Large Deviation Principle for Reflected Poisson driven SDEs in Epidemic Models.

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Abstract

We establish a large deviation principle for a reflected Poisson driven SDE. Our motivation is to study in a forthcoming paper the problem of exit of such a process from the basin of attraction of a locally stable equilibrium associated with its law of large numbers. Two examples are described in which we verify the assumptions that we make to establish the large deviation principle.

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1 Introduction

Consider a compartmental model for an infectious disease which takes the form of the Poisson driven SDE

$$Z^{N,z}(t) := \frac{[Nz]}{N} + \frac{1}{N}\sum_{j=1}^{k} h_j P_j\Big(N\int_0^t \beta_j(Z^N(s))ds\Big), \tag{1}$$

where N is the total size of the population that is assumed to be constant, k is the number of possible types of transitions for a given individual, the d components of $Z^{N,z}(t)$ denote the proportions of individuals in the d distincts compartments at time t, P_j $(j = 1, ..., k)$ are mutually independent standard Poisson processes, $h_j \in \{-1, 0, 1\}^d$ $(j = 1, ..., k)$ are the distinct jump directions with rates $\beta_j(.)$ which are $\mathbb{R}_+-$valued, and such that the solution of (1) remains in the set

$$A := \Big\{z \in \mathbb{R}_+^d : \sum_{i=1}^{d} z_i \le 1\Big\}.$$

Under appropriate assumptions, as $N \to \infty$, $Z^{N,z}(t) \to Y^z(t)$ in probability, locally uniformly in t, where $Y^z(t)$ solves the ODE

$$\frac{dY^z}{dt}(t) = b(Y^z(t)),\ Y^z(0) = z, \tag{2}$$

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with $b(z) = \sum_{j=1}^{k} \beta_j(z)h_j$. We have established in [13] a large deviation principle for the solution of such an SDE.

Our aim in this paper is to establish a large deviation principle for a similar Poisson driven SDE, which is reflected in the interior of some subset O of A. Indeed, let us look at the literature on the exit problem from a domain O of a dynamical system with a small Brownian perturbation. We note that most authors assume that the dynamical system crosses ∂O non-tangentially i.e. $\langle n(z), b(z)\rangle < 0$, for all $z \in \partial O$ where $n(z)$ is the unit outward normal to ∂O at z. On the other hand, Day in [1] considers the case of a "characteristic boundary", i.e. where the dynamical system which begins at a point on the boundary remains there, in other words $\langle n(z), b(z)\rangle = 0$ for all $z \in \partial O$. For instance, O might be the basin of attraction of a locally stable equilibrium of the ODE (2). In order to study the limit as $N \to \infty$ of the exit law of a small Brownian perturbation of the solution of (2) from O, he needs first to study the large deviation of a reflected SDE.

Similarly, we want to study in a forthcoming publication the large deviation of the exit law of a solution of (1) through a characteristic boundary of a stable domain of its law of large numbers limits $Y^z(t)$. Our motivation is the study of the most probable trajectory of $Z^{N,z}(t)$ when exiting the basin of attraction of an endemic situation, which is a locally stable equilibrium of the deterministic model (2).

The proof of the large deviation principle for the reflecting Poisson driven SDE follows the same steps as the proof of the large deviation principle for the original Poisson driven SDE defined by (1), but the arguments are modified when necessary. The paper is organized as follows. The reflected Poisson driven SDE is defined in section 2. The law of large numbers for that reflected SDE is established in section 3. Some preliminary results towards the large deviation are established in section 4. The lower bound is established in section 5, and the upper bound in section 6. Finally, in section 7, we show that our assumptions are satisfied in two examples of epidemics models which we have in mind among the possible applications of our results.

2 The reflected Poisson driven SDE

We will need to consider in this paper the sets $\bar{O}^N = \bar{O} \cap A^{(N)}$ where

$$A^{(N)} = \left\{ z \in A : Nz \in \mathbb{Z}_+^d \right\}.$$

For any $z \in \bar{O}$, we let $[Nz] = ([Nz_1], ..., [Nz_d])$ where $[a]$ denotes the integer part of the real number a. And we define the vector z^N by

$$z^N = \begin{cases} \frac{[Nz]}{N} & \text{if } \frac{[Nz]}{N} \in \bar{O}, \\ \arg\inf\limits_{y\in\bar{O}^{(N)}} |y - z| & \text{otherwise.} \end{cases}$$

We now define the d–dimensional reflected process $\tilde{Z}_t^{N,z}$ by

$$\begin{aligned} \tilde{Z}^N(t) = \tilde{Z}^{N,z}(t) &:= z^N + \frac{1}{N}\sum_{j=1}^{k} h_j Q_t^{N,j} - \frac{1}{N}\sum_{j=1}^{k} h_j \int_0^t \mathbf{1}_{\{\tilde{Z}^N(s-)+\frac{h_j}{N}\notin\bar{O}\}} dQ_s^{N,j} \\ &:= z^N + \frac{1}{N}\sum_{j=1}^{k} h_j \int_0^t \mathbf{1}_{\{\tilde{Z}^N(s-)+\frac{h_j}{N}\in\bar{O}\}} dQ_s^{N,j}, \end{aligned} \tag{3}$$

where the $Q_t^{N,j}$'s are defined by

$$Q_t^{N,j} = P_j\Big(N \int_0^t \beta_j(\tilde{Z}^N(s))ds\Big). \tag{4}$$

We remark that for any $t > 0$, $\tilde{Z}^N(t) \in \bar{O}^{(N)}$ and then a solution of (3) on the time interval $[0, T]$ belongs $a.s.$ to $D_{T,\bar{O}}$, which is the set of functions from $[0, T]$ into $\bar{O}$ and which are right continuous and have left limits. The aim of this paper is to show that the solution of the above reflecting Poisson driven SDE obeys the same large deviation principle with the same "good" rate function as the solution of (1). In the sequel, we denote $\mathbb{P}_z^N$ the probability measure on $D_{T,\bar{O}}$ such that the process $\tilde{Z}^N$ has as initial condition $\tilde{Z}^N(0) = z^N$, where z^N is chosen as we specified above.

The main difficulty to establish our result is that some of the rates β_j vanish on parts of the boundary of the set O. To solve this problem, we make the following assumptions:

Assumption 2.1. *1. $\bar{O}$ is compact and there exists a point z_0 in the interior of O such that each segment joining z_0 and any $z \in \partial O$ does not touch any other point of the boundary ∂O.*

2. *For each $a > 0$ small enough and $z \in \bar{O}$, denoting $z^a = z + a(z_0 - z)$, we assume that there exist two positive constants c_1 and c_2 such that*

$$|z - z^a| \le c_1 a,$$
$$\text{dist}(z^a, \partial O) \ge c_2 a.$$

3. *The rate functions β_j are Lipschitz continuous with the Lipschitz constant C.*

4. *There exist two constants λ_1 and λ_2 such that, whenever $z \in \bar{O}$ is such that $\beta_j(z) < \lambda_1$, $\beta_j(z^a) > \beta_j(z)$ for all $a \in]0, \lambda_2[$.*

5. *There exist constants $\nu \in]0, 1/2[$ and $a_0 > 0$ such that $C_a \ge \exp\{-a^{-\nu}\}$ for all $a < a_0$, where*

$$C_a = \inf_j \inf_{z:\text{dist}(z,\partial O)\ge c_2 a} \beta_j(z).$$

We define $\sigma = \sup_{1\le j\le k} \sup_{z\in A} \beta_j(z)$.

3 The weak law of large numbers

We first assume that ∂O is smooth enough so that the following assumption is satisfied.

Assumption 3.1. *There exists a function $u \in C_b^1(\bar{O})$ which satisfies the following assumptions:*

1. $O = A \cap \{z \in \bar{O} : u(z) > 0\}$, $\partial O = A \cap \{z \in \bar{O} : u(z) = 0\}$.

2. $\nabla u(z) \neq 0$ *for all* $z \in \partial O$.

3. *There exist* C_1, $C_2 > 0$ *such that* $\min\{C_1 \,\text{dist}(z, \partial O), C_2\} \le u(z)$, *for all* $z \in \bar{O}$

4. *$\langle b(z), \nabla u(z)\rangle \geq 0$ for all $z \in \bar{O}$, with $b(z) = \sum_{j=1}^{k} \beta_j(z) h_j$.*

5. *There exits $\rho > 0$ such that $\langle -g_N(z), \nabla u(z)\rangle \geq \rho \sum_{j=1}^{k} \mathbf{1}_{\{z+\frac{h_j}{N}\notin \bar{O}\}}$ for all $z \in \bar{O}$,*

where

$$g_N(z) = \sum_{j=1}^{k} \mathbf{1}_{\{z+\frac{h_j}{N}\notin \bar{O}\}} h_j \beta_j(z).$$

In the remaining of this section we assume that both Assumption 2.1 and Assumption 3.1 are in force.

Lemma 3.2. *Let $(\widetilde{\mathcal{M}}^N(t))_{t\geq 0}$ be the process defined for all $t \geq 0$ by*

$$\widetilde{\mathcal{M}}^N(t) = \sum_{j=1}^{k} \int_0^t h_j \mathbf{1}_{\{\tilde{Z}^N(s^-)+\frac{h_j}{N}\in \bar{O}\}} d\widetilde{Q}_s^{N,j},$$

where

$$\widetilde{Q}_s^{N,j} = \frac{1}{N} Q_s^{N,j} - \int_0^s \beta_j(\tilde{Z}^N(r)) dr.$$

Then $\widetilde{\mathcal{M}}^N(.)$ is a square integrable martingale and for all $T > 0$,

$$\sup_{0\leq t\leq T} |\widetilde{\mathcal{M}}^N(t)| \xrightarrow[N\to+\infty]{\mathbb{P}} 0.$$

Proof. $\widetilde{\mathcal{M}}^N(t)$ is a square integrable martingale since it is a sum of k stochastic integrals of bounded predictable processes with respect to square integrable martingales. We deduce from Doob's inequality, see e.g. [14] ,that

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\leq t\leq T} |\widetilde{\mathcal{M}}^N(t)|^2\Big) &\leq 4\mathbb{E}\Big(|\widetilde{\mathcal{M}}^N(T)|^2\Big) \\ &= 4\mathbb{E}\Big(\langle \widetilde{\mathcal{M}}^N\rangle_T\Big),\end{aligned}$$

where $\langle \mathcal{M}^N\rangle_t$ is the increasing predictable process such that $|\mathcal{M}^N(t)|^2 - \langle \mathcal{M}^N\rangle_t$ is a martingale. We have

$$\begin{aligned}\langle \widetilde{\mathcal{M}}^N\rangle_T &= \int_0^T \sum_{j=1}^{k} |h_j|^2 \mathbf{1}_{\{\tilde{Z}^N(t^-)+\frac{h_j}{N}\in \bar{O}\}} \frac{1}{N^2} N \beta_j(\tilde{Z}^N(t)) dt \\ &\leq \frac{kd\sigma T}{N}.\end{aligned}$$

Thus

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |\widetilde{\mathcal{M}}^N(t)|^2\Big) \leq \frac{kd\sigma T}{N} \to 0, \quad \text{as} \quad N\to\infty.$$

The result follows. □

We note that equation (3) can be re-written as

$$\tilde{Z}^N(t) = z^N + \int_0^t b(\tilde{Z}^N(s))ds - \int_0^t g_N(\tilde{Z}^N(s))ds + \widetilde{\mathcal{M}}^N(t), \tag{5}$$

where for all $z \in A$, $b(z) = \sum_{j=1}^{k} \beta_j(z)h_j$.

Lemma 3.3. *Let $(\widetilde{\mathcal{X}}^N(t))_{t\geq 0}$ be the process defined for all $t \geq 0$ by*

$$\widetilde{\mathcal{X}}^N(t) = \int_0^t \sum_{j=1}^{k} \mathbf{1}_{\left\{\tilde{Z}^N(s)+\frac{h_j}{N}\notin \bar{O}\right\}} ds.$$

Then for all $T > 0$,

$$\widetilde{\mathcal{X}}^N(T) \xrightarrow[N\to+\infty]{\mathbb{P}} 0.$$

Proof. Let u be the function appearing in the Assumption 3.1. Applying Itô's formula to u, we deduce that,

$$\begin{aligned} u(\tilde{Z}^N(t)) =& u(z^N) + \int_0^t \langle \nabla u(\tilde{Z}^N(s)), b(\tilde{Z}^N(s))\rangle ds - \int_0^t \langle \nabla u(\tilde{Z}^N(s)), g_N(\tilde{Z}^N(s))\rangle ds \\ &+ \int_0^t \langle \nabla u(\tilde{Z}^N(s)), d\widetilde{\mathcal{M}}^N(s)\rangle + \sum_{s\leq t} \left[u(\tilde{Z}^N(s)) - u(\tilde{Z}^N(s^-)) - \langle \nabla u(\tilde{Z}^N(s^-)), \Delta\tilde{Z}^N(s)\rangle\right]. \end{aligned}$$

Thus, we can use the Assumption 3.1 4 to deduce

$$u(\tilde{Z}^N(t)) \geq u(z^N) - \int_0^t \langle \nabla u(\tilde{Z}^N(s)), g_N(\tilde{Z}^N(s))\rangle ds + \tilde{\zeta}_t^{N,1} + \tilde{\zeta}_t^{N,2}, \tag{6}$$

where

$$\begin{aligned} \tilde{\zeta}_t^{N,1} &= \int_0^t \langle \nabla u(\tilde{Z}^N(s)), d\widetilde{\mathcal{M}}^N(s)\rangle \\ \tilde{\zeta}_t^{N,2} &= \sum_{s\leq t} \left[u(\tilde{Z}^N(s)) - u(\tilde{Z}^N(s^-)) - \langle \nabla u(\tilde{Z}^N(s^-)), \Delta\tilde{Z}^N(s)\rangle\right]. \end{aligned}$$

Moreover $\sup_{0\leq t\leq T} |\tilde{\zeta}_t^{N,1}| \xrightarrow[N\to+\infty]{\mathbb{P}} 0$. Indeed, again from Doob's inequality,

$$\begin{aligned} \mathbb{E}\Big(\sup_{0\leq t\leq T} |\tilde{\zeta}_t^{N,1}|^2\Big) &\leq 4\mathbb{E}\big(|\tilde{\zeta}_T^{N,1}|^2\big) \\ &\leq 4\mathbb{E}\Big(\langle \tilde{\zeta}^{N,1}\rangle_T\Big). \end{aligned}$$

But

$$\begin{aligned} \langle \tilde{\zeta}^{N,1}\rangle_T &= \int_0^T |\nabla u(\tilde{Z}^N(t))|^2 d\langle \widetilde{\mathcal{M}}^N\rangle_t \\ &= \frac{1}{N}\sum_{j=1}^{k} |h_j|^2 \int_0^T |\nabla u(\tilde{Z}^N(t))|^2 \mathbf{1}_{\left\{\tilde{Z}^N(t)+\frac{h_j}{N}\notin \bar{O}\right\}} \beta_j(\widetilde{Z}^N(t))dt \\ &\leq \frac{kd\sigma K_1 T}{N}. \end{aligned}$$

Then

$$\mathbb{E}\Big(\sup_{0\le t\le T}|\tilde{\zeta}_t^{N,1}|^2\Big)\le \frac{kd\sigma K_1 T}{N}\to 0 \quad \text{as} \quad N\to\infty.$$

We also have $\sup_{0\le t\le T}|\tilde{\zeta}_t^{N,2}| \xrightarrow[N\to+\infty]{\mathbb{P}} 0$. Indeed by Taylor's expansion, if $0\le t\le T$ is a jump time of $\tilde{Z}^N(t)$,

$$u(\tilde{Z}^N(t))-u(\tilde{Z}^N(t^-))=\Big\langle \nabla u\big(\tilde{Z}^N(t^-)+\theta\Delta\tilde{Z}^N(t)\big),\Delta\tilde{Z}^N(t)\Big\rangle \text{ for some random } 0\le\theta\le 1.$$

Consequently, since $u\in C_b^1(\bar{O})$ and $|\Delta\tilde{Z}^N(t)|\le\sqrt{d}/N$,

$$\begin{aligned}
&|u(\tilde{Z}^N(t))-u(\tilde{Z}^N(t^-))-\langle\nabla u(\tilde{Z}^N(t^-)),\Delta\tilde{Z}^N(t)\rangle|\\
&=\Big|\Big\langle\nabla u\big(\tilde{Z}^N(t^-)+\theta\Delta\tilde{Z}^N(t)\big)-\nabla u\big(\tilde{Z}^N(t^-)\big),\Delta\tilde{Z}^N(t)\Big\rangle\Big|\\
&\le\Big|\nabla u\big(\tilde{Z}^N(t^-)+\theta\Delta\tilde{Z}^N(t)\big)-\nabla u\big(\tilde{Z}^N(t^-)\big)\Big|\times|\Delta\tilde{Z}^N(t)|\\
&\le\frac{\sqrt{d}}{N}\sup_{z\in\bar{O},|\theta\Delta z|\le\frac{\sqrt{d}}{N}}|\nabla u(z+\theta\Delta z)-\nabla u(z)|:=\frac{\sqrt{d}}{N}\delta_N, \text{with } \delta_N\to 0 \text{ as } N\to\infty.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sup_{0\le t\le T}|\tilde{\zeta}_t^{N,2}|&=\sup_{t\le T}\Big|\sum_{s\le t}\big[u(\tilde{Z}^N(s))-u(\tilde{Z}^N(s^-))-\big\langle\nabla u(\tilde{Z}^N(s^-)),\Delta\tilde{Z}^N(s)\big\rangle\big]\Big|\\
&\le\sum_{s\le T}\Big|u(\tilde{Z}^N(s))-u(\tilde{Z}^N(s^-))-\big\langle\nabla u(\tilde{Z}^N(s^-)),\Delta\tilde{Z}^N(s)\big\rangle\Big|\\
&\le\frac{\sqrt{d}}{N}\delta_N\sum_{j=1}^k Q_T^{N,j}
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}\Big(\sup_{0\le t\le T}|\tilde{\zeta}_t^{N,2}|\Big)&\le\sqrt{d}\frac{1}{N}\mathbb{E}\Big(\sum_{j=1}^k Q_T^{N,j}\Big)\delta_N\\
&\le C'k\delta_N.
\end{aligned}$$

Let $\delta>0$ and

$$\tilde{\mathcal{B}}^N=\Big\{z\in\bar{O}:\sum_{j=1}^k\mathbf{1}_{\{z+\frac{h_j}{N}\notin\bar{O}\}}>0\Big\}$$

and with the convention that $\inf\emptyset=\infty$, let $T_\delta^{N,1}$ be the stopping time defined by

$$T_\delta^{N,1}:=\inf\Big\{t>0:\tilde{\mathcal{X}}^N(t)\ge 2\delta/3\Big\}\wedge T.$$

By using (6) and the Assumption 3.1 5 we have for any $t\in[T_\delta^{N,1},T]$,

$$u(\tilde{Z}^N(t))\ge\rho\frac{2\delta}{3}+\tilde{\zeta}_t^{N,1}+\tilde{\zeta}_t^{N,2}.$$

We deduce that

$$\inf_{T_\delta^{N,1}<t<T} u(\tilde{Z}^N(t)) \geq \rho\frac{2\delta}{3} + \inf_{T_\delta^{N,1}<t<T}\Big(\tilde{\zeta}_t^{N,1} + \tilde{\zeta}_t^{N,2}\Big).$$

Then

$$\lim_{N\to\infty} \mathbb{P}_z\Big(\inf_{T_\delta^{N,1}<t<T} u(\tilde{Z}^N(t)) \geq \rho\frac{\delta}{3}\Big) = 1. \tag{7}$$

With the convention that $\inf\emptyset = \infty$, let $T_\delta^{N,2}$ the stopping time defined by

$$T_\delta^{N,2} := \inf\Big\{t > T_\delta^{N,1} : \tilde{Z}^N(t) \in \tilde{\mathcal{B}}^N\Big\} \wedge T.$$

As $u = 0$ on the boundary ∂O and ∇u is bounded, there exists a constant ρ' such that if $\tilde{Z}^N(t) \in \tilde{\mathcal{B}}^N$ then $u(\tilde{Z}^N(t)) < \rho'/N$ and then we deduce from (7) that for any $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}_z(T_\delta^{N,2} < T) = 0,$$

and consequently

$$\lim_{N\to\infty} \mathbb{P}_z\Big(\int_0^T \sum_{j=1}^k \mathbf{1}_{\left\{\tilde{Z}^N(s)+\frac{h_j}{N}\notin\bar{O}\right\}} ds > \delta\Big) = 0.$$

□

Theorem 3.4. *Let $\tilde{Z}^N(t)$ be the sequence of processes solution of the reflecting Poisson driven SDE (3) with the initial condition z^N. Assuming that the β_j's are Lipschitz continuous, then for all $T > 0$,*

$$\sup_{t\leq T} |\tilde{Z}^N(t) - Y(t)| \xrightarrow[N\to+\infty]{\mathbb{P}} 0,$$

where $Y(t)$ is the unique solution of (2).

Proof. We have

$$\begin{aligned}
\tilde{Z}^N(t) &= z^N + \frac{1}{N}\sum_{j=1}^k h_j \int_0^t \mathbf{1}_{\{\tilde{Z}^N(s^-)+\frac{h_j}{N}\in\bar{O}\}} dQ_s^{N,j} \\
&= z^N + \int_0^t b(\tilde{Z}^N(s))ds + \widetilde{\mathcal{M}}^N(t) - \int_0^t \sum_{j=1}^k \mathbf{1}_{\left\{\tilde{Z}^N(s)+\frac{h_j}{N}\notin\bar{O}\right\}} \beta_j(\tilde{Z}^N(s))h_j ds,
\end{aligned}$$

where $\widetilde{\mathcal{M}}^N(t)$ is defined as in Lemma 3.2. We define

$$\Phi^N(t) = \widetilde{\mathcal{M}}^N(t) - \int_0^t \sum_{j=1}^k \mathbf{1}_{\left\{\tilde{Z}^N(s)+\frac{h_j}{N}\notin\bar{O}\right\}} \beta_j(\tilde{Z}^N(s))h_j ds$$

and

$$U^N(t) = \tilde{Z}^N(t) - \Phi^N(t).$$

Then

$$|\tilde{Z}^N(t) - Y(t)| \le |U^N(t) - Y(t)| + |\Phi^N(t)|.$$

Moreover we have the following inequalities where the second one follows from the Lipschitz character of the β_j's

$$\begin{aligned}|U^N(t) - Y(t)| &\le |z^N - z| + \int_0^t |b(U^N(s) + \Phi^N(s)) - b(Y(s))|ds \\ &\le |z^N - z| + kC\int_0^t |U^N(s) - Y(s)|ds + kC\int_0^t |\Phi^N(s)|ds.\end{aligned}$$

We now deduce from Gronwall's inequality

$$|U^N(t) - Y(t)| \le \Big(|z^N - z| + kC\int_0^t |\Phi^N(s)|ds\Big)\exp\{kCt\},$$

hence

$$|\tilde{Z}^N(t) - Y(t)| \le \Big(|z^N - z| + kC\int_0^t |\Phi^N(s)|ds\Big)\exp\{kCt\} + |\Phi^N(t)|.$$

Therefore

$$\sup_{0\le t\le T}|\tilde{Z}^N(t) - Y(t)| \le |z^N - z|\exp\{kCT\} + \big(1 + kCT\exp\{kCT\}\big)\sup_{0\le t\le T}|\Phi^N(t)|.$$

Lemmas 3.2 and 3.3 imply that $\sup_{t\le T}|\Phi^N(t)|\overset{\mathbb{P}}{\longrightarrow}0$, as $N\to\infty$. The result follows. □

4 Large Deviations: preliminary results

For all $\phi \in \mathcal{AC}_{T,\bar{O}}$, the subspace of $D_{T,\bar{O}}$ consisting of absolutely continuous functions, let $\mathcal{A}_d(\phi)$ denote the (possibly empty) set of $\mathbb{R}_+^k$-valued Borel measurable functions μ such that

$$\frac{d\phi_t}{dt} = \sum_{j=1}^k \mu_t^j h_j, \quad \text{t a.e.}$$

We define the rate function

$$I_T(\phi) := \begin{cases}\inf_{\mu\in\mathcal{A}_d(\phi)} I_T(\phi|\mu), & \text{if } \phi\in\mathcal{AC}_{T,\bar{O}},\\ \infty, & \text{else,}\end{cases}$$

where

$$I_T(\phi|\mu) = \int_0^T \sum_{j=1}^k f(\mu_t^j, \beta_j(\phi_t))dt$$

with $f(\nu,\omega) = \nu\log(\nu/\omega) - \nu + \omega$. We assume in the definition of $f(\nu,\omega)$ that for all $\nu > 0$, $\log(\nu/0) = \infty$ and $0\log(0/0) = 0\log(0) = 0$.

The following result is a direct consequence of Lemma 4.22 in [10]

Lemma 4.1. *Let F a closed subset of $D_{T,A}$ and $z \in A$. We have*

$$\lim_{\epsilon\to 0} \inf_{y\in A,|y-z|<\epsilon} \inf_{\phi\in F,\phi_0=y} I_T(\phi) = \inf_{\phi\in F,\phi_0=z} I_T(\phi).$$

The next lemma states a large deviation estimate for Poisson random variables.

Lemma 4.2. *Let Y_1,Y_2,...be independent Poisson random variables with mean $\sigma\epsilon$. For all $N \in \mathbb{N}$, let*

$$\bar{Y}^N = \frac{1}{N}\sum_{n=1}^{N} Y_n.$$

For any $s > 0$ there exist $K, \epsilon_0 > 0$ and $N_0 \in \mathbb{N}$ such that with

$$g(\epsilon) = K\sqrt{\log^{-1}(\epsilon^{-1})}, \tag{8}$$

we have

$$\mathbb{P}^N(\bar{Y}^N > g(\epsilon)) < \exp\{-sN\}$$

for all $\epsilon < \epsilon_0$ and $N > N_0$.

Proof. We apply Cramer's theorem, see e.g [2], chapter 2 :

$$\limsup_{N\to\infty} \frac{1}{N}\log(\mathbb{P}^N(\bar{Y}^N > g(\epsilon))) \leq -\inf_{x\geq g(\epsilon)} \Lambda^*_\epsilon(x),$$

where $\Lambda^*_\epsilon(x) = \sup_{\lambda\in\mathbb{R}}\{\lambda x - \Lambda_\epsilon(\lambda)\}$ with

$$\Lambda_\epsilon(\lambda) = \log(\mathbb{E}(e^{\lambda Y_1}) = \sigma\epsilon(e^\lambda - 1).$$

We deduce that

$$\Lambda^*_\epsilon(x) = x\log\frac{x}{\sigma\epsilon} - x + \sigma\epsilon.$$

This last function is convex It reaches its infimum at $x = \sigma\epsilon$ and as $\lim_{\epsilon\to 0}\frac{g(\epsilon)}{\sigma\epsilon} = +\infty$ there exists $\epsilon_1 > 0$ such that $g(\epsilon) > \sigma\epsilon$ for all $\epsilon < \epsilon_1$ and then, with the notation $a_\epsilon \approx b_\epsilon$ meaning that there exists a constant C such that $C^{-1}b_\epsilon \leq a_\epsilon \leq Cb_\epsilon$ for all $\epsilon > 0$,

$$\begin{aligned}
\inf_{x\geq g(\epsilon)} \Lambda^*_\epsilon(x) &= g(\epsilon)\log\frac{g(\epsilon)}{\sigma\epsilon} - g(\epsilon) + \sigma\epsilon \\
&= g(\epsilon)\log(g(\epsilon)) - g(\epsilon)\log(\sigma\epsilon) - g(\epsilon) + \sigma\epsilon \\
&\approx g(\epsilon)\log(1/\epsilon) \\
&\approx K\sqrt{\log(1/\epsilon)} \to \infty \quad \text{as} \quad \epsilon \to 0.
\end{aligned}$$

Then there exists $\epsilon_2 > 0$ such that $\inf_{x\geq g(\epsilon)} \Lambda^*_\epsilon(x) > s$ for all $\epsilon < \epsilon_2$. The lemma follows by choosing $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$. □

5 The Lower Bound

This section is summarized by the following result whose proof is essentially the same as that of Theorem 2.4 in section 2 of [13]. It mainly uses a Girsanov change of probability for doubly stochastic Poisson processes as well as the law of large numbers established in section 3.

Theorem 5.1. *Assume that both Assumption 2.1 and Assumption 3.1 are satisfied. Let $\tilde{Z}^{N,z}$ be the solution of (3) with t restricted to $[0,T]$.*

a) *For $z \in \bar{O}$, $\phi \in D_{T,\bar{O}}$, $\phi_0 = z$, $\eta > 0$ and $\delta > 0$ there exists $N_{\eta,\delta} \in \mathbb{N}$ such that for all $N > N_{\eta,\delta}$*

$$\inf_{y:|y-z|<\delta/2} \mathbb{P}_y\Big(\|\tilde{Z}^N - \phi\|_T < \delta\Big) \geq \exp\{-N(I_T(\phi) + \eta)\}.$$

b) *For any open subset G of $D_{T,\bar{O}}$, the following hold uniformly over $z \in \bar{O}$*

$$\liminf_{\substack{N\to\infty \\ y\to z}} \frac{1}{N} \log \mathbb{P}_y(\tilde{Z}^N \in G) \geq - \inf_{\phi\in G,\phi_0=z} I_T(\phi).$$

6 The Upper Bound

In [13], the upper bound was established as a consequence of a result in [4], which does not apply here. This is why we need to detail the proof of the upper bound.

In this section, we shall assume that both Assumption 2.1 and Assumption 3.1 are in force.

For all $\phi \in D_{T,\bar{O}}$ and $F \subset D_{T,\bar{O}}$, we define $\rho_T(\phi, F) = \inf_{\psi\in F} \|\phi - \psi\|_T$. For $z \in \bar{O}$, $\delta, s > 0$ we define the sets $\Phi_z(s) = \{\psi \in D_{T,\bar{O}} : \psi_0 = z, I_T(\psi) \leq s\}$ and $F_\delta^s(z) = \{\phi \in D_{T,\bar{O}} : \rho_T(\phi, \Phi_z(s)) \geq \delta\}$.

The following Proposition constitutes the main step in the proof of the upper bound.

Proposition 6.1. *For δ, η and $s > 0$ there exists $N_0 \in \mathbb{N}$ such that*

$$\mathbb{P}_z(\tilde{Z}^N \in F_\delta^s(z)) \leq \exp\{-N(s - \eta)\} \tag{9}$$

whenever $N \geq N_0$ and $z \in \bar{O}$.

Proof. Let $\tilde{Z}_a^N(t) = (1-a)\tilde{Z}^N(t) + az_0$ then $\|\tilde{Z}^N - \tilde{Z}_a^N\|_T < c_1 a$ and for all $c_1 a < \delta(d-1)/d$,

$$\mathbb{P}_z(\tilde{Z}^N \in F_\delta^s(z)) \leq \mathbb{P}_z\Big(\rho_T(\tilde{Z}_a^N, \Phi(s)) \geq \frac{\delta}{d}\Big). \tag{10}$$

We now approximate the $\tilde{Z}^N$ by piecewise linear paths. Let $\epsilon > 0$ be such that $T/\epsilon \in \mathbb{N}$. We construct a polygonal approximation of $\tilde{Z}_a^N$ defined for all $t \in [\ell\epsilon, (\ell+1)\epsilon[$ by

$$\Upsilon_t = \Upsilon_t^{a,\epsilon} = \tilde{Z}_a^N(\ell\epsilon)\frac{(\ell+1)\epsilon - t}{\epsilon} + \tilde{Z}_a^N((\ell+1)\epsilon)\frac{t - \ell\epsilon}{\epsilon}.$$

Since $\{\|\tilde{Z}^N_a - \Upsilon\|_T < \frac{\delta}{2d}\} \cap \{\rho_T(\tilde{Z}^N_a, \Phi_z(s)) \geq \frac{\delta}{d}\} \subset \{\rho_T(\Upsilon, \Phi(s)) \geq \frac{\delta}{2d}\}$,

$$\mathbb{P}_z\Big(\rho_T(\tilde{Z}^N_a, \Phi_z(s)) \geq \frac{\delta}{d}\Big) \leq \mathbb{P}_z\Big(\rho_T(\Upsilon, \Phi_z(s)) \geq \frac{\delta}{2d}\Big) + \mathbb{P}_z\Big(\|\tilde{Z}^N_a - \Upsilon\|_T \geq \frac{\delta}{2d}\Big)$$
$$\leq \mathbb{P}_z(I_T(\Upsilon) \geq s) + \mathbb{P}_z\Big(\|\tilde{Z}^N_a - \Upsilon\|_T \geq \frac{\delta}{2d}\Big). \tag{11}$$

We now bound $\mathbb{P}_z(I_T(\Upsilon) \geq s)$. For any choice $\mu \in \mathcal{A}_d(\Upsilon)$ we have $I_T(\Upsilon) \leq I_T(\Upsilon|\mu)$ and

$$\mathbb{P}_z(I_T(\Upsilon) \geq s) \leq \mathbb{P}_z(I_T(\Upsilon|\mu) \geq s). \tag{12}$$

Let $\{\mu^j_t, 1 \leq j \leq k\} \in \mathcal{A}_d(\Upsilon)$ be constant on the intervals $[\ell\epsilon, (\ell+1)\epsilon[$ and equal to

$$\mu^j_t = \frac{1-a}{N\epsilon}\Big[P_j\Big(N\int_0^{(\ell+1)\epsilon} \beta_j(\tilde{Z}^N(s))ds\Big) - P_j\Big(N\int_0^{\ell\epsilon} \beta_j(\tilde{Z}^N(s))ds\Big)\Big]. \tag{13}$$

To control the change of Υ over the intervals of length ϵ, we will use the constant $g(\epsilon)$ from (8) and consider the collection of events $B = \{B_\epsilon\}_{\epsilon>0}$ defined by

$$B_\epsilon = \bigcap_{\ell=0}^{T/\epsilon-1} B^\ell_\epsilon, \text{ with } B^\ell_\epsilon = \Big\{\sup_{\ell\epsilon \leq t_1, t_2 \leq (\ell+1)\epsilon} |\tilde{Z}^N_i(t_1) - \tilde{Z}^N_i(t_2)| \leq g(\epsilon) \quad \text{for} \quad i = 1, ..., d\Big\}.$$

We have

$$\mathbb{P}_z(I_T(\Upsilon|\mu) > s) \leq \mathbb{P}_z(\{I_T(\Upsilon|\mu) > s\} \cap B_\epsilon) + \mathbb{P}_z(B^c_\epsilon). \tag{14}$$

Combining (10), (11), (12) and (14), we deduce that

$$\mathbb{P}_z(\tilde{Z}^N \in F^s_\delta(z)) \leq \mathbb{P}_z(\{I_T(\Upsilon|\mu) > s\} \cap B_\epsilon) + \mathbb{P}_z(B^c_\epsilon) + \mathbb{P}_z\Big(\|\tilde{Z}^N_a - \Upsilon\|_T \geq \frac{\delta}{2d}\Big). \tag{15}$$

The next Lemmas give appropriate upper bounds for the three terms in the right side of (15). The proof of the first one relies upon Lemma 4.2.

Lemma 6.2. *For any $s > 0$ there exists $\epsilon_s > 0$, $N_0 \in \mathbb{N}$ and $K > 0$ such that*

$$\mathbb{P}_z(B^c_\epsilon) + \mathbb{P}_z(\|\tilde{Z}^N_a - \Upsilon\|_T > \delta/2d) < 2\frac{dkT}{\epsilon}\exp\{-sN\} \tag{16}$$

for all $\epsilon < \epsilon_0$, $N > N_0$ and any $z \in \bar{O}$.

Proof. It is enough to show that the two terms on the left of (16) have $\frac{dkT}{\epsilon}\exp\{-sN\}$ as upper bound. For the first terms in that left side, we first remark that for all $j = 1, ..., k$ and $\ell = 1, ..., T/\epsilon$ we can write

$$\int_0^{(\ell+1)\epsilon} \beta_j(\tilde{Z}^N_s)ds < \int_0^{\ell\epsilon} \beta_j(\tilde{Z}^N_s)ds + \sigma\epsilon.$$

Moreover, we have

$$B^c_\epsilon = \bigcup_{i=1,...,d}\ \bigcup_{\ell=1,...,T/\epsilon} \Big\{\sup_{(\ell-1)\epsilon \leq t_1, t_2 \leq \ell\epsilon} |\tilde{Z}^N_i(t_1) - \tilde{Z}^N_i(t_2)| > g(\epsilon)\Big\}.$$

Thus

$$\mathbb{P}_z(B_\epsilon^c) \le \sum_{i=1}^{d} \sum_{\ell=1}^{T/\epsilon} \mathbb{P}_z\Big\{ \sup_{(\ell-1)\epsilon \le t_1, t_2 \le \ell\epsilon} |\tilde{Z}_i^N(t_1) - \tilde{Z}_i^N(t_2)| > g(\epsilon) \Big\}.$$

Using (3) we have

$$\begin{aligned}
&\sup_{(\ell-1)\epsilon \le t_1, t_2 \le \ell\epsilon} |\tilde{Z}_i^N(t_1) - \tilde{Z}_i^N(t_2)| \\
&= \sup_{(\ell-1)\epsilon \le t_1, t_2 \le \ell\epsilon} \Big| \sum_j \frac{h_j^i}{N} \Big[\int_0^{t_1} \big(1 - \mathbf{1}_{\{\tilde{Z}^N(s) + \frac{h_j}{N} \notin \bar{O}\}}\big) dQ_s^{N,j} - \int_0^{t_2} \big(1 - \mathbf{1}_{\{\tilde{Z}^N(s) + \frac{h_j}{N} \notin \bar{O}\}}\big) dQ_s^{N,j} \Big] \Big| \\
&\le \frac{1}{N} \sum_j \Big| \int_{(\ell-1)\epsilon}^{\ell\epsilon} \big(1 - \mathbf{1}_{\{\tilde{Z}^N(s) + \frac{h_j}{N} \notin \bar{O}\}}\big) dQ_s^{N,j} \Big| \\
&\le \frac{1}{N} \sum_j \Big| P_j\Big(N \int_0^{(\ell-1)\epsilon} \beta_j(\tilde{Z}^N(s))ds + N\sigma\epsilon\Big) - P_j\Big(N \int_0^{(\ell-1)\epsilon} \beta_j(\tilde{Z}^N(s))ds\Big) \Big| \\
&\le \frac{1}{N} \sum_j Z_j,
\end{aligned}$$

where Z_j $j = 1, ..., k$ are Poisson random variables with the mean $N\sigma\epsilon$. Then

$$\mathbb{P}_z\Big\{ \sup_{(\ell-1)\epsilon \le t_1, t_2 \le \ell\epsilon} |\tilde{Z}_i^N(t_1) - \tilde{Z}_i^N(t_2)| > g(\epsilon) \Big\} \le k\mathbb{P}^N(N^{-1}Z_1 > g(\epsilon)/k)$$

Since it is a Poisson random variable with mean $N\sigma\epsilon$, Z_1 is the sum of N iid Poisson random variable with mean $\sigma\epsilon$. Hence, from lemma 4.2, for each $s > 0$ there exist constants $K > 0$, $\epsilon_s^1 > 0$ and $N_0 \in \mathbb{N}$ such that

$$\mathbb{P}_z\Big\{ \sup_{(\ell-1)\epsilon \le t_1, t_2 \le \ell\epsilon} |\tilde{Z}_i^N(t_1) - \tilde{Z}_i^N(t_2)| > g(\epsilon) \Big\} \le k \exp\{-sN\}$$

for all $\epsilon < \epsilon_s^1$ and $N > N_0$. Consequently

$$\mathbb{P}_z(B_\epsilon^c) < \frac{dkT}{\epsilon} \exp\{-sN\},$$

which is the first half of (16). We now establish the second half.

We first show that there exist $\epsilon_s \le \epsilon_s^1$ and $N_0 \in \mathbb{N}$ such that for all $\epsilon < \epsilon_s$, $N > N_0$ and any $z \in \bar{O}$,

$$\mathbb{P}_z(\|\tilde{Z}_a^N - \Upsilon\|_T > \delta/2d) < \frac{dkT}{\epsilon} \exp\{-sN\}.$$

We deduce from (3) that for all $t \in [\ell\epsilon, (\ell+1)\epsilon[$

$$
\begin{aligned}
|\tilde{Z}_i^{N,a}(t) - \Upsilon_t^i| &\le \frac{t-\ell\epsilon}{\epsilon}|\tilde{Z}_i^{N,a}((\ell+1)\epsilon) - \tilde{Z}_i^{N,a}(t)| + \frac{(\ell+1)\epsilon - t}{\epsilon}|\tilde{Z}_i^{N,a}(t) - \tilde{Z}_i^{N,a}(\ell\epsilon)| \\
&\le \frac{t-\ell\epsilon}{\epsilon}\sum_j \frac{1}{N}\Big|\int_t^{(\ell+1)\epsilon}(1 - \mathbf{1}_{\{\tilde{Z}^N(s)+\frac{h_j}{N}\notin\bar{O}\}})dQ_s^{N,j}\Big| \\
&+ \frac{(\ell+1)\epsilon - t}{\epsilon}\sum_j \frac{1}{N}\Big|\int_{\ell\epsilon}^{t}(1 - \mathbf{1}_{\{\tilde{Z}^N(s)+\frac{h_j}{N}\notin\bar{O}\}})dQ_s^{N,j}\Big| \\
&\le \sum_j \frac{1}{N}\Big|P_j\Big(N\int_0^{(\ell+1)\epsilon}\beta_j(\tilde{Z}^N(s))ds\Big) - P_j\Big(N\int_0^{\ell\epsilon}\beta_j(\tilde{Z}^N(s))ds\Big)\Big| \\
&\le \frac{1}{N}\sum_j \Big|P_j\Big(N\int_0^{\ell\epsilon}\beta_j(\tilde{Z}^N(s))ds + N\sigma\epsilon\Big) - P_j\Big(N\int_0^{\ell\epsilon}\beta_j(\tilde{Z}^N(s))ds\Big)\Big| \\
&\le \frac{1}{N}\sum_j Z_j,
\end{aligned}
$$

where the Z_j are as in the first part of the proof. Let ϵ_0^2 be the largest ϵ such that $\delta/kd > g(\epsilon)$. Then we have from Lemma 4.2 that for all $\epsilon < \epsilon_0 = \min\{\epsilon_0^1, \epsilon_0^2\}$ and $N > N_0$

$$
\begin{aligned}
\mathbb{P}_z(\|\tilde{Z}^{N,a} - \Upsilon\|_T > \delta) &\le \mathbb{P}_z\Big(\bigcup_{i=1}^d\{|\tilde{Z}_i^{N,a}(t) - \Upsilon_t^i| > \frac{\delta}{d}\} \quad \text{for some} \quad t \in [0,T]\Big) \\
&\le \frac{T}{\epsilon}\max_{0\le\ell\le T/\epsilon-1}\mathbb{P}_z\Big(\bigcup_{i=1}^d\{|\tilde{Z}_i^{N,a}(t) - \Upsilon_t^i| > \frac{\delta}{d}\} \quad \text{for some} \quad t \in [\ell\epsilon, (\ell+1)\epsilon[\Big) \\
&\le \frac{dkT}{\epsilon}\mathbb{P}_z(Z_1/N > \delta/kd) \le \frac{dkT}{\epsilon}\exp\{-sN\}.
\end{aligned}
$$

The result follows. □

It remains to upper bound $\mathbb{P}_z(\{I_T(\Upsilon|\mu) > s\} \cap B_\epsilon)$ from the right hand side of (15). We first deduce from Chebyshev's inequality that for all $0 < \alpha < 1$

$$
\mathbb{P}_z(\{I_T(\Upsilon|\mu) > s\} \cap B_\epsilon) \le \frac{\mathbb{E}_z(\exp\{\alpha N I_T(\Upsilon|\mu)\}\mathbf{1}_{B_\epsilon})}{\exp\{\alpha N s\}}. \tag{17}
$$

In order to conclude the proof of Proposition 6.1, all we need to do is to get an upper bound of the numerator in the right hand side of (17) of the type $\exp\{N\delta\}$, with δ arbitrarily small. This will be achieved in Lemma 6.6. Note that the ideas behind this proof come from [3] and the proof of Theorem 3.2.2, chapter 3 in [7]. We first establish

Lemma 6.3. *For all $0 < \alpha < 1$, $j = 1, ..., k$ and $\ell = 0, ..., T/\epsilon - 1$, there exists W_j which conditionally upon $\mathcal{F}_{\ell\epsilon}^N$ are mutually independent Poisson random variables with respective mean $N\epsilon\beta_\ell^j = N\epsilon(\beta_j(\tilde{Z}^N(\ell\epsilon)) + Cdg(\epsilon))$, such that if*

$$
\Theta_j^\ell = \exp\Big\{\alpha N\int_{\ell\epsilon}^{(\ell+1)\epsilon} f(\mu_t^j, \beta_j(\Upsilon_t))dt\Big\}\mathbf{1}_{B_\epsilon^\ell}
$$

and

$$\Xi_j^\ell = \exp\{\alpha N\epsilon(\sigma + 2Cdg(\epsilon))\}\exp\Big\{\alpha N\epsilon f\Big(\frac{(1-a)W_j}{\epsilon N}, \beta_\ell^{a,j}\Big)\Big\}$$

where $\beta_\ell^{a,j} = (\beta_j(\Upsilon_{\ell\epsilon}) - Cdg(\epsilon))_+$, *then*

$$\Theta_j^\ell \le \Xi_j^\ell \quad a.s \tag{18}$$

Proof. On B_ϵ^ℓ, if $g(\epsilon) < 1$ and $t \in [\ell\epsilon, (\ell+1)\epsilon]$, we have

$$\Big| N\int_{\ell\epsilon}^{(\ell+1)\epsilon} \beta_j(\tilde Z^N(t))dt - N\epsilon\beta_j(\tilde Z^N(\ell\epsilon))\Big| \le N\epsilon Cdg(\epsilon), \quad j = 1, ..., k.$$

If μ_t^j, $j = 1, ..., k$ are defined by (13), we have

$$0 \le \mu_{\ell\epsilon}^j \le \frac{(1-a)W_j}{\epsilon N} \quad \text{a.s.,} \tag{19}$$

where

$$W_j = P_j\Big(N\int_0^{\ell\epsilon}\beta_j(\tilde Z^N(s))ds + \epsilon N(\beta_j(\tilde Z^N(\ell\epsilon)) + Cdg(\epsilon))\Big) - P_j\Big(N\int_0^{\ell\epsilon}\beta_j(\tilde Z^N(s))ds\Big).$$

Moreover on the event B_ϵ^ℓ for all $t \in [\ell\epsilon, (\ell+1)\epsilon]$, if $\beta_\ell^{a,j} = (\beta_j(\Upsilon_{\ell\epsilon}) - Cdg(\epsilon))_+$,

$$\beta_\ell^{a,j} \le \beta_j(\Upsilon_t) \le \beta_\ell^{a,j} + 2Cdg(\epsilon).$$

Hence, again on B_ϵ^ℓ

$$f(\mu_t^j, \beta_j(\Upsilon_t)) \le f(\mu_t^j, \beta_\ell^{a,j}) + 2Cdg(\epsilon).$$

In fact,

$$\begin{aligned} f(\mu_t^j, \beta_j(\Upsilon_t)) &= \mu_t^j \log\frac{\mu_t^j}{\beta_j(\Upsilon_t)} - \mu_t^j + \beta_j(\Upsilon_t) \\ &\le \mu_t^j \log\frac{\mu_t^j}{\beta_\ell^{a,j}} - \mu_t^j + \beta_\ell^{a,j} + 2Cdg(\epsilon) + \mu_t^j \log\frac{\beta_\ell^{a,j}}{\beta_j(\Upsilon_t)} \\ &\le f(\mu_t^j, \beta_\ell^{a,j}) + 2Cdg(\epsilon) \quad \text{since} \quad \log\frac{\beta_\ell^{a,j}}{\beta_j(\Upsilon_t)} < 0. \end{aligned}$$

As $\mu_t^j = \mu_{\ell\epsilon}^j$ is constant over the interval $[\ell\epsilon, (\ell+1)\epsilon[$, we deduce that on B_ϵ^ℓ

$$\exp\Big\{\alpha N\int_{\ell\epsilon}^{(\ell+1)\epsilon} f(\mu_t^j, \beta_j(\Upsilon_t))dt\Big\} \le \exp\{\alpha N\epsilon f(\mu_{\ell\epsilon}^j, \beta_\ell^{a,j}) + 2\alpha NCd\epsilon g(\epsilon)\}. \tag{20}$$

(18) follows from (19), (20) and the convexity of $f(\nu, \omega)$ in ν. □

The next Proposition gives us an upper bound for the conditional expectation of the right hand side of the inequality (18).

Proposition 6.4. *Let* $a = h(\epsilon) = \Big[-\log g^{1/2}(\epsilon)\Big]^{-\frac{1}{\nu}}$ *where* ν *is the constant in the Assumption 2.1 5. For all* $0 < \alpha < 1$ *there exist* ϵ_α, K_α *and* $\tilde{K}$ *such that for all* $\epsilon \le \epsilon_\alpha$ *we have, with* $g(\epsilon)$ *defined by* (8), *for any* $z \in \bar{O}$,

$$\mathbb{E}_z\Big(\exp\Big\{\alpha N\epsilon f\Big(\frac{(1-a)W_j}{\epsilon N}, \beta_\ell^{a,j}\Big)\Big\}|\mathcal{F}_{\ell\epsilon}^N\Big)\Big\} \le K_\alpha \exp\{N\epsilon\tilde{K}(1-\alpha+2h(\epsilon)+2dg(\epsilon))\}.$$

Proof. Conditionally upon $\mathcal{F}_{\ell\epsilon}^N$, W_j is a Poisson random variable with mean $N\epsilon\beta_\ell^j$. Moreover we have, see the definitions of $\beta_\ell^{a,j}$ and β_ℓ^j in the statement of Lemma 6.3,

$$|\beta_\ell^{a,j} - \beta_\ell^j| \le \tilde{C}(a+2dg(\epsilon)).$$

With $\tilde{\epsilon} = \epsilon/(1-a)$ and $\tilde{\alpha} = (1-a)\alpha$, we have

$$\begin{aligned}
&\mathbb{E}_z\Big(\exp\Big\{\alpha N\epsilon f\Big(\frac{(1-a)W_j}{\epsilon N}, \beta_\ell^{a,j}\Big)\Big\}|\mathcal{F}_{\ell\epsilon}^N\Big) = \mathbb{E}_z\Big(\exp\Big\{\alpha N\epsilon f\Big(\frac{W_j}{\tilde{\epsilon}N}, \beta_\ell^{a,j}\Big)\Big\}|\mathcal{F}_{\ell\epsilon}^N\Big)\\
&= \sum_{m\ge 0} \exp\Big\{\alpha N\epsilon f\Big(\frac{m}{\tilde{\epsilon}N}, \beta_\ell^{a,j}\Big)\Big\}\frac{(N\epsilon\beta_\ell^j)^m\exp\{-N\epsilon\beta_\ell^j\}}{m!}\\
&= \sum_{m\ge 0} \exp\Big\{\alpha N\epsilon\Big(\frac{m}{\tilde{\epsilon}N}\log\Big(\frac{m}{\tilde{\epsilon}N\beta_\ell^{a,j}}\Big) - \frac{m}{\tilde{\epsilon}N} + \beta_\ell^{a,j}\Big)\Big\}\frac{(N\epsilon\beta_\ell^j)^m\exp\{-N\epsilon\beta_\ell^j\}}{m!}\\
&\le \exp\{N\epsilon\tilde{C}(a+2dg(\epsilon))\}\sum_{m\ge 0}\frac{m^{\tilde{\alpha}m}\exp\{-\tilde{\alpha}m\}}{m!}(N\epsilon\beta_\ell^{a,j})^{m(1-\tilde{\alpha})}\Big(\frac{\beta_\ell^j}{\beta_\ell^{a,j}}\Big)^m\exp\{-N\epsilon\beta_\ell^{a,j}(1-\alpha)\}\\
&\le \exp\{N\epsilon C_1(a+2dg(\epsilon))\}\sum_{m\ge 0}\frac{m^{\tilde{\alpha}m}\exp\{-\tilde{\alpha}m\}}{m!}(N\epsilon\beta_\ell^{a,j})^{m(1-\tilde{\alpha})}\Big(\frac{\beta_\ell^j}{\beta_\ell^{a,j}}\Big)^m\exp\{-N\epsilon\beta_\ell^{a,j}(1-\tilde{\alpha})\}.
\end{aligned} \tag{21}$$

Since the function $v(x) = x^{m(1-\tilde{\alpha})}\exp\{-2x(1-\tilde{\alpha})\}$ reaches its maximum at $x = m/2$,

$$(N\epsilon\beta_\ell^{a,j})^{m(1-\tilde{\alpha})}\exp\{-2N\epsilon\beta_\ell^{a,j}(1-\tilde{\alpha})\} \le \Big(\frac{m}{2}\Big)^{m(1-\tilde{\alpha})}\exp\{-m(1-\tilde{\alpha})\}.$$

Thus

$$\begin{aligned}
&\sum_{m\ge 0}\frac{m^{\tilde{\alpha}m}\exp\{-\tilde{\alpha}m\}}{m!}(N\epsilon\beta_\ell^{a,j})^{m(1-\tilde{\alpha})}\Big(\frac{\beta_\ell^j}{\beta_\ell^{a,j}}\Big)^m\exp\{-N\epsilon\beta_\ell^{a,j}(1-\tilde{\alpha})\}\\
&\le \exp\{N\epsilon\beta_\ell^{a,j}(1-\tilde{\alpha})\}\sum_{m\ge 0}\frac{m^m\exp\{-m\}}{m!}\Big(\frac{\beta_\ell^j/\beta_\ell^{a,j}}{2^{(1-\tilde{\alpha})}}\Big)^m
\end{aligned} \tag{22}$$

We shall show (see the proof of Lemma 6.5 below) that for all $0 < \alpha < 1$ there exists $\epsilon_\alpha > 0$ such that for all $\epsilon < \epsilon_\alpha$,

$$\frac{\beta_\ell^j}{\beta_\ell^{a,j}} < 2^{(1-\alpha)/2} < 2^{(1-\tilde{\alpha})/2}.$$

Then for ϵ small enough we have

$$\exp\{N\epsilon\beta_\ell^{a,j}(1-\tilde{\alpha})\}\sum_{m\ge 0}\frac{m^m e^{-m}}{m!}\Big(\frac{\beta_\ell^{j,q}/\beta_\ell^{a,j}}{2^{(1-\tilde{\alpha})}}\Big)^m \le e^{N\epsilon\theta(1-\tilde{\alpha})}K_\alpha \tag{23}$$

since the above series converges. The proposition follows from (21), (22) and (23). □

Lemma 6.5. *For all $0 < \alpha < 1$ there exists $\epsilon_\alpha > 0$ such that for all $\epsilon < \epsilon_\alpha$*

$$\frac{\beta_\ell^j}{\beta_\ell^{a,j}} < 2^{(1-\alpha)/2} < 2^{(1-\tilde{\alpha})/2},$$

with again $\tilde{\alpha} = (1-a)\alpha$, a as in the statement of Proposition 6.4.

Proof. We have

$$\frac{\beta_\ell^j}{\beta_\ell^{a,j}} \le \frac{\beta_j(\tilde{Z}^N(\ell\epsilon)) + Cdg(\epsilon)}{(\beta_j(\tilde{Z}^{N,a}(\ell\epsilon)) - Cdg(\epsilon))^+}$$

If $\beta_j(\tilde{Z}^N(\ell\epsilon)) < \lambda_1$ we have using the Assumptions 2.1 4 and 5,

$$\begin{aligned}\frac{\beta_\ell^j}{\beta_\ell^{a,j}} &\le \frac{\beta_j(\tilde{Z}^{N,a}(\ell\epsilon)) + Cdg(\epsilon)}{(\beta_j(\tilde{Z}^{N,a}(\ell\epsilon)) - Cdg(\epsilon))^+} \\ &\le \frac{C_a + Cdg(\epsilon)}{(C_a - Cdg(\epsilon))^+} \le \frac{1 + \frac{Cdg(\epsilon)}{g^{1/2}(\epsilon)}}{\left(1 - \frac{Cdg(\epsilon)}{g^{1/2}(\epsilon)}\right)^+} \to 1 \quad as \quad \epsilon \to 0.\end{aligned}$$

If $\beta_j(\tilde{Z}^N(\ell\epsilon)) \ge \lambda_1$, we have

$$\begin{aligned}\frac{\beta_\ell^j}{\beta_\ell^{a,j}} &\le \frac{\beta_j(\tilde{Z}^N(\ell\epsilon)) + Cdg(\epsilon)}{(\beta_j(\tilde{Z}^N(\ell\epsilon)) - C\bar{C}h(\epsilon) - Cdg(\epsilon))^+} \\ &\le \frac{\lambda_1 + Cdg(\epsilon)}{(\lambda_1 - C\bar{C}h(\epsilon) - Cdg(\epsilon))^+} \to 1 \quad \text{as} \quad \epsilon \to 0.\end{aligned}$$

Then there exists ϵ_α such that $\frac{\beta_\ell^j}{\beta_\ell^{a,j}} < 2^{(1-\alpha)/2} < 2^{(1-\tilde{\alpha})/2}$ for all $\epsilon < \epsilon_\alpha$. □

The next lemma gives us an upper bound for the quantity $\mathbb{E}_z\Big(\exp\{\alpha N I_T(\Upsilon|\mu)\}\mathbf{1}_{B_\epsilon}\Big)$.

Lemma 6.6. *For all $0 < \alpha < 1$ there exist ϵ_α, K_α and $\tilde{K}_1$ such that for all $\epsilon \le \epsilon_\alpha$, we have the following inequality*

$$\mathbb{E}_z\Big(\exp\{\alpha N I_T(\Upsilon|\mu)\}\mathbf{1}_{B_\epsilon}\Big) \le (2K_\alpha)^{\frac{kT}{\epsilon}} \exp\{kNT\tilde{K}_1(1-\alpha+h(\epsilon)+4dg(\epsilon))\}, \qquad (24)$$

for any $z \in \bar{O}$.

Proof. We first deduce from Lemma 6.3 and Proposition 6.4

$$\mathbb{E}_z(\Theta_j^\ell|\mathcal{F}_{\ell\epsilon}^N) \le \mathbb{E}_z(\Xi_j^\ell|\mathcal{F}_{\ell\epsilon}^N) \le K_\alpha \exp\{N\epsilon\tilde{K}_1(1-\alpha+2h(\epsilon)+4dg(\epsilon))\}.$$

Moreover, the Ξ_j^ℓ, $j = 1, ..., k$ are conditionnally independent given $\mathcal{F}_{\ell\epsilon}^N$. So we can take successively the conditional expectations with respect to $\mathcal{F}_{(\frac{T}{\epsilon}-1)\epsilon}^N$, $\mathcal{F}_{(\frac{T}{\epsilon}-2)\epsilon}^N$,...,$\mathcal{F}_{\epsilon}^N$, to conclude that for all $0 < \alpha < 1$ and $\epsilon < \epsilon_\alpha$,

$$\begin{aligned}\mathbb{E}_z\Big(\exp\{\alpha N I_T(\Upsilon|\mu)\}\mathbf{1}_{B_\epsilon}\Big) &= \mathbb{E}_z\Big(\mathbb{E}_z\Big(\prod_{\ell=0}^{\frac{T}{\epsilon}-1}\prod_{j=1}^{k}\Theta_j^\ell|\mathcal{F}_{(\frac{T}{\epsilon}-1)\epsilon}^N\Big)\Big) \le \mathbb{E}_z\Big(\mathbb{E}_z\Big(\prod_{\ell=0}^{\frac{T}{\epsilon}-1}\prod_{j=1}^{k}\Xi_j^\ell|\mathcal{F}_{(\frac{T}{\epsilon}-1)\epsilon}^N\Big)\Big) \\ &\le \mathbb{E}_z\Big(\prod_{\ell=0}^{\frac{T}{\epsilon}-2}\prod_{j=1}^{k}\Xi_j^\ell\mathbb{E}_z\Big(\prod_{j=1}^{k}\Xi_j^{\frac{T}{\epsilon}-1}|\mathcal{F}_{(\frac{T}{\epsilon}-1)\epsilon}^N\Big)\Big) \\ &= (K_\alpha)^{\frac{kT}{\epsilon}} \exp\{kNT\tilde{K}_1(1-\alpha+h(\epsilon)+4dg(\epsilon))\}.\end{aligned}$$

The result follows $\square$

We now conclude the proof of Proposition 6.1. The upper bound of the first term in the right side of (15) is obtained by combining (17) and (24). Indeed, for all $0 < \alpha < 1$, $\epsilon < \min\{\epsilon_0, \epsilon_\alpha, \epsilon_1\}$ and $a = h(\epsilon) = \Big[-\log g^{1/2}(\epsilon)\Big]^{-\frac{1}{\nu}}$,

$$\mathbb{P}_z(\{I_T(\Upsilon|\mu) > s\} \cap B_\epsilon) \le (K_\alpha)^{\frac{kT}{\epsilon}} \exp\{kNT\tilde{K}_1(1-\alpha+h(\epsilon)+4dg(\epsilon))\} \times \exp\{-\alpha N s\} \quad (25)$$

Combining (15), (16) and (25), we have for all $\delta > 0$, α, ϵ and a as above,

$$\mathbb{P}_z(\tilde{Z}^N \in F^s_\delta(z)) \le (K_\alpha)^{\frac{kT}{\epsilon}} \exp\{kNT\tilde{K}_1(1-\alpha+h(\epsilon)+4dg(\epsilon))\} \times \exp\{-\alpha N s\} + \frac{2dTk}{\epsilon} \exp\{-sN\}.$$

Finally, we choose $1-\alpha$ and ϵ small enough to ensure that $kT\tilde{K}_1(1-\alpha+h(\epsilon)+4dg(\epsilon)) < \eta/4$ and $(1-\alpha)s < \eta/4$. We also take N large enough so that $kT\log(K_\alpha)/N\epsilon < \eta/4$ and $\log(2dkT/\epsilon)/N < \eta/4$, hence we deduce that for any $z \in A$,

$$\mathbb{P}_z(\tilde{Z}^N \in F^s_\delta(z)) \le \exp\{-N(s-\eta)\}.$$

$\square$

We now establish the upper bound.

Theorem 6.7. *For any closed subset F of $D_{T,A}$, $z \in A$*

$$\limsup_{\substack{N\to\infty \\ y\to z}} \frac{1}{N} \log \mathbb{P}_y(\tilde{Z}^N \in F) \le -\inf_{\phi\in F, \phi_0 = z} I_T(\phi). \quad (26)$$

Proof. We first assume that $\inf_{\phi\in F,\phi_0=z} I_T(\phi) < \infty$, and let $\gamma > 0$ be arbitrary. By Lemma 4.1, there exists $\epsilon^\gamma > 0$ such that for all $\epsilon < \epsilon^\gamma$,

$$y \in A, |y-z| \le \epsilon \implies \inf_{\phi\in F,\phi_0=y} I_T(\phi) \ge \inf_{\phi\in F,\phi_0=z} I_T(\phi) - \gamma.$$

For $\epsilon < \epsilon^\gamma$, let $s = \inf_{\phi\in F,\phi_0=z} I_T(\phi) - \gamma$,

$$W(\epsilon) = \{\phi \in F : |\phi_0 - z| \le \epsilon\} \quad \text{and} \quad U(\epsilon) = \bigcup_{y\in A, |y-z|\le\epsilon} \Phi_y(s).$$

$W(\epsilon)$ is closed in $D_{T,A}$ and does not intersect the set $U(\epsilon)$, which is compact, see Proposition 4.21 in [10]. By the Hahn-Banach theorem,

$$\delta^\epsilon = \inf_{\phi\in W(\epsilon)} \inf_{\psi\in U(\epsilon)} \|\phi - \psi\|_T > 0.$$

We deduce that for all $\eta > 0$ and any $y \in A$ with $|y - z| \le \epsilon$, there exists $N_0 \in \mathbb{N}$ such that for all $N > N_0$, using (9) for the second inequality,

$$\begin{aligned}\mathbb{P}_y(\tilde{Z}^N \in F) &= \mathbb{P}_y(\tilde{Z}^N \in W(\epsilon)) \\ &\le \mathbb{P}_y(\tilde{Z}^N \in F^s_{\delta^\epsilon}(y)) \\ &\le \exp\{-N(s-\eta)\}.\end{aligned}$$

Consequently, for any $N > N_0$ (where N_0 depends upon ϵ, γ and η) and $|y - z| < \epsilon^\gamma$,

$$\frac{1}{N} \log \mathbb{P}_y(\tilde{Z}^N \in F) \leq - \inf_{\phi \in F, \phi_0 = z} I_T(\phi) + \gamma + \eta.$$

We deduce that

$$\limsup_{\substack{N\to\infty \\ y \to z}} \frac{1}{N} \log \mathbb{P}_y(\tilde{Z}^N \in F) \leq - \inf_{\phi \in F, \phi_0 = z} I_T(\phi) + \gamma + \eta,$$

for any $\nu, \eta > 0$, hence the result in the case $\inf_{\phi \in F, \phi_0 = z} I_T(\phi) < \infty$. Otherwise, for any $s > 0$, there exists $\epsilon > 0$ such that the distance between $W(\epsilon)$ and $U(\epsilon)$ is greater that some $\delta^\epsilon > 0$. Following the above argument, we deduce that

$$\limsup_{\substack{N\to\infty \\ y \to z}} \frac{1}{N} \log \mathbb{P}_y(\tilde{Z}^N \in F) \leq -s + \eta,$$

for any $s, \eta > 0$, from which the result follows. □

We deduce as in [2] Corollary 5.6.15,

Corollary 6.8. *For any open subset F of $D_{T,A}$ and any compact subset K of A,*

$$\limsup_{N\to\infty} \frac{1}{N} \log \sup_{z \in K} \mathbb{P}_z(\tilde{Z}^N \in F) \leq - \inf_{z \in K} \inf_{\phi \in F, \phi_0 = z} I_T(\phi).$$

7 Applications

In this section we present two models of infectious diseases for which the hypothesis 3.1 is verified. Indeed, the characteristic boundary (that we note $\widetilde{\partial O}$) between the basin of attraction of the disease-free equilibrium $\bar{z}$ and the basin of attraction of the endemic equilibrium z^* is the global stable manifold of the saddle point $\tilde{z}$ (another point of endemic equilibrium). We know from [9] (Theorem 1.3.1, page 13) that this boundary is a curve of class C^∞.

7.1 A model with vaccination (SIV)

7.1.1 Description of the model

We consider the so-called $S^N I^N V^N$ model: it is a model with vaccination and demography. In this model, we suppose that a population with constant size N is divided into three compartments namely: $S^N(t)$, $I^N(t)$ and $V^N(t)$ are respectively the number of susceptible, infectious and vaccinated individuals at time t. Figure 1 gives us a good graphical representation of the disease transmission for this model. We assume that susceptibles are vaccinated at rate η and lose their protection at rate θ; the vaccine is not perfect but decreases the rate of infection by a factor $\chi \in [0, 1]$. While each infected infects a given susceptible at rate $\beta S^N(t)/N$, it infects a given vaccinated individual at rate $\chi\beta V^N/N$, where $\beta = r\kappa$, κ being the mean number of individuals met by one infected per unit time,

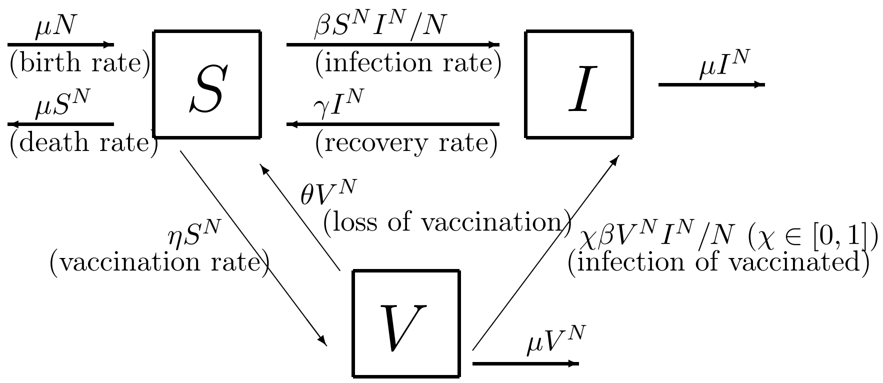

Figure 1: Diagram of the $S^N I^N V^N$ model.

r (resp $r\chi$) is the probability that an encounter between an infected and a susceptible (resp a vaccinated) results in an infection.

The proportions of infectious $i^N(t) = \frac{I^N(t)}{N}$ and vaccinated $v^N(t) = \frac{V^N(t)}{N}$ can be described as follows. Let P_1, P_2, P_3, P_4 , P_5, P_6, P_7 be iid standard Poisson processes:

$$\begin{aligned}
i^N(t) &= i^N(0) + \frac{1}{N}P_1\Big(N\beta\int_0^t i^N(s)(1-i^N(s)-v^N(s))ds\Big) + \frac{1}{N}P_2\Big(N\chi\beta\int_0^t i^N(s)v^N(s)ds\Big)\\
&\quad - \frac{1}{N}P_3\Big(N\gamma\int_0^t i^N(s)ds\Big) - \frac{1}{N}P_6\Big(N\mu\int_0^t i^N(s)ds\Big),\\
v^N(t) &= v^N(0) - \frac{1}{N}P_2\Big(N\chi\beta\int_0^t i^N(s)v^N(s)ds\Big) - \frac{1}{N}P_4\Big(N\theta\int_0^t i^N(s)v^N(s)ds\Big)\\
&\quad + \frac{1}{N}P_5\Big(N\eta\int_0^t (1-i^N(s)-v^N(s))ds\Big) - \frac{1}{N}P_7\Big(N\mu\int_0^t v^N(s)ds\Big).
\end{aligned}$$

If we let

$$\begin{aligned}
h_1 &= (1,0)^\top, \quad \beta_1(z) = \beta z_1(1-z_1-z_2),\\
h_2 &= (1,-1)^\top, \quad \beta_2(z) = \chi\beta z_1 z_2,\\
h_3 &= (-1,0)^\top, \quad \beta_3(z) = \gamma z_1,\\
h_4 &= (0,-1)^\top, \quad \beta_4(z) = \theta z_2,\\
h_5 &= (0,1)^\top, \quad \beta_5(z) = \eta(1-z_1-z_2)\\
h_6 &= (-1,0)^\top, \quad \beta_6(z) = \mu z_1,\\
h_7 &= (0,-1)^\top, \quad \beta_7(z) = \mu z_2,
\end{aligned}$$

our epidemic model takes the form

$$Z^{N,z_0}(t) = \frac{[Nz_0]}{N} + \frac{1}{N}\sum_{j=1}^{7} h_j P_j\Big(N\int_0^t \beta_j(Z^{N,z_0}(s))ds\Big), \tag{27}$$

where $Z^{N,z_0}(t) = (i^N(t), v^N(t))$.

It is easy to show that for all $T > 0$ the process $(Z^{N,z_0}(t))_{0\le t\le T}$ defined by (27) converges almost surely and uniformly on $[0,T]$ as N tends to ∞ to the solution $(i(t), v(t))$

of the following ODE

$$\begin{cases} \frac{dz_1}{dt}(t) = (\beta-\mu-\gamma)z_1(t) - \beta(1-\chi)z_1(t)z_2(t) - \beta z_1^2(t) \\ \frac{dz_2}{dt}(t) = \eta - \eta z_1(t) - (\eta+\mu+\theta)z_2(t) - \chi\beta z_1(t)z_2(t). \end{cases} \tag{28}$$

where $i(t)$, $v(t)$ denote respectively the proportion of infectious and vaccinated at time t. This deterministic SIV model (28) is the two dimensional version of the SIV model studied in [11] by Kribs-Zaleta and Velasco-Hernández (see Theorem 1). They show that if $(\mu+\theta+\chi\eta)^2 < (\mu+\gamma)\chi(1-\chi)\eta$ and $\beta_1 < \beta < \beta_0$ where

$$\begin{aligned} \beta_0 &= (\mu+\gamma)\frac{\mu+\theta+\eta}{\mu+\theta+\chi\eta} \\ \beta_1 &= \mu+\gamma-\frac{\mu+\theta+\chi\eta}{\sigma}+\frac{2}{\chi}\sqrt{(\mu+\gamma)\sigma(1-\chi)\eta}, \end{aligned}$$

then two endemic equilibria $z^* = (z_1^*, z_2^*)$, $\tilde{z} = (\tilde{z}_1, \tilde{z}_2)$ exist, one of which namely z^* is locally stable while $\tilde{z}$ is unstable. These two equilibria are completed with the disease free equilibrium $\bar{z}$ $\left(\bar{z}_1 = 0, \bar{z}_2 = \frac{\eta}{\mu+\theta+\eta}\right)$ which is locally stable. The figure 2 shows the basin of attraction O of the endemic equilibrium z^*. It is delimited by the boundary $\widetilde{\partial O}$ and it contains the point z^*. The first components of the equilibria z^* and $\tilde{z}$ are the solutions of the equation 29 and the second components is given by equation 30 below

$$\begin{cases} D_1x^2 + D_2x + D_3 = 0 \\ \text{where} \quad D_1 = -\beta\chi, D_2 = \chi(\beta-\mu-\gamma) - (\mu+\theta+\chi\eta) \\ \text{and} \quad D_3 = (\mu+\theta+\eta)(1-\frac{\mu+\gamma}{\beta}) - (1-\chi)\eta. \end{cases} \tag{29}$$

$$z_2 = \frac{\eta(1-z_1)}{\mu+\theta+\eta+\beta\chi z_1}. \tag{30}$$

7.1.2 The boundary $\widetilde{\partial O}$ in the SIV model

The aim of this section is to establish that the assumptions 2.1 1 and 2.1 2 are satisfied by the model with vaccination of [11] and with O the basin of attraction of the equilibrium z^*.

We define

- $(\mathcal{D}_{1.1})$ the straight line whose equation reads $(\beta-\mu-\gamma) - \beta(1-\chi)z_2 - \beta z_1 = 0$ i.e

$$z_2 = -\frac{\beta}{\beta(1-\chi)}z_1 + \frac{\beta-\mu-\gamma}{\beta(1-\chi)},$$

- $(\mathcal{H}_1)$ the curve having the equation $\eta - \eta z_1 - (\eta+\mu+\theta)z_2 - \chi\beta z_1 z_2 = 0$ i.e

$$z_2 = \frac{\eta - \eta z_1}{\chi\beta z_1 + (\eta+\mu+\theta)},$$

In order to obtain a parametrization of the characteristic boundary on the interval [0, 2], we make the change of variable $u = t/(1+t)$ and the ODE (28) can be re-written

$$\begin{cases} \frac{dy_1}{du}(u) = \frac{1}{(1-u)^2}[(\beta-\mu-\gamma)y_1(u) - \beta(1-\chi)y_1(u)y_2(u) - \beta y_1^2(u)] \\ \frac{dy_2}{dt}(u) = \frac{1}{(1-u)^2}[\eta - \eta y_1(u) - (\eta+\mu+\theta)y_2(u) - \chi\beta y_1(u)y_2(u)]. \end{cases} \tag{31}$$

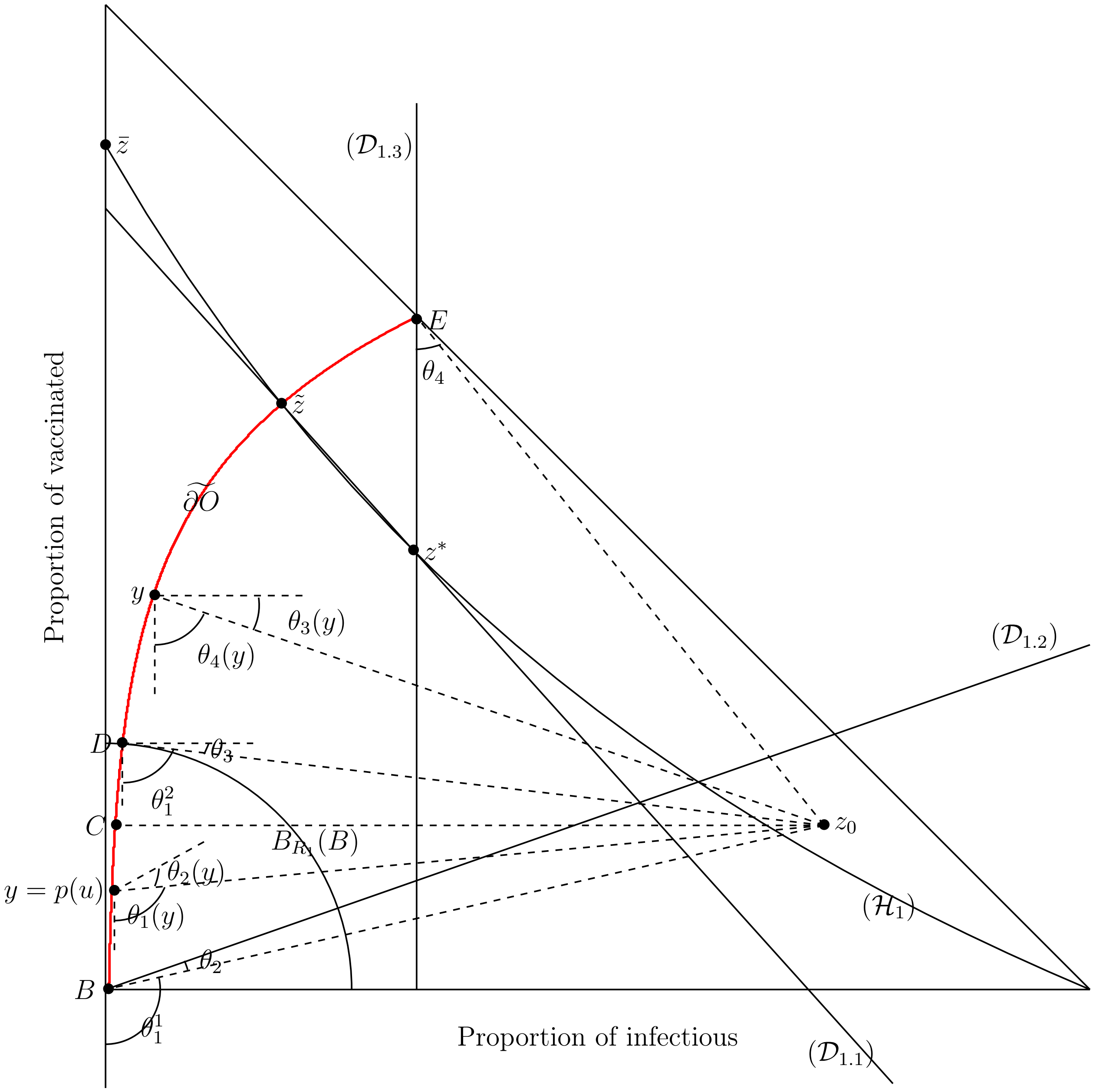

Figure 2: The characteristic boundary $\widetilde{\partial O}$ in the SIV model

Remarks 7.1. • *The part of characteristic boundary $\widetilde{\partial O}_1$ which starts from the point B (different to the origin) to the unstable endemic equilibrium $\tilde{z}$ of the dynamical system* (28) *has as parametrization the solution $p^1(u) = (p_1^1(u), p_2^1(u))_{0\le u<1}$ of* (31) *with the initial condition B.*

- *The part of characteristic boundary $\widetilde{\partial O}_2$ which starts from the extremity E to the unstable endemic equilibrium $\tilde{z}$ of the dynamical system* (28) *(see Figure 2) has as parametrization the solution $p^2(u) = (p_1^2(u), p_2^2(u))_{0\le u<1}$ of the dynamical system defined by* (31) *with as initial condition the point E.*
Thus the characteristic boundary admits as parametric curve $(p(u))_{0\le u\le 2}$ defined by

$$p(u) = \begin{cases} p^1(u) & \text{if} \quad 0 \le u < 1 \\ \tilde{z} \quad \text{if} \quad u = 1 \\ p^2(2-u) & \text{if} \quad 1 < u \le 2. \end{cases}$$

- *As the tangent to the boundary $\widetilde{\partial O}$ at the origin B is vertical, by the continuity there exist a ball $B_{R_1}(B)$ and a constant $\nu > 0$ such that for all $(p_1(u), p_2(u)) \in \widetilde{\partial O} \cap B_{R_1}(B)$,*

$$\frac{\dot{p}_2(u)}{\dot{p}_1(u)} > \nu.$$

In all what follows, $\mathcal{D}_{1.2}$ will be the line having the equation $z_2 = \nu z_1$.

From these remarks we deduce that for all $t \in [0,2]$, $\dot{p}_1(u) \ge 0$ and $\dot{p}_2(u) \ge 0$ since the first part of the characteristic boundary is below both $(\mathcal{D}_{1.1})$ and $\mathcal{H}_1$ and the second part $\widetilde{\partial O}_2$ is above both $(\mathcal{D}_{1.1})$ and $\mathcal{H}_1$.

Now we choose the point $z_0 \in O$ such that the following conditions are satisfied

- z_0 is above $(\mathcal{D}_{1.1})$ and $(\mathcal{H}_1)$,
- z_0 is below $(\mathcal{D}_{1.2})$,
- z_0 is at the right side of $(\mathcal{D}_{1.3})$,
- its second coordinate is smaller than that the point of $\widetilde{\partial O}$ at distance R_1 to B.

It clear that such a point exists.

We now verify the assumptions 2.1 1 and 2.1 2 through the proof of the following Lemma

Lemma 7.2. *The assumption 2.1 1 is satisfied and there exists $\theta \in]0, \pi[$ such that for all $y \in \partial O$ and $a \in]0,1[$,*

$$\text{dist}(y^a, \partial O) \ge a \times \sin(\theta) \times \inf_{v \in \widetilde{\partial O}} |v - z_0|. \tag{32}$$

where $y^a = y + a(z_0 - y)$ and z_0 is chosen above.

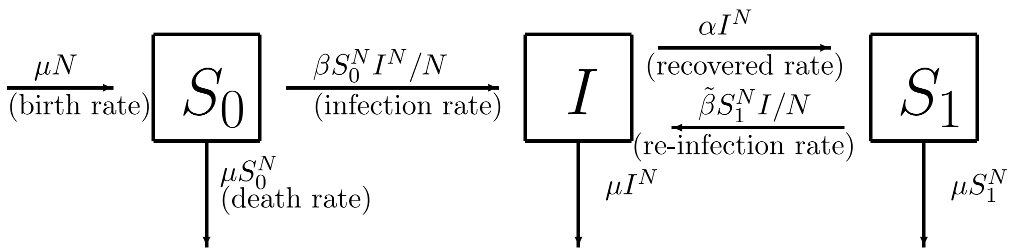

Figure 3: Compartmental diagram of $S_0^N I S_1^N$ model.

Proof. We first note that the only part of boundary ∂O on which (32) could fail is the boundary $\widetilde{\partial O}$. So let's check it on this one. To this end we divide the boundary $\widetilde{\partial O}$ into three parts. The first one goes from B to the intersection point (namely C) of $\widetilde{\partial O}$ and the horizontal line passing through the point z_0. The second part goes from C to the intersection point (namely D) of $\widetilde{\partial O}$ and $B_{R_1}(B)$. And the last goes from D to E.

On the first part of $\widetilde{\partial O}$, we fix a current point y and we denote by $\theta_1(y)$ the angle that the segment which joins the points z_0 and y makes with the vertical line passing through y. We also denote $\theta_2(y)$ the angle that make the same segment with the line with slope ν passing through y. It is not very difficult to see that there exist θ_1^1 and θ_2 such that $\pi/2 \leq \theta_1(y) \leq \theta_1^1 \leq \pi$, $0 < \theta_2 \leq \theta_2(y) \leq \pi/2$ and then $\sin(\theta_1(y)) \geq \sin(\theta_1^1)$ and $\sin(\theta_2(y)) \geq \sin(\theta_2)$(see figure 2).

On the part of the boundary $\widetilde{\partial O}$ from C to D, there exist θ_1^2 such that $0 < \theta_1^2 \leq \theta_1(y) \leq \pi/2$ and $0 < \theta_2 \leq \theta_2(y) \leq \pi/2$ and then $\sin(\theta_1(y)) \geq \sin(\theta_1^2)$ and $\sin(\theta_2(y)) \geq \sin(\theta_2)$.

For the part of $\widetilde{\partial O}$ from D to E, the segment from z_0 to y makes with the horizontal line passing through the point y an angle $\theta_3(y)$ and with the vertical line passing through the point y an angle $\theta_4(y)$. Moreover it is not difficult to remark (see figure 2) that there exist θ_3 and θ_4 with $0 \leq \theta_3 \leq \theta_3(y) \leq \pi/2$, $0 < \theta_4 \leq \theta_4(y) \leq \pi/2$ such that $\sin(\theta_3(y)) \geq \sin(\theta_3)$ and $\sin(\theta_4(y)) \geq \sin(\theta_4)$.

We deduce from the above that for all $y \in \widetilde{\partial O}$,

$$\begin{aligned}
\mathrm{dist}(y^a, \widetilde{\partial O}) &\geq \min_{i=1,2,3,4} \sin(\theta_i(y)) \times |y^a - y| \\
&= a \times \min_{i=1,2,3,4} \sin(\theta_i(y)) \times |y - z_0| \\
&= a \times \min_{i=1,2,3,4} \sin(\theta_i) \times \inf_{v \in \widetilde{\partial O}} |v - z_0|.
\end{aligned}$$

□

7.2 A model with two levels of susceptibility ($S_0 I S_1$)

7.2.1 Description of the model.

The $S_0^N I^N S_1^N$ epidemic model is a model which describes an endemic infection having two levels of susceptibility. In this model, we suppose that a population with constant size N is divided into three compartments namely:

- S_0^N-"the class of naive individuals" who are susceptibles without past infections. i.e individuals who have never been infected and may contract the infection.
- I^N is the class of infectious individuals
- S_1^N is the class of susceptible individuals with at least one past infection. They are also called recovered.

We assume that the probability of infection of an S_0^N type (resp. S_1^N) individual upon contact with an infected individual is r_0 (resp. r_1). If κ is the number of individuals which an infected individual meets per unit time, the rate of infection from S_0 (resp. S_1) compartment is $r_0\kappa S_0^N(t)I^N(t)/N$ (resp. $r_1\kappa S_0^N(t)I^N(t)/N$). Each infected individual recovers at rate α. Each individual dies at rate μ, at which he is replaced in the population by a susceptible individual. The total population size is constant equal to N. The schematic representation of the disease transmission for this model is given by figure 3

The process described above is a continuous Markov Chain with state $(I^N(t), S_1^N(t))$. Let P_1, P_2, P_3, P_4, P_5 be the iid standard Poisson processes, we have

$$\begin{aligned} i^N(t) &= i^N(0) + \frac{1}{N}P_1\Big(N\int_0^t \beta i^N(u)(1-i^N(u)-s_1^N(u))du\Big) - \frac{1}{N}P_2\Big(N\int_0^t \alpha i^N(u)du\Big) \\ &\quad - \frac{1}{N}P_3\Big(N\int_0^t \mu i^N(u)du\Big) + \frac{1}{N}P_4\Big(N\int_0^t r\beta i^N(u)s_1^N(u)du\Big) \qquad (33) \\ s_1^N(t) &= s_1^N(0) + \frac{1}{N}P_2\Big(N\int_0^t \alpha i^N(u)du\Big) - \frac{1}{N}P_4\Big(N\int_0^t r\beta i^N(u)s_1^N(u)du\Big) - \frac{1}{N}P_5\Big(N\int_0^t \mu s_1^N(u)du\Big), \end{aligned}$$

where $i^N(t) = \frac{I^N(t)}{N}$ and $s_1(t) = \frac{S_1^N(t)}{N}$ represent respectively the proportion of infectious and the proportion of susceptibles which have already been sick. Thus if we let $Z^{N,z}(t) = (i^N(t), s_1^N(t))^T$ and

- $\beta_1(x) = \beta x_1(1-x_1-x_2)$, $h_1 = (1,0)^T$
- $\beta_2(x) = \alpha x_1$, $h_2 = (-1,1)^T$
- $\beta_3(x) = \mu x_1$, $h_3 = (-1,0)^T$
- $\beta_4(x) = r\beta x_1 x_2$, $h_4 = (1,-1)^T$
- $\beta_5(x) = \mu x_2$, $h_5 = (0,-1)^T$

The equation (33) can be re-written as

$$Z^{N,z_0}(t) = \frac{[Nz_0]}{N} + \frac{1}{N}\sum_{j=1}^{5} h_j P_j\Big(N\int_0^t \beta_j(Z^{N,z_0}(s))ds\Big). \qquad (34)$$

It is easy to show that for all $T > 0$ the process $(Z^{N,z_0}(t))_{0\leq t\leq T}$ defined by (34) converges almost surely and uniformly on $[0,T]$ as N tends to ∞ to the solution $(i(t), s_1(t))$ of the ODE

$$\begin{cases} \frac{dz_1}{dt}(t) = \beta(1-z_1(t)-z_2(t))z_1(t) - \mu z_1(t) - \alpha z_1(t) + r\beta z_1(t)z_2(t) \\ \frac{dz_2}{dt}(t) = \alpha z_1(t) - \mu z_2(t) - r\beta z_1(t)z_2(t), \end{cases} \qquad (35)$$

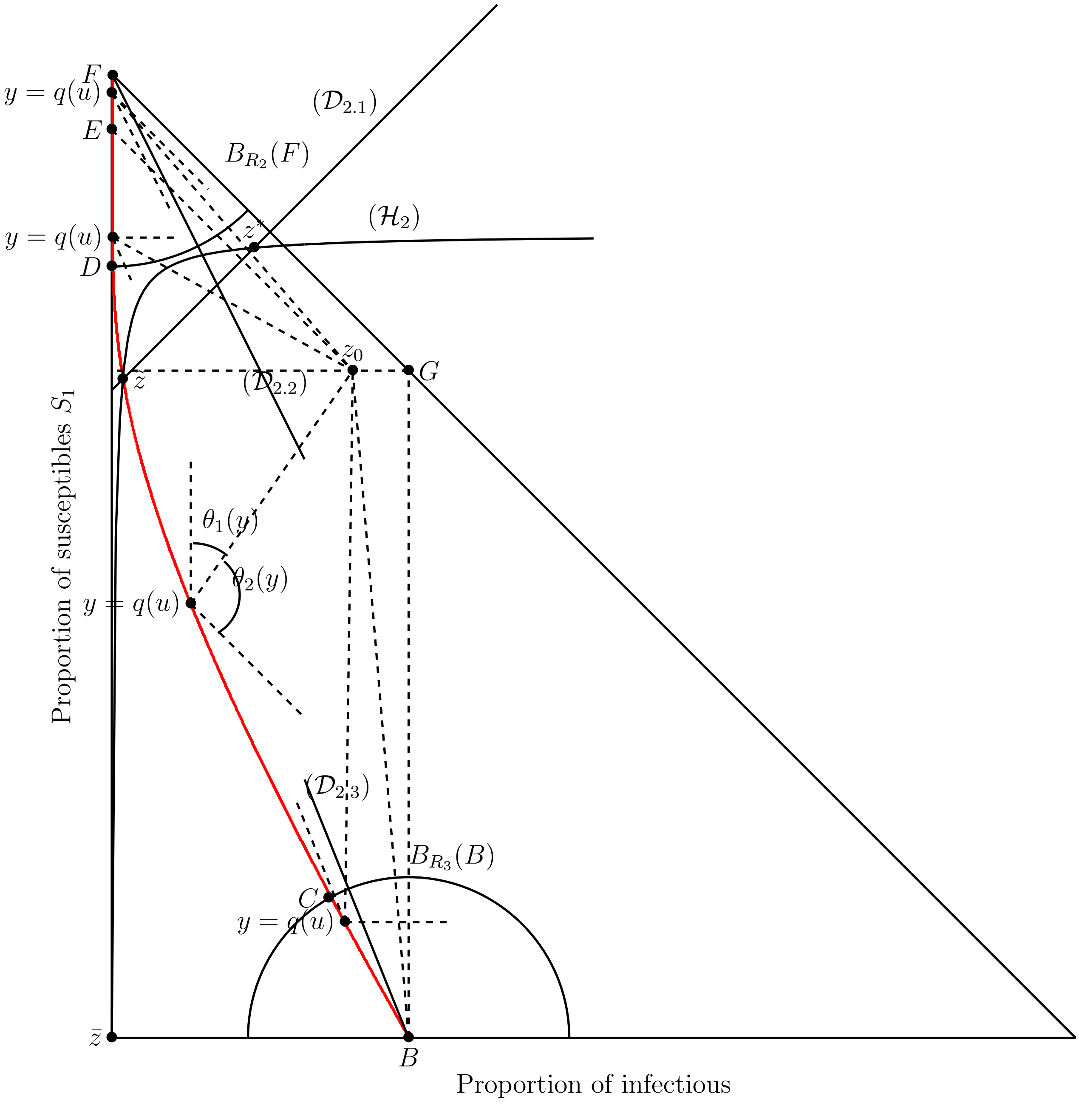

Figure 4: Characteristic boundary $\widetilde{\partial O}$ in the $S_0 I S_1$ model

with the initial condition $(i(0), s_1(0)) = z_0$. Note that the deterministic model defined by (35) is the S_0IS_1 model studied by M. Safan, H. Heesterbeek, and K. Dietz [15]. The basic reproduction number is given by

$$R_0 = \frac{\beta}{\alpha+\mu}, \tag{36}$$

and for $r > 1 + \mu/\alpha$, if we let

$$R_0^* = \frac{\beta^*}{\alpha+\mu} \text{ where } \beta^* = \frac{(\sqrt{\mu(r-1)} + \sqrt{\alpha})^2}{r}, \tag{37}$$

then If the parameters values are chosen in such away that $R_0^* < R_0 < 1$ and $r > 1+\mu/\alpha$, there exist two positive endemic equilibria (EE): the first one z^* defined by (38) which is locally asymptotically stable and the second one $\tilde{z}$ defined by (39) which is unstable. In addition the disease free equilibrium (DFE) $\bar{z} = (0,0)$ is locally asymptotically stable.

$$\begin{cases} z_1^* = \frac{1}{2}\left(\left(1 - \frac{1}{rR_0} - \frac{\mu}{(\alpha+\mu)R_0}\right) + \sqrt{\left(1 - \frac{1}{rR_0} - \frac{\mu}{(\alpha+\mu)R_0}\right)^2 + \frac{4\mu(1-\frac{1}{R_0})}{(\alpha+\mu)rR_0}}\right) \\ z_2^* = \frac{\alpha z_1^*}{\mu + r\beta z_1^*} \end{cases} \tag{38}$$

$$\begin{cases} \tilde{z}_1 &= \frac{1}{2}\left(\left(1 - \frac{1}{rR_0} - \frac{\mu}{(\alpha+\mu)R_0}\right) - \sqrt{\left(1 - \frac{1}{rR_0} - \frac{\mu}{(\alpha+\mu)R_0}\right)^2 + \frac{4\mu(1-\frac{1}{R_0})}{(\alpha+\mu)rR_0}}\right) \\ \tilde{z}_2 &= \frac{\alpha \tilde{z}_1}{\mu + r\beta \tilde{z}_1} \end{cases} \tag{39}$$

7.2.2 The boundary $\widetilde{\partial O}$ in the S_0IS_1 model

In this section, we verify that the assumptions 2.1 1 and 2.1 2 are satisfied for the model $S_0^N I^N S_1^N$. On the figure 4 we see the bassin of attraction O of the equilibrium z^* delimited by the boundary $\widetilde{\partial O}$ and containing the point z^*.

Let

- $(\mathcal{D}_{2.1})$ the straight line having the equation $-(\alpha+\mu-\beta) + \beta(r-1)z_2 - \beta z_1 = 0$ i.e

$$z_2 = \frac{1}{r-1}z_1 + \frac{\alpha+\mu-\beta}{\beta(r-1)},$$

- $(\mathcal{H}_2)$ the curve having the equation $\alpha z_1 - \mu z_2 - r\beta z_1 z_2 = 0$ i.e

$$z_2 = \frac{\alpha z_1}{r\beta z_1 + \mu},$$

In order to obtain a parametrization of the boundary $\widetilde{\partial O}$ on the interval $[0,\ 2]$, we make the change of variable $u = t/(1+t)$ and the ODE (35) can be re-written

$$\begin{cases} \frac{dy_1}{du}(u) = \frac{1}{(1-u)^2}[(\beta-\mu)y_1(u) + \beta(r-1)y_1(u)y_2(u) - \beta y_1^2(u)] \\ \frac{dy_2}{du}(u) = \frac{1}{(1-u)^2}[\alpha y_1(u) - \mu y_2(u) - r\beta y_1(u)y_2(u)]. \end{cases} \tag{40}$$

Remarks 7.3. • *The part of boundary $\widetilde{\partial O}$ which starts from the point $B = (d, 0)$ (intersection point between $\widetilde{\partial O}$ and the horizontal axis) to the unstable endemic equilibrium $\tilde{z}$ of the dynamical system (35) has as parametrization the solution $q^1(u) = (q_1^1(u), q_2^1(u))_{0\le t<1}$ of (40) with the initial condition $B = (d, 0)$.*

- *The part of boundary $\widetilde{\partial O}$ which starts from the unstable endemic equilibrium $\tilde{z}$ of dynamical system (35) to the point F (different to (0,1)) is parametrized by the solution $q^2(u) = (q_1^2(u), q_2^2(u))_{0\le u<1}$ of the ODE defined by (40) with initial condition the point F (see Figure 4).*
  Thus the characteristic boundary admits as parametric curve $(q(u))_{0\le u\le 2}$ defined by

$$q(u) = \begin{cases} q^1(u) & \text{if} \quad 0 \le u < 1 \\ \tilde{z} \quad \text{if} & u = 1 \\ q^2(2-u) & \text{if} \quad 1 < u \le 2. \end{cases}$$

- *As the tangent to the boundary $\widetilde{\partial O}$ at the point F is almost vertical, there exist a ball $B_{R_2}(F)$ and a constant $\omega > 1$ such that for all $(q_1(u), q_2(u)) \in \widetilde{\partial O} \cap B_{R_2}(F)$,*

$$\frac{\dot{q}_2(u)}{\dot{q}_1(u)} < -\omega.$$

  We then defined by $(\mathcal{D}_{2.2})$ the line having for equation $z_2 = -\omega z_1 + 1$.

- *The tangent to the boundary $\widetilde{\partial O}$ at the point $B = (d, 0)$ is a line whose slope is bounded as follows*

$$-\frac{\alpha}{\alpha + \mu - \beta} < \frac{\dot{q}_2}{\dot{q}_1}.$$

  As $\widetilde{\partial O}$ is a continuous curve, there exists a ball $B_{R_3}(B)$ such that for all $(q_1(u), q_2(u)) \in \widetilde{\partial O} \cap B_{R_3}(B)$,

$$-\frac{\alpha}{\alpha + \mu - \beta} < \frac{\dot{q}_2}{\dot{q}_1}.$$

  We also defined by $(\mathcal{D}_{2.3})$ the line having for equation $z_2 = -\frac{\alpha}{\alpha+\mu-\beta} z_1 + \frac{\alpha d}{\alpha+\mu-\beta}$.

From these remarks we deduce that for all $(q_1(u), q_2(u)) \in \widetilde{\partial O}$, $\dot{q}_1(u) \le 0$ and $\dot{q}_2(u) \ge 0$ since the first part of the boundary $\widetilde{\partial O}$ is below of $(\mathcal{D}_{2.1})$ and $\mathcal{H}_2$ and the second part is above of $(\mathcal{D}_{2.1})$ and $\mathcal{H}_2$.

Now we choose the point $z_0 \in \mathring{O}$ such that the following conditions are satisfied

- z_0 is the below of $(\mathcal{D}_{2.1})$ and $(\mathcal{H}_2)$,
- z_0 is on right of $(\mathcal{D}_{2.2})$ and $(\mathcal{D}_{2.3})$,
- z_0 is a point of the horizontal line passing through G (intersection point between the vertical line passing through B and the line whose equation is $z_1 + z_2 = 1$),
- the orthogonal projection of z_0 on the horizontal axis is at a distance smaller than R_3 from point B,

- the projection of z_0 on the vertical axis and parallel to the line whose equation is $z_1 + z_2 = 1$ is at a distance smaller than R_2 from point F.

We rewrite the ODE (35) as $\dot{z}(t) = g(z(t))$. We want to verify that for any $z \in \widetilde{\partial O}$ with $z_2 < \tilde{z}_2$, the vector $g(z)$ points in the sector $(3\pi/4, \pi)$. The reader can then easily verify that the same is true for $-g(z)$ if $z \in \widetilde{\partial O}$ while $z_2 > \tilde{z}_2$.

The fact that $g(z)$ points in the north–west direction follows from $g_1(z) < 0$, $g_2(z) > 0$. It thus remains to prove the

Lemma 7.4. *For any $z \in \widetilde{\partial O}$ with $z_2 < \tilde{z}_2$, $g_1(z) + g_2(z) > 0$.*

Proof $\{z(t),\ t \geq 0\}$ denoting any trajectory of the ODE (35), we define $\xi(t) = z_1(t) + z_2(t)$. It is easy to verify that

$$\frac{d}{dt}\dot{\xi}(t) = -(\mu + \beta z_1(t))\dot{\xi}(t) + \beta(1 - \xi(t))\dot{z}_1(t),$$

hence for any $0 \leq t_0 < t$,

$$\dot{\xi}(t) = \dot{\xi}(t_0)\exp\left(-\int_0^t (\mu + \beta z_1(s))ds\right) + \beta\int_0^t \exp\left(-\int_s^t (\mu + \beta z_1(r))dr\right)(1 - \xi(s))\dot{z}_1(s)ds,$$

and since $g_1(z) < 0$ if $z \in \widetilde{\partial O}$ with $z_2 < \tilde{z}_2$, we have that along the trajectory of (35) from B to $\tilde{z}$, if $\dot{\xi}(t_0) \leq 0$, then $\dot{\xi}(t) < 0$ for all $t > t_0$.

Now, at a point $z = (\tilde{z}_1 + a, \tilde{z}_2 - a)$ for $a \in \mathbb{R}$, we have that

$$\dot{\xi}(t) = (\alpha + \mu - r\beta\tilde{z}_2)a.$$

At the point $(0, \tilde{z}_1 + \tilde{z}_2)$ (that is with $a = -\tilde{z}_1$), we note that $\dot{z}_1(t) = 0$ and $\dot{z}_2(t) = -\mu z_2 < 0$, which implies that $(\alpha + \mu - r\beta\tilde{z}_2) > 0$. Consequently at any point z on the half line $\{(\tilde{z}_1 + a, \tilde{z}_2 - a),\ a > 0\}$, $\dot{\xi}(t) > 0$.

We first conclude from the above two facts that the trajectory of (35) from B to $\tilde{z}$ lies entirely below the half line $\{(\tilde{z}_1 + a, \tilde{z}_2 - a),\ a > 0\}$. Indeed, if that would not be the case, there would be points above that half–line which would be on the left of $\widetilde{\partial O}$, hence a trajectory of (35) starting from such a point would eventually converge to $(0, 0)$, hence cross downward the half line $\{(\tilde{z}_1 + a, \tilde{z}_2 - a),\ a > 0\}$, which is impossible.

Since the trajectory of (35) from B to $\tilde{z}$ lies entirely below the half line $\{(\tilde{z}_1 + a, \tilde{z}_2 - a),\ a > 0\}$, necessarily for any $t_0 > 0$ there exists $t > t_0$ such that $\dot{\xi}(t) > 0$, which, as a consequence of the first statement in the present proof, implies that $\dot{\xi}(t) > 0$ for any $t > 0$, hence the result.

We now deduce the assumption 2.1 1 and 2.1 2 through the proof of the following Lemma.

Lemma 7.5. *The assumption 2.1 1 is satisfied and there exist $\theta \in]0, \pi[$ such that for all $y \in \partial O$*

$$\operatorname{dist}(y^a, \partial O) \geq a \times \sin(\theta) \inf_{v \in \partial O} |v - z_0|,$$

where $y^a = y + a(z_0 - y)$ and z_0 is chosen above.

Proof. We note here that it is enough to show this last inequality over $\widetilde{\partial O}$. Now for $y \in \widetilde{\partial O}$, if $y \in \widetilde{\partial O} \cap B_{R_3}(B)$, we define $\theta_2(y)$ as the angle made by the horizontal line passing through y and the vector from y to z_0 and $\theta_1(y)$ as the angle made by the vector from y to z_0 and the parallel line to $(\mathcal{D}_{2.3})$ passing through y. For the part of $\widetilde{\partial O}$ from C to D, $\theta_2(y)$ is the angle made by the parallel line to the line whose equation is $z_1 + z_2 = 1$ passing through y and the vector from y to z. The angle $\theta_1(y)$ is made by the vector from y to z and the vertical line passing through y. For the part of $\widetilde{\partial O}$ from D to E, $\theta_1(y)$ is made by the vector from y to z_0 and the horizontal line passing through y and $\theta_2(y)$ is the angle between the parallel line to $(\mathcal{D}_{2.2})$ and the vector from y to z_0. In the last part of $\widetilde{\partial O}$ i.e from E to F, $\theta_2(y)$ is made as in the part from D to E and $\theta_1(y)$ is the angle between the vector from y to z_0 and the parallel line to the second bisector passing through y. It is easy to see in each part of $\widetilde{\partial O}$ that there exists $\theta \in]0, \theta[$ such that $\sin(\theta_1(y)), \sin(\theta_2(y)) \geq \sin(\theta)$. And then there exists $\theta \in]0, \pi[$ such that

$$\begin{aligned} \mathrm{dist}(y^a, \widetilde{\partial O}) &\geq \min_{i=1,2} \sin(\theta_i(y)) \times |y^a - y| \\ &= a \times \min_{i=1,2} \sin(\theta_i(y)) \times |y - z_0| \\ &\geq a \times \sin(\theta) \inf_{v \in \widetilde{\partial O}} |v - z_0|. \end{aligned}$$

□

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