On the asymptotic final size distribution of epidemics in heterogeneous populations

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1. Introduction

In this paper, the behaviour of an epidemic of the SIR type in a large closed population is studied. By generalizing the imbedding representation of epidemics of *Scalia-Tomba (1985)*, many of the available results on asymptotic final size distributions of various epidemic models can be obtained and generalized to include heterogeneity of susceptibility to infection.

2. A threshold model for epidemics

Let the susceptible population consist of n individuals, each one characterized by (Q_i, ξ_i), $i = 1, \dots, n$, where $Q =$ "infection threshold" and $\xi =$ "infective power in case of infection".

Let the epidemic start by introducing an initial amount of infection ξ_0 into the population. Individuals with $Q_i \le \xi_0$ are then infected. They add their respective ξ:s to the "infective burden" on the remaining population. Individuals with $\xi_0 < Q_i \le \xi_0 +$ (sum of "first generation" ξ:s) then become infected, and so on.

The above description can be formalized in the following way: let $X_i(t) = 1_{\{Q_i \le t\}}$, $i = 1, \dots, n$, and $X(t) = \sum_{i=1}^n X_i(t)$. The development of generations of infectives can be described by $T_0 := 0$, $a_0 := \xi_0 \to T_1 := X(a_0)$, $a_1 := \xi_0 + \sum_{i=1}^n \xi_i X_i(a_0) \to \dots \to T_{k+1} := X(a_k)$, $a_{k+1} := \xi_0 + \sum_{i=1}^n \xi_i X_i(a_k)$. The succesive values $\{T_k\}$ thus denote the total number of infected individuals up to and including the k:th generation.

Let $a(t) = \sum_{i=1}^n \xi_i 1_{\{X(Q_i) \le t\}}$. We can then write $T_{k+1} = X(\xi_0 + a(T_k))$, $k = 0, 1, \dots$. We can interpret $a(t)$ as the sum of ξ-values for the t individuals with the lowest Q-values (if these are distinct). Let $\widetilde{X}(t) = X(\xi_0 + a(t))$. Then we have $T_0 = 0$, $T_{k+1} = \widetilde{X}(T_k)$, $k = 0, 1, \dots$. We see that $\widetilde{X}(t)$ is non-decreasing, integer-valued and bounded by n. $a(t)$ is also a non-decreasing function. Therefore, the sequence $\{T_k\}$ is non-decreasing and bounded by n. Thus $T_k \nearrow T \le n$ where $T = \min\{t : t = \widetilde{X}(t)\}$. T is the final size of the epidemic in the population.

3. A stochastic model with heterogeneous susceptibility

3.1 Definition and interpretation. Let $\{Q_i\}$ be independent with d.f.:s $\{F_i\}$ and $\{\xi_i\}$ be i.i.d. with d.f. H, independent of $\{Q_i\}$. Let also ξ_0 have the same law as $\xi_1 + \dots + \xi_m$ and be independent of all other quantities. Because of the assumed independencies, the previously defined epidemic model will have the same distributional properties as the following model: let $X(t) = \sum_{i=1}^n 1_{\{Q_i \le t\}}$ and $\xi(t) = \sum_{j=1}^{[t]} \xi_j$, $T_0 = 0$, $T_1 = X(\xi(m))$, $T_{k+1} = X(\xi(m + T_k))$, $\widetilde{X}(t) = X(\xi(t))$. Then $T_{k+1} = \widetilde{X}(m + T_k)$, $k = 0, 1, \dots$, and $T_k \nearrow T = \min\{t : t = \widetilde{X}(m+t)\}$. For future use, define T_* as $T + m$. Then $m \le T_* \le n + m$ and $T_* = \min\{t : t - m = \widetilde{X}(t)\}$. Various considerations now show that the above epidemic model contains several previously studied models, as well as generalizations of these:

1) In *Scalia-Tomba (1985)*, it is shown that the classical Reed-Frost epidemic is obtained by choosing $F_i = \text{geom}(p)$ and $\xi_i = 1$, $\forall i$. By choosing $F_i = F$ and $\xi_i = 1$, $\forall i$, where F is an arbitrary distribution on $\mathbb{N}$, the generalized Reed-Frost process studied in the same paper is obtained.
2) By the same arguments as in 1), the choice $F_i = \text{geom}(p_i)$ (or equivalently $\exp(\theta_i)$ with $p_i = 1 - \exp(-\theta_i)$) and $\xi_i = 1$, $\forall i$, yields a Reed-Frost type epidemic where each individual i has susceptibility p_i to infection (prob. of being infected by an infective individual). It is then convenient to represent the susceptibility "profile " of the population by the empirical distribution of, say, θ-values, i.e. by $G(\theta)$ = proportion of individuals having $\theta_i \leq \theta$.
3) By interpreting ξ as the length of the infectious period of an individual and assuming a constant rate of infectious contacts between each pair of susceptible and infective individuals, we will have a generalization of the classical "general stochastic epidemic" (GSE), for which ξ has an exponential distribution, cf. *Sellke (1983)*. The case corresponding to $F_i = \exp(\theta)$, H arbitrary, has been studied by *Wang (1977)*,*von Bahr and Martin-Löf (1980)* and *Ball (1985)*.
4) The choice $F_i = \exp(\theta_i)$ and $H = \exp(1)$ is thus a natural extension of the GSE to the case of heterogeneous susceptibility. It is worth noting that the classical formulation of the GSE as a Markov process is no longer practical, since individuals will not be equivalent in the heterogeneous case. But, as in *von Bahr and Martin-Löf (1980)*, Markov structure is retained if the actual sets of susceptible and infective individuals of each generation are considered, instead of only their numbers.
5) As long as the distributions $\{F_i\}$ are exponential, it is possible to interpret the model on a contact rate basis, possibly with varying infectious periods. This is possible since the infective action of a number of infective individuals with given infective periods, acting simultaneously on a given susceptible individual, is equivalent to their acting sequentially (as expressed by the $\xi(t)$-process). If the distributions $\{F_i\}$ are arbitrary, the contact rate interpretation in real time may not be made with the same ease. Still, as in *Scalia-Tomba (1985)*, distributions other than exponential may be interpreted as a tendency to have varying susceptibility to infection, depending on actual total epidemic size.

3.2 Asymptotic situation and definitions. Let $n \to \infty$ and consider a sequence of processes $\{X^{(n)}\}$ with respective parameters $\{F_i^{(n)}\}$, $\{m^{(n)}\}$ and $\{\xi_i^{(n)}\}$. Let the d.f. H be fixed. Then we also have

$$X^{(n)}(t) = \sum_{i=1}^{n} 1_{\{Q_i^{(n)} \leq t\}},$$

$$\xi^{(n)}(t) = \sum_{j=1}^{[t]} \xi_j^{(n)},$$

$$\widetilde{X}^{(n)}(t) = X^{(n)}\left(\xi^{(n)}(t)\right),$$

and

$$T_*^{(n)} = \min\left\{t : t - m^{(n)} = \widetilde{X}^{(n)}(t)\right\}.$$

The final size of the epidemic is denoted $T^{(n)} = T_*^{(n)} - m^{(n)}$. Denote $\mathrm{E}(\xi)$ by α and $\mathrm{Var}(\xi)$ by $K < \infty$. Define, for future use, the following quantities:

$$M^{(n)}(t) = \frac{1}{n}\mathrm{E}\left(X^{(n)}(nt)\right) = \frac{1}{n}\sum_{i=1}^{n} F_i^{(n)}(nt),$$

$$C^{(n)}(s,t) = \frac{1}{n}\mathrm{Cov}\left(X^{(n)}(ns), X^{(n)}(nt)\right) = \frac{1}{n}\sum_{i=1}^{n}\left(1 - F_i^{(n)}(n(t \vee s))\right) F_i^{(n)}(n(t \wedge s)),$$

$$Z^{(n)}(t) = \sqrt{n}\left(\frac{X^{(n)}(nt)}{n} - M^{(n)}(t)\right),$$

$$\overline{\xi}^{(n)}(t) = \frac{\xi^{(n)}(nt)}{n},$$

$$\widetilde{Z}^{(n)}(t) = Z^{(n)}\left(\overline{\xi}^{(n)}(t)\right) = \sqrt{n}\left(\frac{\widetilde{X}^{(n)}(nt)}{n} - M^{(n)}\left(\overline{\xi}^{(n)}(t)\right)\right),$$

$$A^{(n)}(t) = \sqrt{n}\left(M^{(n)}\left(\overline{\xi}^{(n)}(t)\right) - M^{(n)}(\alpha t)\right),$$

$$V^{(n)}(t) = \widetilde{Z}^{(n)}(t) + A^{(n)}(t) = \sqrt{n}\left(\frac{X^{(n)}(nt)}{n} - M^{(n)}(\alpha t)\right),$$

$$\mu^{(n)} = \frac{m^{(n)}}{n}, \quad \text{and, finally,}$$

$$\overline{T}^{(n)} = \frac{T_*^{(n)}}{n} = \min\left\{t : t - \mu^{(n)} = \frac{\widetilde{X}^{(n)}(nt)}{n}\right\}$$

$$= \min\left\{t : t - \mu^{(n)} - M^{(n)}(\alpha t) = \frac{V^{(n)}(t)}{\sqrt{n}}\right\}.$$

As $n \to \infty$, we assume that $\mu^{(n)} \to \mu \geq 0$ and that $M^{(n)}(t) \to M(t)$ and $C^{(n)}(s,t) \to C(s,t)$. Some further regularity conditions on convergence and on the limit functions M and C will have to be imposed, but these are explained as the need arises in the subsequent calculations.

3.3 Some preliminary convergence results. By combining the Cramér-Wold device and the Lindeberg CLT, it is easy to show that the finite-dimensional distributions of $Z^{(n)}$ converge to those of a Gaussian process with mean 0 and covariance function $C(s,t)$, subject to e.g. $C(s,t) > 0$, $\forall s,t > 0$. By imposing mild conditions on the continuity of $C(s,t)$ (see e.g. *Cramér and Leadbetter (1967)*, p. 183), such a process exists on $D[0,\infty)$ and has continuous trajectories with probability 1.

Let us consider the relevant processes as random elements on $D[0,\infty)$, endowed with the Skorohod topology (see *Billingsley (1968)*, *Lindwall (1972)*). With a suitable choice of metric, $D[0,\infty)$ becomes a complete, separable, metric space and it can be shown that convergence in distribution of random elements (denoted $Z^{(n)} \Rightarrow Z$) is equivalent to $r_\alpha Z^{(n)} \Rightarrow r_\alpha Z$, $\forall \alpha \in T_Z$, where r_α is the restriction to the interval $[0,\alpha]$, convergence is considered on $D[0,\alpha)$ and T_Z is the set of points at which Z is a.s. continuous (*Lindwall (1972)*).

We already know that the finite-dimensional distributions of $Z^{(n)}$ on $[0,\alpha]$ converge to those of a continuous Gaussian process with covariance function C. The tightness of $\{Z^{(n)}\}$ on $[0,\alpha]$ can be checked by a product moment condition (see *Billingsley (1968)*, p. 128). It can easily be shown that

$$\mathrm{E}\left(\left(Z^{(n)}(t)-Z^{(n)}(t_1)\right)^2\left(Z^{(n)}(t_2)-Z^{(n)}(t)\right)^2\right)\leq 3\left(M^{(n)}(t_2)-M^{(n)}(t_1)\right)^2,$$
$$0\leq t_1\leq t\leq t_2.$$

Letting $w_f(\delta)$ denote the modulus of continuity of f (on $[0,\alpha]$, in this case), tightness of $\{Z^{(n)}\}$ follows if

$$\lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\, w_{M^{(n)}}(\delta)=0.$$

This follows if e.g. $\{M^{(n)}\}$ converge to a continuous M, uniformly on compacts. Thus, denoting the above mentioned Gaussian process by Z, we have shown that $Z^{(n)}\Rightarrow Z$. Similarly, by Donsker's theorem, we have

$$\sqrt{n}\left(\frac{\overline{\xi}^{(n)}(t)-\alpha t}{\sqrt{K}}\right)\Rightarrow W(t)\ ,$$

where $W(t)$ denotes standard Brownian motion on $D[0,\infty)$ (the array $\{\xi_i^{(n)}\}$, $1\leq i\leq n+m^{(n)}$, has to be extended to $1\leq i<\infty$ to allow for $t>1+\mu^{(n)}$). Furthermore, we have (see *Serfozo (1975)*) $\overline{\xi}^{(n)}(t)\Rightarrow \alpha t$, $t\geq 0$, wherefore

$$\widetilde{Z}^{(n)}(t)\Rightarrow Z(\alpha t).$$

Let us now study the distribution of $A^{(n)}$. A condition of the type $\sqrt{n}\left(M^{(n)}-M\right)\to 0$, uniformly on compacts, allows us to equivalently study the process defined as $\sqrt{n}(M(\overline{\xi}^{(n)}(t))-M(\alpha t))$. By using the mean value theorem trajectorywise, the fact that multiplication and addition of random elements are continuous operations and that convergence in probability to a constant function entails convergence in distribution, we may conclude that $A^{(n)}\Rightarrow A$, where A is a Gaussian process with mean 0 and covariance function $KM'(\alpha s)M'(\alpha t)(s\wedge t)$. Again, some mild further condition on the continuity of M' implies that A is a.s. continuous. The same result can be achieved by assuming that $M^{(n)}$ itself is sufficiently differentiable,$\forall n$, with derivatives converging uniformly to those of M. The condition $\sqrt{n}\left(M^{(n)}-M\right)\to 0$, uniformly on compacts, is not necessary, then.

Since $Z^{(n)}$ and $A^{(n)}$ are independent, $\forall n$, and $\overline{\xi}^{(n)}$ converges to a constant function, we can state the above results as

$$\left(Z^{(n)},A^{(n)},\overline{\xi}^{(n)}\right)\Rightarrow(Z,A,\alpha t)$$

on $D^3[0,\infty)$, with the natural product metric. Since all functions in the RHS are a.s. continuous and composition and addition are continuous operations, we finally have

$$V^{(n)}=Z^{(n)}\circ\overline{\xi}^{(n)}+A^{(n)}\Rightarrow V\ ,$$

a continuous Gaussian process with mean 0 and covariance function $C(\alpha s,\alpha t)+KM'(\alpha s)M'(\alpha t)(s\wedge t)$.

3.4 Convergence in distribution of $T^{(n)}$ in the case $\mu > 0$. Let us define

$$\tau^{(n)} = \min\left\{t : t - \mu^{(n)} - M^{(n)}(\alpha t) = 0\right\}, \quad \forall n ,$$

and

$$\tau = \min\left\{t : t - \mu - M(\alpha t) = 0\right\} ,$$

and assume that τ is a true crossing point, i.e. that $\alpha M'(\alpha\tau) < 1$. Then, we have $\tau^{(n)} \to \tau$. Furthermore, since $\sup|V^{(n)}| \xrightarrow{d} \sup|V|$, which is bounded with probability 1, we will also have $\overline{T}^{(n)} \xrightarrow{p} \tau$.

Thus, on $D^2[0,\infty)$, we have $\left(V^{(n)}, \overline{T}^{(n)}\right) \Rightarrow (V, \tau)$ and consequently $V^{(n)} \circ \overline{T}^{(n)} \xrightarrow{d} V(\tau)$, which means

$$\sqrt{n}\left(\frac{\widetilde{X}^{(n)}\left(T_*^{(n)}\right)}{n} - M^{(n)}\left(\alpha\overline{T}^{(n)}\right)\right) \xrightarrow{d} N\left(0, C(\alpha\tau,\alpha\tau) + KM'(\alpha\tau)^2\tau\right).$$

Rewriting the LHS as

$$\begin{aligned}&\sqrt{n}\left(\overline{T}^{(n)} - \mu^{(n)} - M^{(n)}\left(\alpha\overline{T}^{(n)}\right)\right)\\ &\quad = \sqrt{n}\left(\left(\overline{T}^{(n)} - M^{(n)}\left(\alpha\overline{T}^{(n)}\right)\right) - \left(\tau^{(n)} - M^{(n)}\left(\alpha\tau^{(n)}\right)\right)\right),\end{aligned}$$

we finally get

$$\sqrt{n}\left(\overline{T}^{(n)} - \tau^{(n)}\right) \xrightarrow{d} N\left(0, \left(C(\alpha\tau,\alpha\tau) + KM'(\alpha\tau)^2\tau\right)\left(1 - \alpha M'(\alpha\tau)\right)^{-2}\right).$$

3.5 Convergence in distribution of $T^{(n)}$ in the case $m^{(n)} = m, \forall n$. Let us firstly require that two conditions be fulfilled:

1) $\forall \{a_n\} : a_n/n \to 0$, we have $(n/t)M^{(n)}(t/n) \to \lambda \geq 0$, uniformly on $0 \leq t \leq a_n$. λ will typically be $M'(0)$.
2) For any sequence of sets $\{S_n\}$ with cardinalities $|S_n|$ fulfilling $|S_n|/n \to 0$ and $\{a_n\}$ such that $a_n/n \to 0$, we have

$$\frac{1}{t}\sum_{j\in S_n} F_j^{(n)}(t) \to 0 , \quad \text{uniformly for } 0 \leq t \leq a_n.$$

Under these conditions, it is easily verified that the distribution of $\{T_k^{(n)}\}$, $0 \leq k \leq N$, for any fixed $N \geq 0$, converges to that of the succesive cumulated generation sizes in a Galton-Watson process, started by m ancestors, with progeny distribution with generating function $g(s) = \int \exp(-\lambda\xi(1-s))\, dH(\xi)$ and mean $\lambda\alpha$ (see *Ball (1983)* for a similar result). Thus we will have

$$\Pr\left(T^{(n)} = k\right) \to p(k), \quad k \in \mathbb{N} ,$$

where $p(\cdot)$ is the total size distribution in the above mentioned G-W process. For this distribution, it is known that the total probability mass equals γ^m, where $\gamma = 1$ if $\lambda\alpha \leq 1$, but $\gamma < 1$ if $\lambda\alpha > 1$ (γ is the solution closest to 0 of $g(s) = s$).

In the case $\lambda\alpha > 1$, there thus remains the probability mass $1 - \gamma^m > 0$ to account for. We will follow the strategy in *Scalia-Tomba (1985)*, in order to prove that

$$\Pr\left(\frac{T^{(n)} - n\tau}{\sqrt{n}} \in K\right) \to (1 - \gamma^m) \int_K dN(0, v) \ ,$$

K bounded, $N(0, v)$ denoting a normal distribution with mean 0 and variance v equal to that obtained in the previous section for the case $\mu > 0$. Finally, τ is defined as $\min\{t > 0 : t - M(\alpha t) = 0\}$.

Before continuing the demonstration, let us study the meaning of the conditions imposed on $M^{(n)}$ near 0 in a special but interesting case. Assume that

(1) $\quad F_j^{(n)}(t) = 1 - \exp(-\theta_j^{(n)} t/n), \ 1 \leq j \leq n, \text{with} \quad G^{(n)}(\theta) = \#\{\theta_j^{(n)} \leq \theta\}/n.$

Then $M^{(n)}(t) = 1 - \hat{G}^{(n)}(t)$, $\hat{G}^{(n)}$ being the Laplace transform of $G^{(n)}$. The requirement that $M^{(n)} \to M$ is then equivalent to $G^{(n)} \Rightarrow G$, G being a distribution with Laplace transform $1 - M$. The further assumptions on $M^{(n)}$ can easily be seen to mean the uniform integrability of $\{G^{(n)}\}$, with the parameter λ being the expectation of G. The condition that $M^{(n)} \to M$, uniformly on compacts, can be verified by combining the equicontinuity of $\{\{M^{(n)}, M\}\}$ with the pointwise convergence, on bounded intervals.

First, we prove, for any sequence $\{a_n\}$ such that $a_n/n \to 0$ and $a_n \to \infty$, that

$$\lim_{k\to\infty} \lim_{n\to\infty} \Pr\left(k \leq T^{(n)} \leq a_n\right) = 0.$$

We have

(2)
$$\Pr\left(k \leq T_*^{(n)} \leq a_n\right) \leq \sum_{i=k}^{a_n} \Pr\left(\tilde{X}^{(n)}(i) = i - m\right)$$
$$= \sum_{i=k}^{a_n} \int_0^\infty \Pr\left(X^{(n)}(t) = i - m\right) \ dH^{i*}(t).$$

We also have

$$\Pr\left(X^{(n)}(t) = k\right) = \sum_{|S|=k} \prod_{j\in S} F_j^{(n)}(t) \prod_{j\in S^C} \left(1 - F_j^{(n)}(t)\right)$$
$$\leq \frac{t^k}{k!} \left(\frac{n}{t} M^{(n)}\left(\frac{t}{n}\right)\right)^k \exp\left(-t\frac{\sum_{j\in S_*^C} F_j^{(n)}(t)}{t}\right),$$

where S_* is a set describing those j for which $F_j^{(n)}(t)$ is as large as possible, with $|S_*| = k$. Thus we have

$$\Pr\left(X^{(n)}(t) = k\right) \leq \frac{((\lambda + o(1))t)^k}{k!} \exp(-(\lambda + o(1))t) \ ,$$

with $o(1)$ denoting quantities converging uniformly to 0 for $t \in [0, a_n]$, $k \in [0, b_n]$, with a_n/n and $b_n/n \to 0$. Each integral in eq. (2) can be partitioned as follows:

$$\int_0^\infty \Pr\left(X^{(n)}(t) = i - m\right)\, dH^{i*}(t)$$
$$= \int_{|t-\alpha i|\le \epsilon i} (\dots)\, dH^{i*}(t) + \int_{|t-\alpha i|>\epsilon i} (\dots)\, dH^{i*}(t) = I_1 + I_2.$$

Choosing $\epsilon > 0$ so small that $\lambda(\alpha - \epsilon) > 1$, we have

$$\max_{|t-\alpha i|\le \epsilon i} \frac{((\lambda + o(1))t)^{i-m}}{(i-m)!} \exp(-(\lambda + o(1))t) =$$
$$\frac{((\lambda + o(1))(\alpha - \epsilon)i)^{i-m}}{(i-m)!} \exp(-(\lambda + o(1))(\alpha - \epsilon)i) \le$$
$$\exp(-i((\lambda + o(1))(\alpha + \epsilon) - \ln((\lambda + o(1))(\alpha - \epsilon)) - 1)) = z_i.$$

Thus $I_1 \le z_i$ and furthermore $\sum_{i=k}^{a_n} z_i \sim e^{-ck}$, with $c > 0$. By further assuming e.g. that H has a finite fourth moment (this is probably not necessary, however), Tchebyschev's inequality yields that $I_2 \le c/i^2$, since

$$\Pr\left(\left|\sum_{j=1}^{i} \xi_j - \alpha i\right| > ci\right) \le \frac{\mathrm{E}\left(\sum_{j=1}^{i} (\xi_j - \alpha)\right)^4}{c^4 i^4} = \frac{i m^{(4)} + 6K^2\binom{i}{2}}{c^4 i^4} \sim \frac{c}{i^2}.$$

Thus

$$\lim_{n\to\infty} \Pr\left(k \le T_*^{(n)} \le a_n\right) = e^{-ck} + o(k^{-1}).$$

The remaining range of $T_*^{(n)}$ can be studied exactly as in *Scalia-Tomba (1985)*, yielding

$$\lim_{c\to\infty} \lim_{n\to\infty} \Pr\left(\sqrt{n}\left(\overline{T}^{(n)} - \tau\right) \le c\right) = 1 - \gamma^m.$$

The final part of the proof amounts to showing that the limit law of $V^{(n)}$ is unchanged by conditioning on the event $\{T^{(n)} > a_n\}$, for some suitable sequence $\{a_n\}$ such that $a_n \to \infty$ and $a_n/n \to 0$. The conditioning event involves a_n variables of ξ-type and $O(a_n)$ members of the family $\{F_i^{(n)}(t)\}_i$ considered up to time $t \sim O(a_n/n)$. The proof will thus procede exactly as in *Scalia-Tomba (1985)*, by showing that the effect of $O(a_n)$ variables on sums with $O(n)$ terms will be vanishingly small and that thus the limit law of $V^{(n)}$ will be unchanged. The limit law of $\sqrt{n}(\overline{T}_n - \tau)$, conditional on $\{T^{(n)} > a_n\}$, will then be normal with mean 0 and variance as in the previous section.

3.6 Some comments on heterogeneous susceptibility. In the studied model, the law of the Q-variable of an individual can be seen as chosen without replacement from the family $\{F_i^{(n)}\}$. If it were chosen <u>with</u> replacement, the Q-variables would be i.i.d. with law $M^{(n)}$, instead. It is then interesting to note that the branching process approximation is unchanged, the values of γ and τ also, and that the only difference is found in the asymptotic variance v of the limit distribution in case of a "large" outbreak. The variance in the i.i.d. case will be larger, since

$$C^{(n)}(t,t) = M^{(n)}(t) - \frac{1}{n}\sum_{i=1}^{n} F_i^{(n)}(nt)^2 \leq M^{(n)}(t) - M^{(n)}(t)^2 .$$

Consider now the situation outlined earlier, in eq. (1), where $F^i = \exp(\theta_i)$ and $\{\theta_i\}$ have the distribution G. The corresponding model with i.i.d. $\{Q_i\}$ would then have $M(t) = \int 1 - e^{-\theta t}\, dG(\theta)$. In *Scalia-Tomba (1985)*, the risk function $r(t) = M'(t)/\big(1 - M(t)\big)$ was considered as a model of the behaviour of susceptibles at a given epidemic size corresponding to the fraction t of the population (assuming that $\alpha = 1$). However, considering G as a susceptibility distribution, we see that $r(t) = \mathrm{E}_{G_t}(\theta)$, where G_t is the law described by $\{e^{-\theta t} dG(\theta)/(1 - M(t))\}$. Thus $r(t)$ also has the interpretation of average susceptibility among survivors, after a given epidemic size that has modified the susceptibility profile of the population as described by G_t. By applying the Cauchy-Schwartz inequality to $r(t)$, we see that $r(t)$ is nonincreasing, whatever G, corresponding to the intuitive result that average susceptibility decreases as the epidemic progresses, the most susceptible individuals succumbing first in the epidemic, leaving a progressively less susceptible population to face the continued spread. In this notation, the susceptibility profile of the population, after the epidemic has ended, is described by G_τ. It is also worth noting that variation in susceptibility also results in a smaller total epidemic size (smaller τ) than the corresponding epidemic with constant susceptibility $\lambda = textE_G(\theta)$, as seen by applying Jensen's inequality to $M(t)$ and the definition of τ.

References

von Bahr B. and Martin-Löf A., *Threshold limit theorems for some epidemic processes*, Adv. Appl. Prob. **12** (1980), 319–349..

Ball F., *The threshold behaviour of epidemic models*, J. Appl. Prob. **20** (1983), 227–241.

Ball F., *Deterministic and stochastic epidemics with several kinds of susceptibles*, Adv. Appl. Prob **17** (1985)), 1–22..

Billingsley P., "Convergence of Probability Measures," John Wiley, New York, 1968.

Cramér H. and Leadbetter M.R., "Stationary and Related Stochastic Processes," John Wiley, New York, 1967.

Lindvall T., *Weak convergence of probability measures and random functions in the function space D[0,∞)*, J. Appl. Prob. **10** (1973), 109–121.

Sellke T., *On the asymptotic distribution of the size of a stochastic epidemic*, J. Appl. Prob. **20** (1983), 390–394.

Serfozo R.F., *Functional limit theorems for stochastic processes based on embedded processes*, Adv. Appl. Prob. **7** (1975), 123–139.

Scalia-Tomba G., *Asymptotic final size distribution for some chain-binomial processes*, Adv. Appl. Prob. **17** (1985), 477–495.

Wang F.J.S., *Gaussian approximation of some closed stochastic epidemic models*, J. Appl. Prob. **14** (1977), 221–231.

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