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# Epidemic Models with Varying Infectivity on a Refining Spatial Grid—I—The SI Model

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**Abstract:** We consider a space–time SI epidemic model with infection age dependent infectivity and non-local infections constructed on a grid of the torus $\mathbb{T}^d = [0,1)^d$, where the individuals may migrate from node to node. The migration processes in either of the two states are assumed to be Markovian. We establish a functional law of large numbers by letting the initial approximate number of individuals on each node, $N$, to go to infinity and the mesh size of the grid, $\varepsilon$, to go to zero jointly. The limit is a system of parabolic PDE/integral equations. The constraint on the speed of convergence of the parameters $N$ and $\varepsilon$ is that $N\varepsilon^d \to \infty$ as $(N,\varepsilon) \to (+\infty, 0)$.

**Keywords:** epidemic model; varying infectivity; non-local infections; law of large numbers; integral equations

**MSC:** 92D30; 60H30; 60F17

## 1. Introduction

In order to capture the geographic heterogeneity, spatial epidemic models have been well developed, both in discrete and continuous spaces. In discrete space, multi-patch epidemic models have been studied in [1–6], and more recently by [7], where each patch represents a geographic location, and infection may occur within each patch and across distances (for example, due to short travels). See also the multi-patch multi-type epidemic models in [3,8], as well as relevant models in [9–11]. Some of these studies assume migration of individuals among different patches [1,4,5,7,8], while others do not, instead assuming that interactions between patches to induce infection [3,6,10–12]. In continuous space, various PDE models have been developed (see the monographs [13–15] and a survey [16]). There are two well-known models without spatial movement: Kendall's spatial model [17,18] and Diekmann–Thieme's PDE model [19–22], as well as the recent paper [23], which studies an epidemic model with age-dependent infectivity, as in the present paper. Kendall's spatial model is a system of ODEs with a spatial parameter (without spatial partial derivative). It was proved to be the FLLN limit of the multitype Markovian SIR model by Andersson and Djehiche [24], where both the number of types and the population size go to infinity. Diekmann–Thieme's spatial PDE model (with partial derivatives with respect to time and infection-age) has the infection rate depending on the age of infection, as in the PDE model first proposed by Kermack and McKendrick in their 1932 paper [25]. Similar to Kendall's spatial model, there is no partial derivative with respect to the spatial parameter, since there is no movement in space. The Diekmann–Thieme PDE model can be seen as the FLLN limit of a sequence of stochastic models; see [23]. We should also mention the spatial models in continuous space obtained as FLLN limits of stochastic models in continuous space (see [26,27]), where the stochastic model involves a continuous process for the movement of individuals: it is assumed that individual movements follow an Itô diffusion process, and the epidemic models are Markovian.



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In the present paper, we consider an epidemic model on a refining grid of the $d$ dimensional torus $\mathbb{T}^d$. Like in the earlier work [4], the individuals move from one patch to its neighbors according to a random walk. The first novelty of this paper is that the infectivity of each individual is a random function, which evolves with the time elapsed since infection, as first considered in [28], and recently studied in [8,29]. The second novelty is that we allow infection of a susceptible individual by infectious individuals located in distinct patches, and we use a very general rate of infection.

There are two parameters in our model: $N$, which is the order of the number of individuals in each patch; and $\varepsilon$, which is the distance between two neighboring sites. The total number of patches is $\varepsilon^{-d}$, and the total number of individuals in the model is $N\varepsilon^{-d}$. Our goal is to study the limit of the renormalized stochastic finite population model as both $N \to \infty$ and $\varepsilon \to 0$. In this paper, we obtain a convergence result in $L^\infty$ under the restriction that $N\varepsilon^d \to \infty$. In [4], the restriction was much weaker, thanks to clever martingale estimates due to Blount [30]. However, in contradiction with the model in [4], our model is non Markovian, and several of the fluctuating processes are not martingales. As a result, it does not seem possible to extend the techniques of [30] to the situation studied in the present paper.

There are three models in the present paper. The stochastic SDE model parametrized by the pair $(N, \varepsilon)$; the deterministic model, which is an ODE on the patches parametrized by $\varepsilon$ (and is the LLN limit of the first model when $N \to \infty$ with $\varepsilon$ fixed); and the PDE model on the torus $\mathbb{T}^d$, which is the limit of the ODE model as $\varepsilon \to 0$. The convergence of the ODE model to the PDE model exploits standard arguments on semigroups and their approximation, based on some results in [31]. The main new argument in the present paper consists in showing that the difference in $L^\infty$ between the stochastic and the ODE models, which tends to zero as $N \to \infty$ while $\varepsilon$ is fixed according to [8], tends also to zero when $(N, \varepsilon) \to (+\infty, 0)$, provided $N\varepsilon^d \to \infty$.

In this paper, we consider the SI model, S as susceptible, I as infected. An infected individual has an age of infection-dependent infectivity, which we suppose to vanish after some random time. It would be natural to decide that at that time, the individual leaves the I compartment, and enters the R compartment as recovered. For the sake of simplifying our model, we decide that after being infected, an individual remains in the I compartment forever. This does not affect the evolution of the epidemic, since when its infectivity remains zero, an individual from the I compartment does not contribute anymore to the propagation of the illness, just like an individual in the R compartment of an SIR model.

Let us finally comment on the assumptions regarding the age-dependent infectivity. We assume that to each individual who gets infected is attached a random infectivity function, the functions attached to the various individuals being independent and identically distributed (abbreviated i.i.d. below), all having the law of a random function $\lambda$ (the law is different for the initially infected individuals). In this paper, as in [8], we only assume that $\lambda$ belongs a.s. to the Skorohod space of càlàg functions $\mathbf{D}$, and satisfies $0 \leq \lambda(t) \leq \lambda^*$, for some $\lambda^* > 0$. This is weaker than the assumptions made in [29]. The proof in [8] is quite different from the proof in [29]. Here, we use a proof similar to that in [29]. The limitation is that we obtain only the pointwise convergence of the renormalized total infectivity function, while we obtain uniform convergence in $t$ for the proportions of susceptible and infected individuals. We believe that this proof is interesting, due to its simplicity.

Our result allows us to approximate a complicated stochastic model of an epidemic propagating in a population distributed over a large number of patches, by a simpler deterministic PDE model in continuous space. In other words, our result says that, if the population is large, as well as the number of patches, the number of individuals per patch being large as well, then provided the numbers of individuals in the various patches are of the same order, a good approximation of the model is given by a PDE model in continuous space.

The paper is organized as follows. We describe our model in detail in Section 2, in particular, the complex form of the rate of infection. Section 3 is devoted to the mathematical analysis of our model. In Section 3.1, we state the law of large numbers limit as $N \to \infty$, with $\varepsilon$ fixed, referring to [8] for the proof. In Section 3.2, we take the limit as $\varepsilon \to 0$ in the ODE model. In Section 3.3, we study the difference between the stochastic and the ODE model, as $(N, \varepsilon) \to (+\infty, 0)$, and conclude our main result. The next section is a conclusion, where we explain how our results can be used in practice, and describe our future projects for extending the present results. Finally, in Appendix A, we recall both the Duhamel formula, which is an essential tool in several of our proofs, and Kotelenez' extension of Doob's maximal inequality.

## 2. Model Formulation

We consider a total population size $N\varepsilon^{-d}$, initially distributed on the $\varepsilon^{-d}$ nodes of a refining spatial grid $\mathrm{D}_\varepsilon := [0,1)^d \cap \varepsilon\mathbb{Z}^d$, see Figure 1, in which an infection is introduced. Here, $\varepsilon$ is the mesh size of the grid (we assume that $\varepsilon^{-1} \in \mathbb{N}\backslash\{0\}$). We focus our attention to the periodic boundary conditions on the hypercube $[0,1]^d$; that is, our domain is the torus $\mathbb{T}^d := [0,1)^d$. Our results can be extended to a bounded domain of $\mathbb{R}^d$ with smooth boundary, and Neumann boundary conditions.

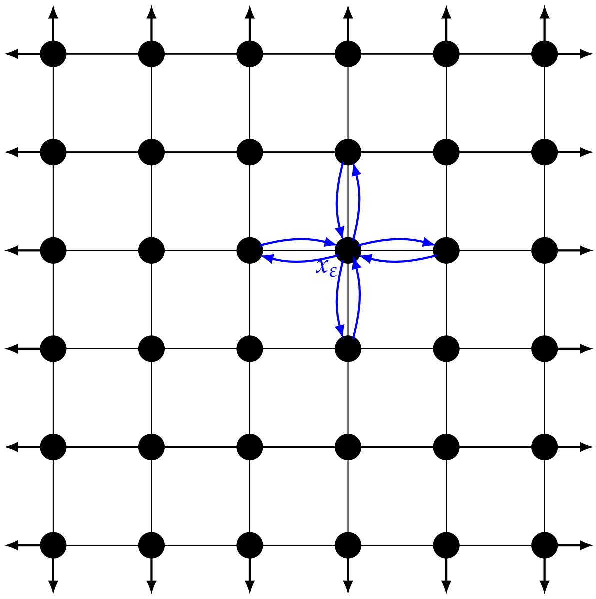


**Figure 1.** Our refining spatial grid $\mathrm{D}_\varepsilon$ in dimension $d = 2$, with periodic boundary conditions. The arrows are drawn to indicate that opposite edges are identified. The arrows at the bottom correspond to the arrows at the top, and those on the left correspond to those on the right. $\varepsilon$ is the mesh size of the grid. The blue arrows around site $x_\varepsilon$ indicate the possible movements of individuals from and to $x_\varepsilon$.

### *2.1. Set-Up and Notations*

We split the population into two subsets $S^{N,\varepsilon}$ and $I^{N,\varepsilon}$. $S^{N,\varepsilon}$ stands for the susceptible individuals, who do not have the disease and who can become infected, while $I^{N,\varepsilon}$ refers to the subset of those individuals who are suffering from the illness or have recovered from the disease.

We shall denote by $x_\varepsilon$ the nodes of the grid $\mathrm{D}_\varepsilon$. $S^{N,\varepsilon}(t, x_\varepsilon)$ denotes the number of susceptible individuals at site $x_\varepsilon$ at time $t$. Let $B^{N,\varepsilon}(t, x_\varepsilon)$ denote the total number of individuals at site $x_\varepsilon$ at time $t$, i.e. $B^{N,\varepsilon}(t, x_\varepsilon) := S^{N,\varepsilon}(t, x_\varepsilon) + I^{N,\varepsilon}(t, x_\varepsilon)$. We define $S^{N,\varepsilon}(t)$ (resp. $I^{N,\varepsilon}(t)$) as the total number of susceptible individuals (resp. infected individuals) at time $t$ in the whole population; that is,

$$S^{N,\varepsilon}(t) := \sum_{x_\varepsilon} S^{N,\varepsilon}(t, x_\varepsilon), \quad \text{and} \quad I^{N,\varepsilon}(t) := \sum_{x_\varepsilon} I^{N,\varepsilon}(t, x_\varepsilon), \quad \forall t \geq 0.$$

We have $B^{N,\varepsilon}(t) := \sum_{x_\varepsilon} B^{N,\varepsilon}(t, x_\varepsilon) = N\varepsilon^{-d}, \quad \forall t \geq 0.$

We now describe our model of varying infectivity. $\{\lambda_{-1}(t),\ t \geq 0\}$ and $\{\lambda_{-1}(t),\ t \geq 0\}$ will denote two mutually independent random functions of time, whose trajectories are in $\mathbf{D}$, the space of càdlàg paths from $\mathbb{R}_+$ into $\mathbb{R}_+$, which we equip with the Skorohod topology, and which are such that $0 \leq \lambda_{-1}(t), \lambda_1(t) \leq \lambda^*$ for all $t \geq 0$, for some constant $\lambda^* > 0$. $\lambda_{-1}(t)$ will be the infectivity at time $t$ of an initially infected individual, while $\lambda_1(t)$ will be the infectivity at time $t$ after his/her infection of an initially susceptible individual. We might assume that an infected individual is first exposed, during a latent period during which $\lambda_1(t) = 0$. Then $\lambda_1(t)$ becomes positive, and eventually the infected individual ceases to be infectious at time $\eta = \sup\{t > 0,\ \lambda_1(t) > 0\}$, after the time of his/her infection. An initially infected individual has been infected at some time in the past, and at time 0, he/she might still be exposed (in which case $\lambda_{-j}(0) = 0$), or infectious (in which case $\lambda_{-j}(0) > 0$), or no longer infectious (in which case $\lambda_{-j}(t) = 0$ for all $t \geq 0$). We think that a continuous function $\lambda(t)$ is a good model of reality. However, we prefer to assume more generally that the trajectories of $\lambda(t)$ are in $\mathbf{D}$, in particular, in order to include in our model the classical case where $\lambda(t)$ jumps from 0 to some fixed deterministic value $\bar{\lambda}$ at some random time, and then jumps back to zero later on.

To each initially infected individual, $1 \leq j \leq I^{N,\varepsilon}(0)$ is attached a random infectivity process $\{\lambda_{-j}(t)\ :\ t \geq 0\}$: $\lambda_{-j}(t)$ is the infectivity at time $t$ of the $j$-th initially infected individual. To each initially susceptible individual, $1 \leq j \leq S^{N,\varepsilon}(0)$ is associated a random infectivity process $\{\lambda_j(t)\ :\ t \geq 0\}$. The initially susceptible individual $j$, who is infected at a random time $\tau_j^{N,\varepsilon}$, has an infectivity of $\lambda_j(t - \tau_j^{N,\varepsilon})$ at time $t$, i.e., $\lambda_j(t)$ is the infectivity at time $t$ after the individual $j$ was infected. We assume that $\lambda_j(t) = 0$ for all $t < 0$ and all $j \in \mathbb{Z}$, and that $\{\lambda_{-j}\ :\ j \geq 1\}$ and $\{\lambda_j\ :\ j \geq 1\}$ are two mutually independent sequences of i.i.d random functions taking values in the interval $[0, \lambda^*]$.

We define the infected periods of newly and initially infected individuals $j > 0$ and $j < 0$, respectively, by the random variables

$$\eta_j := \sup\{t > 0 :\ \lambda_j(t) > 0\},\ j \in \mathbb{Z}\backslash\{0\}.$$

We define $F(t) := \mathbb{P}(\eta_1 \leq t)$, $F_0(t) := \mathbb{P}(\eta_{-1} \leq t)$, the distributions functions of $\eta_j$ for $j \geq 1$ and for $j \leq -1$, respectively. Let $F^c(t) := 1 - F(t)$ and $F_0^c(t) := 1 - F_0(t)$. Moreover, we define

$$\overline{\lambda}(t) := \mathbb{E}[\lambda_1(t)] \ \text{ and } \overline{\lambda}_0(t) := \mathbb{E}[\lambda_{-1}(t)].$$

We assume that susceptible individuals move from patch to patch according to a time-homogenous Markov process $X(t)$ with jump rates $\nu_S/\varepsilon^2$ and transition function

$$p_\varepsilon^{x_\varepsilon, y_\varepsilon}(s,t) = \mathbb{P}(X(t) = y_\varepsilon | X(s) = x_\varepsilon),$$

and while infectious individuals move from patch to patch according to a time-homogeneous Markov process $Y(t)$ with jump rates $\nu_I/\varepsilon^2$ and transition function

$$q_\varepsilon^{x_\varepsilon, y_\varepsilon}(s,t) = \mathbb{P}(Y(t) = y_\varepsilon | Y(s) = x_\varepsilon).$$

$\nu_S$ and $\nu_I$ are positive diffusion coefficients for the susceptible and infected subpopulations, respectively. We assume that those movements of the various individuals are mutually independent.

In addition, we use $X_j^{s,x_\varepsilon}(t)$ (resp. $Y_j^{s,x_\varepsilon}(t)$) to denote the position at time $t$ of the individual $j$ if it is susceptible (resp. infected) during the time interval $(s,t)$, and was in location/node $x_\varepsilon$ at time $s$.

For all $x_\varepsilon \in \mathrm{D}_\varepsilon$, let $V_\varepsilon(x_\varepsilon)$ be the cube centered at the site $x_\varepsilon$ with volume $\varepsilon^d$. Let $\mathrm{H}^\varepsilon \subset L^2(\mathbb{T}^d)$ denote the space of real valued step functions that are constant on each cell $V_\varepsilon(x_\varepsilon)$.

$\Delta_\varepsilon$ is the discrete Laplace operator, defined as follows

$$\Delta_\varepsilon f(x_\varepsilon) = \sum_{i=1}^{d} \varepsilon^{-d}\big[f(x_\varepsilon + \varepsilon e_i) - 2f(x_\varepsilon) + f(x_\varepsilon - \varepsilon e_i)\big], \quad f \in \mathrm{H}^\varepsilon$$

and we define the operators $\Delta_\varepsilon^S f := \nu_S \Delta_\varepsilon f$ and $\Delta_\varepsilon^I f := \nu_I \Delta_\varepsilon f$, $f \in \mathrm{H}^\varepsilon$.

$\Delta$ denotes the d-dimensional Laplace operator, i.e.,

$$\Delta f(x) = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}(x)\,.$$

Let $T_{S,\varepsilon}$ (resp. $T_{I,\varepsilon}$) be the semigroup acting on $\mathrm{H}^\varepsilon$ generated by $\nu_S\Delta_\varepsilon$ (resp. $\nu_I\Delta_\varepsilon$). Similarly, we denote by $T_S$ (resp. $T_I$) the semigroup acting on $L^2(\mathbb{T}^d)$ generated by $\nu_S\Delta$ (resp. $\nu_I\Delta$).

### *2.2. Model Description*

All random variables and processes are defined on a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider an SI epidemic model, where each infectious individual has an infectivity that is randomly varying with the time elapsed since infection. We assume that a susceptible individual in patch $x_\varepsilon$ has contacts with infectious individuals of patch $y_\varepsilon$ at rate $\beta_\varepsilon^{x_\varepsilon,y_\varepsilon}(t)$ at time $t$.

Given a site $x_\varepsilon$, the total force of infection at each time $t$ at site $x_\varepsilon$ is the aggregate infectivity of all the individuals that are currently infectious in this site:

$$\begin{aligned}\mathfrak{F}^{N,\varepsilon}(t,x_\varepsilon) = &\sum_{j=1}^{I^{N,\varepsilon}(0)} \lambda_{-j}(t)\mathbb{1}_{Y_j(t)=x_\varepsilon}\\ &+\sum_{y_\varepsilon}\int_0^t\int_0^\infty\int_{\mathbf{D}}\int_{\mathbf{D}} \lambda(t-s)\mathbb{1}_{u\le S^{N,\varepsilon}(s^-,y_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(s^-,y_\varepsilon)}\mathbb{1}_{Y^{s,y_\varepsilon}(t)=x_\varepsilon}Q^{y_\varepsilon}(ds,du,d\lambda,dY),\end{aligned}$$

where

$$\overline{\Gamma}^{N,\varepsilon}(t,y_\varepsilon) := \frac{1}{N^{1-\gamma}[B^{N,\varepsilon}(t,y_\varepsilon)]^\gamma}\sum_{x_\varepsilon}\beta_\varepsilon^{y_\varepsilon,x_\varepsilon}(t)\mathfrak{F}^{N,\varepsilon}(t,x_\varepsilon)$$

is the force of infection exerted on each susceptible individual in patch $x_\varepsilon$, and $\{Q^{y_\varepsilon}, y_\varepsilon \in D_\varepsilon\}$ are i.i.d. standard Poisson random measures (PRM) on $\mathbb{R}_+^2 \times \mathbf{D}^2$ with intensity $ds \otimes du \otimes d\mathbb{P}_\lambda \otimes d\mathbb{P}_Y$. We assume that $\gamma \in [0,1]$. By an abuse of notation, we denote by $Q^{x_\varepsilon}(ds,du)$ the projection of $Q^{x_\varepsilon}(ds,du,d\lambda,dY)$ on the first two coordinates. Let, with $\Upsilon^{N,\varepsilon}(t,x_\varepsilon) := S^{N,\varepsilon}(t,x_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(t,x_\varepsilon)$,

$$A^{N,\varepsilon}(t,x_\varepsilon) := \int_0^t\int_0^\infty \mathbb{1}_{u\le \Upsilon^{N,\varepsilon}(s^-,x_\varepsilon)}Q^{x_\varepsilon}(ds,du).$$

In what follows, $x_\varepsilon \sim y_\varepsilon$ means that the nodes $x_\varepsilon$ and $y_\varepsilon$ are neighbors (each point of $\mathrm{D}_\varepsilon$ has $2d$ neighbors).

The epidemic dynamic of the model can be described by the following equations

$$\begin{aligned}S^{N,\varepsilon}(t,x_\varepsilon) &= S^{N,\varepsilon}(0,x_\varepsilon) - A^{N,\varepsilon}(t,x_\varepsilon) - \sum_{y_\varepsilon\sim x_\varepsilon}P_S^{x_\varepsilon,y_\varepsilon}\left(\int_0^t\frac{\nu_S}{\varepsilon^2}S^{N,\varepsilon}(s,x_\varepsilon)ds\right) + \sum_{y_\varepsilon\sim x_\varepsilon}P_S^{y_\varepsilon,x_\varepsilon}\left(\int_0^t\frac{\nu_S}{\varepsilon^2}S^{N,\varepsilon}(s,y_\varepsilon)ds\right)\\ I^{N,\varepsilon}(t,x_\varepsilon) &= I^{N,\varepsilon}(0,x_\varepsilon) + A^{N,\varepsilon}(t,x_\varepsilon) - \sum_{y_\varepsilon\sim x_\varepsilon}P_I^{x_\varepsilon,y_\varepsilon}\left(\int_0^t\frac{\nu_I}{\varepsilon^2}I^{N,\varepsilon}(s,x_\varepsilon)ds\right) + \sum_{y_\varepsilon\sim x_\varepsilon}P_I^{y_\varepsilon,x_\varepsilon}\left(\int_0^t\frac{\nu_I}{\varepsilon^2}I^{N,\varepsilon}(s,y_\varepsilon)ds\right),\end{aligned} \tag{1}$$

where $P_S^{x_\varepsilon,y_\varepsilon}$, $P_I^{x_\varepsilon,y_\varepsilon}$, $x_\varepsilon, y_\varepsilon \in \mathrm{D}_\varepsilon$ are mutually independent standard Poisson processes.

In the above equations, $P_S^{x_\varepsilon,y_\varepsilon}$ (resp. $P_I^{x_\varepsilon,y_\varepsilon}$) is the counting process of susceptible (resp. infected) individuals that migrate from the patch $x_\varepsilon$ to $y_\varepsilon$.

In the sequel of this paper, we may use the same notation for different constants (we use the generic notations $c$, $C$ for positive constants). These constants can depend upon some parameters of the model, as long as these are independent of $\varepsilon$ and $N$, and we will not necessarily mention this dependence explicitly. The exact value may change from line to line.

## 3. Model Analysis

### 3.1. Law of Large Numbers as $N \to \infty$, $\varepsilon$ Being Fixed

We consider the renormalized model by dividing the number of individuals in each compartment and at each patch by $N$. Hence, we define

$$\overline{S}^{N,\varepsilon}(t,x_\varepsilon) := \frac{1}{N}S^{N,\varepsilon}(t,x_\varepsilon), \quad \overline{I}^{N,\varepsilon}(t,x_\varepsilon) := \frac{1}{N}I^{N,\varepsilon}(t,x_\varepsilon), \text{ and } \overline{\mathfrak{F}}^{N,\varepsilon}(t,x_\varepsilon) := \frac{1}{N}\mathfrak{F}^{N,\varepsilon}(t,x_\varepsilon).$$

**Assumption 1.** *We make the following assumptions on the initial conditions. We assume that*

**(i)** *there exists a collection of positive numbers* $\{\, \overline{S}^\varepsilon(0,x_\varepsilon),\ \overline{I}^\varepsilon(0,x_\varepsilon),\ x_\varepsilon \in \mathrm{D}_\varepsilon \,\}$ *such that*

$$\sum_{x_\varepsilon}\Big[\overline{S}^\varepsilon(0,x_\varepsilon) + \overline{I}^\varepsilon(0,x_\varepsilon)\Big] = \varepsilon^{-d}\,,$$

*and* $\Big|S^{N,\varepsilon}(0) - N\overline{S}^\varepsilon(0)\Big| \le 1$, $\quad \Big|I^{N,\varepsilon}(0) - N\overline{I}^\varepsilon(0)\Big| \le 1;$

**(ii)** *there exists two continuous functions* $\overline{\mathbf{S}}, \overline{\mathbf{I}} : \mathbb{T}^d \longrightarrow \mathbb{R}_+$ *and two constants* $0 < c_0 < C_0$ *such that* $c_0 \le \overline{\mathbf{S}}(x) \le C_0$, $\overline{\mathbf{I}}(x) \le C_0$ *for all* $x \in \mathbb{T}^d$, $\int_{\mathbb{T}^d}\Big[\overline{\mathbf{S}}(x) + \overline{\mathbf{I}}(x)\Big]dx = 1$, $\int_{\mathbb{T}^d}\overline{\mathbf{I}}(x)dx > 0$, *and*

$$\overline{S}^\varepsilon(0,x_\varepsilon) = \varepsilon^{-d}\int_{V_\varepsilon(x_\varepsilon)}\overline{\mathbf{S}}(x)dx, \quad \overline{I}^\varepsilon(0,x_\varepsilon) = \varepsilon^{-d}\int_{V_\varepsilon(x_\varepsilon)}\overline{\mathbf{I}}(x)dx\,.$$

**(iii)** $\{X_j(0)\,, 1 \le j \le S^{N,\varepsilon}(0)\}$ *and* $\{Y_j(0)\,, 1 \le j \le I^{N,\varepsilon}(0)\}$ *are two mutually independent collections of i.i.d. random variables satisfying* $\mathbb{P}\big(X_j(0) = x_\varepsilon\big) = \dfrac{\overline{S}^\varepsilon(0,x_\varepsilon)}{\overline{S}^\varepsilon(0)}$, *and* $\mathbb{P}\big(Y_j(0) = x_\varepsilon\big) = \dfrac{\overline{I}^\varepsilon(0,x_\varepsilon)}{\overline{I}^\varepsilon(0)}$ *for all* $x_\varepsilon \in \mathrm{D}_\varepsilon$, *where* $\overline{S}^\varepsilon(0) := \sum_{x_\varepsilon}\overline{S}^\varepsilon(0,x_\varepsilon)$ *and* $\overline{I}^\varepsilon(0) := \sum_{x_\varepsilon}\overline{I}^\varepsilon(0,x_\varepsilon)$. *Moreover,* $S^{N,\varepsilon}(0,x_\varepsilon) = \sum_{j=1}^{S^{N,\varepsilon}(0)}\mathbb{1}_{X_j(0)=x_\varepsilon}$ *and* $I^{N,\varepsilon}(0,x_\varepsilon) = \sum_{j=1}^{I^{N,\varepsilon}(0)}\mathbb{1}_{Y_j(0)=x_\varepsilon}$.

**Assumption 2.**

**(i)** *We assume that* $\beta_\varepsilon^{x_\varepsilon,y_\varepsilon}(t) = \beta_t(x_\varepsilon, V_\varepsilon(x_\varepsilon))$, *where* $\beta_t(x,A)$ *is a transition kernel and there exists a constant* $\beta^*$ *such that* $\beta_t(x,\mathbb{T}^d) \le \beta^*$, *for all* $x \in \mathbb{T}^d$ *and* $t \ge 0$.

**(ii)** *there exists a positive constant* $\lambda^* > 0$ *such that* $0 \le \lambda_j(t) \le \lambda^*$, *for all* $j \in \mathbb{Z}\backslash\{0\}$ *and* $t \ge 0$.

Under Assumptions 1 and 2, we have the

**Theorem 1 (Law of Large Numbers: N $\to$ $\infty$, $\varepsilon$ being fixed).** *As* $N \to \infty$, $\Big(\overline{S}^{N,\varepsilon}(t,x_\varepsilon), \overline{\mathfrak{F}}^{N,\varepsilon}(t,x_\varepsilon), \overline{I}^{N,\varepsilon}(t,x_\varepsilon),\ t \ge 0,\ x_\varepsilon \in \mathrm{D}_\varepsilon\Big)$ *converges in* $\mathbf{D}^{3\varepsilon^{-d}}$ *in probability, to the unique solution* $\Big(\overline{S}^\varepsilon(t,x_\varepsilon), \overline{\mathfrak{F}}^\varepsilon(t,x_\varepsilon), \overline{I}^\varepsilon(t,x_\varepsilon),\ t \ge 0,\ x_\varepsilon \in \mathrm{D}_\varepsilon\Big)$ *of the following system of integral equations*

$$
\begin{cases}
\overline{S}^{\varepsilon}(t,x_\varepsilon) = \overline{S}^{\varepsilon}(0,x_\varepsilon) - \int_0^t \overline{S}^{\varepsilon}(s,x_\varepsilon)\overline{\Gamma}^{\varepsilon}(s,x_\varepsilon)ds + \int_0^t \left[\Delta_\varepsilon^S \overline{S}^{\varepsilon}\right](s,x_\varepsilon)ds \\
\overline{\mathfrak{F}}^{\varepsilon}(t,x_\varepsilon) = \overline{\lambda}_0(t)\sum_{y_\varepsilon} \overline{I}^{\varepsilon}(0,y_\varepsilon)q^{y_\varepsilon,x_\varepsilon}(0,t) + \sum_{y_\varepsilon}\int_0^t \overline{\lambda}(t-s)\overline{S}^{\varepsilon}(s,y_\varepsilon)\overline{\Gamma}^{\varepsilon}(s,y_\varepsilon)q^{y_\varepsilon,x_\varepsilon}(s,t)ds \\
\overline{I}^{\varepsilon}(t,x_\varepsilon) = \overline{I}^{\varepsilon}(0,x_\varepsilon) + \int_0^t \overline{S}^{\varepsilon}(s,x_\varepsilon)\overline{\Gamma}^{\varepsilon}(s,x_\varepsilon)ds + \int_0^t \left[\Delta_\varepsilon^I \overline{I}^{\varepsilon}\right](s,x_\varepsilon)ds, \\
\quad t \geq 0,\ x_\varepsilon \in \mathrm{D}_\varepsilon,
\end{cases}
\tag{2}
$$

*where*

$$
\overline{\Gamma}^{\varepsilon}(t,x_\varepsilon) = \frac{1}{\left[\overline{B}^{\varepsilon}(t,x_\varepsilon)\right]^{\gamma}} \sum_{y_\varepsilon} \beta_\varepsilon^{x_\varepsilon,y_\varepsilon}(t)\overline{\mathfrak{F}}^{\varepsilon}(t,y_\varepsilon) \ \ \textit{and} \ \ \overline{B}^{\varepsilon}(t,x_\varepsilon) = \overline{S}^{\varepsilon}(t,x_\varepsilon) + \overline{I}^{\varepsilon}(t,x_\varepsilon).
$$

This Theorem is a special case of Theorem 3.1 in [8], whose proof, which is written for a multi-patch multi-group SIR model, is easily adapted to our case.

### 3.2. Limit as $\varepsilon \to 0$ in the Deterministic Model

Before letting $\varepsilon$ go to zero in the limit system (2) extended to the whole space $\mathbb{T}^d$, we prove some technical lemmas.

**Lemma 1.** *Let $T > 0$. There exists a positive constant $C$ such that $\left\|\overline{S}^{\varepsilon}(t)\right\|_\infty \leq C$ and $\left\|\overline{I}^{\varepsilon}(t)\right\|_\infty \leq C$, for all $\varepsilon > 0$ and $t \in [0\,,\,T]$.*

**Proof.** Using the Duhamel Formula (see (A3) in the Appendix A below), the solution of the first line of (2) reads

$$
\overline{S}^{\varepsilon}(t,x_\varepsilon) = [e^{t\Delta_\varepsilon^S}\overline{S}^{\varepsilon}(0,\cdot)](x_\varepsilon) - \int_0^t e^{(t-s)\Delta_\varepsilon^S}[\overline{S}^{\varepsilon}(s,\cdot)\overline{\Gamma}^{\varepsilon}(s,\cdot)](x_\varepsilon)ds\,.
$$

It is explained in Appendix A below that all entries of the matrix $e^{(t-s)\Delta_\varepsilon^S}$ are non–negative. Clearly, all coordinates of the vector $\overline{S}^{\varepsilon}(s,\cdot)\overline{\Gamma}^{\varepsilon}(s,\cdot)$ are non–negative as well. It then follows from the above formula that

$$
\overline{S}^{\varepsilon}(t,x_\varepsilon) \leq [e^{t\Delta_\varepsilon^S}\overline{S}^{\varepsilon}(0,\cdot)](x_\varepsilon)\,.
$$

As explained in Appendix A,

$$
\left(e^{t\Delta_\varepsilon^S}\right)_{x_\varepsilon,y_\varepsilon} = p_\varepsilon^{x_\varepsilon,y_\varepsilon}(0,t),
$$

which, combined with the last inequality, implies that

$$
\sup_{x_\varepsilon \in D_\varepsilon} \overline{S}^{\varepsilon}(t,x_\varepsilon) = \|\overline{S}^{\varepsilon}(t,\cdot)\|_\infty \leq \|\overline{S}^{\varepsilon}(0,\cdot)\|_\infty \leq C,
$$

as a consequence of Assumption 1 (ii).

We now consider the term $\overline{I}^{\varepsilon}$. First, using the previous estimate, we obtain

$$
\frac{\overline{S}^{\varepsilon}(s,x_\varepsilon)}{\left(\overline{B}^{\varepsilon}(s,x_\varepsilon)\right)^{\gamma}} = \left(\frac{\overline{S}^{\varepsilon}(s,x_\varepsilon)}{\overline{B}^{\varepsilon}(s,x_\varepsilon)}\right)^{\gamma}\left[\overline{S}^{\varepsilon}(s,x_\varepsilon)\right]^{1-\gamma} \leq C(T,\gamma).
$$

Next, we have $\sum_{y_\varepsilon} \beta_\varepsilon^{x_\varepsilon,y_\varepsilon}\overline{\mathfrak{F}}^{\varepsilon}(s,y_\varepsilon) \leq \lambda^* \left\|\overline{I}^{\varepsilon}(s)\right\|_\infty \sum_{y_\varepsilon} \beta_\varepsilon^{x_\varepsilon,y_\varepsilon}(s) \leq \lambda^*\beta^*\left\|\overline{I}^{\varepsilon}(s)\right\|_\infty$. Thus, using the Duhamel formula again,

$$\left\|\overline{I}^{\varepsilon}(t)\right\|_{\infty} \le \left\|\left(T_{I,\varepsilon}(t)\overline{I}^{\varepsilon}(0)\right)\right\|_{\infty} + \int_0^t T_{I,\varepsilon}(t-s)C\left\|\overline{I}^{\varepsilon}(s)\right\|_{\infty} ds$$
$$\le C + C\int_0^t \left\|\overline{I}^{\varepsilon}(s)\right\|_{\infty} ds.$$

The second statement then follows from Gronwall's Lemma. □

**Lemma 2.** *For any $T > 0$, there exists $\varepsilon_0$ and $c > 0$ such that $\overline{B}^{\varepsilon}(t, x_\varepsilon) \ge c$, for all $0 < \varepsilon \le \varepsilon_0$, $x_\varepsilon \in \mathrm{D}_\varepsilon$ and $0 \le t \le T$.*

**Proof.** Let $c$ and $C$ be two positive constants such that $0 < 2c \le \inf_{x_\varepsilon} \overline{S}^{\varepsilon}(0, x_\varepsilon) \le C$. The existence of such constants is a consequence of Assumption 1 (ii). We now define the random time $T_c^\varepsilon$, which is the first time that $\inf_{x_\varepsilon} \overline{S}^{\varepsilon}(t, x_\varepsilon)$ is less than $c$ (and $+\infty$ if this never happens). Formally, $T_c^\varepsilon := \inf\{t > 0\,,\ \inf_{x_\varepsilon} \overline{S}^{\varepsilon}(t, x_\varepsilon) < c\}$. On the interval $[0\,,\,T_c^\varepsilon)$, $\overline{S}^{\varepsilon}(t, x_\varepsilon) \ge c$, $\forall x_\varepsilon \in \mathrm{D}_\varepsilon$. For $t \le T_c^\varepsilon$, we have

$$\overline{\Gamma}^{\varepsilon}(t, x_\varepsilon) = \frac{1}{\left[\overline{B}^{\varepsilon}(t, x_\varepsilon)\right]^{\gamma}} \sum_{y_\varepsilon} \beta_\varepsilon^{x_\varepsilon, y_\varepsilon}(t)\overline{\mathfrak{F}}^{\varepsilon}(t, y_\varepsilon) \le \frac{\lambda^* \beta^*}{c^\gamma}\left\|\overline{I}^{\varepsilon}(t)\right\|_{\infty} := \overline{c},$$
$$\overline{S}^{\varepsilon}(t, x_\varepsilon) \le \overline{S}^{\varepsilon}(0, x_\varepsilon) - \overline{c}\int_0^t \overline{S}^{\varepsilon}(s, x_\varepsilon) + \int_0^t \left[\Delta_\varepsilon^S \overline{S}^{\varepsilon}\right](s, x_\varepsilon) ds.$$

Hence, $e^{\overline{c}t}\overline{S}^{\varepsilon}(t, x_\varepsilon) \ge \inf_{y_\varepsilon} \overline{S}^{\varepsilon}(0, y_\varepsilon) = 2c$, and consequently $T_c^\varepsilon \ge \log 2/\overline{c}$. Then for all $0 \le t \le T_c^\varepsilon$, we have $e^{\overline{c}t}\overline{S}^{\varepsilon}(t, x_\varepsilon) \ge 2c$. So $\overline{S}^{\varepsilon}(t, x_\varepsilon) \ge 2e^{-\overline{c}t}c \ge c$ iff $e^{-\overline{c}t} \ge \frac{1}{2}$.

From Assumption 1 **(ii)** and the fact that $\overline{\mathbf{I}}(0) \ne 0$, there exists a ball $B(x_0, \rho)$ and $a > 0$ such that $\overline{\mathbf{I}}(y) \ge a$, for all $y \in B(x_0, \rho)$. Let us consider the following ODE

$$\frac{d\,u_\varepsilon}{dt} = \nu_I \Delta_\varepsilon u_\varepsilon, \quad u_\varepsilon(0) = a\mathbb{1}_{B(x_0,\rho)}.$$

We have that $u_\varepsilon \longrightarrow u$ in $L^\infty\left([0, T] \times \mathbb{T}^d\right)$ as $\varepsilon \to 0$, where $u$ is the solution of

$$\frac{d\,u}{dt} = \nu_I \Delta u, \quad u(0) = a\mathbb{1}_{B(x_0,\rho)}.$$

For all $\dfrac{\log 2}{\overline{c}} < t \le T$, there exists a positive constant $\underline{c}$, such that $u(t, x) \ge 2\underline{c}$, $\forall x \in \mathbb{T}^d$. Then, there exists $\varepsilon_0 > 0$ such that $\forall \varepsilon \le \varepsilon_0$, $\overline{I}^{\varepsilon}(t, x_\varepsilon) \ge u_\varepsilon(t, x_\varepsilon) \ge \underline{c}$, for all $\dfrac{\log 2}{\overline{c}} < t \le T$. We have shown that $\overline{B}^{\varepsilon}(t, x_\varepsilon) \ge c \wedge \underline{c}$, for all $0 \le t \le T$, $x \in \mathrm{D}_\varepsilon$, $\varepsilon \le \varepsilon_0$. □

We now extend the solution of the system (2) to the whole space $\mathbb{T}^d$. So, we define

$$\overline{\mathbf{S}}^{\varepsilon}(t, x) := \sum_{x_\varepsilon} \overline{S}^{\varepsilon}(t, x_\varepsilon)\mathbb{1}_{V_\varepsilon(x_\varepsilon)}(x),\ \overline{\mathbf{I}}^{\varepsilon}(t, x) := \sum_{x_\varepsilon} \overline{I}^{\varepsilon}(t, x_\varepsilon)\mathbb{1}_{V_\varepsilon(x_\varepsilon)}(x),\ \overline{\mathbf{F}}^{\varepsilon}(t, x) := \sum_{x_\varepsilon} \overline{\mathfrak{F}}^{\varepsilon}(t, x_\varepsilon)\mathbb{1}_{V_\varepsilon(x_\varepsilon)}(x),$$
$$\overline{\mathbf{X}}^{\varepsilon} := (\overline{\mathbf{S}}^{\varepsilon}, \overline{\mathbf{F}}^{\varepsilon}, \overline{\mathbf{I}}^{\varepsilon}).$$

**Theorem 2.** *For all $T \ge 0$, $\sup_{0 \le t \le T}\left\|\overline{\mathbf{X}}^{\varepsilon}(t) - \overline{\mathbf{X}}(t)\right\|_{\infty} \longrightarrow 0$ as $\varepsilon \to 0$, where $\overline{\mathbf{X}} := (\overline{\mathbf{S}}, \overline{\mathbf{F}}, \overline{\mathbf{I}})$ is the unique solution of the following system of parabolic PDE/integral equations.*

$$\begin{cases} \overline{\mathbf{S}}(t,x) = \overline{\mathbf{S}}(0,x) - \int_0^t \overline{\mathbf{S}}(s,x)\overline{\Gamma}(s,x)ds + \int_0^t \left[\Delta^S \overline{\mathbf{S}}\right](s,x)ds, \\ \overline{\mathbf{F}}(t,x) = \overline{\lambda}_0(t)\big(T_I(t)\overline{\mathbf{I}}(0)\big)(x) + \int_0^t \overline{\lambda}(t-s)T_I(t-s)\big(\overline{\mathbf{S}}(s)\overline{\Gamma}(s)\big)(x)ds, \\ \overline{\mathbf{I}}(t,x) = \overline{\mathbf{I}}(0,x) + \int_0^t \overline{\mathbf{S}}(s,x)\overline{\Gamma}(s,x)ds + \int_0^t \left[\Delta^I \overline{\mathbf{I}}\right](s,x)ds, \\ \quad \text{with } \overline{\mathbf{S}}(t,x)\overline{\Gamma}(t,x) = \dfrac{\overline{\mathbf{S}}(t,x)}{\left[\overline{\mathbf{B}}(t,x)\right]^\gamma} \int_{\mathbb{T}^d} \overline{\mathbf{F}}(t,y)\beta(x,dy),\ t \geq 0,\ \ x \in \mathbb{T}^d. \end{cases} \tag{3}$$

*where $T_I$ denotes the semigroup generated by $\nu_I \Delta$.*

Before proving this theorem, we first establish two Propositions.

**Proposition 1.** *Let $T > 0$. If $(\overline{\mathbf{S}}\,,\,\overline{\mathbf{F}}\,,\,\overline{\mathbf{I}})$ is a solution of (3), then for all $0 \leq t \leq T$, there exists $C$, $c > 0$ such that $\left\|\overline{\mathbf{S}}(t)\right\|_\infty \leq C$, $\left\|\overline{\mathbf{I}}(t)\right\|_\infty \leq C$ and $\overline{\mathbf{B}}(t,x) \geq c$, for all $x \in \mathbb{T}^d$.*

**Proof.** The arguments used in the proofs of Lemmas 1 and 2 are easy to transpose to the present situation. □

**Remark 1.** *Let $\mathscr{H}\big(\overline{\mathbf{S}},\overline{\mathbf{I}},\overline{\mathbf{F}}\big)(t,x) := \dfrac{\left[\overline{\mathbf{S}}(t,x) \vee 0\right] \wedge C}{\left[\overline{\mathbf{B}}(t,x) \vee c\right]^\gamma} \int_{\mathbb{T}^d} \beta_t(x,dy)\left[\overline{\mathbf{F}}(t,y) \wedge \lambda^* C\right]$ where $C$ is the upper bound and $c$ the lower bound in Proposition 1, Lemmas 1 and 2. We note hat $\big(\overline{\mathbf{S}}\,,\,\overline{\mathbf{I}}\,,\,\overline{\mathbf{F}}\big)$ is a solution of (3) if it is a solution of the following system*

$$\begin{cases} \overline{\mathbf{S}}(t,x) = \Big(T_S(t)\overline{\mathbf{S}}(0)\Big)(x) - \int_0^t \Big(T_S(t-s)\mathscr{H}\big(\overline{\mathbf{S}}(s),\overline{\mathbf{I}}(s),\overline{\mathbf{F}}(s)\big)\Big)(x)ds, \\ \overline{\mathbf{F}}(t,x) = \overline{\lambda}_0(t)\Big(T_I(t)\overline{\mathbf{I}}(0)\Big)(x) + \int_0^t \overline{\lambda}(t-s)\Big(T_I(t-s)\mathscr{H}\big(\overline{\mathbf{S}}(s),\overline{\mathbf{I}}(s),\overline{\mathbf{F}}(s)\big)\Big)(x)ds, \\ \overline{\mathbf{I}}(t,x) = \Big(T_I(t)\overline{\mathbf{I}}(0)\Big)(x) + \int_0^t \Big(T_I(t-s)\mathscr{H}\big(\overline{\mathbf{S}}(s),\overline{\mathbf{I}}(s),\overline{\mathbf{F}}(s)\big)\Big)(x)ds,\ 0 \leq t \leq T,\ \ x \in \mathbb{T}^d. \end{cases} \tag{4}$$

*Note also that the map $\mathscr{H} : \Big(L^\infty(\mathbb{T}^d)\Big)^3 \longrightarrow L^\infty(\mathbb{T}^d)$ is bounded and globally Lipschitz.*

**Proposition 2.** *The system of Equation (4) has a unique solution.*

**Proof.** The uniqueness of the solution uses the contraction character of the semigroups $T_S$ and $T_I$ on $L^\infty(\mathbb{T}^d)$, and the fact that the map $\mathscr{H}$ is bounded and globally Lipschitz. The existence of the solution can be proved using the Picard iteration procedure. □

We introduce the canonical projection $\mathrm{P}_\varepsilon : L^2(\mathbb{T}^d) \longrightarrow \mathrm{H}^\varepsilon$ given by

$$\varphi \longmapsto \mathrm{P}_\varepsilon\varphi(x) = \varepsilon^{-d} \int_{V_\varepsilon(x_\varepsilon)} \varphi(y)dy \quad \text{if } x \in V_\varepsilon(x_\varepsilon).$$

**Proof of Theorem 2.** Using the fact that the map $\mathscr{H}$ is bounded and globally Lipschitz, we have, provided that $\varepsilon \leq \varepsilon_0$,

$$\left\|\overline{\mathbf{X}}^\varepsilon(t) - \overline{\mathbf{X}}(t)\right\|_\infty \leq C(\lambda^*,\beta^*) \int_0^t \left\|\overline{\mathbf{X}}^\varepsilon(s) - \overline{\mathbf{X}}(s)\right\|_\infty ds + \pi_\varepsilon(t),$$

where $\pi_\varepsilon(t) = \pi_\varepsilon^S(t) + \pi_\varepsilon^I(t) + \pi_\varepsilon^{\mathfrak{F}}(t)$, with

$$\pi_\varepsilon^S(t) = \left\| T_{S,\varepsilon}(t)\overline{\mathbf{S}}^\varepsilon(0) - T_S(t)\overline{\mathbf{S}}(0) \right\|_\infty$$
$$+ \int_0^t \left\| \mathrm{P}_\varepsilon\left( \frac{\overline{\mathbf{S}}(s)}{\left[\overline{\mathbf{B}}(s)\right]^\gamma} \int_{\mathbb{T}^d} \overline{\mathbf{F}}(s,y)\beta_s(.,dy)\right) - \frac{\overline{\mathbf{S}}(s)}{\left[\overline{\mathbf{B}}(s)\right]^\gamma} \int_{\mathbb{T}^d} \overline{\mathbf{F}}(s,y)\beta_s(.,dy) \right\|_\infty ds$$
$$+ \int_0^t \left\| T_{S,\varepsilon}(t-s)\mathrm{P}_\varepsilon\left( \frac{\overline{\mathbf{S}}(s)}{\left[\overline{\mathbf{B}}(s)\right]^\gamma} \int_{\mathbb{T}^d} \overline{\mathbf{F}}(s,y)\beta_s(.,dy)\right) - T_S(t-s)\left( \frac{\overline{\mathbf{S}}(s)}{\left(\overline{\mathbf{B}}(s)\right)^\gamma} \int_{\mathbb{T}^d} \overline{\mathbf{F}}(s,y)\beta_s(.,dy)\right) \right\|_\infty ds,$$

$\pi_\varepsilon^I(t)$ is a quantity similar to $\pi_\varepsilon^S(t)$, with $T_{I,\varepsilon}$ (resp. $T_I$, $\overline{\mathbf{I}}^\varepsilon$ and $\overline{\mathbf{I}}$) in place of $T_{S,\varepsilon}$ (resp. $T_S$, $\overline{\mathbf{S}}^\varepsilon$ and $\overline{\mathbf{S}}$), and

$$\pi_\varepsilon^{\mathfrak{F}}(t) = \lambda^* \left\| T_{I,\varepsilon}(t)\overline{\mathbf{I}}^\varepsilon(0) - T_I(t)\overline{\mathbf{I}}(0) \right\|_\infty$$
$$+ \int_0^t \left\| \mathrm{P}_\varepsilon\left( \frac{\overline{\mathbf{S}}(s)}{\left[\overline{\mathbf{B}}(s)\right]^\gamma} \int_{\mathbb{T}^d} \overline{\mathbf{F}}(s,y)\beta_s(.,dy)\right) - \frac{\overline{\mathbf{S}}(s)}{\left[\overline{\mathbf{B}}(s)\right]^\gamma} \int_{\mathbb{T}^d} \overline{\mathbf{F}}(s,y)\beta_s(.,dy) \right\|_\infty ds$$
$$+ \int_0^t \left\| T_{I,\varepsilon}(t-s)\mathrm{P}_\varepsilon\left( \frac{\overline{\mathbf{S}}(s)}{\left[\overline{\mathbf{B}}(s)\right]^\gamma} \int_{\mathbb{T}^d} \overline{\mathbf{F}}(s,y)\beta_s(.,dy)\right) - T_I(t-s)\left( \frac{\overline{\mathbf{S}}(s)}{\left[\overline{\mathbf{B}}(s)\right]^\gamma} \int_{\mathbb{T}^d} \overline{\mathbf{F}}(s,y)\beta_s(.,dy)\right) \right\|_\infty ds.$$

Then from Gronwall's lemma, $\sup_{0\le t\le T} \left\| \overline{\mathbf{X}}^\varepsilon(t) - \overline{\mathbf{X}}(t) \right\|_\infty \to 0$ follows from $\sup_{0\le t\le T} \pi_\varepsilon(t) \to 0$.

Since the maps $x \longmapsto \overline{\mathbf{S}}(0,x)$, $x \longmapsto \overline{\mathbf{I}}(0,x)$ and $x \longmapsto \frac{\overline{\mathbf{S}}(t,x)}{\left[\overline{\mathbf{B}}(t,x)\right]^\gamma} \int_{\mathbb{T}^d} \overline{\mathbf{F}}(t,y)\beta_t(x,dy)$ are continuous on $\mathbb{T}^d$, and $T_{S,\varepsilon} \longrightarrow T_S$, $T_{I,\varepsilon} \longrightarrow T_I$ as operators on $L^\infty$ as $\varepsilon \to 0$, then $\sup_{0\le t\le T} \pi_\varepsilon(t) \longrightarrow 0$, as $\varepsilon \to 0$ (see Kato [31], Chapter 9, Section 3, Example 3.10). □

*3.3. Limit as $N \to \infty$ and $\varepsilon \to 0$*

In this section, we extend our stochastic model to the whole space $\mathbb{T}^d$ and let both $N \to \infty$ and $\varepsilon \to 0$ in such a way that $N\varepsilon^d \to \infty$. Before stating the main theorem of this section, we first prove some lemmas and propositions.

**Lemma 3.** *There exist two constants $0 < c < C$ such that for all $t \ge 0$, $\varepsilon > 0$ and $x_\varepsilon \in \mathrm{D}_\varepsilon$,*

$$c\varepsilon^d \le \mathbb{P}(X(t) = x_\varepsilon) \le C\varepsilon^d.$$

**Proof.** Define $u^\varepsilon(t,x_\varepsilon) := \mathbb{P}(X(t) = x_\varepsilon)$. We have that $u^\varepsilon(t,x_\varepsilon) = \left(e^{t\left[\Delta_\varepsilon^S\right]^*} u_0^\varepsilon\right)(x_\varepsilon)$. Using the assumption on the initial condition $\mathbb{P}(X(0) = x_\varepsilon)$, then $0 < c\varepsilon^d \le u^\varepsilon(0,x_\varepsilon) \le C\varepsilon^d$, from which we deduce that $0 < c\varepsilon^d \le e^{t\left[\Delta_\varepsilon^S\right]^*} u^\varepsilon(0,x_\varepsilon) \le C\varepsilon^d$; hence, the result. □

**Lemma 4.** *There exits a positive constant $C$ such that for all $0 \le s \le t$, $\varepsilon > 0$ and $x_\varepsilon \in \mathrm{D}_\varepsilon$*

$$\sum_{y_\varepsilon} q_\varepsilon^{y_\varepsilon,x_\varepsilon}(s,t) = 1 \quad \textit{and} \quad \mathbb{P}\big(Y_j(t) = x_\varepsilon\big) \le C\varepsilon^d.$$

**Proof.** The uniform distribution on $\mathrm{D}_\varepsilon$ is invariant for the process $Y(t)$. So, if we start $Y$ at time $s$ with the uniform distribution, i.e., $\mathbb{P}(Y(s) = x_\varepsilon) = \varepsilon^d$, the law of $Y$ at time $t$ is also the uniform law. But

$$\mathbb{P}(Y(t) = x_\varepsilon) = \sum_{y_\varepsilon} \mathbb{P}(Y(s) = y_\varepsilon) q_\varepsilon^{y_\varepsilon,x_\varepsilon}(s,t) \text{ i.e } \varepsilon^d = \varepsilon^d \sum_{y_\varepsilon} q_\varepsilon^{y_\varepsilon,x_\varepsilon}(s,t);$$

thus, $\sum_{y_\varepsilon} q_\varepsilon^{y_\varepsilon,x_\varepsilon}(s,t) = 1$. Finally,

$$\begin{aligned}\mathbb{P}\big(Y_j(t) = x_\varepsilon\big) &= \sum_{y_\varepsilon} \mathbb{P}\big(Y_j(0) = y_\varepsilon\big) q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t) \\ &\leq \sup_{y_\varepsilon} \mathbb{P}\big(Y_j(0) = y_\varepsilon\big) \sum_{y_\varepsilon} q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t).\end{aligned}$$

Hence, the second result follows from the first one and Assumption 1 (**ii**) and (**iii**). □

Let us define $\overline{\mathfrak{F}}_0^{N,\varepsilon}(t,x_\varepsilon) := \frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)} \lambda_{-j}(t)\mathbb{1}_{Y_j(t)=x_\varepsilon}$ and $\overline{\mathfrak{F}}_0^{\varepsilon}(t,x_\varepsilon) := \overline{\lambda}_0(t)\sum_{y_\varepsilon} \overline{I}^\varepsilon(0,y_\varepsilon)$ $q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t)$.

We have the

**Lemma 5.** *Let us assume that* $(N,\varepsilon) \to (\infty,0)$. *Then for all* $T > 0$,

$$\sup_{0\leq t\leq T} \mathbb{E}\left(\left\|\overline{\mathfrak{F}}_0^{N,\varepsilon}(t) - \overline{\mathfrak{F}}_0^{\varepsilon}(t)\right\|_\infty^2\right) \longrightarrow 0, \quad \textit{as} \ \ (N\,,\,\varepsilon) \to (\infty\,,\,0).$$

**Proof.** $\overline{\mathfrak{F}}_0^{N,\varepsilon}(t,x_\varepsilon)$ can be decomposed as follows

$$\overline{\mathfrak{F}}_0^{N,\varepsilon}(t,x_\varepsilon) = \frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)} \big(\lambda_{-j}(t) - \overline{\lambda}_0(t)\big)\mathbb{1}_{Y_j(t)=x_\varepsilon} + \overline{\lambda}_0(t)\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)} \mathbb{1}_{Y_j(t)=x_\varepsilon}.$$

Let us consider the first term. Since $\big(\lambda_{-j}(t)\big)_j$ are independent and identically distributed and independent of $Y_j(t)$, then

$$\begin{aligned}\mathbb{E}\left[\left(\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)} \big(\lambda_{-j}(t) - \overline{\lambda}_0(t)\big)\mathbb{1}_{Y_j(t)=x_\varepsilon}\right)^2\right] &= \frac{1}{N^2}\sum_{j=1}^{I^{N,\varepsilon}(0)} \mathbb{E}\Big[\big|\lambda_{-j}(t) - \overline{\lambda}_0(t)\big|^2\mathbb{1}_{Y_j(t)=x_\varepsilon}\Big] \\ &\leq \frac{1}{N^2}C(\lambda^*)I^{N,\varepsilon}(0)\mathbb{P}(Y_1(t) = x_\varepsilon) \\ &\leq \frac{C(\lambda^*)}{N}\mathbb{P}(Y_1(t) = x_\varepsilon).\end{aligned}$$

Now, since $\sum_{x_\varepsilon} \mathbb{P}(Y_1(t) = x_\varepsilon) = 1$,

$$\begin{aligned}\mathbb{E}\left[\sup_{x_\varepsilon\in \mathrm{D}_\varepsilon}\left(\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)} \big(\lambda_{-j}(t) - \overline{\lambda}_0(t)\big)\mathbb{1}_{Y_j(t)=x_\varepsilon}\right)^2\right] &\leq \sum_{x_\varepsilon}\mathbb{E}\left[\left(\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)} \big(\lambda_{-j}(t) - \overline{\lambda}_0(t)\big)\mathbb{1}_{Y_j(t)=x_\varepsilon}\right)^2\right] \\ &\leq \frac{C(\lambda^*)}{N} \quad \to 0, \end{aligned} \tag{5}$$

as $N \to \infty$. It remains to show that

$$\sup_{x_\varepsilon\in \mathrm{D}_\varepsilon}\left|\overline{\lambda}_0(t)\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)} \mathbb{1}_{Y_j(t)=x_\varepsilon} - \overline{\lambda}_0(t)\sum_{y_\varepsilon}\overline{I}^\varepsilon(0,y_\varepsilon)q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t)\right| \longrightarrow 0, \text{ as } (N,\varepsilon) \longrightarrow (\infty,0).$$

We have

$$\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)} \mathbb{1}_{Y_j(t)=x_\varepsilon} = \frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)} \Big[\mathbb{1}_{Y_j(t)=x_\varepsilon} - \mathbb{P}\big(Y_j(t) = x_\varepsilon\big)\Big] + \frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)} \mathbb{P}\big(Y_j(t) = x_\varepsilon\big).$$

$$\mathbb{E}\left\{\left(\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)}\left[\mathbb{1}_{Y_j(t)=x_\varepsilon}-\mathbb{P}\big(Y_j(t)=x_\varepsilon\big)\right]\right)^2\right\}=\frac{1}{N^2}\sum_{j=1}^{I^{N,\varepsilon}(0)}\mathbb{E}\left(\left|\mathbb{1}_{Y_j(t)=x_\varepsilon}-\mathbb{P}\big(Y_j(t)=x_\varepsilon\big)\right|^2\right)$$
$$\leq\frac{C}{N}\mathbb{P}(Y_1(t)=x_\varepsilon)\,.$$

It follows that

$$\mathbb{E}\left\{\sup_{x_\varepsilon\in\mathrm{D}_\varepsilon}\left(\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)}\left[\mathbb{1}_{Y_j(t)=x_\varepsilon}-\mathbb{P}\big(Y_j(t)=x_\varepsilon\big)\right]\right)^2\right\}\leq\frac{C}{N}\quad\to 0, \tag{6}$$

as $N\to 0$.

Since $\overline{\lambda}_0(t)$ is bounded, it remains to evaluate the quantity $\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)}\mathbb{P}\big(Y_j(t)=x_\varepsilon\big)-\sum_{y_\varepsilon}\overline{I}^\varepsilon(0,y_\varepsilon)q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t)$. We have

$$\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)}\mathbb{P}\big(Y_j(t)=x_\varepsilon\big)=\frac{1}{N}\sum_{y_\varepsilon}\sum_{j=1}^{I^{N,\varepsilon}(0)}\mathbb{P}\big(Y_j(0)=y_\varepsilon\big)q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t);\ \text{thus,}$$

$$\sup_{x_\varepsilon}\left|\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)}\mathbb{P}\big(Y_j(t)=x_\varepsilon\big)-\sum_{y_\varepsilon}\overline{I}^\varepsilon(0,y_\varepsilon)q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t)\right|\leq\frac{1}{N}\sup_{x_\varepsilon}\sum_{y_\varepsilon}q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t)\left|\sum_{j=1}^{I^{N,\varepsilon}(0)}\mathbb{P}\big(Y_j(0)=y_\varepsilon\big)-N\overline{I}^\varepsilon(0,y_\varepsilon)\right|$$
$$\leq\frac{1}{N}\sup_{x_\varepsilon}\sum_{y_\varepsilon}q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t)\frac{\overline{I}^\varepsilon(0,y_\varepsilon)}{\overline{I}^\varepsilon(0)}\left|I^{N,\varepsilon}(0)-N\overline{I}^\varepsilon(0)\right|$$
$$\leq\frac{C}{N}\quad\longrightarrow 0\,. \tag{7}$$

Combining (5)–(7), we finally have

$$\sup_{0\leq t\leq T}\mathbb{E}\left(\sup_{x_\varepsilon\in\mathrm{D}_\varepsilon}\left|\frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)}\lambda_{-j}(t)\mathbb{1}_{Y_j(t)=x_\varepsilon}-\overline{\lambda}_0(t)\sum_{y_\varepsilon}\overline{I}^\varepsilon(0,y_\varepsilon)q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t)\right|^2\right)\longrightarrow 0\,, \tag{8}$$

as $N\to+\infty$. □

Let $\sigma^{N,\varepsilon}$ be the stopping time defined by

$$\sigma^{N,\varepsilon}(\omega):=\inf\{t>0\,,\omega\notin A_{t,\delta}\cap B_{t,\delta}\}, \tag{9}$$

where for all $t\leq T$, $\delta>0$,

$$A_{t,\delta}=\left\{\left\|\int_0^t T_{S,\varepsilon}(t-s)d\mathscr{M}_S^{N,\varepsilon}(s)\right\|_\infty\leq\delta\right\},\quad B_{t,\delta}=\left\{\left\|\int_0^t T_{I,\varepsilon}(t-s)d\widetilde{\mathscr{M}_I}^{N,\varepsilon}(s)\right\|_\infty\leq\delta\right\},$$

with

$$\mathscr{M}_S^{N,\varepsilon}(t,x_\varepsilon)=\sum_{y_\varepsilon\sim x_\varepsilon}\frac{1}{N}M_S^{y_\varepsilon,x_\varepsilon}\left(N\int_0^t\frac{\nu_S}{\varepsilon^2}\overline{S}^{N,\varepsilon}(s,y_\varepsilon)ds\right)-\sum_{y_\varepsilon\sim x_\varepsilon}\frac{1}{N}M_S^{x_\varepsilon,y_\varepsilon}\left(N\int_0^t\frac{\nu_S}{\varepsilon^2}\overline{S}^{N,\varepsilon}(s,x_\varepsilon)ds\right),$$
$$\widetilde{\mathscr{M}_I}^{N,\varepsilon}(t,x_\varepsilon)=\mathscr{M}_I^{N,\varepsilon}(t,x_\varepsilon)+\mathscr{M}_{SI}^{N,\varepsilon}(t,x_\varepsilon),$$

where

$$\mathscr{M}_I^{N,\varepsilon}(t,x_\varepsilon) = \sum_{y_\varepsilon\sim x_\varepsilon} \frac{1}{N} M_I^{y_\varepsilon,x_\varepsilon}\left(N\int_0^t \frac{\nu_I}{\varepsilon^2}\overline{I}^{N,\varepsilon}(s,y_\varepsilon)ds\right) - \sum_{y_\varepsilon\sim x_\varepsilon} \frac{1}{N} M_I^{x_\varepsilon,y_\varepsilon}\left(N\int_0^t \frac{\nu_I}{\varepsilon^2}\overline{I}^{N,\varepsilon}(s,x_\varepsilon)ds\right),$$
$$\mathscr{M}_{SI}^{N,\varepsilon}(t,x_\varepsilon) = \frac{1}{N}\int_0^t\int_0^\infty \mathbb{1}_{u\le S^{N,\varepsilon}(s^-,x_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(s^-,x_\varepsilon)}\overline{Q}^{x_\varepsilon}(ds,du).$$

$\overline{Q}^{x_\varepsilon}(ds,du) := Q^{x_\varepsilon}(ds,du) - dsdu$ is the compensated PRM associated with $Q_\varepsilon^{x_\varepsilon}(ds,du)$, and we have used the notations

$$M_S^{x_\varepsilon,y_\varepsilon}(t) = P_S^{x_\varepsilon,y_\varepsilon}(t) - t, \quad M_I^{x_\varepsilon,y_\varepsilon}(t) = P_I^{x_\varepsilon,y_\varepsilon}(t) - t.$$

Let $\bar{\bar{c}} := \dfrac{\lambda^*\beta^*\left\|\overline{I}^{N,\varepsilon}(t)\right\|_\infty}{c^\gamma}$, where $c$ stands for the bound in Lemma 2. We define the stopping time

$$\tau^{N,\varepsilon} = \inf\left\{t>0\,,\left\|\int_0^t e^{(t-s)\left(\Delta_\varepsilon^S - \bar{\bar{c}} I_d\right)} d\widetilde{\mathscr{M}}_S^{N,\varepsilon}(s)\right\|_\infty \ge \frac{c}{8}\right\},$$

where $I_d$ is the identity operator on $\mathrm{H}^\varepsilon$, and $\widetilde{\mathscr{M}}_S^{N,\varepsilon}(t,x_\varepsilon) := \mathscr{M}_S^{N,\varepsilon}(t,x_\varepsilon) - \mathscr{M}_{SI}^{N,\varepsilon}(t,x_\varepsilon)$.

With those notations, we deduce from (1) that

$$\begin{aligned}
\overline{S}^{N,\varepsilon}(t,x_\varepsilon) &= \overline{S}^{N,\varepsilon}(0,x_\varepsilon) + \int_0^t [\Delta_\varepsilon^S \overline{S}^{N,\varepsilon}](s,x_\varepsilon)ds - \int_0^t \overline{S}^{N,\varepsilon}(s,x_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(s,x_\varepsilon)ds + \widetilde{\mathscr{M}}_S^{N,\varepsilon}(t,x_\varepsilon)\\
\overline{I}^{N,\varepsilon}(t,x_\varepsilon) &= \overline{I}^{N,\varepsilon}(0,x_\varepsilon) + \int_0^t [\Delta_\varepsilon^I \overline{I}^{N,\varepsilon}](s,x_\varepsilon)ds - \int_0^t \overline{S}^{N,\varepsilon}(s,x_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(s,x_\varepsilon)ds + \widetilde{\mathscr{M}}_I^{N,\varepsilon}(t,x_\varepsilon)
\end{aligned} \tag{10}$$

In the proof of the next Proposition, we shall need the following Lemma.

**Lemma 6.** *As* $(N,\varepsilon)\to(\infty,0)$ *in such a way that* $N\varepsilon^d\to\infty$, $\|\overline{S}^{N,\varepsilon}(0,.) - \overline{S}^\varepsilon(0,.)\|_\infty \longrightarrow 0$ *in* $L^2(\Omega)$.

**Proof.** We have

$$\begin{aligned}
\overline{S}^{N,\varepsilon}(0,x_\varepsilon) - \overline{S}^\varepsilon(0,x_\varepsilon) &= \frac{1}{N}\sum_{j=1}^{S^{N,\varepsilon}(0)} \mathbb{1}_{X_j=x_\varepsilon} - \mathbb{P}(X=x_\varepsilon)\overline{S}^\varepsilon(0)\\
&= \overline{S}^\varepsilon(0)\frac{1}{N\overline{S}^\varepsilon(0)}\sum_{j=1}^{S^{N,\varepsilon}(0)}\left[\mathbb{1}_{X_j=x_\varepsilon} - \mathbb{P}(X=x_\varepsilon)\right] + \frac{\mathbb{P}(X=x_\varepsilon)}{N}\left[\overline{S}^{N,\varepsilon}(0) - N\overline{S}^\varepsilon(0)\right].
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[\left|\overline{S}^{N,\varepsilon}(0,x_\varepsilon) - \overline{S}^\varepsilon(0,x_\varepsilon)\right|^2\right] &\le \frac{2}{N^2}\sum_{j=1}^{S^{N,\varepsilon}(0)} Var[\mathbb{1}_{X=x_\varepsilon}] + \frac{2[\mathbb{P}(X=x_\varepsilon)]^2}{N^2}\\
&\le \frac{\overline{S}^\varepsilon(0)}{N}\frac{C}{c}\varepsilon^d + \frac{C\varepsilon^{2d}}{N^2} \le \frac{C'}{N} + \frac{C\varepsilon^{2d}}{N^2}.
\end{aligned}$$

Then

$$\mathbb{E}\left[\sup_{x_\varepsilon\in\mathrm{D}_\varepsilon}|\overline{S}^{N,\varepsilon}(0,x_\varepsilon) - \overline{S}^\varepsilon(0,x_\varepsilon)|^2\right] \le \frac{C'}{N\varepsilon^d} + \frac{C\varepsilon^d}{N^2}.$$

The result follows. □

**Proposition 3.** *For all $T > 0$, there exists $C$ such that for $N$ large enough if $t \leq \sigma^{N,\varepsilon} \wedge T$, then $\left\| \overline{S}^{N,\varepsilon}(t) \right\|_\infty \leq C$ and $\left\| \overline{I}^{N,\varepsilon}(t) \right\|_\infty \leq C$, for all $\varepsilon > 0$. Moreover, there exists $\varepsilon_0 > 0$ and $c_0 > 0$ such that if $t \leq \sigma^{N,\varepsilon} \wedge \tau^{N,\varepsilon} \wedge T$, $\overline{B}^{N,\varepsilon}(t, x_\varepsilon) \geq c_0$, for all $x_\varepsilon \in \mathrm{D}_\varepsilon$, provided $\varepsilon \leq \varepsilon_0$.*

**Proof.** Let us first treat the term $\|\overline{S}^{N,\varepsilon}(t)\|_\infty$.

Using the Duhamel formula (A4) from Appendix A, applied to the first line of (10), we have

$$\overline{S}^{N,\varepsilon}(t, x_\varepsilon) \leq \Big(T_{S,\varepsilon}(t)\overline{S}^{N,\varepsilon}(0,.)\Big)(x_\varepsilon) + \int_0^t \Big(T_{S,\varepsilon}(t-s)d\mathscr{M}_S^{N,\varepsilon}(s,.)\Big)(x_\varepsilon).$$

Since $\overline{S}^{N,\varepsilon}(0, x_\varepsilon) \leq C$, for all $x_\varepsilon \in \mathrm{D}_\varepsilon$, we obtain that for $t \leq \sigma^{N,\varepsilon} \wedge T$,

$$\|\overline{S}^{N,\varepsilon}(t)\|_\infty \leq C + \delta\,.$$

We now consider the term $\left\| \overline{I}^{N,\varepsilon}(t) \right\|_\infty$. Arguing as in the proof of Lemma 1, we have for $t \leq \sigma^{N,\varepsilon} \wedge T$,

$$\begin{aligned}\left\| \overline{I}^{N,\varepsilon}(t) \right\|_\infty &\leq e^{Ct}\left(C + \sup_{0\leq t\leq T}\left\| \int_0^t T_{I,\varepsilon}(t-s)d\widetilde{\mathscr{M}}_I^{N,\varepsilon}(s) \right\|_\infty\right) \\ &\leq (C+\delta)e^{CT}.\end{aligned}$$

We finally consider the term $\overline{B}^{N,\varepsilon}(t, x_\varepsilon)$. It follows from Lemma 6 that $\|\overline{S}^{N,\varepsilon}(0,.) - \overline{S}^{\varepsilon}(0,.)\|_\infty \longrightarrow 0$ and from Lemma 2 that $\overline{S}^{\varepsilon}(0, x_\varepsilon) \geq c$, for all $x_\varepsilon \in \mathrm{D}_\varepsilon$. Then for sufficiently large $N$, $\mathbb{P}\Big(\inf_{x_\varepsilon} \overline{S}^{N,\varepsilon}(0, x_\varepsilon) \geq \frac{c}{2}\Big)$ is close to 1. Let $T_c^{N,\varepsilon} = \inf\Big\{t\,, \inf_{x_\varepsilon} \overline{S}^{N,\varepsilon}(t, x_\varepsilon) < \frac{c}{4}\Big\}$. On the interval $[0\,, T_c^{N,\varepsilon})$, $\overline{S}^{N,\varepsilon}(t, x_\varepsilon) \geq \frac{c}{4}$, $\forall x_\varepsilon \in \mathrm{D}_\varepsilon$. For all $t \leq T_c^{N,\varepsilon} \wedge \sigma^{N,\varepsilon} \wedge T$, we have

$$\overline{\Gamma}^{N,\varepsilon}(t, x_\varepsilon) = \frac{1}{\left[\overline{B}^{N,\varepsilon}(t, x_\varepsilon)\right]^\gamma}\sum_{y_\varepsilon}\beta_\varepsilon^{x_\varepsilon,y_\varepsilon}(t)\overline{\mathfrak{F}}^{N,\varepsilon}(t, y_\varepsilon) \leq \frac{4^\gamma\lambda^*\beta^*\left\| \overline{I}^{N,\varepsilon}(t) \right\|_\infty}{c^\gamma} = \bar{\bar{c}}$$

and moreover, if $t \leq \tau^{N,\varepsilon}$,

$$\begin{aligned}\overline{S}^{N,\varepsilon}(t, x_\varepsilon) &\geq \Big(e^{(\Delta_\varepsilon^S - \bar{\bar{c}} I_d)t}\overline{S}^{N,\varepsilon}(0)\Big)(x_\varepsilon) + \int_0^t \Big(e^{(t-s)(\Delta_\varepsilon^S - \bar{\bar{c}} I_d)}d\widetilde{\mathscr{M}}_S^{N,\varepsilon}(s)\Big)(x_\varepsilon) \\ &\geq \frac{c}{2}e^{-\bar{\bar{c}}t} - \frac{c}{8}.\end{aligned} \tag{11}$$

We note that $\frac{c}{2}e^{-\bar{\bar{c}}t} \geq \frac{c}{4}$ iff $t \leq \frac{\log 2}{\bar{\bar{c}}} = T_{\bar{\bar{c}}}$.

So, on the event $\tau^{N,\varepsilon} \wedge \sigma^{N,\varepsilon} \wedge T \geq T_{\bar{\bar{c}}}$, $\overline{S}^{N,\varepsilon}(t, x_\varepsilon) \geq \frac{c}{8}$, $\forall\, 0 \leq t \leq T_{\bar{\bar{c}}}$ .

For $t > T_{\bar{\bar{c}}}$, $\overline{I}^{N,\varepsilon}(t, x_\varepsilon) \geq \Big(T_{I,\varepsilon}(t)\overline{I}^{N,\varepsilon}(0)\Big)(x_\varepsilon) + \int_0^t \Big(T_{I,\varepsilon}(t-s)d\mathscr{M}_I^{N,\varepsilon}(s)\Big)(x_\varepsilon)$.

We choose $T > T_{\bar{\bar{c}}}$ arbitrary. We know from the proof of Lemma 2 that there exists $\varepsilon_0$ and $\underline{\underline{c}}$ such that $\overline{I}^{\varepsilon}(t, x_\varepsilon) \geq \underline{\underline{c}}$ for all $\varepsilon \leq \varepsilon_0$, $x_\varepsilon \in \mathrm{D}_\varepsilon$ and $\frac{\log 2}{\bar{\bar{c}}} \leq t \leq T$. If we now choose $\delta = \frac{\underline{\underline{c}}}{2}$ in the definition of $\sigma^{N,\varepsilon}$, we deduce that for any $\varepsilon \leq \varepsilon_0$, $x_\varepsilon \in \mathrm{D}_\varepsilon$, $T_{\bar{\bar{c}}} \leq t \leq \sigma^{N,\varepsilon} \wedge T$, $\overline{I}^{N,\varepsilon}(t, x_\varepsilon) \geq \frac{\underline{\underline{c}}}{2}$. □

From now on, we decree that $\sigma^{N,\varepsilon} = 0$ whenever $\inf_{x_\varepsilon} \overline{S}^{N,\varepsilon}(0, x_\varepsilon) < \frac{c}{2}$, or $\varepsilon > \varepsilon_0$.

**Lemma 7.** *Given $T > 0$, there exists $C > 0$ such that for any $t < \tau^{N,\varepsilon} \wedge \sigma^{N,\varepsilon}$, we have*

$$\left\| \overline{S}^{N,\varepsilon}(t)\overline{\Gamma}^{N,\varepsilon}(t) - \overline{S}^{\varepsilon}(t)\overline{\Gamma}^{\varepsilon}(t) \right\|_{\infty} \leq C\Bigg( \left\| \overline{S}^{N,\varepsilon}(t) - \overline{S}^{\varepsilon}(t) \right\|_{\infty} + \left\| \overline{\mathfrak{F}}^{N,\varepsilon}(t) - \overline{\mathfrak{F}}^{\varepsilon}(t) \right\|_{\infty} + \left\| \overline{I}^{N,\varepsilon}(t) - \overline{I}^{\varepsilon}(t) \right\|_{\infty} \Bigg). \quad (12)$$

**Proof.** Note that, using the map $\mathscr{H}$ defined in Remark 1, with a slight modification of the constants, we have

$$\overline{S}^{N,\varepsilon}(t,x_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(t,x_\varepsilon) - \overline{S}^{\varepsilon}(t,x_\varepsilon)\overline{\Gamma}^{\varepsilon}(t,x_\varepsilon) = \mathscr{H}\left(\overline{S}^{N,\varepsilon}, \overline{I}^{N,\varepsilon}, \overline{\mathfrak{F}}^{N,\varepsilon}\right)(t,x_\varepsilon) - \mathscr{H}\left(\overline{S}^{\varepsilon}, \overline{I}^{\varepsilon}, \overline{\mathfrak{F}}^{\varepsilon}\right)(t,x_\varepsilon),$$

and the result then follows from the fact that $\mathscr{H}$ is bounded and globally Lipschitz. □

We define $\omega^{N,\varepsilon}(t) = \omega_S^{N,\varepsilon}(t) + \omega_I^{N,\varepsilon}(t) + \omega_{\mathfrak{F}}^{N,\varepsilon}(t)$, with

$$\begin{aligned} \omega_S^{N,\varepsilon}(t) &= \left\| \overline{S}^{N,\varepsilon}(0) - \overline{S}^{\varepsilon}(0) \right\|_{\infty} + \left\| \int_0^t T_{S,\varepsilon}(t-s)d\widetilde{\mathscr{M}}_S^{N,\varepsilon}(s) \right\|_{\infty}, \\ \omega_I^{N,\varepsilon}(t) &= \left\| \overline{I}^{N,\varepsilon}(0) - \overline{I}^{\varepsilon}(0) \right\|_{\infty} + \left\| \int_0^t T_{I,\varepsilon}(t-s)d\widetilde{\mathscr{M}}_I^{N,\varepsilon}(s) \right\|_{\infty}, \\ \omega_{\mathfrak{F}}^{N,\varepsilon}(t) &= \left\| \overline{\mathfrak{F}}_0^{N,\varepsilon}(t) - \overline{\mathfrak{F}}_0^{\varepsilon}(t) \right\|_{\infty} + \left\| \mathscr{M}_{\mathfrak{F}}^{N,\varepsilon}(t) \right\|_{\infty}, \end{aligned} \quad (13)$$

where

$$\mathscr{M}_{\mathfrak{F}}^{N,\varepsilon}(t,x_\varepsilon) = \frac{1}{N}\sum_{y_\varepsilon}\int_0^t\int_0^\infty\int_{\mathbf{D}}\int_{\mathbf{D}} \lambda(t-s)\mathbb{1}_{u \leq S^{N,\varepsilon}(s^-,y_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(s^-,y_\varepsilon)}\mathbb{1}_{Y^{s,y_\varepsilon}(t)=x_\varepsilon}\overline{Q}^{y_\varepsilon}(ds,du,d\lambda,dY).$$

Note that $\mathscr{M}_{\mathfrak{F}}^{N,\varepsilon}$ is not a martingale.

**Lemma 8.** *As $(N,\varepsilon) \to (\infty,0)$, in such a way that $N\varepsilon^d \to \infty$,*

$$\sup_{0\leq t\leq T} \mathbb{E}\left(\mathbb{1}_{t\leq\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\,[\omega^{N,\varepsilon}(t)]^2\right) \to 0.$$

**Proof.** We shall use the following notation

$$\|\Phi^\varepsilon\|_{\mathtt{H}^\varepsilon} := \left[\sum_{x_\varepsilon}|\Phi^\varepsilon_{x_\varepsilon}|^2\right]^{1/2},$$

for any step function $\Phi^\varepsilon$ ($\Phi^\varepsilon_{x_\varepsilon}$ denoting the value of $\Phi^\varepsilon$ on the cell $V_\varepsilon(x_\varepsilon)$).

Thanks to Theorem 2.1 in P. Kotelenez [32] (see Formula (A6) in Appendix A below), we have

$$\begin{aligned} \mathbb{E}\left[\sup_{t\leq\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\left\|\int_0^t T_{S,\varepsilon}(t-s)d\mathscr{M}_{SI}^{N,\varepsilon}(s)\right\|^2_{\mathtt{H}^\varepsilon}\right] &\leq C\mathbb{E}\left[\left\|\mathscr{M}_{SI}^{N,\varepsilon}(\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T)\right\|^2_{\mathtt{H}^\varepsilon}\right] \\ &\leq \frac{C}{N}\sum_{x_\varepsilon}\mathbb{E}\left(\int_0^T \overline{S}^{N,\varepsilon}(s\wedge\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon},x_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(s\wedge\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon},x_\varepsilon)ds\right). \end{aligned}$$

Provided $t \leq \sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T$, $\overline{\Gamma}^{N,\varepsilon}(t,x_\varepsilon) \leq C(\lambda^*,\beta^*)$ and $\overline{S}^{N,\varepsilon}(t,x_\varepsilon) \leq C$. Then

$$\mathbb{E}\left[\sup_{t\leq\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\left\|\int_0^t T_{S,\varepsilon}(t-s)d\mathscr{M}_{SI}^{N,\varepsilon}(s)\right\|^2_{\mathtt{H}^\varepsilon}\right] \leq C(\lambda^*,\beta^*)\frac{1}{N\varepsilon^d}.$$

Since the $L^\infty$ norm is bounded by the $\mathtt{H}^\varepsilon$ norm, as $(N,\varepsilon) \to (\infty,0)$, provided $N\varepsilon^d \to 0$,

$$\mathbb{E}\left[\sup_{t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\left\|\int_0^t T_{S,\varepsilon}(t-s)d\mathscr{M}_{SI}^{N,\varepsilon}(s)\right\|_\infty^2\right] \longrightarrow 0. \tag{14}$$

The same argument can be used for the term $\left\|\int_0^t T_{S,\varepsilon}(t-s)d\mathscr{M}_S^{N,\varepsilon}(s)\right\|_\infty$. We conclude that as $(N,\varepsilon) \longrightarrow (\infty,0)$, in such a way that $N\varepsilon^d \to 0$,

$$\sup_{t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T} \omega_S^{N,\varepsilon}(t) \longrightarrow 0 \text{ in } L^2(\Omega)\,. \tag{15}$$

A similar proof establishes that

$$\sup_{t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T} \omega_I^{N,\varepsilon}(t) \longrightarrow 0 \text{ in } L^2(\Omega)\,. \tag{16}$$

We now consider $\omega_{\mathfrak{F}}^{N,\varepsilon}(t)$. The convergence to zero of the first term has been established in Lemma 5. We now consider the second term. We have

$$\begin{aligned}
&\sup_{t\le T}\mathbb{E}\left(\mathbb{1}_{t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\sup_{x_\varepsilon}\left|\mathscr{M}_{\mathfrak{F}}^{N,\varepsilon}(t,x_\varepsilon)\right|^2\right)\\
&\quad=\frac{1}{N^2}\sup_{t\le T}\mathbb{E}\left[\mathbb{1}_{t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\sup_{x_\varepsilon}\left(\sum_{y_\varepsilon}\int_0^t\int_0^\infty\int_{\mathbf{D}}\int_{\mathbf{D}}\lambda(t-s)\mathbb{1}_{u\le S^{N,\varepsilon}(s^-,y_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(s^-,y_\varepsilon)}\right.\right.\\
&\qquad\left.\left.\times\,\mathbb{1}_{Y^{s,y_\varepsilon}(t)=x_\varepsilon}\overline{Q}^{y_\varepsilon}(ds,du,d\lambda,dY)\right)^2\right]\\
&\quad\le\frac{1}{N^2}\sum_{x_\varepsilon,y_\varepsilon}\mathbb{E}\int_0^{\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\lambda^2(t-s)S^{N,\varepsilon}(s,y_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(s,y_\varepsilon)q_\varepsilon^{y_\varepsilon,x_\varepsilon}(s,t)ds\\
&\quad\le\frac{(\lambda^*)^2}{N}\sum_{x_\varepsilon}\mathbb{E}\left[\int_0^{\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\sup_{y_\varepsilon}\left|\overline{S}^{N,\varepsilon}(s,y_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(s,y_\varepsilon)\right|\sum_{y_\varepsilon}q_\varepsilon^{y_\varepsilon,x_\varepsilon}(s,t)ds\right]\\
&\quad\le C(\lambda^*)\frac{T}{N\varepsilon^d}.
\end{aligned}\tag{17}$$

The result follows. Note that since $\mathscr{M}_{\mathfrak{F}}^{N,\varepsilon}(t,x_\varepsilon)$ is not a martingale, the result for $\omega_{\mathfrak{F}}^{N,\varepsilon}(t)$ is weaker than (15) and (16). □

Lemma 8 clearly implies

**Lemma 9.** *As $(N,\varepsilon) \longrightarrow (\infty,0)$ in such a way that $N\varepsilon^d \to \infty$, $\mathbb{1}_{t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\int_0^t \omega^{N,\varepsilon}(s)ds \longrightarrow 0$ in probability.*

It remains to establish the next result.

**Lemma 10.** *As $(N,\varepsilon) \to (\infty,0)$, $\mathbb{P}(\sigma^{N,\varepsilon} < T) \longrightarrow 0$ and $\mathbb{P}(\tau^{N,\varepsilon} < T) \longrightarrow 0$.*

**Proof.** We have

$$\begin{aligned}
\mathbb{P}\left(\sigma^{N,\varepsilon} < T\right) \le\ &\mathbb{P}\left(\sup_{t\le\sigma^{N,\varepsilon}\wedge T}\left\|\int_0^t T_{S,\varepsilon}(t-s)d\mathscr{M}_S^{N,\varepsilon}(s)\right\|_\infty \ge \delta/2\right)\\
&+\mathbb{P}\left(\sup_{t\le\sigma^{N,\varepsilon}\wedge T}\left\|\int_0^t T_{I,\varepsilon}(t-s)d\widetilde{\mathscr{M}}_I^{N,\varepsilon}(s)\right\|_\infty \ge \delta/2\right).
\end{aligned}\tag{18}$$

We consider the second term only. The first one is treated similarly. Since

$$\left\| \int_0^t T_{I,\varepsilon}(t-s)d\widetilde{\mathscr{M}}_I^{N,\varepsilon}(s) \right\|_\infty \leq \left\| \int_0^t T_{I,\varepsilon}(t-s)d\mathscr{M}_{SI}^{N,\varepsilon}(s) \right\|_\infty + \left\| \int_0^t T_{I,\varepsilon}(t-s)d\mathscr{M}_I^{N,\varepsilon}(s) \right\|_\infty,$$

from Proposition 3.2 of [4], we have

$$\mathbb{P}\left( \sup_{t \leq \sigma^{N,\varepsilon} \wedge T} \left\| \int_0^t T_{I,\varepsilon}(t-s)d\mathscr{M}_I^{N,\varepsilon}(s) \right\|_\infty \geq \frac{\delta}{2} \right) \leq 4\varepsilon^{-d-2} \exp\left( -\mathtt{a}\frac{\delta^2}{16}N \right) \tag{19}$$

Thanks to the fact that $N\varepsilon^d \longrightarrow 0$, the right hand side, and hence, also the left hand side of (19), tends to 0. By Chebyshev's inequality, we have

$$\mathbb{P}\left( \sup_{t \leq \sigma^{N,\varepsilon} \wedge T} \left\| \int_0^t T_{I,\varepsilon}(t-s)d\mathscr{M}_{SI}^{N,\varepsilon}(s) \right\|_{\mathtt{H}^\varepsilon} \geq \frac{\delta}{2} \right) \leq \frac{4}{\delta^2}\mathbb{E}\left[ \sup_{t \leq \sigma^{N,\varepsilon} \wedge T} \left\| \int_0^t T_{I,\varepsilon}(t-s)d\mathscr{M}_{SI}^{N,\varepsilon}(s) \right\|_{\mathtt{H}^\varepsilon}^2 \right].$$

The right hand side tends to 0, as shown in the proof of Lemma 8. Since the $L^\infty$ norm is bounded by the $\mathtt{H}^\varepsilon$ norm, this finishes the proof that $\mathbb{P}(\sigma^{N,\varepsilon} < T) \to 0$. A similar proof establishes the same result for $\tau^{N,\varepsilon}$. □

We now extend our stochastic process to the whole space $\mathbb{T}^d$. So, we define

$$\overline{\mathbf{S}}^{N,\varepsilon}(t,x) := \sum_{x_\varepsilon} \overline{S}^\varepsilon(t,x_\varepsilon)\mathbb{1}_{V_\varepsilon(x_\varepsilon)}(x), \quad \overline{\mathbf{I}}^{N,\varepsilon}(t,x) := \sum_{x_\varepsilon} \overline{I}^\varepsilon(t,x_\varepsilon)\mathbb{1}_{V_\varepsilon(x_\varepsilon)}(x)$$

$$\overline{\mathbf{B}}^{N,\varepsilon}(t,x) := \sum_{x_\varepsilon} \overline{\mathbf{B}}^\varepsilon(t,x_\varepsilon)\mathbb{1}_{V_\varepsilon(x_\varepsilon)}(x), \quad \overline{\mathbf{F}}^{N,\varepsilon}(t,x) := \sum_{x_\varepsilon} \overline{\mathfrak{F}}^\varepsilon(t,x_\varepsilon)\mathbb{1}_{V_\varepsilon(x_\varepsilon)}(x)$$

and set $\overline{\mathbf{X}}^{N,\varepsilon} := (\overline{\mathbf{S}}^{N,\varepsilon}, \overline{\mathbf{F}}^{N,\varepsilon}, \overline{\mathbf{I}}^{N,\varepsilon})$.

Let us recall the following Gronwall's lemma.

**Lemma 11.** *Let $\phi$ and $\psi$ be two nonegative Borel measurable locally bounded functions on an interval $[0,T)$, with $T < \infty$ and $C$ a non-negative constant. If for all $t \in [0,T)$, the following inequality is satisfied :*

$$\phi(t) \leq C\int_0^t \phi(s)ds + \psi(t), \tag{20}$$

*then $\phi(t) \leq C\int_0^t e^{C(t-s)}\psi(s)ds + \psi(t)$ for all $t \leq T$.*

**Theorem 3.** *As $(N,\varepsilon) \to (\infty,0)$, in such a way that $N\varepsilon^d \to \infty$,*

$$\left\| \overline{\mathbf{X}}^{N,\varepsilon}(t) - \overline{\mathbf{X}}^\varepsilon(t) \right\|_\infty \longrightarrow 0, \text{ in probability}, \quad \forall\, t \geq 0. \tag{21}$$

**Proof.** Since $\left\| \overline{\mathbf{X}}^{N,\varepsilon}(t) - \overline{\mathbf{X}}^\varepsilon(t) \right\|_\infty = \left\| \overline{X}^{N,\varepsilon}(t) - \overline{X}^\varepsilon(t) \right\|_\infty$, it suffices to show that

$$\left\| \overline{X}^{N,\varepsilon}(t) - \overline{X}^\varepsilon(t) \right\|_\infty \longrightarrow 0, \text{ in probability, for all } t \geq 0.$$

We first consider

$$\overline{\mathfrak{F}}^{N,\varepsilon}(t,x_\varepsilon) = \frac{1}{N}\sum_{j=1}^{I^{N,\varepsilon}(0)} \lambda_{-j}(t)\mathbb{1}_{Y_j(t)=x_\varepsilon} + \sum_{y_\varepsilon}\int_0^t \overline{\lambda}(t-s)\overline{S}^{N,\varepsilon}(s,y_\varepsilon)\overline{\Gamma}^{N,\varepsilon}(s,y_\varepsilon)q_\varepsilon^{y_\varepsilon,x_\varepsilon}(s,t)ds + \mathscr{M}_{\mathfrak{F}}^{N,\varepsilon}(t,x_\varepsilon),$$
$$\overline{\mathfrak{F}}^{\varepsilon}(t,x_\varepsilon) = \overline{\lambda}_0(t)\sum_{y_\varepsilon}\overline{I}^{\varepsilon}(0,y_\varepsilon)q_\varepsilon^{y_\varepsilon,x_\varepsilon}(0,t) + \sum_{y_\varepsilon}\int_0^t \overline{\lambda}(t-s)\overline{S}^{\varepsilon}(s,y_\varepsilon)\overline{\Gamma}^{\varepsilon}(s,y_\varepsilon)q_\varepsilon^{y_\varepsilon,x_\varepsilon}(s,t)ds.$$

Exploiting Lemma 7, we have the following: for all $t \le \sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}$

$$\left\|\overline{\mathfrak{F}}^{N,\varepsilon}(t) - \overline{\mathfrak{F}}^{\varepsilon}(t)\right\|_\infty \le \omega_{\mathfrak{F}}^{N,\varepsilon}(t) + C\int_0^t\left(\left\|\overline{S}^{N,\varepsilon}(s)-\overline{S}^{\varepsilon}(s)\right\|_\infty + \left\|\overline{\mathfrak{F}}^{N,\varepsilon}(s)-\overline{\mathfrak{F}}^{\varepsilon}(s)\right\|_\infty + \left\|\overline{I}^{N,\varepsilon}(s)-\overline{I}^{\varepsilon}(s)\right\|_\infty\right)ds. \tag{22}$$

By writing $\overline{S}^{N,\varepsilon}(t,x_\varepsilon)-\overline{S}^{\varepsilon}(t,x_\varepsilon)$ and $\overline{I}^{N,\varepsilon}(t,x_\varepsilon)-\overline{I}^{\varepsilon}(t,x_\varepsilon)$ in their mild semigroup form, and using estimates in Lemmas 1–3 and 7, we obtain, for $t \le \sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T$

$$\left\|\overline{X}^{N,\varepsilon}(t)-\overline{X}^{\varepsilon}(t)\right\|_\infty \le C\int_0^t \left\|\overline{X}^{N,\varepsilon}(s)-\overline{X}^{\varepsilon}(s)\right\|_\infty ds + \omega^{N,\varepsilon}(t). \tag{23}$$

Then, it follows from Gronwall's Lemma 11 that

$$\begin{aligned}\left\|\overline{X}^{N,\varepsilon}(t)-\overline{X}^{\varepsilon}(t)\right\|_\infty &\le C\int_0^t e^{C(t-s)}\omega^{N,\varepsilon}(s)ds + \omega^{N,\varepsilon}(t)\\ &\le Ce^{Ct}\int_0^t \omega^{N,\varepsilon}(s)ds + \omega^{N,\varepsilon}(t), \quad \forall\, t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}.\end{aligned} \tag{24}$$

Consequently, using Lemmas 8–10, for any $t>0$, as $(N,\varepsilon)\to(\infty,0)$, in such a way that $N\varepsilon^d \longrightarrow \infty$,

$$\left\|\overline{X}^{N,\varepsilon}(t)-\overline{X}^{\varepsilon}(t)\right\|_\infty \longrightarrow 0 \quad \text{in probability, } \forall t\ge 0.$$

□

**Theorem 4.** *For all $T>0$, as $(N,\varepsilon)\longrightarrow(\infty,0)$ in such a way that $N\varepsilon^d\to\infty$, we have,*

$$\sup_{0\le t\le T}\left(\left\|\mathbf{\overline{S}}^{N,\varepsilon}(t)-\mathbf{\overline{S}}^{\varepsilon}(t)\right\|_\infty + \left\|\mathbf{\overline{I}}^{N,\varepsilon}(t)-\mathbf{\overline{I}}^{\varepsilon}(t)\right\|_\infty\right) \longrightarrow 0 \ \ \textit{in probability}.$$

**Proof.** In the proof of Theorem 3, we have established the following:

$$\begin{aligned}\left\|\overline{S}^{N,\varepsilon}(t)-\overline{S}^{\varepsilon}(t)\right\|_\infty &\le C\int_0^t\left\|\overline{X}^{N,\varepsilon}(s)-\overline{X}^{\varepsilon}(s)\right\|_\infty ds + \omega_S^{N,\varepsilon}(t)\\ \left\|\overline{I}^{N,\varepsilon}(t)-\overline{I}^{\varepsilon}(t)\right\|_\infty &\le C\int_0^t\left\|\overline{X}^{N,\varepsilon}(s)-\overline{X}^{\varepsilon}(s)\right\|_\infty ds + \omega_I^{N,\varepsilon}(t).\end{aligned} \tag{25}$$

It follows that

$$\begin{aligned}\sup_{0\le t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\left\|\overline{S}^{N,\varepsilon}(t)-\overline{S}^{\varepsilon}(t)\right\|_\infty \le &\sup_{0\le t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T} C\int_0^t\left\|\overline{X}^{N,\varepsilon}(s)-\overline{X}^{\varepsilon}(s)\right\|_\infty ds\\ &+\sup_{0\le t\le\sigma^{N,\varepsilon}\wedge\tau^{N,\varepsilon}\wedge T}\omega_S^{N,\varepsilon}(t).\end{aligned}$$

On the other hand, from (24), for all $t \le \sigma^{N,\varepsilon} \wedge \tau^{N,\varepsilon}$,

$$\left\| \overline{X}^{N,\varepsilon}(t) - \overline{X}^{\varepsilon}(t) \right\|_{\infty} \le Ce^{Ct} \int_0^t \omega^{N,\varepsilon}(s)ds + \omega^{N,\varepsilon}(t). \tag{26}$$

So,we deduce from Lemmas 8–10 and (15) that

$$\sup_{0 \le t \le T} \left\| \overline{S}^{N,\varepsilon}(t) - \overline{S}^{\varepsilon}(t) \right\|_{\infty} \longrightarrow 0 \text{ in probability as } (N, \varepsilon) \longrightarrow (\infty, 0),$$

and the same is true for $\overline{I}^{N,\varepsilon}(t) - \overline{I}^{\varepsilon}(t)$. Thus, the claim follows. □

We can now state our main result.

**Theorem 5.** *For all $T > 0$, as $(N, \varepsilon) \longrightarrow (\infty, 0)$ in such a way that $N\varepsilon^d \to \infty$, we have,*

$$\forall\, t \in [0, T], \quad \left\| \overline{\mathbf{F}}^{N,\varepsilon}(t) - \overline{\mathbf{F}}(t) \right\|_{\infty} \longrightarrow 0, \quad \textit{in probability},$$

*and*

$$\sup_{0 \le t \le T} \left( \left\| \overline{\mathbf{S}}^{N,\varepsilon}(t) - \overline{\mathbf{S}}(t) \right\|_{\infty} + \left\| \overline{\mathbf{I}}^{N,\varepsilon}(t) - \overline{\mathbf{I}}(t) \right\|_{\infty} \right) \longrightarrow 0 \textit{ in probability}$$

*as $(N, \varepsilon) \to (\infty, 0)$ in such a way that $N\varepsilon^d \to \infty$.*

**Proof.** By using the triangle inequality, the first statement follows from Theorems 2 and 3, and the second statement from Theorems 2 and 4. □

## 4. Conclusions

In this paper, we have considered the propagation of an epidemic in a population that is distributed in various patches, the individuals being allowed to move between the patches, and the infection being not necessarily local. Our main result is that, if the population is large, as well as the number of patches, the number of individuals per patch being large as well, then provided that the numbers of individuals in the various patches are of the same order, a good approximation of the model is given by a PDE model in continuous space, which is rather simple.

Note that we also assume that the patches constitute a regular grid in space. It is rather clear that this assumption could be avoided, at the price of modifying the limiting PDE. On the other hand, if the orders of magnitude of the population in the various patches are not the same, then it is not clear how to extend our results. In fact, in such a situation, it is not clear whether an approximate continuous space model can be used, and which approximate model is the proper one.

The main novelty of our work is to allow movements of the individuals, non local infection, and age-dependent infectivity. This last feature allows us to merge the I and R compartments, a recovered individual being for us an infected individual whose infectivity is null. However, there are two drawbacks of the present model. First, we do not follow the evolution of the number of infectious individuals, since we have merged the I and the R compartments. Second, while we distinguish the rate of movements of the S type and the I type individuals, we cannot distinguish the rate between the infectious and the recovered individuals. The reason for studying the SI model in the present paper is that, in our "Varying Infectivity" model, the techniques for proving the convergence as $\varepsilon \to 0$ of the ODE model to the PDE model, which we are using in the SI case, will not be available in the SIR case. One is forced to use different techniques. We will study the extension of the present results to the SIR model in a future publication. But our conviction is that it is worth presenting the results in the SI case, due to the possibility in this case of using classical semigroup techniques.

We also intend to compare in another future work numerical simulations of our models, both the discrete and the continuous space models, with data of a real epidemic, namely the recent Covid epidemic in various parts of the town of Marseille. This will be an occasion to compare our limiting continuous space model with the discrete space model, and to verify that our models do reflect the reality of an epidemic. It will be carried out in cooperation with epidemiologists, who have collected spatial data of the Covid epidemic in Marseille.

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**Conflicts of Interest:** The authors declare that they have no conflict of interest.

## Appendix A

### *Appendix A.1. The Duhamel Formula*

In this paper, we use as an essential tool, the so-called Duhamel formula, which we now present.

Denote by $\{X(t),\ t \geq 0\}$ an $\mathbb{R}^k$-valued function of $t$, solution of the following differential equation:

$$\begin{aligned} \frac{dX}{dt}(t) &= AX(t) + f(t), \\ X(0) &= x, \end{aligned} \tag{A1}$$

where $A$ is an arbitrary $k \times k$ matrix, $f \in L^1_{loc}(\mathbb{R}_+;\mathbb{R}^k)$, and $x \in \mathbb{R}^k$. Then the Duhamel formula says that the unique solution $X(t)$ of this ODE is given as

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A} f(s)ds\,. \tag{A2}$$

Indeed, it is easy to check that the function given by (A2) solves the ODE (A1); hence, it is the unique solution of this ODE. When the forcing term $f$ is given, the Duhamel formula expresses the solution $X(t)$ of the linear ODE (A1) in terms of the initial condition $x$ and $f(t)$. However, if the equation is not linear, and $f(t) = g(t, X(t))$ depends upon the solution, the Duhamel formula becomes

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A} g(s, X(s))ds \tag{A3}$$

and is still valid. We use this Duhamel formula in Lemma 1, in which case, (A1) is the first line of (2), $k = |D_\varepsilon|$, the cardinal of the set $D_\varepsilon$, and $A = \Delta^S_\varepsilon$. Note that $\Delta^S_\varepsilon$ is the infinitesimal generator of the jump Markov process $X(t)$, and

$$\left(e^{(t-s)\Delta^S_\varepsilon}\right)_{x_\varepsilon, y_\varepsilon} = p_\varepsilon^{x_\varepsilon, y_\varepsilon}(s,t) \geq 0,$$

for all $x_\varepsilon, y_\varepsilon \in D_\varepsilon$.

We also have a Duhamel formula for a stochastic equation of the following type:

$$X(t) = X(0) + \int_0^t [AX(s) + g(s, X(s))]ds + M(t),$$

where $M(t)$ is an $\mathbb{R}^k$-valued martingale. Then

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A}g(s, X(s)ds + \int_0^t e^{(t-s)A}dM(s)\,. \tag{A4}$$

Another version of the Duhamel formula is implicitly used in the proof of Proposition 1 for the solution of the first line of the system of Equation (3), which is a parabolic PDE of the form

$$\begin{aligned} &\frac{\partial u}{\partial t}(t) = [\Delta^S u](t,x) - u(t,x)v(t,x),\ t > 0, x \in \mathbb{T}^d, \\ &u(0,x) = u_0(x),\ x \in \mathbb{T}^d\,. \end{aligned} \tag{A5}$$

In this case, the Duhamel formula becomes

$$u(t,x) = [e^{t\Delta^S}u(0,\cdot)](x) - \int_0^t e^{(t-s)\Delta^S}[u(s,\cdot)v(s,\cdot)](x)ds\,.$$

Here, $e^{t\Delta^S}$ denotes the semigroup whose generator is $\Delta^S$. Again, the formula is established by verifying if $u$ satisfies the last formula, then it solves the parabolic PDE. The Duhamel formula says that if $u$ solves the parabolic PDE, then it satisfies the last formula.

### *Appendix A.2. Kotelenez' Inequality*

In Theorem 2.1 of his paper [32], P. Kotelenez has established a maximal inequality of the following type. Let $H$ be a Hilbert space, $M(t)$ an $H$-valued martingale, and $T(t)$ a semigroup of continuous operators on $H$ satisfying for some $c \geq 0$,

$$\|T(t)u\|_H \leq e^{ct}\|u\|_H, \quad \forall u \in H,\ t \geq 0\,.$$

Then, for any $T > 0$, there exists a constant $C_T$ such that, for any stopping time $\tau$ satisfying $\tau \leq T$ a.s.,

$$\mathbb{E}\left(\sup_{0\leq t\leq\tau}\left\|\int_0^t T(t-s)dM(s)\right\|^2\right) \leq C_T\mathbb{E}\Big(\|M(\tau)\|_H^2\Big)\,. \tag{A6}$$

If we replace the semigroup $T(t)$ by the identity operator, then this inequality is the classical Doob's second moment stopped inequality for martingales. Note, however, that $\int_0^t T(t-s)dM(s)$, in general, is not a martingale.

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