$ with compact support) and $h(x)=g(-\log(x))$. $h\in C^2([0,1])$. Let $h_n(x) = \sum_{k=0}^n \binom{n}{k} h(k/n)x^k(1-x)^{n-k}$ be its Bernstein polynomial approximation, which converges uniformly to $h(x)$ on $[0,1]$. Consequently $g_n(x) = \sum_{k=0}^n \binom{n}{k} g(-\log(k/n))e^{-kx}(1-e^{-x})^{n-k}$ is a linear combination of exponential functions with negative exponents, which converges to $g(x)$ uniformly on $\mathbb{R}_+$ as $n\to\infty$. A lengthy but elementary computation shows that $g_n'(x)\to g'(x)$ and $g_n''(x)\to g''(x)$ pointwise. Consequently if L denotes the generator of the Markov process Z^x, we have that $Lg_n(x)\to Lg(x)$. This being true for any $g\in C_K^2(\mathbb{R}_+)$, it is easy to conclude that Z_t^x solves the martingale problem associated to (1.1). The result follows. □

4.3 Proof of Theorem 4.1

We shall prove the statement in the case $n=2$ only. The general proof is very similar. Recall (3.8). We now describe the law of the pair $(Z^{N,x},Z^{N,y})$, for any $0<x<y$. $(Z^{N,x},Z^{N,y})$ jumps

$$\text{from } \Big(\frac{i}{N},\frac{j}{N}\Big) \text{ to } \begin{cases} \left(\frac{i+\ell-1}{N},\frac{j}{N}\right) \text{ at rate } (2cN+d_N)\nu_N(\ell)i + N\sum_{k=1}^i\big(f(\frac{k}{N})-f(\frac{k-1}{N})\big)^+ \\ \left(\frac{i-1}{N},\frac{j}{N}\right) \text{ at rate } (2cN+d_N)\nu_N(0)i + N\sum_{k=1}^i\big(f(\frac{k}{N})-f(\frac{k-1}{N})\big)^- \\ \left(\frac{i}{N},\frac{j+\ell-1}{N}\right) \text{ at rate } (2cN+d_N)\nu_N(\ell)j + N\sum_{k=1}^j\big(f(\frac{i+k}{N})-f(\frac{i+k-1}{N})\big)^+ \\ \left(\frac{i}{N},\frac{j-1}{N}\right) \text{ at rate } (2cN+d_N)\nu_N(0)j + N\sum_{k=1}^j\big(f(\frac{i+k}{N})-f(\frac{i+k-1}{N})\big)^-. \end{cases}$$

Recall (3.5) and (3.7). The process $V^{N,x,y} = Z^{N,y} - Z^{N,x}$ can be expressed as follows

$$\begin{aligned} V_t^{N,x,y} = \frac{[Ny]-[Nx]}{N} &+ \frac{1}{N}\int_0^t \int_{\mathbb{Z}_+} \int_0^{NV_{r-}^{N,x,y}} (z-1)M^{',1,N}(dr,dz,du) \\ &+ \frac{1}{N}\int_0^t \int_{\mathbb{Z}_+} \int_0^{NV_{r-}^{N,x,y}} (z-1)M^{',2,N}(dr,dz,du) \\ &+ \frac{1}{N}\int_0^t \int_{\mathbb{Z}_+} \int_0^{NV_{r-}^{N,x,y}} (z-1)Q^{',N}(dr,dz,du) \\ &+ \frac{1}{N}P_1'\Big(N\int_0^t \sum_{k=1}^{NV_r^{N,x,y}} \Big(f(Z_r^{N,x}+\frac{k}{N}) - f(Z_r^{N,x}+\frac{k-1}{N})\Big)^+ dr\Big) \\ &- \frac{1}{N}P_2'\Big(N\int_0^t \sum_{k=1}^{NV_r^{N,x,y}} \Big(f(Z_r^{N,x}+\frac{k}{N}) - f(Z_r^{N,x}+\frac{k-1}{N})\Big)^- dr\Big), \end{aligned} \tag{4.16}$$

where $M^{',1,N}$, $M^{',2,N}$ and $Q^{',N}$ are Poisson random measures on $(0,\infty)\times\mathbb{Z}_+\times(0,\infty)$, with respective intensities $2cNdr\pi(dz)du$, $d_{-,N}dr\pi^{-,N}(dz)du$ and $d_{+,N}dr\pi^{+,N}(dz)du$ and P_1' and P_2' are standard Poisson processes. Note that $M^{',1,N}$, $M^{',2,N}$, $Q^{',N}$, P_1' and P_2' are mutually independent and are globally independent of $\{Z^{N,x'},\ x'\le x\}$. As previously, note also that $M^{',1,N} + M^{',2,N} \overset{(d)}{=} M^{1,N} + M^{2,N}$ and $Q^{',N} \overset{(d)}{=} Q^N$. Consequently we have the decomposition similar to (3.11)

$$V_t^{N,x,y} = \frac{[Ny]-[Nx]}{N} + \int_0^t [f(Z_r^{N,x} + V_r^{N,x,y}) - f(Z_r^{N,x})]dr + \mathcal{M}_t^{N,x,y} + \mathbf{Q}_t^{N,x,y}, \tag{4.17}$$

where $\mathcal{M}^{N,x,y}$ is a local martingale whose predictable quadratic variation $\langle\mathcal{M}^{N,x,y}\rangle$ is given by

$$\langle\mathcal{M}^{N,x,y}\rangle_t = \int_0^t \left\{\left(2c + \int_0^1 z^2\mu(dz)\right)V_r^{N,x,y} + \frac{1}{N}\|f\|_{N,Z_r^{N,x},V_r^{N,x,y}+Z_r^{N,x}}\right\}dr$$

and where $\mathbf{Q}^{N,x,y}$ is given

$$\mathbf{Q}_t^{N,x,y} = \frac{1}{N}\int_0^t \int_{\mathbb{Z}_+} \int_0^{NV_{r-}^{N,x,y}} (z-1)Q^{',N}(dr,dz,du).$$

Now from an easy adaptation of the arguments of the above results, we deduce the

Proposition 4.14. *For any fixed $0\le x<y$, $V^{N,x,y}\Rightarrow V^{x,y}$ as $N\to\infty$ in $\mathcal{D}([0,\infty);\mathbb{R}_+)$, where $V^{x,y}$ is the unique solution of the following SDE*

$$\begin{aligned} V_t^{x,y} = y - x &+ \int_0^t [f(Z_r^x + V_s^{x,y}) - f(Z_r^x)]dr + \sqrt{2c}\int_0^t \int_0^{V_r^{x,y}} W'(dr,du) \\ &+ \int_0^t \int_0^1 \int_0^{V_{r-}^{x,y}} z\overline{M}'(dr,dz,du) + \int_0^t \int_1^\infty \int_0^{V_{r-}^{x,y}} zM'(dr,dz,du) \end{aligned} \tag{4.18}$$

We now conclude the proof of Theorem 4.1. Fix $0<x<y$. Along a subsequence (denoted as the whole sequence), $(Z^{N,x}, V^{N,x,y}) \Rightarrow (Z^x, V^{x,y})$ in $\mathcal{D}([0,\infty);\mathbb{R}_+^2)$. Consider the two SDEs (1.1) and (4.18), with (W',M') and (W,M) independent. Then for all $\lambda,\mu>0$, with ψ as in (1.3) but with $\beta=0$, the following is a martingale

$$\begin{aligned} e^{-\lambda Z_t^x - \mu V_t^{x,y}} - e^{-\lambda x - \mu(y-x)} &+ \int_0^t [\lambda f(Z_r^x) + \mu\{f(Z_r^x + V_r^{x,y}) - f(Z_r^x)\}]e^{-\lambda Z_r^x - \mu V_r^{x,y}}dr \\ &- \int_0^t \{\psi(\lambda)Z_r^x + \psi(\mu)V_r^{x,y}\}e^{-\lambda Z_r^x - \mu V_r^{x,y}}dr. \end{aligned}$$

An extension of the proof of Proposition 4.13 shows that indeed the limit $(Z_t^x, V_t^{x,y})$ of the sequence $(Z^{N,x}, V^{N,x,y})$ solves that same martingale problem. From the well–known properties of white noise and Poisson random measures, (4.18) can be rewritten as (with the same (W, M) as in (1.1)).

$$V_t^{x,y} = y - x + \int_0^t [f(Z_r^x + V_s^{x,y}) - f(Z_r^x)]dr + \sqrt{2c}\int_0^t \int_{Z_r^x}^{Z_r^x+V_r^{x,y}} W(dr, du)$$
$$+ \int_0^t \int_0^1 \int_{Z_{r-}^x}^{Z_{r-}^x+V_{r-}^{x,y}} z\overline{M}(dr, dz, du) + \int_0^t \int_1^\infty \int_{Z_{r-}^x}^{Z_{r-}^x+V_{r-}^{x,y}} zM(dr, dz, du)$$

The result follows from the above facts and $Z_t^{N,y} = Z_t^{N,x} + V_t^{N,x,y}$, $Z_t^y = Z_t^x + V_t^{x,y}$. □

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Acknowledgments. The authors want to thank the two anonymous Referees, whose remarks allowed them to correct several mistakes and imprecisions.