

LARGE DEVIATION PRINCIPLE FOR EPIDEMIC MODELS.

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Abstract

We consider a general class of epidemic models obtained by applying the random time changes of [8] to a collection of Poisson processes and we show the large deviation principle (LDP) for such models. We generalize the approach followed by Dolgoashinnykh [4] in the case of the SIR epidemic model. Thanks to an additional assumption which is satisfied in many examples, we simplify the recent work by P.Kratz and E.Pardoux [13].

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1. Introduction

In this paper, we are interested in a class of Poisson driven stochastic differential equations which arise in many fields such as chemical kinetics, ecological and epidemics models. We consider a d dimensional processes of the type

$$Z^N(t) := Z^{N,z}(t) := \frac{[Nz]}{N} + \frac{1}{N} \sum_{j=1}^k h_j P_j \left(\int_0^t N \beta_j(Z^N(s)) ds \right), \quad (1)$$

where $(P_j)_{1 \leq j \leq k}$ are i.i.d. standard Poisson processes and the $h_j \in \{-1, 0, 1\}^d$ denote the k distinct jump directions with jump rates $\beta_j(z)$ and $z \in A$, where A is a compact subset of \mathbb{R}^d , which will be assumed to satisfy Assumption 1 below.

In the main application which we have in mind (see our examples at the end of the paper), the components of the vector $Z^N(t)$ are the proportions of the population in the various compartments

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corresponding to the various disease status of the individuals (susceptible, infectious, etc..). In most of those models, the set A is given as

$$A = \left\{ z \in \mathbb{R}_+^d : \sum_{i=1}^d z_i \leq 1 \right\}. \quad (2)$$

We refer the reader to [2] for a presentation of many such epidemics models.

As we shall recall below, it is plain that under mild assumptions, as $N \rightarrow \infty$, $Z^N(t) \rightarrow Y^z(t)$ a.s., locally uniformly for $t > 0$, where $Y^z(t)$ solves the ODE

$$\frac{dY(t)}{dt} = b(Y(t)), Y(0) = z, \quad (3)$$

with $b(y) = \sum_{j=1}^k \beta_j(y) h_j$. In this paper we want to investigate the large deviations from this law of large numbers.

Let us now be more precise about the initial condition $Z^N(0) = [Nz]/N$. In the models we have in mind, since each component of $Z^N(t)$ is a proportion in a population of total population size equal to N , we want $Z^N(t)$ to take its values in the set $A^{(N)} = \{z \in A, Nz \in \mathbb{Z}_+^d\}$. In particular, we want the initial condition $Z^N(0)$ to belong to this set $A^{(N)}$. If that is not the case, some of the components of the vector $Z^N(t)$ may become negative, while jumping from a/N to $(a-1)/N$, $0 < a < 1$, which is not very natural. For that reason, we will use the following convention concerning the initial condition. For some fixed $z \in A$, for $1 \leq i \leq d$, $N \geq 1$, $Z_i^N(0) = [Nz_i]/N$. Consequently the initial condition of the Z^N equation depends upon N .

In all what follows, $D_{T,A}$ denotes the set of functions from $[0, T]$ into A which are right continuous and have left limits at any $t \in [0, T]$ and let $\mathcal{A}_{T,A}$ be the subspace of absolutely continuous functions. $D_{T,A}$ will be equipped with Skorohod's topology, see page 124 in [1].

We denote by \mathcal{B} the Borel σ -field on $D_{T,A}$ and \mathbb{P}_z^N the probability measure on paths with the initial condition $Z^N(0) = [Nz]/N$, defined by

$$\mathbb{P}_z^N(B) = \mathbb{P}(Z^N \in B) \quad \forall B \in \mathcal{B}. \quad (4)$$

Our goal is to show that the probability measures \mathbb{P}_z^N , $N > 1$, satisfy a large deviation principle with a good rate function I_T that we will define below in subsection 2.2. In other words for any

G open subset of $D_{T,A}$ and F closed subset of $D_{T,A}$ with $G \subset F$, we want to show the following inequalities:

$$-\inf_{\phi \in G} I_T(\phi) \leq \liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_z^N(G) \leq \limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}_z^N(F) \leq -\inf_{\phi \in F} I_T(\phi). \quad (5)$$

Large deviation principles is the subject of many treatises, see in particular [3], [5], [9], [11] and [18]. Some of those books study large deviations for Poisson processes, like e.g. [18]. However, in this treatise it is assumed that the rates of the Poisson processes are bounded away from zero, and hence their logarithms are bounded. The case of Poisson processes with vanishing rates is studied in [19]. However their assumptions are not satisfied in our situation, as it is explained in [13]. Our results have already been established in [13]. However, our argument here is simpler, and the proofs are shorter. It is based upon an idea from [4] and our assumptions are slightly different from those in [13]. We also refer to [10] and to [16] for large deviation results concerning specific epidemic models (the latter one for a population of two types whose individuals move in space by jumps).

Our approach forces us to make the following assumption.

Assumption 1. *We suppose that there exists $z_0 \in \mathbb{R}^d$ such that the collection of mappings $\Phi_a : A \mapsto \mathbb{R}^d$ defined by $\Phi_a(z) = z + a(z_0 - z)$, defined for each $0 < a < 1$, is such that $z^a = \Phi_a(z) \in A$ for all $z \in A$, and moreover for some $0 < c_2 < c_1$ and all $z \in A$,*

$$|z - z^a| \leq c_1 a, \quad \text{dist}(z^a, \partial A) \geq c_2 a.$$

We define for all $a > 0$

$$B^a = \left\{ z \in A : \text{dist}(z, \partial A) \geq c_2 a \right\} \text{ and } R^a = \left\{ \phi \in \mathcal{AC}_{T,A} : \phi_t \in B^a \quad \forall t \in [0, T] \right\}.$$

Remark 1. For any convex set A , the Assumption 1 is satisfied with $z_0 \in \mathring{A}$, the interior of A . The same construction is possible for many non necessarily convex sets, provided A is compact, and there is a point z_0 in its interior which is such that for each $z \in \partial A$, the segment joining z_0 and z does not touch any other point of the boundary ∂A . We also note that for A given by (2)

and $z_0 \in A$, the constants c_1, c_2 can be defined by

$$c_1 = \sup_{z \in A} |z - z_0|, \quad c_2 = \sin(\theta_0) \inf_{z \in \partial A} |z - z_0| \leq \inf_{z \in \partial A} |z - z_0| \times \sin(\theta(z)).$$

where $\theta(z)$ is the most acute angle between the tangent to the boundary ∂A at z and the vector $z_0 - z$ and θ_0 is an angle such that for all $z \in \partial A$, $\theta_0 \leq \theta(z) \leq \pi/2$.

For all $a > 0$ we let $C_a = \inf_j \inf_{z \in B^a} \beta_j(z)$ and we formulate our assumptions on the β_j 's.

Assumption 2. 1. *The rate functions β_j are Lipschitz continuous with the Lipschitz constant equal to C .*

2. *For any $1 \leq j \leq k$, $\beta_j(z) > 0$ if $z \in \overset{\circ}{A}$, and β_j is bounded by a positive constant σ .*

3. *There exist two constants λ_1 and λ_2 such that whenever $z \in A$ is such that $\beta_j(z) < \lambda_1$, $\beta_j(z^a) > \beta_j(z)$ for all $a \in]0, \lambda_2[$.*

4. *There exists a constant $\nu \in]0, 1/2[$ such that $\lim_{a \rightarrow 0} a^\nu \log C_a = 0$.*

Note that our assumptions are easily verified in all the classical epidemic models, see below section 5.

Remark 2. We have not made any restriction concerning the set of vectors $\{h_1, \dots, h_k\}$. In all examples which we have in mind, $\{\sum_{j=1}^k \alpha_j h_j, \alpha \in \mathbb{R}_+^k\} = \mathbb{R}^d$, which insures that the process Z_t^N can move in all directions in A . Note that at any rate, there is no restriction as to which β_j 's vanish at some given point $z \in \partial A$. This is a major difference with the assumptions in [19]. Note also that in some sense our assumptions are weaker than those in [13], except for our Assumption 1.

For all $\phi, \psi \in D_{T,A}$ we will define the distance between ϕ and ψ by

$$\|\phi - \psi\|_T = \sup_{t \leq T} |\phi_t - \psi_t|$$

where $|\cdot|$ denotes the Euclidean norm in \mathbb{R}^d .

The paper is structured as follows. In section 1, we formulate the law of large numbers and the Girsanov theorem, we define a good rate function for our large deviation principle and we establish some properties that it satisfies. The proof of the lower bound (first inequality in (5)) is detailed in section 3. In the third one we state the upper bound (third inequality in (5)), which follows from

known results. In the fourth section we apply the large deviations principle to the study of the exit time from a domain, and we give four applications to the time of extinction of an endemic situation in four distinct epidemic models.

2. Some Important Results

We start with equation (1), whose existence and uniqueness is essentially obvious, the solution being constant between its jumps, and Z_t^N has a jump of size h_j/N when $\int_0^t N\beta_j(Z^N(s))ds$ hits a jump time of the Poisson process P_j , $1 \leq j \leq k$. The fact that the process $Z^N(t)$ lives in the compact set A is due to the fact that β_j vanishes on any piece of the boundary ∂A where h_j points towards the exterior of A , see the examples in section 5.

2.1. Law of Large Numbers and Change of Measure

We first recall the law of large numbers, see [15].

Theorem 1. *Let $Z^{N,z}(t)$ be the solution of the Poissonian stochastic differential equation (1) with the initial condition $[Nz]/N$. Assume that Assumption 2.1 holds. Then*

$$\lim_{N \rightarrow \infty} \|Z^{N,z} - Y^z\|_T = 0 \quad a.s.,$$

where $Y^z(\cdot)$ is the unique solution of the ODE (3).

We shall need the following Girsanov theorem. Let Q denote the random number of jumps of Z^N in the interval $[0, T]$, τ_p be the time of the p^{th} jump for $p = 1, \dots, Q$ and define

$$\delta_p(j) = \begin{cases} 1 & \text{if the } p^{th} \text{ jump is in the direction } h_j, \\ 0 & \text{otherwise.} \end{cases}$$

We shall denote $\mathcal{F}_t^N = \sigma\{Z^N(s), 0 \leq s \leq t\}$. Consider another set of rates $\tilde{\beta}_j(z)$, $1 \leq j \leq k$. Combining Theorem III.5.19 from [12] and Theorem 2.4 from [20], we have ($\tilde{\mathbb{P}} \ll \mathbb{P}$ means that $\tilde{\mathbb{P}}$ is absolutely continuous with respect to \mathbb{P} and $Z^N(\tau_p^-) = \lim_{t \rightarrow \tau_p^-, t < \tau_p} Z^N(t)$). The rates $\beta_j(t, z)$ above depend only upon the second variable z . In the next statement, we introduce new rates $\tilde{\beta}_j(t, z)$ which in the next section will depend upon the two variables t and z , and is supposed to

be a.s. continuous at the jump times of $Z^N(t)$.

Theorem 2. Let $\tilde{\mathbb{P}}^N$ denote the law of Z^N when the rates are rates $\tilde{\beta}_j(\cdot)$. Then provided that $\sup_{0 \leq t \leq T, z \in A} \frac{\tilde{\beta}_j(t, z)}{\beta_j(z)} < \infty$, which implies in particular that $\{z : \beta_j(z) = 0\} \subset \{z : \tilde{\beta}_j(t, z) = 0\}$ for all $0 \leq t \leq T$, $\tilde{\mathbb{P}}^N|_{\mathcal{F}_T^N} \ll \mathbb{P}^N|_{\mathcal{F}_T^N}$, and with the convention $\frac{0}{0} = 1$,

$$\xi_T = \frac{d\tilde{\mathbb{P}}^N|_{\mathcal{F}_T^N}}{d\mathbb{P}^N|_{\mathcal{F}_T^N}} = \left(\prod_{p=1}^Q \prod_{j=1}^k \left[\frac{\tilde{\beta}_j(\tau_p, Z^N(\tau_p^-))}{\beta_j(Z^N(\tau_p^-))} \right]^{\delta_p(j)} \right) \exp \left\{ N \sum_{j=1}^k \int_0^T (\beta_j(Z^N(t)) - \tilde{\beta}_j(t, Z^N(t))) dt \right\}. \quad (6)$$

Corollary 1. For any non-negative random variable X ,

$$\mathbb{E}(X) \geq \tilde{\mathbb{E}}(\xi_T^{-1} X)$$

Proof. As $X \geq 0$, we write

$$\mathbb{E}(X) \geq \mathbb{E}(X \mathbf{1}_{\{\xi_T \neq 0\}}) = \tilde{\mathbb{E}}(\xi_T^{-1} X \mathbf{1}_{\{\xi_T \neq 0\}}) = \tilde{\mathbb{E}}(\xi_T^{-1} X).$$

Note that ξ_T^{-1} is well-defined $\tilde{\mathbb{P}}$ -a.s., since $\tilde{\mathbb{P}}(\xi_T = 0) = 0$. □

It is not hard to see that under $\tilde{\mathbb{P}}$, there exist again mutually independent standard Poisson processes \tilde{P}_j , $1 \leq j \leq k$ such that

$$Z^N(t) := Z^{N, z}(t) := \frac{[Nz]}{N} + \frac{1}{N} \sum_{j=1}^k h_j \tilde{P}_j \left(\int_0^t N \tilde{\beta}_j(Z^N(s)) ds \right), \quad (7)$$

2.2. The Rate Function

For all $\phi \in \mathcal{AC}_{T, A}$, let $\mathcal{A}_d(\phi)$ be the set of \mathbb{R}_+^k -valued Borel measurable functions μ which are such that

$$\phi_t = \phi_0 + \sum_{j=1}^k h_j \int_0^t \mu_s^j ds, \quad \text{for all } t > 0.$$

We define the rate function

$$I_T(\phi) := \begin{cases} \inf_{\mu \in \mathcal{A}_d(\phi)} I_T(\phi|\mu), & \text{if } \phi \in \mathcal{AC}_{T,A}; \\ \infty, & \text{otherwise, where} \end{cases}$$

$$I_T(\phi|\mu) = \int_0^T \sum_{j=1}^k f(\mu_t^j, \beta_j(\phi_t)) dt,$$

with $f(\nu, \omega) = \nu \log(\nu/\omega) - \nu + \omega$ and the conventions $\log(\nu/0) = \infty$ for all $\nu > 0$, and $0 \log(0/0) = 0 \log(0) = 0$.

Note that under our standing assumptions the set $\mathcal{A}_d(\phi)$ can be empty for many $\phi \in \mathcal{AC}_{T,A}$. Recall the usual convention that the infimum over an empty set is $+\infty$.

It is shown in [13] that $I_T(\phi) = \int_0^T L(\phi_t, \phi_t') dt$, where for all $z \in A$, $y \in \mathbb{R}^d$, $L(z, y) = \sup_{\theta \in \mathbb{R}^d} \langle \theta, y \rangle - \sum_{j=1}^k \beta_j(z) (e^{\langle \theta, h_j \rangle} - 1)$.

The following Theorem is Proposition 4.23 from [13].

Theorem 3. I_T is a good rate function.

We first establish

Lemma 1. Let $s > 0$, $\phi \in D_{T,A}$ and $\mu \in \mathcal{A}_d(\phi)$ such that $I_T(\phi|\mu) \leq s$. Then for all $0 \leq t_1, t_2 \leq T$ such that $t_2 - t_1 \leq 1/\sigma$,

$$\int_{t_1}^{t_2} \mu_t^j dt \leq \frac{s+1}{-\log(\sigma(t_2 - t_1))}, \quad j = 1, \dots, k.$$

Proof. For all $0 \leq t_1, t_2 \leq T$

$$\int_{t_1}^{t_2} f(\mu_t^j, \beta_j(\phi_t)) dt \leq I_T(\phi|\mu) \leq s,$$

and since, the function $h(x) = x \log(x/\sigma) - x$ is convex in x ,

$$\begin{aligned} h\left(\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \mu_t^j dt\right) &\leq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} h(\mu_t^j) dt \\ &\leq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left(\mu_t^j \log \frac{\mu_t^j}{\beta_j(\phi_t)} - \mu_t^j + \beta_j(\phi_t)\right) dt \\ &\leq \frac{s}{t_2 - t_1}. \end{aligned}$$

It is easy to show that for all $\alpha > 0$, $h(x) \geq \alpha x - \sigma \exp\{\alpha\}$ and then for all $\alpha > 0$

$$\int_{t_1}^{t_2} \mu_t^j dt \leq \frac{1}{\alpha} (s + (t_2 - t_1) \sigma \exp\{\alpha\}).$$

Let $t_2 - t_1 < 1/\sigma$. The result follows by choosing $\alpha = -\log(\sigma(t_2 - t_1))$. \square

For $\phi \in D_{T,A}$ let ϕ^a be defined by $\phi_t^a := \Phi_a(\phi_t)$. We note that $\phi^a \in R^a$.

Lemma 2. *For all $\phi \in D_{T,A}$ we have $\limsup_{a \rightarrow 0} I_T(\phi^a) \leq I_T(\phi)$.*

Proof. It clearly suffices to treat the case where $I_T(\phi) < \infty$. For any $\eta > 0$ there exists $\mu \in \mathcal{A}_d(\phi)$ such that $I_T(\phi|\mu) \leq I_T(\phi) + \eta$. Let $\mu^a = (1 - a)\mu$. Then $\mu^a \in \mathcal{A}_d(\phi^a)$. We will now show that

$$\limsup_{a \rightarrow 0} I_T(\phi^a|\mu^a) \leq I_T(\phi|\mu), \quad (8)$$

which clearly implies the result since

$$\begin{aligned} \limsup_{a \rightarrow 0} I_T(\phi^a) &\leq \limsup_{a \rightarrow 0} I_T(\phi^a|\mu^a) \\ &\leq I_T(\phi|\mu) \\ &\leq I_T(\phi) + \eta. \end{aligned}$$

We note that

$$\begin{aligned} f(\mu_t^{j,a}, \beta_j(\phi_t^a)) &= \mu_t^{j,a} \log \left(\frac{\mu_t^{j,a}}{\beta_j(\phi_t^a)} \right) - \mu_t^{j,a} + \beta_j(\phi_t^a) \\ &= f(\mu_t^j, \beta_j(\phi_t)) + H(a, t, j), \end{aligned}$$

where

$$H(a, t, j) = (1 - a)\mu_t^j \log\left(\frac{\beta_j(\phi_t)}{\beta_j(\phi_t^a)}\right) - a\mu_t^j \log\left(\frac{\mu_t^j}{\beta_j(\phi_t)}\right) + (1 - 2a)\mu_t^j \log(1 - a) + a\mu_t^j.$$

It remains to show that $\limsup_{a \rightarrow 0} \int_0^T H(a, t, j) dt \leq 0$. We first consider the first term in above right-hand side. It follows from Assumption 2.3 that

$$\int_0^T \mu_t^j \log\left(\frac{\beta_j(\phi_t)}{\beta_j(\phi_t^a)}\right) dt \leq \int_0^T \mathbf{1}_{\beta_j(\phi_t) \geq \lambda_1} \log\left(\frac{\beta_j(\phi_t)}{\beta_j(\phi_t^a)}\right) dt,$$

and the right-hand side tends to 0 as $a \rightarrow 0$, since μ_t^j is integrable from our assumption and the Lemme 1. The other terms tend to zero thanks to $I_T(\phi) < \infty$. \square

Lemma 3. *Let $a > 0$ and $\phi \in R^a$ be such that $I_T(\phi) < \infty$. For all $\eta > 0$, there exists $L > 0$, $\phi^L \in R^{a/2}$ such that $\|\phi - \phi^L\|_T < c_1 \frac{a}{2}$, and $\mu^L \in \mathcal{A}_d(\phi^L)$ such that $I_T(\phi^L | \mu^L) \leq I_T(\phi) + \eta$, with $\mu_t^{L,j} < L$, $j = 1, \dots, k$.*

Proof. Let $\eta > 0$ and $\mu \in \mathcal{A}_d(\phi)$ be such that $I_T(\phi | \mu) < I_T(\phi) + \eta/2$. For $L > 0$ let $\mu_t^{L,j} = \mu_t^j \wedge L$ and let ϕ^L be a solution of the ODE

$$\frac{d\phi_t^L}{dt} = \sum_{j=1}^k \mu_t^{L,j} h_j.$$

It follows from the monotone convergence theorem that $\|\phi - \phi^L\|_T \rightarrow 0$ as $L \rightarrow \infty$. Since $\phi \in R^a$, for L large enough, $\phi^a \in R^{a/2}$. It is easy to show that $I_T(\phi^L | \mu^L) \rightarrow I_T(\phi | \mu)$ as $L \rightarrow \infty$. Hence the result. \square

Let $\epsilon > 0$ be such that $T/\epsilon \in \mathbb{N}$ and let ϕ^ϵ be the polygonal approximation of ϕ defined for $t \in [\ell\epsilon, (\ell+1)\epsilon)$ by

$$\phi_t^\epsilon = \phi_{\ell\epsilon} \frac{(\ell+1)\epsilon - t}{\epsilon} + \phi_{(\ell+1)\epsilon} \frac{t - \ell\epsilon}{\epsilon}. \quad (9)$$

Lemma 4. *Let $\eta > 0$ be arbitrary. Let $0 < a < 1$, $\phi \in R^a$ and $\mu \in \mathcal{A}_d(\phi)$ be such that $\mu_t^j < L$, $j = 1, \dots, k$ for some $L > 0$ and $I_T(\phi | \mu) < \infty$. Then there exists a_η such that for all $0 < a \leq a_\eta$ there exists an $\epsilon_a > 0$ and for all $\epsilon < \epsilon_a$ the polygonal approximation ϕ^ϵ belongs to $R^{a/2}$ and $\|\phi - \phi^\epsilon\|_T < c_2 \frac{a}{2} < c_1 \frac{a}{2}$. Moreover, there exists $\mu^\epsilon \in \mathcal{A}_d(\phi^\epsilon)$ such that $\mu_t^{\epsilon,j} < L$, $j = 1, \dots, k$ and*

$$I_T(\phi^\epsilon | \mu^\epsilon) \leq I_T(\phi | \mu) + \eta.$$

Proof. Since ϕ is uniformly continuous on $[0, T]$, there exists $\epsilon_a > 0$ such that for all $\epsilon < \epsilon_a$

$$\sup_{|t-t'| < 2\epsilon} |\phi_t - \phi_{t'}| < c_2 \frac{ae^{-a-\nu}}{4}.$$

Consequently $\|\phi - \phi^\epsilon\|_T < c_2 \frac{a}{2}$ and $\phi^\epsilon \in R^{a/2}$.

For $t \in]\ell\epsilon, (\ell+1)\epsilon[$,

$$\frac{d\phi_t^\epsilon}{dt} = \frac{\phi_{(\ell+1)\epsilon} - \phi_{\ell\epsilon}}{\epsilon} = \frac{1}{\epsilon} \sum_{j=1}^k h_j \int_{\ell\epsilon}^{(\ell+1)\epsilon} \mu_t^j dt.$$

Therefore μ_t^ϵ defined for $t \in [\ell\epsilon, (\ell+1)\epsilon[$ by

$$\mu_t^{\epsilon,j} = \frac{1}{\epsilon} \int_{\ell\epsilon}^{(\ell+1)\epsilon} \mu_r^j dr, j = 1, \dots, k$$

is constant on $[\ell\epsilon, (\ell+1)\epsilon[$ and belongs to $\mathcal{A}_d(\phi^\epsilon)$. We also note that $\mu_t^{\epsilon,j} \leq L$ for all $j = 1, \dots, k$.

Moreover if $0 < \nu \leq L$ and $\omega \geq C_a$ then

$$\left| \frac{\partial f(\nu, \omega)}{\partial \omega} \right| = \left| -\frac{\nu}{\omega} + 1 \right| \leq \frac{L}{C_a} + 1.$$

By the assumption 2.4, there exists $\tilde{a}_\eta > 0$ such that for all $a < \tilde{a}_\eta$

$$\frac{L}{C_a} + 1 \leq Le^{a-\nu} + 1$$

Then for $t \in [\ell\epsilon, (\ell+1)\epsilon[$ and $a < a_0$

$$\begin{aligned} |f(\mu_t^{\epsilon,j}, \beta_j(\phi_t^\epsilon)) - f(\mu_t^{\epsilon,j}, \beta_j(\phi_{\ell\epsilon}))| &\leq \frac{c_2}{4} C(L+1)a = Va/2, \\ |f(\mu_t^j, \beta_j(\phi_t)) - f(\mu_t^j, \beta_j(\phi_{\ell\epsilon}))| &\leq \frac{c_2}{4} C(L+1)a = Va/2. \end{aligned}$$

The above implies that

$$\begin{aligned}
\int_{\ell\epsilon}^{(\ell+1)\epsilon} f(\mu_t^{\epsilon,j}, \beta_j(\phi_t^\epsilon)) dt &\leq \int_{\ell\epsilon}^{(\ell+1)\epsilon} f(\mu_t^{\epsilon,j}, \beta_j(\phi_{\ell\epsilon})) dt + \epsilon Va/2 \\
&= \epsilon f(\mu_{\ell\epsilon}^{\epsilon,j}, \beta_j(\phi_{\ell\epsilon})) + \epsilon Va/2 \\
&\leq \int_{\ell\epsilon}^{(\ell+1)\epsilon} f(\mu_t^j, \beta_j(\phi_{\ell\epsilon})) dt + \epsilon Va/2 \\
&\leq \int_{\ell\epsilon}^{(\ell+1)\epsilon} f(\mu_t^j, \beta_j(\phi_t)) dt + \epsilon Va
\end{aligned}$$

where the second inequality follows from Jensen's inequality. Therefore

$$I_T(\phi^\epsilon | \mu^\epsilon) \leq I_T(\phi | \mu) + VTa.$$

The result follows by choosing $a < \min\{a_0, \eta/VT\}$. \square

3. The Lower Bound

Our proof of the lower bound relies essentially upon the next Lemma.

Lemma 5. *For $z \in A$, $\phi \in \mathcal{AC}_{T,A}$, $\phi_0 = z$, for any $0 < a < 1$ and $\epsilon > 0$ small enough such that the polygonal approximation $\phi^{a,\epsilon}$ of ϕ^a defined as in (9) satisfies $d(\phi^{a,\epsilon}, \partial A) \leq c_2 a/2$, and $\mu^{a,\epsilon} \in \mathcal{A}_d(\phi^{a,\epsilon})$ being chosen as in Lemma 4, the following holds. For any $\eta > 0$ and suitably small $0 < \delta \leq c_2 a/4$, there exist $N_{\eta,\delta} \in \mathbb{N}$ such that for all y , $|y - z| < \delta/2$ and any $N > N_{\eta,\delta}$*

$$\mathbb{P}_y(\|Z^N - \phi^{a,\epsilon}\|_T < \delta) \geq \exp\{-N(I_T(\phi^{a,\epsilon} | \mu^{a,\epsilon}) + \eta)\}.$$

Proof. We first define new rates $\tilde{\beta}_j(t, z) = \mu_t^{a,\epsilon,j} \varkappa(z)$, where $\varkappa \in C(A; [0, 1])$ is such that

$$\varkappa(z) = \begin{cases} 1 & , \text{if } d(z, \partial A) \geq c_2 a/4, \\ 0 & , \text{if } d(z, \partial A) \leq c_2 a/8, \end{cases}$$

so that on the event $\{\|Z^N - \phi^{a,\epsilon}\|_T < \delta\}$, $\tilde{\beta}_j(t, Z_t^N) = \mu_t^{a,\epsilon,j}$ for $0 \leq t \leq T$, while the assumption of Theorem 2 is satisfied. It is clear that the new probability measure $\tilde{\mathbb{P}}_y$ depends upon both a and ϵ , also that dependence will not appear explicitly for the sake of notation simplicity.

With $\frac{8kTLC}{C_a}\delta$, we define the following events B_j^δ , $j = 1, \dots, k$ for controlling the likelihood ratio. For $\gamma > 0$ let

$$B_j^\delta = \left\{ \left| \sum_{p=1}^Q \delta_p(j) \log \left(\frac{\beta_j(Z^N(\tau_p^-))}{\mu_{\lceil \tau_p / \epsilon \rceil}^{a, \epsilon, j}} \right) - N \sum_{\ell=1}^{T/\epsilon} \mu_{\ell\epsilon}^{a, \epsilon, j} \log \left(\frac{\beta_j(\phi_{\ell\epsilon}^{a, \epsilon})}{\mu_{\ell\epsilon}^{a, \epsilon, j}} \right) \right| \leq N\gamma \right\},$$

where Q was first introduced just before Theorem 2.

On the event $\{\|Z^N - \phi^{a, \epsilon}\|_T < \delta\} \cap (\bigcap_{j=1}^k B_j^\delta) = \{\|Z^N - \phi^{a, \epsilon}\|_T < \delta\} \cap B^\delta$, with ξ_T defined by (6),

$$\begin{aligned} \xi_T^{-1} &= \exp \left\{ \sum_{p=1}^Q \sum_{j=1}^k \delta_p(j) \log \left(\frac{\beta_j(Z^N(\tau_p^-))}{\mu_{\tau_p^-}^{a, \epsilon, j}} \right) + N \int_0^T \sum_{j=1}^k (\mu_t^{a, \epsilon, j} - \beta_j(Z^N(t))) dt \right\} \\ &\geq \exp \left\{ -N \sum_{\ell=1}^{T/\epsilon} \sum_{j=1}^k \mu_{\ell\epsilon}^{a, \epsilon, j} \log \left(\frac{\mu_{\ell\epsilon}^{a, \epsilon, j}}{\beta_j(\phi_{\ell\epsilon}^{a, \epsilon})} \right) \epsilon + N \int_0^T \sum_{j=1}^k (\mu_t^{a, \epsilon, j} - \beta_j(Z^N(t))) dt - kN\gamma \right\} \\ &\geq \exp \left\{ -N \sum_{\ell=1}^{T/\epsilon} \sum_{j=1}^k \mu_{\ell\epsilon}^{a, \epsilon, j} \log \left(\frac{\mu_{\ell\epsilon}^{a, \epsilon, j}}{\beta_j(\phi_{\ell\epsilon}^{a, \epsilon})} \right) \epsilon + N \int_0^T \sum_{j=1}^k (\mu_t^{a, \epsilon, j} - \beta_j(\phi_t^{a, \epsilon})) dt - N(kTC\delta + k\gamma) \right\}. \end{aligned}$$

The first inequality is true since the $\mu_t^{a, \epsilon, j}$'s are piecewise constant, and the second one follows from the Lipschitz continuity of the β_j 's. Since the derivative of $\phi^{a, \epsilon}$ is bounded, we can compare the sum in the above exponential and an integral, thus

$$\begin{aligned} \xi_T^{-1} &\geq \exp \left\{ -N \int_0^T \sum_{j=1}^k \left[\mu_t^{a, \epsilon, j} \log \left(\frac{\mu_t^{a, \epsilon, j}}{\beta_j(\phi_t^{a, \epsilon})} \right) - \mu_t^{a, \epsilon, j} + \beta_j(\phi_t^{a, \epsilon}) \right] dt - N[kTC(\delta + \epsilon) + k\gamma] \right\} \\ &= \exp \left\{ -N [I_T(\phi^{a, \epsilon} | \mu^{a, \epsilon}) + kTC(\delta + \epsilon) + k\gamma] \right\} \text{ on the event } \{\|Z^N - \phi^{a, \epsilon}\|_T < \delta\} \cap B^\delta. \end{aligned}$$

Then for any $\eta > 0$, there exists $\delta > 0$ $\epsilon > 0$ small enough, we have

$$\xi_T^{-1} \geq \exp \{-N(I_T(\phi^{a, \epsilon} | \mu^{a, \epsilon}) + \eta/2)\}.$$

Moreover from Corollary 1

$$\begin{aligned} \mathbb{P}_y(\|Z^N - \phi^{a,\epsilon}\|_T < \delta) &\geq \tilde{\mathbb{E}}_y\left(\xi_T^{-1} \cdot \mathbf{1}_{\{\|Z^N - \phi^{a,\epsilon}\|_T < \delta\} \cap B^\delta}\right) \\ &\geq \exp\{-N(I_T(\phi^{a,\epsilon}|\mu^{a,\epsilon}) + \eta/2)\} \tilde{\mathbb{P}}_y(\{\|Z^N - \phi^{a,\epsilon}\|_T < \delta\} \cap B^\delta). \end{aligned}$$

The result is now a consequence of the next Lemma. \square

Lemma 6. For $z \in A$, $\phi \in \mathcal{AC}_{T,A}$, $\phi_0 = z$, for any $0 < a < 1$, $\epsilon > 0$ small enough, $0 < a \leq c_2 a/4$, the polygonal approximation $\phi^{a,\epsilon}$ of ϕ^a has the property that for all y , $|y - z| < \delta/2$

$$\lim_{N \rightarrow \infty} \tilde{\mathbb{P}}_y(\{\|Z^N - \phi^{a,\epsilon}\|_T < \delta\} \cap B^\delta) = 1.$$

Proof. It is enough to prove both that $\lim_{N \rightarrow \infty} \tilde{\mathbb{P}}_y(\|Z^N - \phi^{a,\epsilon}\|_T < \delta) = 1$ and that for all $1 \leq j \leq k$, $\lim_{N \rightarrow \infty} \tilde{\mathbb{P}}_y(\{\|Z^N - \phi^{a,\epsilon}\|_T < \delta\} \cap (B_j^\delta)^c) = 0$. We first sketch the proof of the first limit. With the notation $\tilde{M}_j(t) = \tilde{P}_j(t) - t$, we have that

$$\begin{aligned} Z^N(t) &= \frac{[Ny]}{N} + \sum_{j=1}^k h_j \int_0^t \mu_s^{a,\epsilon,j} ds + \frac{1}{N} \sum_{j=1}^k \tilde{M}_j \left(\int_0^t N \mu_s^{a,\epsilon,j} ds \right) \\ &= \phi_t^{a,\epsilon} + \frac{[Ny]}{N} - z + \frac{1}{N} \sum_{j=1}^k \tilde{M}_j \left(\int_0^t N \mu_s^{a,\epsilon,j} ds \right), \end{aligned}$$

and it follows from the arguments e.g. from [15] that under $\tilde{\mathbb{P}}_y$, the last term on the right tends to 0 a.s. as $N \rightarrow \infty$.

We now establish that $\tilde{\mathbb{P}}_y(\|Z^N - \phi^{a,\epsilon}\|_T < \delta \cap (B_j^\delta)^c) \rightarrow 0$ as $N \rightarrow \infty$. We have $\sup_p |Z^N(\tau_p) - \phi_{\tau_p}^{a,\epsilon}| < \delta$ on $D_\delta = \{\|Z^N - \phi^{a,\epsilon}\|_T < \delta\}$ and we can choose ϵ s.t. $\sup_p |\phi_{\tau_p}^{a,\epsilon} - \phi_{\lceil \tau_p / \epsilon \rceil \epsilon}^{a,\epsilon}| < \delta$ and thus $\sup_p |Z^N(\tau_p) - \phi_{\lceil \tau_p / \epsilon \rceil \epsilon}^{a,\epsilon}| < 2\delta$. Hence on D_δ

$$\begin{aligned} \left| \sum_{p=1}^Q \delta_p(j) \log \left(\frac{\beta_j(Z^N(\tau_p^-))}{\mu_{\lceil \tau_p / \epsilon \rceil \epsilon}^{a,\epsilon,j}} \right) - \sum_{p=1}^Q \delta_p(j) \log \left(\frac{\beta_j(\phi_{\lceil \tau_p / \epsilon \rceil \epsilon}^{a,\epsilon})}{\mu_{\lceil \tau_p / \epsilon \rceil \epsilon}^{a,\epsilon,j}} \right) \right| &\leq \left| \sum_{p=1}^Q \delta_p(j) \log \left(\frac{\beta_j(Z^N(\tau_p^-))}{\beta_j(\phi_{\lceil \tau_p / \epsilon \rceil \epsilon}^{a,\epsilon})} \right) \right| \\ &\leq \frac{2C\delta}{C_a} Q, \end{aligned}$$

since $|\beta_j(Z^N(\tau_p^-)) - \beta_j(\phi_{\lceil \tau_p / \epsilon \rceil \epsilon}^{a,\epsilon})| < 2C\delta$. Q_ℓ denoting the number of jumps of Z^N in the interval

$[(\ell - 1)\epsilon, \ell\epsilon)$, we have (note that on the event $\tau_p \in ((\ell - 1)\epsilon, \ell\epsilon)$, $\lceil \tau_p/\epsilon \rceil = \ell$)

$$\begin{aligned}
& \left| \sum_{p=1}^Q \delta_p(j) \log \left(\frac{\beta_j(Z^N(\tau_p^-))}{\mu_{\lceil \tau_p/\epsilon \rceil \epsilon}^{a,\epsilon,j}} \right) - N \sum_{\ell=1}^{T/\epsilon} \mu_{\ell\epsilon}^{a,\epsilon,j} \log \left(\frac{\beta_j(\phi_{\ell\epsilon}^{a,\epsilon})}{\mu_{\ell\epsilon}^{a,\epsilon,j}} \right) \epsilon \right| \\
& \leq \left| \sum_{p=1}^Q \delta_p(j) \log \left(\frac{\beta_j(\phi_{\lceil \tau_p/\epsilon \rceil \epsilon}^{a,\epsilon})}{\mu_{\lceil \tau_p/\epsilon \rceil \epsilon}^{a,\epsilon,j}} \right) - N \sum_{\ell=1}^{T/\epsilon} \mu_{\ell\epsilon}^{a,\epsilon,j} \log \left(\frac{\beta_j(\phi_{\ell\epsilon}^{a,\epsilon})}{\mu_{\ell\epsilon}^{a,\epsilon,j}} \right) \epsilon \right| \\
& + \left| \sum_{p=1}^Q \delta_p(j) \log \left(\frac{\beta_j(Z^N(\tau_p^-))}{\mu_{\lceil \tau_p/\epsilon \rceil \epsilon}^{a,\epsilon,j}} \right) - \sum_{p=1}^Q \delta_p(j) \log \left(\frac{\beta_j(\phi_{\lceil \tau_p/\epsilon \rceil \epsilon}^{a,\epsilon})}{\mu_{\lceil \tau_p/\epsilon \rceil \epsilon}^{a,\epsilon,j}} \right) \right| \\
& \leq \left| \sum_{\ell=1}^{T/\epsilon} \log \left(\frac{\beta_j(\phi_{\ell\epsilon}^{a,\epsilon})}{\mu_{\ell\epsilon}^{a,\epsilon,j}} \right) \left(\sum_{p=1}^{Q_\ell} \delta_{\ell,p}(j) - N \mu_{\ell\epsilon}^{a,\epsilon,j} \epsilon \right) \right| + \frac{2C\delta}{C_a} Q,
\end{aligned}$$

where

$$\delta_{\ell,p}(j) = \begin{cases} 1, & \text{if the } p\text{-th jump in the time interval } [(\ell - 1)\epsilon, \ell\epsilon) \text{ is in the direction } h_j, \\ 0, & \text{otherwise.} \end{cases}$$

As the rates of jumps are constant on the interval $[(\ell - 1)\epsilon, \ell\epsilon)$ under $\tilde{\mathbb{P}}^N$, $\sum_{p=1}^{Q_\ell} \delta_{\ell,p}(j)$ is the number of jumps of a Poisson process P_j on this interval. So it is a Poisson random variable with mean $N \mu_{\ell\epsilon}^{a,\epsilon,j} \epsilon$. We deduce from Chebyshev's inequality that

$$\tilde{\mathbb{P}}_y \left(\left| \log \left(\frac{\beta_j(\phi_{\ell\epsilon}^{a,\epsilon})}{\mu_{\ell\epsilon}^{a,\epsilon,j}} \right) \left(\sum_{p=1}^{Q_\ell} \delta_{\ell,p}(j) - N \mu_{\ell\epsilon}^{a,\epsilon,j} \epsilon \right) \right| > \frac{N\gamma\epsilon}{2T} \right) \leq \frac{4T^2 \sup_{\ell \leq T/\epsilon} \left(\log^2 \left(\frac{\beta_j(\phi_{\ell\epsilon}^{a,\epsilon})}{\mu_{\ell\epsilon}^{a,\epsilon,j}} \right) N \mu_{\ell\epsilon}^{a,\epsilon,j} \epsilon \right)}{N^2 \gamma^2 \epsilon^2}.$$

As $C_a \leq \beta_j(\phi_t^{a,\epsilon}) \leq \sigma$, $\mu_t^{a,\epsilon,j} \leq L$, we have $\sup_{\ell \leq T/\epsilon} \left(\log^2 \left(\frac{\beta_j(\phi_{\ell\epsilon}^{a,\epsilon})}{\mu_{\ell\epsilon}^{a,\epsilon,j}} \right) \mu_{\ell\epsilon}^{a,\epsilon,j} \right) \leq C(L, a)$. Thus

$$\begin{aligned}
\tilde{\mathbb{P}}_y (\|Z^N - \phi^{a,\epsilon}\|_T < \delta \} \cap (B_j^\delta)^c) & \leq \tilde{\mathbb{P}}_y \left(\left| \sum_{\ell=1}^{T/\epsilon} \log \left(\frac{\beta_j(\phi_{\ell\epsilon}^{a,\epsilon})}{\mu_{\ell\epsilon}^{a,\epsilon,j}} \right) \left(\sum_{p=1}^{Q_\ell} \delta_{\ell,p}(j) - N \mu_{\ell\epsilon}^{a,\epsilon,j} \epsilon \right) \right| + \frac{2C\delta}{C_a} Q > N\gamma \right) \\
& \leq \frac{4T^2 C(L, a)}{N\gamma^2 \epsilon} + \tilde{\mathbb{P}}_y \left(\frac{2C\delta}{C_a} Q \geq \frac{N\gamma}{2} \right).
\end{aligned}$$

Under the probability $\tilde{\mathbb{P}}_y$, Q , the number of jumps during the time interval $[0, T]$ is the sum of T/ϵ

Poisson random variables, the ℓ -th one having the mean $N \sum_{j=1}^k \mu_{\ell\epsilon}^{a,\epsilon,j} \epsilon$. Consequently

$$\tilde{\mathbb{E}}(Q) = \widetilde{Var}(Q) = N \sum_{j=1}^{T/\epsilon} \sum_{j=1}^k \mu_{\ell\epsilon}^{a,\epsilon,j} \epsilon \leq NkTL.$$

We finally deduce from the formula $\gamma = \frac{8kTLC}{C_a} \delta$ that

$$\begin{aligned} \tilde{\mathbb{P}}_y \left(\frac{2C\delta}{C_a} Q \geq \frac{N\gamma}{2} \right) &\leq \tilde{\mathbb{P}}_y(Q - \tilde{\mathbb{E}}(Q) > NkTL) \\ &\leq \frac{1}{NkTL}. \end{aligned}$$

The result follows. □

We now deduce from Lemma 5 the next result, whose proof follows the argument of Lemma 3 in [4]. Note however the local uniform continuity in the initial condition.

Proposition 1. *For $z \in A$, $\phi \in \mathcal{AC}_{T,A}$ with $\phi_0 = z$ and any $\eta > 0$, $\delta > 0$ there exists $N_{\eta,\delta}$ such that for all $N > N_{\eta,\delta}$,*

$$\inf_{y:|y-z|<\delta/2} \mathbb{P}_y(\|Z^N - \phi\|_T < \delta) \geq \exp\{-N(I_T(\phi) + \eta)\}. \quad (10)$$

Proof. For $\delta, \eta > 0$ let $\phi \in \mathcal{AC}_{T,A}$ with $\phi_0 = z$ be such that $I_T(\phi) < \infty$. Then from Lemma 2, if $a_\eta > 0$ is small enough, for all $a < a_\eta$ there exists $\phi^a \in R^a$ with $\|\phi - \phi^a\|_T < c_1 a$ and $I_T(\phi^a) \leq I_T(\phi) + \eta/4$. As $I_T(\phi^a) < \infty$, we deduce from Lemma 3 that there exists $L > 0$ such that $\phi^{a,L} \in R^{a/2}$ satisfies $\|\phi^a - \phi^{a,L}\|_T < c_1 \frac{a}{2}$ and $I_T(\phi^{a,L} | \mu^{a,L}) \leq I_T(\phi^a) + \eta/4$, where $\mu^{a,L} \in \mathcal{A}_d(\phi^{a,L})$ is such that $\mu_t^{a,L,j} < L$, $j = 1, \dots, k$. Now we can deduce from Lemma 4 that for all $\epsilon > 0$ the polygonal approximation $\phi^{a,L,\epsilon}$ of $\phi^{a,L}$ satisfies $\|\phi^{a,L} - \phi^{a,L,\epsilon}\|_T < c_1 \frac{a}{2}$ and $I_T(\phi^{a,L,\epsilon} | \mu^{a,L,\epsilon}) \leq I_T(\phi^{a,L} | \mu^{a,L}) + \eta/4$ where $\mu_t^{a,L,\epsilon,j} < L$, $j = 1, \dots, k$. Choosing

$a < \delta/(4c_1)$, we have

$$\begin{aligned} \inf_{y:|y-z|<\delta/2} \mathbb{P}_y \left(\|Z^N - \phi\|_T < \delta \right) &\geq \inf_{y:|y-z|<\delta/2} \mathbb{P}_y \left(\|Z^N - \phi^{a,L,\epsilon}\|_T < \frac{\delta}{2} \right) \\ &\geq \exp\{-N(I_T(\phi^{a,L,\epsilon}|\mu^{a,L,\epsilon}) + \eta/4)\} \\ &\geq \exp\{-N(I_T(\phi^{a,L}|\mu^{a,L}) + \eta/2)\} \\ &\geq \exp\{-N(I_T(\phi) + \eta)\}, \end{aligned}$$

where we have used Lemma 5 at the third inequality. \square

The following theorem follows rather easily from the previous Proposition.

Theorem 4. *For any open subset G of $D_{T,A}$ and $z \in A$,*

$$\liminf_{\substack{N \rightarrow \infty \\ y \rightarrow z}} \frac{1}{N} \log \mathbb{P}_y^N(G) \geq - \inf_{\phi \in G, \phi_0 = z} I_T(\phi). \quad (11)$$

The next Corollary follows as in [3], Corollary 5.6.15.

Corollary 2. *For any open subset G of $D_{T,A}$ and any compact subset K of A ,*

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \log \inf_{z \in K} \mathbb{P}_z(Z^N \in G) \geq - \sup_{z \in K} \inf_{\phi \in G, \phi_0 = z} I_T(\phi).$$

4. The Upper Bound

If we define $b(x) = \sum_{j=1}^k \beta_j(x)h_j$, $a(x) = 0$, $\mu_x(dv) = \sum_{j=1}^k \beta_j(x)\delta_{h_j}(dv)$, where δ_{h_j} denotes the Dirac measure on \mathbb{R}^d at h_j , we see that we are in the framework of [6], and their assumptions are satisfied. Consequently the following upper bound is a consequence of their Theorem 1.1.

Theorem 5. *For any open subset F of $D_{T,A}$ and any compact subset K of A ,*

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \sup_{z \in K} \mathbb{P}_z(Z^N \in F) \leq - \inf_{z \in K} \inf_{\phi \in F, \phi_0 = z} I_T(\phi).$$

5. Time of exit from a domain

Let O be the domain of attraction of a stable point z^* of the dynamical system (3) and $\widetilde{\partial O}$ be the part of boundary of O that the stochastic system (1) can cross. We need to formulate a theorem which give us an approximate value for the exit time τ_O^N from O for large N as well as the exponential asymptotic of its mean $\mathbb{E}_z(\tau_O^N)$. For the models of infectious disease, τ_O^N is the time to extinction of the disease. This is the most important application of our large deviations result. To this end, for $z, y \in A$ where A is defined by (2), we define the following functionals

$$\begin{aligned} V(z, y, T) &:= \inf_{\phi \in D_{T,A}, \phi_0=z, \phi_T=y} I_T(\phi) \\ V(z, y) &:= \inf_{T>0} V(z, y, T) \\ \bar{V} &:= \inf_{y \in \widetilde{\partial O}} V(z^*, y). \end{aligned}$$

The following theorem is a consequence of the large deviation principle established above, the law of large numbers and some technical arguments. The proof which can found in Section 7 of [13] requires the following technical assumptions:

Assumption 3. 1. For all $z \in O$, the solution $Y^z(\cdot)$ of the ODE (3) satisfies

$$Y^z(t) \in O \quad \text{for all } t > 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} Y^z(t) = z^*.$$

2. $\bar{V} < \infty$.

3. For all $\rho > 0$ there exist constants $T(\rho), \epsilon(\rho) > 0$ with $T(\rho), \epsilon(\rho) \downarrow 0$ as $\rho \downarrow 0$ such that for all $x \in \widetilde{\partial O} \cup \{z^*\}$ and all $z, y \in B_\rho(x) \cap A$ there exists an $\phi = \phi(\rho, z, y) : [0, T(\rho)] \rightarrow A$ with $\phi_0 = z$, $\phi_{T(\rho)} = y$ and $I_{T(\rho)}(\phi) < \epsilon(\rho)$.

4. For all $z \in \widetilde{\partial O}$ there exists an $\delta_0 > 0$ such that for all $\delta < \delta_0$ there exists $z^\delta \in A \setminus \bar{O}$ with $|z - z^\delta| > \delta$.

5. There exists a collection $\{O_\rho, \rho > 0\}$ which is such that

- $\bar{O}_\rho \subset O$ for all $\rho > 0$.
- $d(O_\rho, \widetilde{\partial O}) \rightarrow 0$ as $\rho \rightarrow 0$.

- For all $\rho > 0$, O_ρ satisfies the four above assumptions and for all $z \in \partial O_\rho$, the solution $Y^z(\cdot)$ of the ODE (3) is such that $\lim_{t \rightarrow \infty} Y^z(t) = z^*$.

Note that all these assumptions are satisfied in the infectious disease models which we will present below. The third assumption is not difficult to verify and the fourth one allows to consider a trajectory which crosses the characteristic boundary $\widetilde{\partial O}$, in such a way that all paths in a sufficiently small tube around that trajectory do exit O . That fourth condition is not satisfied in the two first examples below. However, the result applies, thanks to an argument which is detailed in section 7 of [13] for the SIRS model.

Theorem 6. *Under the assumptions 1, 2 and 3, given $\eta > 0$, for all $z \in O$,*

$$\lim_{N \rightarrow \infty} \mathbb{P}_z(\exp\{N(\bar{V} - \eta)\} < \tau_O^N < \exp\{N(\bar{V} + \eta)\}) = 1.$$

Moreover, for all $\eta > 0$, $z \in O$ and N large enough,

$$\exp\{N(\bar{V} - \eta)\} \leq \mathbb{E}_z(\tau_O^N) \leq \exp\{N(\bar{V} + \eta)\}.$$

5.1. The SIS model

We consider a population of fixed size N , which is composed of susceptible and infected individuals. The proportion of infected individuals obeys the SDE

$$I^N(t) = I^N(0) + \frac{1}{N} P_1 \left(N\beta \int_0^t I^N(s)(1 - I^N(s)) ds \right) - \frac{1}{N} P_2 \left(N\alpha \int_0^t I^N(s) ds \right),$$

where β is the rate at which infected individuals infect susceptibles, and α the rate at which an infected individual recovers. In this case $O = \overset{\circ}{A} = (0, 1)$. We assume that $\beta > \alpha$, in which case the law of large number ODE limit

$$\frac{di}{dt}(t) = \beta i(t)(1 - i(t)) - \alpha i(t)$$

has the unique stable equilibrium $i^* = 1 - \alpha/\beta$, which is the endemic equilibrium. Our results that the time taken by the random perturbation to estinguish the disease is of the order of $\exp(N\bar{V})$, where $\bar{V} = \inf_{T>0} \inf_{\phi_0=i^*, \phi_T=0} I_T(\phi)$ (our assumption 3 is not satisfied, but the needed extension is easy

to justify). This is the value function of an optimal control problem, which in this case can be computed explicitly, and we get $\bar{V} = \log(\beta/\alpha) - 1 + \alpha/\beta$. Note that the optimal T is infinite, and \bar{V} depends only upon the ratio $R_0 = \beta/\alpha$, which is called the "basic reproduction number", and it is an increasing function of that ratio. We refer the reader to [2] for more details.

5.2. The SIRS model

In this model the individuals who recover are "retired": they cannot be infected, until they lose their immunity. The model becomes

$$\begin{aligned} I^N(t) &= I^N(0) + \frac{1}{N}P_1\left(N\beta \int_0^t I^N(s)(1 - I^N(s) - R^N(s))ds\right) - \frac{1}{N}P_2\left(N\alpha \int_0^t I^N(s)ds\right), \\ R^N(t) &= R^N(0) + \frac{1}{N}P_1\left(N\alpha \int_0^t I^N(s)ds\right) - \frac{1}{N}P_3\left(N\gamma \int_0^t R^N(s)ds\right). \end{aligned}$$

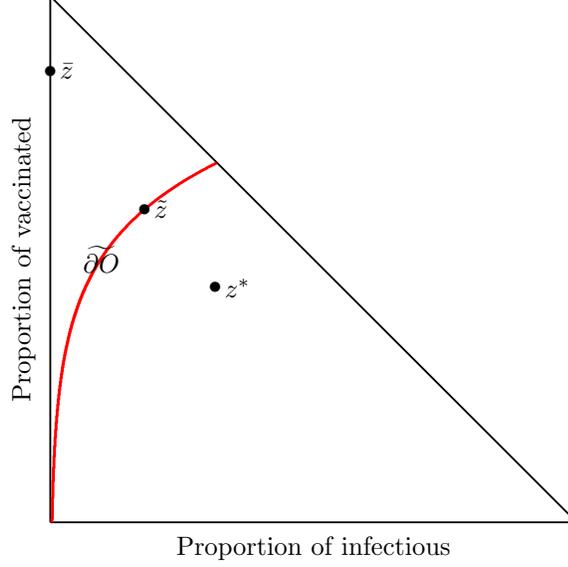
Again, if the basic reproduction number $R_0 = \beta/\alpha > 1$, the law of large numbers ODE

$$\begin{aligned} \frac{di}{dt}(t) &= \beta i(t)(1 - i(t) - r(t)) - \alpha i(t), \\ \frac{dr}{dt}(t) &= \alpha i(t) - \gamma r(t). \end{aligned}$$

has a unique stable endemic equilibrium $(i^*, r^*) = \frac{\beta - \alpha}{\beta(\alpha + \gamma)}(\gamma, \alpha)$. Here $A = \{i \geq 0, r \geq 0 : i + r \leq 1\}$ and $O = \dot{A}$. As shown in [13], although again the assumption 3 is not quite satisfied, Theorem 6 applies. Here unfortunately it does not seem possible to compute explicitly the quantity \bar{V} .

5.3. The SIV model

Consider a model with vaccination for a population of fixed size N , composed of susceptible, infected and vaccinated individuals. We assume that the vaccinated individuals are not fully protected, and can be infected, but this happen at a smaller rate than the infection of susceptibles. They can also loose their partial immunity, and become susceptible again. In each class of individuals, there is the same death rate. The proportions of infected and vaccinated individuals follow the

FIGURE 1: Attraction region O of the equilibrium z^* in SIV model

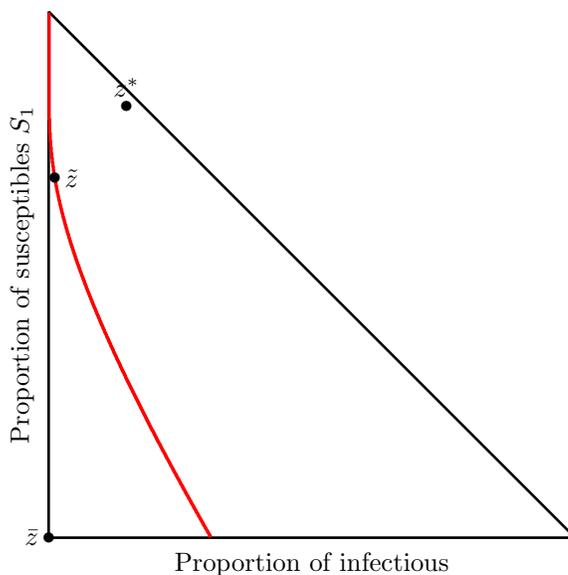
following SDE

$$\begin{aligned}
 I^N(t) &= I^N(0) + \frac{1}{N}P_1\left(N\beta \int_0^t I^N(s)(1 - I^N(s) - V^N(s))ds\right) + \frac{1}{N}P_2\left(N\chi\beta \int_0^t I^N(s)V^N(s)ds\right) \\
 &\quad - \frac{1}{N}P_3\left(N\gamma \int_0^t I^N(s)ds\right) - \frac{1}{N}P_6\left(N\mu \int_0^t I^N(s)ds\right), \\
 V^N(t) &= V^N(0) - \frac{1}{N}P_2\left(N\chi\beta \int_0^t I^N(s)V^N(s)ds\right) - \frac{1}{N}P_4\left(N\theta \int_0^t I^N(s)V^N(s)ds\right) \\
 &\quad + \frac{1}{N}P_5\left(N\eta \int_0^t (1 - I^N(s) - V^N(s))ds\right) - \frac{1}{N}P_7\left(N\mu \int_0^t V^N(s)ds\right).
 \end{aligned}$$

Under appropriate assumptions on the parameters of the model (see [14]), its law of large numbers ODE limit

$$\begin{cases} \frac{di}{dt}(t) = (\beta - \mu - \gamma)i(t) - \beta(1 - \chi)i(t)v(t) - \beta i^2(t) \\ \frac{dv}{dt}(t) = \eta - \eta i(t) - (\eta + \mu + \theta)v(t) - \chi\beta i(t)v(t). \end{cases}$$

has two endemic equilibria $z^* = (z_1^*, z_2^*)$, $\tilde{z} = (\tilde{z}_1, \tilde{z}_2)$. z^* is locally stable while \tilde{z} is unstable. These two equilibria are completed with the disease free equilibrium \bar{z} ($\bar{z}_1 = 0$, $\bar{z}_2 = \frac{\eta}{\mu + \theta + \eta}$) which is locally stable. Figure 1 shows the basin of attraction O of the equilibrium z^* delimited by the characteristic

FIGURE 2: Attraction region O of the equilibrium in S_0IS_1 model

boundary and containing the point z^* . Here $A = \{i \geq 0, v \geq 0 : i + v \leq 1\}$. The assumptions 1, 2 and 3 are satisfied and Theorem 6 applies. Again it is not possible to compute explicitly the quantity \bar{V} but its numerical computation will be the object of a forthcoming publication.

5.4. The S_0IS_1 model

We consider here a model with two levels of susceptibility in which the population has a fixed size N composed of susceptibles S_0 who are individuals who have never been infected and may contract the infection, the infected I and the susceptibles S_1 with at least one past infection. We assume that the infected individuals can recover and become again susceptible (of type S_1), and be infected with a rate which is different from that of the type S_0 individuals. The death rate is the same in all classes. The proportions of infected and vaccinated individuals follow the following

SDE

$$\begin{aligned}
I^N(t) &= I^N(0) + \frac{1}{N}P_1\left(N\beta \int_0^t I^N(s)(1 - I^N(s) - S_1^N(s))ds\right) - \frac{1}{N}P_2\left(N\alpha \int_0^t I^N(s)ds\right) \\
&\quad - \frac{1}{N}P_3\left(N\mu \int_0^t I^N(s)ds\right) + \frac{1}{N}P_4\left(Nr\beta\mu \int_0^t I^N(s)S_1^N(s)ds\right), \\
S_1^N(t) &= S_1^N(0) + \frac{1}{N}P_2\left(N\alpha \int_0^t I^N(s)ds\right) - \frac{1}{N}P_4\left(Nr\beta\mu \int_0^t I^N(s)S_1^N(s)ds\right) \\
&\quad - \frac{1}{N}P_5\left(N\mu \int_0^t S_1^N(s)ds\right).
\end{aligned}$$

If the parameters of this model are chosen in an appropriate way (see [17]), its law of large number ODE limit

$$\begin{cases} \frac{di}{dt}(t) = -(\alpha + \mu - \beta)i(t) + (r - 1)\beta i(t)s_1(t) - \beta i^2(t) \\ \frac{ds_1}{dt}(t) = \alpha i(t) - \mu s_1(t) - r\beta i(t)s_1(t). \end{cases}$$

has two positive endemic equilibria exist; the first one z^* is locally asymptotically stable and the second one \tilde{z} is unstable, in addition to the disease free equilibrium $\bar{z} = (0, 0)$ which is again locally asymptotically stable. Figure 2 shows the basin of attraction O of the equilibrium z^* delimited by the characteristic boundary and containing the point z^* . Here $A = \{i \geq 0, s_1 \geq 0 : i + s_1 \leq 1\}$. The assumptions 1, 2 and 3 are satisfied and Theorem 6 applies.

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