From Brownian motion with a local time drift to Feller's branching diffusion with logistic growth

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Abstract

We give a new proof for the Ray-Knight representation of Feller's branching diffusion with logistic growth in terms of the local times of a reflected Brownian motion H with a drift that is affine linear in the local time accumulated by H at its current level. This proof is inspired by previous work of Norris, Rogers and Williams [5]. The arguments from stochastic analysis complement our recent work [4] which focussed on an approximation by Harris paths that code the genealogies of particle systems.

1 Introduction

The second one of the two classical Ray-Knight theorems (see e.g. [8] or [9]) establishes a close relation between reflected Brownian motion and Feller's branching diffusion. In [4] we proved a generalization, where the role of reflected Brownian motion is taken by the solution of the SDE (4) below, and the Feller branching diffusion is replaced by a Feller branching diffusion with logistic growth [3], see (3) below. As explained in [4] (see also [6]), the process H solving (4) is the exploration process of a forest of random real trees that describes (a version of) the genealogy of a population whose size

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evolution is modeled by the Feller diffusion with logistic growth. The generalized Ray–Knight theorem announced in [6] states how the Feller branching diffusion with logistic growth can be recovered from the exploration process.

Our proof in [4] used an approximation by discrete populations. While that proof is a bit complicated, we believe that the approach of [4] gives a worthy insight into the result, since the way how the genealogy is built and the exploration process codes the genealogical tree of the population is easily understandable at the discrete level.

After the completion of [4], thanks to a hint of J.F. Le Gall, our attention was drawn to work of Norris, Rogers and Williams [5], who provided a generalization of the first Ray–Knight theorem for "Brownian motions with a local time drift" for cases that include the drift appearing in the SDE (4). The technique of [5] helps to obtain the Ray-Knight representation of Feller's branching diffusion with logistic drift purely in terms of stochastic analysis, without discrete approximation. Such a direct proof is accomplished in the present note; the key idea is to use a martingale representation of the Girsanov density (which induces the "local time drift" of the exploration processes $\{H_s\}$) also in the direction of "real time" t, that is with respect to the so called excursion filtration $\{\mathcal{E}_t\}$. We also extend slightly our previous result by establishing an equality between laws of random fields (random functions of time t and ancestral mass x).

In Section 2 we introduce Feller's branching diffusion with logistic growth as a random field $\{Z_t^x\}$. This is a natural set-up for the formulation of our main result, which is given in Section 3 and whose proof is contained in Section 4. The last section gives two remarks concerning a possible shortcut in the proof of the Theorem, and a general version of the second Ray-Knight theorem in the framework of [5].

2 A coupling over the ancestral masses

In this section we define a random field $\{Z_t^x,\ t,x \geq 0\}$ such that for any $x \geq 0$, $Z^x := \{Z_t^x,\ t \geq 0\}$ is a weak solution of the SDE (3) formulated below, i.e. a Feller branching diffusion with logistic growth and ancestral mass x, and for any $t \geq 0$, $x \mapsto \{Z_t^x,\ t,x \geq 0\}$ is non-decreasing. This requires a coupling of the Z^x, $x \geq 0$ which can easily be explained in terms of an an individual-based model considered in [6] and [4]. The idea is to think of the individuals being arranged in a linear order "from left to right", where this order is passed on to the individual's offspring, and where the pairwise fights which induce the negative quadratic term in the logistic drift of the population size are always won by the individual to the left.

To be specific, we define a family of transition probabilities $\mathbf{P}_x, x \geq 0$, on E, where $E = C^c(\mathbb{R}_+, \mathbb{R}_+)$ is the set of continuous mappings from $\mathbb{R}_+$ to $\mathbb{R}_+$ with compact support. For $x > 0$ and $z \in C^c(\mathbb{R}_+, \mathbb{R}_+)$, let $\mathbf{P}_x(z, \cdot)$ be the distribution of $z + Z^{z,x}$, where $Z^{z,x}$ solves

$$Z_t^{x,z} = x + \int_0^t Z_u^{x,z}(\theta - \gamma[Z_u^{x,z} + 2z(u)])du + 2\int_0^t \sqrt{Z_u^{x,z}}dW_u, \qquad (1)$$

with W being a standard Brownian motion. We now show that the family $\mathbf{P}_x, x \geq 0$, satisfies the Chapman-Kolmogorov relations. To this end, we observe that conditioned on $Z^{x,z}$, the random path $V := Z^{y,z+Z^{x,z}}$ solves

$$V_t = y + \int_0^t V_u(\theta - \gamma[V_u + 2(z(u) + Z_u^{x,z})])du + 2\int_0^t \sqrt{V_u}dW'_u \qquad (2)$$

with W' being a standard Brownian motion (independent of W). Consequently, $Z^{x,z} + V$ satisfies

$$Z_t^{x,z} + V_t = x + y + \int_0^t (Z_u^{x,z} + V_u)(\theta - \gamma[Z_u^{x,z} + V_u + 2z(u)])du$$
$$+2\int_0^t \sqrt{Z_u^{x,z}}dW_u + 2\int_0^t \sqrt{V_u}dW'_u.$$

This shows that $z + Z^{x,z} + V$ has distribution $\mathbf{P}_{x+y}(z, \cdot)$, as required.

Definition Let $\{Z^x\}_{x\geq 0}$ be the $C^c(\mathbb{R}_+, \mathbb{R}_+)$-valued Markov chain with transition semigroup $(\mathbf{P}_x)$.

Remark 1 *For each $x > 0$, Z^x solves the SDE*

$$dZ_t^x = \left[\theta Z_t^x - \gamma (Z_t^x)^2\right] dt + 2\sqrt{Z_t^x}dW_t^x, \ Z_0^x = x, \qquad (3)$$

where $\{W_t^x, \ t \geq 0\}$ is a standard Brownian motion. Since the increments $Z^{x+y} - Z^x$, $x, y > 0$, are driven by independent Brownian motions, we have

$$d\langle Z^x, Z^{x+y}\rangle_t = d\langle Z^x, Z^x\rangle_t = Z_t^x \, dt$$

and conseqently

$$d\langle W^x, W^{x+y}\rangle_t = \sqrt{\frac{Z_t^x}{Z_t^{x+y}}}\, dt, \quad \text{with the convention } \frac{0}{0} = 0.$$

3 A Ray-Knight representation

Consider the following reflecting SDE driven by standard Brownian motion B

$$H_s = B_s + \frac{1}{2}L_s(0) + \frac{\theta}{2}s - \gamma \int_0^s L_r(H_r)dr, \ s \geq 0, \tag{4}$$

Here and everywhere below, $\{L_s(t), \ s \geq 0, \ t \geq 0\}$ denotes the local time of the process $\{H_s, \ s \geq 0\}$ accumulated up to time t at level s. Proposition 2, stated and proved in the next section, will ensure (by specializing it to the case $z \equiv 0$) that equation (4) has a unique weak solution, which we assume to be defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Define for any $x > 0$ the stopping time

$$S_x = \inf\{s > 0, \ L_s(0) > x\},$$

and let $\{Z_t^x, \ x, t \geq 0\}$ denote the random field constructed in Section 2.
Our main result is the
Theorem *The two random fields* $\{L_{S_x}(t), \ t, x \geq 0\}$ *and* $\{Z_t^x, \ t, x \geq 0\}$ *have the same law.*

4 Proof of the Theorem

To prepare for the proof of the Theorem, we first fix a $z \in C^c(\mathbb{R}_+, \mathbb{R}_+)$ and consider the SDE

$$H_s^z = B_s + \frac{1}{2}L_s^z(0) + \frac{\theta}{2}s - \gamma \int_0^s \{z(H_r^z) + L_r^z(H_r^z)\}dr, \ s \geq 0, \tag{5}$$

where L^z stands for the local time of H^z. We are going to prove in Subsection 4.1 the

Proposition 2 *The SDE* (5) *has a unique weak solution.*

Suppressing the superscript z, define for any $x > 0$ the stopping time

$$S_x = \inf\{s > 0, \ L_s^z(0) > x\}. \tag{6}$$

The main step in the proof of the Theorem will be to show

Proposition 3 *For* $x > 0$ *and* $z \in C^c(\mathbb{R}_+, \mathbb{R}_+)$ *let* $\{Z_t^{x,z}, \ t \geq 0\}$ *be the solution of* (1). *Then the two processes* $\{L_{S_x}^z(t), \ t \geq 0\}$ *and* $\{Z_t^{x,z}, \ t \geq 0\}$ *have the same law.*

4.1 Proof of Proposition 2

This is an easy adaptation of the arguments in [4], Section 2, which we do explicitly for the reader's convenience, suppressing the superscript z.

Let H denote Brownian motion reflected above 0, i. e.

$$H_s = B_s + \frac{1}{2} L_s(0),$$

where B is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and L is the semimartingale local time of H. Existence of a weak solution to (5) will follow from the existence of a new probability measure $\tilde{\mathbb{P}}$ under which

$$\tilde{B}_s := B_s - \int_0^s \left\{ \frac{\theta}{2} - \gamma \left[z(H_r) + L_r(H_r) \right] \right\} dr, \qquad s \geq 0, \tag{7}$$

is a standard Brownian motion. A sufficient condition for the existence of such a $\tilde{\mathbb{P}}$ is that

$$\mathbb{E} \exp \left(M_s - \frac{1}{2} \langle M \rangle_s \right) = 1, \qquad s \geq 0, \tag{8}$$

with $M_s := \int_0^s \left\{ \frac{\theta}{2} - \gamma \left[z(H_r) + L_r(H_r) \right] \right\} dB_r$.

From Theorem 1.1, chapter 7 (page 152) in [2], a sufficient condition for (8) is that for each $s > 0$ there exists constants $a > 0$ such that

$$\sup_{0 \leq r \leq s} \mathbb{E} \exp(a R_r) < \infty. \tag{9}$$

In our situation, $R_r = \left| \frac{\theta}{2} - \gamma \left[z(H_r) + L_r(H_r) \right] \right|^2$, where z is bounded. Hence all we have to show is

Lemma 1 *Let H be a Brownian motion on $\mathbb{R}_+$ reflected at the origin. Then for all $s > 0$ there exists $\alpha = \alpha(s) > 0$ such that*

$$\sup_{0 \leq r \leq s} \mathbb{E} \left(\exp(\alpha L_r(H_r)^2) \right) < \infty.$$

Proof: Together with a simple scaling argument and a desintegration with respect to H_r, this is immediate from the following

Lemma 2 *Let β be a standard Brownian motion starting at 0, and denote by $L_1(t)$ the local time accumulated by $|\beta|$ at position t up to time 1. There exist constants $a > 0$ and $c > 0$ (not depending on t) such that*

$$\mathbb{E}[e^{a L_1(t)^2} \,|\, |\beta_1| = t] \leq c, \quad t \geq 0. \tag{10}$$

Proof: Denote by $K_1(x)$ the local time of β accumulated up to time 1 at position x. First observe that for $t \geq 0$

$$L_1(t) = K_1(t) + K_1(-t) \text{ a.s.} \tag{11}$$

For deriving (10), by symmetry it suffices to condition under the event $\{\beta_1 = t\}$. Writing $\mathbb{P}^x$ for $\mathbb{P}[\,.\,|\beta_1 = x]$ we conclude from (11) and the Cauchy-Schwarz inequality that for all $a > 0$

$$\mathbb{E}^t[e^{aL_1(t)^2}] \leq \left(\mathbb{E}^t\left[e^{4aK_1(t)^2}\right]\right)^{1/2}\left(\mathbb{E}^t\left[e^{4aK_1(-t)^2}\right]\right)^{1/2}. \tag{12}$$

For $u \leq 1$, the distribution of $K_1(t)$ under $\mathbb{P}^t$ and conditioned under the event that β hits t first at time u, equals the distribution of $\sqrt{1-u}\,K_1(0)$ under $\mathbb{P}^0$. Similarly, for $u_1, u_2 \leq 1$, the distribution of $K_1(-t)$ under $\mathbb{P}^t$ and conditioned under the event that $(\beta_v)_{0\leq v\leq 1}$ hits $-t$ first at time u_1 and last at time u_2, equals the the distribution of $\sqrt{u_2-u_1}\,K_1(0)$ under $\mathbb{P}^0$. Consequently,

$$\mathbb{E}^t\left[e^{4aK_1(t)^2}\right] \leq \mathbb{E}^0\left[e^{4aK_1(0)^2}\right], \qquad \mathbb{E}^t\left[e^{4aK_1(-t)^2}\right] \leq \mathbb{E}^0\left[e^{4aK_1(0)^2}\right]. \tag{13}$$

By a result due to Lévy (see formula (11) in [7]), $K_1(0)$ has under $\mathbb{P}^0$ a Raleigh distribution, i.e.

$$\mathbb{P}^0(K_1(0) > \ell) = e^{-\frac{1}{2}\ell^2}.$$

This means that $K_1^2(0)$ is exponentially distributed, and hence, for suitably small $\delta > 0$, $\mathbb{E}^0\left[e^{\delta K_1(0)^2}\right]$ is finite. Now (10) follows from (12) and (13). ■

So far we have proved existence of a weak solution to (8). Weak uniqueness is easier to prove, since uniqueness is a local property. Let H be a solution to equation (8), and for all $n \geq 1$ let T_n denote the stopping time

$$T_n := \inf\{r > 0 : L_r(H_r) > n\}.$$

By a Girsanov transformation we can change the measure $\mathbb{P}$ into a measure $\bar{\mathbb{P}}$ under which, for all $n \in \mathbb{N}$, the restriction of the process H to the interval $[0, n \wedge T_n]$ is standard Brownian motion reflected above 0. Since $\mathbb{P}$ and $\bar{\mathbb{P}}$ are mutually absolutely continuous, the law of $\{H_{s\wedge n\wedge T_n},\ s \geq 0\}$ under $\mathbb{P}$ is uniquely determined, for each $n \geq 1$. Uniqueness of the law of H solution of (8) then follows, since $T_n \to \infty$ a. s. as $n \to \infty$.

4.2 Proof of Proposition 3

As a by-product of our proof, we will see that the stopping time S_x defined in (6) is finite a.s. A more direct argument for the a.s. finiteness of S_x would make the proof of Proposition 3 even shorter, see the discussion in Subsection 5.1. Since we have not been able to prove this directly, we circumvent this by reflecting the process H^z below the level K, and then let K tend to ∞.

To be specific, for $K > 0$, let H^K be the solution of the SDE

$$H_s^K = B_s + \frac{1}{2}L_s^K(0) - \frac{1}{2}L_s^K(K^-), \ s \geq 0, \tag{14}$$

where L^K stands for the local time of H^K and B is standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In other words, H^K is Brownian motion reflected inside the interval $[0, K]$.

Let us first note that if we define

$$S_x^K = \inf\{s > 0, \ L_s^K(0) > x\},$$

the following result follows readily from Lemma 2.1 in Delmas [1]

Lemma 3 *For any $K > 0$ the processes $\{L_{S_x}(t), \ 0 \leq t \leq K\}$ and $\{L^K_{S_x^K}(t), \ 0 \leq t \leq K\}$ have the same distribution.*

We next define the martingale

$$M_s^K = \int_0^s \left[\frac{\theta}{2} - \gamma\{z(H_r^K) + L_r^K(H_r^K)\}\right] dB_r.$$

The arguments in Section 2 of [4] show here also that for all $s > 0$,

$$\mathbb{E}\exp\left(M_s^K - \frac{1}{2}\langle M^K\rangle_s\right) = 1.$$

Therefore there exists a probability measure $\tilde{\mathbb{P}}^K$ such that for all $s > 0$,

$$\frac{d\tilde{\mathbb{P}}^K}{d\mathbb{P}}\bigg|_{\mathcal{F}_s} = \exp\left(M_s^K - \frac{1}{2}\langle M^K\rangle_s\right).$$

From Girsanov's theorem, under $\tilde{\mathbb{P}}^K$, H^K is a solution of the reflected SDE

$$H_s^K = B_s + \frac{\theta}{2}s - \gamma\int_0^s [z(H_r^K) + L_r^K(H_r^K)]dr + \frac{1}{2}L_s^K(0) - \frac{1}{2}L_s^K(K^-), \ s \geq 0. \tag{15}$$

It is obvious that 0 is a recurrent state both for the solution H^K of (15) and the solution of

$$\bar{H}_s^K = B_s + \frac{\theta}{2}s + \frac{1}{2}\bar{L}_s^K(0) - \frac{1}{2}\bar{L}_s^K(K^-), \ s \geq 0, \tag{16}$$

with $\bar{L}^K$ denoting the local time of $\bar{H}$.
By comparing the solutions of (15) and (16), we see that for fixed ε the conditional probability $\tilde{\mathbb{P}}(L_{s+1}^K(0) - L_s^K(0) > \varepsilon \,|\, \mathcal{F}_s \cap \{H_s^K = 0\})$ can be bounded away from 0 uniformly in s. This shows that

$$\tilde{\mathbb{P}}^K(S_x^K < \infty) = 1. \tag{17}$$

We conclude from Proposition 1.3, Chapter VIII of [8] that

$$\tilde{\mathbb{P}}^K \ll \mathbb{P} \ \text{ on } \mathcal{F}_{S_x^K}, \quad \text{and}$$
$$\frac{d\tilde{\mathbb{P}}^K}{d\mathbb{P}}\Big|_{\mathcal{F}_{S_x^K}} = \exp\left(M_{S_x^K}^K - \frac{1}{2}\langle M^K\rangle_{S_x^K}\right). \tag{18}$$

From Lemma 3 and the second Ray–Knight theorem (see e.g. [8], Thm. XI.2.4) we deduce that under $\mathbb{P}$ the process $\{Z_t^{x,K} := L_{S_x^K}^K(t), \ t \geq 0\}$ is a solution of the SDE

$$dZ_t^{x,K} = 2\sqrt{Z_t^{x,K}}dW_t, \quad Z_0^{x,K} = x$$

killed at time $t = K$. Thus, Proposition 3 follows immediately from Lemma 3 and

Proposition 4 *For any $K > 0$, the process $\{L_{S_x^K}^K(t), \ t \geq 0\}$ is under $\tilde{\mathbb{P}}^K$ a solution of equation* (1)*, killed at time K.*

4.3 Proof of Proposition 4

In this section, $x > 0$ and $K > 0$ are fixed. We start by working with the SDE (14), and take advantage of some of the techniques from [5].

Tanaka's formula gives for $0 \leq t < K$ the identity

$$L_{S_x^K}^K(t) = L_{S_x^K}^K(0) + 2\int_0^{S_x^K} \mathbf{1}_{\{H_s^K \leq t\}} dB_s, \tag{19}$$

since $\mathbf{1}_{\{H_s^K \le t\}} dL_s^K(K) = 0$ and $2H_0^K \wedge t - 2H_{S_x}^K \wedge t = 0$ (both terms vanish). On the other hand, from the second Ray–Knight theorem, $\{L_{S_x^K}^K(t),\ 0 \le t < K\}$ is a $\mathbb{P}$–martingale with quadratic variation given by

$$\langle L_{S_x^K}^K \rangle_t = 4 \int_0^t L_{S_x^K}^K(u) du.$$

Our aim is to identify the drift which

$$Y_t^K := 2 \int_0^{S_x^K} \mathbf{1}_{\{H_s^K \le t\}} dB_s = L_{S_x^K}^K(t) - x \tag{20}$$

acquires under the measure change (18). The key idea for this is to represent the random variable $M_{S_x^K}^K$ showing up in (18) as the terminal outcome ($t = K$) of the process

$$N_t^K = \int_0^{S_x^K} \mathbf{1}_{\{H_s^K \le t\}} \left(\frac{\theta}{2} - \gamma\{z(H_s^K) + L_s^K(H_s^K)\} \right) dB_s, \quad 0 \le t \le K,$$

which with respect to a suitable filtration turns out to be a $\mathbb{P}$-martingale. Following [5], this filtration is obtained as follows: We define for all $0 \le t \le K$ (suppressing the superscript K in the defined quantities)

$$\begin{aligned} A(s,t) &:= \int_0^s \mathbf{1}_{\{H_r^K \le t\}} dr, \quad \tau(s,t) := \inf\{r : \ A(r,t) > s\}, \\ H(s,t) &:= \int_0^t \mathbf{1}_{\{H_r^K \le t\}} dH_r^K, \quad \xi(s,t) := H(\tau(s,t),t), \\ \mathcal{F}(s,t) &:= \sigma(\{\xi(r,t) : \ r \le s\}), \quad \mathcal{E}_t := \mathcal{F}(\infty, t). \end{aligned}$$

It is shown in [10] that $\{\mathcal{E}_t,\ 0 \le t \le K\}$ is a right–continuous filtration, and it follows from Theorem 1 in [5] that $\{N_t^K,\ 0 \le t \le K\}$ is an $(\mathcal{E}_t)$–martingale. Lemma 3 in the same paper yields

$$\langle N^K, Y^K \rangle_t = \int_0^{S_x^K} \mathbf{1}_{\{H_s^K \le t\}} \left(\theta - 2\gamma\{z(H_s^K) + L_s^K(H_s^K)\}\right) ds \tag{21}$$

Reexpressing the r.h.s. of (21) via both the standard and the extended occupation times formula (for a reference to the latter, see e.g. Exercise 1.15 in Chapter VI of [8]) yields

$$\langle N^K, Y^K \rangle_t = \int_0^t [\theta - 2\gamma z(u)] L_{S_x^K}^K(u) du - \gamma \int_0^t \left(L_{S_x^K}^K \right)^2 (u) du. \tag{22}$$

In view of (20) and (22), Girsanov's theorem (see e.g. Theorem 1.4 in [8]) reveals that under $\tilde{\mathbb{P}}^K$,

$$R_t^K := L_{S_x^K}^K(t) - x - \int_0^t [\theta - 2\gamma z(u)] L_{S_x^K}^K(u)du + \gamma \int_0^t \left(L_{S_x^K}^K\right)^2(u)du$$

is a martingale on the interval $0 \le t \le K$. Since the quadratic variation remains unchanged under a Girsanov transformation, we infer that $\langle R^K \rangle_t = 4\int_0^t L_{S_x^K}^K(u)du$, $0 \le t < K$. Consequently, there exists a Brownian motion $\{W_t,\ t \ge 0\}$ such that $L_{S_x^K}^K(t)$ solves for $0 \le t < K$ the SDE (1).

4.4 Completion of the proof of the Theorem

It follows from the description of the law of $\{Z_t^x,\ t \ge 0\}_{x \ge 0}$ made in Section 2 that Z is Markov (as a process indexed by x, with values in the set of continuous paths from $\mathbb{R}_+$ into $\mathbb{R}_+$ with compact support). The fact that $\{L_{S_x}(t),\ t \ge 0\}_{\{x \ge 0\}}$ enjoys the same property follows from the fact that the process $H_r^x := H_{S_x + r}$ solves the SDE (5) with $z(t) = L_{S_x}(t)$ and a Brownian motion B which, from the strong Markov property of Brownian motion, is independent of $\{L_{S_x}(t),\ t \ge 0\}$.

Hence it suffices to prove that for any $0 \le x < x+y$, the conditional law of $L_{S_{x+y}}(\cdot)$ given $L_{S_x}(\cdot)$ equals that of $Z_\cdot^{x+y}$, given $Z_\cdot^x$. Conditioned upon $L_{S_x}(\cdot) = z(\cdot)$, $L_{S_{x+y}}(\cdot) - L_{S_x}(\cdot)$ is the collection of local times accumulated by the solution of (5) up to time S_y, i. e. it has the law of the process $\{L_{S_y}^z(t),\ t \ge 0\}$, while conditionally upon $Z_\cdot^x = z(\cdot)$, the law of $Z_\cdot^{x+y} - Z_\cdot^x$ is that of $Z^{y,z}$, solution of equation (1). Thus, the assertion of the Theorem follows from Proposition 3.

5 Concluding remarks

5.1 A possible shortcut in the proof of Proposition 3

As a direct consequence of Proposition 3 and the occupation times formula (cf. Remark 6.4 in [4]) we deduce (deleting the z for simplicity) that

$$S_x = \int_0^\infty Z_t^x dt.$$

This together with the fact that Feller's branching diffusion with logistic drift dies out a.s. in finite time proves

Lemma 4 *For any $x > 0$, the stopping time S_x defined in* (6) *is finite a.s.*

If we could prove Lemma 4 directly from the SDE (5), then we could simplify our proof of Proposition 3, avoiding the reflection below the arbitrary level K. Here is a hint which tries to explain why Lemma 4 holds. While climbing up, the Brownian motion with positive drift $\theta/2$ accumulates local time at various levels. Sooner or later, it accumulates so much local time around some level in $\mathbb{R}_+$ that the process H governed by (4) starts to go down. It then continues to accumulate local time at various levels, and goes back to zero. After reflection at zero, the next excursions will have already a stronger drift downwards that awaits H. It is remarkable that the recurrence of H to the state 0 holds independently of the relative constellations of the positive parameters θ and γ. Similarly, one may hope to learn about recurrence/transience properties of reflected Brownian motions with more general "local time drifts" from their Ray-Knight transforms.

5.2 A second Ray-Knight theorem for Brownian motion with a local time drift

The equation (5) is of the form

$$H_s = B_s + \frac{1}{2}L_s(0) + \int_0^s g(H_r, L_r(H_r))dr, \ \ s \geq 0, \tag{23}$$

The proof of Proposition 3 shows that $\{L_{S_x}(t), t \geq 0\}$ satisfies the SDE

$$Z_t = x + \int_0^t f(u, Z_u)du + 2\int_0^t \sqrt{Z_u}dW_u \tag{24}$$

with $f(t,\ell) = \int_0^\ell g(t,y)dy$, provided g is such that (23) and (24) have unique weak solutions which arise via Girsanov transformations from the distributions with $g \equiv 0$, and provided $S_x = \inf\{s > 0, \ L_s(0) > x\}$ is finite a.s. We intend to study this more general problem in the future, together with the interpretation of (24) as a model for the evolution of the size of a population.

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