Large Deviation Principle for Poisson driven SDEs in Epidemic Models.

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Abstract

We consider a general class of epidemic models obtained by applying the random time changes of [6] to a collection of Poisson processes and we show the large deviation principle(LDP) for such models. We generalize to a more general situation the approach of followed by Dolgoashinnykh [4] in the case of the SIR epidemic model. Thanks to an additional assumption which is satisfied in many examples, we simplify the recent work by P.Kratz and E.Pardoux [9].

Keywords: Poisson process; Large deviation principle; Law of large number.

AMS classification: 60F10, 60H10, 60J75, 92C60.

Introduction

In this paper, we are interested in a class of Poisson driven stochastic differential equations which arise in many fields such as chemical kinetics, ecological and epidemics models. We consider a d dimensional processes of the type

$$Z^N(t) := Z^{N,z}(t) := \frac{[Nz]}{N} + \frac{1}{N} \sum_{j=1}^{k} h_j P_j \Big(\int_0^t N\beta_j(Z^N(s))ds \Big), \tag{1}$$

where $(P_j)_{1 \le j \le k}$ are i.i.d. standard Poisson processes and the $h_j \in \mathbb{Z}^d$ denote the k distinct jump directions with jump rates $\beta_j(z)$ and $z \in A$, where A is a compact subset of $\mathbb{R}^d$, which will be assumed to satisfy Assumption 0.1 below.

In the main application which we have in mind, the components of the vector $Z^N(t)$ are the proportions of the population in the various compartments corresponding to the various disease status of the individuals (susceptible, infectious, etc..). In most of those models, the set A is given as

$$A = \Big\{ z \in \mathbb{R}_+^d : \sum_{i=1}^{d} z_i \le 1 \Big\}. \tag{2}$$

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We refer the reader to [2] for a presentation of many such epidemics models.

As we shall recall below, it is plain that under mild assumptions, as $N \to \infty$, $Z^N(t) \to Y^z(t)$ a.s., locally uniformly for $t > 0$, where $Y^z(t)$ solves the ODE

$$\frac{dY(t)}{dt} = b(Y(t)), Y(0) = z,$$

where $b(z) = \sum_{j=1}^k \beta_j(z)h_j$. In this paper we want to investigate the large deviations from this law of large numbers.

Let us now be more precise about the initial condition $Z^N(0) = [Nz]/N$. In the models we have in mind, since each component of $Z^N(t)$ is a proportion in a population of total population size equal to N, we want $Z^N(t)$ to take its values in the set $A^{(N)} = \{z \in A, \ Nz \in \mathbb{Z}_+^d\}$. In particular, we want the initial condition $Z^N(0)$ to belong to this set $A^{(N)}$. If that is not the case, some of the components of the vector $Z^N(t)$ may become negative, while jumping from a/N to $(a-1)/N$, $0 < a < 1$, which is not very natural. For that reason, we will use the following convention concerning the initial condition. For some fixed $z \in A$, for $1 \leq i \leq d$, $N \geq 1$, $Z_i^N(0) = [Nz_i]/N$. Consequently the initial condition of the Z^N equation depends upon N.

In all what follows, $D_{T,A}$ denotes the set of functions from $[0,T]$ into A which are right continuous and have left limits at any $t \in [0,T]$ and let $\mathcal{AC}_{T,A}$ be the subspace of absolutely continuous functions.

We denote by $\mathcal{B}$ the Borel σ-field on $D_{T,A}$ and $\mathbb{P}_z^N$ the probability measure on paths whose initial condition $Z^N(0) = [Nz]/N$ defined by

$$\mathbb{P}_z^N(B) = \mathbb{P}(Z^N \in B) \quad \forall B \in \mathcal{B}.$$

Our goal is to show that the probability measures $\mathbb{P}_z^N$, $N > 1$, satisfy a large deviation principle with a good rate function I_T that we will define below in subsection 1.2. In other words for any G open subset of $D_{T,A}$ and F closed subset of $D_{T,A}$ we want to show the following inequalities:

$$- \inf_{\phi \in G} I_T(\phi) \leq \liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}_z(Z^N \in G), \tag{3}$$

$$\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}_z(Z^N \in F) \leq - \inf_{\phi \in F} I_T(\phi). \tag{4}$$

Large deviation principles is the subject of many treatises, see in particular [3], [5], [7], [8] and [11]. Some of those books study large deviations for Poisson processes, like e.g. [11]. However, in this treatise it is assumed that the rates of the Poisson processes are bounded away from zero, and hence their logarithms are bounded. The case of Poisson processes with vanishing rates is studied in [12]. However their assumptions are not satisfied in our situation, as it is explained in [9]. Our result have already been already established in [9]. However, our argument here is simpler, and the proofs are shorter. It is based upon an idea from [4] and forces us to add an assumption, which is satisfied in all examples we have in mind.

That additional assumption is the following.

Assumption 0.1. *We suppose that there exists $z_0 \in \mathbb{R}^d$ such that the collection of mappings $\Phi_a = z + a(z_0 - z)$, defined for each $a > 0$, is such that $z^a = \Phi_a(z) \in A$, and moreover*

$$|z - z^a| \leq c_1 a$$
$$\mathrm{dist}(z^a, \partial A) \geq c_2 a$$

for some $0 < c_2 < c_1$.

We now introduce the sets defined for all $a > 0$ by

$$B^a = \left\{ z \in A : \mathrm{dist}(z, \partial A) \geq c_2 a \right\} \tag{5}$$

and

$$R^a = \left\{ \phi \in \mathcal{AC}_{T,A} : \phi_t \in B^a \quad \forall t \in [0, T] \right\} \tag{6}$$

hence Φ_a maps A into B^a.

Remark 0.2. *For any convex set A, one expects that Assumption 0.1 is satisfied provided $z_0 \in \mathring{A}$. The same construction is possible for many non necessarily convex sets, provided A is compact, and there is a point z_0 in its interior which is such that for each $z \in \partial A$, the segment joining z_0 and z does not touch any other point of the boundary ∂A. We also note that for A given by (2) and $z_0 \in A$, the constants c_1, c_1 can be defined by*

$$c_1 = \sup_{z \in A} |z - z_0|$$
$$c_2 = \sin(\theta_0) \inf_{z \in \partial A} |z - z_0| \leq \inf_{z \in \partial A} |z - z_0| \times \sin(\theta(z)).$$

where $\theta(z)$ is the most acute angle between the boundary ∂A and the vector $z_0 - z$ and θ_0 is a angle such that for all $z \in \partial A$, $\theta_0 \leq \theta(z) \leq \pi/2$. For instance for $d = 2$ (see figure 1), $\theta_0 = \min_{1 \leq \ell \leq 6} \theta_\ell$.

Moreover for all $a > 0$ we define

$$C_a = \inf_j \inf_{z \in B^a} \beta_j(z). \tag{7}$$

We assume that for all $a > 0$, $C_a > 0$ and $\lim_{a \to 0} C_a = 0$.

We now formulate our assumptions on the coefficients β_j.

Assumption 0.3.   *1. The rate functions β_j are Lipschitz continuous with the Lipschitz constant equal to C.*

     2. For any $1 \leq j \leq k$, $\beta_j(z) > 0$ if $z \in \mathring{A}$, and β_j is bounded by a positive constant σ.

     3. There exist two constants λ_1 and λ_2 such that whenever $z \in A$ is such that $\beta_j(z) < \lambda_1$, $\beta_j(z^a) > \beta_j(z)$ for all $a \in]0, \lambda_2[$.

     4. There exists a constant $\nu \in]0, 1/2[$ such that

$$\lim_{a \to 0} a^\nu \log C_a = 0.$$

This means in particular that there exists $a_0 > 0$ such that for all $a < a_0$, $C_a \geq e^{-a^{-\nu}}$.
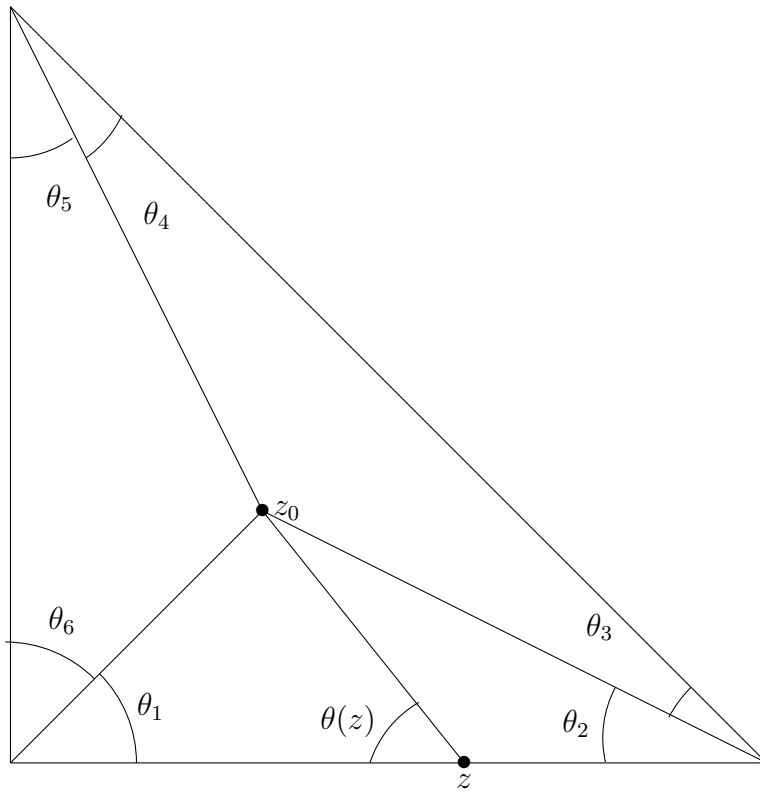
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Figure 1: Domain A

Let us comment on Assumption 0.3. Assumption 0.3.1 is quite standard and ensures in particular that the ODE (8) admits a unique solution. For the compartmental epidemiological models which we are mainly interested in, this assumption is always true because the $\beta_j(z)$ are usually polynomials and A is compact. Also the assumption 0.3.2 is always true because the domain of our process is compact. Assumption 0.3.3 will follow from the fact that close to the boundary, "small" rates are increasing when we follow a direction towards the inside of the domain. Concerning the assumption 0.3.4, such an assumption is true for the models we study because the rates are usually polynomials.

Remark 0.4. *We have not made any restriction concerning the set of vectors $\{h_1, \ldots, h_k\}$. In all examples which we have in mind, the vector space spanned by those vectors is $\mathbb{R}^d$, which insures that the process Z_t^N can move in all directions in A. Note that at any rate, there is no restriction as to which β_j's vanish at some given point $z \in \partial A$. This is a major difference with the assumptions in [12]. Note also that in some sense our assumptions are weaker than those in [9], except for our Assumption 0.1.*

For all $\phi, \psi \in D_{T,A}$ we will define the distance between ϕ and ψ by

$$\|\phi - \psi\|_T = \sup_{t \leq T} |\phi_t - \psi_t|$$

where $|.|$ denotes the Euclidean norm in $\mathbb{R}^d$.

The remainder of this paper is structured as follows. In section 1, we formulate the law of large numbers, we define a good rate function for our large deviation principle and we establish some properties that it satisfies. The second section concerns the proof of the lower bound (3) and the third one the proof of the upper bound (4). The last section of this paper states a result concerning the asymptotic behavior of the exit time from the domain of attraction of a stable point for the dynamical system (8) as well as the exponential asymptotic of its mean $\mathbb{E}_z(\tau_O^N)$. For epidemics models, this exit time is the time of extinction of an endemic disease.

1 Somes Important Results

1.1 Law of Large Number and Change of Measure

We now prove the law of large number.

Theorem 1.1. *Let $Z^{N,z}(t)$ the solution of stochastic differential equation Poissonian (1) with an initial condition $[Nz]/N$. Assume that the assumption 0.3.1 holds. Then*

$$\lim_{N \to \infty} \|Z^{N,z} - Y^z\|_T = 0 \quad a.s.$$

Where $Y^z(.)$ is the solution of the ODE

$$\frac{dY(t)}{dt} := b(Y(t)), Y(0) = z, \tag{8}$$

with an initial condition z and where

$$b(z) := \sum_{j=1}^k \beta_j(z) h_j.$$

Proof. By using the Lipschitz continuity of b, we have with $M_j(t) = P_j(t) - t$, $\widetilde{M}_j^N(t) = \frac{1}{N} P_j(N.t) - t$

$$|Z^N(t) - Y^z(t)| \leq \left| \frac{[Nz]}{N} - z \right| + \int_0^t |b(Z^N(s)) - b(Y^z(s))| ds + \frac{1}{N} \left| \sum_{j=1}^k h_j M_j \left(N \int_0^t \beta_j(Z^N(s)) ds \right) \right|$$

$$\leq \left| \frac{[Nz]}{N} - z \right| + kC\sqrt{d} \int_0^t |Z^N(s) - Y^z(s)| ds + \sqrt{d} \sum_{j=1}^k \left| \widetilde{M}_j^N \left(\int_0^t \beta_j(Z^N(s)) ds \right) \right|$$

$$\leq \left| \frac{[Nz]}{N} - z \right| + kC\sqrt{d} \int_0^t |Z^N(s) - Y^z(s)| ds + k\sqrt{d} \sup_j \sup_{t \leq T} \left| \widetilde{M}_j^N \left(\int_0^t \beta_j(Z^N(s)) ds \right) \right|$$
$$\tag{9}$$

Let $\xi_j^N(t) = \left| \widetilde{M}_j^N \left(\int_0^t \beta_j(Z^N(s)) ds \right) \right|$. From the strong law of large numbers for a Poisson process, we have for all $j = 1, ..., k$

$$\frac{P_j(Nt)}{N} \to t \quad a.s. \quad \text{as} \quad N \to \infty.$$

As we have pointwise convergence of a sequence of increasing function towards a continuous function we can use the second Dini theorem to conclude that this convergence is uniform on any compact time interval, hence for $0 \leq v < \infty$ and $j = 1, ...k$

$$\lim_{N \to \infty} \sup_{u \leq v} |\widetilde{M}_j^N(u)| = 0 \quad a.s.$$

As the β_j are bounded by σ, it follows that

$$\lim_{N \to \infty} \sup_{t \leq T} \xi_j^N(t) = 0 \quad a.s.$$

for $j = 1, ..., k$.
By using by Gronwall's inequality stated above we have

$$|Z_t^N - Y_t^z| \leq k\sqrt{d} \left(\left| \frac{[Nz]}{N} - z \right| + \sup_j \sup_{t \leq T} \xi_j^N(t) \right) \exp\{kC\sqrt{d}t\}$$

and the result follows. $\qquad \square$

We shall need the following Girsanov theorem . Let Q equal to the random number of jumps of Z^N in the interval $[0, T]$, τ_p be the time of the pth jump for $p = 1, ..., Q$ and define

$$\delta_p(j) = \begin{cases} 1 & \text{if the } p^{th} \text{ jump is in the direction } h_j \text{ ,} \\ 0 & \text{otherwise.} \end{cases}$$

We shall denote $\mathcal{F}_t^N = \sigma\{Z^N(s), 0 \leq s \leq t\}$. Consider another set of rates $\tilde{\beta}_j(z)$, $1 \leq j \leq k$. Combining Theorem VI T3 from [1] and Theorem 2.4 from [13], we have

Theorem 1.2. *Let $\widetilde{\mathbb{P}}^N$ denote the law of Z^N when the rates are rates $\tilde{\beta}_j(.)$. Then provided that $\sup_{z \in A} \frac{\tilde{\beta}_j(z)}{\beta_j(z)} < \infty$, which implies in particular that $\{z : \beta_j(z) = 0\} \subset \{z : \tilde{\beta}_j(z) = 0\}$, on the σ-algebra $\mathcal{F}_t^N$, $\widetilde{\mathbb{P}}^N\big|_{\mathcal{F}_T^N} << \mathbb{P}^N\big|_{\mathcal{F}_T^N}$, and*

$$
\xi_T = \xi_T^N = \frac{d\widetilde{\mathbb{P}}^N\big|_{\mathcal{F}_T^N}}{d\mathbb{P}^N\big|_{\mathcal{F}_T^N}}
$$

$$
= \left(\prod_{p=1}^Q \prod_{j=1}^k \left[\frac{\tilde{\beta}_j(Z^N(\tau_p^-))}{\beta_j(Z^N(\tau_p^-))} \right]^{\delta_p(j)} \right) \exp\left\{ N \sum_{j=1}^k \int_0^T (\beta_j(Z^N(t)) - \tilde{\beta}_j(Z^N(t)))dt \right\}. \quad (10)
$$

Corollary 1.3. *For all non-negative measurable function $X \geq 0$,*

$$
\mathbb{E}(X) \geq \tilde{\mathbb{E}}(\xi_T^{-1} X)
$$

Proof. As $X \geq 0$, we write

$$
\mathbb{E}(X) \geq \mathbb{E}(X\mathbf{1}_{\{\xi_T \neq 0\}}) = \tilde{\mathbb{E}}(\xi_T^{-1}X\mathbf{1}_{\{\xi_T \neq 0\}}) = \tilde{\mathbb{E}}(\xi_T^{-1}X).
$$

This last equality comes from the fact that $\widetilde{\mathbb{P}}(\xi_T = 0) = 0$ i.e. ξ_T^{-1} is well-defined $\widetilde{\mathbb{P}}-$almost surely. $\qquad\square$

1.2 The Rate Function

For all $\phi \in \mathcal{AC}_{T,A}$, let $\mathcal{A}_d(\phi)$ the set of vector valued Borel measurable functions μ such that for all $j = 1, ..., k$, $\mu_t^j \geq 0$ and

$$
\frac{d\phi_t}{dt} = \sum_{j=1}^k \mu_t^j h_j, \quad \text{t a.e.}
$$

We define the rate function

$$
I_T(\phi) := \begin{cases} \inf_{\mu \in \mathcal{A}_d(\phi)} I_T(\phi|\mu), & \text{if } \phi \in \mathcal{AC}_{T,A}; \\ \infty, & \text{else.} \end{cases}
$$

where

$$
I_T(\phi|\mu) = \int_0^T \sum_{j=1}^k f(\mu_t^j, \beta_j(\phi_t))dt
$$

with $f(\nu, \omega) = \nu \log(\nu/\omega) - \nu + \omega$. We assume in the definition of $f(\nu, \omega)$ that for all $\nu > 0$, $\log(\nu/0) = \infty$ and $0\log(0/0) = 0\log(0) = 0$.

Note that in the absence of any assumption concerning the vector space spanned by the h_j's, the set $\mathcal{A}_d(\phi)$ can be empty for many $\phi \in \mathcal{AC}_{T,A}$. We of course make use of the usual convention that the infimum over an empty set is infinite.

By using the Legendre-Fenchel transform we define another rate function by

$$
\tilde{I}_T(\phi) := \begin{cases} \int_0^T L(\phi_t, \phi_t')dt & \text{if } \phi \in \mathcal{AC}_{T,A} \\ \infty & \text{else.} \end{cases}
$$

where for all $z \in A$, $y \in \mathbb{R}^d$

$$L(z, y) = \sup_{\theta \in \mathbb{R}^d} \ell(z, y, \theta)$$

with for all $z \in A$, $y \in \mathbb{R}^d$ and $\theta \in \mathbb{R}^d$

$$\ell(z, y, \theta) = \langle \theta, y \rangle - \sum_{j=1}^{k} \beta_j(z)(e^{\langle \theta, h_j \rangle} - 1)$$

We now show the equality between these two definitions of the rate function.

Lemma 1.4. *For all $\phi \in \mathcal{AC}_{T,A}$ and $\mu \in \mathcal{A}_d(\phi)$ we have*

$$\tilde{I}_T(\phi) \leq I_T(\phi|\mu).$$

In particular $\tilde{I}_T(\phi) \leq I_T(\phi)$

Proof. Assume first that for some $B \in \mathcal{B}([0,T])$, with $\int_B dt > 0$ such that for all $t \in B$ there exists $1 \leq j \leq k$ such that $\mu_t^j > 0$ and $\beta_j(\phi_t) = 0$ then $I_T(\phi|\mu) = \infty$ and the inequality is true. We now assume that for almost all $t \in [0, T]$ and for all $j \in 1, ..., k$, $\mu_t^j > 0$ only if $\beta_j(\phi_t) > 0$ then for all $\theta \in \mathbb{R}^d$

$$\begin{aligned}
\ell(\phi_t, \phi_t, \theta) &= \sum_{j=1}^{k} \mu_t^j \langle \theta, h_j \rangle - \beta_j(x)(e^{\langle \theta, h_j \rangle} - 1) \\
&= \sum_{j=1}^{k} g_{\mu_t^j, \beta_j(\phi_t)}(\langle \theta, h_j \rangle) \\
&\leq \sum_{j=1}^{k} g_{\mu_t^j, \beta_j(\phi_t)}\left(\log \frac{\mu_t^j}{\beta_j(\phi_t)} \right) \\
&= \sum_{j=1}^{k} f(\mu_t^j, \beta_j(\phi_t)),
\end{aligned}$$

since $g_{\nu,\beta}(z) = \nu z - \beta(e^z - 1)$ is a function which achieves its maximum at $z = \log \frac{\nu}{\beta}$. $\square$

Lemma 1.5. *For all $\phi \in \mathcal{AC}_{T,A}$,*

$$I_T(\phi) \leq \tilde{I}_T(\phi).$$

Proof. If $\tilde{I}_T(\phi) = \infty$ the inequality is true. We now assume that $\tilde{I}_T(\phi) < \infty$ then for almost all $t \in [0, T]$ we have $L(\phi_t, \phi_t') = \sup_{\theta \in \mathbb{R}^d} \ell(\phi_t, \phi_t', \theta) < \infty$ then by [9] there exists a maximizing sequence $(\theta_n)_n$ of $\ell(\phi_t, \phi_t', .)$ namely $L(\phi_t, \phi_t') = \lim_n \ell(\phi_t, \phi_t', \theta_n)$ and constants s_j such that for all $j = 1, ..., k$,

$$\lim_n \exp\{\langle \theta_n, h_j \rangle\} = s_j.$$

Then we have

$$\lim_n \langle \theta_n, \phi_t' \rangle = L(\phi_t, \phi_t') + \sum_{j:\beta_j(\phi_t)>0} \beta_j(\phi_t)(s_j - 1).$$

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Moreover we differentiate with respect to θ and obtain for all n

$$\nabla_\theta \ell(\phi_t, \phi'_t, \theta_n) = \frac{d\phi_t}{dt} - \sum_{j:\beta_j(\phi_t)>0} \beta_j(\phi_t)h_j \exp\{\langle \theta_n, h_j \rangle\}.$$

As $(\theta_n)_n$ is a maximizing sequence we have for all $t \in [0, T]$

$$\lim_n \nabla_\theta \ell(\phi_t, \phi'_t, \theta_n) = \frac{d\phi_t}{dt} - \sum_{j:\beta_j(\phi_t)>0} \beta_j(\phi_t)s_j h_j = 0.$$

Thus, for almost all $t \in [0, T]$

$$\frac{d\phi_t}{dt} = \sum_{j=1}^k \beta_j(\phi_t)s_j h_j = \sum_{j=1}^k \mu_t^j h_j.$$

Where for almost all $t \in [0, T]$ and $j = 1, ..., k$

$$\mu_t^j = \beta_j(\phi_t)s_j.$$

We deduce that

$$I_T(\phi) \leq I_T(\phi|\mu)$$
$$= \int_0^T \sum_{j=1}^k f(\mu_t^j, \beta_j(\phi_t))dt$$
$$= \int_0^T \sum_{j=1}^k \left\{\mu_t^j \log s_j + \beta_j(\phi_t)(1 - s_j)\right\}dt$$
$$= \int_0^T L(\phi_t, \phi'_t)dt = \tilde{I}_T(\phi).$$

$\square$

The proof of the following theorem can be found in [9].

Theorem 1.6. $I_T = \tilde{I}_T$ *is a good rate function.*

Proof. As the β_j are bounded and continuous, we deduce from Lemma 4.20 in [9] that $\tilde{I}_T$ is lower semicontinuous with respect to Skorokhod's metric on $D_{T,A}$. Therefore the level set $\Phi(s) = \{\phi \in D_{T,A} : \tilde{I}_T(\phi) \leq s\}$ are closed and one can show that those sets are equicontinuous. We also know that A is compact and then the relatively compact subsets of $C([0, T], A)$ are exactly the subsets of equicontinuous functions. Thus the level sets $\Phi(s)$ are compact since they are closed and relatively compact. $\square$

The following result is a direct consequence of Lemma 4.22 in [9]

Lemma 1.7. *Let F a closed subset of $D_{T,A}$ and $z \in A$. We have*

$$\lim_{\epsilon \to 0} \inf_{y \in A, |y-z|<\epsilon} \inf_{\phi \in F, \phi_0 = y} I_T(\phi) = \inf_{\phi \in F, \phi_0 = z} I_T(\phi).$$

Lemma 1.8. *Let $s > 0$, $\phi \in D_{T,A}$ and $\mu \in \mathcal{A}_d(\phi)$ such that $I_T(\phi|\mu) \le s$ then for all $0 \le t_1, t_2 \le T$ such that $t_2 - t_1 \le 1/\sigma$,*

$$\int_{t_1}^{t_2} \mu_t^j dt \le \frac{s+1}{-\log(\sigma(t_2-t_1))} \quad \forall j = 1, ..., k.$$

Proof. We have

$$\int_0^T f(\mu_t^j, \beta_j(\phi_t)) dt \le I_T(\phi|\mu) \le s.$$

moreover, the function $h(x) = x \log(x/\sigma) - x$ is convex in x so that for all $0 \le t_1, t_2 \le T$

$$h\Big(\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \mu_t^j dt\Big) \le \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} h(\mu_t^j) dt$$

$$\le \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \Big(\mu_t^j \log \frac{\mu_t^j}{\beta_j(\phi_t)} - \mu_t^j + \beta_j(\phi_t) \Big) dt$$

$$\le \frac{s}{t_2 - t_1}.$$

It is easy to show that for all $\alpha > 0$, $h(x) \ge \alpha x - \sigma \exp\{\alpha\}$ and then for all $\alpha > 0$

$$\int_{t_1}^{t_2} \mu_t^j dt \le \frac{1}{\alpha}(s + (t_2 - t_1)\sigma \exp\{\alpha\}).$$

Therefore If $t_2 - t_1 < 1/\sigma$ taking $\alpha = -\log(\sigma(t_2 - t_1))$, the result follows. $\qquad\square$

For $\phi \in D_{T,A}$ let ϕ^a defined by $\phi_t^a = (1-a)\phi_t + az_0$ and we have $\phi^a \in R^a$.

Lemma 1.9. *For all $\phi \in D_{T,A}$ we have $\limsup_{a \to 0} I_T(\phi^a) \le I_T(\phi)$.*

Proof. First if $I_T(\phi) = \infty$ the result is easy. If $I_T(\phi) < \infty$, $\forall \eta > 0$ there exists μ such that $I_T(\phi|\mu) \le I_T(\phi) + \eta$. Let $\mu^a = (1-a)\mu$ then $\mu^a \in \mathcal{A}_d(\phi^a)$. We will now show that

$$I_T(\phi^a|\mu^a) \to I_T(\phi|\mu) \quad \text{as} \quad a \to 0, \tag{11}$$

which clearly implies the result since

$$\limsup_{a \to 0} I_T(\phi^a) \le \limsup_{a \to 0} I_T(\phi^a|\mu^a)$$

$$= I_T(\phi|\mu) \le I_T(\phi) + \eta.$$

By the convexity of $f(\nu, \omega)$ in ν and because $0 \le \mu_t^{j,a} \le \mu_t^j$, we have

$$0 \le f(\mu_t^{j,a}, \beta_j(\phi_t^a)) \le f(0, \beta_j(\phi_t^a)) + f(\mu_t^j, \beta_j(\phi_t^a))$$

$$\le \sigma + f(\mu_t^j, \beta_j(\phi_t^a)).$$

Moreover we have

$$f(\mu_t^j, \beta_j(\phi_t^a)) = \mu_t^j \log \frac{\mu_t^j}{\beta_j(\phi_t^a)} - \mu_t^j + \beta_j(\phi_t^a)$$

$$= \mu_t^j \log \frac{\mu_t^j}{\beta_j(\phi_t)} - \mu_t^j + \beta_j(\phi_t) + \mu_t^j \log \frac{\beta_j(\phi_t)}{\beta_j(\phi_t^a)} + \beta_j(\phi_t^a) - \beta_j(\phi_t)$$

$$\le f(\mu_t^j, \beta_j(\phi_t)) + \sigma + \mu_t^j \log \frac{\beta_j(\phi_t)}{\beta_j(\phi_t^a)}.$$

If $\beta_j(\phi_t) < \lambda_1$ then $\beta_j(\phi_t) \leq \beta_j(\phi_t^a)$ and $\log \frac{\beta_j(\phi_t)}{\beta_j(\phi_t^a)} < 0$.

If $\beta_j(\phi_t) \geq \lambda_1$ then using the Lipschitz continuity of the rates β_j we have

$$\log \frac{\beta_j(\phi_t)}{\beta_j(\phi_t^a)} \leq \log \frac{\beta_j(\phi_t)}{\beta_j(\phi_t) - Cc_1 a} \leq \log \frac{\lambda_1}{\lambda_1 - Cc_1 a}$$

$$\leq \log \frac{1}{1 - Cc_1 a/\lambda_1} < \frac{2Cc_1 a}{\lambda_1} < \frac{2Cc_1 c_2}{\lambda_1}.$$

Since $\log(1/(1-x)) < 2x$ for $0 < x < 1/2$; here, we take a small enough to ensure $Cc_1 a < \lambda_1/2$. Finally for all $a < (\lambda_1/2c_1 C) \wedge \lambda_2$

$$0 \leq f(\mu_t^{j,a}, \beta_j(\phi_t^a)) \leq f(\mu_t^j, \beta_j(\phi_t)) + 2\sigma + \frac{2Cc_1 \lambda_2}{\lambda_1} \mu_t^j.$$

By Lemma 1.8 μ_t^j is integrable, we have bounded $f(\mu_t^{j,a}, \beta_j(\phi_t^a))$ for $0 < a < (\lambda_1/2c_1 C) \wedge \lambda_2$ by an integrable function. Moreover $f(\mu_t^{j,a}, \beta_j(\phi_t^a)) \to f(\mu_t^j, \beta_j(\phi_t))$ since first $I_T(\phi) < \infty$ means that for almost all $t \in [0, T]$ and $1 \leq j \leq k$, $\mu_t^j > 0$ only if $\beta_j(\phi_t) > 0$ and then

$$|f(\mu_t^{j,a}, \beta_j(\phi_t^a)) - f(\mu_t^j, \beta_j(\phi_t))| \leq (1-a)\mu_t^j \log(1-a) + |\beta_j(\phi_t^a) - \beta_j(\phi_t)|$$

$$+ |(1-a)\mu_t^j - \mu_t^j| + \left| (1-a)\mu_t^j \log \frac{\mu_t^j}{\beta_j(\phi_t^a)} - \mu_t^j \log \frac{\mu_t^j}{\beta_j(\phi_t)} \right|.$$

The last term of this inequality is either 0 or converge to 0 when a tend to 0. We deduce from, the dominated convergence theorem that

$$\int_0^T f(\mu_t^{j,a}, \beta_j(\phi_t^a)) dt \to \int_0^T f(\mu_t^j, \beta_j(\phi_t)) dt \quad \text{as} \quad a \to 0,$$

from which (11) follows, hence the result. $\qquad \square$

Lemma 1.10. *Let $a > 0$ and $\phi \in R^a$ such that $I_T(\phi) < \infty$. For all $\eta > 0$ there exists $L > 0$ and $\phi^L \in R^{a/2}$ such that $\|\phi - \phi^L\|_T < c_1 \frac{a}{2}$ and $I_T(\phi^L | \mu^L) \leq I_T(\phi) + \eta$ where $\mu^L \in \mathcal{A}_d(\phi^L)$ such that $\mu_t^{L,j} < L$, $j = 1, ..., k$.*

Proof. Let $\eta > 0$ and $\mu \in \mathcal{A}_d(\phi)$ such that $I_T(\phi | \mu) < I_T(\phi) + \eta/2$. For $L > 0$ let $\mu_t^{L,j} = \mu_t^j \wedge L$ and let ϕ^L a solution of the ODE

$$\frac{d\phi_t^L}{dt} = \sum_{j=1}^k \mu_t^{L,j} h_j.$$

We first show that for L sufficiently large ϕ^L is close to ϕ in supnorm. Since μ_t^j is integrable over $[0, T]$ and $0 \leq \mu_t^{L,j} \leq \mu_t^j$, the monotone convergence theorem implies that there exists $L_a > 0$ such that for all $L > L_a$, $j = 1, ..., k$

$$\int_0^T |\mu_t^{L,j} - \mu_t^j| dt < \epsilon_a = c_2 \frac{a}{2k\sqrt{d}}.$$

We deduce that

$$|\phi_t^{L,i} - \phi_t^i| \leq \sum_{j=1}^k |h_j^i| \int_0^T |\mu_t^{L,j} - \mu_t^j| dt < k\epsilon_a$$

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and then we have for all $L > L_a$ $\|\phi^L - \phi\| < c_1 \frac{a}{2}$ since $c_2 < c_1$. As $\phi \in R^a$ the above also ensures that $\phi^L \in R^{a/2}$ since for all $t \in [0, T]$

$$
\begin{aligned}
\operatorname{dist}(\phi_t^L, \partial A) &\geq \operatorname{dist}(\phi_t, \partial A) - |\phi_t^L - \phi_t| \\
&\geq c_2 a - c_2 \frac{a}{2} = c_2 \frac{a}{2}.
\end{aligned}
$$

To show the convergence of $I_T(\phi^L | \mu^L)$ to $I_T(\phi | \mu)$ we need to remark first using the convexity of $f(\nu, \omega)$ in ν that we have

$$
f(\mu_t^{L,j}, \beta_j(\phi_t^L)) \leq f(0, \beta_j(\phi_t^L)) + f(\mu_t^j, \beta_j(\phi_t^L)).
$$

Since $\phi \in R^a$, $C_a \leq \beta_j(\phi_t) \leq \sigma$ and $C_{a/2} \leq \beta_j(\phi_t^L) \leq \sigma$ for all $L > L_a$, notice that

$$
\frac{\partial f(\nu, \omega)}{\partial \omega} = -\frac{\nu}{\omega} + 1
$$

and therefore on the interval $[K_a, \theta]$ where $K_a = C_a \wedge C_{a/2}$

$$
|f(\mu_t^j, \beta_j(\phi_t^L)) - f(\mu_t^j, \beta_j(\phi_t))| < \bar{C}(\mu_t^j + 1)
$$

for some constant $\bar{C} > 0$. Since μ_t^j and $f(\mu_t^j, \beta_j(\phi_t))$ are integrable the dominated convergence theorem implies that

$$
\int_0^T f(\mu_t^{L,j}, \beta_j(\phi_t^L)) dt \to \int_0^T f(\mu_t^j, \beta_j(\phi_t)) dt \quad \text{as} \quad L \to \infty.
$$

$\qquad \square$

Let $\epsilon > 0$ be such that $T/\epsilon \in \mathbb{N}$ and let the ϕ^ϵ be the polygonal approximation of ϕ defined for $t \in [\ell\epsilon, (\ell+1)\epsilon)$ by

$$
\phi_t^\epsilon = \phi_{\ell\epsilon} \frac{(\ell+1)\epsilon - t}{\epsilon} + \phi_{(\ell+1)\epsilon} \frac{t - \ell\epsilon}{\epsilon}. \tag{12}
$$

Lemma 1.11. *For any $\eta > 0$. Let $0 < a < 1$, $\phi \in R^a$ and $\mu \in \mathcal{A}_d(\phi)$ such that $\mu_t^j < L$, $j = 1, ..., k$ for some $L > 0$ and $I_T(\phi | \mu) < \infty$ then there exists a_η such that for all $a < a_\eta$ there exists an $\epsilon_a > 0$ such that for all $\epsilon < \epsilon_a$ the polygonal approximation $\phi^\epsilon \in R^{a/2}$ and $\|\phi - \phi^\epsilon\|_T < c_2 \frac{a}{2} < c_1 \frac{a}{2}$. Moreover, there exists $\mu^\epsilon \in \mathcal{A}_d(\phi^\epsilon)$ such that $\mu_t^{\epsilon,j} < L$, $j = 1, ..., k$ and $I_T(\phi^\epsilon | \mu^\epsilon) \leq I_T(\phi | \mu) + \eta$.*

Proof. Since ϕ is uniformly continuous on $[0, T]$ there exists an ϵ_a such that $\forall \epsilon < \epsilon_a$

$$
\sup_{|t - t'| < 2\epsilon} |\phi_t - \phi_{t'}| < c_2 \frac{a e^{-a^{-\nu}}}{4}
$$

and then $\|\phi - \phi^\epsilon\|_T < c_2 \frac{a}{2}$ and $\phi^\epsilon \in R^{a/2}$ since for all $t \in [0, T]$,

$$
\begin{aligned}
\operatorname{dist}(\phi_t^\epsilon, \partial A) &\geq \operatorname{dist}(\phi_t, \partial A) - |\phi_t^\epsilon - \phi_t| \\
&\geq \operatorname{dist}(\phi_t, \partial A) - |\phi_{\ell\epsilon} - \phi_t| - |\phi_{(\ell+1)\epsilon} - \phi_t| \geq c_2 \frac{a}{2}.
\end{aligned}
$$

For $t \in]\ell\epsilon, (\ell+1)\epsilon[$

$$\frac{d\phi_t^\epsilon}{dt} = \frac{\phi_{(\ell+1)\epsilon} - \phi_{\ell\epsilon}}{\epsilon} = \frac{1}{\epsilon} \sum_{j=1}^{k} h_j \int_{\ell\epsilon}^{(\ell+1)\epsilon} \mu_t^j dt$$

therefore for all $t \in [\ell\epsilon, (\ell+1)\epsilon[$, μ_t^ϵ defined by

$$\mu_t^{\epsilon,j} = \frac{1}{\epsilon} \int_{\ell\epsilon}^{(\ell+1)\epsilon} \mu_t^j dt, j = 1, ..., k$$

is such that $\mu^\epsilon \in \mathcal{A}_d(\phi^\epsilon)$ and is constant over $[\ell\epsilon, (\ell+1)\epsilon[$. We also note that $\mu_t^{\epsilon,j} \leq L$ for all $j = 1, ..., k$. Moreover if $0 < \nu \leq L$ and $\omega \geq C_a$ then

$$\left| \frac{\partial f(\nu, \omega)}{\partial \omega} \right| = |-\frac{\nu}{\omega} + 1| \leq \frac{L}{C_a} + 1.$$

By the assumption 0.3 4, there exists $\tilde{a}_\eta > 0$ such that for all $a < \tilde{a}_\eta$

$$\frac{L}{C_a} + 1 \leq Le^{a^{-\nu}} + 1$$

Then for $t \in [\ell\epsilon, (\ell+1)\epsilon[$ and $a < \bar{a}_\eta, \tilde{a}_\eta$

$$|f(\mu_t^{\epsilon,j}, \beta_j(\phi_t^\epsilon)) - f(\mu_t^{\epsilon,j}, \beta_j(\phi_{\ell\epsilon}))| \leq \frac{1}{2} C(L+1)a = Va$$

$$|f(\mu_t^j, \beta_j(\phi_t)) - f(\mu_t^j, \beta_j(\phi_{\ell\epsilon}))| \leq \frac{1}{2} C(L+1)a = Va.$$

The above imply that

$$\begin{aligned}
\int_{\ell\epsilon}^{(\ell+1)\epsilon} f(\mu_t^{\epsilon,j}, \beta_j(\phi_t^\epsilon))dt &\leq \int_{\ell\epsilon}^{(\ell+1)\epsilon} f(\mu_t^{\epsilon,j}, \beta_j(\phi_{\ell\epsilon}))dt + \epsilon Va \\
&= \epsilon f(\mu_{\ell\epsilon}^{\epsilon,j}, \beta_j(\phi_{\ell\epsilon})) + \epsilon Va \\
&\leq \int_{\ell\epsilon}^{(\ell+1)\epsilon} f(\mu_t^j, \beta_j(\phi_{\ell\epsilon}))dt + \epsilon Va \\
&\leq \int_{\ell\epsilon}^{(\ell+1)\epsilon} f(\mu_t^j, \beta_j(\phi_t))dt + 2Va\epsilon
\end{aligned}$$

where the second inequality follows from Jensen's inequality. Therefore

$$I_T(\phi^\epsilon|\mu^\epsilon) \leq I_T(\phi|\mu) + 2VTa$$

We can now choose $a < \min\{\bar{a}_\eta, \tilde{a}_\eta, \eta/2VT\}$ to have our result. $\qquad\square$

The next lemma states a large deviation estimate for Poisson random variables.

Lemma 1.12. *Let $Y_1, Y_2, ...$ be independent Poisson random variables with mean $\sigma\epsilon$. For all $N \in \mathbb{N}$, let*

$$\bar{Y}^N = \frac{1}{N} \sum_{n=0}^{N} Y_n.$$

For any $s > 0$ there exist $K, \epsilon_0 > 0$ and $N_0 \in \mathbb{N}$ such that taking $g(\epsilon) = K\sqrt{\log^{-1}(\epsilon^{-1})}$ we have

$$\mathbb{P}^N(\bar{Y}^N > g(\epsilon)) < \exp\{-sN\}$$

for all $\epsilon < \epsilon_0$ and $N > N_0$.

Proof. We apply the Gramer's theorem see e.g [3] (chapter 2)

$$\limsup_{N \to \infty} \frac{1}{N} \log(\mathbb{P}^N(\bar{Y}^N > g(\epsilon))) \leq -\inf_{x \geq g(\epsilon)} \Lambda_\epsilon^*(x)$$

where $\Lambda_\epsilon^*(x) = \sup_{\lambda \in \mathbb{R}} \{\lambda x - \Lambda_\epsilon(\lambda)\}$ with

$$\Lambda_\epsilon(\lambda) = \log(\mathbb{E}(e^{\lambda Y_1}) = \sigma\epsilon(e^\lambda - 1).$$

We deduce that

$$\Lambda_\epsilon^*(x) = x \log \frac{x}{\sigma\epsilon} - x + \sigma\epsilon.$$

This last function is convex It reaches its infimum at $x = \sigma\epsilon$ and as $\lim_{\epsilon \to 0} \frac{g(\epsilon)}{\sigma\epsilon} = +\infty$ there exists $\epsilon_1 > 0$ such that $g(\epsilon) > \sigma\epsilon$ for all $\epsilon < \epsilon_1$ and then

$$\inf_{x \geq g(\epsilon)} \Lambda_\epsilon^*(x) = g(\epsilon) \log \frac{g(\epsilon)}{\sigma\epsilon} - g(\epsilon) + \sigma\epsilon$$

$$= g(\epsilon) \log(g(\epsilon)) - g(\epsilon) \log(\sigma\epsilon) - g(\epsilon) + \sigma\epsilon$$

$$\approx K\sqrt{\log(1/\epsilon)} \to \infty \quad \text{as} \quad \epsilon \to 0.$$

Then there exists $\epsilon_2 > 0$ such that $\inf_{x \geq g(\epsilon)} \Lambda_\epsilon^*(x) > s$ for all $\epsilon < \epsilon_2$.

Taking $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$, we have the lemma. $\qquad\square$

2   The Lower Bound

We first prove that for $z \in A$, $\phi \in D_{T,A}$, $\phi_0 = z$ and any $\eta > 0$, $\delta > 0$ there exist $\tilde{\delta} > 0$ and $N_{\eta,\delta}$, such that for all y, $|y - z| < \tilde{\delta}$ and any $N > N_{\eta,\delta}$, we have

$$\mathbb{P}_y(\|Z^N - \phi\|_T < \delta) \geq \exp\{-N(I_T(\phi) + \eta)\}. \tag{13}$$

To this end, it is enough to prove (13) considering $\phi \in \mathcal{AC}_{T,A}$ because the inequality is true when $I_T(\phi) = \infty$. We apply some lemmas of the preceding section to show that it is enough to consider some suitable paths ϕ with the $\mu \in \mathcal{A}_d(\phi)$.

The goal of the next lemma is to establish a crucial inequality to deduce (13).

Lemma 2.1. *For $z \in A$, $\phi \in \mathcal{AC}_{T,A}$, $\phi_0 = z$, there exists a_0 such that for any $a < a_0$, $\epsilon > 0$ the polygonal approximation ϕ^ϵ of ϕ^a defined by*

$$\phi_t^\epsilon = \phi_{\ell\epsilon}^a \frac{(\ell+1)\epsilon - t}{\epsilon} + \phi_{(\ell+1)\epsilon}^a \frac{t - \ell\epsilon}{\epsilon} \quad \forall t \in [\ell\epsilon, (\ell+1)\epsilon[, \tag{14}$$

satisfies the following assertion:

For any $\mu^\epsilon \in \mathcal{A}_d(\phi^\epsilon)$ constant over the time intervals $[\ell\epsilon, (\ell+1)\epsilon[$ and bounded above by some constant $L > 0$, any $\eta > 0$ and suitable small $\delta > 0$ there exist $0 < \tilde{\delta} < \delta$ and $N_{\eta,\delta,\tilde{\delta}} \in \mathbb{N}$ such that for all y, $|y - z| < \tilde{\delta}$ and any $N > N_{\eta,\delta,\tilde{\delta}}$

$$\mathbb{P}_y(\|Z^N - \phi^\epsilon\|_T < \delta) \geq \exp\{-N(I_T(\phi^\epsilon|\mu^\epsilon) + \eta)\}.$$

Proof. Note that μ^ϵ can be chosen as in Lemma 1.11. We define the events B_j, $j = 1, ..., k$ for controlling the likelihood ratio. For $\gamma > 0$ let

$$B_j = \Big\{ \Big| \sum_{p=1}^Q \delta_p(j) \log \Big(\frac{\beta_j(Z^N(\tau_p^-))}{\mu_{\lfloor \tau_p/\epsilon \rfloor \epsilon}^{\epsilon,j}} \Big) - N \sum_{\ell=1}^{T/\epsilon} \mu_{\ell\epsilon}^{\epsilon,j} \log \Big(\frac{\beta_j(\phi_{\ell\epsilon}^\epsilon)}{\mu_{\ell\epsilon}^{\epsilon,j}} \Big) \epsilon \Big| \leq N\gamma \Big\}$$

Where Q was introduced first above theorem 1.2.

In what follows we put $\tilde{\beta}_j(Z^N(t)) = \mu_t^{\epsilon,j}$ and we have on $\{\|Z^N - \phi^\epsilon\|_T < \delta\} \cap (\bigcap_{j=1}^k B_j) = \{\|Z^N - \phi^\epsilon\|_T < \delta\} \cap B$, ξ_T defined by (10),

$$\xi_T^{-1} = \exp\Big\{ \sum_{p=1}^Q \sum_{j=1}^k \delta_p(j) \log \Big(\frac{\beta_j(Z^N(\tau_p^-))}{\mu_{\tau_p^-}^{\epsilon,j}} \Big) + N \int_0^T \sum_{j=1}^k (\mu_t^{\epsilon,j} - \beta_j(Z^N(t))) dt \Big\}$$

$$\geq \exp\Big\{ -N \sum_{\ell=1}^{T/\epsilon} \sum_{j=1}^k \mu_{\ell\epsilon}^{\epsilon,j} \log \Big(\frac{\mu_{\ell\epsilon}^{\epsilon,j}}{\beta_j(\phi_{\ell\epsilon}^\epsilon)} \Big) \epsilon + N \int_0^T \sum_{j=1}^k (\mu_t^{\epsilon,j} - \beta_j(Z^N(t))) dt - kN\gamma \Big\}$$

$$\geq \exp\Big\{ -N \sum_{\ell=1}^{T/\epsilon} \sum_{j=1}^k \mu_{\ell\epsilon}^{\epsilon,j} \log \Big(\frac{\mu_{\ell\epsilon}^{\epsilon,j}}{\beta_j(\phi_{\ell\epsilon}^\epsilon)} \Big) \epsilon + N \int_0^T \sum_{j=1}^k (\mu_t^{\epsilon,j} - \beta_j(\phi_t^\epsilon)) dt - N(kTC\delta + k\gamma) \Big\}$$

We note here that the first inequality is true because the $\mu_t^{\epsilon,j}$ is constant on the intervals $[\ell\epsilon, (\ell+1)\epsilon[$ and the second one come from the Lipschitz continuity of the rates β_j. Since the derivative of ϕ^ϵ is bounded, we have

$$\xi_T^{-1} \geq \exp\Big\{ -N \int_0^T \sum_{j=1}^k \Big[\mu_t^{\epsilon,j} \log \Big(\frac{\mu_t^{\epsilon,j}}{\beta_j(\phi_t^\epsilon)} \Big) - \mu_t^{\epsilon,j} + \beta_j(\phi_t^\epsilon) \Big] dt - N[kTC(\delta + \epsilon) + k\gamma] \Big\}$$

$$\geq \exp\Big\{ -N \big(I_T(\phi^\epsilon | \mu^\epsilon) + [kTC(\delta + \epsilon) + k\gamma] \big) \Big\} \quad \text{on the event} \quad \{\|Z^N - \phi^\epsilon\|_T < \delta\} \cap B.$$

Then for any $\eta > 0$, there exists $\delta > 0$ $\epsilon > 0$ and $\gamma > 0$ such that for N large enough we have

$$\xi_T^{-1} \geq \exp\{-N(I_T(\phi^\epsilon | \mu^\epsilon) + \eta/2)\}$$

Moreover from corollary 1.3

$$\mathbb{P}_y(\|Z^N - \phi^\epsilon\|_T < \delta) \geq \widetilde{\mathbb{E}}\Big(\xi_T^{-1} \cdot \mathbf{1}_{\{\|Z^N - \phi^\epsilon\|_T < \delta\}} \Big)$$

$$\geq \widetilde{\mathbb{E}}_y\Big(\xi_T^{-1} \cdot \mathbf{1}_{\{\{\|Z^N - \phi^\epsilon\|_T < \delta\} \cap B\}} \Big)$$

$$\geq \exp\{-N(I_T(\phi^\epsilon | \mu^\epsilon) + \eta/2)\} \widetilde{\mathbb{P}}_y(\{\|Z^N - \phi^\epsilon\|_T < \delta\} \cap B)$$

$\square$

To conclude this proof it is enough to establish the following lemma:

Lemma 2.2. *For $z \in A$, $\phi \in \mathcal{AC}_{T,A}$, $\phi_0 = z$, there exists a_0 such that for any $a < a_0$, $\epsilon > 0$ the polygonal approximation ϕ^ϵ of ϕ^a defined by (14) has the property that there exists $\tilde{\delta} > 0$ such that for all y, $|y - z| < \tilde{\delta}$*

$$\lim_{N \to \infty} \widetilde{\mathbb{P}}_y(\{\|Z^N - \phi^\epsilon\|_T < \delta\} \cap B) = 1$$

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Proof. It is enough to prove that $\lim_{N\to\infty} \widetilde{\mathbb{P}}_y(\|Z^N - \phi^\epsilon\|_T < \delta) = 1$ and that for all $1 \leq j \leq k$, $\lim_{N\to\infty} \widetilde{\mathbb{P}}_y(\{\|Z^N - \phi^\epsilon\|_T < \delta\} \cap B_j^c) = 0$. The first limit follows from Theorem 1.1 for processes under the probability $\widetilde{\mathbb{P}}_y$ provided that we choose a_0 and $\tilde{\delta} < \delta/2$ in suitable way. We now establish that $\widetilde{\mathbb{P}}_y(\|Z^N - \phi^\epsilon\|_T < \delta \cap B_j^c) \to 0$ as $N \to \infty$, for any $1 \leq j \leq k$.

We have $\sup_p |Z^N(\tau_p) - \phi^\epsilon_{\tau_p}| < \delta$ on $\{\|Z^N - \phi^\epsilon\|_T < \delta\}$ and we can choose ϵ small enough such that $\sup_p |\phi^\epsilon_{\tau_p} - \phi^\epsilon_{\lfloor \tau_p/\epsilon \rfloor \epsilon}| < \delta$ and thus $\sup_p |Z^N(\tau_p) - \phi^\epsilon_{\lfloor \tau_p/\epsilon \rfloor \epsilon}| < 2\delta$.

Note that we have on $\{\|Z^N - \phi^\epsilon\|_T < \delta\}$

$$\Big| \sum_{p=1}^{Q} \delta_p(j) \log\Big(\frac{\beta_j(Z^N(\tau_p^-))}{\mu^{\epsilon,j}_{\lfloor \tau_p/\epsilon \rfloor \epsilon}}\Big) - \sum_{p=1}^{Q} \delta_p(j) \log\Big(\frac{\beta_j(\phi^\epsilon_{\lfloor \tau_p/\epsilon \rfloor \epsilon})}{\mu^{\epsilon,j}_{\lfloor \tau_p/\epsilon \rfloor \epsilon}}\Big)\Big| \leq \Big| \sum_{p=1}^{Q} \delta_p(j) \log\Big(\frac{\beta_j(Z^N(\tau_p^-))}{\beta_j(\phi^\epsilon_{\lfloor \tau_p/\epsilon \rfloor \epsilon})}\Big)\Big|$$
$$\leq \frac{2CQ\delta}{C_a}$$

since $|\beta_j(Z^N(\tau_p^-)) - \beta_j(\phi^\epsilon_{\lfloor \tau_p/\epsilon \rfloor \epsilon})| < 2C\delta$. Let m_ℓ the number of jumps in the interval $[(\ell-1)\epsilon, \ell\epsilon[$ we have

$$\Big| \sum_{p=1}^{Q} \delta_p(j) \log\Big(\frac{\beta_j(Z^N(\tau_p^-))}{\mu^{\epsilon,j}_{\lfloor \tau_p/\epsilon \rfloor \epsilon}}\Big) - N \sum_{\ell=1}^{T/\epsilon} \mu^{\epsilon,j}_{\ell\epsilon} \log\Big(\frac{\beta_j(\phi^\epsilon_{\ell\epsilon})}{\mu^{\epsilon,j}_{\ell\epsilon}}\Big)\epsilon\Big|$$
$$\leq \Big| \sum_{p=1}^{Q} \delta_p(j) \log\Big(\frac{\beta_j(\phi^\epsilon_{\lfloor \tau_p/\epsilon \rfloor \epsilon})}{\mu^{\epsilon,j}_{\lfloor \tau_p/\epsilon \rfloor \epsilon}}\Big) - N \sum_{\ell=1}^{T/\epsilon} \mu^{\epsilon,j}_{\ell\epsilon} \log\Big(\frac{\beta_j(\phi^\epsilon_{\ell\epsilon})}{\mu^{\epsilon,j}_{\ell\epsilon}}\Big)\epsilon\Big|$$
$$+ \Big| \sum_{p=1}^{Q} \delta_p(j) \log\Big(\frac{\beta_j(Z^N(\tau_p^-))}{\mu^{\epsilon,j}_{\lfloor \tau_p/\epsilon \rfloor \epsilon}}\Big) - \sum_{p=1}^{Q} \delta_p(j) \log\Big(\frac{\beta_j(\phi^\epsilon_{\lfloor \tau_p/\epsilon \rfloor \epsilon})}{\mu^{\epsilon,j}_{\lfloor \tau_p/\epsilon \rfloor \epsilon}}\Big)\Big|$$
$$\leq \Big| \sum_{\ell=1}^{T/\epsilon} \log\Big(\frac{\beta_j(\phi^\epsilon_{\ell\epsilon})}{\mu^{\epsilon,j}_{\ell\epsilon}}\Big)\Big(\sum_{p=1}^{m_\ell} \delta_p(j) - N\mu^{\epsilon,j}_{\ell\epsilon}\epsilon\Big)\Big| + \frac{2CQ\delta}{C_a}.$$

As the rate of jumps are constant on the interval $[(\ell-1)\epsilon, \ell\epsilon[$ under $\widetilde{\mathbb{P}}^N$, $\sum_{p=1}^{m_\ell} \delta_p(j)$ is the number of jumps of a Poisson process P_j on this interval. So it is a Poisson random variable with mean $N\mu^{\epsilon,j}_{\ell\epsilon}\epsilon$. We deduce from Chebyshev's inequality that

$$\widetilde{\mathbb{P}}_y\Big(\Big| \log\Big(\frac{\beta_j(\phi^\epsilon_{\ell\epsilon})}{\mu^{\epsilon,j}_{\ell\epsilon}}\Big)\Big(\sum_{p=1}^{m_\ell} \delta_p(j) - N\mu^{\epsilon,j}_{\ell\epsilon}\epsilon\Big)\Big| > \frac{N\gamma\epsilon}{2T}\Big) \leq \frac{4T^2 \sup_{\ell \leq T/\epsilon}\Big(\log^2\Big(\frac{\beta_j(\phi^\epsilon_{\ell\epsilon})}{\mu^{\epsilon,j}_{\ell\epsilon}}\Big)N\mu^{\epsilon,j}_{\ell\epsilon}\epsilon\Big)}{N^2\gamma^2\epsilon^2}.$$

As $C_a \leq \beta_j(\phi^\epsilon_t) \leq \sigma$ and $\mu^{\epsilon,j}_t \leq L$ we have $\sup_{\ell \leq T/\epsilon}\Big(\log^2\Big(\frac{\beta_j(\phi^\epsilon_{\ell\epsilon})}{\mu^{\epsilon,j}_{\ell\epsilon}}\Big)\mu^{\epsilon,j}_{\ell\epsilon}\Big) \leq C(L,a)$. Thus

$$\widetilde{\mathbb{P}}_y(\|Z^N - \phi^\epsilon\|_T < \delta\} \cap B_j^c) \leq \widetilde{\mathbb{P}}_y\Big(\Big| \sum_{\ell=1}^{T/\epsilon} \log\Big(\frac{\beta_j(\phi^\epsilon_{\ell\epsilon})}{\mu^{\epsilon,j}_{\ell\epsilon}}\Big)\Big(\sum_{p=1}^{m_\ell} \delta_p(j) - N\mu^{\epsilon,j}_{\ell\epsilon}\epsilon\Big)\Big| + \frac{2CQ\delta}{C_a} > N\gamma\Big)$$
$$\leq \frac{4T^2 C(L,a)}{N\gamma^2\epsilon} + \widetilde{\mathbb{P}}_y\Big(\frac{2CQ\delta}{C_a} \geq \frac{N\gamma}{2}\Big).$$

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The number of jumps during the period time T under the probability $\widetilde{\mathbb{P}}_z$ is the sum of T/ϵ Poisson random variables with mean $N \sum_{j=1}^{k} \mu_{\ell\epsilon}^{\epsilon,j} \epsilon$. we take $\gamma = \frac{8C\delta}{C_a} \sum_{\ell=1}^{T/\epsilon} \sum_{j=1}^{k} \mu_{\ell\epsilon}^{\epsilon,j} \epsilon$ where δ is chosen such that δ/C_a is small. Therefore, as long as $\sum_{\ell=1}^{T/\epsilon} \sum_{j=1}^{k} \mu_{\ell\epsilon}^{\epsilon,j} > 0$, the law of large number for Poisson variables give us

$$\widetilde{\mathbb{P}}_y\Big(\frac{2CQ\delta}{C_a} \geq \frac{N\gamma}{2}\Big) = \widetilde{\mathbb{P}}_y\Big(\frac{Q}{N} \geq 2 \sum_{\ell=1}^{T/\epsilon} \sum_{j=1}^{k} \mu_{\ell\epsilon}^{\epsilon,j} \epsilon\Big) \to 0$$

as $N \to \infty$. $\qquad\square$

We now deduce from Lemma 2.1 the next result follows the argument from in the proof of Lemma 3 in [4].

Proposition 2.3. *For $z \in A$, $\phi \in \mathcal{AC}_{T,A}$, $\phi_0 = z$ and any $\eta > 0$, $\delta > 0$ there exist $\tilde{\delta} > 0$ and $N_{\eta,\delta}$ such that for all $N > N_{\eta,\delta}$,*

$$\inf_{y:|y-z|<\tilde{\delta}} \mathbb{P}_y(\|Z^N - \phi\|_T < \delta) \geq \exp\{-N(I_T(\phi) + \eta)\}. \tag{15}$$

Proof. For $\delta, \eta > 0$ let $\phi \in \mathcal{AC}_{T,A}$, $\phi_0 = z$ such that $I_T(\phi) < \infty$ then using Lemma 1.9 we have that there exists $a_\eta > 0$ such that for all $a < a_\eta$ there exists $\phi^a \in R^a$ such that $\|\phi - \phi^a\|_T < c_1 a$ and $I_T(\phi^a) \leq I_T(\phi) + \eta/4$. As $I_T(\phi^a) < \infty$ using the lemma 1.10 we deduce that there exists $L > 0$ and $\phi^{a,L} \in R^{a/2}$ is such that $\|\phi^a - \phi^{a,L}\|_T < c_1 \frac{a}{2}$ and $I_T(\phi^{a,L}|\mu^{a,L}) \leq I_T(\phi^a) + \eta/4$ where $\mu^{a,L} \in \mathcal{A}_d(\phi^{a,L})$ such that $\mu_t^{a,L,j} < L$, $j = 1, ..., k$. Now we can deduce from Lemma 1.11 that for all $\epsilon > 0$ the polygonal approximation $\phi^{a,L,\epsilon}$ of $\phi^{a,L}$ satisfies $\|\phi^{a,L} - \phi^{a,L,\epsilon}\|_T < c_1 \frac{a}{2}$ and $I_T(\phi^{a,L,\epsilon}|\mu^{a,L,\epsilon}) \leq I_T(\phi^{a,L}|\mu^{a,L}) + \eta/4$ where $\mu^{a,L,\epsilon} \in \mathcal{A}_d(\phi^{a,L,\epsilon})$ is such that $\mu_t^{a,L,\epsilon,j} < L$, $j = 1, ..., k$. Now we choose a such that $2c_1 a < \delta/2$ and we have

$$\begin{aligned}
\inf_{y:|y-z|<\tilde{\delta}} \mathbb{P}_y\Big(\|Z^N - \phi\|_T < \delta\Big) &\geq \inf_{y:|y-z|<\tilde{\delta}} \mathbb{P}_y\Big(\|Z^N - \phi^a\|_T < \frac{\delta}{2} + c_1 a\Big) \\
&\geq \inf_{y:|y-z|<\tilde{\delta}} \mathbb{P}_y\Big(\|Z^N - \phi^{a,L}\|_T < \frac{\delta}{2} + c_1 \frac{a}{2}\Big) \\
&\geq \inf_{y:|y-z|<\tilde{\delta}} \mathbb{P}_y\Big(\|Z^N - \phi^{a,L,\epsilon}\|_T < \frac{\delta}{2}\Big) \\
&\geq \exp\{-N(I_T(\phi^{a,L,\epsilon}|\mu^{a,L,\epsilon}) + \eta/4)\} \\
&\geq \exp\{-N(I_T(\phi^{a,L}|\mu^{a,L}) + \eta/2)\} \\
&\geq \exp\{-N(I_T(\phi^a) + 3\eta/4)\} \\
&\geq \exp\{-N(I_T(\phi) + \eta)\}
\end{aligned}$$

where we have used the lemma 2.1 at the 5th inequality. $\qquad\square$

We finish the proof of the lower bound by the following theorem

Theorem 2.4. *For any open subset G of $D_{T,A}$ and $z \in A$,*

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}_z^N(G) \geq - \inf_{\phi \in G, \phi_0 = z} I_T(\phi). \tag{16}$$

Proof. Note that in fact (13) and (16) are equivalent. We only have to show that (16) follows from (13). To this end let $I = \inf_{\phi \in G, \phi_0 = z} I_T(\phi) < \infty$ then, for $\eta > 0$ there exists a $\phi^\eta \in G$, $\phi_0^\eta = z$ such that $I_T(\phi^\eta) \leq I + \eta$. Moreover we can choose $\delta = \delta(\phi^\eta)$ small enough such that $\{\|Z^N - \phi^\eta\|_T < \delta\} \subset G$. And then $\mathbb{P}_z(\|Z^N - \phi^\eta\|_T < \delta) \leq \mathbb{P}_z^N(G)$. This implies from the inequality (13) that for all $\eta > 0$,

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}_z^N(G) \geq \liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}_z(\|Z^N - \phi^\eta\|_T < \delta)$$
$$\geq -I_T(\phi^\eta)$$
$$\geq -I - \eta$$

and then

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}_z^N(G) \leq -I.$$

$\square$

Corollary 2.5. *For any open subset G of $D_{T,A}$ and any compact subset K of A,*

$$\liminf_{N \to \infty} \frac{1}{N} \log \inf_{z \in K} \mathbb{P}_z(Z^N \in G) \geq -\sup_{z \in K} \inf_{\phi \in G, \phi_0 = z} I_T(\phi).$$

Proof. The arguments are the same as in the proof of Corollary 5.6.15 in [3]. Let

$$I_K := \sup_{z \in K} \inf_{\phi \in G, \phi_0 = z} I_T(\phi).$$

For $\eta > 0$ fix, let $I_K^\eta := \max\{I_K + \eta, \eta^{-1}\}$. Then from Proposition 2.3 it follows that for any $z \in K$, there exists a N_z such that for all $N > N_z$ and $y \in B(z, \frac{1}{N_z})$,

$$\frac{1}{N} \log \mathbb{P}_y(Z^N \in G) \geq -\inf_{\phi \in G, \phi_0 = z} I_T(\phi) \geq -I_K^\eta.$$

And then

$$\frac{1}{N} \log \inf_{y \in B(z, \frac{1}{N_z})} \mathbb{P}_y(Z^N \in G) \geq -I_K^\eta.$$

As K is compact, there exits a finite sequence $(z_i)_{1 \leq i \leq m} \subset K$ such that $K \subset \bigcup_{i=1}^m B(z_i, \frac{1}{N_{z_i}})$. Then for $N \geq \max_{1 \leq i \leq m} N_{z_i}$,

$$\frac{1}{N} \log \inf_{y \in K} \mathbb{P}_y(Z^N \in G) \geq -I_K^\eta.$$

It first remains to take $\liminf$ as $N \to \infty$ and then let η tend to 0 to have result. $\square$

3 The Upper Bound

For all $\phi \in D_{T,A}$, and $F \subset D_{T,A}$ we define

$$\rho_T(\phi, F) = \inf_{\psi \in F} \|\phi - \psi\|_T. \tag{17}$$

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For $z \in A$, $\delta, s > 0$ we define the set

$$F_\delta^s = \{\phi \in D_{T,A} : \rho_T(\phi, \Phi(s)) \geq \delta\},$$

where $\Phi(s) = \{\psi \in D_{T,A} : I_T(\psi) \leq s\}$.

We start by proving the following Proposition which will be enough to conclude the upper bound.

Proposition 3.1. *For $z \in A$, δ, η and $s > 0$ there exists $N_0 \in \mathbb{N}$ such that*

$$\mathbb{P}_z^N(F_\delta^s) \leq \exp\{-N(s - \eta)\} \tag{18}$$

whenever $N \geq N_0$.

Proof. Let $Z_a^N(t) = (1-a)Z^N(t) + az_0$ then $\|Z^N - Z_a^N\| < c_1 a$ and for all $c_1 a < \delta(d-1)/d$ we have

$$\mathbb{P}_z^N(F_\delta^s) = \mathbb{P}_z\Big(\rho_T(Z^N, \Phi(s)) \geq \delta\Big)$$
$$\leq \mathbb{P}_z\Big(\rho_T(Z_a^N, \Phi(s)) \geq \frac{\delta}{d}\Big). \tag{19}$$

We now approximate the paths Z^N by smoother paths. Let $\epsilon > 0$ be such that $T/\epsilon \in \mathbb{N}$. We construct a polygonal approximation of Z_a^N defined for all $t \in [\ell\epsilon, (\ell+1)\epsilon[$ by

$$\Upsilon_t = \Upsilon_t^{a,\epsilon} = Z_a^N(\ell\epsilon)\frac{(\ell+1)\epsilon - t}{\epsilon} + Z_a^N((\ell+1)\epsilon)\frac{t - \ell\epsilon}{\epsilon}.$$

The event $\{\|Z_a^N - \Upsilon\|_T < \frac{\delta}{2d}\} \cap \{\rho_T(Z_a^N, \Phi(s)) \geq \frac{\delta}{d}\}$ is contained in $\{\rho_T(\Upsilon, \Phi(s)) \geq \frac{\delta}{2d}\}$ and

$$\mathbb{P}_z\Big(\rho_T(Z_a^N, \Phi(s)) \geq \frac{\delta}{d}\Big) = \mathbb{P}_z\Big(\rho_T(Z_a^N, \Phi(s)) \geq \frac{\delta}{d}; \|Z_a^N - \Upsilon\|_T < \frac{\delta}{2d}\Big)$$
$$+ \mathbb{P}_z\Big(\rho_T(Z_a^N, \Phi(s)) \geq \frac{\delta}{d}; \|Z_a^N - \Upsilon\|_T \geq \frac{\delta}{2d}\Big)$$
$$\leq \mathbb{P}_z\Big(\rho_T(\Upsilon, \Phi(s)) \geq \frac{\delta}{2d}\Big) + \mathbb{P}_z\Big(\|Z_a^N - \Upsilon\|_T \geq \frac{\delta}{2d}\Big)$$
$$\leq \mathbb{P}_z(I_T(\Upsilon) \geq s) + \mathbb{P}_z\Big(\|Z_a^N - \Upsilon\|_T \geq \frac{\delta}{2d}\Big) \tag{20}$$

We now bound $\mathbb{P}_z(I_T(\Upsilon) \geq s)$. For any choice $\mu \in \mathcal{A}_d(\Upsilon)$ we have $I_T(\Upsilon) \leq I_T(\Upsilon|\mu)$ and

$$\mathbb{P}_z(I_T(\Upsilon) \geq s) \leq \mathbb{P}_z(I_T(\Upsilon|\mu) \geq s). \tag{21}$$

Let μ_t^j, $j = 1, ..., k$ be constant on the intervals $[\ell\epsilon, (\ell+1)\epsilon[$ and equal to

$$\mu_t^j = \frac{1-a}{N\epsilon}\Big[P_j\Big(N\int_0^{(\ell+1)\epsilon} \beta_j(Z^N(s)ds\Big) - P_j\Big(N\int_0^{\ell\epsilon} \beta_j(Z^N(s)ds\Big)\Big] \tag{22}$$

Since Υ is piecewise linear, for $t \in]\ell\epsilon, (\ell+1)\epsilon[$

$$\frac{d\Upsilon_t}{dt} = \frac{(1-a)}{\epsilon}(Z^N((\ell+1)\epsilon) - Z^N(\ell\epsilon)) = \sum_{j=1}^k \mu_t^j h_j.$$

Then the μ_t^j given by (22) belong to $\mathcal{A}_d(\Upsilon)$.

To control the change in Υ over the intervals of length ϵ define $g(\epsilon) = K\sqrt{\log^{-1}(\epsilon^{-1})}$ where $K > 0$ is fixed, and define a collection of events $B = \{B_\epsilon\}_{\epsilon > 0}$

$$B_\epsilon = \bigcap_{\ell=0}^{T/\epsilon - 1} B_\epsilon^\ell$$

where

$$B_\epsilon^\ell = \left\{ \sup_{\ell\epsilon \leq t_1, t_2 \leq (\ell+1)\epsilon} |Z_i^N(t_1) - Z_i^N(t_2)| \leq g(\epsilon) \quad \text{for} \quad i = 1, ..., d \right\}.$$

We have

$$\mathbb{P}_z(I_T(\Upsilon|\mu) > s) \leq \mathbb{P}_z(\{I_T(\Upsilon|\mu) > s\} \cap B_\epsilon) + \mathbb{P}(B_\epsilon^c). \tag{23}$$

Combining (19), (20), (21) and (23) we deduce that

$$\mathbb{P}_z^N(F_\delta^s) \leq \mathbb{P}_z(\{I_T(\Upsilon|\mu) > s\} \cap B_\epsilon) + \mathbb{P}(B_\epsilon^c) + \mathbb{P}_z\left(\|Z_a^N - \Upsilon\|_T \geq \frac{\delta}{2d}\right). \tag{24}$$

Now, we find appropriate upper bounds for the three terms in the right side of (24).

Lemma 3.2. *For any $s > 0$ there exists $\epsilon_0 > 0$, $N_0 \in \mathbb{N}$ and $K > 0$ such that*

$$\mathbb{P}_z(B_\epsilon^c) < \frac{dkT}{\epsilon} \exp\{-sN\} \tag{25}$$

for all $\epsilon < \epsilon_0$ and $N > N_0$ where $g(\epsilon) = K\sqrt{\log^{-1}(\epsilon^{-1})}$.

Proof. For all $j = 1, ..., k$ and $\ell = 1, ..., T/\epsilon$ we can write

$$\int_0^{(\ell+1)\epsilon} \beta_j(Z_s^N)ds < \int_0^{\ell\epsilon} \beta_j(Z_s^N)ds + \sigma\epsilon.$$

Moreover, we have

$$B_\epsilon^c = \bigcup_{i=1,...,d} \bigcup_{\ell=1,...,T/\epsilon} \left\{ \sup_{(\ell-1)\epsilon \leq t_1, t_2 \leq \ell\epsilon} |Z_i^N(t_1) - Z_i^N(t_2)| > g(\epsilon) \right\}.$$

Thus

$$\mathbb{P}_z(B_\epsilon^c) \leq \sum_{i=1}^{d} \sum_{\ell=1}^{T/\epsilon} \mathbb{P}\left\{ \sup_{(\ell-1)\epsilon \leq t_1, t_2 \leq \ell\epsilon} |Z_i^N(t_1) - Z_i^N(t_2)| > g(\epsilon) \right\}.$$

Using (1) and denoting by $Z_i^N(.)$ the ith coordinate of $Z^N(.)$ we have, since $|h_j^i| \leq 1$ for

all $1 \leq j \leq k$, $1 \leq i \leq d$,

$$\sup_{(\ell-1)\epsilon \leq t_1, t_2 \leq \ell\epsilon} |Z_i^N(t_1) - Z_i^N(t_2)|$$

$$= \sup_{(\ell-1)\epsilon \leq t_1, t_2 \leq \ell\epsilon} \Big| \sum_j \frac{h_j^i}{N} \Big[P_j \Big(N \int_0^{t_1} \beta_j(Z^N(s))ds \Big) - P_j \Big(N \int_0^{t_2} \beta_j(Z^N(s))ds \Big) \Big] \Big|$$

$$\leq \frac{1}{N} \sum_j \Big[P_j \Big(N \int_0^{\ell\epsilon} \beta_j(Z^N(s))ds \Big) - P_j \Big(N \int_0^{(\ell-1)\epsilon} \beta_j(Z^N(s))ds \Big) \Big]$$

$$\leq \frac{1}{N} \sum_j \Big[P_j \Big(N \int_0^{(\ell-1)\epsilon} \beta_j(Z^N(s))ds + N\sigma\epsilon \Big) - P_j \Big(N \int_0^{(\ell-1)\epsilon} \beta_j(Z^N(s))ds \Big) \Big]$$

$$\leq \frac{1}{N} \sum_j Z_j.$$

Where Z_j $j = 1, ..., k$ are independent Poisson random variables with mean $N\sigma\epsilon$. Then

$$\mathbb{P}_z \Big\{ \sup_{(\ell-1)\epsilon \leq t_1, t_2 \leq \ell\epsilon} |Z_i^N(t_1) - Z_i^N(t_2)| > g(\epsilon) \Big\} \leq k\mathbb{P}_z(N^{-1}Z_1 > g(\epsilon)/k)$$

And it follows from lemma 1.12 that there exist a constants $K > 0$, $\epsilon_0 > 0$ and $N_0 \in \mathbb{N}$ such that

$$\mathbb{P}_z \Big\{ \sup_{(\ell-1)\epsilon \leq t_1, t_2 \leq \ell\epsilon} |Z_i^N(t_1) - Z_i^N(t_2)| > g(\epsilon) \Big\} \leq k \exp\{-sN\}$$

For all $\epsilon < \epsilon_0$ and $N > N_0$. And then

$$\mathbb{P}_z(B_\epsilon^c) < \frac{dkT}{\epsilon} \exp\{-sN\}.$$

$\square$

The upper bound for $\mathbb{P}_z \Big(\|Z_a^N - \Upsilon\|_T \geq \frac{\delta}{2d} \Big)$ in (24) is given by the following Lemma:

Lemma 3.3. *For all $\delta, s > 0$ there exist $\epsilon_1 > 0$, $N_0 \in \mathbb{N}$ such that*

$$\mathbb{P}_z(\|Z_a^N - \Upsilon\|_T > \delta/2d) < \frac{dkT}{\epsilon} \exp\{-sN\}, \tag{26}$$

for all $\epsilon < \epsilon_1$ and $N > N_0$.

Proof. Using (1) we write for all $t \in [\ell\epsilon, (\ell+1)\epsilon[$

$$|Z_{a,i}^N(t) - \Upsilon_t^i| \leq \sum_j \frac{1}{N} \Big[P_j \Big(N \int_0^{(\ell+1)\epsilon} \beta_j(Z^N(s))ds \Big) - P_j \Big(N \int_0^{\ell\epsilon} \beta_j(Z^N(s))ds \Big) \Big]$$

$$\leq \frac{1}{N} \sum_j \Big[P_j \Big(N \int_0^{\ell\epsilon} \beta_j(Z^N(s))ds + N\sigma\epsilon \Big) - P_j \Big(N \int_0^{\ell\epsilon} \beta_j(Z^N(s))ds \Big) \Big]$$

$$\leq \frac{1}{N} \sum_j Z_j$$

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where the Z_j are as in the proof of the last lemma. Let ϵ_2 be the maximal ϵ such that $\delta/2kd^2 > g(\epsilon)$. Then we have from lemma 1.12 that for all $\epsilon < \epsilon_1 = \min\{\epsilon_0, \epsilon_2\}$ and $N > N_0$

$$
\begin{aligned}
\mathbb{P}_z(\|Z_a^N - \Upsilon\|_T > \delta/2d) &\leq \mathbb{P}_z\Big(\bigcup_{i=1}^d \{|Z_{a,i}^N(t) - \Upsilon_t^i| > \frac{\delta}{2d^2}\} \quad \text{for some} \quad t \in [0, T]\Big) \\
&\leq \frac{T}{\epsilon} \max_{0 \leq \ell \leq T/\epsilon - 1} \mathbb{P}_z\Big(\bigcup_{i=1}^d \{|Z_{a,i}^N(t) - \Upsilon_t^i| > \frac{\delta}{2d^2}\} \quad \text{for some} \quad t \in [\ell\epsilon, (\ell+1)\epsilon[\Big) \\
&\leq \frac{dkT}{\epsilon} \mathbb{P}_z(Z_1/N > \delta/2kd^2) \leq \frac{dkT}{\epsilon} \exp\{-sN\}.
\end{aligned}
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

It remains to upper bound $\mathbb{P}_z(\{I_T(\Upsilon|\mu) > s\} \cap B_\epsilon)$. We first deduce from Chebyshev's inequality that for all $0 < \alpha < 1$

$$
\mathbb{P}_z(\{I_T(\Upsilon|\mu) > s\} \cap B_\epsilon) \leq \frac{\mathbb{E}_z(\exp\{\alpha N I_T(\Upsilon|\mu)\}\mathbf{1}_{B_\epsilon})}{\exp\{\alpha N s\}}. \tag{27}
$$

In order to finish the proof of Proposition 3.1, all we need to do is to get an upper bound of the numerator in the right hand side of (27) of the type $\exp\{N\delta\}$, with δ arbitrarily small. This will be achieved in the Lemma 3.7. Note that the ideas behind this proof come from [4] and the proof of Theorem 3.2.2, chapter 3 in [8]. We first establish

Lemma 3.4. *For all $0 < \alpha < 1$, $j = 1, ..., k$ and $\ell = 0, ..., T/\epsilon - 1$, there exist Z_j^- and Z_j^+ which conditionally upon $\mathcal{F}_{\ell\epsilon}^N$ are Poisson random variables with mean $N\epsilon\beta_\ell^{j-} = N\epsilon(\beta_j(Z^N(\ell\epsilon)) - Cdg(\epsilon))_+$ and $N\epsilon\beta_\ell^{j+} = N\epsilon(\beta_j(Z^N(\ell\epsilon)) + Cdg(\epsilon))$ respectively such that if*

$$
\Theta_j^\ell = \exp\Big\{\alpha N \int_{\ell\epsilon}^{(\ell+1)\epsilon} f(\mu_t^j, \beta_j(\Upsilon_t))dt\Big\}\mathbf{1}_{B_\epsilon^\ell}
$$

and

$$
\begin{aligned}
\Xi_j^\ell = \exp\{2\alpha N Cdg(\epsilon)\epsilon\} \times \Big[&\exp\Big\{\alpha N\epsilon f\Big(\frac{(1-a)Z_j^-}{\epsilon N}, \beta_\ell^{a,j}\Big)\Big\} \\
&+ \exp\Big\{\alpha N\epsilon f\Big(\frac{(1-a)Z_j^+}{\epsilon N}, \beta_\ell^{a,j}\Big)\Big\}\Big]
\end{aligned}
$$

with $\beta_\ell^{a,j} = (\beta_j(\Upsilon_{\ell\epsilon}) - Cdg(\epsilon))_+$, then

$$
\Theta_j^\ell \leq \Xi_j^\ell \quad a.s \tag{28}
$$

Proof. On B_ϵ^ℓ, with ϵ such that $g(\epsilon) < 1$ and $t \in [\ell\epsilon, (\ell+1)\epsilon]$, using the Lipshitz continuity of the rates β_j we have

$$
|\beta_j(Z^N(t)) - \beta_j(Z^N(\ell\epsilon))| \leq C|Z^N(t) - Z^N(\ell\epsilon)| \leq Cdg(\epsilon), \quad j = 1, ..., k
$$

Then we have

$$
\Big|N \int_{\ell\epsilon}^{(\ell+1)\epsilon} \beta_j(Z^N(t))dt - N\epsilon\beta_j(Z^N(\ell\epsilon))\Big| \leq N\epsilon Cdg(\epsilon), \quad j = 1, ..., k.
$$

As μ_t^j, $j = 1, ..., k$ are given by (22), we can write

$$\frac{(1-a)Z_j^-}{\epsilon N} \leq \mu_{\ell\epsilon}^j \leq \frac{(1-a)Z_j^+}{\epsilon N} \quad \text{a.s.} \tag{29}$$

where for example

$$Z_j^- = P_j\Big(N \int_0^{\ell\epsilon} \beta_j(Z^N(s))ds + \epsilon N(\beta_j(Z^N(\ell\epsilon)) - Cdg(\epsilon))_+\Big) - P_j\Big(N \int_0^{\ell\epsilon} \beta_j(Z^N(s))ds\Big)$$

$$Z_j^+ = P_j\Big(N \int_0^{\ell\epsilon} \beta_j(Z^N(s))ds + \epsilon N(\beta_j(Z^N(\ell\epsilon)) + Cdg(\epsilon))\Big) - P_j\Big(N \int_0^{\ell\epsilon} \beta_j(Z^N(s))ds\Big).$$

Moreover it is easy to see that on B_ϵ^ℓ we have

$$\max_{1 \leq i \leq d} |\Upsilon_t^i - \Upsilon_{\ell\epsilon}^i| < (1-a)g(\epsilon) < g(\epsilon) \quad \text{for} \quad t \in [\ell\epsilon, (\ell+1)\epsilon].$$

And then

$$|\beta_j(\Upsilon_t) - \beta_j(\Upsilon_{\ell\epsilon})| \leq C|\Upsilon_t - \Upsilon_{\ell\epsilon}| \leq Cdg(\epsilon)$$

we deduce that

$$\beta_j(\Upsilon_t) \geq (\beta_j(\Upsilon_{\ell\epsilon}) - Cdg(\epsilon))_+ = \beta_\ell^{a,j}$$

and

$$\beta_j(\Upsilon_t) \leq \beta_j(\Upsilon_{\ell\epsilon}) + Cdg(\epsilon) = \beta_\ell^{a,j} + 2Cdg(\epsilon).$$

Thus

$$f(\mu_t^j, \beta_j(\Upsilon_t)) = \mu_t^j \log \frac{\mu_t^j}{\beta_j(\Upsilon_t)} - \mu_t^j + \beta_j(\Upsilon_t)$$

$$\leq \mu_t^j \log \frac{\mu_t^j}{\beta_\ell^{a,j}} - \mu_t^j + \beta_\ell^{a,j} + 2Cdg(\epsilon)$$

$$\leq f(\mu_t^j, \beta_\ell^{a,j}) + 2Cdg(\epsilon).$$

As $\mu_t^j = \mu_{\ell\epsilon}^j$ is constant over the interval $[\ell\epsilon, (\ell+1)\epsilon[$, we deduce that on B_ϵ^ℓ

$$\exp\Big\{\alpha N \int_{\ell\epsilon}^{(\ell+1)\epsilon} f(\mu_t^j, \beta_j(\Upsilon_t))dt\Big\} \leq \exp\{\alpha N\epsilon f(\mu_{\ell\epsilon}^j, \beta_\ell^{a,j}) + 2\alpha NCd\epsilon g(\epsilon)\}. \tag{30}$$

From (29), (30) and the convexity of $f(\nu, \omega)$ in ν we deduce the inequality of lemma. □

The next proposition gives us a bound for the conditional expectation of the right hand side of the inequality (28).

Proposition 3.5. *Let* $a = h(\epsilon) = \Big[-\log g^{1/2}(\epsilon)\Big]^{-\frac{1}{\nu}}$ *where* ν *is given by the assumption* 0.3 4. *For all* $0 < \alpha < 1$ *there exist* ϵ_α, K_α *and* $\tilde{K}$ *such that for all* $\epsilon \leq \epsilon_\alpha$ *we have*

$$\max_{q=-,+} \Big\{\mathbb{E}_z\Big(\exp\Big\{\alpha N\epsilon f\Big(\frac{(1-a)Z_j^q}{\epsilon N}, \beta_\ell^{a,j}\Big)\Big\}|\mathcal{F}_{\ell\epsilon}^N\Big)\Big\}$$

$$\leq K_\alpha \exp\{N\epsilon\tilde{K}(1 - \alpha + 2h(\epsilon) + 2dg(\epsilon))\}.$$

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Proof. Conditionally upon $\mathcal{F}_{\ell\epsilon}^N$, Z_j^q is a Poisson variable with mean $N\epsilon\beta_\ell^{j,q}$. Moreover we have by the definition

$$\max\{|\beta_\ell^{a,j} - \beta_\ell^{j-}|, |\beta_\ell^{a,j} - \beta_\ell^{j+}|\} \leq \tilde{C}(a + 2dg(\epsilon))$$

let $\tilde{\epsilon} = \epsilon/(1-a)$ and $\tilde{\alpha} = (1-a)\alpha$ then we have

$$\mathbb{E}_z\Big(\exp\Big\{\alpha N\epsilon f\Big(\frac{(1-a)Z_j^q}{\epsilon N}, \beta_\ell^{a,j}\Big)\Big\}|\mathcal{F}_{\ell\epsilon}^N\Big) = \mathbb{E}_z\Big(\exp\Big\{\alpha N\epsilon f\Big(\frac{Z_j^q}{\tilde{\epsilon}N}, \beta_\ell^{a,j}\Big)\Big\}|\mathcal{F}_{\ell\epsilon}^N\Big)$$

$$= \sum_{m\geq 0}\exp\Big\{\alpha N\epsilon f\Big(\frac{m}{\tilde{\epsilon}N}, \beta_\ell^{a,j}\Big)\Big\}\frac{(N\epsilon\beta_\ell^{j,q})^m\exp\{-N\epsilon\beta_\ell^{j,q}\}}{m!}$$

$$= \sum_{m\geq 0}\exp\Big\{\alpha N\epsilon\Big(\frac{m}{\tilde{\epsilon}N}\log\Big(\frac{m}{\tilde{\epsilon}N\beta_\ell^{a,j}}\Big) - \frac{m}{\tilde{\epsilon}N} + \beta_\ell^{a,j}\Big)\Big\}\frac{(N\epsilon\beta_\ell^{j,q})^m\exp\{-N\epsilon\beta_\ell^{j,q}\}}{m!}$$

$$\leq \exp\{N\epsilon\tilde{C}(a + 2dg(\epsilon))\}\sum_{m\geq 0}\frac{m^{\tilde{\alpha}m}\exp\{-\tilde{\alpha}m\}}{m!}(N\epsilon\beta_\ell^{a,j})^{m(1-\tilde{\alpha})}\Big(\frac{\beta_\ell^{j,q}}{\beta_\ell^{a,j}}\Big)^m\exp\{-N\epsilon\beta_\ell^{a,j}(1-\alpha)\}$$

$$\leq \exp\{N\epsilon C_1(a + 2dg(\epsilon))\}\sum_{m\geq 0}\frac{m^{\tilde{\alpha}m}\exp\{-\tilde{\alpha}m\}}{m!}(N\epsilon\beta_\ell^{a,j})^{m(1-\tilde{\alpha})}\Big(\frac{\beta_\ell^{j,q}}{\beta_\ell^{a,j}}\Big)^m\exp\{-N\epsilon\beta_\ell^{a,j}(1-\tilde{\alpha})\}.$$
$$\tag{31}$$

Moreover the function $v(x) = x^{m(1-\tilde{\alpha})}\exp\{-2x(1-\tilde{\alpha})\}$ reaches its maximum at $x = m/2$ thus we have

$$x^{m(1-\tilde{\alpha})}\exp\{-2x(1-\tilde{\alpha})\} \leq \Big(\frac{m}{2}\Big)^{m(1-\tilde{\alpha})}\exp\{-m(1-\tilde{\alpha})\} \quad \forall x$$

In particular

$$(N\epsilon\beta_\ell^{a,j})^{m(1-\tilde{\alpha})}\exp\{-2N\epsilon\beta_\ell^{a,j}(1-\tilde{\alpha})\} \leq \Big(\frac{m}{2}\Big)^{m(1-\tilde{\alpha})}\exp\{-m(1-\tilde{\alpha})\}.$$

Thus

$$\sum_{m\geq 0}\frac{m^{\tilde{\alpha}m}\exp\{-\tilde{\alpha}m\}}{m!}(N\epsilon\beta_\ell^{a,j})^{m(1-\tilde{\alpha})}\Big(\frac{\beta_\ell^{j,q}}{\beta_\ell^{a,j}}\Big)^m\exp\{-N\epsilon\beta_\ell^{a,j}(1-\tilde{\alpha})\}$$

$$\leq \exp\{N\epsilon\beta_\ell^{a,j}(1-\tilde{\alpha})\}\sum_{m\geq 0}\frac{m^m\exp\{-m\}}{m!}\Big(\frac{\beta_\ell^{j,q}/\beta_\ell^{a,j}}{2^{(1-\tilde{\alpha})}}\Big)^m \tag{32}$$

Let us admit for a moment,

Lemma 3.6. *For all $0 < \alpha < 1$ there exists $\epsilon_\alpha > 0$ such that for all $\epsilon < \epsilon_\alpha$ and $q = -, +$,*

$$\frac{\beta_\ell^{j,q}}{\beta_\ell^{a,j}} < 2^{(1-\alpha)/2} < 2^{(1-\tilde{\alpha})/2}.$$

Then for ϵ small enough we have

$$\exp\{N\epsilon\beta_\ell^{a,j}(1-\tilde{\alpha})\}\sum_{m\geq 0}\frac{m^m e^{-m}}{m!}\Big(\frac{\beta_\ell^{j,q}/\beta_\ell^{a,j}}{2^{(1-\tilde{\alpha})}}\Big)^m$$

$$\leq e^{N\epsilon\theta(1-\tilde{\alpha})}\sum_{m\geq 0}\frac{m^m e^{-m}}{m!}\Big(\frac{1}{2^{(1-\alpha)/2}}\Big)^m \qquad (33)$$

$$= e^{N\epsilon\theta(1-\tilde{\alpha})}K_\alpha.$$

Since the series above converges. We deduce from (31), (32) and (33) that

$$\mathbb{E}_z\Big(\exp\Big\{\alpha N\epsilon f\Big(\frac{(1-a)Z_j^q}{\epsilon N},\beta_\ell^{a,j}\Big)\Big\}|\mathcal{F}_{\ell\epsilon}^N\Big)\leq K_\alpha\exp\{N\epsilon C_2(1-\alpha+a)\}\exp\{N\epsilon\tilde{C}(a+cdg(\epsilon))\}$$

$$\leq K_\alpha\exp\{N\epsilon\tilde{K}(1-\alpha+2h(\epsilon)+2dg(\epsilon))\}.$$

$\square$

Proof of lemma 3.6
For $q=-$ we have
$$\frac{\beta_\ell^{j,-}}{\beta_\ell^{a,j}}\leq\frac{\beta_j(Z^N(\ell\epsilon))}{\beta_j(Z^{N,a}(\ell\epsilon))-Cdg(\epsilon)}$$

If $\beta_j(Z^N(\ell\epsilon))<\lambda_1$ we have using the assumptions 0.3 3 and 0.3 4

$$\frac{\beta_\ell^{j,-}}{\beta_\ell^{a,j}}\leq\frac{\beta_j(Z^{N,a}(\ell\epsilon))}{\beta_j(Z^{N,a}(\ell\epsilon))-Cdg(\epsilon)}\leq\frac{C_a}{C_a-Cdg(\epsilon)}$$

$$\leq\frac{1}{1-\frac{Cdg(\epsilon)}{g^{1/2}(\epsilon)}}\to 1\quad\text{as}\quad\epsilon\to 0.$$

If $\beta_j(Z^N(\ell\epsilon))\geq\lambda_1$, we have

$$\frac{\beta_\ell^{j,-}}{\beta_\ell^{a,j}}\leq\frac{\beta_j(Z^N(\ell\epsilon))}{\beta_j(Z^N(\ell\epsilon))-C\bar{C}a-Cdg(\epsilon)}\leq\frac{\lambda_1}{\lambda_1-C\bar{C}h(\epsilon)-Cdg(\epsilon)}$$

$$\to 1\quad\text{as}\quad\epsilon\to 0.$$

And for $q=+$ We have
$$\frac{\beta_\ell^{j,+}}{\beta_\ell^{a,j}}\leq\frac{\beta_j(Z^N(\ell\epsilon))+Cdg(\epsilon)}{\beta_j(Z^{N,a}(\ell\epsilon))-Cdg(\epsilon)}$$

If $\beta_j(Z^N(p\epsilon))<\lambda_1$, thank to the assumptions 0.3 3 and 0.3 4

$$\frac{\beta_\ell^{j,+}}{\beta_\ell^{a,j}}\leq\frac{\beta_j(Z^{N,a}(\ell\epsilon))+Cdg(\epsilon)}{\beta_j(Z^{N,a}(\ell\epsilon))-Cdg(\epsilon)}$$

$$\leq\frac{C_a+Cdg(\epsilon)}{C_a-Cdg(\epsilon)}\leq\frac{1+\frac{Cdg(\epsilon)}{g^{1/2}(\epsilon)}}{1-\frac{Cdg(\epsilon)}{g^{1/2}(\epsilon)}}\to 1\quad as\quad\epsilon\to 0.$$

If $\beta_j(Z^N(\ell\epsilon)) \geq \lambda_1$, we have

$$\frac{\beta_\ell^{j,+}}{\beta_\ell^{a,j}} \leq \frac{\beta_j(Z^N(\ell\epsilon)) + Cdg(\epsilon)}{\beta_j(Z^N(\ell\epsilon)) - C\bar{C}h(\epsilon) - Cdg(\epsilon)}$$

$$\leq \frac{\lambda_1 + Cdg(\epsilon)}{\lambda_1 - C\bar{C}h(\epsilon) - Cdg(\epsilon)} \to 1 \quad \text{as} \quad \epsilon \to 0.$$

Thus, there exists ϵ_α such that $\frac{\beta_\ell^{j,q}}{\beta_\ell^{a,j}} < 2^{(1-\alpha)/2} < 2^{(1-\tilde{\alpha})/2}$ for all $\epsilon < \epsilon_\alpha$.

The next lemma gives us a upper bound for the quantity $\mathbb{E}_z\Big(\exp\{\alpha N I_T(\Upsilon|\mu)\}\mathbf{1}_{B_\epsilon}\Big)$.

Lemma 3.7. *For all $0 < \alpha < 1$ there exist ϵ_α, K_α and $\tilde{K}_1$ such that for all $\epsilon \leq \epsilon_\alpha$, we have the following inequality*

$$\mathbb{E}_z\Big(\exp\{\alpha N I_T(\Upsilon|\mu)\}\mathbf{1}_{B_\epsilon}\Big) \leq (2K_\alpha)^{\frac{kT}{\epsilon}} \exp\{kNT\tilde{K}_1(1 - \alpha + h(\epsilon) + 4dg(\epsilon))\} \qquad (34)$$

Proof. We first remark using the lemma 3.4 and the proposition 3.5 that

$$\mathbb{E}_z(\Theta_j^\ell|\mathcal{F}_{\ell\epsilon}^N) \leq \mathbb{E}_z(\Xi_j^\ell|\mathcal{F}N_{\ell\epsilon}) \leq 2K_\alpha \exp\{N\epsilon\tilde{K}_1(1 - \alpha + 2h(\epsilon) + 4dg(\epsilon))\}.$$

Moreover, the Ξ_j^ℓ, $j = 1, ..., k$ are conditionnally independent given $\mathcal{F}_{\ell\epsilon}^N$. So we can take the iterative conditional expectations with respect to $\mathcal{F}_{(\frac{T}{\epsilon}-1)\epsilon}^N$, $\mathcal{F}_{(\frac{T}{\epsilon}-2)\epsilon}^N$, ..., $\mathcal{F}_\epsilon^N$, to get that for all $0 < \alpha < 1$ and $\epsilon < \epsilon_\alpha$

$$\mathbb{E}_z\Big(\exp\{\alpha N I_T(\Upsilon|\mu)\}\mathbf{1}_{B_\epsilon}\Big) = \mathbb{E}_z\Big(\prod_{\ell=0}^{\frac{T}{\epsilon}-1} \exp\Big\{\alpha N \int_{\ell\epsilon}^{(\ell+1)\epsilon} \sum_j f(\mu_t^j, \beta_j(\Upsilon_t))dt\Big\}\mathbf{1}_{B_\epsilon^\ell}\Big)$$

$$= \mathbb{E}_z\Big(\mathbb{E}_z\Big(\prod_{\ell=0}^{\frac{T}{\epsilon}-1}\prod_{j=1}^k \Theta_j^\ell|\mathcal{F}_{(\frac{T}{\epsilon}-1)\epsilon}^N\Big)\Big) \leq \mathbb{E}_z\Big(\mathbb{E}_z\Big(\prod_{\ell=0}^{\frac{T}{\epsilon}-1}\prod_{j=1}^k \Xi_j^\ell|\mathcal{F}_{(\frac{T}{\epsilon}-1)\epsilon}^N\Big)\Big)$$

$$\leq \mathbb{E}_z\Big(\prod_{\ell=0}^{\frac{T}{\epsilon}-2}\prod_{j=1}^k \Xi_j^\ell \mathbb{E}_z\Big(\prod_{j=1}^k \Xi_j^{\frac{T}{\epsilon}-1}|\mathcal{F}_{(\frac{T}{\epsilon}-1)\epsilon}^N\Big)\Big)$$

$$\leq \prod_{p=0}^{\frac{T}{\epsilon}-1} (2K_\alpha)^k \exp\{kN\epsilon\tilde{\tilde{C}}(1 - \alpha + h(\epsilon) + 4dg(\epsilon))\}$$

$$= (2K_\alpha)^{\frac{kT}{\epsilon}} \exp\{kNT\tilde{K}_1(1 - \alpha + h(\epsilon) + 4dg(\epsilon))\}.$$

$\square$

The upper bound for the first term in the right side of (24) is obtained by combining (27) and (34) and we have for all $0 < \alpha < 1$, $\epsilon < \min\{\epsilon_0, \epsilon_\alpha, \epsilon_1\}$ and $a = h(\epsilon) = \Big[-\log g^{1/2}(\epsilon)\Big]^{-\frac{1}{\nu}}$,

$$\mathbb{P}_z(\{I_T(\Upsilon|\mu) > s\} \cap B_\epsilon) \leq (2K_\alpha)^{\frac{kT}{\epsilon}} \exp\{kNT\tilde{K}_1(1 - \alpha + h(\epsilon) + 4dg(\epsilon))\} \times \exp\{-\alpha Ns\} \qquad (35)$$

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The end of the proof of the proposition 3.1 can be done by Combining (24), (25), (26) and (35). We have thus for all $\delta > 0$, $0 < \alpha < 1$, $\epsilon < \min\{\epsilon_0, \epsilon_\alpha, \epsilon_1\}$ and $a = h(\epsilon) = \left[-\log g^{1/2}(\epsilon)\right]^{-\frac{1}{\nu}}$,

$$\mathbb{P}_z(F_\delta^s) \leq (2K_\alpha)^{\frac{kT}{\epsilon}} \exp\{kNT\tilde{K}_1(1-\alpha+h(\epsilon)+4dg(\epsilon))\} \times \exp\{-\alpha Ns\} + \frac{2dTk}{\epsilon} \exp\{-sN\}.$$

Finally, we take $1-\alpha$ and ϵ small enough to ensure that $kT\tilde{K}_1(1-\alpha+h(\epsilon)+4dg(\epsilon)) < \eta/4$ and $(1-\alpha)s < \eta/4$. We also take N large enough so that $kT\log(2K_\alpha)/N\epsilon < \eta/4$ and $\log(2dkT/\epsilon)/N < \eta/4$ and we have

$$\mathbb{P}_z(F_\delta^s) \leq \exp\{-N(s-3\eta/4)\} + \frac{2dT}{\epsilon} \exp\{-sN\}$$
$$\leq \frac{dkT}{\epsilon}.\exp\{-N(s-3\eta/4)\} \leq \exp\{-N(s-\eta)\}.$$

$\square$

We conclude the proof of the upper bound by the following theorem

Theorem 3.8. *For any closed subset F of $D_{T,A}$ and $z \in A$*

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_z^N(F) \leq - \inf_{\phi \in F, \phi_0 = z} I_T(\phi). \tag{36}$$

Proof. All we need to prove is that the inequality (18) implies (36). To this end let $F \in D_{T,A}$ be a closed set, choose $\eta > 0$ and let

$$s = \inf_{\phi \in F, \phi_0 = z} I_T(\phi) - \eta/2.$$

The closed set $F_z = \{\phi \in F : \phi_0 = z\}$ does not intersect the set $\Phi(s)$, which is compact since I_T is a good rate function. Therefore

$$\delta = \inf_{\phi \in F_z} \inf_{\psi \in \Phi(s)} \|\phi - \psi\|_T > 0.$$

We use the inequality (18) to have for any δ, η and $s > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $N > N_0$,

$$\mathbb{P}_z^N(F) \leq \mathbb{P}_z^N(F_\delta^s)$$
$$\leq \exp\{-N(s-\eta/2)\}$$
$$\leq \exp\left\{-N\left(\inf_{\phi \in F, \phi_0 = z} I_T(\phi) - \eta\right)\right\},$$

then

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_z^N(F) \leq - \inf_{\phi \in F, \phi_0 = z} I_T(\phi).$$

$\square$

Remark 3.9. *We can also show without difficulty that (36) implies (18).*

Corollary 3.10. *For any open subset F of $D_{T,A}$ and any compact subset K of A,*

$$\limsup_{N \to \infty} \frac{1}{N} \log \sup_{z \in K} \mathbb{P}_z(Z^N \in F) \leq - \inf_{z \in K} \inf_{\phi \in F, \phi_0 = z} I_T(\phi).$$

Proof. The arguments are the same as in the proof of Corollary 5.6.15 in [3]. Let

$$I_K := \inf_{z \in K} \inf_{\phi \in F, \phi_0 = z} I_T(\phi).$$

For $\eta > 0$ fix, let $I_K^\eta := \min\{I_K - \eta, \eta^{-1}\}$. Then from Lemma 1.7 it follows that for any $z \in K$, there exists a N_z such that for all $N > N_z$ and $y \in B(z, \frac{1}{N_z})$,

$$\inf_{\phi \in F, \phi_0 = y} I_T(\phi) \geq \inf_{\phi \in F, \phi_0 = z} I_T(\phi) - \eta \geq I_K^\eta.$$

Therefore we have from (36) that

$$\frac{1}{N} \log \mathbb{P}_y(Z^N \in F) \leq - \inf_{\phi \in F, \phi_0 = y} I_T(\phi) \leq -I_K^\eta.$$

And then

$$\frac{1}{N} \log \sup_{y \in B(z, \frac{1}{N_z})} \mathbb{P}_y(Z^N \in F) \leq -I_K^\eta.$$

As K is compact, there exits a finite sequence $(z_i)_{1 \leq i \leq m} \subset K$ such that $K \subset \bigcup_{i=1}^m B(z_i, \frac{1}{N_{z_i}})$. Then for $N \geq \max_{1 \leq i \leq m} N_{z_i}$,

$$\frac{1}{N} \log \sup_{y \in K} \mathbb{P}_y(Z^N \in F) \leq -I_K^\eta.$$

It first remains to take $\limsup$ as $N \to \infty$ and then let η tend to 0 to have result. $\qquad \square$

4 Time of exit from a domain

Let O the domain of attraction of a stable point z^* of the dynamical system (8) and $\widetilde{\partial O}$ be the part of boundary of O that the stochastic system (1) can cross. We need to formulate a theorem which give us an approximate value for the exit time τ_O^N from O for large N as well as the exponential asymptotic of its mean $\mathbb{E}_z(\tau_O^N)$. For models of infectious disease we have in mind, τ_O^N is an approximate value of the time to extinction of the disease. This is the most important application of our large deviations result. To this end, for $z, y \in A$ where A is defined by (2), we define the following functionals

$$V(z, y, T) := \inf_{\phi \in D_{T,A}, \phi_0 = z, \phi_T = y} I_T(\phi)$$

$$V(z, y) := \inf_{T > 0} V(z, y, T)$$

$$V_{\widetilde{\partial O}} := \inf_{y \in \widetilde{\partial O}} V(z^*, y).$$

The theorem is a consequence of the large deviation principle established above, the law of large numbers and some technical arguments. The proof which can found in Section 7 of [9] requires the following technical assumptions:

Assumption 4.1. *1. For all $z \in O$, the solution $Y^z(.)$ of the ODE (8) satisfies*

$$Y^z(t) \in O \quad \text{for all} \quad t > 0 \quad \text{and} \quad \lim_{t \to \infty} Y^z(t) = z^*.$$

2. $V_{\widetilde{\partial O}} < \infty$.

3. For all $\rho > 0$ there exist constants $T(\rho), \epsilon(\rho) > 0$ with $T(\rho), \epsilon(\rho) \downarrow 0$ as $\rho \downarrow 0$ such that for all $x \in \widetilde{\partial O} \cup \{z^*\}$ and all $z, y \in B_\rho(x) \cap A$ there exists an $\phi = \phi(\rho, z, y)$: $[0, T(\rho)] \to A$ with $\phi_0 = z$, $\phi_{T(\rho)} = y$ and $I_{T(\rho)}(\phi) < \epsilon(\rho)$.

4. For all $z \in \widetilde{\partial O}$ there exists an $\delta_0 > 0$ such that for all $\delta < \delta_0$ there exists $z^\delta \in A \setminus \bar{O}$ with $|z - z^\delta| > \delta$.

5. There exists a collection $\{O_\rho, \rho > 0\}$ which is such that

- $\overline{O_\rho} \subset O$ *for all $\rho > 0$.*
- $d(O_\rho, \widetilde{\partial O}) \to 0$ *as $\rho \to 0$.*
- *For all $\rho > 0$, O_ρ satisfies the four above assumptions and for all $z \in \partial O_\rho$, the solution $Y^z(.)$ of the ODE (8) is such that $\lim_{t \to \infty} Y^z(t) = z^*$.*

Note that all these assumptions are satisfied in the infectious disease models that we have in mind. The third assumption is not difficult to verify and the fourth one allows to consider a trajectory which crosses the characteristic boundary $\widetilde{\partial O}$, in such a way that all paths in a sufficiently small tube around that trajectory do exit O.

Theorem 4.2. *Given $\eta > 0$, for all $z \in O$,*

$$\lim_{N \to \infty} \mathbb{P}_z \big(\exp\{N(V_{\widetilde{\partial O}} - \eta)\} < \tau_O^N < \exp\{N(V_{\widetilde{\partial O}} + \eta)\} \big) = 1.$$

Moreover, for all $\eta > 0$, $z \in O$ and N large enough,

$$\exp\{N(V_{\widetilde{\partial O}} - \eta)\} \leq \mathbb{E}_z(\tau_O^N) \leq \exp\{N(V_{\widetilde{\partial O}} + \eta)\}.$$

References

[1] Pierre Brémaud *Point processes and Queues*, Springer Verlag, New York, 1981.

[2] Tom Britton and Etienne Pardoux Stochastic epidemics in a homogeneous community, to appear, 2016.

[3] Amir Dembo and Ofer Zeitouni. *Large deviations techniques and applications.* Springer-Verlag, New York, 2009.

[4] Regina Dolgoarshinnykh. Sample path large deviations for sirs epidemic processes, preprint.

[5] Paul Dupuis and Richard S. Ellis. *A weak convergence approach to the theory of large deviations.* Wiley series in probability and statistics. Wiley, New York, 1997.

[6] Stuart N. Ethier and Thomas G. Kurtz. *Markov processes: characterization and convergence.* Wiley Series in Probability and Statistics. John Wiley & Sons, 2005.

[7] Jin Feng and Thomas G. Kurtz. *Large deviations for stochastic processes.* Mathematical Surveys and Monographs **131**, American Mathematical Society, Providence, 2006.

[8] Mark I. Freidlin and Alexander D. Wentzell. *Random perturbations of dynamical systems*, 3d ed. Grundlehren des Mathematischen Wissenschaften **260**, Springer, New York, 2012.

[9] Peter Kratz and Etienne Pardoux. Large deviation for infections diseases. *arxiv:1602.02803*, 2016.

[10] Thomas G. Kurtz. Strong approximation theorems for density dependent Markov chains. *Stochastic Processes and their Applications*, **6**(3):223–240, 1978.

[11] Adam Shwartz and Alan Weiss. *Large Deviations for Performance Analysis.* Chapman Hall, London, 1995.

[12] Adam Shwartz and Alan Weiss. Large deviations with diminishing rates. *Mathematics of Operations Research*, **30**(2):281–310, 2005.

[13] Alexander Sokol and Niels Richard Hansen Exponential martingales and change of measure for counting processes. *Stochastic Analysis and Applications*, **33**, 823–843, 2015.