Muller's ratchet clicks in finite time

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Abstract

We consider the accumulation of deleterious mutations in asexual population, a phenomenom known as Muller's ratchet, using the continuous time model proposed by [3]. We show that for any parameter $\lambda > 0$ (the rate at which mutations occur), for any $\alpha > 0$ (the toxicity of the mutations) and for any N, the ratchet clicks a.s. in finite time. That is to say the minimum number of deleterious mutation in the population goes to infinity a.s.

1 Introduction

In natural evolution, deleterious mutations occur much more frequently than beneficial ones. Since the last category is always favored by selection, one may wonder about the advantage of sexual reproduction over the asexual type. The answer is simple : in an asexually reproducing population, each individual always inherits all the deleterious mutations of his ancestor (except if another mutation occurs at the same place; but this event is rare and we will not consider it) whereas in sexual reproduction, recombinations occur, which allow an individual to take part of a chromosome from each of his parents, therefore permitting him to get rid of deleterious mutations. Muller's ratchet can be used as an attempt to translate this phenomenom in a mathematical model, thus explaining the advantage of sexual reproduction [7] . If one considers the best class (the group of fittest individuals) in a given population, Muller's ratchet is said to click when the best class gets empty. Since beneficial mutations do not occur in this model, it means that all the children of the best class (if there are any) have mutated.

The first model for Muller's ratchet due to Haigh [4] can be described as follow. Consider a population of fixed sized N which evolves in discrete time. Only deleterious mutations happen. Denoting by $0 \leq \alpha \leq 1$ the deleterious strength of the mutations, and $\lambda > 0$ the rate at which they occur, each new generation is constituted as follows : each individual chooses a parent from the previous generation, in such a way that the probability to choose a father with k deleterious mutations is (we denote by N_k the number of such individuals in the previous generation) :

$$\frac{(1-\alpha)^k}{\sum_{k=0}^N N_k(1-\alpha)^k}.$$

Next each newborn gains K deleterious mutations, where K a Poisson random variable with parameter λ. It is immediate to see that this model clicks a.s. in finite time, indeed at each generation, with probability $(1-\exp(-\frac{\lambda}{\alpha}))^N$ all the individuals mutate, which induces the click.

The following Fleming Viot model in continuous time has been proposed by A. Etheridge, P. Pfaffelhuber and A. Wakolbinger in [3] :

N denotes the size of the population;
$X_k(t)$ the proportion of individuals with k deleterious mutations at time t;
λ is the mutation rate;
α is the fitness decrease due to each mutation;
$\{B_{k,\ell}, k > \ell \geq 0\}$ are independent Brownian motions, and $B_{k,\ell} = -B_{\ell,k}$;
$M_1 = \sum_{k\in\mathbb{N}} kX_k$ the mean number of mutation in the population,
$M_\ell = \sum_{k\in\mathbb{N}}(k-M_1)^\ell X_k$ the ℓ–th centered moment, $\forall \ell \geq 2$.

The Fleming–Viot model for Muller's ratchet in continuous time is given by the following infinite set of SDEs

$$\begin{cases} dX_k = [\alpha(M_1-k)X_k + \lambda(X_{k-1}-X_k)]\,dt + \displaystyle\sum_{\ell\geq 0,\ell\neq 0}\sqrt{\frac{X_kX_\ell}{N}}dB_{k,\ell} \\ \qquad = [\alpha(M_1-k)X_k + \lambda(X_{k-1}-X_k)[\,dt + \sqrt{\dfrac{X_k(1-X_k)}{N}}dB_k \\ X_k(0) = x_k;\ k\geq 0. \end{cases} \tag{1.1}$$

This system of SDEs is well posed provided the initial condition belongs to

$$\mathcal{X} = \{(x_k)_{k\in\mathbb{Z}_+} \in \mathbb{R}_+^{\mathbb{Z}_+}, \sum_{k\geq 0} x_k = 1, \text{ and } \sum_{k\geq 0} e^{N\alpha k}x_k < \infty\},$$

as will be explained in section 2. We equip this set with the distance

$$d(x,y) = \sum_{k\geq 0} e^{\alpha N k} \mid x_k - y_k \mid,$$

which makes it a complete metric space. If X is a $\mathcal{X}$ valued solution of (1.1), we can write the equation for M_1 :

$$dM_1(t) = (\lambda - \alpha M_2(t))dt + \sqrt{\frac{M_2(t)}{N}}dB_t.$$

We define $T_0 = \inf\{t > 0, X_0(t) = 0\}$.

The purpose of the present work is to show that this model of Muller's ratchet is bound to click in finite time, that is to say $T_0 < \infty$ a.s. . We are going to prove the following theorem :

Theorem 1 *For any choice of initial condition in $\mathcal{X}$, let $(X_k(t))_{k\in\mathbb{Z}_+}$ the solution of (1.1). Then $\mathbb{P}(T_0 < \infty) = 1$.*

We will in fact prove a stronger result, namely

Theorem 2 *For any choice of initial condition in $\mathcal{X}$, let $(X_k(t))_{k\in\mathbb{Z}_+}$ the solution of (1.1). Then $\mathbb{E}(T_0) < \infty$.*

There are several difficulties in this model. First, it is an infinite system of SDEs which cannot be reduced to a finite dimensional system. Only X_0 and M_1 enter the coefficients of the equation for X_0, but the equation for M_1 brings in the second moment M_2. The system of SDEs for the M_k's is infinite as well, the M_ℓ up to order $\ell = 2k$ enter the coefficients of the equation for M_k, and there is no known solution to it (except in the deterministic case $N = +\infty$, which is solved in [3]). In addition, one has $d\langle X_0, M_1\rangle = -\frac{M_1 X_0}{N}dt$. But there is no easy relation between X_0 and M_1, except that $X_0 + M_1 \geq 1$, and $(X_0 = 1) \Rightarrow (M_1 = 0)$. But we could have $X_0 \to 0$ and $M_1 \to \infty$. Last but not least, the diffusion coefficient in dX_k is not a Lipschitz function of X_k at 0 and 1, and it vanishes at those two points.

In order to prove the theorem, we will use a three steps proof. First, in section 3 we will show that M_1 cannot grow too fast with a good probability, and we will deduce that for a specific set of initial conditions, the ratchet does click with a strictly positive probability p_{fin}, in a given interval of time.

Next, we show in section 4 that the product $X_0 M_1^2$ is bound to come back under $\frac{2(\lambda+1)}{\alpha}$ after any time, and we use all the previous results to deduce that M_1 is also bound to return under $\beta = \frac{\lambda}{\alpha}$ after any time, as long as the ratchet does not click.

Finally in section 5 we prove that each time M_1 gets below β, the ratchet clicks with a positive probability in a prescribed interval of time. We then conclude with the help of the strong Markov property.

In section 6 we show how the proof of Theorem 1 can be turned into a proof of Theorem 2. The reader may wonder why we do not prove Theorem 2 from the very beginning, and first prove a weaker result. The reason is that the difference between the two proofs is essentially that while proving Theorem 1, we prove that as long as the ratchet has not clicked, M_1 is bound to return below the value β, i. e. the drift of X_0 is bound to become non–positive, which is an interesting result in itself, while the proof of Theorem 2 is based on the same strategy, but with β replaced by a much less explicit quantity.

We shall essentially work with the two dimensional process $\{X_0(t), M_1(t)\}$, and we shall use the equation for X_1 only in one place, namely in Lemma 5.1 in order to show that X_0 does not get stuck near the value 1. We shall make use of the three following equations.

$$
\begin{aligned}
dX_0 &= (\alpha M_1 - \lambda)\, X_0 dt + \sqrt{\frac{X_0(1-X_0)}{N}} dB_0 \\
dX_1 &= (\alpha(M_1-1)X_1 + \lambda(X_0 - X_1))\, dt + \sqrt{\frac{X_1(1-X_1)}{N}} dB_1 \\
dM_1(t) &= (\lambda - \alpha M_2(t))dt + \sqrt{\frac{M_2(t)}{N}} dB_t,
\end{aligned}
$$

This system is not closed, since M_2 enters the coefficients of the last equation. However, the crucial remark is that it will not be necessary to estimate M_2, in order to estimate M_1. This is due to the fact that the M_1–equation takes the form $dM_1(t) = \lambda dt + dZ_t$, where $Z_t = W(A_t) - \alpha N A_t$, if $A_t := \int_0^t M_2(s)/N ds$ and $\{W(t),\ t \geq 0\}$ is a standard Brownian motion. The larger M_2 is, the more likely Z_t is negative, which produces a smaller M_1. This means that we should be able to estimate M_1, without having to estimate M_2, which is done below in Lemma 3.2 and 4.3.

Section 2 is devoted to some preliminary results on our system of SDEs.

2 Preliminary results

We first state a minor variant of the weak existence and uniqueness for the solution of our system, which is due to [2]. Indeed, a slight modification of the arguments in [2] (see [1] for details) yields the following result.

Proposition 2.1 *The infinite system of SDEs (1.1) has a unique weak solution for any initial condition in $\mathcal{X}$, in the sense that the associated martingale problem is well posed.*

Proposition 2.1 relies upon the following Lemma, for which we will provide a proof :

Lemma 2.2 *If $X(0) \in \mathcal{X}$, then $\forall t \geq 0$ $X(t) \in \mathcal{X}$ a. s.*

We first establish :

Lemma 2.3 *Let X be a $\mathbb{Z}_+$-valued random variable, and write $x_k = \mathbb{P}(X = k)$, $k \geq 0$. Suppose that Y is another $\mathbb{Z}_+$-valued random variable, whose law is given by $\mathbb{P}(Y = k) = \frac{F(k)x_k}{\sum_{k\in\mathbb{Z}_+} F(k)x_k}$, where $F : \mathbb{Z}_+ \Rightarrow \mathbb{R}_+^\star$ is an increasing function, such as*

$$\sum_{k\in\mathbb{Z}_+} F(k)\, x_k < \infty$$

Then

$$\mathbb{E}(X) \leq \mathbb{E}(Y).$$

Proof : The case where F is constant is trivial, since in that case $y_k = x_k$, $\forall k \geq 0$.

Now if F is non-constant, from the hypothesis we have that: $\frac{F(k)}{\sum_{j\in\mathbb{Z}_+} F x_j} - 1$ is an increasing non constant fonction such that

$$\sum_{k\in\mathbb{Z}_+} \left(\frac{F(k)}{\sum_{j\in\mathbb{Z}_+} F(j)\, x_j} - 1 \right) x_k = 0$$

Hence we have : $\exists \ell \in \mathbb{Z}_+$ such as $\forall n < \ell \leq k$, $\frac{F(k)}{\sum_{j\geq 0} F(j)x_j} - 1 \geq 0$; $\frac{F(n)}{\sum_{n\geq 0} F(n)x_n} - 1 \leq 0$ and

$$\sum_{k=\ell}^{k=\infty} \left(\frac{F(k)}{\sum_{j\geq 0} F(j)\, x_j} - 1 \right) x_k = -\sum_{k=0}^{\ell-1} \left(\frac{F(k)}{\sum_{j\geq 0} F(j)\, x_j} - 1 \right) x_k.$$

Then

$$\sum_{k\geq 0} k \left(\frac{F(k)}{\sum_{j\geq 0} F(j)\, x_j} - 1 \right) x_k = \sum_{k=\ell}^{k=\infty} k \left(\frac{F(k)}{\sum_{j\geq 0} F(j)\, x_j} - 1 \right) x_k + \sum_{k=0}^{\ell-1} k \left(\frac{F(k)}{\sum_{j\geq 0} F(j)\, x_j} - 1 \right) x_k.$$

the first term in positive and the second negative, so the above right–hand side is bounded from below by

$$\ell \sum_{k=\ell}^{k=\infty} \left(\frac{F(k)}{\sum_{j\geq 0} F(j)\, x_j} - 1 \right) x_k + (\ell - 1) \sum_{k=0}^{\ell-1} \left(\frac{F(k)}{\sum_{j\geq 0} F(j)\, x_j} - 1 \right) x_k,$$

which is non–negative. ♢

We deduce the following Corollary, where again $(x_k, k \geq 0)$ is the law of X.

Corollary 2.4 $\forall C, \rho > 0$ *We have the following inequality :*

$$\left(\sum_{j\geq 0} j x_j \right) \left(\sum_{k\geq 0} \left(e^{\rho k} \wedge C\right) x_k \right) \leq \sum_{j\geq 0} j \left(e^{\rho j} \wedge C\right) x_j.$$

Proof : We may divide by $\left(\sum_{k\geq 0} \left(e^{\rho k} \wedge C\right) x_k\right)$, which is strictly positive and finite. We see that the result is equivalent to

$$\sum_{j\geq 0} j x_j \leq \sum_{j\geq 0} j \frac{\left(e^{\rho j} \wedge C\right)}{\sum_{k\geq 0} \left(e^{\rho k} \wedge C\right) x_k} x_j,$$

which follows from the previous lemma with $F(k) = \left(e^{\rho k} \wedge C\right)$.

♢

Now let $X = (X_k(t), k \geq 0, t \geq 0)$ be the solution of (1.1). We define

$$\Psi(t, \rho) = \mathbb{E} \left(\sum_{k\geq 0} X_k(t)\, e^{\rho k} \right)$$

and $\forall C > 0$

$$\Psi_C(t, \rho) = \mathbb{E} \left(\sum_{k\geq 0} X_k(t) \left(e^{\rho k} \wedge C\right) \right)$$

We now have

Lemma 2.5 *Let X be a solution of (1.1). Then for all $t \geq 0$, $\rho \geq 0$,*

$$\Psi(t, \rho) \leq \Psi(0, \rho)\, e^{\lambda(e^{\rho}-1)t}.$$

Proof : Let

$$\Phi(t,\rho) = \sum_{k\geq 0} X_k(t)\, e^{\rho k},$$
$$\Phi_C(t,\rho) = \sum_{k\geq 0} X_k(t)\,(e^{\rho k}\wedge C).$$

We deduce from Ito's formula

$$\Psi_C(t,\rho) = \Psi_C(0,\rho) + \mathbb{E}\int_0^t \sum_{k\in\mathbb{Z}} \left(\lambda\left(x_{k-1}(r) - x_k(r)\right) + s\left(-k + \sum_{j\geq 0} j x_j(r)\right) x_k(r)\right)\left(e^{\rho k}\wedge C\right) dr$$

$$\leq \Psi_C(0,\rho) + \mathbb{E}\int_0^t \left(\left(\lambda\left(e^{\rho}\Phi_C(r) - \Phi_C(r)\right) - s\left(\sum_{k\geq 0} k x_k(r)\left(e^{\rho k}\wedge C\right) + \sum_{j\geq 0} j x_j(r)\,\Phi_C(r)\right)\right)\right) dr$$

because we work with $\rho > 0$, so $Ce^{-\rho} \leq C$.

Moreover, thanks to the previous Corollary, we have

$$\left(\sum_{j\geq 0} j x_j\right)\left(\sum_{k\geq 0}\left(e^{\rho k}\wedge C\right) x_k\right) \leq \sum_{j\geq 0} j\left(e^{\rho j}\wedge C\right) x_j$$

that is to say

$$\sum_{j\geq 0} j x_j(r)\,\Phi_C(r) - \sum_{j\geq 0} j\left(e^{\rho j}\wedge C\right) x_j \leq 0$$

Since our functions are bounded, we can invert $\mathbb{E}$ and $\int$,

$$\Psi_C(t,\rho) \leq \Psi_C(0,\rho) + \int_0^t \left(\lambda\left(e^{\rho} - 1\right)\right)\Psi_C(r,\rho)\, dr$$

The result is a consequence of the Gromwall inequality, and the monotone convergence Theorem. ♢

Lemma 2.2 now follows.

Our processes are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with a filtration $(\mathcal{F}_t, t \geq 0)$ which is such that for each $k, \ell \geq 0$ $\{B_{k,\ell}(t),\ t \geq 0\}$ is a $\mathcal{F}_t$–Brownian motion. We denote by $\mathcal{P}$ the corresponding σ-algebra of predictable subsets of $\mathbb{R}_+ \times \Omega$.

From the weak existence and uniqueness, we deduce that our system has the strong Markov property, using a very similar proof as in Theorem 6.2.2 from [8].

In the next sections, we will use the following comparison theorem several times. This Lemma can be proved exactly as the comparison Theorem 3.7 found in chapter IX of [6].

Lemma 2.6 *Let B_t be a standard $\mathcal{F}_t$–Brownian motion, T a stopping time, σ be a 1/2 Hölder function, $b_1 : \mathbb{R} \to \mathbb{R}$ a Lipschitz function and $b_2 : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is $\mathcal{P} \otimes B(\mathbb{R})$ measurable function. Consider the two SDEs*

$$\begin{cases} dY_1(t) = b_1(Y_1(t))dt + \sigma(Y_1(t))dB_t, \\ Y_1(0) = y_1; \end{cases} \tag{2.1}$$

$$\begin{cases} dY_2(t) = b_2(t, Y_2(t))dt + \sigma(Y_2(t))dB_t, \\ Y_2(0) = y_2. \end{cases} \tag{2.2}$$

Let Y_1 (resp Y_2) be a solution of (2.1) (resp (2.2)). If $y_1 \leq y_2$ (resp $y_2 \leq y_1$) and outside a measurable subset of Ω of probability zero, $\forall t \in [0,T]$, $\forall x \in \mathbb{R}$, $b_1(x) \leq b_2(t,x)$ (resp $b_1(x) \geq b_2(t,x)$). Then a. s. $\forall t \in [0,T]$, $Y_1(t) \leq Y_2(t)$ (resp $Y_1(t) \geq Y_2(t)$).

3 The result for a specific set

First, we start with a trivial lemma which will be used several times below :

Lemma 3.1 *Let $E, F \in \mathcal{F}$. Then $\mathbb{P}(E \cap F) \geq \mathbb{P}(E) + \mathbb{P}(F) - 1$.*

Proof : Clearly

$$1 \geq \mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F).$$

$\diamondsuit$

Now first we will show that M_1 cannot grow too fast :

Lemma 3.2 *For all $c > 0$, $t > 0$,*

$$\mathbb{P}\left(\sup_{0 \leq r \leq t'} M_1(r+t) - M_1(t) \leq \lambda t' + c\right) \geq 1 - exp(-2\alpha N c) > 0$$

Proof : We define $Z^t_{t+s} = \int_t^{t+s} \sqrt{\frac{M_2(r)}{N}} dB_r - \alpha \int_t^{s+t} M_2(r)dr,$. We note that, for any $t > 0$, $\{exp(2\alpha N Z^t_{t+u})\}_{u \geq 0}$ is both a local martingale and a supermartingale. We also have

$$\sup_{0 \leq s \leq t'} M_1(s+t) - M_1(t) \leq \sup_{0 \leq s \leq t'} Z^t_{s+t} + \lambda t'.$$

And $\forall c > 0$

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq u \leq t'} Z^t_{t+u} \geq c\right) &\leq \mathbb{P}\left(\sup_{0 \leq u \leq t'} exp\left(2\alpha N Z^t_{t+u}\right) \geq exp\left(2\alpha N c\right)\right) \\ &\leq exp\left(-2\alpha N c\right) < 1 \end{aligned}$$

where we have taken advantage of the fact that $exp\left(2\alpha N Z^t_{u+t}\right)$ is a local martingale and of Doob's inequality. Then

$$\mathbb{P}\left(\sup_{0 \leq r \leq t'} M_1\left(r+t\right) - M_1\left(t\right) \leq \lambda t' + c\right) \geq 1 - exp\left(-2\alpha N c\right) > 0$$

$\diamondsuit$

Note that we have in fact $\mathbb{P}\left(\sup_{u \geq 0} Z^t_{t+u} \geq c\right) \leq exp\left(-2\alpha N c\right) < 1$

We choose an arbitrary value $m > 0$ for $M_1(0)$, which will remain the same throughout this document (for example one could choose $m = 1$), $\overline{\varepsilon} = \frac{1}{10N\alpha}$, and let us define

$$t'_3 = \frac{\overline{\varepsilon} N}{3\lambda} = \frac{1}{30\lambda\alpha},$$

$$m_{max} = m + \lambda A(t'_3) + \frac{\overline{\varepsilon}}{6},$$

where $A(t) = \frac{1}{4N} \int_0^t (1 - X_0(s))ds$,

$$p_2 = exp(-\alpha N \frac{\overline{\varepsilon}}{6}) = \exp(-\frac{1}{60}),$$

$$\mu = \frac{\overline{\varepsilon}}{6m_{max}} \wedge \frac{\overline{\varepsilon}}{4} \wedge \frac{1}{10},$$

and let δ be a real number, which will be specified below, such that $\delta \leq \frac{1}{10} \wedge \frac{\overline{\varepsilon}}{m}$.

Now let Y_0 be the solution of the following SDE :

$$\begin{cases} dY_0(t) = dt + 2\sqrt{Y_0(t)}dB_0 \\ Y_0(0) = \delta \end{cases} \tag{3.1}$$

We will show that starting with $X_0(0) = x_0 \leq \delta$, $M_1(0) = m_1 \leq m$, and as long as $X_0 M_1 < 2\overline{\varepsilon}$, we can compare $X_0(t)$ with the solution of (3.1). In this part we will use three time scales. To help the reading, we will note with a prime all the times expressed for the simplified system (3.1), that is to say t'_3 and T'_μ.

Lemma 3.3 *Let* $T_{min} = \inf\{t > 0, X_0(t)M_1(t) > 2\overline{\varepsilon}$ *or* $X_0(t) > \delta + \mu\}$. *Then provided* $X_0(0) = x_0 < \delta$, $\forall t \in [0, T_{min}]$, *we have* $X_0(t) \leq Y_0(A(t))$ *where* Y_0 *solves the SDE* (3.1) *and* $A(t) = \frac{1}{4}\int_0^t \frac{1-X_0(s)}{N} ds$.

Note that for $0 \leq t \leq T_{min}$, $\frac{t}{5N} \leq A(t) \leq \frac{t}{4N}$ because $\frac{4}{5} \leq 1 - X_0 \leq 1$ (thanks to the choices of μ and δ, and $1 - X_0 \geq 1 - \delta - \mu \geq 1 - \frac{1}{10} - \frac{1}{10} \geq \frac{4}{5}$) and t'_3 has been chosen in such way that $A(t'_3) \leq \frac{\overline{\varepsilon}}{12\lambda}$.
PROOF : We define $\sigma(t) = \inf\{u > 0, A(u) \geq t\}$ and $\tilde{X}_0(t) = X_0(\sigma(t))$ (resp $\tilde{M}_1(t) = M_1(\sigma(t))$). Then there exists a standard Brownian motion W_t such that

$$d\tilde{X}_0(t) = (\alpha\tilde{M}_1(t) - \lambda)\tilde{X}_0(t)\frac{4N}{1 - \tilde{X}_0(t)}dt + 2\sqrt{\tilde{X}_0(t)}dW_t$$

Since $\tilde{M}_1(t)\tilde{X}_0(t) \leq 2\overline{\varepsilon}$ and $\overline{\varepsilon} = \frac{1}{10N\lambda}$, then $\forall\ t \leq A(T_{min})$, we have

$$(\alpha\tilde{M}_1(t) - \lambda)\tilde{X}_0(t)\frac{4N}{1 - \tilde{X}_0(t)} \leq 1$$

Then, using Lemma 2.6, we obtain the conclusion. $\diamondsuit$

Next we will prove that Y_0 can reach zero. The following Lemma exploits an argument from [5]

Lemma 3.4 *Let* $Y_0(t)$ *be the solution of (3.1)* , $T'_0 = \inf\{t > 0, Y_0(t) = 0\}$, $T'_{\tilde{\mu}} = \inf\{t > 0, Y_0(t) = \delta + \tilde{\mu}\}$. *Then* $\forall p < 1$, $\forall \tilde{\mu} > 0$, $\exists \delta > 0$ *such that*

$$\mathbb{P}(T'_0 \leq t'_3 \wedge T'_{\tilde{\mu}}) \geq p$$

PROOF : Let

$$\begin{aligned}\tilde{Y}(t) &= \delta \exp\left(-t + 2B_0(t)\right),\\ D(t) &= \int_0^t \tilde{Y}(s)ds,\\ \rho(t) &= \inf\{s > 0, D(s) > t\}.\end{aligned}$$

We have

$$Y_0(t) = \tilde{Y}(\rho(t))$$
$$T'_0 = D(\infty) < \infty$$

Then,

$$\begin{aligned}
&\mathbb{P}(T'_0 \leq t'_3 \wedge T'_{\tilde{\mu}}) \\
&= \mathbb{P}\left(\left\{\int_0^\infty \exp(-t + 2B_0(t))dt \leq \frac{t'_3}{\delta}\right\} \cap \left\{\sup_{t\geq 0} \exp(-t + 2B_0(t)) \leq \frac{\delta + \tilde{\mu}}{\delta}\right\}\right) \\
&\longrightarrow 1,
\end{aligned}$$

as $\delta \to 0$, since $\sup_{t\geq 0} \exp(-t + 2B_0(t)) < \infty$ a.s.

$\diamondsuit$

Then we can obtain the δ we'll be using from now on. Let δ' the largest value of δ such that Lemma 3.4 holds, with $p = p_2$ and $\tilde{\mu} = \mu$ (which is a function of m_{max}) as given above. We choose

$$\delta = \delta' \wedge \frac{1}{10} \wedge \frac{\overline{\varepsilon}}{m}$$

.

Thanks to Lemma 3.4, Y_0 will not become greater than $\delta + \mu$ and will reach 0 with probability p_2 before the time $t'_3 \wedge T'_\mu$. Then X_0 will do the same before time $A(t'_3) \wedge A(T'_\mu)$, provided that $X_0(t)M_1(t) \leq 2\overline{\varepsilon}\ \forall 0 \leq t \leq A(t'_3) \wedge A(T'_\mu)$. Hence the fact that $T_0 < A(t'_3)$ with positive probability, provided $x_0 \leq \delta$ and $M_1(0) \leq m$ will follow from the above results and

Lemma 3.5 *If $X_0(0) \leq \delta$ and $M_1(0) \leq m$, then we have*

$$\mathbb{P}\left(\sup_{0\leq t\leq A(t'_3\wedge T'_\mu)} X_0(t)M_1(t) \leq 2\overline{\varepsilon}\right) = p_3 > 1 - p_2.$$

PROOF : We use Lemma 3.2. Consider the event

$$E_{m,t'_3,\tilde{\varepsilon}} = \left\{\sup_{0\leq t\leq A(t'_3)\wedge A(T'_\mu)} M_1(t) \leq m + \lambda A(t'_3) + \frac{\overline{\varepsilon}}{6}\right\}.$$

We have

$$\begin{aligned}\mathbb{P}(E_{m,t_3',\tilde{\varepsilon}}) &\geq \mathbb{P}\left(\sup_{0\leq t\leq A(t_3')} M_1(t) \leq m_1 + \lambda A(t_3') + \frac{\overline{\varepsilon}}{6}\right) \\ &\geq 1 - exp(-\alpha N \frac{\overline{\varepsilon}}{3}) = 1 - exp(-\frac{1}{30}).\end{aligned}$$

Since $X_0(t) \leq \delta + \mu$ for $t \leq A(T'_\mu)$, on the event $E_{m,t_3',\tilde{\varepsilon}}$,

$$\begin{aligned}\sup_{0\leq t\leq A(t_3')\wedge A(T'_\mu)} X_0(t)M_1(t) &\leq (\delta+\mu)(m + \lambda A(t_3') + \frac{\overline{\varepsilon}}{6}) \\ &\leq \delta m + \mu m + \lambda A(t_3') + \frac{\overline{\varepsilon}}{6} \\ &\leq \overline{\varepsilon} + \frac{\overline{\varepsilon}}{6} + \frac{\overline{\varepsilon}}{12} + \frac{\overline{\varepsilon}}{6} \\ &\leq 2\overline{\varepsilon},\end{aligned}$$

where we have used the fact that $\lambda + \mu \leq 1$ for the second inequality. $\diamondsuit$

Combining Lemma 3.1, Lemma 3.3, Lemma 3.5 and Lemma 3.4, denoting $t_3 = A(t_3')$, we deduce the

Corollary 3.6 $\exists p_{fin} \geq p_3 + p_2 - 1 > 0$ *such as,*

$$\mathbb{P}(T_0 < t + t_3 \mid X_0(t) \leq \delta, M_1(t) \leq m) \geq p_{fin} > 0$$

Now we will extend this result to the following larger set of initial conditions. δ and m being defined above (in particular such that $\delta m \leq \overline{\varepsilon}$), we consider the set

$$\mathcal{I} = \{(x, m_1) \in [0;1] \times \mathbb{R}_+, x \leq \delta, x m_1 \leq \delta m\}$$

Thanks to the previous result, we only need to consider the case $m_1 \geq m$. Let (x_0', m_1') be a point in the set $\mathcal{I}$. First, let us consider the point (δ, m). From the previous section, starting from (δ, m), the process (X_0, M_1) has a strictly positive probability to reach 0 before the time $t_3 = A(t_3')$. We will show that the process starting from (x_0', m_1') has a larger probability to reach 0 before time t_3, which will extend the previous result.

Let $C = \frac{m_1'}{m} \geq 1$. Then we have $x_0' \leq \frac{\delta}{C}$.

Now we will use the same reasoning as in Lemma 3.4 with a few modifications. Indeed, since the probability that $Y_0(t)$ reaches 0 before a prescribed time is decreasing in δ, we increase this probability by starting from $Y_0(0) = x'_0 = \delta' \leq \frac{\delta}{C}$. since $C \geq 1$. We will use this new value. Moreover, the starting point satisfies $x'_0 m'_1 \leq \overline{\varepsilon}$. The only thing which is worse than with the starting point (δ, m) is the fact that m'_1 is greater than m, hence a greater m_{max}. But this only appears in one place : in the definition of μ.

Note that if we define $m'_{max} = m'_1 + \lambda t_3 + \frac{\overline{\varepsilon}}{6}$, the maximum reached by M'_1, we have :

$$m'_{max} \leq C m_{max}$$
$$\mathbb{P}(\sup_{0 \leq t \leq t_3} M'_1(t) \leq M'_{max}) \geq 1 - exp(-\alpha N \frac{\overline{\varepsilon}}{3})$$

By the definition of μ, if we define μ' with m'_{max} instead of m_{max} (i.e. $\mu' = \frac{\overline{\varepsilon}}{6m'_{max}} \wedge \frac{\overline{\varepsilon}}{4} \wedge \frac{1}{10},$) we have $\mu' \geq \frac{\mu}{C}$. But if we look at the proof of Lemma 3.4, we have, since $\frac{t'_3}{\delta'} \geq \frac{Ct'_3}{\delta} \geq \frac{t'_3}{\delta}$ and $\frac{\delta'+\mu'}{\delta'} = 1 + \frac{\mu'}{\delta'} \geq 1 + \frac{\mu}{\delta}$,

$$\begin{aligned} \mathbb{P}(T'_0 \leq t'_3 \wedge T'_{\mu'}) &\geq \mathbb{P}\left(\left\{\int_0^\infty exp(-t + 2B_1(t))dt \leq \frac{t'_3}{\delta'}\right\} \cap \left\{\sup_{t\geq 0} exp(-t + 2B_1(t)) \leq \frac{\delta' + \mu'}{\delta'}\right\}\right) \\ &\geq \mathbb{P}\left(\left\{\int_0^\infty exp(-t + 2B_1(t))dt \leq \frac{t'_3}{\delta}\right\} \cap \left\{\sup_{t\geq 0} exp(-t + 2B_1(t)) \leq \frac{\delta + \mu}{\delta}\right\}\right) \end{aligned}$$

Hence we have a larger probability to reach zero starting from (x', m'_1) rather than from (δ, m), which concludes the proof.

$\diamondsuit$

We sum up in the following Proposition the results obtained in this section, with $\varepsilon = \delta m$ (note that $\varepsilon \leq \overline{\varepsilon}$).

Proposition 3.7 *Let $X(t) = (X_k(t))_{k \in \mathbb{Z}_+}$ be the solution of the initial model, and M_1 its mean as defined in section 1. Then $\exists p_{fin} > 0$ and t_3 such that*

$$\mathbb{P}(T_0 < t + t_3 | X_0(t) \leq \delta, X_0(t) M_1(t) \leq \varepsilon) \geq p_{fin} > 0.$$

4 A recurrence property of M_1

With the help of the results proved in the previous section, we will now prove some results on M_1. We will show that as long that as the ratchet has not clicked, M_1 is bound to return under some specified value. This particular point will be important in the sequel.

We begin with the following lemma, which is true for any probability on $\mathbb{Z}_+$. It will be crucial for establishing one of our first estimates.

Lemma 4.1 *Let p be a probability on $\mathbb{Z}_+$, and let $x_k = p(k)$, $m_1 = \sum_{k\geq 0} kx_k$ and $m_2 = \sum_{k\geq 0}(k-m_1)^2 x_k$. Then*

$$m_2 \geq (1-x_0)m_2 \geq x_0 m_1^2.$$

PROOF : If $x_0 = 1$, $m_1 = m_2 = 0$ and the result is true. So it suffices to study the case $x_0 < 1$. By Jensen's inequalities we have

$$\left(\sum_{k\geq 1} \frac{x_k}{1-x_0} k\right)^2 \leq \sum_{k\geq 1} \frac{x_k}{1-x_0} k^2$$

with equality if and only if there exists only one $k \geq 1$ such as $x_k > 0$. Then :

$$\left(\sum_{k\geq 1} x_k k\right)^2 \leq (1-x_0) \sum_{k\geq 1} x_k k^2,$$

that is

$$m_1^2 \leq (1-x_0) \sum_{k\geq 1} x_k k^2,$$

hence

$$\begin{aligned} x_0 m_1^2 &\leq (1-x_0) \sum_{k\geq 1} x_k k^2 - (1-x_0) m_1^2 \\ x_0 m_1^2 &\leq (1-x_0) \left(\sum_{k\geq 1} x_k k^2 - m_1^2\right) \\ x_0 m_1^2 &\leq (1-x_0) m_2. \end{aligned}$$

◇

Now, on our current model, we introduce the stopping time

$$H_\lambda^t := \inf\{s \geq t,\ X_0(s)M_1(s)^2 \leq 2\frac{\lambda+1}{\alpha}\},$$

and we note $H_\lambda = H_\lambda^0$.

Our next claim is

Proposition 4.2 *For any stopping time T, we have $H_\lambda^T < +\infty$ a.s.*

The Proposition follows from the strong Markov property and

Lemma 4.3 *Suppose that $X_0(0)M_1(0)^2 > 2\frac{\lambda+1}{s}$. Then $H_\lambda < \infty$ a. s.*

Proof : On the interval $[0, H_\lambda]$,we have from Lemma 4.1

$$-\frac{\alpha}{2}M_2 \leq -\frac{\alpha}{2}X_0M_1^2 \leq -(\lambda+1),$$

and M_1 is bounded from above by

$$\begin{aligned} M_1(t) &\leq M_1(0) - \int_0^t \left(1 + \frac{\alpha}{2}M_2(t)\right)dt + \int_0^t \sqrt{\frac{M_2(t)}{N}}dB_t, \\ &\leq M_1(0) - t - \int_0^t \left(\frac{\alpha}{2}M_2(t)\right)dt + \int_0^t \sqrt{\frac{M_2(t)}{N}}dB_t. \end{aligned} \tag{4.1}$$

Since M_1 cannot become negative, it now suffice to show that

$$Z_t := \int_0^t \sqrt{\frac{M_2(r)}{N}}dB_r - \frac{\alpha}{2}\int_0^t M_2(r)dr$$

is bounded from above a.s. If we define $C(t) = \frac{1}{N}\int_0^t M_2(s)ds$, we have $Z_t = W(C(t)) - \frac{N}{2}\alpha C(t)$ where W is a standard Brownian motion.

Now, if $C(\infty) = \infty$ then $\lim_{t\to\infty} Z_t = -\infty$, hence Z_t is bounded from above. Or else $C(\infty) < \infty$, and we have $\sup_{t>0} \|Z_t\| = \sup_{0<s<C(\infty)} \|W(s) - \frac{N}{2}\alpha s\| < \infty$ a.s. $\diamondsuit$

Now we will finally be able to prove that M_1 always return below $\beta := \lambda/\alpha$, as long as the ratchet does not click. Let

$$S_\beta^t = \inf\{s > t, M_1(s) \leq \beta\}.$$

Then we will prove the following lemma :

Lemma 4.4 *$\forall t > 0$, we have $\mathbb{P}(T_0 \wedge S^t_\beta < \infty) = 1$*

PROOF : In order to simplify the notations, we treat the case t=0. First, we let $\delta_{inf} = \delta \wedge \frac{\varepsilon^2\alpha}{4(\lambda+1)}$ (recall that $\varepsilon = \delta m$).

Now we introduce the process Y^s_t, defined $\forall s \geq 0$, $\forall t \geq s$ which is the solution of the following system :

$$\begin{cases} dY^s_t = \sqrt{\dfrac{Y^s_t(1-Y^s_t)}{N}} dB_0(t), t \geq s \\ Y^s_s = \delta_{inf}. \end{cases} \tag{4.2}$$

We define for any $0 \leq u \leq 1$

$$R^s_u = inf\{t \geq s, Y^s_t = u\},$$

We have $R^s_0 \wedge R^s_1 < +\infty$ a.s. and $\mathbb{P}(R^s_1 < R^s_0) > 0$. Indeed; $\forall a \in (0, \delta_{inf})$, by the non–degeneracy Y^s_t gets out of $[a, 1-a]$ in finite time. Then if we choose a small enough (using the same reasoning as in Lemma 3.4), we have a chance p'_{fin} to reach 0 before a time $V > 0$ as soon as we start below a (the same with 1 and starting above $\geq 1-a$ by symetry). Hence the result, using the strong Markov property, as this situation happens infinitely many time as long as the process doesn't reach 0 or 1. Note that using the Green function, one can in fact prove that $\mathbb{E}(R^s_0 \wedge R^s_1) + \infty)$.

From this we deduce that $\exists K > 0, p > 0$ such as $\mathbb{P}(R^s_1 \leq K \wedge R^s_0) \geq p > 0$. In particular $\mathbb{P}(R^s_1 \leq K) \geq p > 0$.

We use $L = K \vee t_3$. (t_3 from Proposition 3.7). We define the following sequence of stopping times :

$$U_0 = \inf\left\{ s > t, X_0(s)M_1^2(s) \leq 2\frac{\lambda+1}{\alpha} \right\},$$

and $\forall n \geq 1$,

$$U_n = \inf\left\{ s > U_{n-1} + L, X_0(s)M_1^2(s) \leq 2\frac{\lambda+1}{\alpha} \right\}.$$

For all $n \geq 0$, U_n is a.s. finite, thanks to Proposition 4.2.

Now, at U_0 : Either $X_0(U_0) \leq \delta_{inf}$ $(\leq \delta)$, then

$$\begin{aligned} X_0M_1 &= \sqrt{X_0M_1^2 \times X_0} \\ &\leq \sqrt{2\frac{\lambda+1}{\alpha}\frac{\varepsilon^2\alpha}{4(\lambda+1)}} \\ &< \varepsilon \end{aligned}$$

Then we can use Proposition 3.7, and we have $\mathbb{P}(T_0 \leq U_0 + L) = p_{fin} > 0$.

Or else $X_0(U_0) > \delta_{inf}$. And in that case there are two possibilities : Either $\inf_{U_0 \leq s \leq U_0+L} M_1(t) \geq \beta$. In that case we have $(\alpha M_1 - \lambda)X_0 \geq 0$, and then we can deduce from Lemma 2.6 that $X_0(s) \geq Y_s^{U_0}$. Then $\mathbb{P}(T_1 \leq U_0 + L) \geq p > 0$. But if $X_0(s) = 1$, then $M_1(s) = 0$. Hence $\mathbb{P}(S_\beta \leq U_0 + L) \geq p > 0$. In the other case $\inf_{U_0 \leq s \leq U_0+L} M_1(t) < \beta$, hence $S_\beta \leq U_0 + L$.

To conclude, we have

$$\mathbb{P}(T_0 \wedge S_\beta^t = +\infty) \leq \mathbb{P}(T_0 \wedge S_\beta^t \geq U_0 + L) \leq 1 - q,$$

with $q = p \wedge p_{fin}$.

It follows from the Markov property of the process $X = (X_k)$ and a repetition of the above argument with U_0 replaced by U_1 that

$$\mathbb{P}(T_0 \wedge S_\beta^t = +\infty) \leq \mathbb{P}(T_0 \wedge S_\beta^t \geq U_1 + L) \leq (1 - q)^2$$

.

Indeed, $\forall \ell \geq 0$, $\mathbb{P}(T_0 \wedge S_\beta^t > U_\ell + L) \leq 1 - q$. If we define

$$A = \left\{ (x_k)_k \in \mathcal{X}, x_0(\sum_{k\geq 0} kx_k)^2 \leq 2\frac{\lambda+1}{\alpha} \right\},$$

then

$$\begin{aligned}
\mathbb{P}(T_0 \wedge S_\beta^t > U_1 + L) &= \mathbb{P}(T_0 \wedge S_\beta^t > U_0 + L, T_0 \wedge S_\beta^t > U_1 + L) \\
&\leq \mathbb{P}(T_0 \wedge S_\beta^t > U_0 + L) \sup_{x\in A} \mathbb{P}(T_0 \wedge S_\beta^t > U_1 + L \mid X(U_1) = x) \\
&\leq (1 - q)^2
\end{aligned}$$

Iterating tyhis argument, we deduce that

$$\mathbb{P}(T_0 \wedge S_\beta^t = +\infty) = 0.$$

◊

5 Reaching the special set from any initial condition

Now we will show that starting from an initial condition $((x_k)_{k\in\mathbb{Z}_+}, m_1)$ with $m_1 \leq \beta$ the process has a probability bounded below by p_{final} to click before a given time. Since the process is Markovian and this situation occurs infinitely many times as long as the ratchet has not clicked, we will conclude that $\mathbb{P}(T_0 < +\infty) = 1$.

In this part we note $(x_k)_{k\geq 0}$ the initial condition of our system, and we suppose that $m_1 = \sum_{k\geq 0} kx_k \leq \beta$.

One of the difficulties we have to face is that the quadratic variation of X_0 is $\frac{X_0(1-X_0)}{N}$, which is not bounded from below near 1 and 0. We need to study three separate cases.

5.1 $x_0 \in (x_{max}; 1]$

The following lemma will show that if X_0 starts too close to 1, it will quickly go under x_{max} :

Lemma 5.1 *Let $t_1 = \frac{8}{\lambda^2}$ and*

$$x_{max} = max\left\{\frac{9}{10}, \frac{3\lambda+5\alpha}{5(\lambda+\alpha)}, 1-\frac{2}{\lambda}\right\}.$$

Then if $X_0(0) > x_{max}$, then

$$\mathbb{P}(\inf_{s<t_1} X_0(s) \leq x_{max}) \geq 1 - \exp(-N) > 0$$

PROOF : Let $T_{x_{max}} = \inf\{s \geq 0, X_0(s) \leq x_{max}\}$. On the time interval $[0, T_{x_{max}}]$, we have

$$X_0(s) > x_{max} \geq \frac{3\lambda+5\alpha}{5(\lambda+\alpha)}.$$

Since $X_1 \leq 1 - X_0$, on the same interval we have $X_1(s) \leq \frac{2\lambda}{5(\lambda+\alpha)}$,

$$\begin{aligned}
\alpha M_1 X_1 + \lambda X_0 - (\lambda+\alpha)X_1 \geq& \lambda X_0 - (\lambda+\alpha)\frac{2\lambda}{5(\lambda+\alpha)} \\
\geq& \lambda X_0 - \frac{2\lambda}{5} \\
\geq& \frac{\lambda}{2},
\end{aligned}$$

since also $X_0(s) > 0,9\ \forall s \in [0, T_{\delta_1}]$

Hence $X_1(s) \geq Y_1(s)$ when $s \in [0, T_{x_{max}}]$, where Y_1 is the solution of the SDE

$$\begin{cases} dY_1(s) = \dfrac{\lambda}{2}ds + \sqrt{\dfrac{Y_1(1-Y_1)}{N}}dB_1(s), \\ Y_1(0) = 0, \end{cases} \tag{5.1}$$

where we stop Y_1 as soon as it reaches 1.

Since $Y_1(1-Y_1) \leq \frac{1}{4}$, we have (with $C = \frac{2}{\lambda}$, and $\gamma > 0$ to be chosen below)

$$\begin{aligned}
&\mathbb{P}\left(\int_0^{t_1}\sqrt{\frac{Y_1(1-Y_1)}{N}}dB_1 < -C\right) \\
&= \mathbb{P}\left(-\int_0^{t_1}\sqrt{\frac{Y_1(1-Y_1)}{N}}dB_1 > C\right) \\
&= \mathbb{P}\left(exp\left(-\gamma\int_0^{t_1}\sqrt{\frac{Y_1(1-Y_1)}{N}}dB_1 - \int_0^{t_1}\frac{\gamma^2 Y_1(s)(1-Y_1(s))}{2N}ds\right)\right. \\
&\qquad\qquad\qquad\left. > exp\left(\gamma C - \int_0^{t_1}\frac{\gamma^2 Y_1(s)(1-Y_1(s))}{2N}ds\right)\right) \\
&\leq \mathbb{P}\left(exp\left(-\gamma\int_0^{t_1}\sqrt{\frac{Y_1(1-Y_1)}{N}}dB_1 - \int_0^{t_1}\frac{\gamma^2 Y_1(s)(1-Y_1(s))}{2N}ds\right) > exp\left(\gamma C - \frac{\gamma^2}{8N}t_1\right)\right) \\
&\leq exp\left(-\gamma C + \frac{\gamma^2}{8N}t_1\right) \\
&\leq exp\left(\frac{-2NC^2}{t_1}\right) = exp\left(-N\right)
\end{aligned}$$

where the last line is obtained by choosing $\gamma = \frac{4CN}{t_1}$ which minimizes the previous quantity, and the one before is Chebychev's inequality. Hence

$$\mathbb{P}\left(\int_0^{t_1}\sqrt{\frac{Y_1(1-Y_1)}{N}}dB_1 \geq -C\right) \geq 1 - exp\left(-N\right) > 0$$

Now, since

$$\int_0^{t_1}\frac{\lambda}{2}ds = \frac{4}{\lambda} = 2C$$

we have

$$\left\{\int_0^{t_1}\sqrt{\frac{Y_1(1-Y_1)}{N}}dB_1 \geq -C\right\} \subset \{T_{x_{max}} < t_1\},$$

which implies that

$$\mathbb{P}\left(T_{x_{max}} \leq t_1\right) \geq 1 - exp(-N)$$

Hence the conclusion. ♢

We need to control M_1 on the same time interval of length t_1. Using Lemma 3.2 we will deduce the following Proposition :

Proposition 5.2 *Let* $x_{max} = max\{\frac{9}{10}, \frac{3\lambda+5\alpha}{5(\lambda+\alpha)}, 1-\frac{2}{\lambda}\}$, $t_1 = \frac{8}{\lambda^2}$, $\varepsilon_0 = \frac{1}{2\alpha N}ln\left(\frac{2}{1-\exp(-N)}\right)$ *and* $\beta' = \beta + \lambda t_1 + \varepsilon_0$. *If* $X_0(0) > x_{max}$ *and* $M_1(0) < \beta$ *, then*

$$\mathbb{P}\left(\{T_{x_{max}} \leq t_1\} \cap \{M_1(T_{x_{max}}) \leq \beta'\}\right) = p_{init} > 0$$

Proof : From Lemma 5.1,

$$\mathbb{P}(T_{x_{max}} \leq t_1) \geq 1 - \exp(-N)$$

,

$$\mathbb{P}(M_1(T_{x_{max}}) \leq \beta') \geq 1 - exp(-2\alpha N\varepsilon_0)$$

due to Lemma 3.2. Those two inequalities together with Lemma 3.1 imply

$$\begin{aligned}\mathbb{P}\left(\{T_{x_{max}} \leq t_1\} \cap \{M_1(T_{x_{max}}) \leq \beta'\}\right) &\geq 1 - exp(-N) - exp(-\alpha 2N\varepsilon_0)\\ &\geq \frac{1-exp(-N)}{2} := p_{init}\end{aligned}$$

♢

So even if we started with $(X_0(0), M_1(0))$ such as $X_0(0) \geq x_{max}$ and $M_1(0) \leq \beta$, we obtain before time t_1 with probability at least $p_{init} > 0$ a new initial condition $X_0 \leq x_{max}$ and $M_1 \leq \beta'$, so we can resume with the next case.

5.2 $X_0 \le x_{max}$ but either $X_0 > \delta$ or $X_0 M_1 > \varepsilon$

The idea of this subsection is to show that X_0 can go from x_{max} to a $\delta' < \delta$, using small steps of size $\delta_1, \delta_2, \ldots$ in finite time, and during this evolution, M_1 stays small enough to have at the end $X_0 M_1 \le \varepsilon$, with a strictly positive probability p_{trans}.

We start by showing some inequalities

Lemma 5.3 *Let $\{V_t,\ t \ge 0\}$ be a standard Brownian motion, and $c > 0$ a constant. Then for any $t > 0$, $\tilde{\delta} > 0$, $\tilde{\mu} > 0$,*

$$\mathbb{P}\left(\inf_{0\le s\le t}\{cs + V_s\} \le -\tilde{\delta}, \sup_{0\le s\le t}\{cs + V_s\} \le \tilde{\mu}\right) \ge 1 - \sqrt{\frac{2}{\pi}}\left(\frac{\tilde{\delta}}{\sqrt{t}} + c\sqrt{t}\right) - 2\exp\left[-\frac{1}{2}\left(\frac{\tilde{\mu}}{\sqrt{t}} - c\sqrt{t}\right)^2\right].$$

Proof : Using Lemma 3.1, the result follows from the two following computations. We have, with Z denoting a $N(0,1)$ random variable,

$$\begin{aligned}\mathbb{P}\left(\inf_{0\le s\le t}\{cs + V_s\} \le -\tilde{\delta}\right) &\ge \mathbb{P}\left(\inf_{0\le s\le t} V_s \le -\tilde{\delta} - ct\right) \\ &= \mathbb{P}\left(\sup_{0\le s\le t} V_s \ge \tilde{\delta} + ct\right) \\ &= 2\mathbb{P}(V_t \ge \tilde{\delta} + ct) \\ &= 1 - \mathbb{P}\left(|Z| \le \frac{\tilde{\delta}}{\sqrt{t}} + c\sqrt{t}\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathbb{P}\left(\sup_{0\le s\le t}(cs + V_s) \le \tilde{\mu}\right) &\ge \mathbb{P}\left(\sup_{0\le s\le t} V_s \le \tilde{\mu} - ct\right) \\ &= 1 - \mathbb{P}\left(\sup_{0\le s\le t} V_s \ge \tilde{\mu} - ct\right) \\ &= 1 - 2\mathbb{P}(V_t \ge \tilde{\mu} - ct).\end{aligned}$$

$$
\begin{aligned}
\mathbb{P}(V_t \geq \tilde{\mu} - ct) &= \mathbb{P}\left(Z \geq \frac{\tilde{\mu}}{\sqrt{t}} - c\sqrt{t}\right) \\
&= \mathbb{P}\left(\exp(\gamma Z - \gamma^2/2) \geq \exp\left(\gamma\left[\frac{\tilde{\mu}}{\sqrt{t}} - c\sqrt{t}\right] - \frac{\gamma^2}{2}\right)\right) \\
&\leq \exp\left(-\gamma\left[\frac{\tilde{\mu}}{\sqrt{t}} - c\sqrt{t}\right] + \frac{\gamma^2}{2}\right)
\end{aligned}
$$

Choosing $\gamma = \tilde{\mu}/\sqrt{t} - c\sqrt{t}$, we conclude from the above computations that

$$
\mathbb{P}\left(\sup_{0\leq s\leq t}(cs + V_s) \leq \tilde{\mu}\right) \geq 1 - 2\exp\left[-\frac{1}{2}\left(\frac{\tilde{\mu}}{\sqrt{t}} - c\sqrt{t}\right)^2\right].
$$

◊

We will choose from now on

$$
\tilde{\varepsilon} = \frac{\log(4)}{2\alpha N}, \quad \text{so that } e^{-2N\alpha\tilde{\varepsilon}} = \frac{1}{4}. \tag{5.2}
$$

We are going to argue like in section 3. We will use the same notations A and σ again for the time change (but they are not the same) . We start from $(X_0, M_1) = (x, \beta')$, where $0 < x \leq x_{max} < 1$ and $\beta < \beta'$. We choose $0 < \tilde{\mu} = \frac{1-x_{max}}{2}$ and aim at proving that X_0 will go down to δ' in a finite number of steps, while staying below $x + \tilde{\mu}$ (so that $1 - X_0(t) \geq a := \frac{1-x_{max}}{2}$), and while M_1 does not go too far on the right, all that with positive probability.

We start with the SDE

$$
dX_0(t) = (\alpha M_1(t) - \lambda)X_0(t)dt + \sqrt{\frac{X_0(t)[1 - X_0(t)]}{N}}dB_0.
$$

Let

$$
\begin{aligned}
A(t) &:= \int_0^t \frac{X_0(s)[1 - X_0(s)]}{N}ds, \quad \text{and} \\
\sigma(t) &:= \inf\{s > 0, A(s) > t\}.
\end{aligned}
$$

Since

$$
\int_0^{\sigma(t)} \frac{X_0(s)(1 - X_0(s))}{N}ds = t,
$$

we deduce that

$$\frac{d\sigma(t)}{dt} = \frac{N}{\tilde{X}_0(t)(1-\tilde{X}_0(t))}, \quad \text{provided we let}$$
$$\tilde{X}_0(t) := X_0(\sigma(t)).$$

Finally

$$\sigma(t) = \int_0^t \frac{N}{\tilde{X}_0(s)(1-\tilde{X}_0(s))} ds,$$

and if we let

$$\tilde{M}_1(t) := M_1(\sigma_t),$$

we deduce from the above SDE for the process X_0 that

$$\tilde{X}_0(t) = x + N\int_0^t \frac{\alpha\tilde{M}_1(s) - \lambda}{1-\tilde{X}_0(s)} ds + B_t,$$

where B_t is a new standard Brownian motion (we use the same notation as above, which is an abuse).

At the k–th step of our iterative procedure, we let $\tilde{X}_0$ start from $x - \sum_{j=1}^{k-1}\delta_j$, and we stop the process $\tilde{X}_0$ at the first time that it reaches the level $x - \sum_{j=1}^{k}\delta_j$. We will choose not only the sequence δ_k, but also the sequence s_k in such a way that we can deduce from Lemma 5.3 that for each $1 \le k \le K$ (K to be defined below),

$$\mathbb{P}\left(\inf_{0\le s\le s_k}\{\Theta_k s + B_s\} \le -\delta_k, \sup_{0\le s\le s_k}\{\Theta_k s + B_s\} \le \tilde{\mu}\right) > \frac{1}{3}, \tag{5.3}$$

where, with $s'_k := \sigma(s_k)$,

$$\Theta_k = \frac{N\alpha}{a}\left(\beta' + k\tilde{\varepsilon} + \lambda\sum_{j=1}^{k} s'_j\right),$$

so that we have from Lemma 3.2 and our choice of $\tilde{\varepsilon}$ that

$$\mathbb{P}(\sup_{0\le s\le s'_k} M_1(s) \le \Theta_k \Big| M_1(0) \le \Theta_{k-1}) \ge 3/4. \tag{5.4}$$

Our result will follow from a combination of (5.3) and (5.4), provided we show that we can choose the two sequences δ_k and s_k for $k \ge 1$ in such a way that not only (5.3) holds, but also that there exists $K < \infty$ such that

$$x - \sum_{k=1}^{K}\delta_k \le \delta'.$$

Since during the k–th step we are considering the event that $X_0(t) \leq x + \tilde{\mu}$ i. e. $1 - X_0(t) \geq a$, and also $X_0(t) \geq x - \sum_{j=1}^{k} \delta_j$, we have that

$$s_k' \leq \frac{N}{a(x - \sum_{j=1}^{k} \delta_j)} s_k,$$

so that a bona fide choice of Θ_k in terms of $\{\delta_j, s_j,\ 1 \leq j \leq k\}$ is

$$\Theta_k := N\frac{\alpha}{a}\left[\beta' + k\tilde{\varepsilon} + N\frac{\lambda}{a}\sum_{j=1}^{k} \frac{s_j}{x - \sum_{i=1}^{j} \delta_i}\right].$$

We first want to insure that (the reason for 0.4 will be made clear below)

$$\frac{\delta_k}{\sqrt{s_k}} + \Theta_k\sqrt{s_k} \leq 0.4,$$

which we achieve by requesting both that

$$\delta_k = 0.2\sqrt{s_k} \tag{5.5}$$

and

$$\Theta_k\sqrt{s_k} \leq 0.2 \Leftrightarrow s_k \leq \left(\frac{0.2}{\Theta_k}\right)^2. \tag{5.6}$$

On the other hand, we shall also request that for each $k \geq 1$,

$$\frac{\delta_k}{x - \sum_1^k \delta_j} \leq 1 \Leftrightarrow \delta_k \leq \frac{1}{2}(x - \sum_{j=1}^{k-1} \delta_j).$$

It follows from (5.5) and $\Theta_k \geq N\frac{\alpha\beta'}{a}$ combined with (5.6) that with

$$C_N = N\frac{\alpha\beta'}{a} \quad \text{and } D_N = \frac{N}{a}\left(\alpha\tilde{\varepsilon} + \frac{\lambda}{\beta'}\right),$$

$$\begin{aligned} C_N \leq \Theta_k &\leq C_N + 25\frac{N^2\lambda\alpha}{a^2}\left(\sup_{1\leq j\leq k} \delta_j\right)k + k\tilde{\varepsilon}\frac{N\alpha}{a} \\ &\leq C_N + D_N k, \end{aligned}$$

since we have $\forall j \geq 0$

$$\begin{aligned} \delta_j = 0.2\sqrt{s_j} &\leq \frac{(0.2)^2}{\Theta_j} \\ &\leq \frac{1}{25}\frac{a}{N\alpha\beta'} \end{aligned}$$

Finally this leads to choosing, with $\kappa \leq \frac{1}{25}$ to be chosen below

$$\delta_k = \inf\left(\frac{\kappa}{(C_N + D_N k)}, \frac{1}{2}(x - \sum_{j=1}^{k-1} \delta_j)\right) \tag{5.7}$$
$$s_k = 25\delta_k^2.$$

Then we have

Lemma 5.4 $\exists K > 0$, $\forall k > K$,

$$\delta_k = \frac{1}{2}(x - \sum_{j=1}^{k-1} \delta_j).$$

Proof : Since $\sum_{k\geq 0} \frac{\kappa}{(C_N+D_N k)} = +\infty$, $\exists K' > 0$ such as $\sum_{k=2}^{K'} \frac{\kappa}{(C_N+D_N k)} > 1$. Then $\exists 2 \leq K \leq K'$ such as $\inf\left(\frac{\kappa}{(C_N+D_N K)}, \frac{1}{2}(x - \sum_{j=1}^{K-1} \delta_j)\right) = \frac{1}{2}(x - \sum_{j=1}^{K-1} \delta_j)$. Then by recurrence, if we have the previous equality at rank k, for the rank $k+1$ we have

$$\begin{aligned} \frac{\frac{\kappa}{(C_N+D_N(k+1))}}{\delta_k} &\geq \frac{\frac{\kappa}{(C_N+D_N(k+1))}}{\frac{\kappa}{(C_N+D_N k)}} \\ &> \frac{1}{2} \text{ since } k \geq 2 \end{aligned}$$

That is to say

$$\begin{aligned} \frac{\kappa}{(C_N + D_N(k+1))} &> \frac{1}{2}\delta_k = \frac{1}{4}(x - \sum_{j=1}^{k-1} \delta_j) \\ &> \frac{1}{2}(x - \sum_{j=1}^{k} \delta_j) \end{aligned}$$

Hence $\delta_{k+1} = \inf\left(\frac{\kappa}{(C_N+D_N(k+1))}, \frac{1}{2}(x - \sum_{j=1}^{k} \delta_j)\right) = \frac{1}{2}(x - \sum_{j=1}^{k} \delta_j)$. ◊

This means that at each $k > K$, $\tilde{X}_0$ progresses by a step equal to half the remaining distance to zero. Consequently $\exists c > 0$ $x_k = x - \sum_{j=1}^{k} \delta_j \leq c2^{-k}$. We are looking for the smallest integer $\overline{k}$ such that $c2^{-\overline{k}} \leq \delta'$, which implies that

$$\overline{k} - 1 \leq \left[\frac{\log(c) - \log(\delta')}{\log(2)}\right].$$

Since moreover $\Theta_{\overline{k}} \leq (25\delta_{\overline{k}})^{-1}$, there exists a constant c' such that $\delta' \times \Theta_{\overline{k}} \leq c'\delta' \log(\frac{1}{\delta'})$. Hence there exists a $\delta' \leq \delta$ (which depends only upon C_N, D_N, c' which are constants) such that at the end of the $\overline{k}$–th step, both $X_0 \leq \delta$ and $X_0 M_1 \leq \varepsilon$. We just need to check that the probability of the previous path is bounded below by a positive constant.

Given the choice that we have made for $\tilde{\varepsilon}$, it suffices to make sure that

$$\sqrt{\frac{2}{\pi}}\left(\frac{\delta_k}{\sqrt{s_k}} + \Theta_k\sqrt{s_k}\right) < 1/3, \ \forall k \geq 1, \tag{5.8}$$

as well as

$$2\exp\left[-\frac{1}{2}\left(\frac{\tilde{\mu}}{\sqrt{s_k}} - \Theta_k\sqrt{s_k}\right)^2\right] < 1/3, \ \forall k \geq 1. \tag{5.9}$$

Since $3^{-1}\sqrt{\pi/2} > 0.4$, (5.5)+(5.6) implies (5.8).

On the other hand, (5.9) is equivalent to

$$\left(\frac{\tilde{\mu}}{\sqrt{s_k}} - \Theta_k\sqrt{s_k}\right)^2 > 2\log 6,$$

which follows from $\kappa \leq \frac{\sqrt{2\log 6 + 4C_N\tilde{\mu}} - \sqrt{2\log 6}}{10}$ (where κ is the constant which appears in :eqrefkappa). We therefore choose

$$\kappa = \frac{1}{25} \wedge \frac{\sqrt{2\log 6 + 4C_N\tilde{\mu}} - \sqrt{2\log 6}}{10}.$$

We can now conclude that

Proposition 5.5 *Supose that $X_0(0) \leq x_{max}$ and $M_1(0) \leq \beta'$. Let*

$$T_{\delta'} = \inf\{s > 0, \ X_0(s) \leq \delta'\}.$$

Then

$$\mathbb{P}\left(T_{\delta'} \leq t_2, \ X_0(T_{\delta'}) \times M_1(T_{\delta'}) \leq \varepsilon\right) \geq \left(\frac{1}{12}\right)^{\tilde{k}_{max}} := p_{trans},$$

with $t_2 = 25\overline{k}_{max}$.

In this statement, $\overline{k}_{max}$ is the number of steps needed to reach δ' in the above procedure, while starting from x_{max}.

PROOF : It follows from (5.4), (5.8), (5.9), Lemma 5.3 and again Lemma 3.1 that the k–th step in the above procedure happens with probability at least 1/12. It remains to exploit the Markov property, like at the end of the proof of Lemma 4.4. ♢

5.3 Conclusion

From Proposition 3.7, starting at the end of the previous path, we have a probability p_{fin} to reach 0 during an interval of time of length t_3.

So to sump up, using again the Markovian properties of the system, we have

Proposition 5.6 *For any finite stopping time T, if $M_1(T) \leq \beta$, then*

$$\mathbb{P}(T_0 < T + t_1 + t_2 + t_3) \geq p_{fin} p_{trans} p_{ini} > 0$$

Moreover Lemma 4.4 implies that this situation will happen infinitely many times as long as the ratchet does not click, hence the proof of Theorem 1, exploiting again the Markov property of the solution of (1.1).

6 $E(T_0) < +\infty$

This final section is devoted to the proof of Theorem 2.

We first note that the reasoning of section 5 can be done with any initial value ρ for M_1, instead of β. That is to say, with $S_\rho^t = inf\,\{s > t, M_1(s) \leq \rho\}$ (and $S_\rho = S_\rho^0$),

Lemma 6.1 $\exists\ t_1^\rho, t_2^\rho, t_3^\rho < \infty$, *and* $p_{ini}^\rho, p_{trans}^\rho, p_{fin}^\rho > 0$ *such that*

$$\mathbb{P}(T_0 < S_\rho^t + t_1^\rho + t_2^\rho + t_3^\rho) \geq p_{ini}^\rho p_{trans}^\rho p_{fin}^\rho$$

Now let us choose $\rho = \frac{\varepsilon}{\delta} \vee \frac{2\lambda}{\alpha}$. We have :

Lemma 6.2 *Let $K = L + t_3$ (L to be defined below). Then $\exists \tilde{p} > 0$, such that for any initial condition in the set $\mathcal{X}$,*

$$\mathbb{P}(T_0 \wedge S_\rho \leq K) \geq \tilde{p}$$

PROOF : We are going to argue like in the proof of Lemma 4.4. We introduce the process $\{Y_s, s \geq 0\}$, which is the solution of the following system :

$$\begin{cases} dY_s = \dfrac{\alpha\varepsilon}{2} ds + \sqrt{\dfrac{Y_s(1-Y_s)}{N}} dB_0(s) \\ Y_0 = 0 \end{cases} \tag{6.1}$$

Let for any $0 \leq u \leq 1$

$$R_u = inf\{s \geq 0, Y_s = u\}.$$

Since $\frac{\alpha\varepsilon}{2} > 0$ we deduce that $\exists L > 0, p > 0$ such as $\mathbb{P}(R_1 \leq L) \geq p > 0$. We use $K = L + t_3$. (t_3 from Proposition 3.7).

Now there are several possibilities :

Either $\inf_{0\leq s\leq L} M_1(s) \leq \rho$, then $S_\rho < L < K$.

Or else $\inf_{0\leq s\leq L} M_1(s) \geq \rho$. Then either $\inf_{0\leq s\leq L} X_0(s)M_1(s) \leq \varepsilon$, then $\exists t < L$ such as $X_0(t)M_1(t) \leq \varepsilon$ (which implies $X_0(t) \leq \delta$, because $M_1(t) \geq \rho \geq \frac{\varepsilon}{\delta}$). In that case we can use Proposition 3.7, and we have $\mathbb{P}(T_0 \leq K) = p_{fin} > 0$, which implies $\mathbb{P}(T_0 \wedge S_\rho \leq K) = p_{fin} > 0$,

Or else we have both $\inf_{0\leq s\leq L} M_1(s) \geq \rho$ and $\inf_{0\leq s\leq L} X_0(s)M_1(s) \geq \varepsilon$. In that last sub–case we have (since $X_0 \geq \frac{\varepsilon}{M_1}$, and $\alpha M_1 - \lambda \geq \lambda > 0$)

$$\begin{aligned}\inf_{0\leq s\leq L}(\alpha M_1(s) - \lambda)X_0(s) &\geq \inf_{0\leq s\leq L} \varepsilon(\alpha - \frac{\lambda}{M_1(s)})\\ &\geq \frac{\alpha\varepsilon}{2},\end{aligned}$$

and consequently we can use the comparison theorem (Lemma 2.6), which implies that $\forall s \in [0, L]$, $X_0(s) \geq Y_s$. Then $\mathbb{P}(T_1 \leq L) \geq p > 0$. But when X_0 hits 1, M_1 hits 0. Hence $\mathbb{P}(S_\rho \leq L) \geq p > 0$.

We may now conclude that there exists $\tilde{p} > 0$ such that

$$\mathbb{P}(T_0 \wedge S_\rho \leq K) \geq \tilde{p}$$

$\diamondsuit$

We deduce from the two above Lemma :

Corollary 6.3 *There exists $\overline{K} < \infty$, and $\overline{p} > 0$ such that, for any initial condition in $\mathcal{X}$,*

$$\mathbb{P}(T_0 \leq \overline{K}) \geq \overline{p}.$$

We can now conclude.

Proof of Theorem 2 We deduce from Corollary 6.3 and the strong Markov property that for all $n \geq 0$, $\mathbb{P}(T_0 > n\overline{K}) \leq (1 - \overline{p})^n$. Consequently

$$\begin{aligned}\mathbb{E}(T_0) &= \sum_{n=0}^{\infty} \int_{nK}^{(n+1)K} \mathbb{P}(T > t)dt\\ &\leq \sum_{n=0}^{\infty} K\mathbb{P}(T > nK)\\ &= \frac{K}{\overline{p}}\end{aligned}$$

◊

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