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QUENCHED LARGE DEVIATIONS FOR ONE DIMENSIONAL NONLINEAR FILTERING*

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Abstract. Consider the standard, one dimensional, nonlinear filtering problem for diffusion processes observed in small additive white noise: $dX_t = b(X_t)dt + dB_t$, $dY_t^\varepsilon = \gamma(X_t)dt + \varepsilon dV_t$, where $B_\cdot, V_\cdot$ are standard independent Brownian motions. Denote by $q_1^\varepsilon(\cdot)$ the density of the law of Ξ_1 conditioned on $\sigma(Y_t^\varepsilon : 0 \le t \le 1)$. We provide "quenched" large deviation estimates for the random family of measures $q_1^\varepsilon(x)dx$: there exists a continuous, explicit mapping $\bar{\mathcal{J}} : \mathbb{R}^2 \to \mathbb{R}$ such that for almost all $B_\cdot, V_\cdot$, $\bar{\mathcal{J}}(\cdot, X_1)$ is a good rate function, and for any measurable $G \subset \mathbb{R}$,

$$-\inf_{x\in G^o} \bar{\mathcal{J}}(x, X_1) \le \liminf_{\varepsilon\to 0} \varepsilon \log \int_G q_1^\varepsilon(x)dx \le \limsup_{\varepsilon\to 0} \varepsilon \log \int_G q_1^\varepsilon(x)dx \le -\inf_{x\in \bar{G}} \bar{\mathcal{J}}(x, X_1).$$

Key words. nonlinear filtering, large deviations

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1. Introduction and statement of results.

Consider the following one dimensional filtering problem, where the signal process $X_\cdot$ and the observation process $Y_\cdot^\varepsilon$, parametrized by a "small noise intensity" ε, are

$$\begin{aligned} dX_t &= b(X_t)dt + dB_t, \quad X_0 \sim p_0(\cdot), \\ dY_t^\varepsilon &= h(X_t)dt + \varepsilon dV_t. \end{aligned} \tag{1.1}$$

Here, $B_\cdot, V_\cdot$ are independent standard one dimensional Brownian motions, and the functions b, h, p_0 satisfy the following assumptions:[1]

(A-1) b, h, b', h' are Lipschitz functions,
(A-2) $h'(\cdot) \ge h_0 > 0$,
(A-3) $|\log p_0(x) - \log p_0(y)| \le c(1 + |x| + |y|)|x - y|$, $x, y \in \mathbb{R}$, and p_0 is uniformly bounded.

For technical reasons, we need to impose the following additional restriction:

(A-4) $h'b, h'h, h'', hb$ are Lipschitz functions and $\lim_{|x|\to\infty} h''(x) = 0$.

(A-4) implies that, outside large compacts, the observation function h is essentially linear. Let $\Omega_1 = \Omega_2 = C([0, 1]; \mathbb{R})$, $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F}_i$ be the Borel σ-algebra on Ω_i,

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[1]Due to the one dimensional nature of our model, no generality is lost in assuming the diffusion coefficient of the signal process to be one. Indeed, if the signal process satisfies $d\Xi_t = \beta(\Xi_t)dt + \sigma(\Xi_t)dB_t$, with σ uniformly bounded away from zero, then the transformation $X_t = \bar{G}(\Xi_t)$, with $\bar{G}(x) = \int_0^x (1/\sigma)(u)du$, allows one to rewrite the problem in the form (1.1).

$i = 1, 2$, and $\mathcal{F}$ be the Borel σ-algebra on Ω; let P_1, P_2 denote the Wiener measure on Ω_1, Ω_2, and $P = P_1 \otimes P_2$. We define $B_t(\omega) = \omega_1(t)$, $V_t(\omega) = \omega_2(t)$, $0 \le t \le 1$. The pair (B, V) is then distributed according to P. The solution (X, Y^ε) of the SDE (1.1) is then an $\mathcal{F}$-measurable, $C([0,1]; \mathbb{R}^2)$-valued, random variable.

Let $\mu_t^\varepsilon(\cdot)$ denote the conditional law of X_t conditioned on $\mathcal{Y}_t^\varepsilon = \sigma\{Y_s^\varepsilon, 0 \le s \le t\}$, which we consider as an $\mathcal{F}$-measurable map from Ω to $M_1(\mathbb{R})$, the space of probability measures on $\mathbb{R}$. Note that μ_t^ε is in fact measurable with respect to the ε-dependent σ-algebra $\mathcal{Y}_t^\varepsilon \subset \mathcal{F}$.

It is known that μ_t^ε is absolutely continuous, with $\mu_t^\varepsilon(dx) = q_t^\varepsilon(x)dx$, and that as $\varepsilon \to 0$, the conditional law $\mu_1^\varepsilon(dx) = q_1^\varepsilon(x)dx$ of X_1, given $\mathcal{Y}_1^\varepsilon$, converges to the Dirac measure δ_{X_1}. (All these facts can be found, e.g., in [7].) In particular, X_1 is measurable with respect to the limiting σ-algebra $\mathcal{Y}_1^0$, since h is one-to-one. It is known from the results of Picard [7] that the conditional law μ_1^ε has a variance of order ε and can be well approximated by a Gaussian law, which is given by an extended Kalman filter.

Our goal in this paper is to establish a large deviations result in the following sense. Let G be a measurable subset of $\mathbb{R}$. By the above remarks, we know that on the event $\{X_1 \notin \overline{G}\}$, $\mu_1^\varepsilon(G) \to 0$, P-almost surely. It turns out that it goes to zero at exponential speed, i.e., roughly like $\exp[-c_1(G)/\varepsilon]$. What is the value of $c_1(G) = -\lim \varepsilon \log \mu_1^\varepsilon(G)$ (if this limit exists), the "rate function" that tells us at which speed the quantity $P(X_1 \in G \,|\, \mathcal{Y}_1^\varepsilon)$ goes to zero, whenever $X_1 \notin \overline{G}$? Clearly $c_1(G)$ must depend on X_1 (at least intuitively through its distance to $\overline{G}$), and we shall see that this is indeed the case. There is no surprise in the fact that $c_1(\cdot)$ is random, since it tells us at which exponential speed the random measures μ_1^ε converge to the random measure $\mu_1^0 = \delta_{X_1}$. Our results show that c_1 does not depend on anything else, in the sense that, conditionally on $\sigma(X_1)$, it is P-almost surely constant.

We call our result "quenched" (borrowing that terminology from the theory of random media), meaning that the randomness of the observation process is frozen. One could also discuss a "semiquenched" large deviations statement by computing the P_1-almost sure limit (if it exists) of

$$\varepsilon \log \int \int_G q_1^\varepsilon(x + X_1) dx dP_2,$$

while an "annealed" large deviations result would describe the asymptotic behavior of

$$\varepsilon \log E \int_G q_1^\varepsilon(x + X_1) dx.$$

Finally, one could also consider large deviations questions at the level of the conditional measure itself, for example questions concerning the rate of decay of probabilities of the form $P(q_1^\varepsilon(x)dx \in A)$, with A a measurable subset of the space of probability measures on $\mathbb{R}$. We hope to study all these elsewhere.

Let us now state our result. Define

$$\bar{\mathcal{J}}(x, X_1) = \int_{X_1}^{x} (h(y) - h(X_1)) dy.$$

Our main result is the following theorem. For standard definitions concerning the large deviation principle (LDP), see [3]. For a set $G \subset \mathbb{R}$, we denote by G^o its interior and by $\bar{G}$ its closure.

THEOREM 1.1. *Assume* (A-1)–(A-4). *Then the family of (random) probability measures $q_1^\varepsilon(x)dx$ satisfies a quenched LDP (on the space $\mathbb{R}$ equipped with the standard Euclidean norm) with continuous good rate function $\bar{\mathcal{J}}(\cdot, X_1)$. That is, for any measurable set $G \subset \mathbb{R}$,*

$$-\inf_{x\in G^o} \bar{\mathcal{J}}(x, X_1) \le \liminf_{\varepsilon\to 0} \varepsilon \log \int_G q_1^\varepsilon(x)dx \le \limsup_{\varepsilon\to 0} \varepsilon \log \int_G q_1^\varepsilon(x)dx$$
$$\le -\inf_{x\in\bar{G}} \bar{\mathcal{J}}(x, X_1), \quad P-a.s. \tag{1.2}$$

In fact, we have the estimate, valid for any fixed compact set $K_0 \subset \mathbb{R}$,

$$\lim_{\varepsilon\to 0} \sup_{x\in K_0} |\varepsilon \log q_1^\varepsilon(x) + \bar{\mathcal{J}}(x, X_1)| = 0, \quad P-a.s. \tag{1.3}$$

(It will be obvious from the proof that the fixed time 1 can be replaced by any fixed time $t \in (0, \infty)$; that is, the statement of Theorem 1.1 remains true with q_t^ε and X_t replacing q_1^ε and X_1.)

Remarks.

1. In the particular case $h(x) = x$, Theorem 1.1 can be deduced from the results of [10].

2. The reader could wonder why the statement (1.2) is equivalent to the large deviations principle on $\mathbb{R}$ for P-almost ω, since in (1.2), the null set on which the statement does not hold true may depend on G. Note, however, that once the inequalities in (1.2) hold true for each interval $G = (a, b)$ on a set of full measure $\Omega_{a,b}$, we can set

$$\Omega' = \cap_{a,b\in\mathbb{Q}} \Omega_{a,b}$$

and conclude that $P(\Omega') = 1$ while (1.2) holds true for all $\omega \in \Omega'$ and all open intervals G with rational endpoints. Since the latter are a base for the topology on $\mathbb{R}$, one concludes (see, e.g., [3, Theorem 4.1.11]) that the full LDP holds for each $\omega \in \Omega'$.

We conclude this introduction with some comments about previous work and possible applications and extensions of our result. Our motivation for the study of the large deviations of the optimal filter is their utility in a variety of applications such as tracking (see [9]) or the study of the filter memory length (see [1]). In the one dimensional linear observation case studied in [10], precise pointwise estimates can be derived by comparison with the linear filtering problem, whose (Gaussian) solution is known explicitly. In contrast, here, the main tool used in the proof of Theorem 1.1 is the representation, due to Picard [7], of the density q_1^ε in terms of an auxiliary suboptimal filter, and the availability of good estimates on the performance of this suboptimal filter. These results are not available in the general multidimensional case. When they are, e.g., in the setup discussed in [8], we believe our analysis can be carried through. Hence, while our result is presently limited to one dimension, we expect that its multidimensional extension to the case where the dimensions of the state and observation coincide, and the observation function is one-to-one, could be deduced from the results of [8]. Extension to the case where the dimension of the observation is smaller than the dimension of the state (which is the most relevant case for applications) would require completely new additional ideas, since the result would be of a completely different nature (the limiting measure is no longer necessarily a Dirac measure, and even when it is, the convergence to the Dirac measure is at different speeds for different coordinates).

We finally note that Hijab [4] has derived a (path) quenched large deviations for the conditional density for systems in which both the signal and the observation noises are small. This is related, by a time change, to looking at short times (of order εT) of the filtering equations

$$dX_t^\varepsilon = \frac{1}{\varepsilon}\bar{b}(X_t^\varepsilon)dt + dB_t, \quad X_0^\varepsilon = x,$$
$$dY_t^\varepsilon = h(X_t^\varepsilon)dt + \varepsilon dV_t.$$

(Hijab's results are not stated in this way, but are equivalent to the description given here. Note that his setup is more general than ours in that it applies to the multi-dimensional setup and allows for general regular diffusion coefficients.) Hijab's results are not directly comparable with the LDP we derive here because of the different time interval on which they apply, and also because of the different type of conditioning. (His statement looks at the conditional density as a continuous functional of the observation trajectory, and considers the LDP when this trajectory is frozen. It is thus not directly applicable as a quenched statement.)

We refer the reader to [5] for a general introduction to stochastic calculus and stochastic differential equations, and to [6] for an exposition of nonlinear filtering theory.

Convention. Throughout the paper, when relevant, we make explicit on what parameters constants depend, even if the actual value of the constant may change from line to line. When nothing explicit is mentioned, i.e., a generic constant C is used, it is understood that it may depend on the trajectories $\{X.\}$, $\{V.\}$, but not on ε. For $\infty > t > 0$, we use the notation $||f||_t = \sup_{s\le t}|f(s)|$, with $||f|| := ||f||_{1/\varepsilon}$. Finally, we use θ^t to denote the shift operator, e.g., $\theta^t\tilde{m}(\cdot) = \tilde{m}(t+\cdot)$.

2. Picard's formulation and a path integral.

The filtering problem we are going to analyze is (1.1), and the assumptions (A-1)–(A-4) will be assumed to hold throughout the paper. We also note that since nothing is changed (in terms of the filtering problem) by adding a constant to the observation function h, we may and will assume throughout the paper that $h(0) = 0$.

It is known from the results of Picard [7] that the conditional law $q_1^\varepsilon(x)dx$ has a small variance, and that there exist finite dimensional filters that provide good approximations of the unknown state. We shall now recall the formula derived by Picard [7] for $q_1^\varepsilon(x)$, which was used there to study approximate filters. It will be an essential tool for our large deviation results.

Define the approximate filter

$$dM_t^\varepsilon = b(M_t^\varepsilon)dt + \frac{1}{\varepsilon}(dY_t^\varepsilon - h(M_t^\varepsilon)dt),$$

with $M_0^\varepsilon = 0$, and let $\bar{m}_s = M_{1-s}^\varepsilon$ and $\tilde{m}_s = \bar{m}_{\varepsilon s}$, $s \in [0, 1/\varepsilon]$.

One of the main contributions of Picard in [7, Proposition 4.2] was to express the conditional density $q_1^\varepsilon(x)$ in terms of the law of an auxiliary process $\{\bar{X}_{1-t}^x, 0 \le t \le 1\}$, which fluctuates backward in time, starting at time 1 from the position x, around the trajectory of the approximate filter $M_\cdot^\varepsilon$. Performing a time change and a Girsanov transformation, Picard's result can be rewritten as follows.[2] Define the process

$$d\tilde{Z}_s^{\varepsilon,x} = \left[-h(\tilde{Z}_s^{\varepsilon,x}) + \tilde{m}_s h'(\tilde{Z}_s^{\varepsilon,x}) - \varepsilon b(\tilde{Z}_s^{\varepsilon,x})\right] ds + \sqrt{\varepsilon}d\tilde{W}_s, \quad \tilde{Z}_0^{\varepsilon,x} = x,$$

[2]For completeness, and since the computations involved are somewhat lengthy, we present the derivation in an appendix at the end of the paper.

with $\widetilde{W}$ a standard Brownian motion, independent of $B_\cdot, V_\cdot$. Throughout, we let $\mathbb{E}$ and $\mathbb{P}$ denote expectations and probabilities with respect to the law of the Brownian motion $\widetilde{W}_\cdot$. Then a version of the conditional density of X_1, given $\mathcal{Y}_1^\varepsilon$, is given by

$$q_1^\varepsilon(x) = \frac{\rho_1^\varepsilon(x)}{\int_{\mathbb{R}} \rho_1^\varepsilon(x)dx}, \tag{2.1}$$

where

$$\rho_1^\varepsilon(x) := e^{-F(x,\tilde{m}_0)/\varepsilon}\mathbb{E}\left[\exp\left(I_\varepsilon(\tilde{Z}^{\varepsilon,x}_{1/\varepsilon},0) + \int_0^{1/\varepsilon} g_1(\tilde{Z}^{\varepsilon,x}_s,\tilde{m}_s)ds + \frac{1}{\varepsilon}\int_0^{1/\varepsilon} g_2(\tilde{Z}^{\varepsilon,x}_s,\tilde{m}_s)ds\right)\right], \tag{2.2}$$

and

$$\begin{aligned}
F(z,m) &= \int_0^z (h(y) - h(m))dy - mh(z) + h(m)z,\\
I_\varepsilon(z,m) &= \log p_0(z) + \frac{1}{\varepsilon}F(z,m),\\
g_1(z,m) &= -mh'(z)b(z) + \frac{mh''(z)}{2} + h(z)b(z) - \frac{h'(z)}{2} - \varepsilon b'(z) - h(z)b(m),\\
g_2(z,m) &= h(z)h(m) - \frac{h^2(m)}{2} - mh(z)h'(z) + \frac{m^2h'(z)^2}{2}.
\end{aligned}$$

Note that assumptions (A-1)–(A-4) ensure that, for each given m, $g_1(\cdot,m)$, $g_2(\cdot,m)$ are Lipschitz functions with Lipschitz constant uniformly bounded for m in compacts.

It is important to note that, above and throughout the paper, expressions of the form $\mathbb{E}(\cdot)$ may still be random, due to their possible dependence on $B_\cdot, V_\cdot$. Thus, any equality between such expressions is to be understood in an almost sure sense. We will not explicitly mention this in what follows.

Equipped with (2.2), one is tempted to apply standard tools of large deviations theory, viz. the large deviations principle for $\tilde{Z}^{\varepsilon,x}_\cdot$ and Varadhan's lemma, to the analysis of the exponential rate of decay of the $\mathbb{P}$ expectation in (2.2). This temptation is quenched when one realizes that, in fact, the rate of growth of ρ_1^ε is exponential in $1/\varepsilon^2$, and it is only after normalization that one can hope to obtain the relevant $1/\varepsilon$ asymptotics. This fact, unfortunately, makes the analysis slightly more subtle. In the next section, we present several lemmas, whose proof is deferred to section 4, and show how to deduce Theorem 1.1 from these lemmas. Before closing this section, however, we state the following easy a priori estimates. Recall that, according to our convention, $||X||_1 = \sup_{s\le 1}|X_s|$.

LEMMA 2.1. *$||X||_1 < \infty$, P-almost surely,*

$$|||\tilde{m}||| := \limsup_{\varepsilon\to 0} \sup_{t\in[0,1/\varepsilon]} |\tilde{m}_t| < \infty, \quad P-a.s.,$$

and for $T_\varepsilon = \log(1/\varepsilon)$, $|||\tilde{m}_X||| := \sup_{s\in[0,T_\varepsilon]}|\tilde{m}_s - X_1|$,

$$\limsup_{\varepsilon\to 0} |||\tilde{m}_X||| = 0, \quad P-a.s. \tag{2.3}$$

Further, there exists a constant $C_{V,X}$ depending only on $\{X_\cdot, V_\cdot\}$ such that

$$\sup_{s\in[0,T_\varepsilon]} |\tilde{m}_s - X_1| \le C_{V,X}/\sqrt{T_\varepsilon}, \quad P-a.s.$$

Proof of Lemma 2.1. The statement that $||X||_1 < \infty$ is part of the statement concerning existence of solutions to the SDE (1.1). Next, we prove that

$$\limsup_{\varepsilon\to 0} \sup_{t\le 1} |M_t^\varepsilon| < \infty. \tag{2.4}$$

Indeed, fix constants $C = C(||X||_1)$ and ε_0 such that $h(y) - h(x) + \sup_{\varepsilon\le\varepsilon_0} \varepsilon b(x) < 0$ for all $x \ge C$ and $|y| \le ||X||_1$ (this is always possible because b, h are Lipschitz and $h' > h_0$). Define the stopping times $\tau_0 = 0,\ \theta_0 = 0$ and

$$\tau_i = \inf\{t > \theta_{i-1} : M_t^\varepsilon = C\}, \qquad \theta_i = \inf\{t > \tau_i : M_t^\varepsilon = C + 1\}.$$

By definition, $M_t^\varepsilon \le C + 1$ for $t \in [\tau_i, \theta_i]$, while for $t \in [\theta_i, \tau_{i+1}]$ it holds that for all $\varepsilon < \varepsilon_0$,

$$M_t^\varepsilon = M_{\theta_i}^\varepsilon + \int_{\theta_i}^t \left[b(M_s^\varepsilon) + \frac{1}{\varepsilon}(h(X_s) - h(M_s^\varepsilon)) \right] ds + V_t - V_{\theta_i} \le C + 1 + 2||V||_1.$$

We conclude that $\sup_{t\le 1} M_t^\varepsilon \le C+1+2||V||_1 < \infty$ for all $\varepsilon < \varepsilon_0$. A similar argument shows that $\inf_{t\le 1} M_t^\varepsilon \ge -(C + 1 + 2\|V\|_1)$.

To see the stated convergence of $\tilde{m}_s$ to X_1, recall that X_t and V_t are almost surely Hölder(η) continuous for all $\eta < 1/2$. Fix $t_0 = 1 - 2\varepsilon T_\varepsilon$, $t_1 = 1 - \varepsilon T_\varepsilon$, $\delta_\varepsilon = 1/\sqrt{T_\varepsilon}$, and write $Y_t = M_t^\varepsilon - X_1$. With these notations,

$$Y_t = Y_{t_0} + \int_{t_0}^t \left[b(M_s^\varepsilon) + \frac{h(X_s) - h(X_1)}{\varepsilon} \right] ds + \frac{1}{\varepsilon}\int_{t_0}^t (h(X_1) - h(M_s^\varepsilon)) ds + (V_t - V_{t_0}).$$

By the first part of the lemma, it holds that $|Y_{t_0}| \le C$. We first show that for some $\tau \in (t_0, t_1)$ it holds that $|Y_\tau| \le \delta_\varepsilon$. Indeed, assume without loss of generality that $Y_{t_0} > \delta_\varepsilon$. Then, by the Hölder property of $X_\cdot$ and $V_\cdot$, it holds that

$$\sup_{t\in(t_0,t_1)} |V_t - V_{t_0}| \le C(\varepsilon T_\varepsilon)^\eta, \qquad \sup_{t\in(t_0,t_1)} |X_t - X_{t_0}| \le C(\varepsilon T_\varepsilon)^\eta.$$

Hence, if a τ as defined above does not exist, then necessarily, using the Lipschitz continuity of h,

$$-C \le C_1 \varepsilon T_\varepsilon \left(1 + \frac{(\varepsilon T_\varepsilon)^\eta}{\varepsilon} \right) - h_0 \delta_\varepsilon T_\varepsilon + C_1(\varepsilon T_\varepsilon)^\eta,$$

which is clearly impossible unless $\varepsilon \ge \varepsilon_0$ for some $\varepsilon_0 > 0$. Now, for $\tau < t \le 1$ we claim that it is impossible that $Y_t > 2\delta_\varepsilon$. Indeed, let $\theta' = \inf\{\tau < t \le 1 : Y_t = 2\delta_\varepsilon\}$. Repeating the argument above, we now obtain that if such a θ' exists, it must hold that for some $\theta < 2\varepsilon T_\varepsilon$,

$$\delta_\varepsilon \le C_1\theta + C_1\frac{\theta^{\eta+1}}{\varepsilon} + C_1\theta^\eta - \frac{h_0\delta_\varepsilon\theta}{\varepsilon},$$

which again is impossible, unless $\varepsilon \ge \varepsilon_0'$, for some $\varepsilon_0' > 0$. The case of $Y_t < -2\delta_\varepsilon$ for some $t > t_0$ being handled similarly, the conclusion follows. ∎

3. Auxiliary lemmas and proof of Theorem 1.1. Let us set $J_\varepsilon(x) := \rho_1^\varepsilon(x)e^{F(x,\tilde{m}_0)/\varepsilon}$ and

$$\bar{L}_\varepsilon(x,t) = \exp\left(\int_0^t \left(g_1(\tilde{Z}_s^{\varepsilon,x},\tilde{m}_s) + \frac{1}{\varepsilon}g_2(\tilde{Z}_s^{\varepsilon,x},\tilde{m}_s)\right)ds\right) \tag{3.1}$$

and

$$L_\varepsilon(x,t) = \exp(I_\varepsilon(\tilde{Z}_t^{\varepsilon,x},0))\bar{L}_\varepsilon(x,t). \tag{3.2}$$

Although both $\bar{L}_\varepsilon(x,t)$ and $L_\varepsilon(x,t)$ depend on the path $\tilde{m}_\cdot$, we omit this dependence when no confusion occurs, while $L_\varepsilon(x,t,m_\cdot)$ will denote the quantity $L_\varepsilon(x,t)$ with $\tilde{m}_\cdot$ replaced by $m_\cdot$, and similarly for $\bar{L}_\varepsilon$.

The following are the auxiliary lemmas alluded to above. The proof of the first, Lemma 3.1, is standard, combining large deviations estimates for solutions of SDEs (see, e.g., [2, Theorem 2.13, p. 91]) with Varadhan's lemma (see, e.g., [3, Theorem 4.3.1, p. 137]), and is omitted.

LEMMA 3.1 (finite horizon LDP). *Fix $T < \infty$ and a compact $K \subset\subset \mathbb{R}$. Define*

$$I_T(x,z) := \sup_{\phi\in H^1:\phi_0=x,\phi_T=z} \int_0^T g_2(\phi_s,X_1)ds - \frac{1}{2}\int_0^T \left[\dot{\phi}_s + h(\phi_s) - X_1h'(\phi_s)\right]^2 ds.$$

Then, uniformly in $x,z\in K$, P-almost surely,

$$\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0}\left|\varepsilon\log\mathbb{E}\left[\bar{L}_\varepsilon(x,T)\mathbf{1}_{\{|\tilde{Z}_T^{\varepsilon,x}-z|<\delta\}}\right] - I_T(x,z)\right| = 0.$$

It is worth noting the following simpler representation of $I_T(x,z)$:

$$\begin{aligned} I_T(x,z) = \sup_{\phi\in H^1:\phi_0=x,\phi_T=z}\Bigg[&X_1(h(z)-h(x)) - h(X_1)(z-x) \\ &- \frac{1}{2}\int_0^T \left[\dot{\phi}_s - (h(X_1)-h(\phi_s))\right]^2 ds\Bigg]. \end{aligned} \tag{3.3}$$

From this representation, the following is immediate:

$$I_T(X_1,X_1) = 0, \tag{3.4}$$

and, with $V_T(x) := I_T(x,X_1)$, it holds that

$$V_T(x) \to_{T\to\infty} -X_1h(x) + h(X_1)x. \tag{3.5}$$

This, and standard large deviations considerations, give the next result.

COROLLARY 3.2. *Fix a compact set $K \subset\subset \mathbb{R}$. Then uniformly in $x,z\in K$, P-almost surely,*

$$\begin{aligned} &\limsup_{T\to\infty}\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0}\Big|\varepsilon\log\mathbb{E}\left[\bar{L}_\varepsilon(x,T)\mathbf{1}_{\{|\tilde{Z}_T^{\varepsilon,x}-z|<\delta/2\}}\mathbf{1}_{\{|\tilde{Z}_{T/2}^{\varepsilon,x}-X_1|<\delta/2\}}\right] \\ &\qquad - h(X_1)x + h(x)X_1 - I_{T/2}(X_1,z)\Big| \\ &= \limsup_{T\to\infty}\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0}\left|\varepsilon\log\mathbb{E}\left[\bar{L}_\varepsilon(x,T)\mathbf{1}_{\{|\tilde{Z}_T^{\varepsilon,x}-z|<\delta/2\}}\mathbf{1}_{\{|\tilde{Z}_{T/2}^{\varepsilon,x}-X_1|<\delta/2\}}\right] - I_T(x,z)\right| \\ &= \limsup_{T\to\infty}\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0}\left|\varepsilon\log\mathbb{E}\left[\bar{L}_\varepsilon(x,T)\mathbf{1}_{\{|\tilde{Z}_T^{\varepsilon,x}-z|<\delta/2\}}\right] - I_T(x,z)\right| = 0. \end{aligned}$$

The key to the proof of Theorem 1.1 is a localization procedure that allows one to restrict attention to compact (in time and space) subsets. A first coarse step in that direction is provided by the next two lemmas.

LEMMA 3.3 (coarse localization 1). *For each* $\eta > 0$ *there exist constants* $M_1 = M_1(|||\tilde{m}|||, \eta, |X_1|)$ *and* $\varepsilon_{00} = \varepsilon_{00}(|||\tilde{m}|||, \eta, |X_1|)$ *such that, for all* $\varepsilon < \varepsilon_{00}$,

$$\int \rho_1^\varepsilon(x)\mathbf{1}_{\{|x|>M_1/\sqrt{\varepsilon}\}}dx \le e^{-\eta/\varepsilon} \inf_{|x|<1} \rho_1^\varepsilon(x) \le e^{-\eta/\varepsilon} \int \rho_1^\varepsilon(x)\mathbf{1}_{\{|x|\le M_1/\sqrt{\varepsilon}\}}dx, \quad P-a.s. \tag{3.6}$$

LEMMA 3.4 (coarse localization 2). *For each* $\eta > 0$ *and* M_1, ε_{00} *as in Lemma* 3.3, *there exist constants* $M_i = M_i(|||\tilde{m}|||, \eta, |X_1|)$, $i = 2, 3$, *with* $M_3 \le M_2$ *and* $\varepsilon_0 = \varepsilon_0(|||\tilde{m}|||, \eta, |X_1|) < \varepsilon_{00}$, *such that for all* $\varepsilon < \varepsilon_0$, *uniformly in* $|x| \le M_1/\sqrt{\varepsilon}$,

$$J_\varepsilon(x) \le 2\mathbb{E}\left[L_\varepsilon\left(x, \frac{1}{\varepsilon}\right)\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,x}|| \le M_3/\varepsilon\}}\right], \tag{3.7}$$

and uniformly in $|z| \le M_3/\varepsilon$, $T < 1/\varepsilon$,

$$\mathbb{E}\left[L_\varepsilon\left(z, \frac{1}{\varepsilon} - T, \theta^T\tilde{m}\right)\right] \le 2\mathbb{E}\left[L_\varepsilon\left(z, \frac{1}{\varepsilon} - T, \theta^T\tilde{m}\right)\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,z}||_{1/\varepsilon-T} \le M_2/\varepsilon\}}\right]. \tag{3.8}$$

The following comparison lemma is also needed.

LEMMA 3.5. *There exists a function* $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$, *depending on* $|||\tilde{m}|||, |X_1|, \eta$ *only, with* $g(\delta) \to_{\delta\to 0} 0$, *and an* $\varepsilon_1 = \varepsilon_1(|||\tilde{m}|||, X_1, \eta) < \varepsilon_0$ *such that for all* $\varepsilon < \varepsilon_1$, $t \in [1/2\varepsilon, 1/\varepsilon]$, *and* $|x|, |y| \le M_3/\varepsilon$, $|x - y| < \delta$,

$$\varepsilon \log\left(\frac{\mathbb{E}(L_\varepsilon(x, t, \theta^{1/\varepsilon-t}\tilde{m})\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,x}||_t \le M_2/\varepsilon\}})}{\mathbb{E}(L_\varepsilon(y, t, \theta^{1/\varepsilon-t}\tilde{m})\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,y}||_t \le M_2/\varepsilon\}})}\right) \le g(\delta), \tag{3.9}$$

and there exists a constant $C_1(|||\tilde{m}|||, X_1, \eta)$ *such that, for all* $\varepsilon < \varepsilon_1$,

$$\sup_{t\in[1/2\varepsilon,1/\varepsilon]} \varepsilon \left|\log\left(\frac{\mathbb{E}\left[L_\varepsilon(x, t, \theta^{1/\varepsilon-t}\tilde{m})\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,x}||_t \le M_2/\varepsilon\}}\right]}{\mathbb{E}\left[L_\varepsilon(X_1, t, \theta^{1/\varepsilon-t}\tilde{m})\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,X_1}||_t \le M_2/\varepsilon\}}\right]}\right)\right| \le C_1(1 + |x|). \tag{3.10}$$

The last step needed in order to carry out the localization procedure is the following.

LEMMA 3.6 (localization). *Fix a sequence* T_ε *as in Lemma* 2.1. *Then there exist constants* $C_i = C_i(|||\tilde{m}|||, M_1, M_2, M_3, X_1) > 0$, $i \ge 2$, *and* $\varepsilon_2 = \varepsilon_2(|||\tilde{m}|||, M_1, M_2, M_3, X_1) < \varepsilon_1$ *such that, for all* $\varepsilon < \varepsilon_2$, $|x| \le M_1/\sqrt{\varepsilon}$, $|z| \le M_3/\varepsilon$, $\delta < 1$, *and* $1 \le T \le T_\varepsilon$,

$$\begin{aligned}&\mathbb{E}\left[\bar{L}_\varepsilon(x, T)\mathbf{1}_{\{|\tilde{Z}_T^{\varepsilon,x} - z| < \delta\}}\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,x}||_T \le M_3/\varepsilon\}}\right] \\ &\quad \le \exp\left(\frac{C_2}{\varepsilon} - \frac{C_3(|z| - |x|)_+^2}{\varepsilon} + \frac{C_4(|x| + |z|)}{\varepsilon}\right),\end{aligned} \tag{3.11}$$

and, uniformly for $|z - X_1| < 1, |x - X_1| < 1$,

$$\mathbb{E}\left[\bar{L}_\varepsilon(x, T)\mathbf{1}_{\{|\tilde{Z}_T^{\varepsilon,x} - z| < \delta\}}\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,x}||_T \le M_3/\varepsilon\}}\right] \ge \exp\left(-\frac{C_2}{\varepsilon}\right). \tag{3.12}$$

We may now proceed to the proof of Theorem 1.1, as a consequence of the above lemmas. Fix an $\eta > 0$ as in Lemma 3.3, and for $\delta > 0$, $T > 0$ to be chosen below, with $T < T_\varepsilon$, T_ε as in Lemma 2.1, define

$$\begin{aligned}\tilde{J}_\varepsilon(x) &= \mathbb{E}\left(L_\varepsilon\left(x, \frac{1}{\varepsilon}\right)\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,x}||_T \le M_3/\varepsilon, ||\tilde{Z}^{\varepsilon,x}|| \le M_2/\varepsilon\}}\right)\\ &= \sum_{i=-M_3/\varepsilon\delta}^{M_3/\varepsilon\delta} \mathbb{E}\left(L_\varepsilon\left(x, \frac{1}{\varepsilon}\right)\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,x}|| \le M_2/\varepsilon, ||\tilde{Z}^{\varepsilon,x}||_T \le M_3/\varepsilon, |\tilde{Z}^{\varepsilon,x}_T - i\delta| \le \delta/2\}}\right)\\ &=: \sum_{i=-M_3/\varepsilon\delta}^{M_3/\varepsilon\delta} \tilde{J}_{\varepsilon,T}(x, i\delta). \qquad (3.13)\end{aligned}$$

Set $\mathcal{Z}^{\varepsilon,x}_T = \sigma(\tilde{Z}^{\varepsilon,x}_t, t \le T)$. Using the Markov property, and the fact that $M_3 < M_2$, one may write, for $|z| < M_3/\varepsilon$,

$$\begin{aligned}\tilde{J}_{\varepsilon,T}(x,z) = \mathbb{E}\Bigg[&\bar{L}_\varepsilon(x,T)\mathbf{1}_{\{|\tilde{Z}^{\varepsilon,x}_T - z| \le \delta/2\}}\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,x}||_T \le M_3/\varepsilon\}}\\ &\cdot \mathbb{E}\left(L_\varepsilon\left(\tilde{Z}^{\varepsilon,x}_T, \frac{1}{\varepsilon} - T, \theta^T\tilde{m}\right)\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,x}|| \le M_2/\varepsilon\}} \mid \mathcal{Z}^{\varepsilon,x}_T\right)\Bigg]. \qquad (3.14)\end{aligned}$$

Applying (3.9) and the Markov property, it follows that on the event $\{|\tilde{Z}^{\varepsilon,x}_T - z| \le \delta/2\} \cap \{||\tilde{Z}^{\varepsilon,x}||_T \le M_3/\varepsilon\}$, one has for $\varepsilon < \varepsilon_1$, and $|x| \le M_1/\sqrt{\varepsilon}$, $|z| \le M_3/\varepsilon$,

$$\begin{aligned}&\mathbb{E}\left(L_\varepsilon\left(\tilde{Z}^{\varepsilon,x}_T, \frac{1}{\varepsilon} - T, \theta^T\tilde{m}\right)\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,x}|| \le M_2/\varepsilon\}} \mid \mathcal{Z}^{\varepsilon,x}_T\right)\\ &\quad= \mathbb{E}\left(L_\varepsilon\left(\tilde{Z}^{\varepsilon,x}_T, \frac{1}{\varepsilon} - T, \theta^T\tilde{m}\right)\mathbf{1}_{\{\sup_{T \le t \le 1/\varepsilon}|\tilde{Z}^{\varepsilon,x}_t| \le M_2/\varepsilon\}} \mid \mathcal{Z}^{\varepsilon,x}_T\right)\\ &\quad\le e^{g(\delta)/\varepsilon}\mathbb{E}\left(L_\varepsilon\left(z, \frac{1}{\varepsilon} - T, \theta^T\tilde{m}\right)\mathbf{1}_{\{\sup_{0 \le 1/\varepsilon - T}|\tilde{Z}^{\varepsilon,z}_t| \le M_2/\varepsilon\}}\right)\\ &\quad= e^{g(\delta)/\varepsilon}\mathbb{E}\left(L_\varepsilon\left(z, \frac{1}{\varepsilon} - T, \theta^T\tilde{m}\right)\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,z}||_{1/\varepsilon - T} \le M_2/\varepsilon\}}\right).\end{aligned}$$

Substituting in (3.14), one concludes that for all $\varepsilon < \varepsilon_1$ and $|x| \le M_1/\sqrt{\varepsilon}$, $|z| \le M_3/\varepsilon$,

$$\begin{aligned}\tilde{J}_{\varepsilon,T}(x,z)e^{-g(\delta)/\varepsilon} &\le \mathbb{E}\left[\bar{L}_\varepsilon(x,T)\mathbf{1}_{\{|\tilde{Z}^{\varepsilon,x}_T - z| \le \delta/2\}}\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,x}||_T \le M_3/\varepsilon\}}\right]\\ &\qquad\cdot \mathbb{E}\left[L_\varepsilon\left(z, \frac{1}{\varepsilon} - T, \theta^T\tilde{m}\right)\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,z}||_{1/\varepsilon - T} \le M_2/\varepsilon\}}\right]\\ &:= \hat{J}_{\varepsilon,T}(x,z) \le \tilde{J}_{\varepsilon,T}(x,z)e^{g(\delta)/\varepsilon}. \qquad (3.15)\end{aligned}$$

Next, using (3.10) in the first inequality and Lemma 3.6 in the second, it follows that for all $\varepsilon < \varepsilon_2$; $T \in (1, T_\varepsilon)$, T_ε as in Lemma 2.1; and some constants C_i independent of T, ε,

$$\begin{aligned}\hat{J}_{\varepsilon,T}(x,z) &\leq \mathbb{E}\left[\bar{L}_\varepsilon(x,T)\mathbf{1}_{\{|\tilde{Z}^{\varepsilon,x}_T - z|\leq\delta/2\}}\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,x}||_T\leq M_3/\varepsilon\}}\right] \\ &\quad\cdot\mathbb{E}\left[L_\varepsilon\left(X_1,\frac{1}{\varepsilon}-T,\theta^T\tilde{m}\right)\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,X_1}||_{1/\varepsilon-T}\leq M_2/\varepsilon\}}\right]e^{C_1(|z|+1)/\varepsilon} \\ &\leq \exp\left(\frac{C_2}{\varepsilon}-\frac{C_3(|z|-|x|)_+^2}{\varepsilon}+\frac{C_5(|x|+|z|)}{\varepsilon}\right) \\ &\quad\cdot\mathbb{E}\left[L_\varepsilon\left(X_1,\frac{1}{\varepsilon}-T,\theta^T\tilde{m}\right)\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,X_1}||_{1/\varepsilon-T}\leq M_2/\varepsilon\}}\right].\end{aligned} \tag{3.16}$$

Similarly, for all $\varepsilon<\varepsilon_2$ and $|x-X_1|\leq 1$, $|z-X_1|\leq 1$,

$$\hat{J}_{\varepsilon,T}(x,z)\geq\exp\left(-\frac{C_2}{\varepsilon}\right)\mathbb{E}\left[L_\varepsilon\left(X_1,\frac{1}{\varepsilon}-T,\theta^T\tilde{m}\right)\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,X_1}||_{1/\varepsilon-T}\leq M_2/\varepsilon\}}\right]. \tag{3.17}$$

We next note that, due to the quadratic growth of $F(x,X_1)$ as $|x|\to\infty$, there exists a compact set $\mathcal{K}_1$, depending on $|||m|||, X_1, \eta, C_i$ only, such that

$$\begin{aligned}&\sup_{(x,z)\in(\mathcal{K}_1\times\mathcal{K}_1)^c}\frac{C_2}{\varepsilon}-\frac{C_3(|z|-|x|)_+^2}{\varepsilon}+\frac{C_5(|x|+|z|)}{\varepsilon}-\frac{F(x,X_1)}{\varepsilon} \\ &\qquad\leq-\frac{F(X_1,X_1)}{\varepsilon}-\frac{C_2}{\varepsilon}.\end{aligned} \tag{3.18}$$

Thus, using (3.16) in the first inequality, (3.18) in the second, and (3.17) in the third,

$$\begin{aligned}&\sup_{\substack{|x|\leq M_1/\sqrt{\varepsilon},|z|\leq M_3/\varepsilon,\\(x,z)\in(\mathcal{K}_1\times\mathcal{K}_1)^c}}\hat{J}_{\varepsilon,T}(x,z)e^{-F(x,X_1)/\varepsilon} \\ &\leq\mathbb{E}\left[L_\varepsilon\left(X_1,\frac{1}{\varepsilon}-T,\theta^T\tilde{m}\right)\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,X_1}||_{1/\varepsilon-T}\leq M_2/\varepsilon\}}\right] \\ &\quad\cdot\sup_{\substack{|x|\leq M_1/\sqrt{\varepsilon},|z|\leq M_3/\varepsilon,\\(x,z)\in(\mathcal{K}_1\times\mathcal{K}_1)^c}}\exp\left(\frac{C_2}{\varepsilon}-\frac{C_3(|z|-|x|)_+^2}{\varepsilon}+\frac{C_5(|x|+|z|)}{\varepsilon}-\frac{F(x,X_1)}{\varepsilon}\right) \\ &\leq\mathbb{E}\left[L_\varepsilon\left(X_1,\frac{1}{\varepsilon}-T,\theta^T\tilde{m}\right)\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,X_1}||_{1/\varepsilon-T}\leq M_2/\varepsilon\}}\right]\exp\left(-\frac{C_2}{\varepsilon}-\frac{F(X_1,X_1)}{\varepsilon}\right) \\ &\leq\hat{J}_{\varepsilon,T}(X_1,X_1)e^{-F(X_1,X_1)/\varepsilon}.\end{aligned} \tag{3.19}$$

It follows by substituting (3.19) into (3.15) that for all ε small enough and any $T\in(0,T_\varepsilon)$,

$$\sup_{|x|\leq M_1/\sqrt{\varepsilon},|z|\leq M_3/\varepsilon}\tilde{J}_{\varepsilon,T}(x,z)e^{-F(x,X_1)/\varepsilon}\leq e^{2g(\delta)/\varepsilon}\sup_{x\in\mathcal{K}_1,z\in\mathcal{K}_1}\tilde{J}_{\varepsilon,T}(x,z)e^{-F(x,X_1)/\varepsilon}. \tag{3.20}$$

We may, by enlarging $\mathcal{K}_1$ if necessary, assume also that $[-1,1]\subset\mathcal{K}_1$. With η and $\mathcal{K}_1$ as above, next choose T large enough, δ small enough (with $g(\delta)<\eta/8$), and $\varepsilon_3(\delta,T,\eta,|||\tilde{m}|||,|||\tilde{m}_X|||,X_1)<\varepsilon_2$ such that, for all $\varepsilon<\varepsilon_3$, the following hold:

- The errors in the expression in Corollary 3.2 and in (3.5) are each bounded above by $\eta/8$, uniformly in $x,z\in\mathcal{K}_1$.
- $|F(x,\tilde{m}_0)-F(x,X_1)|\leq\eta/8$, uniformly in $x\in\mathcal{K}_1$ (which is possible by Lemma 2.1 and the uniform continuity of $F(x,\cdot)$ for x in compacts).

- $\varepsilon \log 2 \le \eta/8$.
- $\varepsilon \log(2M_3/\varepsilon\delta) \le \eta/8$.

Hence, for $x \in \mathcal{K}_1$ and all $\varepsilon < \varepsilon_3$,

$$\begin{aligned}
\varepsilon \log \rho_1^\varepsilon(x) &= -F(x,\tilde m_0) + \varepsilon \log \mathbb{E}\left(L_\varepsilon\left(x,\frac{1}{\varepsilon}\right)\right) \quad \texttt{by (2.2)} \\
&\le -F(x,\tilde m_0) + \varepsilon \log \mathbb{E}\left(L_\varepsilon\left(x,\frac{1}{\varepsilon}\right)\mathbf{1}_{\{||\tilde Z^{\varepsilon,x}|| \le M_3/\varepsilon\}}\right) + \varepsilon \log 2 \quad \texttt{by (3.7)} \\
&\le -F(x,X_1) + \varepsilon \log \mathbb{E}\left(L_\varepsilon\left(x,\frac{1}{\varepsilon}\right)\mathbf{1}_{\{||\tilde Z^{\varepsilon,x}|| \le M_3/\varepsilon\}}\right) + \frac{\eta}{4} \quad \texttt{by } \varepsilon < \varepsilon_3 \\
&\le -F(x,X_1) + \varepsilon \log \tilde J_\varepsilon(x) + \frac{\eta}{4} \quad \texttt{by (3.13)} \\
&\le -F(x,X_1) + \varepsilon \log \sup_{z\in\mathcal{K}_1} \tilde J_{\varepsilon,T}(x,z) + \frac{\eta}{2} \quad \texttt{by (3.13) and (3.20)} \\
&\le -F(x,X_1) + \sup_{z\in\mathcal{K}_1}\Bigg[\varepsilon \log \mathbb{E}(\bar L_\varepsilon(x,T)\mathbf{1}_{\{|\tilde Z_T^{\varepsilon,x} - z| \le \delta/2\}}) \\
&\quad + \varepsilon \log \mathbb{E}\left(L_\varepsilon\left(z,\frac{1}{\varepsilon}-T,\theta^T\tilde m\right)\mathbf{1}_{\{||\tilde Z^{\varepsilon,z}||_{1/\varepsilon - T} \le M_2/\varepsilon\}}\right)\Bigg] + \frac{5\eta}{8} \quad \texttt{by (3.15)} \\
&\le -F(x,X_1) + \sup_{z\in\mathcal{K}_1}\Bigg[h(X_1)x - h(x)X_1 + I_{T/2}(X_1,z) \\
&\qquad + \varepsilon \log \mathbb{E}\left(L_\varepsilon\left(z,\frac{1}{\varepsilon}-T,\theta^T\tilde m\right)\mathbf{1}_{\{||\tilde Z^{\varepsilon,z}||_{1/\varepsilon - T} \le M_2/\varepsilon\}}\right)\Bigg] \\
&\quad + \frac{7\eta}{8} \quad \texttt{by Corollary 3.2} \\
&\le h(X_1)x - h(x)X_1 - F(x,X_1) + \eta \\
&\quad + \sup_{z\in\mathcal{K}_1}\left[I_{T/2}(X_1,z) + \varepsilon \log \mathbb{E}\left(L_\varepsilon\left(z,\frac{1}{\varepsilon}-T,\theta^T\tilde m\right)\right)\right] \\
&=: -\bar{\mathcal{J}}(x,X_1) + \eta + \mathcal{C}_\varepsilon, \qquad (3.21)
\end{aligned}$$

where $\mathcal{C}_\varepsilon$ depends only on ε, and not on x, and is defined by the last equality. Similarly, for all $x \in \mathcal{K}_1$ and all $\varepsilon < \varepsilon_3$,

$$\begin{aligned}
\varepsilon \log \rho_1^\varepsilon(x) &= -F(x,\tilde m_0) + \varepsilon \log \mathbb{E}\left(L_\varepsilon\left(x,\frac{1}{\varepsilon}\right)\right) \quad \texttt{by (2.2)} \\
&\ge -F(x,\tilde m_0) + \varepsilon \log \mathbb{E}\left(L_\varepsilon\left(x,\frac{1}{\varepsilon}\right)\mathbf{1}_{\{||\tilde Z^{\varepsilon,x}||_T \le M_3/\varepsilon, ||\tilde Z^{\varepsilon,x}|| \le M_2/\varepsilon\}}\right) \\
&\ge -F(x,X_1) + \varepsilon \log \mathbb{E}\left(L_\varepsilon\left(x,\frac{1}{\varepsilon}\right)\mathbf{1}_{\{||\tilde Z^{\varepsilon,x}||_T \le M_3/\varepsilon, ||\tilde Z^{\varepsilon,x}|| \le M_2/\varepsilon\}}\right) \\
&\quad - \frac{\eta}{4} \quad \texttt{by } \varepsilon < \varepsilon_3 \\
&= -F(x,X_1) + \varepsilon \log \tilde J_\varepsilon(x) - \frac{\eta}{4} \quad \texttt{by definition} \\
&\ge -F(x,X_1) + \varepsilon \log \sup_{z\in\mathcal{K}_1} \tilde J_{\varepsilon,T}(x,z) - \frac{\eta}{4} \quad \texttt{by definition}
\end{aligned}$$

$$
\begin{aligned}
&\geq -F(x,X_1) + \sup_{z\in\mathcal{K}_1}\left[\varepsilon\log\mathbb{E}(\bar{L}_\varepsilon(x,T)\mathbf{1}_{\{|\tilde{Z}^{\varepsilon,x}_T - z|\leq\delta/2\}})\right.\\
&\qquad\left. + \varepsilon\log\mathbb{E}\left(L_\varepsilon\left(z,\frac{1}{\varepsilon}-T,\theta^T\tilde{m}\right)\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,z}||_{1/\varepsilon-T}\leq M_2/\varepsilon\}}\right)\right] - \frac{5\eta}{8} \quad \texttt{by (3.15)}\\
&\geq -F(x,X_1) + \sup_{z\in\mathcal{K}_1}\left[h(X_1)x - h(x)X_1 + I_{T/2}(X_1,z)\right.\\
&\qquad\qquad\left. + \varepsilon\log\mathbb{E}\left(L_\varepsilon\left(z,\frac{1}{\varepsilon}-T,\theta^T\tilde{m}\right)\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,z}||_{1/\varepsilon-T}\leq M_2/\varepsilon\}}\right)\right]\\
&\qquad - \frac{7\eta}{8} \quad \texttt{by Corollary(3.2)}\\
&\geq h(X_1)x - h(x)X_1 - F(x,X_1) - \eta\\
&\qquad + \sup_{z\in\mathcal{K}_1}\left[I_{T/2}(X_1,z) + \varepsilon\log\mathbb{E}\left(L_\varepsilon\left(z,\frac{1}{\varepsilon}-T,\theta^T\tilde{m}\right)\right)\right]\\
&= -\bar{\mathcal{J}}(x,X_1) - \eta + \mathcal{C}_\varepsilon, \qquad (3.22)
\end{aligned}
$$

where $\mathcal{C}_\varepsilon$ is the same as in (3.21). Since $\bar{\mathcal{J}}(\cdot,X_1)$ is continuous and $\bar{\mathcal{J}}(X_1,X_1)=0$, it follows from (3.22) that

$$
\liminf_{\varepsilon\to 0}\varepsilon\log\int_{\mathbb{R}}\rho_1^\varepsilon(x)dx - \mathcal{C}_\varepsilon \geq -2\eta. \qquad (3.23)
$$

On the other hand, for $\varepsilon < \varepsilon_3$,

$$
\begin{aligned}
\varepsilon\log\int_{\mathbb{R}}\rho_1^\varepsilon(x)dx &\leq \varepsilon\log(1+e^{-\eta/\varepsilon}) + \varepsilon\log\int_{|x|\leq M_1/\sqrt{\varepsilon}}\rho_1^\varepsilon(x)dx \quad \texttt{by Lemma 3.3}\\
&\leq \varepsilon\log(1+e^{-\eta/\varepsilon}) + \varepsilon\log 2 + \varepsilon\log\left(\frac{2M_3}{\varepsilon\delta}\right)\\
&\qquad + \sup_{|x|\leq M_1/\sqrt{\varepsilon},|z|\leq M_3/\varepsilon}\varepsilon\log\left(\tilde{J}_{\varepsilon,T}(x,z)e^{-F(x,X_1)/\varepsilon}\right)\\
&\qquad\qquad \texttt{by Lemma 3.4 and (3.13)}\\
&\leq \frac{5\eta}{8} + \sup_{x,z\in\mathcal{K}_1}\varepsilon\log\left(\tilde{J}_{\varepsilon,T}(x,z)e^{-F(x,X_1)/\varepsilon}\right) \quad \texttt{by (3.20)}\\
&\leq \frac{5\eta}{8} + \varepsilon\log\left(\sup_{x\in\mathcal{K}_1}\rho_1^\varepsilon(x)\right)\\
&\leq 2\eta + \mathcal{C}_\varepsilon - \inf_x\bar{\mathcal{J}}(x,X_1) = 2\eta + \mathcal{C}_\varepsilon \qquad (3.24)\\
&\qquad\qquad \texttt{by (3.21) and } \bar{\mathcal{J}}(x,X_1)\geq 0.
\end{aligned}
$$

Consider now an open ball $B(x_0,r)\subset\mathbb{R}$. Then, using (3.24) in the first inequality and (3.22) in the last,

$$
\begin{aligned}
\liminf_{\varepsilon\to 0}\varepsilon\log\int_{B(x_0,r)}q_1^\varepsilon(x)dx &= \liminf_{\varepsilon\to 0}\left[\varepsilon\log\int_{B(x_0,r)}\rho_1^\varepsilon(x)dx - \varepsilon\log\int_{\mathbb{R}}\rho_1^\varepsilon(x)dx\right]\\
&\geq \liminf_{\varepsilon\to 0}\left[\varepsilon\log\int_{B(x_0,r)}\rho_1^\varepsilon(x)dx - \mathcal{C}_\varepsilon - 2\eta\right]\\
&\geq -\bar{\mathcal{J}}(x_0,X_1) - 3\eta.
\end{aligned}
$$

Since η is arbitrary, one deduces that

$$\liminf_{\varepsilon\to 0} \varepsilon \log \int_{B(x_0,r)} q_1^\varepsilon(x)dx \geq -\bar{\mathcal{J}}(x_0, X_1). \tag{3.25}$$

To see the complementary upper bound for the ball $B(x_0, r)$, enlarge $\mathcal{K}_1$ if necessary so that $B(x_0, r) \subset \mathcal{K}_1$ (decreasing ε_3 above as a byproduct). Then, using (3.23) in the first inequality and (3.21) in the last,

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \varepsilon \log \int_{B(x_0,r)} q_1^\varepsilon(x)dx &= \limsup_{\varepsilon\to 0} \left[\varepsilon \log \int_{B(x_0,r)} \rho_1^\varepsilon(x)dx - \varepsilon \log \int_{\mathbb{R}} \rho_1^\varepsilon(x)dx\right] \\
&\leq \limsup_{\varepsilon\to 0} \left[\varepsilon \log \int_{B(x_0,r)} \rho_1^\varepsilon(x)dx - \mathcal{C}_\varepsilon + 2\eta\right] \\
&\leq -\sup_{x\in B(x_0,r)} \bar{\mathcal{J}}(x, X_1) + 3\eta + \limsup_{\varepsilon\to 0} \varepsilon \log(2r).
\end{aligned}$$

Since η is arbitrary, the above, (3.25), and the continuity of $\bar{\mathcal{J}}(\cdot, X_1)$ imply that

$$\lim_{r\to 0} \limsup_{\varepsilon\to 0} \varepsilon \log \int_{B(x_0,r)} q_1^\varepsilon(x)dx = \lim_{r\to 0} \liminf_{\varepsilon\to 0} \varepsilon \log \int_{B(x_0,r)} q_1^\varepsilon(x)dx = \bar{\mathcal{J}}(x_0, X_1).$$

Next, [3, Theorem 4.1.11], the above, Remark 2 following Theorem 1.1, and the continuity of $\bar{\mathcal{J}}(\cdot, X_1)$ imply that the weak LDP holds for the sequence of (random) measures $\mu_1^\varepsilon(dx) = q_1^\varepsilon(x)dx$ on $\mathbb{R}$. To prove the full large deviations principle, it remains, by [3, Lemma 1.2.8], to prove the exponential tightness of the sequence μ_1^ε. That is, for each given L we must find a constant C_L such that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \int_{[-L,L]^c} q_1^\varepsilon(x)ds < -L. \tag{3.26}$$

Since the proof of (3.26) uses some estimates from the proof of Lemma 3.3, to avoid repetitions we postpone it to the end of section 4.

Finally, we note that (1.3) is an immediate consequence of the estimates (3.21), (3.22), (3.24), and (3.23). ◻

4. Proofs of auxiliary lemmas. Throughout this section, C denotes a positive constant that depends on $|||\tilde{m}|||, |||\tilde{m}_X|||, C_{V,X}, X$ only, and whose value may change from line to line.

Proof of Lemma 3.3. The right-hand inequality is a trivial consequence of the left-hand one. To prove the latter, we first need an upper bound for the left-hand side of (3.6). A subsequent, easily derived lower bound on the middle term will conclude the proof. Define the function

$$H(x) = \int_0^x h(y)dy. \tag{4.1}$$

We note that

$$I_\varepsilon(\tilde{Z}^{\varepsilon,x}_{1/\varepsilon}, 0) - \frac{F(x, \tilde{m}_0)}{\varepsilon} = \frac{1}{\varepsilon}\left(H(\tilde{Z}^{\varepsilon,x}_{1/\varepsilon}) - H(x)\right) + \log p_0(\tilde{Z}^{\varepsilon,x}_{1/\varepsilon}) + \frac{\tilde{m}_0 h(x)}{\varepsilon}.$$

We first rewrite the $\tilde{Z}_t^{\varepsilon,x}$ equation as

$$\tilde{Z}_t^{\varepsilon,x} = x + \int_0^t [-h(\tilde{Z}_s^{\varepsilon,x}) + g(s, \tilde{Z}_s^{\varepsilon,x})]ds + \sqrt{\varepsilon}\tilde{W}_t,$$

and next deduce from Itô's formula that

$$\begin{aligned} &H(\tilde{Z}_{1/\varepsilon}^{\varepsilon,x}) - H(x) \\ &\quad = \int_0^{1/\varepsilon} \left[- h^2(\tilde{Z}_s^{\varepsilon,x}) + (hg)(s, \tilde{Z}_s^{\varepsilon,x}) + \frac{\varepsilon}{2} h'(\tilde{Z}_s^{\varepsilon,x})\right] ds + \sqrt{\varepsilon} \int_0^{1/\varepsilon} h(\tilde{Z}_s^{\varepsilon,x}) d\tilde{W}_s. \end{aligned}$$

It now follows from (2.2) and the (uniform in m in compacts) linear growth of $g_1(z,m)$ and $g_2(z,m)$ in z that for some C (depending on $|||\tilde{m}|||$ and X only) and all $\varepsilon \le 1$, $\delta > 0$,

$$\begin{aligned} &\rho_1^\varepsilon(x) \le \exp\left[\frac{C}{\varepsilon^2} + \frac{\tilde{m}_0 h(x)}{\varepsilon}\right] \left(\mathbb{E}\left[p_0(\tilde{Z}_{1/\varepsilon}^{\varepsilon,x})\right]^{\frac{1+\delta}{\delta}}\right)^{\frac{\delta}{1+\delta}} \\ &\times \left(\mathbb{E}\exp\left[\frac{1+\delta}{\sqrt{\varepsilon}} \int_0^{1/\varepsilon} h(\tilde{Z}_s^{\varepsilon,x}) d\tilde{W}_s - \frac{1+\delta}{\varepsilon} \int_0^{1/\varepsilon} h^2(\tilde{Z}_s^{\varepsilon,x}) ds + \frac{C}{\varepsilon} \int_0^{1/\varepsilon} |\tilde{Z}_s^{\varepsilon,x}| ds\right]\right)^{\frac{1}{1+\delta}}. \end{aligned}$$

Now, provided $\delta < 1$, we have $1 + \delta > \frac{(1+\delta)^2}{2}$, and thus there exists a $p > 1$ and a $p' > 0$ such that

$$1 + \delta = \frac{p(1+\delta)^2}{2} + p'.$$

Thus, with $q = p/(p-1)$,

$$\begin{aligned} &\left(\mathbb{E}\exp\left[\frac{1+\delta}{\sqrt{\varepsilon}} \int_0^{1/\varepsilon} h(\tilde{Z}_s^{\varepsilon,x}) d\tilde{W}_s - \frac{(1+\delta)^2 p}{2\varepsilon} \int_0^{1/\varepsilon} h^2(\tilde{Z}_s^{\varepsilon,x}) ds \right.\right. \\ &\qquad\qquad \left.\left. - \frac{p'}{\varepsilon} \int_0^{1/\varepsilon} h^2(\tilde{Z}_s^{\varepsilon,x}) ds + \frac{C}{\varepsilon} \int_0^{1/\varepsilon} |\tilde{Z}_s^{\varepsilon,x}| ds\right]\right)^{\frac{1}{1+\delta}} \\ &\le \left(\mathbb{E}\exp\left[\frac{p(1+\delta)}{\sqrt{\varepsilon}} \int_0^{1/\varepsilon} h(\tilde{Z}_s^{\varepsilon,x}) d\tilde{W}_s - \frac{(1+\delta)^2 p^2}{2\varepsilon} \int_0^{1/\varepsilon} h^2(\tilde{Z}_s^{\varepsilon,x}) ds\right]\right)^{\frac{1}{p(1+\delta)}} \\ &\quad \times \left(\mathbb{E}\exp\left[-\frac{p'q}{2\varepsilon} \int_0^{1/\varepsilon} h^2(\tilde{Z}_s^{\varepsilon,x}) ds + \frac{Cq}{\varepsilon} \int_0^{1/\varepsilon} |\tilde{Z}_s^{\varepsilon,x}| ds\right]\right)^{\frac{1}{q(1+\delta)}} \\ &= \left(\mathbb{E}\exp\left[-\frac{p'q}{2\varepsilon} \int_0^{1/\varepsilon} h^2(\tilde{Z}_s^{\varepsilon,x}) ds + \frac{Cq}{\varepsilon} \int_0^{1/\varepsilon} |\tilde{Z}_s^{\varepsilon,x}| ds\right]\right)^{\frac{1}{q(1+\delta)}}. \end{aligned}$$

Since $h(z)^2 \ge h_0^2 z^2$ (recall that $h(0) = 0!$), there exist $C(\delta) > 0$, $C_1(\delta)$ such that $p'qh(z)^2/2 - Cq|z| \ge C(\delta)z^2 - C_1(\delta)$, and hence, with $C_2(\delta) = C + C_1(\delta)\delta/p(1+\delta)$

(all constants here being positive and depending on $|||\tilde{m}|||, X$ only!),

$$
\begin{aligned}
\rho_1^\varepsilon(x) &\le \exp\left[\frac{C_2(\delta)}{\varepsilon^2} + \frac{\tilde{m}_0 h(x)}{\varepsilon}\right] \left(\mathbb{E}\left[p_0(\tilde{Z}^{\varepsilon,x}_{1/\varepsilon})\right]^{\frac{1+\delta}{\delta}}\right)^{\frac{\delta}{1+\delta}} \\
&\quad\times \left(\mathbb{E}\exp\left[-\frac{C(\delta)}{\varepsilon}\int_0^{1/\varepsilon} |\tilde{Z}^{\varepsilon,x}_s|^2 ds\right]\right)^{\frac{\delta}{q(1+\delta)}} \\
&\le \exp\left[\frac{C_3(\delta)}{\varepsilon^2}\right]\left(\mathbb{E}\exp\left[-\frac{C(\delta)}{\varepsilon}\int_0^{1/\varepsilon} |\tilde{Z}^{\varepsilon,x}_s|^2 ds\right]\right)^{\frac{\delta}{q(1+\delta)}}.
\end{aligned}
\tag{4.2}
$$

It thus remains to estimate the last factor in the above right-hand side. Define $\tau = \inf\{t > 0 : |\tilde{Z}^{\varepsilon,x}_s| < x/2\}$ and fix $\eta > 0$. We claim that for some $\eta > 0$ small enough, it holds that for some $C_\eta > 0$, x_0 and all $|x| \ge x_0$,

$$
\mathbb{P}(\tau < \eta) \le \exp\left[-\frac{C_\eta x^2}{\varepsilon}\right]. \tag{4.3}
$$

Assume (4.3), which will be proved below, and note that on the event $\{\tau \ge \eta\}$ we have that $\inf_{s\in(0,\eta]} |\tilde{Z}^{\varepsilon,x}_s| > x/2$. We deduce from (4.2)

$$
\rho_1^\varepsilon(x) \le \exp\left[\frac{C_3(\delta)}{\varepsilon^2}\right] \times \left(\exp\left[-\frac{C_\eta x^2}{\varepsilon}\right] + \exp\left[-\frac{C(\delta)x^2\eta\delta}{4q(1+\delta)\varepsilon}\right]\right), \tag{4.4}
$$

from which one easily concludes the bound

$$
\rho_1^\varepsilon(x) \le \exp\left[\frac{C_4(\delta)}{\varepsilon^2} - \frac{Cx^2}{\varepsilon}\right] \tag{4.5}
$$

for some constants $C_4(\delta)$ and C depending on $\delta, |||\tilde{m}|||, X$ only.

On the other hand, define the event

$$
A_C = \left\{\sup_{t\in(0,1/\varepsilon)} \sqrt{\varepsilon}|\tilde{W}_t| \le C\right\}.
$$

Then there exists a constant $C_3 > 0$ depending on C such that $\mathbb{P}(A_C) \ge C_3$. Note that on the event A_C, because $h'(\cdot) > 0$ and h, b are Lipschitz, Gronwall's inequality implies that $\sup_{|x|\le 1, s\le 1/\varepsilon} |\tilde{Z}^{\varepsilon,x}_s| \le C'$ for some constant C' depending on $C, \tilde{m}, X$ only. Thus, on the event A_C,

$$
\left| I_\varepsilon(\tilde{Z}^{\varepsilon,x}_{1/\varepsilon}, 0) + \int_0^{1/\varepsilon} g_1(\tilde{Z}^{\varepsilon,x}_s, \tilde{m}_s)ds + \frac{1}{\varepsilon}\int_0^{1/\varepsilon} g_2(\tilde{Z}^{\varepsilon,x}_s, \tilde{m}_s)ds \right| \le \frac{C_4}{\varepsilon^2},
$$

where C_4 depends only on $\tilde{m}, X$ and the constants in Assumptions (A-1)–(A-4). Hence (cf. (2.2)), there exists a constant C_2 (again, depending on the same quantities only) such that, uniformly in $|x| < 1$,

$$
\rho_1^\varepsilon(x) \ge \exp\left[-\frac{C_2}{\varepsilon^2}\right]. \tag{4.6}
$$

Equations (4.6) and (4.5) complete the proof of the lemma, once we prove (4.3).

Toward this end, assume without loss of generality that $x > 0$, and set $\hat{h} = 2\sup_{y>0} h'(y)$. Using the Itô formula, one has

$$\begin{aligned}(4.7)\quad &\tilde{Z}_t^{\varepsilon,x} e^{\hat{h}t} \\ &= x + \int_0^t \left(\hat{h}\tilde{Z}_s^{\varepsilon,x} - h(\tilde{Z}_s^{\varepsilon,x}) + \tilde{m}_s h'(\tilde{Z}_s^{\varepsilon,x}) - \varepsilon b(\tilde{Z}_s^{\varepsilon,x})\right) e^{\hat{h}s} ds + \sqrt{\varepsilon}\int_0^t e^{\hat{h}s} d\tilde{W}_s.\end{aligned}$$

Hence, denoting $C_3 = |||\tilde{m}||| \sup_x h'(x)$, it follows that the event $\{\tau < \eta\}$ is contained in the event

$$\left\{\sup_{t\in(0,\eta)} \left|\sqrt{\varepsilon}\int_0^t e^{\hat{h}s} d\tilde{W}_s\right| \geq x - C_3 \frac{e^{\hat{h}\eta}-1}{\hat{h}} - \frac{xe^{\hat{h}\eta}}{2}\right\} \subset \left\{\sup_{t\in(0,\eta)} \left|\sqrt{\varepsilon}\int_0^t e^{\hat{h}s} d\tilde{W}_s\right| \geq \frac{x}{4}\right\} =: B$$

if one chooses η small enough and x large enough. We have that

$$\mathbb{P}(B) \leq 4\exp\left(-\frac{Cx^2}{\varepsilon}\right)$$

for some constant C, which completes the proof of (4.3). □

Proof of Lemma 3.4. We prove only (3.7), the proof of (3.8) being similar. All we need to show is that for all $\varepsilon \leq \varepsilon_0$, $|x| \leq M_1/\sqrt{\varepsilon}$, and some M_2,

$$(4.8)\qquad \mathbb{E}\left[L_\varepsilon\left(x, \frac{1}{\varepsilon}\right) \mathbf{1}_{\{\|\tilde{Z}^{\varepsilon,x}\| > M_2/\varepsilon\}}\right] \leq \mathbb{E}\left[L_\varepsilon\left(x, \frac{1}{\varepsilon}\right)\mathbf{1}_{\{\|\tilde{Z}^{\varepsilon,x}\| \leq M_2/\varepsilon\}}\right].$$

We first bound the left-hand side of (4.8) for $\varepsilon \leq 1$. Recall the function H introduced in (4.1), and apply Itô's formula to develop $H(\tilde{Z}_t^{\varepsilon,x})$ between $t = 0$ and $t = 1/\varepsilon$, obtaining

$$\begin{aligned}\log L_\varepsilon\left(x, \frac{1}{\varepsilon}\right) - \frac{H(x)}{\varepsilon} &= -\frac{1}{2\varepsilon}\int_0^{1/\varepsilon} |h(\tilde{Z}_t^{\varepsilon,x}) - h(\tilde{m}_t)|^2 dt - \frac{1}{2\varepsilon}\int_0^{1/\varepsilon} |h(\tilde{Z}_t^{\varepsilon,x})|^2 dt \\ &\quad + \frac{1}{\sqrt{\varepsilon}}\int_0^{1/\varepsilon} h(\tilde{Z}_t^{\varepsilon,x}) d\tilde{W}_t + \int_0^{1/\varepsilon} g_{3,\varepsilon}(\tilde{Z}_t^{\varepsilon,x}, \tilde{m}_t) dt + \log p_0(\tilde{Z}_{1/\varepsilon}^{\varepsilon,x}),\end{aligned}$$

where

$$g_{3,\varepsilon}(z,m) = g_1(z,m) - b(z)h(z) + \frac{1}{2}h'(z) + \frac{1}{2\varepsilon}m^2(h'(z))^2.$$

Note that $\log p_0(\cdot)$ is bounded above, and

$$|g_{3,\varepsilon}(z,\tilde{m}_t)| \leq C\left(\frac{1}{\varepsilon} + |z|\right).$$

Now since, for any $p > 1$,

$$\mathbb{E}\left[\exp\left(-\frac{p^2}{2\varepsilon}\int_0^{1/\varepsilon} |h(\tilde{Z}_t^{\varepsilon,x})|^2 dt + \frac{p}{\sqrt{\varepsilon}}\int_0^{1/\varepsilon} h(\tilde{Z}_t^{\varepsilon,x}) d\tilde{W}_t\right)\right] = 1,$$

it follows from Hölder's inequality that for any $q > p > 1$ satisfying $1/p + 1/q = 1$,

$$\begin{aligned} &e^{-H(x)/\varepsilon}\ \mathbb{E}\left[L_\varepsilon\left(x, \frac{1}{\varepsilon}\right)\mathbf{1}_{\{\|\tilde{Z}^{\varepsilon,x}\| > M_2/\varepsilon\}}\right] \\ &\leq \left(\mathbb{E}\left[\mathbf{1}_{\{\|\tilde{Z}^{\varepsilon,x}\| > M_2/\varepsilon\}} \exp\left(C\int_0^{1/\varepsilon}\left(\frac{1}{\varepsilon} + |\tilde{Z}_t^{\varepsilon,x}|^2\right)dt\right)\right.\right. \\ &\left.\left.\times \exp\left(-\frac{q}{2\varepsilon}\int_0^{1/\varepsilon}|h(\tilde{Z}_t^{\varepsilon,x}) - h(\tilde{m}_t)|^2 dt + \frac{p}{2\varepsilon}\int_0^{1/\varepsilon}|h(\tilde{Z}_t^{\varepsilon,x})|^2 dt\right)\right]\right)^{1/q}, \end{aligned} \tag{4.9}$$

where $C > 0$. However, note that, due to $h' \geq h_0$, there exists a constant C depending on $|||\tilde{m}|||$ such that

$$\sup_{z\in\mathbb{R}, |m|\leq|||\tilde{m}|||} |z|^2 - \frac{q}{2}|h(z) - h(m)|^2 + \frac{p}{2}|h(z)|^2 \leq C.$$

Substituting this into (4.9), one deduces that

$$e^{-H(x)/\varepsilon}\mathbb{E}\left[L_\varepsilon\left(x, \frac{1}{\varepsilon}\right)\mathbf{1}_{\{\|\tilde{Z}^{\varepsilon,x}\| > M_2/\varepsilon\}}\right] \leq \left(\mathbb{E}\left[\mathbf{1}_{\{\|\tilde{Z}^{\varepsilon,x}\| > M_2/\varepsilon\}} \exp\left(\frac{C}{\varepsilon^2}\right)\right]\right)^{1/q}. \tag{4.10}$$

(Recall that the value of C may change from line to line!)

We prove below that, provided M_2 is large enough, there exists a $c > 0$ such that

$$\mathbb{E}\left[\mathbf{1}_{\{\|\tilde{Z}^{\varepsilon,x}\| > M_2/\varepsilon\}}\right] \leq \exp\left(-\frac{c}{\varepsilon^3}\right). \tag{4.11}$$

Combined with (4.10), this implies that, uniformly in $|x| \leq M_1/\sqrt{\varepsilon}$,

$$\mathbb{E}\left[L_\varepsilon\left(x, \frac{1}{\varepsilon}\right)\mathbf{1}_{\{\|\tilde{Z}^{\varepsilon,x}\| > M_2/\varepsilon\}}\right] \leq \exp\left(-\frac{c}{\varepsilon^3}\right). \tag{4.12}$$

To see (4.11), let $H = \sup|h'|$, define $\theta_0 = 0$, and let

$$\tau_i = \inf\left\{t > \theta_{i-1} : |\tilde{Z}_t^{\varepsilon,x}| > \frac{M_2}{2\varepsilon}\right\}, \quad \theta_i = \inf\left\{t > \tau_i : |\tilde{Z}_t^{\varepsilon,x}| < \frac{M_2}{4\varepsilon}\right\}.$$

Setting $f(z, m) = -h(z) + mh'(z) - \varepsilon b(z)$, we have that, for $|z| \in [M_2/4\varepsilon, M_2/\varepsilon]$, $t \leq 1/\varepsilon$, and ε small enough, it holds that $h_0 M_2/8\varepsilon \leq |f(z, \tilde{m}_t)| \leq 2HM_2/\varepsilon$ and $\text{sign}f(z, \tilde{m}_t) = -\text{sign}(z)$. Then, choosing $\eta = (16H)^{-1}$ for each i, it holds that

$$\begin{aligned} \mathbb{P}\left(\theta_i - \tau_i < \eta, \sup_{t\in[\tau_i,\theta_i]}|\tilde{Z}_t^{\varepsilon,x}| < \frac{M_2}{\varepsilon}\right) &\leq \mathbb{P}\left(\sqrt{\varepsilon}\sup_{0\leq t\leq\eta}|W_t| \geq \frac{M_2}{4\varepsilon} - \frac{2H\eta M_2}{\varepsilon}\right) \\ &\leq \mathbb{P}\left(\sqrt{\varepsilon}\sup_{0\leq t\leq\eta}|W_t| \geq \frac{M_2}{8\varepsilon}\right) \\ &\leq \exp\left(-\frac{cM_2^2}{\varepsilon^3\eta}\right). \end{aligned} \tag{4.13}$$

Similarly

$$\begin{aligned} \mathbb{P}\left(\theta_i - \tau_i \geq \eta, |\tilde{Z}_{\tau_i+\eta}^{\varepsilon,x}| \geq \frac{M_2}{2\varepsilon}\right) &\leq \mathbb{P}\left(\sqrt{\varepsilon}W_\eta \geq \frac{h_0 M_2\eta}{8\varepsilon}\right) \\ &\leq \exp\left(-\frac{cM_2^2\eta}{\varepsilon^3}\right) \end{aligned} \tag{4.14}$$

and

$$\mathbb{P}\left(\sup_{t\in[\tau_i,(\tau_i+\eta)\wedge\theta_i]}|\tilde{Z}_t^{\varepsilon,x}|>\frac{M_2}{\varepsilon}\right)\le\mathbb{P}\left(\sqrt{\varepsilon}\sup_{0\le t\le\eta}|W_t|\ge\frac{M_2}{2\varepsilon}\right)\le\exp\left(-\frac{cM_2^2}{\varepsilon^3\eta}\right). \tag{4.15}$$

Hence, using (4.13), (4.14), and (4.15),

$$\mathbb{E}\left[\mathbf{1}_{\{\|\tilde{Z}^{\varepsilon,x}\|>M_2/\varepsilon\}}\right]\le\frac{1}{\varepsilon\eta}\left(\exp\left(-\frac{cM_2^2}{\varepsilon^3\eta}\right)+\exp\left(-\frac{cM_2^2}{\varepsilon^3}\right)+\exp\left(-\frac{cM_2^2\eta}{\varepsilon^3\eta}\right)\right),$$

completing the proof of (4.11).

We now turn to the lower bound of the right-hand side of (4.8). Let, with $M_1'=M_1+1$,

$$\varepsilon_0=1\wedge\left(\frac{M_2}{M_1'}\right)^2.$$

For $\varepsilon\le\varepsilon_0$,

$$\left\{\|\tilde{Z}^{\varepsilon,x}\|\le\frac{M_1'}{\sqrt{\varepsilon}}\right\}\subset\left\{\|\tilde{Z}^{\varepsilon,x}\|\le\frac{M_2}{\varepsilon}\right\},$$

so that for some $c'>0$

$$\mathbb{E}\left[L_\varepsilon\left(x,\frac{1}{\varepsilon}\right)\mathbf{1}_{\{\|\tilde{Z}^{\varepsilon,x}\|\le M_2/\varepsilon\}}\right]\ge\mathbb{E}\left[L_\varepsilon\left(x,\frac{1}{\varepsilon}\right)\mathbf{1}_{\{\|\tilde{Z}^{\varepsilon,x}\|\le M_1'/\sqrt{\varepsilon}\}}\right]\ge\exp\left(-\frac{c'}{\varepsilon^{5/2}}\right)\mathbb{P}\left(\|\tilde{Z}^{\varepsilon,x}\|\le\frac{M_1'}{\sqrt{\varepsilon}}\right). \tag{4.16}$$

Finally (4.8) follows from (4.12), (4.16), and the estimate

$$\mathbb{P}\left(\|\tilde{Z}^{\varepsilon,x}\|\le\frac{M_1'}{\sqrt{\varepsilon}}\right)\ge\mathbb{P}\left(\sqrt{\varepsilon}\|\tilde{W}\|\le C\right)\ge c''>0. \qquad \square$$

Proof of Lemma 3.5. Note first that because of (A-4), there exists a constant $\kappa=\kappa(|||\tilde{m}|||)$ such that for all $z\notin[-\kappa,\kappa]$, all $\varepsilon<1/\kappa$, all $|m|\le|||\tilde{m}|||$, and all z',

$$\Delta(z,z',m)=-h(z)+h(z')+m[h'(z)-h'(z')]-\varepsilon[b(z)-b(z')]$$

satisfies $\operatorname{sign}(\Delta(z,z',m))=\operatorname{sign}(z'-z)$, while $|\Delta(z,z',m)|\ge h_0|z-z'|/2$.

Assume, without loss of generality, that $x<y$. Fix, for δ given, a smooth, even, nonnegative function $c(z)$ such that $c(|z|)$ is nonincreasing, $c(z)=\sqrt{\delta}$ for $|z|\le\kappa$, and $c(z)=0$ for $|z|>2\kappa$, with $||c'||\le 10\sqrt{\delta}$. Define next the diffusions

$$d\xi_s^1=[-h(\xi_s^1)+\tilde{m}_sh'(\xi_s^1)-\varepsilon b(\xi_s^1)+c(\xi_s^1)\mathbf{1}_{\{\tau>s\}}]ds+\sqrt{\varepsilon}dB_s,\quad \xi_0^1=x,$$
$$d\xi_s^2=[-h(\xi_s^2)+\tilde{m}_sh'(\xi_s^2)-\varepsilon b(\xi_s^2)]dt+\sqrt{\varepsilon}dB_s,\quad \xi_0^2=y,$$

where B is a Brownian motion independent of the process $\tilde{m}$, and $\tau=\min\{t:\xi_t^1=\xi_t^2\}\wedge 1/\varepsilon$. Note that ξ^2 coincides in distribution with $\tilde{Z}^{\varepsilon,y}$, whereas the law of ξ^1

is absolutely continuous with respect to the law of $\tilde{Z}^{\varepsilon,x}$ with the Radon–Nikodym derivative given by

$$\begin{aligned}\Lambda &= \exp\left(\frac{1}{\varepsilon}\int_0^\tau c(\xi_s^1)d\xi_s^1 - \frac{1}{2\varepsilon}\int_0^\tau c^2(\xi_s^1)ds - \frac{1}{\varepsilon}\int_0^\tau c(\xi_s^1)g(s,\xi_s^1)ds\right)\\ &= \exp\left(\frac{1}{\varepsilon}[\bar{c}(\xi_\tau^1)-\bar{c}(\xi_0^1)] - \frac{1}{2\varepsilon}\int_0^\tau c^2(\xi_s^1)ds - \frac{1}{\varepsilon}\int_0^\tau c(\xi_s^1)g(s,\xi_s^1)ds - \frac{1}{2}\int_0^\tau c'(\xi_s^1)ds\right),\end{aligned} \tag{4.17}$$

where $g(s,z) = -h(z) + \tilde{m}_s h'(z) - \varepsilon b(z)$ and $\bar{c}(z) = \int_0^z c(y)dy$.

Next, note that with $\zeta_s = \xi_s^1 - \xi_s^2$, and using that $x < y$, it holds that $\zeta_s \le 0$ for all s, while by definition, $|\zeta_0| \le \delta$. Hence, by the definition of $c(\cdot)$ and of κ, it holds that for all $\delta < \delta_1(\kappa, |||m|||)$,

$$\frac{d\zeta_s}{ds} \ge -\frac{h_0\zeta_s}{2} + \frac{c(\xi_s^1)\mathbf{1}_{s<\tau}}{2},$$

from which one concludes that $\zeta_s \ge -\delta e^{-hs/2}$. In particular, this implies that for all such δ,

$$\int_0^\tau c(\xi_s^1)\mathbf{1}_{\{\tau>s\}}ds = \int_0^\tau c(\xi_s^1)ds \le C\delta$$

for some constant $C = C(\kappa, |||\tilde{m}|||)$. Since $c(z) = 0$ for $|z| > 2\kappa$, and since $|g(s,z)|$ is bounded uniformly in $s \le 1/\varepsilon$ and $|z| \le 2\kappa$ (by a bound that depends only on $|||\tilde{m}|||$), the last inequality implies that

$$\left|\int_0^\tau c(\xi_s^1)g(s,\xi_s^1)ds\right| \le C\delta,$$

again for some constant C depending on $\kappa, |||\tilde{m}|||$ only. Finally, note that

$$\int_0^\tau c^2(\xi_s^1)ds \le \sqrt{\delta}\int_0^\tau c(\xi_s^1)ds \le C\delta^{3/2},$$

and that $|\bar{c}(z)| \le 2\kappa\sqrt{\delta}$. Substituting back into (4.17) and recalling that $\kappa = \kappa(|||\tilde{m}|||)$, one concludes the existence of a constant $C_2 = C_2(|||\tilde{m}|||)$ such that for all $\delta < \delta_1$,

$$e^{-C_2\sqrt{\delta}/\varepsilon} \le \Lambda \le e^{C_2\sqrt{\delta}/\varepsilon}. \tag{4.18}$$

Therefore, with $\mathbb{E}_B$ denoting expectation with respect to $B.$, and using the bound on Λ in the second inequality, and the Lipschitz property of g_1, g_2 together with the exponential decay of ζ_s in the third, and omitting the dependence on $\theta^{1/\varepsilon - t}\tilde{m}$

everywhere, it holds, for all $t > 1/2\varepsilon$, that

(4.19)
$$\begin{aligned}
&\mathbb{E}L_\varepsilon(x,t) \le 2\mathbb{E}L_\varepsilon(x,t)\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,x}||<M_2/\varepsilon\}}\\
&= 2\mathbb{E}_B\left(\mathbf{1}_{\{||\xi^1||<M_2/\varepsilon\}}\Lambda^{-1}\exp\left(I_\varepsilon(\xi_t^1,0) + \int_0^t\left(g_1(\xi_s^1,\tilde{m}_s) + \frac{1}{\varepsilon}g_2(\xi_s^1,\tilde{m}_s)\right)ds\right)\right)\\
&\le 2\mathbb{E}_B\Bigg(\mathbf{1}_{\{||\xi^2||<(M_2+1)/\varepsilon\}}\ \exp\Bigg(\frac{C_2\sqrt{\delta}}{\varepsilon} + I_\varepsilon(\xi_t^2+\zeta_t,0)\\
&\qquad\qquad + \int_0^t\left(g_1(\xi_s^2+\zeta_s,\tilde{m}_s) + \frac{1}{\varepsilon}g_2(\xi_s^2+\zeta_s,\tilde{m}_s)\right)ds\Bigg)\Bigg)\\
&\le 2\mathbb{E}_B\left(\exp\left(\frac{C_3\sqrt{\delta}}{\varepsilon} + I_\varepsilon(\xi_t^2,0) + \int_0^t\left(g_1(\xi_s^2,\tilde{m}_s) + \frac{1}{\varepsilon}g_2(\xi_s^2,\tilde{m}_s)\right)ds\right)\right)\\
&= 2\mathbb{E}\left(\exp\left(\frac{C_3\sqrt{\delta}}{\varepsilon} + I_\varepsilon(\tilde{Z}_t^{\varepsilon,y},0) + \int_0^t\left(g_1(\tilde{Z}_s^{\varepsilon,y},\tilde{m}_s) + \frac{1}{\varepsilon}g_2(\tilde{Z}_s^{\varepsilon,y},\tilde{m}_s)\right)ds\right)\right)\\
&= 2\exp\left(\frac{C_3\sqrt{\delta}}{\varepsilon}\right)\mathbb{E}L_\varepsilon(y,t)\\
&\le 4\exp\left(\frac{C_3\sqrt{\delta}}{\varepsilon}\right)\mathbb{E}\left(\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,y}||<M_2/\varepsilon\}}L_\varepsilon(y,t)\right),
\end{aligned}$$

yielding (3.9) for $x < y$ and $\delta < \delta_1$, with $g(\delta) = C_3\sqrt{\delta}$. Further, the same computation gives

$$4\mathbb{E}\left(L_\varepsilon(x,t)\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,x}||<M_2/\varepsilon\}}\right) \ge \exp\left(\frac{-C_3\sqrt{\delta}}{\varepsilon}\right)\mathbb{E}\left(L_\varepsilon(y,t)\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,y}||<M_2/\varepsilon\}}\right),$$

yielding, by exchanging the roles of x and y, (3.9) for $x > y$ and $\delta < \delta_1$ with the same $g(\delta)$. Finally, for $\delta > \delta_1$, iterate this procedure to obtain (3.9) with $g(\delta) = C_3\sqrt{\delta\wedge\delta_1}\lceil\delta/\delta_1\rceil$. Substituting $y = X_1$ into the latter version of (3.9) then gives (3.10). ∎

Proof of Lemma 3.6. Throughout the proof, we fix once and for all the sequence T_ε. All constants C_i used in the proof may depend on the choice of the sequence but not explicitly on ε.

We begin with the proof of (3.11). Using Girsanov's theorem, one finds that with $\bar{Z}_t^{\varepsilon,x} = x + \sqrt{\varepsilon}\tilde{W}_t$,

$$\begin{aligned}
&\mathbb{E}\left[\bar{L}_\varepsilon(x,T)\mathbf{1}_{\{|\tilde{Z}_T^{\varepsilon,x}-z|<\delta\}}\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,x}||_T\le M_3/\varepsilon\}}\right]\\
&= \mathbb{E}\Bigg[\mathbf{1}_{\{|\bar{Z}_T^{\varepsilon,x}-z|<\delta\}}\mathbf{1}_{\{||\bar{Z}^{\varepsilon,x}||_T\le M_3/\varepsilon\}}\\
&\quad\cdot\exp\Bigg(\frac{1}{\varepsilon}\int_0^T\left[-h(\bar{Z}_s^{\varepsilon,x}) + \tilde{m}_s h'(\bar{Z}_s^{\varepsilon,x}) - \varepsilon b(\bar{Z}_s^{\varepsilon,x})\right]d\bar{Z}_s^{\varepsilon,x}\\
&\qquad - \frac{1}{\varepsilon}\int_0^T\Bigg(\frac{[h(\bar{Z}_s^{\varepsilon,x}) - h(\tilde{m}_s)]^2}{2} + \frac{b^2(\bar{Z}_s^{\varepsilon,x})\varepsilon^2}{2} + \varepsilon b(\bar{Z}_s^{\varepsilon,x})h(\bar{Z}_s^{\varepsilon,x})\\
&\qquad\qquad - \varepsilon h'(\bar{Z}_s^{\varepsilon,x})b(\bar{Z}_s^{\varepsilon,x})\tilde{m}_s - \varepsilon g_1(\bar{Z}_s^{\varepsilon,x},\tilde{m}_s)\Bigg)ds\Bigg)\Bigg].
\end{aligned}\tag{4.20}$$

We consider the different terms in (4.20) separately. Note first that one may, exactly as in the course of the proof of Lemma 3.5, move from starting point x to starting point X_1 in the right-hand side of (4.20), with the effect of picking up a term bounded by $\exp(C|x|/\varepsilon)$ and widening the allowed region where $\bar{Z}_T^{\varepsilon,x}$ need to be; namely, for all $T_\varepsilon \geq T > 1$, the right-hand side of (4.20) is bounded by

$$\exp\left(\frac{C_1 + C_2|x|}{\varepsilon}\right) \mathbb{E}\Bigg[\mathbf{1}_{\{|\bar{Z}_T^{\varepsilon,X_1} - z| < \delta + |x| + |X_1|\}} \mathbf{1}_{\{||\bar{Z}^{\varepsilon,X_1}||_T \leq |x| + (M_3+1)/\varepsilon\}}$$

$$\cdot \exp\left(\frac{1}{\varepsilon}\int_0^T \Big[-h(\bar{Z}_t^{\varepsilon,X_1}) + \tilde{m}_s h'(\bar{Z}_s^{\varepsilon,X_1}) - \varepsilon b(\bar{Z}_s^{\varepsilon,X_1})\Big]\, d\bar{Z}_s^{\varepsilon,X_1}\right)\Bigg]. \tag{4.21}$$

An integration by parts gives that

$$-\int_0^T h(\bar{Z}_t^{\varepsilon,X_1}) d\bar{Z}_t^{\varepsilon,X_1} = -\bar{\mathcal{J}}(\bar{Z}_T^{\varepsilon,X_1}, X_1) - h(X_1)(\bar{Z}_T^{\varepsilon,X_1} - X_1) + \frac{\varepsilon}{2}\int_0^T h'(\bar{Z}_t^{\varepsilon,X_1}) dt,$$

and hence, on the event $\{|\bar{Z}_T^{\varepsilon,X_1} - z| < \delta + |x| + |X_1|\}$, it holds that

$$-\int_0^T h(\bar{Z}_t^{\varepsilon,X_1}) d\bar{Z}_t^{\varepsilon,X_1} \leq -C(|z| - |x| - |X_1| - \delta)_+^2 + C. \tag{4.22}$$

Similarly, with $B(z) = \int_{X_1}^z b(x)dx$,

$$\int_0^T b(\bar{Z}_s^{\varepsilon,X_1}) d\bar{Z}_s^{\varepsilon,X_1} = B(\bar{Z}_T^{\varepsilon,X_1}) - \frac{\varepsilon}{2}\int_0^T b'(\bar{Z}_s^{\varepsilon,X_1}) ds \leq C(|z|^2 + |x|^2 + 1). \tag{4.23}$$

Finally, rewrite

$$\int_0^T \tilde{m}_s h'(\bar{Z}_s^{\varepsilon,X_1}) d\bar{Z}_s^{\varepsilon,X_1} = X_1 \int_0^T h'(\bar{Z}_s^{\varepsilon,X_1}) d\bar{Z}_s^{\varepsilon,X_1} + \int_0^T (\tilde{m}_s - X_1) h'(\bar{Z}_s^{\varepsilon,X_1}) d\bar{Z}_s^{\varepsilon,X_1}.$$

The first stochastic integral in the above expression is handled exactly as in (4.23), and substituting into (4.21) one concludes that the right-hand side of (4.20) is bounded by

$$\begin{aligned}
&\exp\left(\frac{C + C(|x| + |z|) - C(|z| - |x|)_+^2}{\varepsilon}\right) \mathbb{E}\left[\exp\left(\frac{1}{\varepsilon}\int_0^T (\tilde{m}_s - X_1) h'(\bar{Z}_s^{\varepsilon,X_1}) d\bar{Z}_s^{\varepsilon,X_1}\right)\right] \\
&\quad \leq \exp\left(\frac{C + C(|x| + |z|) - C(|z| - |x|)_+^2}{\varepsilon} + \frac{1}{2\varepsilon}\int_0^{T_\varepsilon} C|\tilde{m}_s - X_1|^2 ds\right) \\
&\quad \leq \exp\left(\frac{C + C(|x| + |z|) - C(|z| - |x|)_+^2}{\varepsilon}\right),
\end{aligned}$$

where in the last inequality we have used the last part of Lemma 2.1. This completes the proof of (3.11).

The proof of (3.12) proceeds along similar lines. The starting point is the change of measure leading to (4.20). Define the function

$$\Psi_t = \begin{cases} x + 2(X_1 - x)t, & t \leq \frac{1}{2}, \\ X_1, & T - \frac{1}{2} > t \geq \frac{1}{2}, \\ z + 2(z - X_1)(t - T), & T \geq t \geq T - \frac{1}{2}. \end{cases}$$

Let D denote the event

$$D := \left\{ \sup_{t \le T} |\bar{Z}_t^{\varepsilon,x} - \Psi_t| < \sqrt{\varepsilon} \right\}.$$

We will prove below that, for $|x - X_1| \le 1$ and $T < T_\varepsilon$, there exists a constant C independent of T and ε such that

$$\mathbb{P}(D) \ge e^{-C/\varepsilon}. \tag{4.24}$$

We can clearly bound the right-hand side of (4.20) from below by

$$\mathbb{E}\Bigg[\mathbf{1}_{\{|\bar{Z}_T^{\varepsilon,x} - z| < \delta\}} \mathbf{1}_{\{||\bar{Z}^{\varepsilon,x}||_T \le M_3/\varepsilon\}} \mathbf{1}_D \cdot \exp\Bigg(\frac{1}{\varepsilon} \int_0^T \left[-h(\bar{Z}_t^{\varepsilon,x}) + \tilde{m}_s h'(\bar{Z}_s^{\varepsilon,x}) - \varepsilon b(\bar{Z}_s^{\varepsilon,x}) \right] d\bar{Z}_s^{\varepsilon,x} - \frac{1}{\varepsilon} \int_0^T \bigg(\frac{[h(\bar{Z}_t^{\varepsilon,x}) - h(\tilde{m}_t)]^2}{2} + \frac{b^2(\bar{Z}_t^{\varepsilon,x})\varepsilon^2}{2} + \varepsilon b(\bar{Z}_s^{\varepsilon,x}) h(\bar{Z}_s^{\varepsilon,x}) - \varepsilon h'(\bar{Z}_s^{\varepsilon,x}) b(\bar{Z}_s^{\varepsilon,x}) \tilde{m}_s - \varepsilon g_1(\bar{Z}_s^{\varepsilon,x}, \tilde{m}_s) \bigg) ds \Bigg) \Bigg].$$

We now assume that (4.24) and $|z - X_1| \le 1$ hold. Then using the same integration by parts as in the proof of the upper bound, one concludes that the right-hand side of (4.20) is bounded from below by

$$\mathbb{E}\left[\mathbf{1}_D \exp\left(\frac{-C}{\varepsilon} + \frac{1}{\varepsilon} \int_0^T (\tilde{m}_s - X_1) h'(\bar{Z}_s^{\varepsilon,x}) d\bar{Z}_s^{\varepsilon,x} \right) \right]. \tag{4.25}$$

However, since

$$\mathrm{Var}\left(\int_0^T (\tilde{m}_s - X_1) h'(\bar{Z}_s^{\varepsilon,x}) d\bar{Z}_s^{\varepsilon,x} \right) \le C\varepsilon,$$

one gets, using Chebyshev's inequality, that

$$\mathbb{P}\left[\int_0^T (\tilde{m}_s - X_1) h'(\bar{Z}_s^{\varepsilon,x}) d\bar{Z}_s^{\varepsilon,x} < -c \right] \le \exp\left(-\frac{C_2 c^2}{\varepsilon} \right).$$

Hence,

$$\mathbb{P}\left[\int_0^T (\tilde{m}_s - X_1) h'(\bar{Z}_s^{\varepsilon,x}) d\bar{Z}_s^{\varepsilon,x} < -c | D \right] \le \frac{\exp\left(-\frac{C_2 c^2}{\varepsilon} \right)}{\mathbb{P}(D)} \le \frac{1}{2}$$

if c is chosen large, where in the last inequality we used (4.24). In particular, it follows that

$$\mathbb{E}\left[\exp\left(\frac{1}{\varepsilon} \int_0^T (\tilde{m}_s - X_1) h'(\bar{Z}_s^{\varepsilon,x}) d\bar{Z}_s^{\varepsilon,x} \right) \,\middle|\, D \right] \ge \exp\left(-\frac{C}{\varepsilon} \right)$$

for some $C > 0$. Substituting back into (4.25), the required lower bound follows.

It thus remains only to prove (4.24). This, however, is immediate from a martingale argument: first, perform the change of measure making $S_t := \bar{Z}_t^{\varepsilon,x} - \Psi_t$ into a Brownian motion of variance ε. Then, for $1 \le T \le T_\varepsilon$,

$$\mathbb{P}(D) = \mathbb{E}\left(\mathbf{1}_{\{\sup_{t\le T}|S_t|\le\sqrt{\varepsilon}\}} \exp\left(-\frac{1}{\varepsilon}\int_0^T \dot{\Psi}_t dS_t - \frac{1}{2\varepsilon}\int_0^T \dot{\Psi}_t{}^2 dt\right)\right).$$

Integrating the stochastic integral by parts and using that $\dot{\Psi}(t) = 0$ for $t \in (1/2, T - 1/2)$, (4.24) follows, which completes the proof of the lemma. $\square$

Proof of (3.26). We let $\eta > 0$ as before. Note first that, by (4.5) and (4.6), there is a constant M depending on $|||\tilde{m}|||$ only such that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \int_{[-M/\sqrt{\varepsilon}, M/\sqrt{\varepsilon}]^c} q_1^\varepsilon(x)dx = -\infty. \tag{4.26}$$

We may and will in what follows assume that $M = M_1$, where M_1 is defined in Lemma 3.3, and we use M_3 and M_2 as in Lemma 3.4.

Next, set ε_4 such that $\varepsilon_4 \log 2 < \eta/8$ and $\varepsilon \log(2M_3/\varepsilon\delta) \le \eta/8$ for $\varepsilon < \varepsilon_4$. Repeating the arguments in (3.21), without using the compact set $\mathcal{K}_1$, one has for $\varepsilon < \varepsilon_4$ and $|x| \le M_1/\sqrt{\varepsilon}$,

$$\begin{aligned}
\varepsilon \log \rho_1^\varepsilon(x) &\le -F(x, \tilde{m}_0) + \varepsilon \log \tilde{J}_\varepsilon(x) + \frac{\eta}{4} \quad \texttt{as in (3.21)} \\
&\le -F(x, \tilde{m}_0) + \varepsilon \log \sup_{|z|\le M_3/\varepsilon} \hat{J}_{\varepsilon,T}(x,z) + \frac{\eta}{2} \quad \texttt{by (3.13) and (3.15)} \\
&\le -F(x, \tilde{m}_0) + \frac{\eta}{2} + C_2 - C_3(|z| - |x|)_+^2 + C_5(|x| + |z|) \\
&\quad + \varepsilon \log \mathbb{E}\left[L_\varepsilon\left(X_1, \frac{1}{\varepsilon} - T, \theta^T \tilde{m}\right)\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,X_1}||_{1/\varepsilon - T} \le M_2/\varepsilon\}}\right].
\end{aligned} \tag{4.27}$$

A similar argument shows that for $|x - X_1| < 1$ and some constant C_6 depending only on $X, |||\tilde{m}|||$,

$$\begin{aligned}
\varepsilon \log \rho_1^\varepsilon(x) &\ge -F(X_1, X_1) - C_6 \\
&\quad + \varepsilon \log \mathbb{E}\left[L_\varepsilon\left(X_1, \frac{1}{\varepsilon} - T, \theta^T \tilde{m}\right)\mathbf{1}_{\{||\tilde{Z}^{\varepsilon,X_1}||_{1/\varepsilon - T} \le M_2/\varepsilon\}}\right].
\end{aligned} \tag{4.28}$$

Fixing now an L, and using as in (3.18) the uniform quadratic growth of $F(x,m)$ as $|x| \to \infty$ and $|m| < |||\tilde{m}|||$, one finds a compact set $\mathcal{K}^L$ such that

$$\begin{aligned}
&\sup_{|m|<|||\tilde{m}|||} \sup_{x\in(\mathcal{K}^L)^c, z\in\mathbb{R}} \frac{C_2}{\varepsilon} - \frac{C_3(|z| - |x|)_+^2}{\varepsilon} + \frac{C_5(|x| + |z|)}{\varepsilon} - F(x,m) \\
&\quad \le -F(X_1, X_1) - \frac{C_6 + L}{\varepsilon},
\end{aligned} \tag{4.29}$$

and hence, from (4.27) and (4.28), for $x \in (\mathcal{K}^L)^c \cap [-M_1/\sqrt{\varepsilon}, M_1/\sqrt{\varepsilon}]$,

$$\varepsilon \log \rho_1^\varepsilon(x) \le \inf_{|y - X_1|\le 1} \varepsilon \log \rho_1^\varepsilon(y) - L. \tag{4.30}$$

Hence,

(4.31)

$$\begin{aligned}\limsup_{\varepsilon\to 0}\varepsilon\log\int_{(\mathcal{K}^L)^c} q_1^\varepsilon(x)dx &= \limsup_{\varepsilon\to 0}\varepsilon\log\int_{(\mathcal{K}^L)^c\cap[-M_1/\sqrt{\varepsilon},M_1/\sqrt{\varepsilon}]} q_1^\varepsilon(x)dx \quad \texttt{by (4.26)}\\ &\le \limsup_{\varepsilon\to 0}\left[\varepsilon\log\int_{(\mathcal{K}^L)^c\cap[-M_1/\sqrt{\varepsilon},M_1/\sqrt{\varepsilon}]}\rho_1^\varepsilon(x)dx - \varepsilon\log\int_{[X_1-1,X_1+1]}\rho_1^\varepsilon(x)dx\right]\\ &\le \limsup_{\varepsilon\to 0}\left[\varepsilon\log\left(\frac{2M_1}{\sqrt{\varepsilon}}\right) + \inf_{|y-X_1|\le 1}\varepsilon\log\rho_1^\varepsilon(y) - L - \inf_{|y-X_1|\le 1}\varepsilon\log\rho_1^\varepsilon(y)\varepsilon\log 2\right]\\ &\qquad\qquad \texttt{by (4.30)}\\ &\le -L\,.\end{aligned}$$

This completes the proof. □

Appendix. Derivation of (2.1). We first recall Picard's theorem [7, Proposition 4.2]: under the assumptions of the current paper and with the same notation, a version of the conditional unnormalized density is given by

$$\tilde{q}(1,x) = \exp\left\{\frac{1}{2\varepsilon^2}\int_0^1 h^2(\bar{m}_s)ds - \frac{1}{\varepsilon}F(x,\tilde{m}_0)\right\}\tilde{\mathbb{E}}'\left[\exp\rho_1^{y,x}\right], \tag{A.1}$$

where

$$\begin{aligned}\rho_1^{x,y} &= \log p_0(\bar{X}_1^x) + \frac{1}{\varepsilon}F(\bar{X}_1^x,0) - \frac{1}{\varepsilon}\int_0^1 h(\bar{m}_s)d\bar{X}_s^x - \frac{1}{\varepsilon}\int_0^1 h(\bar{X}_s^x)b(\bar{m}_s)ds\\ &\quad+\frac{1}{\varepsilon}\int_0^1 \bar{m}_s h'(\bar{X}_s^x)d\bar{X}_s^x + \frac{1}{2\varepsilon}\int_0^1 \bar{m}_s h''(\bar{X}_s^x)ds\\ &\quad+\frac{1}{\varepsilon}\int_0^1\left[b(\bar{X}_s^x)(h(\bar{X}_s^x)-h(\bar{m}_s)) - \frac{1}{2}h'(\bar{X}_s^x) - \varepsilon b'(\bar{X}_s^x)\right]ds,\\ d\bar{X}_s^x &= -\frac{1}{\varepsilon}(h(\bar{X}_s^x)-h(\bar{m}_s))ds - b(\bar{X}_s^x)ds + dW_s, \quad \bar{X}_0^x = x,\end{aligned}$$

$W.$ is a Brownian motion, and $\tilde{\mathbb{E}}'$ denotes expectation with respect to this Brownian motion. Performing a time change $t\mapsto\varepsilon t$ and setting $\tilde{W}_t = \frac{1}{\sqrt{\varepsilon}}W_{\varepsilon t}$, we have that $\tilde{W}_t$ is again a standard Brownian motion and, with $\bar{X}_t^{\varepsilon,x} = \bar{X}_{\varepsilon t}^x$,

$$\begin{aligned}\rho_1^{x,y} &= \log p_0(\bar{X}_{1/\varepsilon}^{\varepsilon,x}) + \frac{1}{\varepsilon}F(\bar{X}_{1/\varepsilon}^{\varepsilon,x},0) - \frac{1}{\varepsilon}\int_0^{1/\varepsilon} h(\tilde{m}_s)d\bar{X}_s^{\varepsilon,x} - \int_0^{1/\varepsilon} h(\bar{X}_s^{\varepsilon,x})b(\tilde{m}_s)ds\\ &\quad+\frac{1}{\varepsilon}\int_0^{1/\varepsilon}\tilde{m}_s h'(\bar{X}_s^{\varepsilon,x})d\bar{X}_s^{\varepsilon,x} + \frac{1}{2}\int_0^{1/\varepsilon}\tilde{m}_s h''(\bar{X}_s^{\varepsilon,x})ds\\ &\quad+\int_0^{1/\varepsilon}\left[b(\bar{X}_s^{\varepsilon,x})(h(\bar{X}_s^{\varepsilon,x})-h(\tilde{m}_s)) - \frac{1}{2}h'(\bar{X}_s^{\varepsilon,x}) - \varepsilon b'(\bar{X}_s^{\varepsilon,x})\right]ds,\\ d\bar{X}_s^{\varepsilon,x} &= -(h(\bar{X}_s^{\varepsilon,x})-h(\tilde{m}_s))ds - \varepsilon b(\bar{X}_s^{\varepsilon,x})ds + \sqrt{\varepsilon}d\tilde{W}_s\,, \quad \bar{X}_0^{\varepsilon,x} = x,\end{aligned}$$

and

$$\tilde{q}(1,x) = \exp\left\{\frac{1}{2\varepsilon}\int_0^{1/\varepsilon} h^2(\tilde{m}_s))ds - \frac{1}{\varepsilon}F(x,\tilde{m}_0)\right\}\tilde{\mathbb{E}}\left[\exp\rho_1^{y,x}\right], \tag{A.2}$$

where the expectation now is with respect to the Brownian motion $\tilde{W}_t$.

Observe next that, by Girsanov's theorem, the law of the process $\bar{X}_t^{\varepsilon,x}$ is absolutely continuous with respect to that of the process $\tilde{Z}_t^{\varepsilon,x}$, with the Radon–Nikodym derivative given by

$$\begin{aligned} e^{\Lambda} = \exp \Bigg[\frac{1}{\varepsilon} \int_0^{1/\varepsilon} & [h(\tilde{m}_s) - \tilde{m}_s h'(\tilde{Z}_s^{\varepsilon,x})] d\tilde{Z}_s^{\varepsilon,x} \\ & - \frac{1}{2\varepsilon} \int_0^{1/\varepsilon} [h(\tilde{Z}_s^{\varepsilon,x}) - h(\tilde{m}_s) + \varepsilon b(\tilde{Z}_s^{\varepsilon,x})]^2 ds \\ & + \frac{1}{2\varepsilon} \int_0^{1/\varepsilon} [h(\tilde{Z}_s^{\varepsilon,x}) - \tilde{m}_s h'(\tilde{Z}_s^{\varepsilon,x}) + \varepsilon b(\tilde{Z}_s^{\varepsilon,x})]^2 ds \Bigg]. \end{aligned} \tag{A.3}$$

Hence, with $\mathbb{E}$ denoting expectations with respect to the Brownian motion $\tilde{W}_t$ appearing in the definition of $\tilde{Z}_t^{\varepsilon,x}$, (A.2) transforms to

$$\tilde{q}(1,x) = \exp\left\{ \frac{1}{2\varepsilon} \int_0^{1/\varepsilon} h^2(\tilde{m}_s)) ds - \frac{1}{\varepsilon} F(x, \tilde{m}_0) \right\} \mathbb{E} \exp[\Lambda_1(x)],$$

where

$$\begin{aligned} \Lambda_1(x) = \Lambda &+ \log p_0(\tilde{Z}_{1/\varepsilon}^{\varepsilon,x}) + \frac{1}{\varepsilon} F(\tilde{Z}_{1/\varepsilon}^{\varepsilon,x}, 0) - \frac{1}{\varepsilon} \int_0^{1/\varepsilon} h(\tilde{m}_s) d\tilde{Z}_s^{\varepsilon,x} - \int_0^{1/\varepsilon} h(\tilde{Z}^{\varepsilon,x}) b(\tilde{m}_s) ds \\ &+ \frac{1}{\varepsilon} \int_0^{1/\varepsilon} \tilde{m}_s h'(\tilde{Z}_s^{\varepsilon,x}) d\tilde{Z}_s^{\varepsilon,x} + \frac{1}{2} \int_0^{1/\varepsilon} \tilde{m}_s h''(\tilde{Z}_s^{\varepsilon,x}) ds \\ &+ \int_0^{1/\varepsilon} \left[b(\tilde{Z}_s^{\varepsilon,x})(h(\tilde{Z}_s^{\varepsilon,x}) - h(\tilde{m}_s)) - \frac{1}{2} h'(\tilde{Z}_s^{\varepsilon,x}) - \varepsilon b'(\tilde{Z}_s^{\varepsilon,x}) \right] ds \\ = \log p_0&(\tilde{Z}_{1/\varepsilon}^{\varepsilon,x}) + \frac{1}{\varepsilon} F(\tilde{Z}_{1/\varepsilon}^{\varepsilon,x}, 0) + \int_0^{1/\varepsilon} g_1(\tilde{Z}_s^{1/\varepsilon}, \tilde{m}_s) ds + \frac{1}{\varepsilon} \int_0^{1/\varepsilon} g_2(\tilde{Z}_s^{1/\varepsilon}, \tilde{m}_s) ds. \end{aligned}$$

Since $\int_0^{1/\varepsilon} h^2(\tilde{m}_s) ds$ does not depend on x, taking

$$\rho_1^{\varepsilon}(x) = \tilde{q}(1,x) \exp\left\{ -\frac{1}{2\varepsilon} \int_0^{1/\varepsilon} h^2(\tilde{m}_s)) ds \right\}$$

gives a version of the unnormalized conditional density that coincides with (2.2). □

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