

# From exploration paths to mass excursions – variations on a theme of Ray and Knight

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**Abstract.** Based on an intuitive approach to the Ray-Knight representation of Feller’s branching diffusion in terms of Brownian excursions we survey a few recent developments around exploration and mass excursions. One of these is Bertoin’s “tree of alleles with rare mutations” [6], seen as a tree of excursions of Feller’s branching diffusion. Another one is a model of a population with individual reproduction, pairwise fights and emigration to ever new colonies, conceived as a tree of excursions of Feller’s branching diffusion with logistic growth [14]. Finally, we report on a Ray-Knight representation of Feller’s branching diffusion with logistic growth in terms of a reflected Brownian motion whose drift depends on the local time accumulated at its current level [19, 27].

**2010 Mathematics Subject Classification.** Primary 60J70; Secondary 60J80, 60J55.

**Keywords.** Ray-Knight representation, Harris path, Feller’s branching diffusion, Itô’s excursion measure, exploration process, local time, Feller branching with logistic growth.

## 1. Introduction

The two classical theorems of Ray and Knight (see e.g. [24], [32] or [33]) give beautiful connections between Brownian excursions (described by Itô’s excursion measure) and excursions of Feller’s branching diffusion.

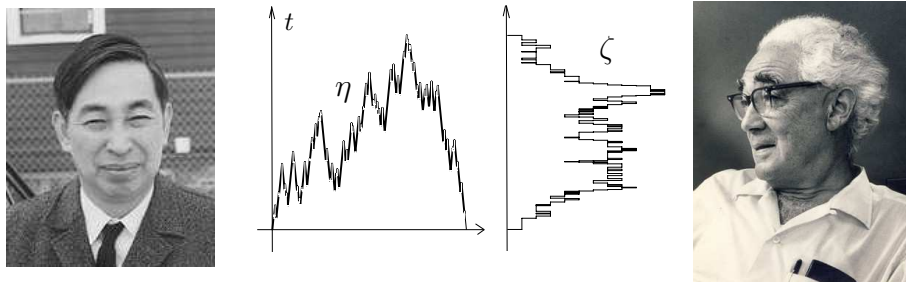


Figure 1. Itô meets Feller. Sketch of a Brownian excursion and the corresponding excursion of a Feller branching diffusion.

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\*Research supported in part by the Agence Nationale de la Recherche under grant ANR-06-BLAN-0113.

†Research partly supported by Deutsche Forschungsgemeinschaft (FOR 498).

Here is an informal statement of the second Ray-Knight theorem: *The time which a (suitably stopped) reflected Brownian motion spends near level  $t$  (and which is formally captured by its local time at  $t$ ), viewed as a process in  $t$ , is a Feller branching diffusion.* Let's go for the trees in the forest: The reflected Brownian motion is the concatenation of many *Brownian excursions*, and the random path of the Feller branching diffusion is a sum of many *Feller excursions* (we will come back to this in Section 4). And indeed, as adumbrated in Figure 1, the just described "Ray-Knight mapping" works also on these building blocks, and maps a Brownian excursion into a Feller excursion.

A nice way to understand the Ray-Knight mapping is to interpret the Brownian excursion as the *exploration path* of a tree, and the Feller excursion as *width profile* of the same tree. This interpretation, and the mapping from exploration excursions to width profiles (or *mass excursions*, is most easily conceived in a (pre-limit) situation of binary trees in continuous time. We will review this in Section 2. There, we will also point to a few historic landmarks and give some more hints to the literature.

In Section 3 we will state a continuous-time version of the Harris representation of binary Galton-Watson trees in terms of continuous and piecewise linear exploration paths whose slopes change at constant rate.

In Section 4 we describe the result of a scaling limit. This takes the exploration paths to reflected Brownian motion (exploring a forest of continuum trees) and the rescaled mass processes to a Feller branching diffusion. With the right scaling the image of Itô's excursion measure under this mapping is the excursion measure of Feller's branching diffusion, which is an *excursion measure from an exit boundary* as defined by Pitman and Yor [30]. Poisson processes and subordinators will play a central role. Section 5 deals with subcritical branching by a Girsanov reweighting of both the exploration and the mass excursion measures.

In Sections 6–8 we give a brief synopsis of a few recent developments in the framework of "exploration and mass excursions". Intended symmetries in the presentation are captured by the following tableau (where FBD stands for Feller's branching diffusion):

Sections 4 and 5: Ray-Knight representation of FBD	Section 6: Trees of excursions of FBD
Section 8: Ray-Knight representation of FBD with logistic growth	Section 7: Trees of excursions of FBD with logistic growth

In Section 6 we focus on Bertoin's *trees of alleles in branching processes with rare neutral mutations*, with all mutations leading to ever new types. In the scaling limit studied in [6], the *tree of alleles* can be viewed as a rooted tree all of whose nodes have a countable out-degree. The root is labelled by a *subcritical* Feller branching diffusion, and all the other nodes are labelled by subcritical Feller

excursions, where given that the label of the parent is a mass path  $z = (z_t)$ , the labels of its children are a Poisson population of mass excursions with intensity measure  $A(z)\bar{Q}$ , with  $A(z) = \int_0^\infty z_t dt$  being the “size” of  $z$  and  $\bar{Q}$  being the subcritical Feller excursion measure. As we will review in Section 6, the total size of the generations in the “tree of alleles” is then a (discrete time, continuum mass) branching process which can be represented as an iteration of independent copies of inverse Gaussian subordinators.

Without changing the mathematics of this model, one may think of a geographically instead of a genetically structured population and replace the concept of “mutation to an ever new type” by that of “migration to an ever new colony”. Interesting extensions of this model have been considered. In [7], Bertoin allows for dependencies between the number of emigrant and “homebody” children, otherwise leaving the independence in the individual reproduction untouched. Another extension (which includes the model with local competition discussed in Section 7) is to replace the excursion measure  $\bar{Q}$  of the subcritical Feller branching by the excursion measure  $Q$  of some other diffusion on  $\mathbb{R}_+$ , but with the same emigration mechanism as in the model described at the beginning of Section 6. This is the class of *Virgin Island models* studied by Hutzenthaler [14].

The last two sections feature *Feller’s branching diffusion with logistic growth*, a process which has been studied in detail by Lambert [18]. In Section 7 we review the Virgin Island model in which the measure  $Q$  that governs the tree of colony sizes is the excursion measure of Feller’s branching diffusion with logistic growth. In an individual-based interpretation, this process incorporates supercritical reproduction and pairwise fights between individuals within each colony.

At first sight, the Feller branching diffusion with logistic growth does not lend itself to a Ray-Knight representation, because the competition between individuals destroys the “branching property”, i.e. the independence in the reproduction. (Other than in Section 7, we now focus on the situation within one colony.) In Section 8, however, we will provide such a representation, by introducing an order among the individuals and decreeing that the pairwise fights are always won by the individual “to the left”. As we will see, this results in an exploration process which is a reflected Brownian motion with constant upward drift plus a downward drift which is proportional to the local time accumulated at the current level. The exploration path encodes a forest of countably many continuous trees in the same way as reflected Brownian motion does in the critical Feller branching case, with a sampling from the exploration time axis corresponding to a sampling from the leaves in the forest, see [23]. With the above-mentioned “left-right rule” for the individual fights, the excursions which come later in the exploration tend to be smaller – the trees to the right are “under attack from those to the left”.

In this exposé our aim is to explain concepts and ideas on an intuitive rather than a thoroughly formal level. To this end we sometimes resort to a verbal description and refrain from giving full and rigorous proofs.

## 2. Harris paths and tree profiles

With a binary tree in continuous time one can associate (like in Figure 2) two *excursions from zero*. One is the *exploration excursion*  $\eta$  which arises by traversing the tree at a constant speed and recording the height as a function of the exploration time  $s$ . The other is the *mass excursion*  $\zeta$  which gives the profile of the tree, i.e. the number of extant branches as a function of the real time  $t$ .

The idea to establish a correspondence between planar (rooted) trees and paths by traversing the vertices of the tree and recording the height (i.e. the distance of the root) as a function of the “exploration time” goes back to Theodore Harris ([13], cf. [28] ch. 6). Following Pitman and Winkel [29] we name such an exploration excursion a *Harris path*. Later we will also consider the concatenation of such excursions, which describe the exploration of a forest of trees (and is called Harris path as well). For the moment, let us consider one single tree.

The number  $\zeta_t$  of branches extant in the tree at time  $t$  equals half the number of the level  $t$ -crossings of the Harris path  $\eta$ , which in turn equals the number of excursions of  $\eta$  above height  $t$ .

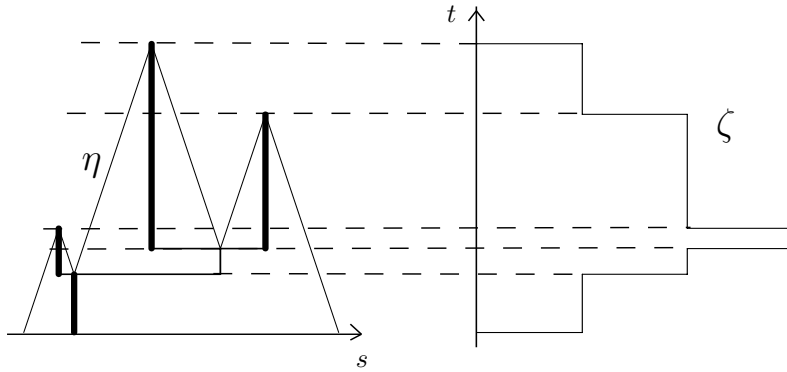


Figure 2. Left: A binary tree and its Harris path (exploration excursion)  $\eta$ . Right: The tree profile (mass excursion)  $\zeta$ .

Let us agree (for the moment) on a traversal speed 2. This results in slope  $\pm 2$  of the Harris path, and consequently half the number of its level  $t$ -crossings can be read off as

$$\zeta_t = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^\infty \mathbf{1}_{\{t < \eta_s < t + \varepsilon\}} ds. \quad (1)$$

With the chosen slope  $\pm 2$  of the Harris path, it is clear that the total branch length of the tree equals the total time that is needed to traverse the tree. In particular, the integrated mass excursion equals the length of the exploration excursion, i.e.

$$A(\zeta) := \int_0^\infty \zeta_t dt = \inf\{s > 0 : \eta(s) = 0\} =: R(\eta). \quad (2)$$

If the tree is random, say critical binary Galton-Watson with branching rate  $\sigma^2$ , then the durations of the successive periods of increase and decrease of the exploration turn out to be i.i.d. exponential with parameter  $\sigma^2/2$ , see Section 3.

In an  $(N^2, N)$ -scaling as described in Section 4, the concatenation of i.i.d. copies of rescaled exploration excursions  $\eta^N$  converges, as  $N \rightarrow \infty$ , to a reflected Brownian motion, and the sum of i.i.d. copies of rescaled mass excursions  $\zeta^N$  converges to a Feller branching diffusion. Both limiting objects can be represented in terms of Poisson populations, with the intensity measures being Itô's excursion measure on one side and the excursion measure of Feller's branching diffusion (as described in [30]) on the other. In this way, "Itô meets Feller", as the two did in Princeton in 1954, two years after the appearance of Harris' paper [13] with its section on "walks and trees".

In 1963 Daniel Ray and Frank Knight published their papers [31] and [17] which contain the essence of what is now known as the two Ray-Knight theorems. The relation (1), which persists in the scaling limit and allows to read off the mass excursion as a local time process of the exploration excursion (see Section 4), is at the heart of this.

Further landmarks in exploring the connection between Feller branching processes and Brownian excursions are the work of Kawazu and Watanabe [16] and of Neveu and Pitman [25]. In Aldous' 1991-93 trilogy [3], the *continuum random tree* shaped up as a central object. It arises as a limit of rescaled Galton-Watson trees and plays in the realm of random trees a role similar to that of Brownian motion in the classical invariance principle. New limit objects, called *Lévy trees*, appear as soon as heavy-tailed offspring distributions come into play. For the description and the analysis of these trees as well, exploration and "height" processes are an important tool. Yet another pioneering development has been Le Gall's *Random snake* [21], which, based on the idea of exploration processes, provides a representation of Dawson and Watanabe's *super-Brownian motion* (and other measure-valued branching processes, [8]) as a continuum-tree-indexed Markov motion. For this and further extensions, we refer to the monographs of Duquesne and Le Gall [10], Evans [11] and Pitman [28], and to the survey papers [22, 23] by Le Gall.

### 3. A discrete Ray-Knight theorem

In this section we state a version of the Ray-Knight theorem for Harris paths. The central observation is Lemma 3.1, which (in the critical case) traces back to [20]. Our proof, which can be easily adapted to a non-critical binary branching like that of Lemma 8.1 below, is similar to that of a more general result by Geiger and Kersting ([12], Thm. 2.1), who, however, use exploration paths with downward jumps. See also [29] and [4] for other variants of the proof of Lemma 3.1.

Consider a binary critical Galton-Watson tree in continuous time with branching rate (or variance parameter)  $\sigma^2$ , called  $T(\sigma^2)$  for short. Think of each branch having an  $\text{Exp}(\sigma^2/2)$ -distributed lifetime and carrying a rate  $\sigma^2/2$ -Poisson process

of birth time points. When the death clock rings, the branch terminates (in a leaf of the tree), when a birth clock rings, then a new branch, and hence a new independent subtree, starts, say, to the right of the mother branch.

The tree is traversed with constant speed 2 in the following “depth first search” manner: Start from the root and follow the leftmost branch up to its leaf, then turn and go down. Let  $B_1 > B_2 > \dots > B_K$  be the time points of births along the leftmost branch, written in descending order. If  $K = 0$ , that is if there are no birth points along the leftmost branch, then go down to height 0 and stop. Otherwise, turn at height  $B_1$  and enter the branch born there, proceeding in the analogous way as before, now using the birth time points along that branch. When coming down to height  $B_1$  again, proceed downwards to height  $B_2$  if  $K > 1$  (and then turn and enter the branch born at time  $B_2$ , and so on), otherwise go down to height 0 and stop there.

**Lemma 3.1.** *The exploration process of the tree  $\mathbb{T}(\sigma^2)$  constructed in the just described way is in distribution equal to an excursion  $E$  from 0 of a process with continuous, piecewise linear paths with slopes  $\pm 2$ , starting at height 0 with positive slope, changing slope at rate  $\sigma^2$ , and dying at its first return to 0.*

*Proof.* Whenever the exploration process moves upwards, it traverses, independently of its past, an  $\text{Exp}(\sigma^2/2)$ -distributed height before changing slope. Now consider a downward piece of the exploration process. The birth points along the branches of the tree form a Poisson process with intensity  $\sigma^2/2$ . The same is true for the yet unexplored birth points on the path between any point in the tree and the root, independently of the previous exploration. Hence, if the height of the current point is  $t$ , the distance travelled down from this point is distributed as  $\min(T, t)$ , where  $T$  is an  $\text{Exp}(\sigma^2/2)$ -random variable.  $\square$

For an  $\mathbb{R}_+$ -valued path  $h = (h_u, u \geq 0)$ , we put

$$\Lambda_s(t, h) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^s \mathbf{1}_{\{t < h_u < t + \varepsilon\}} du \quad (3)$$

provided the right hand side exists, and define  $\Lambda(t, h) := \Lambda_\infty(t, h)$ .

In view of (1) and (3) we thus obtain from Lemma 3.1 the following

**Corollary 3.2.** *For a random excursion  $E$  as in Lemma 3.1,  $\Lambda(\cdot, E)$  has the same distribution as the profile (or “mass excursion”) of the random tree  $\mathbb{T}(\sigma^2)$ , and hence is a critical binary Galton-Watson process with branching rate  $\sigma^2$  and one initial individual.*

#### 4. Brownian scaling: Itô meets Feller

Now let us have a look at a scaling by  $N$  which, as  $N \rightarrow \infty$ , takes a sequence of Galton-Watson processes into Feller’s branching diffusion. Time is speeded up by the factor  $N$ , mass is scaled down by the same factor  $N$ , and there are  $\lfloor Nx \rfloor$

initial individuals instead of one, with  $x$  being a positive real number. As to the exploration paths, this results in a concatenation of  $\lfloor Nx \rfloor$  exploration excursions. In view of (1) and (3), the downscaling of the mass is achieved by speeding up the exploration by a factor  $N$ , which results in slopes  $\pm 2N$ . Measured in real time, the rate of change of the slope is  $\frac{\sigma^2}{2}N$ , and measured in exploration time, it is  $\sigma^2 N^2$ .

**Definition 4.1.** For  $N \in \mathbb{N}$  let  $H^N$  be a continuous, piecewise linear process with slopes  $\pm 2N$ , starting in 0 with positive slope, changing slope at rate  $\sigma^2 N^2$  and reflected at 0. Moreover, for  $x > 0$ , let  $H^{N,x}$  be the path  $H^N$  stopped at the time  $S_x^N$  when completing  $\lfloor Nx \rfloor$  excursions from 0. In other words,  $S_x^N$  equals the smallest  $s$  for which  $\Lambda_s(0, H^N) \geq \lfloor Nx \rfloor / N$ .

**Remark 4.2.** Due to Lemma 3.1,  $H^{N,x}$  is equal in distribution to the exploration path of a rescaled Galton-Watson forest consisting of  $\lfloor xN \rfloor$  trees. Hence, analogous to Corollary 3.2, the “level counts” of the rescaled Harris path are equal in distribution to a rescaled Galton-Watson process, i.e.

$$(\Lambda(t, H^{N,x}))_{t \geq 0} \stackrel{d}{=} \left( \frac{1}{N} \Gamma_{tN}^{\lfloor Nx \rfloor} \right)_{t \geq 0}, \quad (4)$$

where  $(\Gamma_t^k)_{t \geq 0}$  is a critical binary Galton-Watson process with branching rate  $\sigma^2$  and  $k$  initial individuals.

The sequence of paths  $H^N$  as described in Definition 4.1 converges, as  $N \rightarrow \infty$ , in distribution to a reflected Brownian motion with variance parameter  $4/\sigma^2$ . (Note that the expected time between two consecutive changes of the slope of  $H^N$  is  $\Delta_N := 1/(\sigma^2 N^2)$ , and the variance of the height difference is  $(2N)^2(1/((\sigma^2 N^2))^2) = (4/\sigma^2)\Delta_N$ .) Although local times are *not* continuous functionals of the paths, it is possible to take the limit  $N \rightarrow \infty$  in (4), see e.g. [4]. This is one road to the classical

**Ray-Knight Theorem.** *Let  $H$  be reflected Brownian motion with variance parameter  $4/\sigma^2$ . For  $x \geq 0$  define*

$$S_x := \inf\{s > 0 : \Lambda_s(0, H) \geq x\} \quad (5)$$

and put  $H^x := (H_s)_{0 \leq s \leq S_x}$  and  $\Lambda(H^x) := (\Lambda_{S_x}(t, H))_{t \geq 0}$ . Then

$$\Lambda(H^x) \stackrel{d}{=} Z^x. \quad (6)$$

Here,  $Z^x$  is a *critical Feller branching diffusion* with variance parameter  $\sigma^2$ , i.e. a weak solution of the SDE

$$dZ_t = \sigma \sqrt{Z_t} dW_t, \quad Z_0 = x, \quad (7)$$

with  $W$  a standard Brownian motion.

The quantity  $\Lambda_s(t, H)$  gives one way to measure the time which the path  $H$  spends at level  $t$  up to time  $s$ . An alternative way to do this is via the *semimartingale*

local time  $L_s(t, H)$  (see [32] Ch. VI). For unit variance ( $4/\sigma^2 = 1$ ),  $L_s(t, H) = \Lambda_s(t, H)$  a.s. ([32] Cor. VI.1.9).  $L$  and  $\Lambda$  obey the scalings  $L_s(t, kH) = kL_s(t, H)$ ,  $\Lambda_s(t, kH) = \frac{1}{k}\Lambda_s(t, H)$  for  $k > 0$ . Consequently,  $L$  and  $\Lambda$  are related via

$$L_s(t, H) = \frac{4}{\sigma^2}\Lambda_s(t, H) \quad \text{a.s.}, \quad (8)$$

which corresponds to the *occupation times formula*, see e.g. [32] Cor. VI.1.6.

Note that  $H$  can be represented as  $H = \frac{2}{\sigma}|\beta|$ , with  $\beta$  a standard Brownian motion. By Tanaka's formula, one has  $|\beta_s| = B_s + L_s(0, \beta)$  for a standard Brownian motion  $B$ . Since  $L_s(0, |\beta|) = 2L_s(0, \beta)$  and because of the scaling of  $L_s(0, H)$  we obtain

$$H_s = \frac{2}{\sigma}B_s + \frac{1}{2}L_s(0, H). \quad (9)$$

Let  $n$  be Itô's excursion measure of Brownian motion, and  $n_+$  its restriction to  $\mathcal{E}_+$ , the set of  $[0, \infty)$ -valued excursions. The intensity measure for the excursion representation of (9) on the  $\Lambda(0, H)$ -axis is given by

$$\tilde{n} := \frac{2}{\sigma}n_+(\frac{2}{\sigma}\eta \in \cdot). \quad (10)$$

In other words: Let  $(\xi_i, \eta_i)$  be the points of a Poisson process on  $\mathbb{R}_+ \times \mathcal{E}_+$  with intensity measure  $dx \otimes \tilde{n}$  and write  $\biguplus_{\xi_i \leq x} \eta_i$  for the *concatenation* of all the excursions  $\eta_i$  with  $\xi_i \leq x$ , constructed as in [32] Proposition XII.2.5. Put  $H^x := (H_s)_{0 \leq s \leq S_x}$  with  $S_x$  defined in (5). Then

$$H^x \stackrel{d}{=} \biguplus_{\xi_i \leq x} \eta_i. \quad (11)$$

The prefactor  $2/\sigma$  in (10) comes from the scaling relation  $\Lambda_s(0, \frac{2}{\sigma}|\beta|) = \frac{\sigma}{2}\Lambda_s(0, |\beta|)$ . Indeed, because of the relation  $L_s(0, |\beta|) = 2L_s(0, \beta)$ , the measure  $n_+$  is the intensity measure for the excursion representation of reflected Brownian motion  $|\beta|$  on the  $L(0, |\beta|)$  (=  $\Lambda(0, |\beta|)$ )-axis.

Clearly,  $\Lambda\left(\biguplus_{\xi_i \leq x} \eta_i\right) = \sum_{\xi_i \leq x} \Lambda(\eta_i)$ . Combining this with (11), we arrive at the following re-formulation of the Ray-Knight representation (6):

$$\sum_{\xi_i \leq x} \Lambda(\eta_i) \stackrel{d}{=} Z^x \quad (12)$$

Let  $\tilde{Q}$  be the image of  $\tilde{n}$  under the mapping  $\eta \mapsto \zeta := \Lambda(\eta)$ , i.e.,

$$\tilde{Q} = \tilde{n}(\Lambda(\eta) \in \cdot). \quad (13)$$

Then, by the Poisson mapping theorem,  $(\xi_i, \zeta_i) := (\xi_i, \Lambda(\eta_i))$  is a Poisson point process with intensity measure  $dx \otimes \tilde{Q}$ . Thus, (12) translates into

$$Z^x \stackrel{d}{=} \sum_{\xi_i \leq x} \zeta_i, \quad x \geq 0, \quad (14)$$



which is a representation of Feller's branching diffusion in terms of a *path-valued subordinator* that decomposes  $Z^x$  with respect to the ancestral mass. In particular, (14) renders the so-called *branching property* of Feller's branching diffusion:  $Z^{x+x'} \stackrel{d}{=} Z^x + Z^{x'}$ , with  $Z^x$  and  $Z^{x'}$  independent.

The measure  $\tilde{Q}$  can be understood as the Lévy measure of the path-valued subordinator (14) (or also as the *canonical measure* of the infinitely divisible random measure  $Z_t^x dt$ ). We claim that

$$\tilde{Q}(\cdot) = \lim_{x \rightarrow 0} \frac{1}{x} \mathbf{P}(Z^x \in (\cdot)), \quad (15)$$

which identifies  $\tilde{Q}$  as the *excursion measure* of Feller's branching diffusion (7) in the sense of Pitman and Yor (see [30] Sec. 4 and [14] Sec. 9).

To see (15) it suffices to look at the random variables  $\langle f, Z^x \rangle := \int_0^\infty f(t) Z_t^x dt$  for continuous functions  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with compact support that vanish on  $[0, \varepsilon]$  for some  $\varepsilon > 0$ . Because  $\tilde{Q}(\zeta_\varepsilon > 0) < \infty$ , only a finite (Poisson) number of summands contribute to  $\langle f, Z^x \rangle$ . As  $x \rightarrow 0$ , the probability that more than one summand contributes is  $o(x)$ , hence  $\mathbf{P}(\langle f, Z^x \rangle \in (\cdot)) = x\tilde{Q}(\langle f, Z^x \rangle \in (\cdot)) + o(x)$ .

The ( $\sigma$ -finite) measure  $\tilde{Q}$  is Markovian, having the semigroup of (7). Let us emphasize, however, that for the dynamics (7), other than in the classical Itô excursion theory, the point 0 is not regular but absorbing.

As a preparation for the next two sections we compute for the area under Feller's branching diffusion the decomposition that corresponds to the representation (14). In other words, we compute the Poisson representation of the infinitely divisible random variable  $\int_0^\infty Z_t^x dt$ , with  $Z^x$  being the solution of (7). Here, a crucial observation is the identity

$$R(\eta) = A(\zeta) \quad (16)$$

for  $\zeta = \Lambda(\eta)$ , with  $R(\eta)$  being the length of the exploration excursion  $\eta$  and  $A(\zeta)$  the area under the mass excursion  $\zeta$ , cf. (2). This identity follows with the choice  $f \equiv 1$  from the occupation times formula ([32] VI.1.6)

$$\int_0^{R(\eta)} f(\eta_s) ds = \int_0^\infty f(t) \Lambda_{R(\eta)}(t, \eta) dt.$$

We recall that the image of  $n$  under the mapping  $\eta \mapsto R(\eta)$  is

$$\rho(da) := n(R(\eta) \in da) = \frac{1}{\sqrt{2\pi a^3}} da, \quad (17)$$

see [32] Prop. XII 2.8. This  $\rho$  is the Lévy measure of the *inverse Brownian local time at 0*, which is a  $\frac{1}{2}$ -stable subordinator. Using (14), (2) and (5) we obtain

$$A(Z^x) := \int_0^\infty Z_t^x dt \stackrel{d}{=} \sum_{\xi_i \leq x} A(\zeta_i) = \sum_{\xi_i \leq x} R(\eta_i) = S_x. \quad (18)$$

Hence  $(A(Z^x))$  as well as  $(S_x)$  is a subordinator with Lévy measure

$$\tilde{Q}(A(\zeta) \in da) = \tilde{n}(R(\eta) \in da) = \frac{2}{\sigma} n_+(R(\eta) \in da) = \frac{1}{\sigma} \rho(da), \quad (19)$$

where we used (17) and the fact that  $n_+ \circ R^{-1} = \frac{1}{2} n \circ R^{-1}$  in the last equality of (19).

Thus, due to Lévy's representation of Brownian local time as current maximum of a Brownian motion ([32] Thm VI 2.3), the distribution of  $S_x$  equals that of the time at which a standard Brownian motion first hits the level  $x/\sigma$  (or equivalently, the time at which a Brownian motion with variance parameter  $\sigma^2$  first hits the level  $x$ .)

## 5. Subcritical branching: reweighting the excursions

As a preparation for Section 6 we analyze the Ray-Knight representation of a *subcritical* Feller branching diffusion  $Z^x$  that satisfies

$$dZ_t = \sigma \sqrt{Z_t} dW_t - cZ_t dt, \quad Z_0 = x. \quad (20)$$

for a fixed  $c > 0$ .

Let us first discuss the dynamics of the exploration process. The subcriticality leads to a decrease of the birth rate, and hence to a downward drift in the exploration process. To figure out what this drift is, let us revert to the (continuous time, discrete mass) picture described at the beginning of Section 4. There, the rate of birth points along the branches (in real time after speeding up by the factor  $N$ ) was  $\frac{\sigma^2}{2}N$ , and now it is  $\frac{\sigma^2}{2}N - c$ . Due to the exploration speed  $2N$ , the rate (in exploration time) of change from a downwards to an upwards slope is thus  $\sigma^2 N^2 - 2cN$ . The rate from an upwards to a downwards slope remains unaffected and is  $\sigma^2 N^2$ . In the limit  $N \rightarrow \infty$  this leads to the drift  $-\frac{2c}{\sigma^2} ds$  and the quadratic variation  $\frac{4}{\sigma^2} ds$ , and to the exploration process being a reflected Brownian motion governed by the equation

$$H_s = \frac{2}{\sigma} \left( B_s - \frac{c}{\sigma} s \right) + \frac{1}{2} L_s(0, H). \quad (21)$$

The excursion measure  $\bar{n}$  governing (21) is (10) multiplied by a Girsanov density, up to time  $R(\eta) \wedge s$ ,  $s > 0$ , is  $\exp \left( - \int_0^{R(\eta) \wedge s} \frac{c}{\sigma} d\eta_r - \frac{1}{2} \int_0^{R(\eta) \wedge s} \left( \frac{c}{\sigma} \right)^2 dr \right)$ . As  $s \rightarrow \infty$ , this converges to  $g(R(\eta)) := \exp \left( -\frac{1}{2} \left( \frac{c}{\sigma} \right)^2 R(\eta) \right)$ ; hence

$$\frac{d\bar{n}}{d\tilde{n}}(\eta) = g(R(\eta)). \quad (22)$$

Because of (13), (16) and (22), the excursion measure  $\bar{Q}$  of the  $c$ -subcritical Feller branching diffusion arises by reweighting that of the critical Feller branching diffusion with the Girsanov density  $g(A(\zeta))$ :

$$\frac{d\bar{Q}}{d\tilde{Q}}(\zeta) = g(A(\zeta)). \quad (23)$$

This can also be seen without recurring to the exploration excursions: the Girsanov density which introduces the  $c$ -subcriticality for a Feller branching diffusion is

$$\exp\left(-\int_0^\infty \frac{cZ_t}{\sigma\sqrt{Z_t}} dW_t - \frac{1}{2} \int_0^\infty -\frac{(cZ_t)^2}{\sigma^2 Z_t} dt\right),$$

which for an excursion  $Z = \zeta$  equals  $g(A(\zeta)) = \exp\left(-\frac{1}{2} \left(\frac{c}{\sigma}\right)^2 A(\zeta)\right)$ .

To obtain the Lévy measure of the subordinator  $(S_x)$  given by (5) and (18), but now in the  $c$ -subcritical case, we have to multiply (19) by the Girsanov factor  $g(a)$ . This Lévy measure is thus given by

$$\nu(da) := g(a) \frac{1}{\sigma} \rho(da) = \exp\left(-\frac{1}{2} \left(\frac{c}{\sigma}\right)^2 a\right) \frac{1}{\sigma} \frac{1}{\sqrt{2\pi a^3}} da. \quad (24)$$

The distribution of  $S_x$  is explicit. Indeed, again due to Lévy's representation of local time, the distribution of  $S_x$  equals the distribution of the time at which a Brownian motion with variance parameter  $\sigma^2$  and drift  $c$  first hits the level  $x$ . This distribution is known as the *inverse Gaussian* with parameters  $x/c$  and  $x^2/\sigma^2$ , and has density

$$\mathbf{P}(S_x \in da) = \frac{x}{\sqrt{2\pi\sigma^2 a^3}} \exp\left(-\frac{(ca-x)^2}{2\sigma^2 a}\right). \quad (25)$$

## 6. Bertoin's "Trees of alleles with rare mutations"

In the second paper of a recent trilogy [5, 6, 7] on trees of alleles and trees of colonies in branching processes, Jean Bertoin considers a critical Galton-Watson process in discrete generations with  $\lfloor Nx \rfloor$  ancestors, offspring variance  $\sigma^2$ , and probability  $c/N$  that an individual at its birth acquires a new mutation never seen so far in the population, which it then inherits to all its descendants (see in particular Section 4 of [6]). The evolution is neutral in the sense that all the individuals, irrespective of their type, have the same reproduction law.

The situation is thus similar to the so called *infinite sites model* in population genetics, with the *type* or *allele* of an individual being the set of all mutations it carries. In this way one obtains a *tree of alleles*: the root consists of all individuals that descend from the ancestors without any mutation. Each child of the root consists of a mutant child  $\chi$  of one of these non-mutant individuals, plus that part of  $\chi$ 's offspring that carries no additional mutation. Bertoin investigates the process of the total sizes of the alleles and shows that as  $N \rightarrow \infty$  this process, when divided by  $N^2$ , converges to a *continuous state branching process with discrete generations and reproduction measure  $\nu$*  given by (24). We give a brief intuitive explanation of this along the lines of the previous section.

With the Brownian scaling discussed at the beginning of Section 4, and with one of the two factors  $N$  taken for rescaling the mass and the other one for rescaling the time, the rescaled mass process of the non-mutant individuals converges to the

$c$ -subcritical Feller branching diffusion  $Z^{(0)} := Z^x$  following (20). Write  $A^{(0)} := \int_0^\infty Z_t^{(0)} dt$  for the total non-mutant (or “wild-type”) mass. The mass  $cA^{(0)}$ , which is lost from the non-mutants due to mutation, serves as ancestral mass for another  $c$ -subcritical Feller branching diffusion  $Z^{(1)}$ . In this way, one obtains inductively a sequence  $Z^{(k)}$ ,  $k = 1, 2, \dots$  of  $c$ -subcritical Feller branching diffusions, obeying (20) with  $Z^{(k)}$ ,  $W^{(k)}$  and  $cA^{(k-1)} := c \int_0^\infty Z_t^{(k-1)} dt$  in place of  $Z$ ,  $W$  and  $x$ , and with  $Z^{(k)}$  independent of  $(Z, Z^{(1)}, \dots, Z^{(k-1)})$ , given  $A^{(k-1)}$ . The process  $\mathbb{A} = (A^{(0)}, A^{(1)}, \dots)$  is an  $\mathbb{R}_+$ -valued Markov chain, with  $A^{(k)}$  describing the sum of the sizes of the (countably many)  $k$ -th step mutant alleles.

In view of equation (18),  $A^{(0)}$  has the same distribution as  $S_x$ , where  $S = (S_\ell)_{\ell \geq 0}$  is a subordinator with Lévy measure  $\nu$  defined in (24). Consequently the Markov chain  $\mathbb{A}$  can be represented as an iteration of independent subordinators. For this, let  $S^{(0)}, S^{(1)}, S^{(2)}, \dots$  be independent copies of  $S$ , and put  $M_0 := S_x^{(0)}$ ,

$$M_1 := S_{cM_0}^{(1)}, \dots, M_k := S_{cM_{k-1}}^{(k)}, \dots \quad (26)$$

The process  $\mathbb{A} = (A^{(0)}, A^{(1)}, \dots)$  (which describes the total generation sizes in the tree of alleles) then obeys

$$(A^{(0)}, A^{(1)}, \dots) \stackrel{d}{=} (M_0, M_1, \dots). \quad (27)$$

Thus,  $\mathbb{A}$  is a continuous state branching process with discrete generations (a so-called *Jirina process*).

Indeed, for each  $k$ ,  $M_k$  is a sum of jumps of  $S^{(k)}$ , and for  $k \geq 1$  each of these jumps “stems” from a jump of  $S^{(k-1)}$ . By forgetting the structure of  $M_0$  (and thus decreeing that all the summands of  $M_1$  stem from  $M_0$ ), but keeping track of the genealogy the subordinator jumps in the later generations, one arrives at a tree whose nodes are labelled with the jump sizes, with  $M_0$  at its root. This is what Bertoin calls the *tree indexed continuous state branching process (CSBP) with reproduction measure  $\nu$* .

As the main result of [6], Bertoin proves that in the regime described at the beginning of this section, the rescaled tree of alleles  $N^{-2}\mathcal{A}^N$  converges in the sense of finite dimensional distributions to the just described tree indexed CSBP.

We also mention recent related work of Abraham and Delmas [1] in the framework of continuous state branching processes. Intuitively, the forest of non-mutant trees (which makes up the original allele) arises from a forest with a critical offspring dynamics by *pruning*, i.e. “cutting off” the mutant individuals together with their entire offspring. This procedure is iterated when passing from the  $k$ -th step mutants to the  $(k+1)$ -st step ones. In this sense, the individual genealogy that underlies this model fits into a general framework of *pruned trees*, see [2] and references therein.

There is also a geographic (instead of a genetic) interpretation of the tree of mass excursions: instead of types, one may think of colonies (or islands), with “mutation to an ever new type” becoming “emigration to an ever new island”. It is this picture which Bertoin adopts in [7]. There, he also considers a situation

in which a *bivariate subordinator*  $(T(\ell), Y(\ell))_{\ell \geq 0}$  (instead of the pair  $(S_\ell, S_{c\ell})_{\ell \geq 0}$ ) appears in an analogue of (26). The jumps of  $Y$  occur at the same points  $\ell$  as those of  $T$ , the pair of jump sizes being governed by a Lévy measure on  $\mathbb{R}_+ \times \mathbb{R}_+$ . With an appropriate scaling, the bivariate subordinator allows to describe dependencies between the numbers of “homebody” and emigrant children in a large population limit, see Theorem 2 in [7]. With  $(T^{(0)}, Y^{(0)}), (T^{(1)}, Y^{(1)}), \dots$  being i.i.d. copies of  $(T, Y)$ , the analogue of (26) and (27) becomes

$$A_x^{(0)} := T^{(0)}(x), \quad A_x^{(1)} := T^{(1)}(Y^{(0)}(x)), \quad A_x^{(2)} := T^{(2)}(Y^{(1)}(Y^{(0)}(x))), \dots \quad (28)$$

Let us write  $R(x) := T^{(0)}(x) + T^{(1)}(Y^{(0)}(x)) + T^{(2)}(Y^{(1)}(Y^{(0)}(x))) \dots$ . Both  $T(\ell)_{\ell \geq 0}$  and  $(R(\ell))_{\ell \geq 0}$  are subordinators, hence  $T(\ell)$  and  $R(\ell)$  are sums over countably many jumps. We denote the populations of jump sizes in  $T(x)$  and in  $R(x)$  by  $\mathcal{J}_x$  and  $\mathcal{C}_x$ , noting that both  $\mathcal{J} = (\mathcal{J}_\ell)_{\ell \geq 0}$  and  $\mathcal{C} = (\mathcal{C}_\ell)_{\ell \geq 0}$  are measure-valued subordinators. A decomposition of  $\mathcal{C}_x$  with respect to the “first generation colonies” gives

$$\mathcal{C}_x \stackrel{d}{=} \mathcal{J}_x + \tilde{\mathcal{C}}_{Y(x)}, \quad (29)$$

with  $(\tilde{\mathcal{C}}_\ell)$  an independent copy of  $(\mathcal{C}_\ell)$ . This is a re-formulation of Bertoin’s stochastic fixed point equation in [7], Theorem 1, which there is expressed as an integral equation involving the Lévy measures of  $(T, Y)$  and of  $\mathcal{C}$ .

In the next section we will discuss another extension of the “tree of alleles” model, again formulated in a geographic framework with emigration to ever new colonies. In this model, the current population size in a colony will have an impact on the individual death rate. The population size in one colony, as a function of the ancestral mass  $x$ , is then no subordinator any more. Still, due to the “Virgin Island assumption”, the tree of colonies will be described by an iteration of subordinators exactly as in (26), with the subordinators  $S^{(1)}, S^{(2)}, \dots$  figuring in (26) being independent copies of a subordinator. While this subordinator need not have a representation like (5), its Lévy measure still is the image of an excursion measure  $Q$  under the mapping  $\zeta \mapsto A(\zeta)$ , see formula (31) below.

## 7. Feller branching with logistic growth, and Virgin Islands

As an additional ingredient to the stochastic dynamics in (20) we now add a nonlinear drift. For simplicity we assume it to be *logistic*, thus considering

$$dZ_t = \sigma \sqrt{Z_t} dW_t + (\theta Z_t - \gamma Z_t^2) dt - c Z_t dt, \quad Z_0 = x. \quad (30)$$

In a population model, the additional drift terms  $\theta Z_t dt$  and  $-\gamma Z_t^2 dt$  describe a supercritical reproduction and a killing due to pairwise fights; we will elaborate more on this in the next section. In a geographic picture, the drift term  $-c Z_t dt$  results from *emigration*. The SDE (30) describes the evolution of the population size in the “mother colony”, and the total mass that emigrates from the mother

colony is  $c \int_0^\infty Z_t dt =: cA^{(0)}$ . The *Virgin Island assumption* is that each migration (of an infinitesimal mass  $cZ_t dt$ ) is to a new colony, where  $cZ_t dt$  becomes a potential ancestral mass.

Thus,  $cA^{(0)} =: cM_0$  serves as the (random) time argument in a subordinator  $\tilde{S}$  with Lévy measure  $\mu$  defined by

$$\mu(da) = Q(A(\zeta) \in da), \quad (31)$$

where the mass excursion measure  $Q$  is again defined by (15), but now with  $Z$  following (30) instead of (20). Proceeding inductively like in (26), we now arrive at a *tree of colonies*. This is the *Virgin Island model* studied by M. Hutzenthaler in [14], also for more general drift and diffusion coefficients than those in (30). Figure 3, adapted from [14], symbolizes a “tree of mass excursions” (embedded in time) which could either be a tree of alleles (as in the previous section) or a tree of colonies.

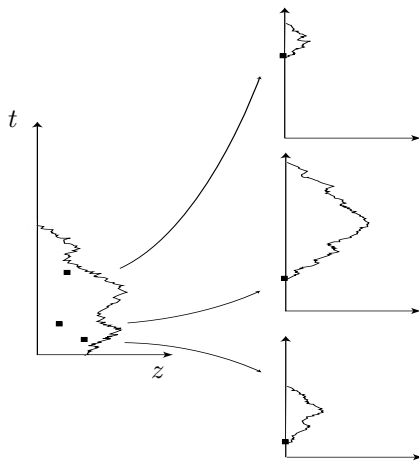


Figure 3. The root and three of its (countably many) children in a tree of excursions.

In [14], the analogy between this time embedding and a (continuous-mass generalization of a) Crump-Mode-Jagers branching process is elaborated. In the Feller branching case, a similar construction has been carried out in [9] as a basis for a superprocess with dependent spatial motion and interactive immigration.

Let us emphasize again that for the logistic Feller branching (30) there is no subordinator representation as in (14), since the non-linear drift in (30) destroys the independence in the reproduction (and the infinite divisibility of  $Z_t$ ). However, due to the Virgin Island assumption, which makes the evolution of the different colonies independent of each other, the tree of colony sizes still does have a representation in terms of an iteration of independent copies of a subordinator  $\tilde{S}$ , and hence as a tree-indexed continuous state branching process as described after

formula (27). In the case discussed there (that is for  $\theta = \gamma = 0$ ), the Lévy measure of these subordinators was given by (24). In the present case (with  $\gamma > 0$ ) there is no explicit formula for the Lévy measure of the subordinators except the equality (31). However, there is still a handy criterion that allows to decide whether the total mass (summed over all colonies) in the Virgin Island model is finite a.s. This happens if and only if  $\mathbf{E}\tilde{S}_{ca_0} \leq a_0$  for all  $a_0 \in \mathbb{R}_+$ , or equivalently iff

$$c \int_0^\infty a \mu(da) \leq 1. \quad (32)$$

Because of (31), the first moment of  $\mu$  is

$$\int_0^\infty a \mu(da) = \int_0^\infty a Q(A(\zeta) \in da) = \int_0^\infty dt \int_0^\infty y Q(\zeta_t \in dy), \quad (33)$$

with the last equality due to Fubini. Remarkably,

$$\int_0^\infty dt Q(\zeta_t \in dy) = m(y) dy, \quad (34)$$

where

$$m(y) = \frac{1}{\sigma^2 y/2} \exp\left(\int_0^y \frac{z(\theta - c - \gamma z)}{\sigma^2 z/2} dz\right) \quad (35)$$

is the *speed density* of (30) (in its adequate norming). Indeed, both sides of (34) are invariant measures for the semigroup of (30), and therefore must be proportional to each other. That they are in fact equal follows e.g. from Lemma 9.8 in [14]; our (34) is equation (178) in [14]. Putting (33) – (35) together we see that (32) is equivalent to

$$c \int_0^\infty \exp\left((\theta - c)y - \gamma \frac{\sigma^2}{4} y^2\right) dy \leq 1, \quad (36)$$

which thus characterizes the a.s. finiteness of the total mass in the Virgin Island model obtained from (30).

It turns out that the Virgin Island model is most favorable for survival in the limit  $t \rightarrow \infty$  when compared with models with the same local population dynamics (30) but a possibly different migration mechanism. To be specific, fix for  $d \in \mathbb{N}$  probability weights  $m_k$ ,  $k \in \mathbb{Z}^d$ , and consider the system of interacting diffusions

$$dZ_{k,t} = \sigma \sqrt{Z_{k,t}} dW_{k,t} + (\theta Z_{k,t} - \gamma Z_{k,t}^2) dt + c \left( \sum_j m_{j-k} Z_{j,t} - Z_{k,t} \right) dt, \quad (37)$$

$$Z_{k,0} = x \delta_{0k}, \quad k \in \mathbb{Z}^d,$$

where  $W_k$ ,  $k \in \mathbb{Z}^d$ , are independent standard Brownian motions. Using a self-duality of the solution of (37) and a comparison between (37) (now with a spatially homogeneous initial configuration) and a *mean field model*, it is shown ([15],

Thm. 1, Thm. 3 and Cor. 4) that under the assumption (36) the total mass process  $\sum_{k \in \mathbb{Z}^d} Z_{k,t}$  from (37) hits 0 in finite time a.s., irrespective of the choice of the weights  $m_k$ . In [14] a direct comparison of (37) with the corresponding Virgin Island model is announced also for more general diffusion and drift coefficients, in which case no self-duality would be available.

We conclude this section with a representation of the random variable  $\int_0^\infty Z_t^x dt$ , with  $Z^x$  being the solution of (30). This uses a time change introduced by Lambert in [18]. Consider the additive functional

$$A_t = \int_0^t Z_s^x ds,$$

and the associated time change

$$\alpha_t = \inf\{s > 0, A_t > s\}.$$

As noted in [18], the process  $Y_t^x := Z_{\alpha_t}^x$  is an Ornstein–Uhlenbeck process, solving the SDE

$$dY_t^x = (\theta - c - \gamma Y_t^x)dt + \sigma dB_t, \quad Y_0^x = x,$$

with this identification is valid only for  $0 \leq t \leq \tau_x$ , where  $\tau_x := \inf\{t > 0, Y_t^x = 0\}$ . Let  $T_x$  be the extinction time of the logistic Feller process  $Z_t^x$ . We clearly have  $\alpha_{\tau_x} = T_x$ , and consequently

$$\tau_x = \int_0^\infty Z_t^x dt.$$

In the particular case  $\gamma = \theta = 0$ , this identity ties in with the remark at the end of Section 5.

## 8. A Ray-Knight representation of logistic Feller branching

As mentioned in the previous section, the Feller branching diffusion with logistic growth, which follows the SDE (30), has no subordinator representation as in (14), nor has it a Ray-Knight representation in terms of a concatenation of independent exploration excursions. This is due to the non-linear term on the r.h.s. of (30) that comes from the pairwise fights in which one of the two fighters is killed. However, by breaking the symmetry between the individuals we will manage to bring an exploration process and a Ray-Knight representation back into the picture.

To explain the strategy, we consider a “discrete mass - continuous time” approximation of (30) and its exploration process. As in Section 4, for  $N \in \mathbb{N}$ , the approximation will be given by the total mass  $Z^{(N)}$  of a population of individuals, each of which has mass  $1/N$ . The initial mass is  $Z_0^{(N)} = \lfloor Nx \rfloor / N$ , and  $Z^{(N)}$  follows a Markovian jump dynamics: from its current state  $k/N$ ,

$$Z^{(N)} \text{ jumps to } \begin{cases} (k+1)/N & \text{at rate } kN\sigma^2/2 + k\theta \\ (k-1)/N & \text{at rate } kN\sigma^2/2 + k(k-1)\gamma/N. \end{cases} \quad (38)$$



The quadratic death term  $k(k-1)\gamma/N$  can be attributed to each of the  $k(k-1)/2$  pairs fighting at rate  $2\gamma$ , with each of the fights being lethal for one of the two individuals. The dynamics of the total mass will not be affected if we view the individuals alive at time  $t$  as being arranged “from left to right”, and decree that each of the pairwise fights is won by the individual to the left. In this way we arrive at the “death rate due to fights”  $2\gamma\mathcal{L}_i(t)/N$  for individual  $i$ , where  $\mathcal{L}_i(t)$  denotes the number of contemporaneous individuals to the left of individual  $i$  at time  $t$ . In this way we grow a forest of  $\lfloor Nx \rfloor$  trees, with all the individuals being under attack from the contemporaneans to their left. This forest is explored with speed  $2N$  in the way as described in Section 3, leading to slopes  $\pm 2N$  of the exploration path  $H^N$ . For an individual  $i$  living at real time  $t$  and being explored in an upward piece of  $H^N$  at exploration time  $s$ , the exploration process  $H^N$  experiences a rate of change from positive to negative slope which is increased by the pairwise fights. This additional rate of change from positive to negative slope is  $2\gamma\mathcal{L}_i(t)/N$  in real time, and  $4\gamma\mathcal{L}_i(t)$  in exploration time. In terms of the “local time” (3) this can be expressed as  $4\gamma N\Lambda_s(H_s^N, H^N)$ , since the number of contemporaneans to the left of the individual  $i$  is  $\mathcal{L}_i(t) = N\Lambda_s(t, H^N)$ . In the same way as we arrived at Lemma 3.1 in the case  $\theta = \gamma = 0$ , we can now identify the stochastic dynamics of  $s \mapsto H_s^N$ :

**Lemma 8.1.** *The exploration path  $s \mapsto H_s^N$  obeys the following dynamics:*

- At time  $s = 0$ ,  $H^N$  starts at height 0 and with slope  $2N$ .
- When  $H^N$  moves upwards, its slope jumps from  $2N$  to  $-2N$  at rate  $N^2\sigma^2 + 4\gamma N\Lambda_s(H_s^N, H^N)$ .
- When  $H^N$  moves downwards, its slope jumps from  $-2N$  to  $2N$  at rate  $N^2\sigma^2 + 2N\theta$ .
- Whenever  $H^N$  reaches 0, its slope jumps from  $-2N$  to  $2N$ , i.e.  $H^N$  is reflected at 0.

Write

$$S_x^N = \inf\{s : \Lambda_s^N(0, H^N) \geq \lfloor Nx \rfloor / N\}. \quad (39)$$

for the first time at which  $H^N$  completes  $\lfloor Nx \rfloor$  excursions. Just as we obtained from Lemma 3.1 the discrete Ray-Knight representation for a Galton-Watson process (Corollary 3.2), we obtain in the following Corollary of Lemma 8.1 a similar representation for the Galton-Watson process with logistic growth:

**Corollary 8.2.** *Let  $H^N$  be a stochastic process following the dynamics specified in Lemma 8.1. Then  $t \mapsto \Lambda_{S_x^N}(t, H^N)$  follows the jump dynamics (38).*

In [19] we prove that the sequence of processes  $H^N$  converges, as  $N \rightarrow \infty$ , to the weak solution of

$$H_s = \frac{2}{\sigma}B_s + \frac{2\theta}{\sigma^2}s - \gamma \int_0^s L_r(H_r, H)dr + \frac{1}{2}L_s(0, H), \quad s \geq 0. \quad (40)$$

with  $B$  a standard Brownian motion and  $L_s(t, H)$  the semimartingale local time of  $H$ , with  $L$  and  $\Lambda$  connected by (8). One ingredient in this proof is the following

**Proposition 8.3** ([19]). *For  $\sigma > 0$ ,  $\theta, \gamma \geq 0$ , the stochastic integral equation (40) has a unique weak solution. This solution is obtained by a Girsanov transform from a reflected Brownian motion with variance parameter  $\sigma^2/4$ .*

A main result of [19] is a Ray-Knight representation of Feller's branching diffusion with logistic growth in terms of a reflected Brownian motion  $H$  which experiences a constant positive drift plus a negative drift that is proportional to the local time spent by  $H$  so far at its current level:

**Theorem 8.4.** *Let  $H$  be the solution of (40), and for  $x > 0$  let  $S_x$  be defined by (5). Then  $(\Lambda_{S_x}(t, H))_{t \geq 0}$  is a Feller branching diffusion with logistic growth, i.e. a weak solution of (30) (with  $c = 0$ ).*

The strategy in [19] to prove this theorem is to justify the passage to the limit  $N \rightarrow \infty$  in Corollary 8.2. As a conclusion from this theorem, we obtain an analogue of the representation (14) for the logistic Feller process (30), but now as a path-valued Markovian jump process rather than a path-valued subordinator. To see this, let  $(\xi_i, \eta_i)$  be the point process of excursions on  $\mathbb{R}_+ \times \mathcal{E}$ , where  $\mathcal{E}$  is the space of excursions from 0, and such that  $(H_s)_{0 \leq s \leq S_x}$  is the concatenation of the excursions  $\eta_i$  with  $\xi_i \leq x$ . For  $\eta \in \mathcal{E}$ , let  $\zeta(\eta) := \Lambda_{R(\eta)}(\cdot, \eta)$  be the image of  $\eta$  under the Ray-Knight mapping. Then another way to state Theorem 8.4 is that

$$\sum_{i: \xi_i \leq x} \zeta(\eta_i) \stackrel{d}{=} Z^x, \quad (41)$$

where  $Z^x$  solves (30) (with  $c = 0$ ).

Again using the device that “those to the left win against those to the right” (which of course again is no dictum politicum) we can identify the transition probabilities of the path-valued Markov process  $(Z^x)_{x \geq 0}$ . The following is readily checked:

**Remark 8.5.** For  $x > 0$  let  $Z^x$  be a solution of (30) with  $c = 0$ . For a given path  $z = (z_t)_{t \geq 0}$  and for  $\varepsilon > 0$ , let  $X^\varepsilon(z)$  be a solution of

$$dX_t = \sigma \sqrt{X_t} dW_t^{(x, x+\varepsilon)} + ((\theta - 2z_t \gamma)X_t - \gamma X_t^2) dt, \quad X_0 = \varepsilon, \quad (42)$$

where the standard Wiener process  $W^{(x, x+\varepsilon)}$  is independent from the Wiener process  $W$  in (30). Then  $Z^{x+\varepsilon} := Z^x + X^\varepsilon(Z^x)$  is a weak solution of

$$dZ_t = \sigma \sqrt{Z_t} dW_t^{(0, x+\varepsilon)} + \theta Z_t - \gamma Z_t^2 dt, \quad Z_0 = x + \varepsilon.$$

We conjecture that the Markov process  $(Z^x)_{x \geq 0}$  has a “jump kernel”  $Q^z$  that is given by Pitman and Yor's excursion measure of the diffusion process (42), i.e.

$$Q^z = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \mathbf{P}(X^\varepsilon(z) \in \cdot).$$

With  $z$  as on the l.h.s. of (41),  $Q^z$  would then be the image under the Ray-Knight mapping  $\eta \mapsto \zeta = \Lambda(\eta)$  of the conditional intensity kernel of the point process  $(\xi_i, \eta_i)$ , given its restriction to  $[0, x] \times \mathcal{E}_+$ .

**Acknowledgement.** We are grateful to Jean Bertoin, Jean-François Delmas, Martin Hutzenthaler, Ute Lenz, Arno Siri-Jégousse and an anonymous referee for many stimulating and helpful remarks.

**Note added in proof.** Following a similar route as in [26] we were recently able to prove Theorem 8.4 also directly by a Girsanov argument, see [27]. We thank J.F. Le Gall for pointing us to [26].

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