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Adapted solution of a backward stochastic differential equation

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Abstract: Let $\{W_t;\ t\in[0,1]\}$ be a standard k-dimensional Wiener process defined on a probability space $(\Omega,\ \mathcal{F},\ P)$, and let $\{\mathcal{F}_t\}$ denote its natural filtration. Given a $\mathcal{F}_1$ measurable d-dimensional random vector X, we look for an *adapted* pair of processes $\{x(t),\ y(t);\ t\in[0,1]\}$ with values in $\mathbf{R}^d$ and $\mathbf{R}^{d\times k}$ respectively, which solves an equation of the form:

$$x(t)+\int_t^1 f(s,\ x(s),\ y(s))\,\mathrm{d}s+\int_t^1[g(s,\ x(s))+y(s)]\,\mathrm{d}W_s=X.$$

A linearized version of that equation appears in stochastic control theory as the equation satisfied by the adjoint process. We also generalize our results to the following equation:

$$x(t)+\int_t^1 f(s,\ x(s),\ y(s))\,\mathrm{d}s+\int_t^1 g(s,\ x(s),\ y(s))\,\mathrm{d}W_s=X$$

under rather restrictive assumptions on g.

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1. Introduction

The equation for the adjoint process in optimal stochastic control (see Bensoussan [1], Bismut [2], Haussmann [4], Kushner [6]) is a linear version of the following equation:

$$x(t)+\int_t^1 f(s,\ x(s),\ y(s))\,\mathrm{d}s+\int_t^1[g(s,\ x(s))+y(s)]\,\mathrm{d}W(s)=X \tag{1.1}$$

where $\{W(t),\ t\in[0,1]\}$ is a standard k-dimensional Wiener process defined on $(\Omega,\ \mathcal{F},\ P)$, $\{\mathcal{F}_t,\ t\in[0,1]\}$ is its natural filtration (i.e. $\mathcal{F}_t=\sigma(W(s),\ 0\leqslant s\leqslant t)$), X is a given $\mathcal{F}_1$ measurable d-dimensional random vector, and f maps $\Omega\times[0,1]\times\mathbb{R}^d\times\mathbb{R}^{d\times k}$ into $\mathbb{R}^d$. f is assumed to be $\mathcal{P}\otimes\mathcal{B}_d\otimes\mathcal{B}_{d\times k}/\mathcal{B}_d$ measurable, where $\mathcal{P}$ denotes the σ-algebra of $\mathcal{F}_t$-progressively measurable subsets of $\Omega\times[0,1]$. We assume moreover that f is uniformly Lipschitz with respect to both x and y. We are looking for a pair $\{x(t),\ y(t);\ t\in[0,1]\}$ with values in $\mathbb{R}^d\times\mathbb{R}^{d\times k}$ which we require to be $\{\mathcal{F}_t\}$ adapted. Note that it is the freedom of choosing the process $\{y(t)\}$ which will allow us to find an adapted solution. Indeed, in case $y\equiv 0$ and X is deterministic, the above equation would have a unique $\mathcal{F}^t$-adapted solution $\{x(t)\}$, where $\mathcal{F}^t=\sigma(W(s)-W(t);\ t\leqslant s\leqslant 1)$. Note also that the following relation exists between $\{x(t)\}$, $\{y(t)\}$ and $\{W(t)\}$:

$$y(t)=\frac{\mathrm{d}}{\mathrm{d}t}\langle x,\ W\rangle_t-g(t,\ x(t)),\quad t\text{ a.e.}$$

where $\langle x,\ W\rangle_t$ denotes the joint quadratic variation process between x and W.

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Our main result will be an existence and uniqueness result for an adapted pair $\{x(t), y(t); t \in [0, 1]\}$ which solves (1.1). We expect that our result will prove to be useful in optimal stochastic control.

We shall also consider the more general equation

$$x(t) + \int_t^1 f(s, x(s), y(s))\, ds + \int_t^1 g(s, x(s), y(s))\, dW(s) = X \tag{1.2}$$

where $g : \Omega \times [0, 1] \times \mathbb{R}^d \times \mathbb{R}^{d\times k} \to \mathbb{R}^{d\times k}$ and is $\mathscr{P} \otimes \mathscr{B}_d \otimes \mathscr{B}_{d\times k}/\mathscr{B}_{d\times k}$ measurable, and satisfies a rather restrictive assumption which implies in particular that the mapping $y \to g(s, x, y)$ is a bijection for any (s, ω, x).

The paper is organized as follows: in Section 2, we study a version of equation (1.1) where f and g do not depend on x, in Section 3 we study equation (1.1), and in Section 4 we study equation (1.2). Let us close this section with the introduction of some notations that will be used throughout the present paper. $M^2(0, 1; \mathbb{R}^d)$ (resp. $M^2(0, 1; \mathbb{R}^{d\times k})$) will denote the set of $\mathbb{R}^d$-valued (resp. $\mathbb{R}^{d\times k}$-valued) processes which are $\mathscr{F}_t$-progressively measurable, and are square integrable over $\Omega \times (0, 1)$ with respect to $P \times \lambda$ (here λ denotes Lebesgue measure over [0, 1]). For $x \in \mathbb{R}^d$, $|x|$ will denote its Euclidean norm. An element $y \in \mathbb{R}^{d\times k}$ will be considered as a $d \times k$ matrix; note that its Euclidean norm is given by $|y| = \sqrt{\mathrm{Tr}(yy^*)}$, and $(y, z) = \mathrm{Tr}(yz^*)$.

2. A simplified version of equation (1.1)

As a preparation for the study of equation (1.1), we consider in this section simplified versions of that equation. Let us first prove:

Lemma 2.1. *Given $X \in L^2(\Omega, \mathscr{F}_1, P; \mathbb{R}^d)$, $f \in M^2(0, 1, \mathbb{R}^d)$ and $g \in M^2(0, 1; \mathbb{R}^{d\times k})$, there exists a unique pair $(x, y) \in M^2(0, 1\ \mathbb{R}^d) \times M^2(0, 1; \mathbb{R}^{d\times k})$ such that*

$$x(t) + \int_t^1 f(s)\, ds + \int_t^1 [g(s) + y(s)]\, dW(s) = X, \quad 0 \leqslant t \leqslant 1. \tag{2.1}$$

Proof. Define

$$x(t) = E^{\mathscr{F}_t}\left[X - \int_t^1 f(s)\, ds\right], \quad 0 \leqslant t \leqslant 1.$$

It follows from a well-known martingale representation theorem (see e.g. Karatzas and Shreve [5], p. 182) that there exists $\bar{y} \in M^2(0, 1; \mathbb{R}^{d\times k})$ such that

$$E^{\mathscr{F}_t}\left[X - \int_0^1 f(s)\, ds\right] = x(0) + \int_0^t \bar{y}(s)\, dW(s).$$

We finally define $y(t) = \bar{y}(t) - g(t)$, $0 \leqslant t \leqslant 1$. It is easily seen that the pair (x, y) which we have constructed solves (2.1), and that any solution to (2.1) takes the above form. □

We now consider the equation

$$x(t) + \int_t^1 f(s, y(s))\, ds + \int_t^1 [g(s) + y(s)]\, dW(s) = X \tag{2.2}$$

where now $f : \Omega \times (0, 1) \times \mathbb{R}^{d\times k} \to \mathbb{R}^d$ is $\mathscr{P} \otimes \mathscr{B}_{dk}/\mathscr{B}_d$ measurable ($\mathscr{P}$ denotes the σ-algebra of progressively measurable subsets of $\Omega \times (0, 1)$) with the property that

$$f(\cdot, 0) \in M^2(0, 1; \mathbb{R}^d) \tag{2.3i}$$

and there exists $c > 0$ such that

$$|f(t, y_1) - f(t, y_2)| \leqslant c\,|y_1 - y_2| \tag{2.3ii}$$

for any y_1, $y_2 \in \mathbb{R}^{d\times k}$, and (t, ω) a.e. Note that (2.3i) + (2.3ii) imply that $f(\cdot, y(\cdot)) \in M^2(0, 1; \mathbb{R}^d)$ whenever $y \in M^2(0, 1; \mathbb{R}^{d\times k})$.

Proposition 2.2. *Let $X \in L^2(\Omega, \mathscr{F}_1, P; \mathbb{R}^d)$, $g \in M^2(0, 1; \mathbb{R}^{d\times k})$ and $f : \Omega \times (0, 1) \times \mathbb{R}^{d\times k} \to R^d$ be a mapping satisfying the above requirements, in particular* (2.3). *Then there exists a unique pair $(x, y) \in M^2(0, 1; \mathbb{R}^d) \times M^2(0, 1; \mathbb{R}^{d\times k})$ which satisfies* (2.2).

Proof. *Uniqueness.* Let (x_1, y_1) and (x_2, y_2) be two solutions. It follows from Itô's formula applied to $|x_1(s) - x_2(s)|^2$ from $s = t$ to $s = 1$ that

$$\begin{aligned} &|x_1(t) - x_2(t)|^2 + \int_t^1 |y_1(s) - y_2(s)|^2 \, ds \\ &= -2\int_t^1 (f(s, y_1(s)) - f(s, y_2(s)), x_1(s) - x_2(s)) \, ds \\ &\quad -2\int_t^1 (x_1(s) - x_2(s), [y_1(s) - y_2(s)] \, dW_s) \end{aligned}$$

It follows from well-known inequalities for semi-martingales that $\sup_{0\leqslant t\leqslant 1} |x_1(t) - x_2(t)|^2$ is P-integrable. Since moreover $y_1 - y_2 \in M^2(0, 1; \mathbb{R}^{d\times k})$, the above stochastic integral is P-integrable and has zero expectation. We obtain

$$\begin{aligned} &E|x_1(t) - x_2(t)|^2 + E\int_t^1 |y_1(s) - y_2(s)|^2 \, ds \\ &= -2E\int_t^1 (f(s, y_1(s)) - f(s, y_2(s)), x_1(s) - x_2(s)) \, ds \\ &\leqslant \tfrac{1}{2}E\int_t^1 |y_1(s) - y_2(s)|^2 \, ds + 2c^2 E\int_t^1 |x_1(s) - x_2(s)|^2 \, ds. \end{aligned}$$

Uniqueness now follows from Gronwall's Lemma.

Existence. With the help of Lemma 2.1, we define an approximating sequence by a kind of Picard iteration. Let $y_0(t) \equiv 0$, and $\{(x_n(t), y_n(t));\ 0 \leqslant t \leqslant 1\}_{n\geqslant 1}$ be a sequence in $M^2(0, 1; \mathbb{R}^d) \times M^2(0, 1; \mathbb{R}^{d\times k})$ defined recursively by

$$x_n(t) + \int_t^1 f(s, y_{n-1}(s)) \, ds + \int_t^1 [g(s) + y_n(s)] \, dW(s) = X. \tag{2.4$_n$}$$

Using again the Itô formula and the same inequalities as above, we get (with $K = 2c^2$)

$$\begin{aligned} &E(|x_{n+1}(t) - x_n(t)|^2) + E\int_t^1 |y_{n+1}(s) - y_n(s)|^2 \, ds \\ &\leqslant KE\int_t^1 |x_{n+1}(s) - x_n(s)|^2 \, ds + \tfrac{1}{2}E\int_t^1 |y_n(s) - y_{n-1}(s)|^2 \, ds. \end{aligned}$$

Define $u_n(t) = E\int_t^1 |x_n(s) - x_{n-1}(s)|^2 \, ds$ and $v_n(t) = E\int_t^1 |y_n(s) - y_{n-1}(s)|^2 \, ds$, for $n \geqslant 1$ ($x_0(t) \equiv 0$). The last inequality implies that

$$-\frac{d}{dt}(u_{n+1}(t) e^{Kt}) + e^{Kt} v_{n+1}(t) \leqslant \tfrac{1}{2} e^{Kt} v_n(t). \tag{2.5}$$

Integrating from t to 1, we obtain

$$u_{n+1}(t) + \int_t^1 e^{K(s-t)} v_{n+1}(s) \, ds \leqslant \tfrac{1}{2}\int_t^1 eK(s-t) v_n(s) \, ds.$$

It follows in particular that

$$\int_0^1 e^{Kt} v_{n+1}(t) \, dt \leqslant 2^{-n}\bar{c} \, e^K$$

with $\bar{c}=E\int_0^1|y_1(t)|^2\,\mathrm{d}t=\sup_{0\leqslant t\leqslant 1}v_1(t)$; but also then

$$u_{n+1}(0)\leqslant 2^{-n}\bar{c}\,\mathrm{e}^K. \tag{2.6}$$

However, from (2.5) and $(\mathrm{d}/\mathrm{d}t)u_{n+1}(t)\leqslant 0$, we deduce that

$$v_{n+1}(0)\leqslant Ku_{n+1}(0)+\tfrac{1}{2}v_n(0)\leqslant 2^{-n}\bar{K}+\tfrac{1}{2}v_n(0)$$

with $\bar{K}=\bar{c}K\,\mathrm{e}^K$, from which follows immediately

$$v_{n+1}(0)\leqslant 2^{-n}\big(n\bar{K}+v_1(0)\big). \tag{2.7}$$

Since the square roots of the right hand sides of (2.6) and (2.7) are summable series, we have that $\{x_n\}$ (resp. $\{y_n\}$) is a Cauchy sequence in $M^2(0,1;\mathbb{R}^d)$ (resp. in $M^2(0,1;\mathbb{R}^{d\times k})$). Then from $(2.4)_n$, $\{x_n\}$ is also a Cauchy sequence in $L^2(\Omega;\ C(0,1;\mathbb{R}^d))$, and passing to the limit in $(2.4)_n$ as $n\to\infty$, we obtain that the pair (x, y) defined by

$$x=\lim_{n\to\infty}x_n,\qquad y=\lim_{n\to\infty}y_n$$

solves equation (2.2).

3. Equation (1.1)

We can now study the equation

$$x(t)+\int_t^1 f(s,x(s),y(s))\,\mathrm{d}s+\int_t^1[g(s,x(s))+y(s)]\,\mathrm{d}W(s)=X \tag{3.1}$$

where $f:\Omega\times(0,1)\times\mathbb{R}^d\times\mathbb{R}^{d\times k}\to\mathbb{R}^d$ is $\mathscr{P}\otimes\mathscr{B}_d\otimes\mathscr{B}_{dk}/\mathscr{B}_d$ measurable and $g:\Omega\times(0,1)\times\mathbb{R}^d\to\mathbb{R}^{d\times k}$ is $\mathscr{P}\otimes\mathscr{B}_d/\mathscr{B}_{dk}$ measurable, with the properties that

$$f(\cdot,0,0)\in M^2(0,1;\mathbb{R}^d),\qquad g(\cdot,0,0)\in M^2(0,1;\mathbb{R}^{d\times k}) \tag{3.2i}$$

and there exists $c>0$ such that

$$|f(t,x_1,y_1)-f(t,x_2,y_2)|\leqslant c(|x_1-x_2|+|y_1-y_2|), \tag{3.2ii}$$

$$|g(t,x_1)-g(t,x_2)|\leqslant c|x_1-x_2| \tag{3.2iii}$$

for all $x,\ x_1,\ x_2\in R^d,\ y,\ y_1,\ y_2\in\mathbb{R}^{d\times k}$, (t,ω) a.e. Note that (3.2i) + (3.2ii) + (3.2iii) imply that $f(\cdot,x(\cdot),y(\cdot))\in M^2(0,1;\mathbb{R}^d)$ and $g(\cdot,x(\cdot),y(\cdot))\in M^2(0,1;\mathbb{R}^{d\times k})$, whenever $x\in M^2(0,1;\mathbb{R}^d)$, $y\in M^2(0,1;\mathbb{R}^{d\times k})$.

Theorem 3.1. *Given $X\in L^2(\Omega,\mathscr{F}_1,P;\mathbb{R}^d)$, f and g given as above and satisfying in particular* (3.2), *there exists a unique pair $(x,y)\in M^2(0,1;\mathbb{R}^d)\times M^2(0,1;\mathbb{R}^{d\times k})$ which solves equation* (3.1).

Proof. *Uniqueness.* Let (x_1,y_1) and (x_2,y_2) be two solutions in $M^2(0,1;\mathbb{R}^d)\times M^2(0,1;\mathbb{R}^{d\times k})$. By a similar argument as that in Proposition 2.2,

$$\begin{aligned}
&E|x_1(t)-x_2(t)|^2+E\int_t^1|y_1(s)-y_2(s)|^2\,\mathrm{d}s\\
&\quad=-2E\int_t^1\big(f(s,x_1(s),y_1(s))-f(s,x_2(s),y_2(s)),\,x_1(s)-x_2(s)\big)\,\mathrm{d}s\\
&\qquad-E\int_t^1|g(s,x_1(s))-g(s,x_2(s))|^2\,\mathrm{d}s\\
&\qquad-2E\int_t^1\big(g(s,x_1(s))-g(s,x_2(s)),\,y_1(s)-y_2(s)\big)\,\mathrm{d}s\\
&\quad\leqslant\bar{c}E\int_t^1|x_1(s)-x_2(s)|^2\,\mathrm{d}s+\tfrac{1}{2}E\int_t^1|y_1(s)-y_2(s)|^2\,\mathrm{d}s
\end{aligned}$$

for a certain constant $\bar{c}$. The result follows.

Existence. We will now construct an approximating sequence using a Picard type iteration with the help of Proposition 2.2. Let $x_0(t) \equiv 0$, and $\{(x_n(t), y_n(t));\ 0 \leqslant t \leqslant 1\}_{n \geqslant 1}$ be a sequence in $M^2(0, 1; \mathbb{R}^d) \times M^2(0, 1; \mathbb{R}^{d\times k})$ defined recursively by

$$x_n(t) + \int_t^T f(s, x_{n-1}(s), y_n(s))\,\mathrm{d}s + \int_t^T [g(s, x_{n-1}(s)) + y_n(s)]\,\mathrm{d}W(s) = X, \quad 0 \leqslant t \leqslant 1. \tag{3.3}$$

Using a by now 'usual' procedure, we obtain

$$E(|x_{n+1}(t) - x_n(t)|^2) + \tfrac{1}{2}E\int_t^1 |y_{n+1}(s) - y_n(s)|^2\,\mathrm{d}s$$
$$\leqslant c\Big(E\int_t^1 |x_{n+1}(s) - x_n(s)|^2\,\mathrm{d}s + E\int_t^1 |x_n(s) - x_{n-1}(s)|^2\,\mathrm{d}s\Big). \tag{3.4}$$

Define $u_n(t) = E\int_t^1 |x_n(s) - x_{n-1}(s)|^2\,\mathrm{d}s$. It follows from (3.3):

$$-\frac{\mathrm{d}u_{n+1}}{\mathrm{d}t}(t) - cu_{n+1}(t) \leqslant cu_n(t), \quad u_{n+1}(1) = 0,$$

or

$$u_{n+1}(t) \leqslant c\int_t^1 \mathrm{e}^{c(s-t)} u_n(s)\,\mathrm{d}s.$$

Iterating that inequality, we deduce that

$$u_{n+1}(0) \leqslant \frac{(c\,\mathrm{e}^c)^n}{n!} u_1(0).$$

This, together with (3.4), implies that $\{x_n\}$ is a Cauchy sequence in $M^2(0, 1; \mathbb{R}^d)$ and $\{y_n\}$ a Cauchy sequence in $M^2(0, 1; \mathbb{R}^{d\times k})$. Then, from (3.3), $\{x_n\}$ converges also in $L^2(\Omega;\ C(0, 1; \mathbb{R}^d))$. It then follows from (3.3) that

$$x = \lim_{n\to\infty} x_n, \qquad y = \lim_{n\to\infty} y_n$$

solves equation (3.1). □

4. Equation (1.2)

We now consider the equation

$$x(t) + \int_t^1 f(s, x(s), y(s))\,\mathrm{d}s + \int_t^1 g(s, x(s), y(s))\,\mathrm{d}W(s) = X \tag{4.1}$$

where $X \in L^2(\Omega, \mathscr{F}_1, P; \mathbb{R}^d)$, $f: \Omega \times (0, 1) \times \mathbb{R}^d \times \mathbb{R}^{d\times k} \to \mathbb{R}^d$ is $\mathscr{P} \otimes \mathscr{B}_d \otimes \mathscr{B}_{d\times k}/\mathscr{B}_d$ measurable and $g: \Omega \times (0, 1) \times \mathbb{R}^d \times \mathbb{R}^{d\times k} \to \mathbb{R}^{d\times k}$ is $\mathscr{P} \otimes \mathscr{B}_d \otimes \mathscr{B}_{d\times k}/\mathscr{B}_{d\times k}$ measurable, and they satisfy:

$$f(\cdot, 0, 0) \in M^2(0, 1; \mathbb{R}^d), \qquad g(\cdot, 0, 0) \in M^2(0, 1; \mathbb{R}^{d\times k}); \tag{4.2i}$$

there exists $c > 0$ such that

$$|f(t, x_1, y_1) - f(t, x_1, y_2)| + |g(t, x_1, y_1) - g(t, x_2, y_2)| \leqslant c(|x_1 - x_2| + |y_1 - y_2|) \tag{4.2ii}$$

for all $x_1, x_2 \in \mathbb{R}^d$, $y_1, y_2 \in \mathbb{R}^{d\times k}$, (t, ω) a.e.; and there exists $\alpha > 0$ such that

$$|g(t, x, y_1) - g(t, x, y_2)| \geqslant \alpha |y_1 - y_2| \tag{4.2iii}$$

for all $x \in \mathbb{R}^d$, y_1, $y_2 \in \mathbb{R}^{d\times k}$, (t, ω) a.e. Note that (4.2iii) is satisfied is the case of equation (3.1), with $\alpha = 1$. Moreover, (4.2ii) + (4.2iii) implies that for all $x \in \mathbb{R}^d$ and (t, ω) a.e.,

$$y \to g(t, x, y)$$

is a bijection from $\mathbb{R}^{d\times k}$ onto itself. Indeed, the one-to-one property follows at once from (4.2ii) and the onto property from continuity (4.2ii), injectivity, and the fact that

$$\lim_{|y|\to+\infty} |g(t, x, y)| = +\infty$$

for all $x \in \mathbb{R}^d$, (t, ω) a.e. (see the comment following the proof of Theorem 4.1 in Protter [7]). We have:

Theorem 4.1. *Under the above conditions on X, f and g, in particular* (4.2), *there exists a unique pair* $(x, y) \in M^2(0, 1; \mathbb{R}^d) \times M^2(0, 1; \mathbb{R}^{d\times k})$ *which solves equation* (4.1).

Proof. The proof is an adaptation, with essentially obvious changes, of the proofs of the previous results. We just indicate the three steps, and explain the one new argument which is needed in the first step. The first step consists in studying the equation

$$x(t) + \int_t^1 f(s)\,\mathrm{d}s + \int_t^1 g(s, y(s))\,\mathrm{d}W(s) = X \tag{4.3}$$

where g satisfies the simplified version of (4.2) obtained by suppressing the dependence in x. The second step solves the equation

$$x(t) + \int_t^1 f(s, y(s))\,\mathrm{d}s + \int_t^1 g(s, y(s))\,\mathrm{d}W(s) = X$$

where f and g satisfy the simplified version of (4.2) obtained by suppressing the dependence in x, and the third step solves equation (4.1). Let us only consider equation (4.3). From Lemma 2.1, there exists a unique pair $(x, \bar{y})$ such that

$$x(t) + \int_t^1 f(s)\,\mathrm{d}s + \int_t^1 \bar{y}(s)\,\mathrm{d}W(s) = X.$$

It remains to show that given $\bar{y} \in M^2(0, 1; \mathbb{R}^{d\times k})$, there exists a unique $y \in M^2(0, 1; \mathbb{R}^{d\times k})$ such that

$$g(t, y(t)) = \bar{y}(t) \quad (t, \omega) \text{ a.e.}$$

It follows from the properties of g that for any $(t, \omega, y) \in [0, 1] \times \Omega \times \mathbb{R}^{d\times k}$, there exists a unique element $\phi_t(\omega, y)$ of $\mathbb{R}^{d\times k}$ such that

$$g(t, \omega, \phi_t(\omega, y)) = y.$$

It remains only to show that ϕ is $\mathscr{P} \otimes \mathscr{B}_{d\times k}/\mathscr{B}_{d\times k}$ measurable.

We can, without loss of generality, assume that $\Omega = C([0, 1]; \mathbb{R}^k)$, $W_t(\omega) = \omega(t)$ and $\mathscr{F}_1$ is the Borel field over Ω. Note that the mapping

$$G(t, \omega, y) = (t, \omega, g(t, \omega, y))$$

is a bijection from $\mathscr{E} = [0, 1] \times \Omega \times \mathbb{R}^{d\times k}$ into itself. Since $\mathscr{E}$ is a complete and separable metric space, it follows from Theorem 10.5, page 506 in Ethier and Kurtz [3] that G^{-1} is Borel measurable, i.e. $\mathscr{B}([0, 1]) \otimes \mathscr{F}_1 \otimes \mathscr{B}_{d\times k}$ measurable. By considering for each t the restriction of the same map to $[0, t] \times C([0, t]; \mathbb{R}^{d\times k}) \times \mathbb{R}^{d\times k}$, we obtain that G^{-1} is $\mathscr{P} \otimes \mathscr{B}_{d\times k}$ measurable, which proves the above assertion for ϕ. □

Remark 4.2. It would be nice to solve the above equations for an arbitrary (non-necessarily square integrable) final condition X. That extension of our results would follow from the following result. Let X_1,

$X_2 \in L^2(\Omega, \mathscr{F}_1, P; \mathbb{R}^d)$, and let $\{x_1(t)\}$, $\{x_2(t)\}$ denote the corresponding solutions of equation (4.1). Then $x_1 = x_2$ on $[0, 1] \times \{X_1 = X_2\}$. However, even in the simplest case of equation (2.1) with $f \equiv 0$ and $g \equiv 0$,

$$x_i(t) = E(X_i/\mathscr{F}_t), \quad i = 1, 2,$$

and the above result does not hold, except if $\{X_1 = X_2\} \in \mathscr{F}_0$.

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