Muller's ratchet clicks in finite time

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Abstract

We consider the accumulation of deleterious mutations in an asexual population, a phenomenon known as Muller's ratchet, using the continuous time model proposed in [4]. We show that for any parameter $\lambda > 0$ (the rate at which mutations occur), for any $\alpha > 0$ (the toxicity of the mutations) and for any size $N > 0$ of the population, the ratchet clicks a.s. in finite time. That is to say the minimum number of deleterious mutations in the population goes to infinity a.s.

1   Introduction

In natural evolution, deleterious mutations occur much more frequently than beneficial ones. Since the last category is always favored by selection, one may wonder about the advantage of sexual reproduction over the asexual type. The answer has been proposed : in an asexually reproducing population, each individual always inherits all the deleterious mutations of his ancestor (except if another mutation occurs at the same locus on the genome; but this event is rare and we will not consider it) whereas in sexual reproduction, recombinations occur, which allow an individual to take part of a chromosome from each of his parents, therefore giving him a chance to get rid of deleterious mutations. Muller's ratchet can be used as an attempt to translate this phenomenon in a mathematical model, thus explaining the advantage of sexual reproduction [7]. If one considers the best class (the group of fittest individuals) in a given asexual population, Muller's ratchet is said to click when the best class becomes empty. Since beneficial mutations do not occur in this model, it means that all the individuals of the best class have mutated.

The first model for Muller's ratchet due to Haigh [5] can be described as follows. Consider an asexual population of fixed sized N which evolves in discrete time, with a multiplicative selection model. Only deleterious mutations occur. Denoting by $0 \leq \alpha \leq 1$ the deleterious strength of the mutations, and $\lambda > 0$ the rate at which

they occur, every generation is constituted as follows : each individual chooses a parent from the previous generation, in such a way that the probability of choosing a specific father with k deleterious mutations is (we denote by N_k the number of such individuals in the previous generation) :

$$\frac{(1-\alpha)^k}{\sum_{k=0}^{N} N_k(1-\alpha)^k}.$$

Next each newborn gains K deleterious mutations, where K a Poisson random variable with parameter λ. It is immediate to see that this model clicks a.s. in finite time. Indeed at each generation, with probability $(1-\exp(-\lambda))^N$ all the individuals mutate, which induces the click.

There are three parameters in our model :

N is the size of the population,

λ is the mutation rate,

α is the fitness decrease due to each mutation.

The Fleming–Viot model for Muller's ratchet proposed by A. Etheridge, P. Pfaffelhuber and A. Wakolbinger in [4] consists in the following infinite set of SDEs for the $X_k(t)$'s, $k \geq 0$, where $X_k(t)$ denotes the proportion of individuals in the population who carry exactly k deleterious mutations at time t (with $X_{-1} \equiv 0$) :

$$\begin{cases} dX_k(t) = [\alpha(M_1(t) - k)X_k(t) + \lambda(X_{k-1}(t) - X_k(t))] \, dt + \sum_{\ell \geq 0, \ell \neq k} \sqrt{\frac{X_k(t)X_\ell(t)}{N}} dB_{k,\ell}(t) \\ X_k(0) = x_k; \ k \geq 0, \end{cases}$$

$$(1.1)$$

where $\{B_{k,\ell}, k > \ell \geq 0\}$ are independent Brownian motions, $B_{k,\ell} = -B_{\ell,k}$; and $M_1(t) = \sum_{k \geq 0} kX_k(t)$.

The first term in the drift models the selective effect of the deleterious mutations. Those individuals who cary less (resp. more) mutations than the average number of mutations in the population have a selective advantage (resp. disadvantage). The second term in the drift reflects the effect of the accumulation of mutations : at rate λ, individuals carrying $k-1$ mutations gain a k–th mutation, they jump into the k–class, and at the same rate individuals carrying k mutation gain a $k+1$–th mutation, they jump out of the k–class. The diffusion term reflects the resampling effect of the birth events, where the factor $N^{-1/2}$ can be understood as being equivalent to the rescaling of time $t \to t/N$, if N is the "effective population size", which is natural in Kingman's coalescent [6]. For the equivalence between the present model and a more intuitive look–down model à la Donnelly–Kurtz, we refer the reader to [1].

We will show in section 2 that the infinite dimensional of SDEs (1.1) is well posed provided we choose the initial condition $x = (x_k, \ k \geq 0) \in \mathcal{X}_\delta$ for some $\delta > 0$, where

$$\mathcal{X}_\delta := \left\{ x \in [0,1]^\infty, \quad \sum_{k=0}^\infty x_k = 1, \quad \sum_{k=0}^\infty k^{2+\delta} x_k < \infty \right\}.$$

We define $T_0 = \inf\{t > 0, X_0(t) = 0\}$. The purpose of the present work is to show that this model of Muller's ratchet is bound to click in finite time, that is to say $T_0 < \infty$ a.s. . We are going to prove the following theorem :

Theorem 1 *For any $\delta > 0$, for any choice of initial condition in $\mathcal{X}_\delta$, let $(X_k(t))_{k \in \mathbb{Z}_+}$ be the solution of* (1.1). *Then $\mathbb{P}(T_0 < \infty) = 1$.*

We will in fact prove a stronger result, namely

Theorem 2 *For any $\delta > 0$, for any choice of initial condition in $\mathcal{X}_\delta$, let $(X_k(t))_{k \in \mathbb{Z}_+}$ be the solution of* (1.1). *Then there exists $\overline{\rho} > 0$ such that $\mathbb{E}\left[\exp(\rho T_0)\right] < \infty$, for all $0 < \rho < \overline{\rho}$.*

Clearly, a model for Muller's ratchet must have the property that the ratchet clicks in finite time. In a sense our result says that the Etheridge–Pfaffelhuber–Wakolbinger model for Muller's ratchet is a reasonable model, in the sense that it exhibits a.s. clicking, as the computer simulations had already shown, see [4]. Note that once the zero class is empty, the 1–class takes its place, and some time later a second click happens, at which time both the zero class and the 1–class become empty, and so on. Of course, we would like to know more about the time it takes for the ratchet to click. Here we show that its expectation is finite. We hope to get more precise information in some future work.

There are several difficulties in this model. First, it is an infinite system of SDEs which cannot be reduced to a finite dimensional system. Only X_0 and M_1 enter the coefficients of the equation for X_0, but the equation for M_1 brings in the second moment M_2. The system of SDEs for the centered moments of all orders is infinite as well, the moments of order up to $\ell = 2k$ enter the coefficients of the equation for k–th centered moment, and there is no known solution to it (except in the deterministic case $N = +\infty$, which is solved in [4]). In addition, one has $d \langle X_0, M_1 \rangle = -\frac{M_1 X_0}{N} dt$. But there is no easy relation between X_0 and M_1, except that $X_0 + M_1 \geq 1$, and $(X_0 = 1) \Rightarrow (M_1 = 0)$. But we could have $X_0 \to 0$ and $M_1 \to \infty$. Last but not least, the diffusion coefficient in dX_k is not a Lipschitz function of X_k at 0 and 1, and it vanishes at those two points.

In order to prove the theorem, we will use a three steps proof. First, in section 3 we will show that M_1 cannot grow too fast with a good probability, and we will deduce that for a specific set of initial conditions, the ratchet does click with a strictly positive probability p_{fin}, in a given interval of time.

Next, we show in section 4 that the product $X_0 M_1^2$ is bound to come back under $\frac{2(\lambda+1)}{\alpha}$ after any time, and we use all the previous results to deduce that M_1 is also bound to return under $\beta = \frac{\lambda}{\alpha}$ after any time, as long as the ratchet does not click.

Finally in section 5 we prove that each time M_1 gets below β, the ratchet clicks with a positive probability in a prescribed interval of time. We then conclude with the help of the strong Markov property.

In section 6 we show how the proof of Theorem 1 can be modified into a proof of Theorem 2. The reader may wonder why we do not prove Theorem 2 from the very beginning, and first prove a weaker result. The reason is that the difference between the two proofs is essentially that while proving Theorem 1, we prove that as long as the ratchet has not clicked, M_1 is bound to return below the value β, i.e. the drift of X_0 is bound to become non–positive, which is an interesting result in itself, while the proof of Theorem 2 is based on the same strategy, but with β replaced by a much less explicit quantity.

We shall essentially work with the two dimensional process $\{X_0(t), M_1(t)\}$, and we shall use the equation for X_1 only in one place, namely in Lemma 5.1 in order to show that X_0 does not get stuck near the value 1. This means that we shall make use only of the three following equations.

$$dX_0(t) = (\alpha M_1(t) - \lambda) X_0(t) dt + \sqrt{\frac{X_0(t)(1 - X_0(t))}{N}} dB_0(t)$$

$$dX_1(t) = (\alpha(M_1(t) - 1)X_1(t) + \lambda(X_0(t) - X_1(t))) dt + \sqrt{\frac{X_1(t)(1 - X_1(t))}{N}} dB_1(t)$$

$$dM_1(t) = (\lambda - \alpha M_2(t)) dt + \sqrt{\frac{M_2(t)}{N}} dB(t).$$

The three Brownian motions B_0, B_1 and B are standard Brownian motions. But they are not independent, and the three dimensional process $(B_0(t), B_1(t), B(t))$ is not a Gaussian process. But this will play no role in our analysis. This system is not closed, since M_2 enters the coefficients of the last equation. However, the crucial remark is that it will not be necessary to estimate M_2, in order to estimate M_1. This is due to the fact that the M_1–equation takes the form $dM_1(t) = \lambda dt + dZ_t$, where $Z_t = W(A_t) - \alpha N A_t$, if $A_t := N^{-1} \int_0^t M_2(s) ds$ and $\{W(t), \ t \geq 0\}$ is a standard Brownian motion. The larger M_2 is, the more likely Z_t is negative, which produces

a smaller M_1. This means that we should be able to estimate M_1, without having to estimate M_2, which is done below in Lemma 3.2 and 4.3.

2   Preliminary results

The aim of this section is to establish a weak existence and uniqueness result for the infinte system of SDEs (1.1), under the following condition on the initial condition

$$x \in \mathcal{X}_\delta, \quad \text{for some } \delta > 0, \text{ where}$$

$$\mathcal{X}_\delta := \left\{ x \in [0,1]^\infty, \quad \sum_{k=0}^\infty x_k = 1, \quad \sum_{k=0}^\infty k^{2+\delta} x_k < \infty \right\}.$$

We equip this set with the topology under which a probability $x^n = (x_k^n, \ k \geq 0)$ on $\mathbb{Z}_+$ converges to $x = (x_k, \ k \geq 0)$ if both it converges weakly, and $\sup_n \sum_{k \geq 0} k^{2+\delta} x_k^n < \infty$. More precisely, we will prove in this section

Theorem 3 *If the initial condition x belongs to $\mathcal{X}_\delta$, for some $\delta > 0$, then (1.1) has a unique weak solution $X(t) = \{X_k(t), \ k \geq 0\}$ which is a. s. continuous with values in $\mathcal{X}_\delta$.*

Remark 2.1 *Previous results on this system of SDEs assume that the probability x on $\mathbb{Z}_+$ possesses exponential moments of arbitrary order, see [3], or of some order, see [8]. This assumption is naturally requested if one wants to be able to write equations for arbitrary moments of the random measure $X(t)$ on $\mathbb{Z}_+$. However, we will need only to make sure that $M_1(t)$ and $M_2(t)$ have finite expectation, and for that purpose our weaker condition will be sufficient.*

We start with the case $\alpha = 0$.

2.1   The case $\alpha = 0$

Proposition 2.2 *Suppose that $\alpha = 0$. Then, for any initial condition $x \in \mathcal{X}_\delta$, (1.1) has a unique weak solution $X(t) = \{X_k(t), \ k \geq 0\}$ which is a. s. continuous with value in $\mathcal{X}_\delta$, and is such that for each $\lambda, \delta > 0$, there exists a locally bounded function $C_{\lambda,\delta}(t)$ such that*

$$\mathbb{E} \sum_{k=0}^\infty k^{2+\delta} X_k(t) \leq C_{\lambda,\delta}(t). \tag{2.1}$$

PROOF : Let us rewrite our system of SDEs in the particular case $\alpha = 0$ (again it is written with the convention that $X_{-1}(t) \equiv 0$) in the form

$$
\begin{cases}
dX_k(t) = \lambda(X_{k-1}(t) - X_k(t))dt + \mathcal{M}_k(t), \ k \geq 0; \\
\langle \mathcal{M}_k, \mathcal{M}_\ell \rangle_t = N^{-1} \int_0^t X_k(s)(\delta_{k,\ell} - X_\ell(s))ds, \ k, \ell \geq 0; \\
X_k(0) = x_k, \ k \geq 0;
\end{cases}
\tag{2.2}
$$

where the $\mathcal{M}_k(t)$'s are continuous martingales, and $\langle \mathcal{M}_k, \mathcal{M}_\ell \rangle$ stands for the joint quadratic variation of the two martingales $\mathcal{M}_k$ and $\mathcal{M}_\ell$. We can apply the result of Theorem 2.1 in [10], which insures that (2.2) has a unique weak solution. The facts that $X_k(t) \geq 0$, for all $k \geq 0$, $t \geq 0$, a.s. and $\sum_{k \geq 0} X_k(t) = 1$ for all $t \geq 0$ a.s. follow from the results in [10], but we give a proof of the second fact for the convenience of the reader.

Summing the k–th equation in the system (2.2) from $k = 0$ to $k = K$ yields

$$
\sum_{k=0}^{K} X_k(t) = \sum_{k=0}^{K} x_k - \lambda \int_0^t X_K(s)ds + \mathcal{M}_{0,K}(t),
$$

where $\mathcal{M}_{0,K}(t)$ is a continuous martingale whose quadratic variation is given by

$$
\langle \mathcal{M}_{0,K} \rangle_t = N^{-1} \int_0^t \left(\sum_{k=0}^{K} X_k(s) \right) \left(1 - \sum_{k=0}^{K} X_k(s) \right) ds.
$$

We first deduce that

$$
\mathbb{E} \left(\sum_{k=0}^{K} X_k(t) \right) \leq \sum_{k=0}^{K} x_k,
$$

$$
\mathbb{E} \left(\sum_{k=0}^{\infty} X_k(t) \right) \leq \sum_{k=0}^{\infty} x_k,
$$

where the second inequality follows from the first by monotone convergence. It then follows that $\int_0^t X_K(s)ds \to 0$ as $K \to \infty$, while by dominated convergence

$$
\int_0^t \left(\sum_{k=0}^{K} X_k(s) \right) \left(1 - \sum_{k=0}^{K} X_k(s) \right) ds \to \int_0^t \left(\sum_{k=0}^{\infty} X_k(s) \right) \left(1 - \sum_{k=0}^{\infty} X_k(s) \right) ds
$$

a.s. as $K \to \infty$. It follows from these considerations that the process $Y(t) := \sum_{k=0}^{\infty} X_k(t)$ solves the SDE

$$Y(t) = 1 + N^{-1/2} \int_0^t \sqrt{Y(s)[1 - Y(s)]} d\tilde{B}_s,$$

for some standard Brownian motion $\tilde{B}$. This clearly shows that $Y \equiv 1$ a.s. We next have

$$\mathbb{E}\left(\sum_{k=0}^{K} k X_k(t)\right) = \sum_{k=0}^{K} k x_k + \lambda \mathbb{E} \int_0^t \sum_{k=0}^{K} (k X_{k-1}(s) - k X_k(s))\, ds,$$

$$\mathbb{E}\left(\sum_{k=0}^{\infty} k X_k(t)\right) \leq \sum_{k=0}^{\infty} k x_k + \lambda t,$$

since $\sum_{j=0}^{K-1} X_j(s) \leq 1$. Furthermore, using this last inequality in the last step below,

$$\mathbb{E}\left(\sum_{k=0}^{K} k^2 X_k(t)\right) = \sum_{k=0}^{K} k^2 x_k + \lambda \mathbb{E} \int_0^t \sum_{k=0}^{K} \left(k^2 X_{k-1}(s) - k^2 X_k(s)\right) ds$$

$$\leq \sum_{k=0}^{K} k^2 x_k + \lambda \mathbb{E} \int_0^t \sum_{j=0}^{K-1} (2j+1) X_j(s) ds,$$

$$\mathbb{E}\left(\sum_{k=0}^{\infty} k^2 X_k(t)\right) \leq \sum_{k=0}^{\infty} k^2 x_k + (\lambda + \lambda^2) t + 2\lambda t \sum_{k=0}^{\infty} k x_k.$$

Let us now suppose that $0 < \delta \leq 1$, and we exploit the fact that $k^{2+\delta} - (k-1)^{2+\delta} \leq (k-1)^{1+\delta} \leq (k-1)^2$. We then deduce that

$$\mathbb{E}\left(\sum_{k=0}^{K} k^{2+\delta} X_k(t)\right) = \sum_{k=0}^{K} k^{2+\delta} x_k + \lambda \mathbb{E} \int_0^t \sum_{k=0}^{K} \left(k^{2+\delta} X_{k-1}(s) - k^{2+\delta} X_k(s)\right) ds$$

$$\leq \sum_{k=0}^{K} k^{2+\delta} x_k + \lambda \mathbb{E} \int_0^t \sum_{j=1}^{K-1} j^2 X_j(s) ds$$

$$\mathbb{E}\left(\sum_{k=0}^{\infty} k^{2+\delta} X_k(t)\right) \leq C_2(\lambda, t)$$

from the last estimate. If $\delta > 1$, we need to estimate the third moment in terms of the second, then the fourth in terms of the third, ..., and finally the $2+\delta$–th in terms of the $2 + \lfloor \delta \rfloor$–th.

So far we have proved that $X(t) \in \mathcal{X}_\delta$ a. s. for all $t \geq 0$. We now prove that in fact a. s., $X(t) \in \mathcal{X}_\delta$ for all $t \geq 0$. Our next argument will be very similar to an argument in [8]. For any $m \geq 1$, $t \geq 0$, let

$$N_{m,\delta}(t) := \sum_{k=0}^{\infty} \inf(k,m)^{2+\delta} X_k(t).$$

It is easy to check that $\{N_{m,\delta}(t),\ t \geq 0\}$ is a positive submartingale, to which we can apply Doob's inequality, which, together with the monotone convergence theorem, yields that for any $K, T > 0$,

$$
\begin{aligned}
\mathbb{P}\left(\sup_{0 \leq t \leq T} \sum_{k=0}^{\infty} k^{2+\delta} X_k(t) > K \right) &= \lim_{m \to \infty} \mathbb{P}\left(\sup_{0 \leq t \leq T} N_{m,\delta}(t) > K \right) \\
&\leq \lim_{m \to \infty} K^{-1} \mathbb{E}\left[N_{m,\delta}(t) \right] \\
&= K^{-1} \mathbb{E}\left[\sum_{k=0}^{\infty} k^{2+\delta} X_k(T) \right] \\
&\leq K^{-1} C_{\lambda,\delta}(T),
\end{aligned}
$$

where we have used (2.1) for the last inequality. It now follows that for all $T > 0$,

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} \sum_{k=0}^{\infty} k^{2+\delta} X_k(t) < \infty \right) = 1.$$

The a. s. continuity with values in $\mathcal{X}_\delta$ is now easy to check. $\diamond$

We next want to establish the equation for the first moment $M_1(t) := \sum_{k \geq 1} k X_k(t)$. This equation will involve the process $M_2(t) = \sum_{k \geq 1} k^2 X_k(t) - [M_1(t)]^2$. We know by now that those quantities are well defined and finite.

Proposition 2.3 *The first moment solves the SDE*

$$dM_1(t) = \lambda dt + d\mathcal{M}(t),$$

where $\{\mathcal{M}(t),\ t \geq 0\}$ is a continuous martingale satisfying

$$\langle \mathcal{M}, \mathcal{M} \rangle_t = N^{-1} \int_0^t M_2(s) ds,$$

and for any $k \geq 0$,

$$\langle \mathcal{M}, \mathcal{M}_k \rangle_t = N^{-1} \int_0^t (k - M_1(s)) X_k(s) ds.$$

PROOF : For any $K > 1$, let $M_{1,K}(t) := \sum_{k=1}^{K} k X_k(t)$. We have readily

$$M_{1,K}(t) = M_{1,K}(0) + \lambda \int_0^t \sum_{j=0}^{K-1} X_j(s) ds - \lambda \int_0^t K X_K(s) ds + \mathcal{M}_{1,K}(t),$$

where $\mathcal{M}_{1,K}(t)$ is a continuous martingale, with

$$d\langle \mathcal{M}_{1,K} \rangle_t = N^{-1} \left(\sum_{k=1}^{K} k^2 X_k(t) - [M_{1,K}(t)]^2 \right) dt.$$

It follows from (2.1) that

$$\mathbb{E} \int_0^t K X_K(s) ds \to 0$$

as $K \to \infty$. Consequently, all terms in the above equation converge as $K \to \infty$, yielding that

$$M_1(t) = M_1(0) + \lambda t + \mathcal{M}(t),$$

where $\mathcal{M}(t)$ is a continuous martingale as follows from the next lemma, which is such that

$$d\langle \mathcal{M}, \mathcal{M} \rangle_t = N^{-1} M_2(t) dt.$$

Moreover, if $1 \le k \le K$,

$$d\langle M_{1,K}, X_k \rangle_t = N^{-1}[k X_k(t)(1 - X_k(t)) - \sum_{\ell \ne k, \ell \le K} \ell X_k(t) X_\ell(t)] dt$$

$$= N^{-1} X_k(t)[k - \sum_{\ell \le K} \ell X_\ell(t)] dt.$$

The second part of the result follows, by letting $K \to \infty$. $\Diamond$

To complete this last proof, we need to establish

Lemma 2.4 *The collection of processes $\{\mathcal{M}_{1,K}(t),\ t \ge 0\}_{K \ge 1}$ is tight in $C([0, +\infty))$.*

PROOF : From the Corollary of Theorem 7.4 page 83 in [2] and Chebychef's inequality, it suffices to prove that for each $T > 0$ there exists a constant $C(\delta, T)$ such that for all $0 \le s < t \le T$,

$$\mathbb{E}\left[|\mathcal{M}_{1,K}(t) - \mathcal{M}_{1,K}(s)|^{2+\delta} \right] \le C(T)|t - s|^{1+\delta/2}. \tag{2.3}$$

From the well–known Davis–Burkholder–Gundy inequality (see e. g. [9]), there exists a constant $c(\delta)$ such that

$$\mathbb{E}\left[|\mathcal{M}_{1,K}(t) - \mathcal{M}_{1,K}(s)|^{2+\delta}\right] \le c(\delta)\mathbb{E}\left[(\langle\mathcal{M}_{1,K}\rangle_t - \langle\mathcal{M}_{1,K}\rangle_s)^{1+\delta/2}\right].$$

Forgetting a power of N, we have, using Jensen's inequality in two distinct instances,

$$\mathbb{E}\left[\left(\int_s^t \sum_{k\ge 0} k^2 X_k(r)dr\right)^{1+\delta/2}\right] \le (t-s)^{\delta/2}\mathbb{E}\int_s^t\left(\sum_{k\ge 0}k^2 X_k(r)\right)^{1+\delta/2}dr$$

$$\le (t-s)^{\delta/2}\mathbb{E}\int_s^t\sum_{k\ge 0}k^{2+\delta}X_k(r)dr.$$

(2.3) follows by combining the two last estimates with (2.1). $\diamond$

2.2 The general case

We can now prove Theorem 3. We first proceed with the

PROOF OF EXISTENCE We now introduce a Girsanov transformation. $\{B(t),\ t \ge 0\}$ denoting the Brownian motion which appeared in Proposition 2.3, for any $\alpha > 0$, let

$$Z_\alpha(t) := \exp\left(-\alpha\sqrt{N}\int_0^t\sqrt{M_2(s)}dB_s - \frac{\alpha^2 N}{2}\int_0^t M_2(s)ds\right).$$

It is easily seen that

$$Z_\alpha(t) = \exp\left(N\alpha\left[M_1(0) + \lambda t - M_1(t) - \frac{\alpha}{2}\int_0^t M_2(s)ds\right]\right)$$

$$\le \exp\left(N\alpha\left[M_1(0) + \lambda t\right]\right).$$

It is now clear that $\{Z_\alpha(t),\ t \ge 0\}$ is a martingale, and consequently there exists a unique probability measure $\mathbb{P}^\alpha$ on $(\Omega, \mathcal{F})$, such that for all $t > 0$,

$$\left.\frac{d\mathbb{P}^\alpha}{d\mathbb{P}}\right|_{\mathcal{F}_t} = Z_\alpha(t).$$

It now follows from Girsanov's theorem that there exist $\mathbb{P}^\alpha$–standard Brownian motions $\{B^\alpha(t),\ t \ge 0\}$ and $\{B_k^\alpha(t),\ t \ge 0\}$ for all $k \ge 0$, such that

$$\int_0^t\sqrt{\frac{M_2(s)}{N}}dB(s) = -\alpha\int_0^s M_2(s)ds + \int_0^t\sqrt{\frac{M_2(s)}{N}}dB^\alpha(s),\ \text{and for all } k \ge 0,$$

$$\int_0^t\sqrt{X_k(s)(1 - X_k(s))N}dB_k(s) = \int_0^t \alpha(M_1(s) - k)X_k(s)ds + \int_0^t\sqrt{X_k(s)(1 - X_k(s))N}dB_k^\alpha(s).$$

Consequently under $\mathbb{P}^{\alpha}$, we have proved weak existence to our infinite dimensional system (1.1). We can now turn to the

PROOF OF UNIQUENESS We now exploit again Girsanov's theorem to prove weak uniqueness. Consider for some $\alpha, \delta > 0 > 0$ any $\mathcal{X}_{\delta}$–valued solution of our SDE, which we rewrite as

$$X_k(t) = x_k + \int_0^t \left[\alpha(M_1(s) - k)X_k(s) + \lambda(X_{k-1}(s) - X_k(s)) \right] ds + \mathcal{M}_k(t), \ k \geq 0;$$

$$M_1(t) = \sum_{k \geq 0} kx_k + \int_0^t \left[\lambda - \alpha M_2(s) \right] ds + \mathcal{M}(t),$$

where for $k, \ell \geq 0$,

$$\langle \mathcal{M}_k, \mathcal{M}_\ell \rangle_t = N^{-1} \int_0^t X_k(s)(\delta_{k,\ell} - X_\ell(s)) ds,$$

$$\langle \mathcal{M}_k, \mathcal{M} \rangle_t = N^{-1} \int_0^t X_k(s)((k - M_1(s)) ds,$$

$$.\langle \mathcal{M}, \mathcal{M} \rangle_t = N^{-1} \int_0^t M_2(s) ds,$$

defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{Q}^{\alpha}$ denote the probability law of our solution on the space $C([0, +\infty); \mathcal{X}_{\delta})$, and define, for $t \geq 0$,

$$Y_{\alpha}(t) = \exp\left(\alpha \sqrt{N} \mathcal{M}_t - \frac{\alpha^2 N}{2} \int_0^t M_2(s) ds \right).$$

For each $n \geq 1$, let
$$\tau_n := \inf\left\{ t > 0, \ \int_0^t M_2(s) ds > n \right\}.$$

It is not hard to show that for each $n \geq 1$, the probability measure $\mathbb{Q}$ defined on $C([0, +\infty); \mathcal{X}_{\delta})$ equipped with its Borel σ–field, by

$$\left. \frac{d\mathbb{Q}}{d\mathbb{Q}^{\alpha}} \right|_{\mathcal{F}_{\tau_n}} = Y_{\alpha}(\tau_n)$$

coincides with the law of the unique weak solution of (2.2) up to time τ_n. Hence the restriction of $\mathbb{Q}^{\alpha} = (Y_{\alpha}(\tau_n))^{-1} \cdot \mathbb{Q}$ to the σ–algebra $\mathcal{F}_{\tau_n}$ coincides with the law of the solution which we have constructed above. Since $\tau_n \to \infty$ a.s., weak uniqueness is proved.

2.3 A comparison theorem for one–dimensional SDEs

We state a result, which will be useful later in this paper. Our processes are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with a filtration $(\mathcal{F}_t, t \geq 0)$ which is such that for each $k, \ell \geq 0$ $\{B_{k,\ell}(t),\ t \geq 0\}$ is a $\mathcal{F}_t$–Brownian motion. We denote by $\mathcal{P}$ the corresponding σ-algebra of predictable subsets of $\mathbb{R}_+ \times \Omega$.

From the weak existence and uniqueness, we deduce that our system has the strong Markov property, using a very similar proof as in Theorem 6.2.2 from [11]. Indeed, the proof of that results exploits weak uniqueness of the martingale problem, together with the measurability of the law of the solution, with respect to the starting point. In our case that mapping is easily shown to be continuous.

In the next sections, we will use the following comparison theorem several times. This Lemma can be proved exactly as the comparison Theorem 3.7 from chapter IX of [9].

Lemma 2.5 *Let $B(t)$ be a standard $\mathcal{F}_t$–Brownian motion, T a stopping time, σ be a 1/2 Hölder function, $b_1 : \mathbb{R} \to \mathbb{R}$ a Lipschitz function and $b_2 : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is $\mathcal{P} \otimes B(\mathbb{R})$ measurable function. Consider the two SDEs*

$$\begin{cases} dY_1(t) = b_1(Y_1(t))dt + \sigma(Y_1(t))dB(t), \\ \quad Y_1(0) = y_1; \end{cases} \tag{2.4}$$

$$\begin{cases} dY_2(t) = b_2(t, Y_2(t))dt + \sigma(Y_2(t))dB(t), \\ \quad Y_2(0) = y_2. \end{cases} \tag{2.5}$$

Let Y_1 (resp Y_2) be a solution of (2.4) (resp (2.5)). If $y_1 \leq y_2$ (resp $y_2 \leq y_1$) and outside a measurable subset of Ω of probability zero, $\forall t \in [0, T]$, $\forall x \in \mathbb{R}$, $b_1(x) \leq b_2(t, x)$ (resp $b_1(x) \geq b_2(t, x)$). Then a.s. $\forall t \in [0, T]$, $Y_1(t) \leq Y_2(t)$ (resp $Y_1(t) \geq Y_2(t)$).

3 The result for a specific set of initial conditions

From

$$\mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F) = \mathbb{P}(E \cup F) \leq 1,$$

we deduce the following trivial lemma which will be used several times below :

Lemma 3.1 *Let $E, F \in \mathcal{F}$. Then $\mathbb{P}(E \cap F) \geq \mathbb{P}(E) + \mathbb{P}(F) - 1$.*

Now first we show that M_1 cannot grow too fast :

Lemma 3.2 *For all $c > 0$, $t > 0$, $t' > 0$,*

$$\mathbb{P}\left(\sup_{0 \le r \le t'} M_1(t + r) - M_1(t) \le \lambda t' + c\right) \ge 1 - \exp(-2\alpha Nc)$$

PROOF : We define $Z_{t+s}^t = \int_t^{t+s} \sqrt{\frac{M_2(r)}{N}} dB_r - \alpha \int_t^{s+t} M_2(r) dr,$. We note that, for any $t > 0$, $\{\exp(2\alpha N Z_{t+u}^t), \ u \ge 0\}$ is both a local martingale and a super–martingale. We also have

$$\sup_{0 \le s \le t'} M_1(t + s) - M_1(t) \le \sup_{0 \le s \le t'} Z_{t+s}^t + \lambda t'.$$

But for all $c > 0$,

$$\mathbb{P}\left(\sup_{0 \le u \le t'} Z_{t+u}^t \ge c\right) \le \mathbb{P}\left(\sup_{0 \le u \le t'} \exp\left(2\alpha N Z_{t+u}^t\right) \ge \exp\left(2\alpha Nc\right)\right)$$

$$\le \exp\left(-2\alpha Nc\right)$$

where we have taken advantage of the fact that $\exp\left(2\alpha N Z_{t+u}^t\right)$ is a local martingale and of Doob's inequality. Then

$$\mathbb{P}\left(\sup_{0 \le r \le t'} M_1(t + r) - M_1(t) \le \lambda t' + c\right) \ge 1 - \exp\left(-2\alpha Nc\right).$$

$$\Diamond$$

Note that we have in fact $\mathbb{P}\left(\sup_{u \ge 0} Z_{t+u}^t \ge c\right) \le \exp\left(-2\alpha Nc\right)$.

We choose an arbitrary value $m > 0$ for $M_1(0)$, which will remain the same throughout this document (for example one could choose $m = 1$), and we define

$$\bar{\varepsilon} = \frac{1}{10N\alpha}, \quad t_3' = \frac{\bar{\varepsilon}N}{3\lambda} = \frac{1}{30\lambda\alpha}, \tag{3.1}$$

$$m_{\max} = m + \lambda A(t_3') + \frac{\bar{\varepsilon}}{6},$$

where $A(t) = \frac{1}{4N} \int_0^t (1 - X_0(s)) ds,$

$$p_2 = \exp(-\alpha N \frac{\bar{\varepsilon}}{6}) = \exp(-\frac{1}{60}), \tag{3.2}$$

$$\mu = \frac{\bar{\varepsilon}}{6m_{\max}} \wedge \frac{\bar{\varepsilon}}{4} \wedge \frac{1}{10}, \tag{3.3}$$

and let δ be a real number, which will be specified below, such that $\delta \leq \frac{1}{10} \wedge \frac{\bar{\varepsilon}}{m}$.

Now let Y_0 be the solution of the following SDE :

$$\begin{cases} dY_0(t) = dt + 2\sqrt{Y_0(t)}dW(t) \\ Y_0(0) = \delta \end{cases} \tag{3.4}$$

with W a standard Brownian motion.

We will show that starting with $X_0(0) = x_0 \leq \delta$, $M_1(0) = m_1 \leq m$, and as long as $X_0 M_1 < 2\bar{\varepsilon}$ and X_0 remains small enough, we can compare $X_0(t)$ with the solution of (3.4).

Lemma 3.3 *For any $\delta > 0$, $\bar{\varepsilon}$ given as in (3.1), μ as in (3.3), let*

$$T_{min} = \inf\{t > 0, X_0(t)M_1(t) > 2\bar{\varepsilon} \text{ or } X_0(t) > \delta + \mu\}.$$

Then provided that $X_0(0) = x_0 < \delta$, if $A(t) := \frac{1}{4}\int_0^t \frac{1-X_0(s)}{N}ds$, there exists a standard Brownian motion W such that the corresponding solution Y_0 of (3.4) satisfies

$$X_0(t) \leq Y_0(A(t)), \quad \forall t \in [0, T_{min}].$$

PROOF : We first note that for $0 \leq t \leq T_{min}$, $\frac{t}{5N} \leq A(t) \leq \frac{t}{4N}$ because $\frac{4}{5} \leq 1-X_0 \leq 1$ (thanks to the choices of μ and δ, and $1 - X_0 \geq 1 - \delta - \mu \geq 1 - \frac{1}{10} - \frac{1}{10} \geq \frac{4}{5}$) and t_3' has been chosen in such way that $A(t_3') \leq \frac{\bar{\varepsilon}}{12\lambda}$.

Define $\sigma(t) = \inf\{u > 0, A(u) \geq t\}$ and $\tilde{X}_0(t) = X_0(\sigma(t))$ (resp $\tilde{M}_1(t) = M_1(\sigma(t))$). Then there exists a standard Brownian motion W_t such that

$$d\tilde{X}_0(t) = (\alpha\tilde{M}_1(t) - \lambda)\tilde{X}_0(t)\frac{4N}{1 - \tilde{X}_0(t)}dt + 2\sqrt{\tilde{X}_0(t)}dW_t$$

But whenever $t \leq A(T_{min})$,

$$(\alpha\tilde{M}_1(t) - \lambda)\tilde{X}_0(t)\frac{4N}{1 - \tilde{X}_0(t)} \leq \frac{4\alpha N\tilde{M}_1(t)\tilde{X}_0(t)}{1 - \tilde{X}_0(t)}$$

$$\leq 1,$$

because the numerator on the right is less than or equal to 4/5, while the denominator is bigger than or equal to the same figure.

The result then follows from Lemma 2.5. $\diamondsuit$

Next we will prove that Y_0 reaches zero with positive probability on a fixed time interval. For any $\alpha \in \mathbb{R}$, we define

$$T_\alpha' = \inf\{t > 0, \ Y_0(t) = \alpha\}.$$

Lemma 3.4 *Let $Y_0(t)$ be the solution of (3.4). For all $p < 1$, $\tilde{\mu} > 0$, there exists $\delta > 0$ such that with t'_3 defined as in (3.1),*

$$\mathbb{P}(T'_0 \leq t'_3 \wedge T'_{\delta + \tilde{\mu}}) \geq p$$

PROOF : Let

$$\tilde{Y}(t) = \delta \exp\left(-t + 2W(t)\right),$$

$$D(t) = \int_0^t \tilde{Y}(s)ds,$$

$$\rho(t) = \inf\{s > 0, D(s) > t\}.$$

It is too hard to show that

$$\tilde{Y}(\rho(t)) = \delta \exp\left(-\int_0^t \frac{ds}{\tilde{Y}(\rho(s))} + 2\int_0^t \frac{dW(s)}{\sqrt{\tilde{Y}(\rho(s))}}\right).$$

It now follows from Ito's formula that the process $\{Y(t) := \tilde{Y}(\rho(t)), \ t \geq 0\}$ is the unique solution of the Y_0–SDE, hence $Y_0(t) = \tilde{Y}(\rho(t))$, $t \geq 0$. We deduce that $T'_0 = D(\infty) < \infty$, and

$$\mathbb{P}(T'_0 \leq t'_3 \wedge T'_{\delta+\tilde{\mu}})$$

$$= \mathbb{P}\left(\left\{\int_0^\infty \exp(-t + 2W(t))dt \leq \frac{t'_3}{\delta}\right\} \cap \left\{\sup_{t \geq 0} \exp(-t + 2W(t)) \leq \frac{\delta + \tilde{\mu}}{\delta}\right\}\right)$$

$$\to 1,$$

as $\delta \to 0$, since $\sup_{t \geq 0} \exp(-t + 2B_0(t)) < \infty$ a.s.

$\diamondsuit$

Now we can choose the value of δ which we will be using from now on. Let δ' be the largest value of δ such that Lemma 3.4 holds, with $p = p_2$ defined in (3.2) and $\tilde{\mu} = \mu$ (which is a function of $m_{\max}$) as defined by (3.3). We choose (recall that the value of $\bar{\varepsilon}$ has been precised in (3.1))

$$\delta = \delta' \wedge \frac{1}{10} \wedge \frac{\bar{\varepsilon}}{m}. \tag{3.5}$$

Thanks to Lemma 3.4, when starting at time 0 from δ, Y_0 reaches 0 with probability p_2 before time $t'_3 \wedge T'_{\delta+\mu}$. Then X_0 will do the same before time $A(t'_3) \wedge A(T'_{\delta+\mu})$, provided that $X_0(t)M_1(t) \leq 2\bar{\varepsilon}$, $\forall 0 \leq t \leq A(t'_3) \wedge A(T'_{\delta+\mu})$. Hence the fact that $T_0 < A(t'_3)$ with positive probability, provided $x_0 \leq \delta$ and $M_1(0) \leq m$ will follow from the above results and

Lemma 3.5 *If $X_0(0) \leq \delta$ and $M_1(0) \leq m$, then we have (again with $\bar{\varepsilon}$ and t_3' given by (3.1))*

$$\mathbb{P}\left(\sup_{0 \leq t \leq A(t_3' \wedge T_{\delta+\mu}')} X_0(t) M_1(t) \leq 2\bar{\varepsilon}\right) = p_3 > 1 - p_2.$$

PROOF : We use Lemma 3.2. Consider the event

$$E_{m,t_3',\bar{\varepsilon}} = \left\{\sup_{0 \leq t \leq A(t_3') \wedge A(T_{\delta+\mu}')} M_1(t) \leq m + \lambda A(t_3') + \frac{\bar{\varepsilon}}{6}\right\}.$$

We have

$$\mathbb{P}(E_{m,t_3',\bar{\varepsilon}}) \geq \mathbb{P}\left(\sup_{0 \leq t \leq A(t_3')} M_1(t) \leq m_1 + \lambda A(t_3') + \frac{\bar{\varepsilon}}{6}\right)$$

$$\geq 1 - \exp(-\alpha N \frac{\bar{\varepsilon}}{3}) = 1 - \exp(-\frac{1}{30}).$$

Since $X_0(t) \leq \delta + \mu$ for $t \leq A(T_{\delta+\mu}')$, on the event $E_{m,t_3',\bar{\varepsilon}}$,

$$\sup_{0 \leq t \leq A(t_3') \wedge A(T_{\delta+\mu}')} X_0(t) M_1(t) \leq (\delta + \mu)(m + \lambda A(t_3') + \frac{\bar{\varepsilon}}{6})$$

$$\leq \delta m + \mu m + \lambda A(t_3') + \frac{\bar{\varepsilon}}{6}$$

$$\leq \bar{\varepsilon} + \frac{\bar{\varepsilon}}{6} + \frac{\bar{\varepsilon}}{12} + \frac{\bar{\varepsilon}}{6}$$

$$\leq 2\bar{\varepsilon},$$

where we have used the fact that $\delta + \mu \leq 1$ for the second inequality. $\diamondsuit$

Combining Lemma 3.1, Lemma 3.3, Lemma 3.5 and Lemma 3.4, denoting $t_3 = A(t_3')$, we deduce the

Corollary 3.6 $\exists p_{fin} \geq p_3 + p_2 - 1 > 0$ *such that*

$$\mathbb{P}(T_0 \leq t_3 | X_0(0) \leq \delta, M_1(0) \leq m) \geq p_{fin} > 0.$$

We will now prove the same result, but for a larger set of initial conditions. m being as fixed above, and δ as in (3.5), let

$$\mathcal{I} = \{(x_0, m_1) \in [0, 1] \times \mathbb{R}_+, x_0 \leq \delta, x_0 m_1 \leq \delta m\}$$

We now prove the (p_{fin} is as defined in Corallary 3.6)

Proposition 3.7 *For any $(x_0, m_1) \in \mathcal{I}$,*

$$\mathbb{P}(T_0 \leq t_3 | X_0(0) = x_0, M_1(0) = m_1) \geq p_{\mathit{fin}}.$$

PROOF : Thanks to the previous Corollary, we only need to consider the case $m_1 > m$. Let (x_0, m_1) be a point in the set $\mathcal{I}$. First, let us consider the point (δ, m). From the previous section, starting from (δ, m), the process (X_0, M_1) has a strictly positive probability to reach 0 before the time $t_3 = A(t_3')$. We will show that the process starting from (x_0, m_1) has a larger probability to reach 0 before time t_3, which will prove the Proposition.

Let $C = \frac{m_1}{m} \geq 1$. Then we have $x_0 \leq \frac{\delta}{C}$.

Now we will use the same reasoning as in Lemma 3.4 with a few modifications. Indeed, since the probability that $Y_0(t)$ reaches 0 before a prescribed time is decreasing in δ, we increase this probability by starting from $Y_0(0) = x_0 = \delta' \leq \frac{\delta}{C}$. since $C \geq 1$. We will use this new value. Moreover, the starting point satisfies $x_0 m_1 \leq \bar{\varepsilon}$. The only thing which is worse than with the starting point (δ, m) is the fact that m_1 is greater than m, hence a greater $m_{\max}$. But this only appears in one place : in the definition of μ.

Note that if we define $m_{\max}' = m_1 + \lambda t_3 + \frac{\bar{\varepsilon}}{6}$, we deduce from Lemma 3.2

$$\mathbb{P}(\sup_{0 \leq t \leq t_3} M_1(t) \leq m_{\max}') \geq 1 - \exp(-\alpha N \frac{\bar{\varepsilon}}{3}).$$

We define μ' similarly as μ in (3.3), but with $m_{\max}$ replaced by $m_{\max}'$, hence since $m_{\max}' \leq C m_{\max}$, $\mu' \geq \frac{\mu}{C}$. But if we look at the proof of Lemma 3.4, we have, since $\frac{t_3'}{\delta'} \geq \frac{C t_3'}{\delta} \geq \frac{t_3'}{\delta}$ and $\frac{\delta' + \mu'}{\delta'} = 1 + \frac{\mu'}{\delta'} \geq 1 + \frac{\mu}{\delta}$,

$$\mathbb{P}(T_0 \leq t_3) \geq \mathbb{P}(T_0' \leq t_3' \wedge T_{\delta' + \mu'}')$$

$$\geq \mathbb{P}\left(\left\{\int_0^\infty \exp(-t + 2W(t))dt \leq \frac{t_3'}{\delta'}\right\} \cap \left\{\sup_{t \geq 0} \exp(-t + 2W(t)) \leq \frac{\delta' + \mu'}{\delta'}\right\}\right)$$

$$\geq \mathbb{P}\left(\left\{\int_0^\infty \exp(-t + 2W(t))dt \leq \frac{t_3'}{\delta}\right\} \cap \left\{\sup_{t \geq 0} \exp(-t + 2W(t)) \leq \frac{\delta + \mu}{\delta}\right\}\right)$$

Hence we have a larger probability to reach zero starting from (x_0, m_1) rather than from (δ, m), which concludes the proof.

$\Diamond$

We sum up in the following Proposition the results obtained in this section, with $\varepsilon = \delta m$ (recall that m has been chosen arbitrarily, δ is prescribed by (3.5), and note that $\varepsilon \leq \bar{\varepsilon}$).

Proposition 3.8 *Let* $X(t) = (X_k(t))_{k \in \mathbb{Z}_+}$ *be the solution of the initial model, and* M_1 *its mean as defined in section 1. Then* $\exists p_{fin} > 0$ *and* t_3 *such that for any* $t \geq 0$,

$$\mathbb{P}(T_0 \leq t + t_3 | X_0(t) \leq \delta, X_0(t)M_1(t) \leq \varepsilon) \geq p_{fin} > 0.$$

4   A recurrence property of M_1

With the help of the results proved in the previous section, we will now prove some results on M_1. We will show that as long that as the ratchet has not clicked, M_1 is bound to return under some specified value. This particular point will be important in the sequel.

We begin with the following lemma, which is true for any probability on $\mathbb{Z}_+$. It will be crucial for establishing one of our first estimates.

Lemma 4.1 *Let* p *be a probability on* $\mathbb{Z}_+$, *and let* $x_k = p(k)$, $m_1 = \sum_{k \geq 0} k x_k$ *and* $m_2 = \sum_{k \geq 0} (k - m_1)^2 x_k$. *Then*

$$m_2 \geq (1 - x_0)m_2 \geq x_0 m_1^2.$$

PROOF : If $x_0 = 1$, $m_1 = m_2 = 0$ and the result is true. So it suffices to study the case $x_0 < 1$. By Jensen's inequalities we have

$$\left(\sum_{k \geq 1} \frac{x_k}{1 - x_0} k \right)^2 \leq \sum_{k \geq 1} \frac{x_k}{1 - x_0} k^2$$

with equality if and only if there exists only one $k \geq 1$ such that $x_k > 0$. Then :

$$\left(\sum_{k \geq 1} x_k k \right)^2 \leq (1 - x_0) \sum_{k \geq 1} x_k k^2,$$

that is

$$m_1^2 \leq (1 - x_0) \sum_{k \geq 1} x_k k^2,$$

hence

$$x_0 m_1^2 \leq (1 - x_0) \sum_{k \geq 1} k^2 x_k - (1 - x_0)m_1^2$$
$$= (1 - x_0)m_2.$$

$$\diamondsuit$$

We now introduce the stopping time

$$H_\lambda^t := \inf\{s \geq t, \ X_0(s)M_1(s)^2 \leq 2\frac{\lambda+1}{\alpha}\},$$

and we define $H_\lambda = H_\lambda^0$.

Our next claim is

Proposition 4.2 *For any stopping time T, we have $H_\lambda^T < +\infty$ a.s.*

The Proposition follows from the strong Markov property and

Lemma 4.3 *Suppose that $X_0(0)M_1(0)^2 > 2\frac{\lambda+1}{\alpha}$. Then $H_\lambda < \infty$ a.s.*

PROOF : On the interval $[0, H_\lambda]$, we have from Lemma 4.1

$$-\frac{\alpha}{2}M_2 \leq -\frac{\alpha}{2}X_0M_1^2 \leq -(\lambda+1),$$

hence M_1 is bounded from above by

$$\begin{aligned}
M_1(t) &\leq M_1(0) - \int_0^t \left(1 + \frac{\alpha}{2}M_2(s)\right)ds + \int_0^t \sqrt{\frac{M_2(s)}{N}}dB_s, \\
&\leq M_1(0) - t - \frac{\alpha}{2}\int_0^t M_2(s)ds + \int_0^t \sqrt{\frac{M_2(s)}{N}}dB_s.
\end{aligned} \tag{4.1}$$

We will show next that

$$Z_t := \int_0^t \sqrt{\frac{M_2(r)}{N}}dB_r - \frac{\alpha}{2}\int_0^t M_2(r)dr$$

is bounded from above a.s. This will imply the result, since on the event $\{H_\lambda = +\infty\}$, (4.1) holds for all $t > 0$, which would imply that M_1 eventually becomes negative, and this is absurd.

If we define $C(t) = \frac{1}{N}\int_0^t M_2(s)ds$, we have $Z_t = W(C(t)) - \frac{\alpha N}{2}C(t)$ where W is a standard Brownian motion.

Now, if $C(\infty) = \infty$ then $\lim_{t\to\infty} Z_t = -\infty$, hence Z_t is bounded from above. Or else $C(\infty) < \infty$, and we have $\sup_{t>0}|Z_t| = \sup_{0<s<C(\infty)}|W(s) - \frac{\alpha N}{2}s| < \infty$ a.s. $\diamondsuit$

Now we will finally be able to prove that M_1 always return below $\beta := \lambda/\alpha$, as long as the ratchet does not click. Let

$$S_\beta^t = \inf\{s > t, M_1(s) \leq \beta\}.$$

Then we will prove the following lemma :

Lemma 4.4 $\forall t > 0$, *we have* $\mathbb{P}(T_0 \wedge S_\beta^t < \infty) = 1$

PROOF : In order to simplify the notations, we treat the case $t = 0$. First, we let $\delta_{\inf} = \delta \wedge \frac{\varepsilon^2 \alpha}{4(\lambda+1)}$ (recall that $\varepsilon = \delta m$).

Now we introduce the process Y_t^s, defined $\forall s \geq 0$, $\forall t \geq s$ which is the solution of the following system :

$$\begin{cases} dY_t^s = \sqrt{\dfrac{Y_t^s(1 - Y_t^s)}{N}}\, dB_0(t), t \geq s \\ Y_s^s = \delta_{\inf}. \end{cases} \tag{4.2}$$

We define for any $0 \leq u \leq 1$

$$R_u^s = \inf\{t \geq s, Y_t^s = u\},$$

We have

$$\begin{cases} R_0^s \wedge R_1^s < +\infty \text{ a.s.} \\ \mathbb{P}(R_1^s < R_0^s) > 0. \end{cases} \tag{4.3}$$

Indeed, for all $a \in (0, \delta_{\inf})$, by the non–degeneracy of the diffusion coefficient, Y_t^s gets out of $[a, 1 - a]$ in finite time. Then if we choose a small enough (using the same reasoning as in Lemma 3.4), we have a chance p'_{fin} to reach 0 before a time $V > 0$ as soon as we start below a (the same with 1 and starting above $\geq 1 - a$ by symmetry).

Then, using the strong Markov property, this situation would repeat itself infinitely many times, would Y_t^s never reach 0 or 1, hence the first line of (4.3). The second line is essentially obvious. Note that using an argument based upon Green's functions, one can in fact prove that $\mathbb{E}(R_0^s \wedge R_1^s) < +\infty$.

From this we deduce that there exist $K > 0$, $p > 0$ such that $\mathbb{P}(R_1^s \leq K \wedge R_0^s) \geq p > 0$. In particular $\mathbb{P}(R_1^s \leq K) \geq p > 0$.

We define $L = K \vee t_3$, where t_3 is as in Proposition 3.8, and the following sequence of stopping times :

$$U_0^t = \inf\left\{s > t, X_0(s)M_1^2(s) \leq 2\frac{\lambda+1}{\alpha}\right\},$$

and $\forall n \geq 1$,

$$U_n^t = \inf\left\{s > U_{n-1}^t + L, X_0(s)M_1^2(s) \leq 2\frac{\lambda+1}{\alpha}\right\}.$$

For all $n \geq 0$, U_n^t is a.s. finite, thanks to Proposition 4.2.

Now, at time U_0^t : either $X_0(U_0^t) \leq \delta_{\inf}$ ($\leq \delta$), then

$$X_0(U_0^t)M_1(U_0^t) = \sqrt{X_0(U_0^t)M_1^2(U_0^t) \times X_0(U_0^t)}$$

$$\leq \sqrt{2\frac{\lambda+1}{\alpha}\frac{\varepsilon^2\alpha}{4(\lambda+1)}}$$

$$< \varepsilon$$

In that case we deduce from Proposition 3.8 that $\mathbb{P}(T_0 \leq U_0^t + L) = p_{fin} > 0$.

Or else $X_0(U_0^t) > \delta_{\inf}$. And in that case there are two possibilities : either $\inf_{U_0^t \leq s \leq U_0^t+L} M_1(s) \geq \beta$, in which case we have $\inf_{U_0^t \leq s \leq U_0^t+L}(\alpha M_1(s) - \lambda)X_0(s) \geq 0$, and then we deduce from Lemma 2.5 that $X_0(s) \geq Y_s^{U_0^t}$. Consequently $\mathbb{P}(T_1 \leq U_0^t + L) \geq p > 0$, where $T_1 = \{t \geq 0, X_0(t) = 1\}$. But if $X_0(s) = 1$, then $M_1(s) = 0$. Hence $\mathbb{P}(S_\beta \leq U_0 + L) \geq p > 0$. In the other case $\inf_{U_0^t \leq s \leq U_0^t+L} M_1(s) < \beta$, hence in that case also $S_\beta \leq U_0^t + L$.

To conclude, we have

$$\mathbb{P}(T_0 \wedge S_\beta^t = +\infty) \leq \mathbb{P}(T_0 \wedge S_\beta^t \geq U_0^t + L) \leq 1 - q,$$

with $q = p \wedge p_{fin}$.

It follows from the strong Markov property of the process $X = (X_k, \ k \geq 0)$, repeating this argument with U_0^t replaced by U_1^t that

$$\mathbb{P}(T_0 \wedge S_\beta^t = +\infty) \leq \mathbb{P}(T_0 \wedge S_\beta^t \geq U_1^t + L) \leq (1 - q)^2$$

.

Iterating the above agument, we have for all $\ell \geq 0$,

$$\mathbb{P}(T_0 \wedge S_\beta^t > U_\ell^t + L) \leq (1 - q)^\ell.$$

We have proved that

$$\mathbb{P}(T_0 \wedge S_\beta^t = +\infty) = 0.$$

$\diamondsuit$

5 Reaching the special set from any initial condition

Now we will show that starting from an initial condition $((x_k)_{k\in\mathbb{Z}_+}, m_1)$ with $m_1 \leq \beta$ the process has a probability bounded below by p_{final} to click before a given time.

Since the process is Markovian and this situation repeats itself as long as the ratchet has not clicked, we will conclude that $\mathbb{P}(T_0 < +\infty) = 1$.

In this part we denote by $(x_k)_{k \geq 0}$ the initial condition of our system, and we suppose that $m_1 = \sum_{k \geq 0} k x_k \leq \beta$.

One of the difficulties we have to face is that the quadratic variation of X_0 is $\frac{X_0(1-X_0)}{N}$, which is not bounded away from 0, near 1 and 0. We need to study three separate cases.

The first case will be described in terms of the constant

$$x_{\max} = \max\left\{ \frac{9}{10}, \frac{3\lambda + 5\alpha}{5(\lambda + \alpha)}, 1 - \frac{2}{\lambda} \right\}. \tag{5.1}$$

5.1 $x_0 \in (x_{\max}; 1]$

The following lemma will show that if X_0 starts too close to 1, it will quickly go under $x_{\max}$:

Lemma 5.1 Let $t_1 = \frac{8}{\lambda^2}$. If $X_0(0) > x_{\max}$, then

$$\mathbb{P}(\inf_{s < t_1} X_0(s) \leq x_{\max}) \geq 1 - \exp(-N).$$

PROOF : Let $T_{x_{\max}} = \inf\{s \geq 0, X_0(s) \leq x_{\max}\}$. On the time interval $[0, T_{x_{\max}})$, we have

$$X_0(s) > x_{\max} \geq \frac{3\lambda + 5\alpha}{5(\lambda + \alpha)}.$$

Since $X_1 \leq 1 - X_0$, on the same interval we have $X_1(s) \leq \frac{2\lambda}{5(\lambda+\alpha)}$ while $X_0(s) > \frac{9}{10}$, hence

$$\alpha M_1(s) X_1(s) + \lambda X_0(s) - (\lambda + \alpha) X_1(s) \geq \lambda X_0(s) - (\lambda + \alpha) \frac{2\lambda}{5(\lambda + \alpha)}$$

$$\geq \frac{\lambda}{2}.$$

Then $X_1(s) \geq Y_1(s)$ for $s \in [0, T_{x_{\max}}]$, where Y_1 is the solution of the SDE

$$\begin{cases} dY_1(s) = \frac{\lambda}{2} ds + \sqrt{\frac{Y_1(1 - Y_1)}{N}} dB_1(s), \\ Y_1(0) = 0, \end{cases} \tag{5.2}$$

where we stop Y_1 as soon as it reaches 1.

We have

$$\mathbb{P}\left(\int_0^{t_1} \sqrt{\frac{Y_1(1-Y_1)}{N}} dB_1 < -C\right)$$

$$= \mathbb{P}\left(-\int_0^{t_1} \sqrt{\frac{Y_1(1-Y_1)}{N}} dB_1 > C\right)$$

$$\leq \mathbb{P}\left(\exp\left(-\gamma \int_0^{t_1} \sqrt{\frac{Y_1(1-Y_1)}{N}} dB_1 - \int_0^{t_1} \frac{\gamma^2 Y_1(s)(1-Y_1(s))}{2N} ds\right) > \exp\left(\gamma C - \frac{\gamma^2}{8N} t_1\right)\right)$$

$$\leq \exp\left(-\gamma C + \frac{\gamma^2}{8N} t_1\right)$$

$$\leq \exp\left(\frac{-2NC^2}{t_1}\right) = \exp\left(-N\right),$$

where the first inequality follows from $Y_1(s)(1-Y_1(s)) \leq 1/4$, the second one is Chebychev's inequality, the third one follows by choosing $\gamma = 4CN/t_1$, and finally the last identity by choosing $C = 2/\lambda$. Hence

$$\mathbb{P}\left(\int_0^{t_1} \sqrt{\frac{Y_1(1-Y_1)}{N}} dB_1 \geq -\frac{2}{\lambda}\right) \geq 1 - \exp\left(-N\right) > 0$$

Now, since

$$\int_0^{t_1} \frac{\lambda}{2} ds = \frac{4}{\lambda},$$

and on $[0, T_{x_{\max}})$, $X_0(s) > 1 - 2/\lambda$, hence $X_1(s) < 2/\lambda$, we have the inclusion

$$\left\{\int_0^{t_1} \sqrt{\frac{Y_1(1-Y_1)}{N}} dB_1 \geq -\frac{2}{\lambda}\right\} \subset \{T_{x_{\max}} < t_1\},$$

which implies that

$$\mathbb{P}\left(T_{x_{\max}} \leq t_1\right) \geq 1 - \exp(-N),$$

hence the conclusion.   ◇

We need to control M_1 on the same time interval of length t_1. Using Lemma 3.2 we will deduce the following Proposition :

Proposition 5.2 *Let again $x_{\max}$ be given by (5.1), $t_1 = \frac{8}{\lambda^2}$, $\varepsilon_0 = \frac{1}{2\alpha N} \ln \left(\frac{2}{1-\exp(-N)} \right)$ and $\beta' = \beta + \lambda t_1 + \varepsilon_0$. If $X_0(0) > x_{\max}$ and $M_1(0) < \beta$, then*

$$\mathbb{P}\left(\{T_{x_{\max}} \le t_1\} \cap \{M_1(T_{x_{\max}}) \le \beta'\} \right) = p_{init} > 0$$

PROOF : It follows from Lemma 5.1 and Lemma 3.2

$$\mathbb{P}(T_{x_{\max}} \le t_1) \ge 1 - \exp(-N)$$

,

$$\mathbb{P}(M_1(T_{x_{\max}}) \le \beta') \ge 1 - \exp(-2\alpha N \varepsilon_0)$$

Those two inequalities together with Lemma 3.1 imply

$$\mathbb{P}\left(\{T_{x_{\max}} \le t_1\} \cap \{M_1(T_{x_{\max}}) \le \beta'\} \right) \ge 1 - \exp(-N) - \exp(-2\alpha N \varepsilon_0),$$
$$\ge \frac{1 - \exp(-N)}{2} := p_{init}.$$

$$\diamond$$

So even if we started with $(X_0(0), M_1(0))$ such that $X_0(0) \ge x_{\max}$ and $M_1(0) \le \beta$, we obtain before time t_1 with probability at least $p_{init} > 0$ a new initial condition $X_0 \le x_{\max}$ and $M_1 \le \beta'$, so we can resume with the next case.

5.2 $X_0 \le x_{\max}$ but either $X_0 > \delta$ or $X_0 M_1 > \varepsilon$

The idea of this subsection is to show that with a strictly positive probability p_{trans}, both X_0 goes from $x_{\max}$ to a $\delta' < \delta$ in finite time, and during the same time interval, M_1 stays small enough so that at the end $X_0 M_1 \le \varepsilon$.
    We start by showing some inequalities

Lemma 5.3 *Let $\{V_t, \ t \ge 0\}$ be a standard Brownian motion, and $c > 0$ a constant. Then for any $t > 0$, $\tilde{\delta} > 0$, $\tilde{\mu} > 0$,*

$$\mathbb{P}\left(\inf_{0 \le s \le t} \{cs + V_s\} \le -\tilde{\delta}, \ \sup_{0 \le s \le t} \{cs + V_s\} \le \tilde{\mu} \right)$$
$$\ge 1 - \sqrt{\frac{2}{\pi}} \left(\frac{\tilde{\delta}}{\sqrt{t}} + c\sqrt{t} \right) - 2\exp\left[-\frac{1}{2} \left(\frac{\tilde{\mu}}{\sqrt{t}} - c\sqrt{t} \right)^2 \right].$$

PROOF : Using Lemma 3.1, the result follows from the two following computations. We have, with Z denoting a $N(0,1)$ random variable,

$$\mathbb{P}\left(\inf_{0\leq s\leq t}\{cs+V_s\}\leq-\tilde{\delta}\right)\geq\mathbb{P}\left(\inf_{0\leq s\leq t}V_s\leq-\tilde{\delta}-ct\right)$$
$$=\mathbb{P}\left(\sup_{0\leq s\leq t}V_s\geq\tilde{\delta}+ct\right)$$
$$=2\mathbb{P}(V_t\geq\tilde{\delta}+ct)$$
$$=1-\mathbb{P}\left(|Z|\leq\frac{\tilde{\delta}}{\sqrt{t}}+c\sqrt{t}\right)$$
$$\geq 1-\sqrt{\frac{2}{\pi}}\left(\frac{\tilde{\delta}}{\sqrt{t}}+c\sqrt{t}\right).$$

On the other hand,

$$\mathbb{P}\left(\sup_{0\leq s\leq t}(cs+V_s)\leq\tilde{\mu}\right)\geq\mathbb{P}\left(\sup_{0\leq s\leq t}V_s\leq\tilde{\mu}-ct\right)$$
$$=1-\mathbb{P}\left(\sup_{0\leq s\leq t}V_s\geq\tilde{\mu}-ct\right)$$
$$=1-2\mathbb{P}\left(Z\geq\frac{\tilde{\mu}}{\sqrt{t}}-c\sqrt{t}\right),$$

and

$$\mathbb{P}\left(Z\geq\frac{\tilde{\mu}}{\sqrt{t}}-c\sqrt{t}\right)=\mathbb{P}\left(\exp(\gamma Z-\gamma^2/2)\geq\exp\left(\gamma\left[\frac{\tilde{\mu}}{\sqrt{t}}-c\sqrt{t}\right]-\frac{\gamma^2}{2}\right)\right)$$
$$\leq\exp\left(-\gamma\left[\frac{\tilde{\mu}}{\sqrt{t}}-c\sqrt{t}\right]+\frac{\gamma^2}{2}\right).$$

Choosing $\gamma=\tilde{\mu}/\sqrt{t}-c\sqrt{t}$, we conclude from the above computations that

$$\mathbb{P}\left(\sup_{0\leq s\leq t}(cs+V_s)\leq\tilde{\mu}\right)\geq 1-2\exp\left[-\frac{1}{2}\left(\frac{\tilde{\mu}}{\sqrt{t}}-c\sqrt{t}\right)^2\right].$$

$\diamondsuit$

We will choose from now on

$$\tilde{\varepsilon}=\frac{\log(4)}{2\alpha N},\quad\text{so that }e^{-2N\alpha\tilde{\varepsilon}}=\frac{1}{4}.\tag{5.3}$$

We start from $(X_0, M_1) = (x, \beta')$, where $0 < x \leq x_{\max} < 1$ (recall the definition (5.1) of $x_{\max}$) and $\beta < \beta'$. Let $0 < \tilde{\mu} = \frac{1-x_{\max}}{2}$. We are going to prove that, with positive probability, X_0 goes down to δ' in a finite number of steps, while staying below $x + \tilde{\mu}$ (so that $1 - X_0(t) \geq a := \frac{1-x_{\max}}{2}$), and while M_1 remains under control.

Considering the SDE

$$dX_0(t) = (\alpha M_1(t) - \lambda)X_0(t)dt + \sqrt{\frac{X_0(t)[1 - X_0(t)]}{N}}dB_0,$$

let

$$A(t) := \int_0^t \frac{X_0(s)[1 - X_0(s)]}{N}ds, \quad \text{and}$$
$$\sigma(t) := \inf\{s > 0, A(s) > t\}.$$

Since

$$\int_0^{\sigma(t)} \frac{X_0(s)(1 - X_0(s))}{N}ds = t,$$

we deduce that

$$\frac{d\sigma(t)}{dt} = \frac{N}{\tilde{X}_0(t)(1 - \tilde{X}_0(t))}, \quad \text{provided we let}$$
$$\tilde{X}_0(t) := X_0(\sigma(t)).$$

Finally

$$\sigma(t) = \int_0^t \frac{N}{\tilde{X}_0(s)(1 - \tilde{X}_0(s))}ds,$$

and if we let

$$\tilde{M}_1(t) := M_1(\sigma(t)),$$

we deduce from the above SDE for the process X_0 that

$$\tilde{X}_0(t) = x + N \int_0^t \frac{\alpha \tilde{M}_1(s) - \lambda}{1 - \tilde{X}_0(s)}ds + B(t),$$

where $B(t)$ is a new standard Brownian motion (we use the same notation as above, which is a slight abuse).

At the k–th step of our iterative procedure, $k \geq 1$, we let $\tilde{X}_0$ start from $x - \sum_{j=1}^{k-1} \delta_j$, and we stop the process $\tilde{X}_0$ at the first time that it reaches the level $x - \sum_{j=1}^{k} \delta_j$. We will choose not only the sequence δ_k, but also the sequence s_k in such a

way that we can deduce from Lemma 5.3 that for each $1 \leq k \leq K$ (K to be defined below),

$$\mathbb{P}\left(\inf_{0 \leq s \leq s_k}\{\Theta_k s + B_s\} \leq -\delta_k, \sup_{0 \leq s \leq s_k}\{\Theta_k s + B_s\} \leq \tilde{\mu}\right) > \frac{1}{3}, \qquad (5.4)$$

where, with $s_k' := \sigma(s_k)$ and, again with $\tilde{\varepsilon}$ defined by (5.3) and $a = (1 - x_{\max})/2$,

$$\Theta_k = \frac{N\alpha}{a}\left(\beta' + k\tilde{\varepsilon} + \lambda \sum_{j=1}^{k} s_j'\right),$$

so that we deduce from Lemma 3.2 and our choice of $\tilde{\varepsilon}$ that

$$\mathbb{P}(\sup_{0 \leq s \leq s_k'} M_1(s) \leq \Theta_k | M_1(0) \leq \Theta_{k-1}) \geq 3/4. \qquad (5.5)$$

Our result will follow from a combination of (5.4) and (5.5), provided we show that we can choose the two sequences δ_k and s_k for $k \geq 1$ in such a way that not only (5.4) holds, but also that there exists $K < \infty$ such that

$$x - \sum_{k=1}^{K} \delta_k \leq \delta'.$$

Since during the k–th step we are considering the event that $X_0(t) \leq x + \tilde{\mu}$ i.e. $1 - X_0(t) \geq a$, and also $X_0(t) \geq x - \sum_{j=1}^{k} \delta_j$, we have that

$$s_k' \leq \frac{N}{a(x - \sum_{j=1}^{k} \delta_j)} s_k,$$

so that a bona fide choice of Θ_k in terms of $\{\delta_j, s_j, \ 1 \leq j \leq k\}$ is

$$\Theta_k := N\frac{\alpha}{a}\left[\beta' + k\tilde{\varepsilon} + N\frac{\lambda}{a}\sum_{j=1}^{k} \frac{s_j}{x - \sum_{i=1}^{j} \delta_i}\right].$$

We first want to insure that (the reason for 0.4 will be made clear below)

$$\frac{\delta_k}{\sqrt{s_k}} + \Theta_k\sqrt{s_k} \leq 0.4,$$

which we achieve by requesting both that

$$\delta_k = 0.2\sqrt{s_k} \qquad (5.6)$$

and

$$\Theta_k \sqrt{s_k} \le 0.2 \Leftrightarrow s_k \le \left(\frac{0.2}{\Theta_k}\right)^2. \tag{5.7}$$

On the other hand, we shall also request that for each $j \ge 1$,

$$\frac{\delta_j}{x - \sum_1^j \delta_i} \le 1 \Leftrightarrow \delta_j \le \frac{1}{2}(x - \sum_{i=1}^{j-1} \delta_i).$$

This combined with (5.6) implies that

$$\frac{s_j}{x - \sum_{i=1}^j \delta_i} \le 25\delta_j.$$

Consequently, if we choose $C_N = N\alpha\beta'/a$, we deduce that

$$C_N \le \Theta_k \le C_N + 25\frac{N^2\lambda\alpha}{a^2}\left(\sup_{1 \le j \le k} \delta_j\right)k + k\tilde{\varepsilon}\frac{N\alpha}{a}.$$

Moreover, a combination of (5.6) and (5.7) yields

$$\delta_j = 0.2\sqrt{s_j} \le \frac{(0.2)^2}{\Theta_j}$$

$$\le \frac{1}{25}\frac{a}{N\alpha\beta'},$$

and from the above inequality follows

$$\Theta_k \le C_N + D_N k, \tag{5.8}$$

$$\text{with } D_N = \frac{N}{a}\left(\alpha\tilde{\varepsilon} + \frac{\lambda}{\beta'}\right).$$

Finally this leads us to choose

$$\delta_k = \inf\left(\frac{\kappa}{(C_N + D_N k)}, \frac{1}{2}(x - \sum_{j=1}^{k-1} \delta_j)\right), \tag{5.9}$$

$$s_k = 25\delta_k^2.$$

In order to make sure that (5.7) is satisfied, in view of (5.6), we will choose below $\kappa \le \frac{1}{25}$.

We now have

Lemma 5.4 $\exists K > 0$, $\forall k > K$,

$$\delta_k = \frac{1}{2}(x - \sum_{j=1}^{k-1} \delta_j).$$

PROOF : We first show that for $k \geq 2$,

$$\frac{1}{2}(x - \sum_{j=1}^{k-1} \delta_j) \leq \frac{\kappa}{(C_N + D_N k)} \Rightarrow \frac{1}{2}(x - \sum_{j=1}^{k} \delta_j) < \frac{\kappa}{(C_N + D_N(k+1))}.$$

Indeed, if the above left inequality holds, then

$$\frac{\frac{\kappa}{(C_N + D_N(k+1))}}{\frac{1}{2}(x - \sum_{j=1}^{k-1} \delta_j)} \geq \frac{\frac{\kappa}{(C_N + D_N(k+1))}}{\frac{\kappa}{(C_N + D_N k)}} > \frac{1}{2},$$

where the last inequality follows easily from $k \geq 2$. Consequently

$$\frac{\kappa}{(C_N + D_N(k+1))} > \frac{1}{2}\delta_k$$

$$= \frac{1}{4}\left(x - \sum_{j=1}^{k-1} \delta_j\right)$$

$$= \frac{1}{2}\left(x - \sum_{j=1}^{k} \delta_j\right).$$

Finally, since there exists $K' \geq 1$ such that

$$x - \sum_{j=1}^{K'} \frac{\kappa}{(C_N + D_N j)} < 0,$$

for some $k \leq K'$,

$$\frac{\kappa}{(C_N + D_N k)} > \frac{1}{2}\left(x - \sum_{j=1}^{k} \delta_j\right).$$

$\diamond$

This means that at each $k > K$, $\tilde{X}_0$ progresses by a step equal to half the remaining distance to zero. Consequently $\exists c > 0$ $x_k = x - \sum_{j=1}^{k} \delta_j \leq c2^{-k}$. We are looking for the smallest integer $\bar{k}$ such that $c2^{-\bar{k}} \leq \delta'$, which implies that

$$\bar{k} - 1 < \frac{\log(c) - \log(\delta')}{\log(2)} \leq \bar{k}.$$

Consequently, since we may as well assume that $\delta' \leq 1/2$,

$$\overline{k} \leq 1 + \frac{\log(c)}{\log(2)} + \frac{\log(1/\delta')}{\log(2)}$$

$$\leq [\log(2)]^{-1}\left(2 + \frac{\log(c)}{\log(2)}\right)\log\left(\frac{1}{\delta'}\right).$$

Combining this estimate with (5.8), we deduce that there exists a constant C'_N such that

$$\Theta_{\overline{k}} \leq C'_N \log\left(\frac{1}{\delta'}\right).$$

Hence there exists a $\delta' \leq \delta \wedge 1/2$ such that $\delta'\Theta_{\overline{k}} \leq \varepsilon$. If we now check that the probability of the previous path is bounded below by a positive constant, we will have that with a positive constant, at the end of the $\overline{k}$–th step, both $X_0 \leq \delta' \leq \delta$, and $M_1 \leq \Theta_{\overline{k}}$, hence $X_0 M_1 \leq \varepsilon$, which puts us in a position to apply Proposition 3.8.

Given the choice that we have made for $\tilde{\varepsilon}$, it suffices to make sure that

$$\sqrt{\frac{2}{\pi}}\left(\frac{\delta_k}{\sqrt{s_k}} + \Theta_k\sqrt{s_k}\right) < 1/3, \ \forall k \geq 1, \tag{5.10}$$

as well as

$$2\exp\left[-\frac{1}{2}\left(\frac{\tilde{\mu}}{\sqrt{s_k}} - \Theta_k\sqrt{s_k}\right)^2\right] < 1/3, \ \forall k \geq 1. \tag{5.11}$$

Since $3^{-1}\sqrt{\pi/2} > 0.4$, (5.6)+(5.7) implies (5.10).

On the other hand, (5.11) is equivalent to

$$\left(\frac{\tilde{\mu}}{\sqrt{s_k}} - \Theta_k\sqrt{s_k}\right)^2 > 2\log 6. \tag{5.12}$$

But we have

Lemma 5.5 *A sufficient condition for* (5.12) *is that*

$$\kappa \leq 1 \wedge \frac{C_N\tilde{\mu}}{25 + 10\log 6}. \tag{5.13}$$

PROOF : It follows from (5.13)

$$\tilde{\mu}(C_N + D_N k) > (25 + 10\log 6)\kappa$$

$$\geq \left(\frac{25\Theta_k}{C_N + D_N k} + 10\log 6 \right)\kappa$$

$$\geq \frac{25\Theta_k}{C_N + D_N k}\kappa^2 + 10\log 6\kappa,$$

$$\frac{\tilde{\mu}}{5\kappa/(C_N + D_N k)} > \Theta_k \frac{5\kappa}{C_N + D_N k} + 2\log 6.$$

Finally (5.12) follows from the last inequality, (5.9) and (5.6).   ◇

We therefore choose

$$\kappa = \frac{1}{25} \wedge \frac{C_N \tilde{\mu}}{25 + 10\log 6}.$$

We can now conclude that

Proposition 5.6 *Suppose that $X_0(0) \leq x_{\max}$ and $M_1(0) \leq \beta'$. Let*

$$T_{\delta'} = \inf\{s > 0, \ X_0(s) \leq \delta'\}.$$

Then

$$\mathbb{P}\left(T_{\delta'} \leq t_2, \ X_0(T_{\delta'}) \times M_1(T_{\delta'}) \leq \varepsilon \right) \geq \left(\frac{1}{12} \right)^{\tilde{k}_{\max}} := p_{trans},$$

with $t_2 = 25\overline{k}_{\max}$, and $\overline{k}_{\max}$ is the number of steps needed to reach δ' in the above procedure, while starting from $x_{\max}$.

PROOF : It follows from (5.5), (5.10), (5.11), Lemma 5.3 and again Lemma 3.1 that the k–th step in the above procedure happens with probability at least $1/12$. It remains to exploit the Markov property, like at the end of the proof of Lemma 4.4. ◇

5.3   Conclusion

From Proposition 3.8, starting at the end of the previous path, we have a probability p_{fin} to reach 0 during an interval of time of length t_3.

So to sum up, using again the strong Markov property of the system, we have

Proposition 5.7 *For any finite stopping time T, if $M_1(T) \leq \beta$, then*

$$\mathbb{P}(T_0 < T + t_1 + t_2 + t_3) \geq p_{fin}p_{trans}p_{init} > 0$$

Moreover Lemma 4.4 implies that this situation will happen infinitely many times as long as the ratchet does not click, which implies Theorem 1, exploiting again the strong Markov property of the solution of (1.1).

6   Proof of Theorem 2

This final section is devoted to the proof of Theorem 2.

We first note that the reasoning of section 5 can be done with any initial value ρ for M_1, instead of β. That is to say, with $S^t_\rho = \inf\{s > t, M_1(s) \leq \rho\}$ (and $S_\rho = S^0_\rho$),

Lemma 6.1 $\exists\ t^\rho_1, t^\rho_2, t^\rho_3 < \infty$, *and* $p^\rho_{init}, p^\rho_{trans}, p^\rho_{fin} > 0$ *such that*

$$\mathbb{P}(T_0 < S^t_\rho + t^\rho_1 + t^\rho_2 + t^\rho_3) \geq p^\rho_{init}p^\rho_{trans}p^\rho_{fin}$$

Choosing $\rho = \frac{\varepsilon}{\delta} \vee \frac{2\lambda}{\alpha}$, we have :

Lemma 6.2 *There exists K, $\tilde{p} > 0$, such that for any initial condition in the set $\mathcal{X}_\delta$,*

$$\mathbb{P}(T_0 \wedge S_\rho \leq K) \geq \tilde{p}$$

PROOF : We are going to argue like in the proof of Lemma 4.4. We introduce the process $\{Y_s, s \geq 0\}$, which is the solution of the following system :

$$\begin{cases} dY_s = \dfrac{\alpha\varepsilon}{2}ds + \sqrt{\dfrac{Y_s(1 - Y_s)}{N}}dB_0(s) \\ Y_0 = 0 \end{cases} \tag{6.1}$$

For any $0 \leq u \leq 1$, let

$$R_u = \inf\{s \geq 0, Y_s = u\}.$$

Since $\frac{\alpha\varepsilon}{2} > 0$ we deduce that there exist $L > 0$, $p > 0$ such that $\mathbb{P}(R_1 \leq L) \geq p > 0$. We choose $K = L + t_3$, where t_3 hax been defined in Proposition 3.8.

Now there are several possibilities :

Case 1. $\inf_{0 \leq s \leq L} M_1(s) \leq \rho$, then $S_\rho < L < K$.

Case 2a. $\inf_{0 \le s \le L} M_1(s) \ge \rho$ and $\inf_{0 \le s \le L} X_0(s)M_1(s) \le \varepsilon$. Then there exists $t < L$ such that $X_0(t)M_1(t) \le \varepsilon$ (which implies $X_0(t) \le \delta$, because $M_1(t) \ge \rho \ge \frac{\varepsilon}{\delta}$). In that case we can use Proposition 3.8, and we have $\mathbb{P}(T_0 \le K) = p_{fin} > 0$, which implies $\mathbb{P}(T_0 \wedge S_\rho \le K) = p_{fin} > 0$,

Case 2b. $\inf_{0 \le s \le L} M_1(s) \ge \rho$ and $\inf_{0 \le s \le L} X_0(s)M_1(s) \ge \varepsilon$. In that last case we have (using first $X_0 \ge \frac{\varepsilon}{M_1}$ combined with $\alpha M_1 - \lambda \ge \lambda > 0$, and next $-\frac{\lambda}{M_1(s)} \ge -\frac{\alpha}{2}$)

$$
\inf_{0 \le s \le L}(\alpha M_1(s) - \lambda)X_0(s) \ge \inf_{0 \le s \le L} \varepsilon\left(\alpha - \frac{\lambda}{M_1(s)}\right)
$$
$$
\ge \frac{\alpha \varepsilon}{2},
$$

and consequently we can use the comparison theorem (Lemma 2.5), which implies that $\forall s \in [0, L]$, $X_0(s) \ge Y_s$. Then $\mathbb{P}(T_1 \le L) \ge p > 0$. But when X_0 hits 1, M_1 hits 0. Hence $\mathbb{P}(S_\rho \le L) \ge p > 0$.

We may now conclude that there exists $\tilde{p} > 0$ such that

$$
\mathbb{P}(T_0 \wedge S_\rho \le K) \ge \tilde{p}
$$

$\diamondsuit$

We deduce from the two above Lemmas :

Corollary 6.3 *There exists $\overline{K} < \infty$, and $\overline{p} > 0$ such that, for any initial condition in $\mathcal{X}$,*
$$
\mathbb{P}(T_0 \le \overline{K}) \ge \overline{p}.
$$

We can now proceed with the
PROOF OF THEOREM 2 We deduce from Corollary 6.3 and the strong Markov property that for all $n \ge 0$, $\mathbb{P}(T_0 > n\overline{K}) \le (1 - \overline{p})^n$. Consequently

$$
\mathbb{E}[e^{\rho T_0}] = \sum_{n=0}^{\infty} e^{(n+1)\rho\overline{K}} \mathbb{P}(n\overline{K} \le T_0 \le (n+1)\overline{K})
$$
$$
\le \sum_{n=0}^{\infty} e^{(n+1)\rho\overline{K}} (1 - \overline{p})^n
$$
$$
= e^{\rho\overline{K}} \sum_{n=0}^{\infty} \left(e^{\rho\overline{K}}(1 - \overline{p})\right)^n
$$
$$
< \infty,
$$

provided $\log(1 - \bar{p}) + \rho \overline{K} < 0$, in other words $\rho < \bar{\rho} := -\log(1 - \bar{p})/\overline{K}$. ◇

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