Stochastic Partial Differential Equations

Lectures given in Fudan University, Shanghaï, April 2007

É. Pardoux
Marseille, France

Contents

Chapter 1

Introduction and Motivation

1.1 Introduction

We shall study in these lectures parabolic PDEs, which will be mostly non linear. The general type of equations we have in mind is of the form

$$\frac{\partial u}{\partial t}(t,x) = F(t,x,u(t,x),Du(t,x),D^2u(t,x)) + G(t,x,u(t,x),Du(t,x)) \overset{\cdot}{W}(t,x),$$

or in the *semi linear* case

$$\frac{\partial u}{\partial t}(t,x) = \Delta u + f(t,x,u(t,x)) + g(t,x,u(t,x)) \overset{\cdot}{W}(t,x).$$

We shall make precise what we mean by $\overset{\cdot}{W}(t,x)$. We shall distinguish two cases

1. $\overset{\cdot}{W}$ is white noise in time and colored noise in space.

2. $\overset{\cdot}{W}$ is white both in time and in space.

In both cases, we can define $\overset{\cdot}{W}$ in the distributional sense, as a centered Gaussian process, indexed by test functions $h : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$:

$$\overset{\cdot}{W} = \{\overset{\cdot}{W}(h);\ h \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^d)\},$$

whose covariance is given by

$$\begin{aligned} \mathbb{E}\left(\overset{\cdot}{W}(h)\overset{\cdot}{W}(k)\right) &= \int_0^\infty dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\ h(t,x)k(t,y)\varphi(x-y) \ \text{ in case 1} \\ &= \int_0^\infty dt \int_{\mathbb{R}^d} dx\ h(t,x)k(t,x) \ \text{ in case 2.} \end{aligned}$$

Here φ is a "reasonable" kernel, which might blow up to infinity at 0. Note that the first formula converges to the second one, if we let φ converge to the Dirac mass at 0. On the other hand, the solution of a PDE of the form

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + f(t,x,u(t,x))$$

can be considered

1. either as a real valued function of (t,x);

2. or else as a function of t with values in an infinite dimensional space of functions of x (typically a Sobolev space).

Likewise, in the case of an SPDE of one of the above types, we can consider the solution

1. either as a one dimensional random field, solution of a multiparameter SDE;

2. or else as a stochastic process indexed by t, and taking values in an infinite dimensional function space, solution of an infinite dimensional SDE.

There are several serious difficulties in the study of SPDEs, which are due to the lack of regularity with respect to the time variable, and the interaction between the regularity in time and the regularity in space. As a result, as we will see, the theory of nonlinear SPDEs driven by space–time white noise, and with second order PDE operators, is limited to the case of a one dimensional space variable. Also, there is no really satisfactory theory of strongly nonlinear SPDEs. See the work of Lions and Souganidis on viscosity solutions of SPDEs, so far essentially unpublished.

1.2 Motivation

We now introduce several models from various fields, which are expressed as SPDEs.

1.2.1 Turbulence

Several mathematicians and physicists have advocated that the Navier–Stokes equation with additive white noise forcing is a relevant model for turbulence. This equation in dimension $d = 2$ or 3 reads

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \nu \Delta u(t,x) + \displaystyle\sum_{i=1}^{d} u_i(t,x) \frac{\partial u}{\partial x_i}(t,x) + \frac{\partial W}{\partial t}(t,x) \\ u(0,x) = u_0(x), \end{cases}$$

where $u(t,x) = (u_1(t,x), \dots, u_d(t,x))$ is the velocity of the fluid at time t and point x. The noise term is often choosen of the form

$$W(t,x) = \sum_{k=1}^{\ell} W^k(t) e_k(x),$$

where $\{W^1(t), \dots, W^\ell(t),\ t \geq 0\}$ are mutually independent standard Brownian motions.

1.2.2 Population dynamics, population genetics

The following model has been proposed by D. Dawson in 1972, for the evolution of the density of a population

$$\frac{\partial u}{\partial t}(t,x) = \nu \frac{\partial^2 u}{\partial x^2}(t,x) + \alpha \sqrt{u}(t,x)\, \dot{W}\,(t,x),$$

where $\dot{W}$ is a space–time white noise. In this case, one can derive closed equations for the first two moments

$$m(t,x) = \mathbb{E}[u(t,x)], \quad V(t,x,y) = \mathbb{E}[u(t,x)u(t,y)].$$

One can approach this SPDE by a model in discrete space as follows. $u(t,i)$, $i \in \mathbf{Z}$ denotes the number of individuals in the colony i at time t. Then

- $\frac{\alpha^2}{2} u(t,i)$ is both the birth and the death rate;
- $\nu u(t,i)$ is the emigration rate, both from i to $i-1$ and to $i+1$.

W. Fleming has proposed an analogous model in population genetics, where the term $\alpha\sqrt{u}$ is replaced by $\alpha\sqrt{u(1-u)}$.

1.2.3 Neurophysiology

The following model has been proposed by J. Walsh [27], in order to describe the propagation of an electric potential in a neuron (which is identified with the interval $[0, L]$).

$$\frac{\partial V}{\partial t}(t,x) = \frac{\partial^2 V}{\partial x^2}(t,x) - V(t,x) + g(V(t,x))\, \dot{W}\,(t,x).$$

Here again $\dot{W}\,(t,x)$ denotes a space–time white noise.

1.2.4 Evolution of the curve of interest rate

This model has been studied by R. Cont in 1998. Let $\{u(t,x),\ 0 \leq x \leq L, t \geq 0\}$ the interest rate for a loan at time t, and duration x. We let

$$u(t,x) = r(t) + s(t)(Y(x) + X(t,x)),$$

where $Y(0) = 0,\ Y(L) = 1;\ X(t,0) = 0,\ X(t,L) = 1;\ \{(r(t), s(t)),\ t \geq 0\}$ is a two dimensional diffusion process, and X solves the following parabolic SPDE

$$\frac{\partial X}{\partial t}(t,x) = \frac{k}{2}\frac{\partial^2 X}{\partial x^2}(t,x) + \frac{\partial X}{\partial x}(t,x) + \sigma(t, X(t,x))\, \dot{W}\,(t,x).$$

Several authors have proposed a first order parabolic SPDE (i. e. the above equation for X with $k = 0$), with a finite dimensional noise.

1.2.5 Non Linear Filtering

Consider the $\mathbb{R}^{d+k}$–valued process $\{(X_t, Y_t)\, t \geq 0\}$, solution of the system of SDEs

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t b(s, X_s, Y)ds + \int_0^t f(s, X_s, Y)dV_s + \int_0^t g(s, X_s, Y)dW_s \\ Y_t = \displaystyle\int_0^t h(s, X_s, Y)ds + W_t, \end{cases}$$

where the coefficients b, f, g and h may depend at each time s upon the whole past of Y before time s. We are interested in the evolution in t of the conditionnal law of X_t, given $\mathcal{F}_t^Y = \sigma\{Y_s,\ 0 \leq s \leq t\}$. It is known that if we denote by $\{\sigma_t,\ t \geq 0\}$ the measure–valued process solution of the Zakai equation

$$\sigma_t(\varphi) = \sigma_0(\varphi) + \int_0^t \sigma_s(L_{sY}\varphi)ds + \sum_{\ell=1}^{k} \int_0^t \sigma_s(L_{sY}^{\ell}\varphi)dY_s^{\ell},\ t \geq 0, \varphi \in C_b^{\infty}(\mathbb{R}^d)$$

where σ_0 denotes the law of X_0, and, if $a = ff^* + gg^*$,

$$L_{sY}\varphi(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x,Y)\frac{\partial^2\varphi}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(t,x,Y)\frac{\partial \varphi}{\partial x_i}(x),$$

$$L_{sY}^{\ell}\varphi(x) = h_{\ell}(t,x,Y)\varphi(x) + \sum_{j=1}^{d} g_{i\ell}(t,x,)\frac{\partial \varphi}{\partial x_i}(x)$$

then

$$\mathbb{E}(\varphi(X_t)|\mathcal{F}_t) = \frac{\sigma_t(\varphi)}{\sigma_t(1)},$$

i. e. σ_t, is equal, up to a normalization factor, to the conditionnal law of X_t, given $\mathcal{F}_t$, see e. g. [22]. Note that whenever the random measure σ_t posseses a density $p(t,x)$, the latter satisfies the following SPDE

$$\begin{aligned} dp(t,x) = &\left(\frac{1}{2}\sum_{i,j}\frac{\partial^2(a_{ij}p)}{\partial x_i \partial x_j}(t,x,Y)dt - \sum_i \frac{\partial(b_i p)}{\partial x_i}(t,x,Y)\right)dt \\ &+ \sum_{\ell}\left(h_{\ell}p(t,x,Y) - \sum_i \frac{\partial(g_{i\ell}p)}{\partial x_i}(t,x,Y)\right)dY_t^{\ell}. \end{aligned}$$

1.2.6 Movement by mean curvature in random environment

Suppose that each point of a hypersurface in $\mathbb{R}^d$ moves in the direction normal to the hypersurface, with a speed gien by

$$dV(x) = v_1(Du(x), u(x))dt + v_2(u(x)) \circ dW_t,$$

where $\{W_t,\ t \geq 0\}$ is a one–dimensional standard Brownian motion, and the notation $\circ$ means that the stochastic integral is understood in the Stratonovich sense.

The hypersurface at time t is a level set of the function $\{u(t,x),\ x \in \mathbb{R}^d\}$, where u solves a nonlinear SPDE of the form

$$du(t,x) = F(D^2u, Du)(t,x)dt + H(Du)(t,x) \circ dW_t,$$

where

$$F(X,p) = \mathrm{tr}\left[\left(I - \frac{p \otimes p}{|p|^2}\right) X\right], \quad H(p) = \alpha|p|.$$

This is our unique example of a strongly nonlinear SPDE, which cannot be studied with the methods presented in these notes. It is one of the motivating examples for the study of viscosity solutions of SPDEs, see Lions, Souganidis [12].

1.2.7 Hydrodynamic limit of particle systems

The following model has been proposed by L. Bertini and G. Giacomin [2]. The idea is to describe the movement of a curve in $\mathbb{R}^2$ which is the interface between e. g. water and ice. The true model should be in $\mathbb{R}^3$, but this is an interesting simplified model.

Consider first a discrete model, where the set of interfaces is the set

$$\Lambda = \{\xi \in \mathbf{Z}^{\mathbf{Z}},\ |\xi(x+1) - \xi(x)| = 1,\ \forall x \in \mathbf{Z}\}.$$

We describe the infinitesimal generator of the process of interest as follows. For any $\varepsilon > 0$, we define the infinitesimal generator

$$\begin{aligned} L_\varepsilon(\xi) = \sum_{x \in \mathbf{Z}} \big[& c_\varepsilon^+(x,\xi)\{f(\xi + 2\delta_x) - f(\xi)\} \\ & + c_\varepsilon^-(x,\xi)\{f(\xi - 2\delta_x) - f(\xi)\}\big], \end{aligned}$$

where

$$\delta_x(y) = \begin{cases} 0, & \text{if } y \neq x; \\ 1, & \text{if } y = x; \end{cases}$$

$$c_\varepsilon^+(x,\xi) = \begin{cases} \frac{1}{2} + \sqrt{\varepsilon}, & \text{if } \xi(x) = \frac{\xi(x-1)+\xi(x+1)}{2} - 1; \\ 0, & \text{if not}; \end{cases}$$

$$c_\varepsilon^-(x,\xi) = \begin{cases} \frac{1}{2}, & \text{if } \xi(x) = \frac{\xi(x-1)+\xi(x+1)}{2} + 1; \\ 0, & \text{if not}. \end{cases}$$

Define $\{\xi^\varepsilon_t,\ t \geq 0\}$ as the jump Markov process with generator L^ε, and

$$u_\varepsilon(t,x) = \sqrt{\varepsilon}\left(\xi_{t/\varepsilon^2}\left(\frac{x}{\varepsilon}\right) - \left(\frac{1}{2\varepsilon^{3/2}} - \frac{1}{24\varepsilon^{1/2}}\right)t\right),$$

then we have the following result

Theorem 1.2.1. *If $\sqrt{\varepsilon}\xi^\varepsilon_0\left(\frac{x}{\varepsilon}\right) \Rightarrow u_0(x)$, and some technical conditions are met, then $u_\varepsilon(t,x) \Rightarrow u(t,x)$, where u solves (at least formally) the following SPDE*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) - \dfrac{1}{2}\left|\dfrac{\partial u}{\partial x}(t,x)\right|^2 + \dot{W}\,(t,x), \\ u(0,x) = u_0(x), \end{cases}$$

where $\dot{W}$ denotes the space–time white noise.

The last SPDE is named the KPZ equation, after Kardar, Parisi, Zhang. Note that if we define $v(t,x) = \exp[-u(t,x)]$, we have the following equation for v

$$\frac{\partial v}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(t,x) - v(t,x)\,\dot{W}\,(t,x).$$

If we regularize $\dot{W}$ in space, then we construct corresponding sequences v_n and u_n, which satisfy

$$\frac{\partial v_n}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 v_n}{\partial x^2}(t,x) - v_n(t,x)\,\dot{W}_n\,(t,x),$$

and

$$\frac{\partial u_n}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 u_n}{\partial x^2}(t,x) - \frac{1}{2}\left(\left|\frac{\partial u_n}{\partial x}(t,x)\right|^2 - c_n\right) + \dot{W}_n\,(t,x),$$

where $c_n \to 0$, as $n \to \infty$.

1.2.8 Fluctuations of an interface on a wall

Funaki and Olla [8] have proposed the following model in discrete space for the fluctuations of the microscopic height of an interface on a wall (the interface is forced to stay above the wall)

$$\begin{cases} dv_N(t,x) = -\left[V'(v_N(t,x)-v_N(t,x-1)) + V'(v_N(t,x)-v_N(t,x+1)\right] dt \\ \qquad\qquad + dW(t,x) + dL(t,x), \quad t \geq 0, x \in \Gamma = \{1,2,\ldots,N-1\}, \\ v_N(t,x) \geq 0, \quad L(t,x) \text{ is nondecreasing in } t, \text{ for all } x \in \Gamma \\ \displaystyle\int_0^\infty v_N(t,x) dL(t,x) = 0, \text{ for all } x \in \Gamma \\ v_N(t,0) = v_N(t,N) = 0, \quad t \geq 0, \end{cases}$$

where $V \in C^2(\mathbb{R})$, is symetric and V'' is positive, bounded and bounded away from zero, and $\{W(t,1),\ldots,W(t,N-1),\ t \geq 0\}$ are mutually independent standard Brownian motions. The above is a coupled system of reflected SDEs. Assuming that $v_N(0,\cdot)$ is a randomvector whose law is the invariant distribution of the solution of that system of reflected SDEs, one considers the rescaled macroscopic height

$$\overline{v}_N(t,x) = \frac{1}{N}\sum_{y\in\Gamma} v_N(N^2t,y)\mathbf{1}_{[y/N-1/2N,y/N+1/2N]}(x), \quad 0 \leq x \leq 1,$$

which here converges to 0, as $N \to \infty$. Now the fluctuations, defined by

$$u_N(t,x) = \frac{1}{\sqrt{N}}\sum_{y\in\Gamma} v_N(N^2t,y)\mathbf{1}_{[y/N-1/2N,y/N+1/2N]}(x), \quad 0 \leq x \leq 1,$$

converge, as $N \to \infty$, towards the solution of the reflected stochastic heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \nu \dfrac{\partial^2 u}{\partial x^2}(t,x) + \dot{W}(t,x) + \xi(t,x) \\ u(t,x) \geq 0, \ \xi \text{ is a random measure }, \displaystyle\int_{\mathbb{R}_+\times[0,1]} u(t,x)\xi(dt,dx) = 0 \\ u(t,0) = u(t,1) = 0, \end{cases}$$

where $\dot{W}(t,x)$ stands for the "space–time" white noise, and ν is a constant which is in particular a function of V. Note that this reflected stochastic heat equation has been studied in Nualart, P. [20].

Chapter 2

SPDEs as infinite dimensional SDEs

2.1 Itô calculus in Hilbert space

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a probability space equipped with a filtration $(\mathcal{F}_t)$ which is supposed to be right continuous and such that $\mathcal{F}_0$ contains all the $\mathbb{P}$–null sets of $\mathcal{F}$.

Martingales Let H be a Hilbert space, and $\{M_t,\ 0 \le t \le T\}$ be a continuous H–valued martingale, which is such that $\sup_{0 \le t \le T} \mathbb{E}(\|M_t\|^2) < \infty$.

Then $\{\|M_t\|^2,\ 0 \le t \le T\}$ is a continuous real–valued submartingale, and there exists a unique continuous increasing $\mathcal{F}_t$–adapted process $\{\langle M \rangle_t,\ 0 \le t \le T\}$ such that $\{\|M_t\|^2 - \langle M \rangle_t,\ 0 \le t \le T\}$ is a martingale.

We denote by $\{M_t \otimes M_t,\ 0 \le t \le T\}$ the $\mathcal{L}^1_+(H)$–valued process defined by

$$((M_t \otimes M_t)h, k)_H = (M_t, h)_H \times (M_t, k)_H,$$

$h, k \in H$. We have used the notation $\mathcal{L}^1_+(H)$ to denote the set of self–adjoint semi–definite linear positive trace–class operators from H into itself. We have the following Theorem, whose last assertion is due to Métivier and Pistone

Theorem 2.1.1. *To any continuous square integrable H–valued martingale $\{M_t,\ 0 \le t \le T\}$, we can associate a unique continuous adapted increasing $\mathcal{L}^1_+(H)$–valued process $\{\langle\langle M \rangle\rangle_t,\ 0 \le t \le T\}$ such that $\{M_t \otimes M_t - \langle\langle M \rangle\rangle_t,\ 0 \le$*

$t \le T\}$ is a martingale. Moreover, there exists a unique predictable $\mathcal{L}^1_+(H)$–valued process $\{Q_t,\ 0 \le t \le T\}$ such that

$$\langle\langle M \rangle\rangle_t = \int_0^t Q_s d\langle M \rangle_s, \quad 0 \le t \le T.$$

Note that since Tr is a linear operator,

$$\mathrm{Tr}(M_t \otimes M_t - \langle\langle M \rangle\rangle_t) = \|M_t\|^2 - \mathrm{Tr}\langle\langle M \rangle\rangle_t$$

is a real valued martingale, hence $\mathrm{Tr}\langle\langle M \rangle\rangle_t = \langle M \rangle_t$. Consequenty, we have that $\langle M \rangle_t = \int_0^t \mathrm{Tr} Q_s d\langle M \rangle_s$, and

$$\mathrm{Tr} Q_t = 1, \quad t \text{ a. e.}, \quad \text{a. s.} \tag{2.1}$$

Example 2.1.2. ***H–valued Wiener process*** *Let $\{B^k_t,\ t \ge 0,\ k \in \mathbb{N}\}$ be a collection of mutually independent standard scalar Brownian motions, and $Q \in \mathcal{L}^1_+(H)$. If $\{e_k,\ k \in \mathbb{N}\}$ is an orthonormal basis of H. Then the process*

$$W_t = \sum_{k \in \mathbb{N}} B^k_t Q^{1/2} e_k, \quad t \ge 0$$

is an H–valued square integrable martingale, with $\langle W \rangle_t = TrQ \times t$, and $Q_t = Q/TrQ$. It is called an H–valued Wiener process, or Brownian motion.

Conversely, if $\{M_t,\ 0 \le t \le T\}$ is a continuous H–valued martingale, such that $\langle M \rangle_t = c \times t$ and $Q_t = Q$, where $c \in \mathbb{R}_+$ and $Q \in \mathcal{L}^1_+(H)$ are deterministic, then $\{M_t,\ 0 \le t \le T\}$ is an H–valued Wiener process (this is an infinite dimensional version of a well–known theorem due to P. Lévy).

Example 2.1.3. Cylindrical Brownian motion *This should be called a "counter–example", rather than an example. Let again $\{B^k_t,\ t \ge 0,\ k \in \mathbb{N}\}$ be a collection of mutually independent standard scalar Brownian motions, and $\{e_k,\ k \in \mathbb{N}\}$ an orthonormal basis of H. Then the series*

$$W_t = \sum_{k \in \mathbb{N}} B^k_t e_k$$

does not converge in H. In fact it converges in any larger space K such that the injection from H into K is Hilbert–Schmidt. We shall call such a process a cylindrical Wiener process on H (which does not take its values in H !). Formally, $\langle\langle W \rangle\rangle_t = tI$, which is not trace class !

Stochastic integral with respect to an H–valued martingale Let $\{\varphi_t,\ 0 \le t \le T\}$ be a predictable H–valued process such that

$$\int_0^T (Q_t\varphi_t, \varphi_t)_H d\langle M\rangle_t < \infty \quad \text{a. s.}$$

Then we can define the stochastic integral

$$\int_0^t (\varphi_s, dM_s)_H, \quad 0 \le t \le T.$$

More precisely, we have that

$$\int_0^t (\varphi_s, dM_s)_H = \lim_{n\to\infty} \sum_{i=1}^{n-1} \left(\frac{1}{t_i^n - t_{i-1}^n} \int_{t_{i-1}^n}^{t_i^n} \varphi_s ds, M_{t_{i+1}^n \wedge t} - M_{t_i^n \wedge t} \right)_H,$$

with for example $t_i^n = iT/n$. The above limit holds in probability.

The process $\{\int_0^t (\varphi_s, dM_s)_H,\ 0 \le t \le T\}$ is a continuous $\mathbb{R}$–valued local martingale, with

$$\langle \int_0^{\cdot} (\varphi_s, dM_s)_H \rangle_t = \int_0^t (Q_s\varphi_s, \varphi_s)_H d\langle M\rangle_s,$$

and if moreover

$$\mathbb{E} \int_0^T (Q_t\varphi_t, \varphi_t)_H d\langle M\rangle_t < \infty,$$

then the above stochastic integral is a square integrable martingale.

Stochastic integral with respect to a cylindrical Brownian motion Let again $\{\varphi_t,\ 0 \le t \le T\}$ be a predictable H–valued process, and we suppose now that

$$\int_0^T \|\varphi_t\|_H^2 dt < \infty \quad \text{a. s.}$$

It is then not very difficult to show that

$$\int_0^t (\varphi_s, dW_s) = \lim_{n\to\infty} \sum_{k=1}^n \int_0^t (\varphi_s, e_k) dB_s^k$$

exists as a limit in probability.

Itô formula Let $\{X_t\}$, $\{V_t\}$ and $\{M_t\}$ be H–valued processes, where

- $X_t = X_0 + V_t + M_t, \quad t \geq 0,$
- $\{V_t\}$ is a bounded variation process with $V_0 = 0$,
- $\{M_t\}$ is a local martingale with $M_0 = 0$.

Let moreover $\Phi : H \to \mathbb{R}$ be such that $\Phi \in C^1(H; \mathbb{R})$, and for any $h \in H$, $\Phi''(h)$ exists in the Gateau sense, and moreover $\forall Q \in \mathcal{L}^1(H)$, the mapping $h \to \mathrm{Tr}(\Phi''(h)Q)$ is continuous. Then we have

$$\begin{aligned}\Phi(X_t) = \Phi(X_0) &+ \int_0^t (\Phi'(X_s), dV_s) + \int_0^t (\Phi'(X_s), dM_s) \\ &+ \frac{1}{2}\int_0^t \mathrm{Tr}(\Phi''(X_s)Q_s) d\langle M\rangle_s\end{aligned}$$

Example 2.1.4. *The case where $\Phi(h) = \|h\|_H^2$ will be important in what follows. In that case, we have*

$$\|X_t\|^2 = \|X_0\|^2 + 2\int_0^t (X_s, dV_s) + 2\int_0^t (X_s, dM_s) + \langle M\rangle_t,$$

since here $\Phi''/2 = I$, and $TrQ_s = 1$, see (2.1).

2.2 SPDE with additive noise

This is the simplest case, where the existence–uniqueness theory needs almost no more than the theory of deterministic PDEs. We are motivated by the two following examples :

1. The heat equation with additive noise. Let us consider our last example from section 1.2.8, but whithout the reflection, i. e. the SPDE (here in arbitrary dimension, $x \in D \subset \mathbb{R}^d$)

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \nu \Delta u(t,x) + \dfrac{\partial W}{\partial t}(t,x), \quad t \geq 0,\ x \in D \\ u(0,x) = u_0(x), \quad u(t,x) = 0,\ t \geq 0,\ x \in \partial D, \end{cases}$$

   where $\{W(t,x),\ t \geq 0, x \in D\}$ denotes a Wiener process with respect to the time variable, with arbitrary correlation in the spatial variable (possibly white in space).

2. The two–dimensional Navier–Stokes equation with additive finite dimensional noise. Its vorticity formulation is as follows

$$\begin{cases} \dfrac{\partial \omega}{\partial t}(t,x) + B(\omega,\omega)(t,x) = \nu\Delta\omega(t,x) + \dfrac{\partial W}{\partial t}(t,x) \\ \omega(0,x) = \omega_0(x), \end{cases}$$

where $x = (x_1, x_2) \in \mathbf{T}^2$, the two-dimensional torus $[0, 2\pi] \times [0, 2\pi]$, $\nu > 0$ is the viscosity constant, $\frac{\partial W}{\partial t}$ is a white–in–time stochastic forcing of the form

$$W(t,x) = \sum_{k=1}^{\ell} W_k(t)e_k(x),$$

where $\{W_1(t), \ldots, W_\ell(t)\}$ are mutually independent standard Brownian motions and

$$B(\omega,\tilde{\omega}) = \sum_{i=1}^{2} u_i(x)\frac{\partial\tilde{\omega}}{\partial x_i}(x)$$

where $u = \mathcal{K}(\omega)$. Here $\mathcal{K}$ is the Biot-Savart law which in the two-dimentsional periodic setting can be expressed

$$\mathcal{K}(\omega) = \sum_{k\in\mathbf{Z}^2_*} \frac{k^\perp}{|k|^2}\left[\beta_k\cos(k\cdot x) - \alpha_k\sin(k\cdot x)\right] \tag{2.2}$$

where $k^\perp = (-k_2, k_1)$ and $\omega(t,x) = \sum_{k\in\mathbf{Z}^2_*} \alpha_k\cos(k\cdot x) + \beta_k\sin(k\cdot x)$ with $\mathbf{Z}^2_* = \{(j_1, j_2) \in \mathbf{Z}^2 : j_2 \geq 0, |j| > 0\}$.

Let us start with some results on PDEs, sketching two different approaches.

2.2.1 The semi–group approach to linear parabolic PDEs

First consider the following abstract linear parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t) = Au(t),\ t \geq 0 \\ u(0) = u_0, \end{cases}$$

where A is (possibly unbounded) linear operator in some Hilbert space H, i. e. A maps its domain $D(A) \subset H$ into H. Suppose that $u_0 \in H$, and we are looking for a solution which should take its values in H. For each $t > 0$, the mapping $u_0 \to u(t)$ is a linear mapping $P(t) \in \mathcal{L}(H)$, and the mappings $\{P(t),\ t \geq 0\}$ form a semigroup, in the sense that $P(t+s) = P(t)P(s)$. A is called the infinitesimal generator of this semigroup. Suppose now that $H = L^2(D)$, where D is some domain in $\mathbb{R}^d$. Then the linear operator $P(t)$ has a kernel $p(t,x,y)$ such that $\forall h \in L^2(D)$,

$$[P(t)h](x) = \int_D p(t,x,y)h(y)dy.$$

Example 2.2.1. *If $D = \mathbb{R}^d$, and $A = \frac{1}{2}\Delta$, then*

$$p(t,x,y) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x-y|^2}{2t}\right).$$

Consider now the PDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t) = Au(t) + f(t),\ t \geq 0 \\ u(0) = u_0, \end{cases}$$

where $f(\cdot)$ is an H–valued function of t. The solution of this last equation is given by the *variation of constants formula*

$$u(t) = P(t)u_0 + \int_0^t P(t-s)f(s)ds.$$

Consider now the parabolic equation with additive white noise, i. e.

$$\begin{cases} \dfrac{du}{dt}(t) = Au(t) + \dfrac{dW}{dt}(t),\ t \geq 0 \\ u(0) = u_0, \end{cases} \tag{2.3}$$

where $\{W(t),\ t \geq 0\}$ is an H–valued Wiener process. Then the variation of constants formula, generalized to this situation, yields the following formula for $u(t)$:

$$u(t) = P(t)u_0 + \int_0^t P(t-s)dW(s),$$

in terms of a Wiener integral. In the case $H = L^2(D)$, $W(t) = W(t, x)$ and this formula can be rewritten in terms of the kernel of the semigroup as follows

$$u(t,x) = \int_D p(t,x,y)u_0(y)dy + \int_0^t \int_D p(t-s,x,y)W(ds,y)dy.$$

In the case of the cylindrical Wiener process, i. e. if the equation is driven by space–time white noise, then the above formula takes the form

$$u(t,x) = \int_D p(t,x,y)u_0(y)dy + \int_0^t \int_D p(t-s,x,y)W(ds,dy),$$

where $\{W(t,x),\ t \geq 0, x \in D\}$ denotes the so–called Brownian sheet, and the above is a two–parameter stochastic integral, which we will discuss in more detail in chapter 3. We just considered a case where $W(t)$ does not take its values in H.

Let us now discuss the opposite case, where $W(t)$ takes its values not only in H, but in fact in $D(A)$. Then considering again the equation (2.3), and defining $v(t) = u(t) - W(t)$, we have the following equation for v :

$$\begin{cases} \dfrac{dv}{dt}(t) = Av(t) + AW(t) \\ v(0) = u_0, \end{cases}$$

which can be solved ω by ω, whithout any stochastic integration.

2.2.2 The variational approach to linear and nonlinear parabolic PDEs

We now sketch the variational approach to deterministic PDEs, which was developped among others by J. L. Lions. We first consider the case of

Linear equations From now on, A will denote an extension of the unbounded operator from the previous section. That is, instead of considering

$$A\ :\ D(A) \longrightarrow H,$$

we shall consider

$$A\ :\ V \longrightarrow V',$$

where

$$D(A) \subset V \subset H \subset V'.$$

More precisely, the framework is as follows.

H is a separable Hilbert space. We shall denote by $|\cdot|_H$ or simply by $|\cdot|$ the norm in H and by $(\cdot,\cdot)_H$ or simply $(\cdot,\cdot)$ its scalar product. Let $V \subset H$ be a reflexive Banach space, which is dense in H, with continuous injection. We shall denote by $\|\cdot\|$ the norm in V. We shall identify H with its dual H', and consider H' as a subspace of the dual V' of V, again with continuous injection. We then have the situation

$$V \subset H \simeq H' \subset V'.$$

More precisely, we assume that the duality pairing $\langle\cdot,\cdot\rangle$ between V and V' is such that whenever $u \in V$ and $v \in H \subset V'$, $\langle u, v\rangle = (u,v)_H$. Finally, we shall denote by $\|\cdot\|_*$ the norm in V', defined by

$$\|v\|_* = \sup_{u\in V,\ \|u\|\leq 1} \langle u, v\rangle.$$

We can whithout loss of generality assume that whenever $u \in V$, $|u| \leq \|u\|$. It then follows (exercise) that if again $u \in V$, $\|u\|_* \leq |u| \leq \|u\|$.

Now suppose an operator $A \in \mathcal{L}(V, V')$ is given, which is assumed to satisfy the following *coercivity* assumption :

$$\begin{cases} \exists \lambda, \alpha > 0 \ \text{ such that } \ \forall u \in V, \\ 2\langle Au, u\rangle + \alpha\|u\|^2 \leq \lambda|u|^2, \end{cases}$$

Example 2.2.2. *Let D be an open domain in $\mathbb{R}^d$. We let $H = L^2(D)$ and $V = H^1(D)$, where*

$$H^1(D) = \{u \in L^2(D);\ \frac{\partial u}{\partial x_i} \in L^2(D),\ i = 1, \ldots, d\}.$$

Equipped with the scalar product

$$((u,v)) = \int_D u(x)v(x)dx + \sum_{i=1}^d \int_D \frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_i}(x)dx,$$

$H^1(D)$ is a Hilbert space, as well as $H^1_0(D)$, which is the closure in $H^1(D)$ of the set $C^\infty_K(D)$ of smooth functions with support in a compact subset of D. We now let

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}.$$

$\Delta \in \mathcal{L}(H^1(D), [H^1(D)]')$, and also $\Delta \in \mathcal{L}(H^1_0(D), [H^1_0(D)]')$. Note that provided that the boundary ∂D of D is a little bit smooth, $H^1_0(D)$ can be identified with the closed subset of $H^1(D)$ consisting of those functions which are zero on the boundary ∂D (one can indeed make sense of the trace of $u \in H^1(D)$ on the boundary ∂D. $[H^1_0(D)]' = H^{-1}(D)$, where any element of $H^{-1}(D)$ can be put in the form

$$f + \sum_{i=1}^{d} \frac{\partial g_i}{\partial x_i},$$

where $f, g_1, \dots, g_d \in L^2(D)$.

We consider the linear parabolic equation

$$\begin{cases} \dfrac{du}{dt}(t) = Au(t) + f(t), \ t \geq 0; \\ u(0) = u_0. \end{cases} \tag{2.4}$$

We have the

Theorem 2.2.3. *If $A \in \mathcal{L}(V, V')$ is coercive, $u_0 \in H$ and $f \in L^2(0, T; V')$, then the equation* (2.4) *has a unique solution $u \in L^2(0, T; V)$, which also belongs to $C([0, T]; H)$.*

We first need to show the following interpolation result

Lemma 2.2.4. *If $u \in L^2(0, T; V)$, $t \to u(t)$ is absolutely continuous with values in V', and $\frac{du}{dt} \in L^2(0, T; V')$, then $u \in C([0, T]; H)$ and*

$$\frac{d}{dt}|u(t)|^2 = 2\langle \frac{du}{dt}(t), u(t) \rangle, \ t \text{ a. e.}$$

Proof of Theorem 2.2.3 **Uniqueness** Let u, $v \in L^2(0, T; V)$ two solutions of equation (2.4). Then the difference $u - v$ solves

$$\begin{aligned} &\frac{d(u-v)}{dt}(t) = A(u(t) - v(t)), \\ &u(0) - v(0) = 0. \end{aligned}$$

Then from the Lemma,

$$|u(t)-v(t)|^2 = 2\int_0^t \langle A(u(s)-v(s)), u(s)-v(s)\rangle ds \le \lambda \int_0^t |u(s)-v(s)|^2 ds,$$

and Gronwall's lemma implies that $u(t)-v(t)=0$, $\forall t \ge 0$.

Existence We use a Galerkin approximation. Let $\{e_k,\ k\ge 1\}$ denote an orthonormal basis of H, made of elements of V. For each $n \ge 1$, we define

$$V_n = \text{span}\{e_1, e_2, \dots, e_n\}.$$

For all $n \ge 1$, there exists a function $u_n \in C([0,T]; V_n)$ such that for all $1 \le k \le n$,

$$\begin{aligned}\frac{d}{dt}(u_n(t), e_k) &= \langle Au_n(t), e_k\rangle + \langle f(t), e_k\rangle,\\ (u_n(0), e_k) &= (u_0, e_k).\end{aligned}$$

u_n is the solution of a finite dimensional linear ODE. We now prove the following uniform estimate

$$\sup_n \left[\sup_{0\le t\le T} |u_n(t)|^2 + \int_0^T \|u_n(t)\|^2 dt\right] < \infty. \tag{2.5}$$

It is easily seen that

$$|u_n(t)|^2 = \sum_{k=1}^n (u_0, e_k)^2 + 2\int_0^t \langle Au_n(s) + f(s), u_n(s)\rangle ds.$$

Hence we deduce from the coercivity of A that

$$|u_n(t)|^2 + \alpha \int_0^t \|u_n(s)\|^2 ds \le |u_0|^2 + \int_0^T \|f(s)\|_*^2 ds + (\lambda+1)\int_0^t |u_n(s)|^2 ds,$$

and (2.5) follows from Gronwall's lemma.

Now there exists a subsequence, which, by an abuse of notation, we still denote $\{u_n\}$, which converges in $L^2(0,T;V)$ weakly to some u. Since A is linear and continuous from V into V', it is also continuous for the weak topologies, and taking the limit in the approximating equation, we have a solution of (2.4). □

Let us now indicate how this approach can be extended to

Nonlinear equations Suppose now that $A : V \to V'$ is a nonlinear operator satisfying again the coercivity assumption. We can repeat the first part of the above proof. However, taking the limit in the approximating sequence is now much more involved. The problem is the following. While a continuous linear operator is continuous for the weak topologies, a nonlinear operator which is continuous for the strong topologies, typically fails to be continuous with respect to the weak topologies.

In the framework which has been exposed in this section, there are two possible solutions, which necessitate two different assumptions.

1. **Monotonicity**. If we assume that the non linear operator A satisfies in addition the condition

$$\langle A(u) - A(v), u - v\rangle \leq \lambda |u - v|^2,$$

together with some boundedness condition of the type $\|A(u)\|_* \leq c(1+\|u\|)$, and some continuity condition, then the above difficulty can be solved. Indeed, following the proof in the linear case, we show both that $\{u_n\}$ is a bounded sequence in $L^2(0,T;V)$ and that $\{A(u_n)\}$ is a bounded sequence in $L^2(0,T;V')$. Hence there exists a subsequence, still denoted the same way, along which $u_n \to u$ in $L^2(0,T;V)$ weakly, and $A(u_n) \to \xi$ weakly in $L^2(0,T;V')$. It remains to show that $\xi = A(u)$. Let us explain the argument, in the case where the monotonicity assumption is satisfied with $\lambda = 0$. Then we have that for all $v \in L^2(0,T;V)$,

$$\int_0^T \langle A(u_n(t)) - A(v(t)), u_n(t) - v(t)\rangle dt \leq 0.$$

The above expression can be developped into four terms, three of which converge whithout any difficulty to the wished limit. The only difficulty is with the term

$$\int_0^T \langle A(u_n(t)), u_n(t)\rangle dt = \frac{1}{2}\left[|u_n(T)|^2 - \sum_{k=1}^{n}(u_0, e_k)^2\right] - \int_0^T \langle f(t), u_n(t)\rangle dt.$$

Two of the three terms of the right hand side converge. The first one DOES NOT. But it is not hard to show that the subsequence can be choosen in such a way that $u_n(T) \to u(T)$ in H weakly, and the mapping which to a vector in H associates the square of its norm is convex and strongly continuous, hence it is the upper envelope of linear continuous (hence also weakly continuous) mappings, hence it is l. s. c. with respect to the weak topology of H, hence

$$\text{liminf}_n |u_n(T)|^2 \geq |u(T)|^2,$$

and consequently we have that, again for all $v \in L^2(0,T;V)$,

$$\int_0^T \langle \xi(t) - A(v(t)), u(t) - v(t) \rangle dt \leq 0.$$

We now choose $v(t) = u(t) - \theta w(t)$, with $\theta > 0$, divide by θ, and let $\theta \to 0$, yielding

$$\int_0^T \langle \xi(t) - A(u(t)), w(t) \rangle dt \leq 0.$$

Since w is an arbitrary element of $L^2(0,T,;V)$, the left hand side must vanish, hence $\xi \equiv A(u)$.

Example 2.2.5. *The simplest example of an operator which is monotone in the above sense is an operator of the form*

$$A(u)(x) = \Delta u(x) + f(u(x)),$$

where $f\ : \mathbb{R} \to \mathbb{R}$ *is the sum of a Lipschitz and a decreasing function.*

2. **Compactness** We now assume that the injection from V into H is compact (in the example $V = H^1(D)$, $H = L^2(D)$, this implies that D be bounded). Note that in the preceding arguments, there was no serious diffculty in proving that the sequence $\{\frac{du_n}{dt}\}$ is bounded in $L^2(0,T;V')$. But one can show the following compactness Lemma (see Lions [11]) :

   Lemma 2.2.6. *Let the injection from* V *into* H *be compact. If a sequence* $\{u_n\}$ *is bounded in* $L^2(0,T;V)$*, while the sequence* $\{\frac{du_n}{dt}\}$ *is bounded in* $L^2(0,T;V')$*, then one can extract a subsequence of the sequence* $\{u_n\}$*, which converges strongly in* $L^2(0,T;H)$*.*

   Let us explain how this Lemma can be used in the case of the Navier–Stokes equation. The nonlinear term is the sum of terms of the form $u_i(t,x)\frac{\partial u}{\partial x_i}$, i. e. the product of a term which converges strongly with a term which converges weakly, i. e. one can take the limit in that product.

PDE with additive noise Let us now consider the parabolic PDE

$$\begin{cases} \dfrac{du}{dt}(t) = A(u(t)) + f(t) + \dfrac{dW}{dt}(t), \ t \geq 0; \\ u(0) = u_0. \end{cases}$$

If we assume that the trajectories of the Wiener process $\{W(t)\}$ belong to $L^2(0, T; V)$, then we can define $v(t) = u(t) - W(t)$, and note that v solves the PDE with random coefficents

$$\begin{cases} \dfrac{dv}{dt}(t) = A(v(t) + W(t)) + f(t), \ t \geq 0; \\ u(0) = u_0, \end{cases}$$

which can again be solved ω by ω, whithout any stochastic integration. However, we want to treat equations driven by a noise which does not necessarily takes its values in V, and also may not be additive.

2.3 Variational approach to SPDEs

The framework is the same as in the last subsection.

2.3.1 Monotone – coercive SPDEs

Let $A : V \to V'$ and for each $k \geq 1$, $B_k : V \to H$, so that $B = (B_k, \ k \geq 1) : V \to \mathcal{H} = \ell^2(H)$.

We make the following four basic assumptions :

Coercivity

$$(H1) \begin{cases} \exists \alpha > 0, \lambda, \nu \ \text{ such that } \ \forall u \in V, \\ 2\langle A(u), u\rangle + |B(u)|^2_{\mathcal{H}} + \alpha\|u\|^2 \leq \lambda|u|^2 + \nu, \end{cases}$$

Monotonicity

$$(H2) \begin{cases} \exists \lambda > 0 \ \text{ such that } \ \forall u, v \in V, \\ 2\langle A(u) - A(v), u - v\rangle + |B(u) - B(v)|^2_{\mathcal{H}} \leq \lambda|u - v|^2. \end{cases}$$

Linear growth

$$(H3) \quad \exists c > 0 \ \text{ such that } \ \|A(u)\|_* \leq c(1 + \|u\|), \ \forall u \in V,$$

Weak continuity

$$(H4)\begin{cases}\forall u,v,w\in V,\\ \text{the mapping } \lambda\to\langle A(u+\lambda v),w\rangle\ \text{ is continuous from } \mathbb{R} \text{ into } \mathbb{R}.\end{cases}$$

Note that

$$|B(u)|^2_{\mathcal{H}}=\sum_{k=1}^{\infty}|B_k(u)|^2,\quad |B(u)-B(v)|^2_{\mathcal{H}}=\sum_{k=1}^{\infty}|B_k(u)-B_k(v)|^2.$$

We want to study the equation

$$\begin{aligned}u(t)&=u_0+\int_0^t A(u(s))ds+\int_0^t B(u(s))dW_s\\ &=u_0+\int_0^t A(u(s))ds+\sum_{k=1}^{\infty}\int_0^t B_k(u(s))dW^k_s,\end{aligned}\tag{2.6}$$

where $u_0\in H$, and $\{W_t=(W^k_t,\ k=1,2,\ldots),\ t\geq 0\}$ is a sequence of mutually independent $\mathcal{F}_t$–standard scalar Brownian motions. We shall look for a solution u whose trajectories should satisfy $u\in L^2(0,T;V)$, for all $T>0$. Hence $A(u(\cdot))\in L^2(0,T;V')$, for all $T>0$. In fact, the above equation can be considered as an equation in the space V', or equivalently we can write the equation in the so–called *weak form*

$$(u(t),v)=(u_0,v)+\int_0^t\langle A(u(s)),v\rangle ds+\int_0^t(B(u(s)),v)dW_s,\ \forall v\in V,t\geq 0,\tag{2.7}$$

where the stochatic integral term should be interpreted as

$$\int_0^t(B(u(s)),v)dW_s=\sum_{k=1}^{\infty}\int_0^t(B_k(u(s)),v)dW^k_s.$$

Remark 2.3.1. *Since $|u|\leq\|u\|$, it follows from $(H1)$ + $(H3)$ that for some constant c', $|B(u)|_{\mathcal{H}}\leq c'(1+\|u\|)$.*

We can w. l. o. g. assume that λ is the same in $(H1)$ and in $(H2)$. In fact one can always reduce to the case $\lambda=0$, since $v=e^{-\lambda t/2}u$ solves the same equation, with A replaced

$$e^{-\lambda t/2}A(e^{\lambda t/2}\cdot)-\frac{\lambda}{2}I,$$

and B replaced by

$$e^{-\lambda t/2}B(e^{\lambda t/2}\cdot),$$

and in many cases this new pair satisfies $(H1)$ and $(H2)$ with $\lambda = 0$.

Remark 2.3.2. *We can replace in $(H1)$ $\|u\|^2$ by $\|u\|^p$, with $p > 2$, provided we replace $(H3)$ by*

$$(H3)_p \quad \exists c > 0 \ \text{ such that } \ \|A(u)\|_* \le c(1 + \|u\|^{p-1}), \ \forall u \in V.$$

This modified set of assumptions is well adapted for treating certain non linear equations, see the last example in the next subsection. Note that the operator A can be the sum of several A_i's with different associated p_i's.

We can now state the main result of this section.

Theorem 2.3.3. *Under the assumptions $(H1)$, $(H2)$, $(H3)$ and $(H4)$, if $u_0 \in H$, there exists a unique adapted process $\{u(t),\ t \ge 0\}$ whose trajectories belong a. s. for any $T > 0$ to the space $L^2(0,T;V) \cap C([0,T];H)$, which is a solution to equation (2.6).*

An essential tool for the proof of this Theorem is the following ad hoc Itô formula:

Lemma 2.3.4. *Let $u_0 \in H$, $\{u(t),\ 0 \le t \le T\}$ and $\{v(t),\ 0 \le t \le T\}$ be adapted processes with trajectories in $L^2(0,T;V)$ and $L^2(0,T;V')$ respectively, and $\{M_t,\ 0 \le t \le T\}$ be a continuous H–valued local martingale, such that*

$$u(t) = u_0 + \int_0^t v(s)ds + M_t.$$

Then

(i) $u \in C([0,T];H)$ a. s.

(ii) the following formula holds $\forall 0 \le t \le T$ and a. s.

$$|u(t)|^2 = |u_0|^2 + 2\int_0^t \langle v(s), u(s)\rangle ds + 2\int_0^t (u(s), dM_s) + \langle M\rangle_t.$$

Proof: **Proof of (ii)** Since V is dense in H, there exists an orthonormal basis $\{e_k,\ k \geq 1\}$ of H with each $e_k \in V$. For the sake of this proof, we shall assume that V is a Hilbert space, and that the above basis is also orthogonal in V. Also these need not be true, it holds in many interesting examples. The general proof is more involved that the one which follows. We have, with the notation $M_t^k = (M_t, e_k)$,

$$\begin{aligned}
|u(t)|^2 &= \sum_k (u(t), e_k)^2 \\
&= \sum_k \left[(u_0, e_k)^2 + 2\int_0^t \langle v(s), e_k\rangle (e_k, u(s))ds + 2\int_0^t (u(s), e_k)dM_s^k + \langle M^k\rangle_t \right] \\
&= |u_0|^2 + 2\int_0^t \langle v(s), u(s)\rangle ds + 2\int_0^t (u(s), dM_s) + \langle M\rangle_t.
\end{aligned}$$

Proof of (i) It clearly follows from our assumptions that $u \in C([0,T]; V')$ a. s. Moreover, from (ii), $t \to |u(t)|$ is a. s. continuous. It suffices to show that $t \to u(t)$ is continuous into H equipped with its weak topology, since whenever $u_n \to u$ in H weakly and $|u_n| \to |u|$, then $u_n \to u$ in H strongly (easy exercise, exploiting the fact that H is a Hilbert space). Now, clearly $u \in L^\infty(0,T;H)$ a. s., again thanks to (ii). Now let $h \in H$ and a sequence $t_n \to t$, as $n \to \infty$ be arbitrary. All we have to show is that $(u(t_n), h) \to (u(t), h)$ a. s. Let $\{h_m,\ m \geq 1\} \subset V$ be such that $h_m \to h$ in H, as $m \to \infty$. Let us choose $\varepsilon > 0$ arbitrary, and m_0 large enough, such that

$$\sup_{0\leq t\leq T} |u(t)| \times |h - h_m| \leq \varepsilon/2, \quad m \geq m_0.$$

It follows that

$$\begin{aligned}
|(u(t), h) - (u(t_n), h)| &\leq |(u(t), h - h_{m_0})| + |(u(t) - u(t_n), h_{m_0})| + |(u(t_n), h - h_{m_0})| \\
&\leq \|u(t) - u(t_n)\|_* \times \|h_{m_0}\| + \varepsilon,
\end{aligned}$$

hence

$$\text{limsup}_n |(u(t), h) - (u(t_n), h)| \leq \varepsilon,$$

and the result follows from the fact that ε is arbitrary. □

We give a further result, which will be needed below.

Lemma 2.3.5. *Under the assumptions of Lemma 2.3.4, and given a function Φ from H into $\mathbb{R}$, which satisfies all the assumptions from the Itô formula* *in section 2.1, plus the fact that $\Phi'(u) \in V$ whenever $u \in V$, and that the mapping $u \to \Phi'(u)$ is continuous from V into V equipped with the weak topology, and for some c, all $u \in V$,*

$$\|\Phi'(u)\| \leq c(1 + \|u\|).$$

Then we have the Itô formula

$$\Phi(X_t) = \Phi(X_0) + \int_0^t \langle v_s, \Phi'(X_s)\rangle ds + \int_0^t (\Phi'(X_s), dM_s)$$
$$+ \frac{1}{2}\int_0^t Tr(\Phi''(X_s)Q_s)d\langle M\rangle_s$$

PROOF OF THEOREM 2.3.3 **Uniqueness** Let $u, v \in L^2(0,T;V) \cap C([0,T];H)$ a. s. be two adapted solutions. For each $n \geq 1$, we define the stopping time

$$\tau_n = \inf\{t \leq T;\ |u(t)|^2 \vee\ |v(t)|^2 \vee \int_0^t (\|u(s)\|^2 + \|v(s)\|^2)ds \geq n\}.$$

We note that $\tau_n \to \infty$ a. s., as $n \to \infty$. Now we apply Lemma 2.3.4 to the difference $u(t) - v(t)$, which satisfies

$$u(t) - v(t) = \int_0^t [A(u(s)) - A(v(s))]ds + \int_0^t [B(u(s)) - B(v(s))]dW_s.$$

Clearly $M_t = \int_0^t [B(u(s)) - B(v(s))]dW_s$ is a local martingale, and $\langle M\rangle_t = \int_0^t |B(u(s)) - B(v(s))|^2_{\mathcal{H}} ds$. Hence we have

$$|u(t) - v(t)|^2 = 2\int_0^t \langle A(u(s)) - A(v(s)), u(s) - v(s)\rangle ds$$
$$+ 2\int_0^t (u(s) - v(s), B(u(s)) - B(v(s)))dW_s$$
$$+ \int_0^t |B(u(s)) - B(v(s))|^2_{\mathcal{H}} ds$$

If we write that identity with t replaced by $t \wedge \tau_n = \inf(t, \tau_n)$, it follows from the first part of Remark 2.3.1 that the stochastic integral

$$\int_0^{t\wedge\tau_n} (u(s) - v(s), B(u(s)) - B(v(s)))dW_s$$

is a martingale with zero mean. Hence taking the expectation and exploiting the monotonicity assumption $(H2)$ yields

$$\begin{aligned}\mathbb{E}[|u(t\wedge\tau_n)-v(t\wedge\tau_n)|^2] &= 2\mathbb{E}\int_0^{t\wedge\tau_n}\langle A(u(s))-A(v(s)),u(s)-v(s)\rangle ds \\ &+\mathbb{E}\int_0^{t\wedge\tau_n}|B(u(s))-B(v(s))|^2_{\mathcal{H}}ds \\ &\le \lambda\mathbb{E}\int_0^{t\wedge\tau_n}|u(s)-v(s)|^2ds \\ &\le \lambda\mathbb{E}\int_0^t |u(s\wedge\tau_n)-v(s\wedge\tau_n)|^2ds,\end{aligned}$$

hence from Gronwall's Lemma, $u(t\wedge\tau_n)-v(t\wedge\tau_n)=0$ a. s., for all $0\le t\le T$ and all $n\ge 1$. Uniqueness is proved.
Existence We use a Galerkin approximation. Again, $\{e_k,\ k\ge 1\}$ denotes an orthonormal basis of H, made of elements of V. For each $n\ge 1$, we define

$$V_n=\text{span}\{e_1,e_2,\dots,e_n\}.$$

The two main steps in the proof of existence is contained in the two following Lemmas :

Lemma 2.3.6. *For all $n\ge 1$, there exists an adapted process $u_n\in C([0,T];V_n)$ a. s. such that for all $1\le k\le n$,*

$$(u_n(t),e_k)=(u_0,e_k)+\int_0^t\langle A(u_n(s)),e_k\rangle ds+\sum_{\ell=1}^n\int_0^t(B_\ell(u_n(s)),e_k)dW_s^\ell. \quad (2.8)$$

Lemma 2.3.7.

$$\sup_n\mathbb{E}\left[\sup_{0\le t\le T}|u_n(t)|^2+\int_0^T\|u_n(t)\|^2dt\right]<\infty.$$

Let us admit for a moment these two Lemmas, and continue the proof of the Theorem. Lemma 2.3.7 tells us that the sequence $\{u_n,\ n\ge 1\}$ is bounded in $L^2(\Omega;C([0,T];H)\cap L^2(\Omega\times[0,T];V)$. It then follows from our assumptions that

1. the sequence $\{A(u_n),\ n\ge 1\}$ is bounded in $L^2(\Omega\times[0,T];V')$;

2. the sequence $\{B(u_n),\ n \geq 1\}$ is bounded in $L^2(\Omega \times [0,T]; \mathcal{H})$.

Hence there exists a subsequence of the original sequence (which, by an abuse of notation, we do not distinguish from the original sequence), such that

$$\begin{aligned} u_n &\rightharpoonup u \ \text{ in } L^2(\Omega; L^2(0,T;V) \cap L^\infty(0,T;H)) \\ A(u_n) &\rightharpoonup \xi \ \text{ in } L^2(\Omega \times (0,T); V') \\ B(u_n) &\rightharpoonup \eta \ \text{ in } L^2(\Omega \times (0,T); \mathcal{H}) \end{aligned}$$

weakly (and in fact weakly $\star$ in the L^∞ space). It is now easy to let $n \to \infty$ in equation (2.8), and deduce that for all $t \geq 0$, $k \geq 1$,

$$(u(t), e_k) = (u_0, e_k) + \int_0^t \langle \xi(s), e_k \rangle ds + \sum_{\ell=1}^{\infty} \int_0^t (\eta_\ell(s), e_k) dW_s^\ell. \tag{2.9}$$

It thus remains to prove that

Lemma 2.3.8. *We have the identities $\xi = A(u)$ and $\eta = B(u)$.*

We now need to prove the three Lemmas.

Proof of Lemma 2.3.6 If we write the equation for the coefficients of $u_n(t)$ in the basis of V_n, we obtain a usual finite dimensional Itô equation, to which the classical theory does not quite apply, since the coefficients of that equation need not be Lipschitz. However, several results allow us to treat the present situation. We shall not discuss this point further, since it is technical, and in all the examples we have in mind, the coefficients of the approximate finite dimensional equation are locally Lipschitz, which the reader can as well assume for convenience.

Proof of Lemma 2.3.7 We first show that

$$\sup_n \left[\sup_{0 \leq t \leq T} \mathbb{E}(|u_n(t)|^2) + \mathbb{E} \int_0^T \|u_n(s)\|^2 ds \right] < \infty. \tag{2.10}$$

From equation (2.8) and Itô's formula, we deduce that for all $1 \leq k \leq n$,

$$\begin{aligned} (u_n(t), e_k)^2 = (u_0, e_k)^2 &+ 2 \int_0^t (u_n(s), e_k) \langle A(u_n(s)), e_k \rangle ds \\ &+ 2 \sum_{\ell=1}^n \int_0^t (u_n(s), e_k)(B_\ell(u_n(s)), e_k) dW_s^\ell + \sum_{\ell=1}^n \int_0^t (B_\ell(u_n(s)), e_k)^2 ds \end{aligned}$$

Summing from $k=1$ to $k=n$, we obtain

$$\begin{aligned}|u_n(t)|^2 &= \sum_{k=1}^{n}(u_0, e_k)^2 + 2\int_0^t \langle A(u_n(s)), u_n(s)\rangle ds \\ &\quad + 2\sum_{\ell=1}^{n}\int_0^t (B_\ell(u_n(s)), u_n(s))dW_s^\ell + \sum_{\ell=1}^{n}\sum_{k=1}^{n}\int_0^t (B_\ell(u_n(s)), e_k)^2 ds,\end{aligned} \tag{2.11}$$

from which we deduce that

$$\begin{aligned}|u_n(t)|^2 &\le |u_0|^2 + 2\int_0^t \langle A(u_n(s)), u_n(s)\rangle ds \\ &\quad + 2\sum_{\ell=1}^{n}\int_0^t (B_\ell(u_n(s)), u_n(s))dW_s^\ell + \int_0^t |B(u_n(s))|_{\mathcal{H}}^2 ds,\end{aligned} \tag{2.12}$$

Now we take the expectation in the above inequality :

$$\mathbb{E}(|u_n(t)|^2) \le |u_0|^2 + 2\mathbb{E}\int_0^t \langle A(u_n(s)), u_n(s)\rangle ds + \mathbb{E}\int_0^t |B(u_n(s))|_{\mathcal{H}}^2 ds,$$

and combine the resulting inequality with the assumption $(H1)$, yielding

$$\mathbb{E}\left(|u_n(t)|^2 + \alpha\int_0^t \|u_n(s)\|^2 ds\right) \le |u_0|^2 + \lambda\mathbb{E}\int_0^t |u_n(s)|^2 ds + \nu t. \tag{2.13}$$

Combining with Gronwall's Lemma, we conclude that

$$\sup_n \sup_{0\le t\le T} \mathbb{E}(|u_n(t)|^2) < \infty, \tag{2.14}$$

and combining the last two inequalities, we deduce that

$$\sup_n \mathbb{E}\int_0^T \|u_n(t)\|^2 dt < \infty. \tag{2.15}$$

The estimate (2.10) follows from (2.14) + (2.15). We now take the sup over t in (2.12), yielding

$$\begin{aligned}\sup_{0\le t\le T}|u_n(t)|^2 &\le |u_0|^2 + 2\int_0^T |\langle A(u_n(s)), u_n(s)\rangle| ds \\ &\quad + 2\sup_{0\le t\le T}\left|\sum_{\ell=1}^{n}\int_0^t (B_\ell(u_n(s)), u_n(s))dW_s^\ell\right| + \int_0^T |B(u_n(s))|_{\mathcal{H}}^2 ds.\end{aligned} \tag{2.16}$$

Now the Davis–Burkholder–Gundy inequality tells us that

$$\begin{aligned}
&\mathbb{E}\left[2\sup_{0\le t\le T}\left|\sum_{\ell=1}^{n}\int_0^t (B_\ell(u_n(s)),u_n(s))dW_s^\ell\right|\right]\\
&\le c\mathbb{E}\sqrt{\sum_{\ell=1}^{n}\int_0^T (B_\ell(u_n(t)),u_n(t))^2dt}\\
&\le c\mathbb{E}\left[\sup_{0\le t\le T}|u_n(t)|\sqrt{\int_0^T |B(u_n(t))|^2_{\mathcal{H}}dt}\right]\\
&\le \frac{1}{2}\mathbb{E}\left(\sup_{0\le t\le T}|u_n(t)|^2\right)+\frac{c^2}{2}\mathbb{E}\int_0^T |B(u_n(t))|^2_{\mathcal{H}}dt
\end{aligned}$$

Combining (2.16) with the assumption $(H1)$ and this last inequality, we deduce that

$$\mathbb{E}\left(\sup_{0\le t\le T}|u_n(t)|^2\right)\le 2|u_0|^2+c'\mathbb{E}\int_0^T(1+|u_n(t)|^2dt.$$

The result follows from this and (2.14).

Proof of Lemma 2.3.8 We are going to exploit the monotonicity assumption $(H2)$, which for simplicity we assume to hold with $\lambda=0$ (this is in fact not necessary, but is also not a restriction). $(H2)$ with $\lambda=0$ implies that for all $v\in L^2(\Omega\times(0,T);V)$ and all $n\ge 1$,

$$2\mathbb{E}\int_0^T\langle A(u_n(t)-A(v(t)),u_n(t)-v(t)\rangle dt+\mathbb{E}\int_0^T|B(u_n(t))-B(v(t))|^2_{\mathcal{H}}dt\le 0. \tag{2.17}$$

Weak convergence implies that

$$\begin{aligned}
\int_0^T\langle A(u_n(t)),v(t)\rangle dt &\rightharpoonup \int_0^T\langle \xi(t),v(t)\rangle dt,\\
\int_0^T\langle A(v(t)),u_n(t)\rangle dt &\rightharpoonup \int_0^T\langle A(v(t)),u(t)\rangle dt,\\
\int_0^T(B(u_n(t)),B(v(t)))_{\mathcal{H}}dt &\rightharpoonup \int_0^T(\eta(t),B(v(t)))_{\mathcal{H}}dt.
\end{aligned}\tag{2.18}$$

in $L^2(\Omega)$ weakly. Suppose we have in addition the inequality

$$\begin{aligned} &2\mathbb{E}\int_0^T \langle \xi(t), u(t)\rangle dt + \mathbb{E}\int_0^T |\eta(t)|^2_{\mathcal{H}} dt \\ &\leq \text{liminf}_{n\to\infty}\mathbb{E}\left[2\int_0^T \langle A(u_n(t)), u_n(t)\rangle dt + \int_0^T |B(u_n(t)|^2_{\mathcal{H}} dt\right]. \end{aligned} \tag{2.19}$$

It follows from (2.17), (2.18) and (2.19) that for all $v \in L^2(\Omega \times (0,T); V)$,

$$2\mathbb{E}\int_0^T \langle \xi(t) - A(v(t)), u(t) - v(t)\rangle dt + \mathbb{E}\int_0^T |\eta(t) - B(v(t))|^2_{\mathcal{H}} dt \leq 0. \tag{2.20}$$

We first choose $v = u$ in (2.20), and deduce that $\eta \equiv B(u)$. Moreover (2.20) implies that

$$\mathbb{E}\int_0^T \langle \xi(t) - A(v(t)), u(t) - v(t)\rangle dt \leq 0.$$

Next we choose $v(t) = u(t) - \theta w(t)$, with $\theta > 0$ and $w \in L^2(\Omega \times (0,T); V)$. After division by θ, we obtain the inequality

$$\mathbb{E}\int_0^T \langle \xi(t) - A(u(t) - \theta w(t)), w(t)\rangle dt \leq 0.$$

We now let $\theta \to 0$, and thanks to the assumption $(H4)$, we deduce that

$$\mathbb{E}\int_0^T \langle \xi(t) - A(u(t)), w(t)\rangle dt \leq 0, \quad \forall w \in L^2(\Omega \times (0,T); V).$$

It clearly follows that $\xi \equiv A(u)$.

It remains to establish the inequality (2.19). It follows from (2.11) that

$$2\mathbb{E}\int_0^T \langle A(u_n(t)), u_n(t)\rangle dt + \mathbb{E}\int_0^T |B(u_n(t)|^2_{\mathcal{H}} dt \geq \mathbb{E}\left[|u_n(T)|^2 - |u_n(0)|^2\right],$$

and from Lemma 2.3.4 applied to $u(t)$ satisfying (2.9) that

$$2\mathbb{E}\int_0^T \langle \xi(t), u(t)\rangle dt + \mathbb{E}\int_0^T |\eta(t)|^2_{\mathcal{H}} dt = \mathbb{E}\left[|u(T)|^2 - |u_0|^2\right].$$

Hence (2.19) is a consequence of the inequality

$$\mathbb{E}\left[|u(T)|^2 - |u_0|^2\right] \leq \text{liminf}_{n\to\infty}\mathbb{E}\left[|u_n(T)|^2 - |u_n(0)|^2\right].$$

But clearly $u_n(0) = \sum_{k=1}^{n}(u_0, e_k)e_k \to u_0$ in H. Hence the result will follow from the convexity of the mapping $\rho \to \mathbb{E}(|\rho|^2)$ from $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; H)$ into $\mathbb{R}$, provided we show that $u_n(T) \to u(T)$ in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$ weakly. Since the sequence $\{u_n(T),\ n \geq 1\}$ is bounded in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$, we can w. l. o. g. assume that the subsequence has been choosen in such a way that $u_n(T)$ converges weakly in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$ as $n \to \infty$. On the other hand, for any n_0 and $v \in V_{n_0}$, whenever $n \geq n_0$,

$$(u_n(T), v) = (u_0, v) + \int_0^T \langle A(u_n(t)), v\rangle dt + \sum_{\ell=1}^{n} \int_0^T (B_\ell(u_n(t)), v) dW_t^\ell.$$

The right–hand side converges weakly in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ towards

$$(u_0, v) + \int_0^T \langle \xi(t), v\rangle dt + \sum_{\ell=1}^{\infty} \int_0^T (\eta_\ell(t), v) dW_t^\ell = (u(T), v).$$

The result follows.

2.3.2 Examples

A simple example We start with a simple example, which will illustrate the coercivity condition. Consider the following parabolic "bilinear" SPDE with space dimension equal to one, driven by a one dimensional Wiener process, namely

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x) + \theta \frac{\partial u}{\partial x}(t,x)\frac{dW}{dt}(t);\ u(0,x) = u_0(x).$$

The coercivity condition, when applied to this SPDE, yields the restriction $|\theta| < 1$. Under that assumption, the solution, starting from $u_0 \in H$, is in V for a. e. $t > 0$, i. e. we have the regularization effect of a parabolic equation.

When $\theta = 1$ (resp. $\theta = -1$), we deduce from Itô's formula the explicit solution $u(t,x) = u_0(x + W(t))$ (resp. $u(t,x) = u_0(x - W(t))$). It is easily seen that in this case the regularity in x of the solution is the same at each time $t > 0$ as it is at time 0. This should not be considered as a parabolic equation, but rather as a first order hyperbolic equation.

What happens if $|\theta| > 1$? We suspect that solving the SPDE in that case raises the same type of difficulty as solving a parabolic equation (like the heat equation) backward in time.

Note that the above equation is equivalent to the following SPDE in the Stratonovich sense

$$\frac{\partial u}{\partial t}(t,x) = \frac{1-\theta^2}{2}\frac{\partial^2 u}{\partial x^2}(t,x) + \theta\frac{\partial u}{\partial x}(t,x)\circ\frac{dW}{dt}(t);\ u(0,x) = u_0(x),$$

which perhaps explains better the above discussion.

Zakai's equation We look at the equation for the density p in the above example 1.2.5. We assume that the following are bounded functions defined on $\mathbb{R}^d$: $a, b, h, g, \frac{\partial a_{ij}}{\partial x_j}, \frac{\partial g_{i\ell}}{\partial x_i}$, for all $1 \le i, j \le d$, $1 \le \ell \le k$. The equation for p is of the form

$$\frac{\partial p}{\partial t}(t,x) = Ap(t,x) + \sum_{\ell=1}^{k} B_\ell p(t,x)\frac{dW_\ell}{dt}(t),$$

if we let

$$Au = \frac{1}{2}\sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_i \frac{\partial}{\partial x_i}\left(\left(\sum_j \frac{1}{2}\frac{\partial a_{ij}}{\partial x_j} - b_i\right)u\right)$$

and

$$B_\ell = -\sum_i g_{i\ell}\frac{\partial u}{\partial x_i} + \left(h_\ell - \sum_i \frac{\partial g_{i\ell}}{\partial x_i}\right)u.$$

We note that

$$\begin{aligned}2\langle Au, u\rangle + \sum_{\ell=1}^{k}|B_\ell u|^2 = &\sum_{i,j}\int_{\mathbb{R}^d}(gg^* - a)_{ij}(x)\frac{\partial u}{\partial x_i}(x)\frac{\partial u}{\partial x_j}(x)dx \\ &+ \sum_i \int_{\mathbb{R}^d} c_i(x)\frac{\partial u}{\partial x_i}(x)u(x)dx + \int_{\mathbb{R}^d} d(x)u^2(x)dx.\end{aligned}$$

Whenever $ff^*(x) > \beta I > 0$ for all $x \in \mathbb{R}^d$, the coercivity assumption is satisfied with any $\alpha < \beta$, some $\lambda > 0$ and $\nu = 0$. Note that it is very natural that the ellipticity assumption concerns the matrix ff^*. Indeed, in the particular case where $h \equiv 0$, we observe the Wiener process W, so the uncertainty in the conditionnal law of X_t given $\mathcal{F}_t^Y$ depends on the diffusion matrix ff^* only. The case whithout the restriction that ff^* be elliptic can be studied, but we need some more regularity of the coefficients.

Nonlinear examples One can always add a term of the form

$$f_1(t,x,u) + f_2(t,x,u)$$

to $A(u)$, provided $u \to f_1(t,x,u)$ is decreasing for all (t,x), and $f_2(t,x,u)$ is Lipchitz in u, with a uniform Lipschitz constant independent of (t,x). Note that a typical decreasing f_1 is given by

$$f_1(t,x,u) = -c(t,x)|u|^{p-2}u, \quad \text{provided that } c(t,x) \geq 0.$$

Similarly, one can add to $B(u)$ a term $g(t,x,u)$, where g have the same property as f_2.

Another nonlinear example The following operator (with $p > 2$)

$$A(u) = \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) - |u|^{p-2}u$$

possesses all the required properties, if we let $H = L^2(\mathbb{R}^d)$,

$$V = W^{1,p}(\mathbb{R}^d) = \{u \in L^p(\mathbb{R}^d),\ \frac{\partial u}{\partial x_i} \in L^p(\mathbb{R}^d), i = 1, \ldots, d\}$$

and $V' = W^{-1,q}(\mathbb{R}^d)$, where $1/p + 1/q = 1$.

2.3.3 Coercive SPDEs with compactness

We keep the assumptions $(H1)$ and $(H3)$ from the previous subsection, and we add the following conditions.

Sublinear growth of B

$$(H5) \begin{cases} \exists c, \delta > 0 \ \text{ such that } \forall u \in V, \\ |B(u)|_{\mathcal{H}} \leq c(1 + \|u\|^{1-\delta}) \end{cases}$$

Compactness

$$(H6) \quad \text{The injection from } V \text{ into } H \text{ is compact.}$$

Continuity

$$(H7) \begin{cases} u \to A(u) \text{ is continuous from } V_{\text{weak}} \cap H \text{ into } V'_{\text{weak}} \\ u \to B(u) \text{ is continuous from } V_{\text{weak}} \cap H \text{ into } \mathcal{H} \end{cases}$$

We now want to formulate our SPDE as a martingale problem. We choose

$$\Omega = C([0,T]; H_{\text{weak}}) \cap L^2(0,T;V) \cap L^2(0,T;H),$$

which we equip with the sup of the topology of uniform convergence with values in H equipped with its weak topology, the weak topology of $L^2(0,T;V)$, and the strong topology of $L^2(0,T;H)$. Moreover we let $\mathcal{F}$ be the associated Borel σ–field. For $0 \le t \le T$, let Ω_t denote the same space as Ω, but with T replaced by t, and Π_t be the projection from Ω into Ω_t, which to a function defined on the interval $[0,T]$ associates its restriction to the interval $[0,t]$. Now $\mathcal{F}_t$ will denote the smallest sub–σ–field of $\mathcal{F}$, which makes the projection Π_t measurable, when Ω_t is equipped with its own Borel σ–field. From now on , in this subection, we define $u(t,\omega) = \omega(t)$. Let us formulate the

Definition 2.3.9. *A probability $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a solution to the martingale problem associated with the SPDE* (2.6) *whenever*

(i) $\mathbb{P}(u(0) = u_0) = 1$;

(ii) the process

$$M_t := u(t) - u(0) - \int_0^t A(u(s))ds$$

is a continuous H–valued $\mathbb{P}$–martingale with associated increasing process

$$\langle\langle M \rangle\rangle_t = \int_0^t B(u(s))B^*(u(s))ds.$$

There are several equivalent formulations of (ii). Let us give the formulation which we will actually use below. Let $\{e_i,\ i = 1, 2, \ldots\}$ be an orthonormal basis of H, with $e_i \in V$, $\forall i \ge 1$.

(ii)' For all $i \ge 1$, $\varphi \in C_b^2(\mathbb{R})$, $0 \le s \le t$, Φ_s continuous, bounded and $\mathcal{F}_s$–measurable mapping from Ω into $\mathbb{R}$,

$$\mathbb{E}_{\mathbb{P}}\left((M_t^{i,\varphi} - M_s^{i,\varphi})\Phi_s\right) = 0, \quad \text{where}$$

$$\begin{aligned} M_t^{i,\varphi} &= \varphi[(u(t), e_i)] - \varphi[(u_0, e_i)] - \int_0^t \varphi'[(u(s), e_i)]\langle A(u(s)), e_i\rangle ds \\ &+ \frac{1}{2}\int_0^t \varphi''[(u(s), e_i)](BB^*(u(s))e_i, e_i)ds. \end{aligned}$$

This formulation of a martingale problem for solving stochastic differential equations was first introduced by Stroock and Varadhan fo solving finite dimensinal SDEs, and by Viot [26] for solving SPDEs. It is his results which we present here.

We first note that if we have a solution to the SPDE, its probability law on Ω solves the martingale problem. Conversely, if we have a solution to the martingale problem, then we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and an H–valued process $\{u(t),\ 0 \le t \le T\}$ defined on it, with trajectories in $L^2(0,T;V)$, such that

$$u(t) = u_0 + \int_0^t A(u(s))ds + M_t,$$

where $\{M_t,\ 0 \le t \le T\}$ is a continuous H–valued martingale, and

$$\langle\langle M \rangle\rangle_t = \int_0^t B(u(s))B^*(u(s))ds.$$

It follows from a representation theorem similar to a well–known result in finite dimension that there exists, possibly on a larger probability space, a Wiener process $\{W(t),\ t \ge 0\}$ such that (2.6) holds. A solution of the martingale problem is called a *weak solution* of the SPDE, in the sense that one can construct a pair $\{(u(t), W(t)),\ t \ge 0\}$ such that the second element is a Wiener process, and the first solves the SPDE driven by the second, while until now we have given ourselves $\{W(t),\ t \ge 0\}$, and we have found the corresponding solution $\{u(t),\ t \ge 0\}$.

We next note that whenever a SPDE is such that it admits at most one strong solution (i. e., to each given Wiener process W, we can associate at most one solution u of the SPDE driven by W), then the martingale problem has also at most one solution.

We now prove the

Theorem 2.3.10. *Under the assumptions* $(H1)$*,* $(H3)$*,* $(H5)$*,* $(H6)$ *and* $(H7)$*, there exists a solution* $\mathbb{P}$ *to the martingale problem, i. e. which satisfies (i) and (ii).*

Proof: We start with the same Galerkin approximation as we have used before. Again $\{e_1, \ldots, e_n, \ldots\}$ is an orthonormal basis of H, with each $e_n \in V$,

$V_n = \mathrm{span}\{e_1, \ldots, e_n\}$

$\pi_n =$ the orthogonal projection operator in H upon V_n.

We first note that for each $n \geq 1$, there exists a probability measure $\mathbb{P}_n$ on $(\Omega, \mathcal{F})$ such that

$(0)_n$ $\mathrm{Supp}(\mathbb{P}_n) \subset C([0,T]; V_n)$;

$(i)_n$ $\mathbb{P}_n(u(0) = \pi_n u_0) = 1$

$(ii)_n$ $\forall i \leq n, \varphi \in C_b^2(\mathbb{R}), 0 \leq s \leq t \leq T$,

$$\mathbb{E}_n \left((M_t^{i,\varphi} - M_s^{i,\varphi})\Phi_s \right) = 0, \quad \text{where}$$

$\{M_t^{i,\varphi}\}$ and Φ_s are defined exactly as in condition (ii) and (ii)' of Definition 2.3.9.

Indeed, the existence of each P_n is obtained by solving a finite dimensional martingale problem (or a finite dimensional SDE). This works whithout any serious difficulty, and we take this result for granted.

Let us accept for a moment the

Lemma 2.3.11. *The sequence of probability measures $\{\mathbb{P}_n,\ n = 1, 2, \ldots\}$ on Ω is tight.*

We shall admit the fact (which has been proved by M. Viot in his thesis) that Prohorov's theorem is valid in the space Ω. This is not obvious, since Ω is not a Polish space, but it is true. Hence we can extract from the sequence $\{P_n,\ n = 1, 2, \ldots\}$ a subsequence, which as an abuse of notation we still denote $\{P_n\}$, such that $\mathbb{P}_n \Rightarrow \mathbb{P}$. Now $\mathbb{P}$ satisfies clearly (i), and the mapping

$$\omega \to (M_t^{i,\varphi}(\omega) - M_s^{i,\varphi}(\omega))\Phi_s(\omega)$$

is continuous from Ω into $\mathbb{R}$. Moreover, it follows from the coercivity assumption $(H1)$ that the estimate

$$\sup_n \mathbb{E}_n \left[\sup_{0 \leq t \leq T} |u(t)|^2 + \int_0^T \|u(t)\|^2 dt \right] < \infty \tag{2.21}$$

from Lemma 2.3.7 is still valid. Now this plus the conditions $(H3)$ and $(H5)$ implies that there exists some $p > 1$ (the exact value of p depends upon the value of δ in condition $(H5)$ such that

$$\sup_n \mathbb{E}_n \left[|M_t^{i,\varphi} - M_s^{i,\varphi}|^p \right] < \infty.$$

Hence

$$\mathbb{E}_n\left((M_t^{i,\varphi} - M_s^{i,\varphi})\Phi_s\right) \to \mathbb{E}\left((M_t^{i,\varphi} - M_s^{i,\varphi})\Phi_s\right),$$

and condition (ii) is met. It remains to proceed to the

PROOF OF LEMMA 2.3.11 (SKETCH): Let us denote by

- τ_1 the weak topology on $L^2(0,T;V)$,
- τ_2 the uniform topology on $C([0,T];H_{\text{weak}})$,
- τ_3 the strong topology on $L^2(0,T;H)$.

It suffices to show that the sequence $\{\mathbb{P}_n,\ n \geq 1\}$ is τ_i–tight successively for $i = 1, 2, 3$. We choose

$$K = \left\{u,\ \sup_{0\leq t\leq T} |u(t)| \leq \ell,\ \int_0^T \|u(t)\|^2 dt \leq k\right\}.$$

From (2.21), $\mathbb{P}_n(K^c)$ can be made arbitrarily small by choosing ℓ and k large enough.

1. τ_1–tightness. K is relatively compact for the weak topology τ_1, since it is a bounded set of $L^2(0,T;V)$, which is a reflexive Banach space.

2. τ_2–tightness. We need to show that K is relatively compact for the topology τ_2. For this, it suffices to show that for all $h \in H$ with $|h| = 1$, the set of functions

$$\{t \to (u(t), h), \quad u \in K\}$$

is a compact subset of $C([0,T])$. Since $u \in K$ implies that $\sup_{0\leq t\leq T} |u(t)| \leq \ell$, it is sufficient to prove that for any $r > 0$, $v \in V$ with $\|v\| = r$, the set of functions

$$\{t \to (u(t), v), \quad u \in K\}$$

is a compact subset of $C([0,T])$. Now $\sup_{0\leq t\leq T} |(u(t), v)|$ is well controlled. So, using Arzela–Ascoli's theorem, it suffices to control uniformly the modulus of continuity of $\{t \to (u(t), v)\}$ uniformly in $u \in K$.

But

$$
\begin{aligned}
(u(t), v) &= (u_0, v) + \int_0^t \langle A(u(s)), v\rangle ds + M_t^v, \text{ and} \\
\mathbb{E}_n \left| \int_s^t \langle A(u(r)), v\rangle dr \right| &\le \|v\| \sqrt{t-s} \sqrt{\mathbb{E}_n \int_0^T \|A(u(r))\|_*^2 dr} \\
&\le c\|v\|\sqrt{t-s}, \\
\mathbb{E}_n \left(\sup_{s\le r\le t} |M_r^v - M_s^v|^{2p} \right) &\le c_p |v|^p \mathbb{E}_n \left(\left| \int_s^t (BB^*(u(r))e_i, e_i) dr \right|^p \right) \\
&\le c_p |v|^p (t-s)^{p\delta} \left(\mathbb{E}_n \int_0^T (1 + \|u(r)\|^2) dr \right)^{p(1-\delta)},
\end{aligned}
$$

for all $p > 0$, δ being the constant from the condition $(H5)$.

3. τ_3–tightness. We just saw in fact that we can control the modulus of continuity of $\{t \to u(t)\}$ as a V'–valued function under $\mathbb{P}_n$. Recall the bound
$$\mathbb{E}_n \int_0^T \|u(t)\|^2 dt \le c.$$
It remains to exploit the next Lemma.

Lemma 2.3.12. *Given that the injection from V into H is compact, from any sequence $\{u_n,\ n \ge 1\}$ which is both bounded in $L^2(0,T;V) \cap L^\infty(0,T;H)$ and equicontinuous as V'–valued functions, and such that the sequence $\{u_n(0)\}$ converges strongly in H, one can extract a subsequence which converges in $L^2(0,T;H)$ strongly.*

We first prove the following

Lemma 2.3.13. *To each $\varepsilon > 0$, we can associate $c(\varepsilon) \in \mathbb{R}$ such that for all $v \in V$,*
$$|v| \le \varepsilon\|v\| + c(\varepsilon)\|v\|_*.$$

Proof: If the result was not true, one could find $\varepsilon > 0$ and a sequence $\{v_n,\ n \ge 1\} \subset V$ such that for all $n \ge 1$,
$$|v_n| \ge \varepsilon\|v_n\| + n\|v_n\|_*.$$

We define $u_n = |v_n|^{-1}v_n$. Then we have that

$$1 = |u_n| \geq \varepsilon\|u_n\| + n\|u_n\|_*.$$

This last inequality show both that the sequence $\{u_n,\ n \geq 1\}$ is bounded in V, and converges to 0 in V'. Hence, from the compactnes of the injection from V into H, $u_n \to u$ in H strongly, and necessarily $u = 0$. But this contradicts the fact that $|u_n| = 1$ for all n.

Proof of Lemma 2.3.12: From the equicontinuity in V' and the fact that $u_n(0) \to u_0$ in H, there is a subsequence which converges in $C([0,T];V')$, hence also in $L^2(0,T;V')$, to u, and clearly $u \in L^2(0,T;V)$. Now from Lemma 2.3.13, to each $\varepsilon > 0$, we can associate $c'(\varepsilon)$ such that

$$\begin{aligned}\int_0^T |u_n(t)-u(t)|^2 dt &\leq \varepsilon \int_0^T \|u_n(t)-u(t)\|^2 dt + c'(\varepsilon)\int_0^T \|u_n(t)-u(t)\|_*^2 dt\\ &\leq \varepsilon C + c'(\varepsilon)\int_0^T \|u_n(t)-u(t)\|_*^2 dt\end{aligned}$$

$$\text{limsup}_n \int_0^T |u_n(t)-u(t)|^2 dt \leq\ C\varepsilon,$$

and the result follows fom the fact that ε can be chosen arbitrarily small.

2.4 Semilinear SPDEs

We want now to concentrate on the following class of SPDEs

$$\begin{cases}\dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\displaystyle\sum_{ij}\dfrac{\partial}{\partial x_j}\left(a_{ij}(t,x)\dfrac{\partial u}{\partial x_i}\right)(t,x) + \sum_i b_i(t,x)\dfrac{\partial u}{\partial x_i}(t,x)\\ \qquad\qquad + f(t,x;u(t,x))\\ \qquad\qquad + \displaystyle\sum_k\left(\sum_i g_{ki}(t,x)\dfrac{\partial u}{\partial x_i}(t,x) + h_k(t,x;u(t,x))\right)\dfrac{dW^k}{dt}(t)\\ u(0,x) = u_0(x)\end{cases} \tag{2.22}$$

Under the following standard assumptions

- $\exists\alpha > 0$ such that $\overline{a} = a - \sum_k g_{k\cdot}g_{k\cdot} \geq \alpha I$;
- $2[f(t,x;r) - f(t,x;r')](r-r') + \sum_k |h_k(t,x;r) - h_k(t,x;r')|^2 \leq \lambda |r-r'|^2$;
- $r \longrightarrow f(t,x;r)$ is continuous;
- $rf(t,x;r) + \sum_k |h_k(t,x;r)|^2 \leq C(1+|r|^2)$,

equation (2.22) has a unique solution with trajectories in $C([0,T];L^2(\mathbb{R}^d)) \cap L^2(0,T;H^1(\mathbb{R}^d))$.

Let us now give conditions under which the solution remains non negative.

Theorem 2.4.1. *Assume that $u_0(x) \geq 0$, for a. e. x, and for a. e. t and x, $f(t,x;0) \geq 0$, $h_k(t,x;0) = 0$, for all k. Then*

$$u(t,x) \geq 0, \quad \forall t \geq 0,\ x \in \mathbb{R}^d.$$

Proof: Let us consider the new equation

$$\left\{\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \frac{1}{2}\sum_{ij}\frac{\partial}{\partial x_j}\left(a_{ij}(t,x)\frac{\partial u}{\partial x_i}\right)(t,x) + \sum_i b_i(t,x)\frac{\partial u^+}{\partial x_i}(t,x) + f(t,x;u^+(t,x)) \\ &= \sum_k\left(\sum_i g_{ki}(t,x)\frac{\partial u}{\partial x_i}(t,x) + h_k(t,x;u^+(t,x))\right)\frac{dW^k}{dt}(t) \end{aligned}\right. \tag{2.23}$$

Existence and uniqueness for this new equation follows almost the same arguments as for equation (2.22). We exploit the fact that the mapping $r \to r^+$ is Lipschitz. Moreover, we can w. l. o. g. assume that the $\partial b_i/\partial x_i$'s are bounded functions, since from the result of the theorem with smooth coefficients will follow the general result, by taking the limit along a converging sequence of smooth coefficients. However, it is not hard to show that, with this additional assumption, the mapping

$$u \to \sum_i b_i(t,x)\frac{\partial u^+}{\partial x_i}$$

is compatible with the coercivity and monotonicity of the pair of operator appearing in (2.23). If we can show that the solution of (2.23) is non negative, then it will be the unique solution of (2.22), which then will be non negative.

Let $\varphi \in C^2(\mathbb{R})$ be convex and such that

$$\begin{cases} \bullet\ \varphi(r) = 0, \quad \text{for } r \geq 0; \\ \bullet\ \varphi(r) > 0, \quad \text{for } r < 0; \\ \bullet\ 0 \leq \varphi(r) \leq Cr^2 \quad \forall r; \\ \bullet\ -c|r| \leq \varphi'(r) \leq 0 \quad \forall r; \\ \bullet\ 0 \leq \varphi''(r) \leq C \quad \forall r. \end{cases}$$

Intuitively, φ is a regularization of $(r^-)^2$. Let now $\Phi : L^2(\mathbb{R}) \to \mathbb{R}$ be defined by

$$\Phi(u) = \int_{\mathbb{R}^d} \varphi(u(x))dx.$$

We have $\Phi'(h) = \varphi'(h(\cdot))$, which is well defined as an element of $L^2(\mathbb{R}^d)$, since $|\varphi'(x)| \leq c|x|$, and $\Phi''(h) = \varphi''(h(\cdot))$, it belongs to $\mathcal{L}(L^2(\mathbb{R}^d))$, since $|\varphi''(x)| \leq C$. We let

$$\begin{aligned} Au &= \frac{1}{2}\sum_{ij} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \sum_i b_i(t,x)\frac{\partial u^+}{\partial x_i} + f(u^+) \\ B_k u &= \sum_i g_{ki}\frac{\partial u}{\partial x_i} + h_k(u^+) \end{aligned}$$

It follows from the Itô formula from Lemma 2.3.5 that

$$\begin{aligned} \Phi(u(t)) = \Phi(u_0) &+ \int_0^t \langle A(u(s)), \varphi'(u(s))\rangle ds \\ &+ \sum_k \int_0^t \left(B_k(u(s)), \varphi'(u(s))\right) dW_s^k \\ &+ \frac{1}{2}\sum_k \int_0^t \left(B_k(u(s)), \varphi''(u(s))B_k(u(s))\right) ds, \end{aligned}$$

Now $\Phi(u_0) = 0$, and

$$\begin{aligned}
\mathbb{E}\Phi(u(t)) &= -\frac{1}{2}\mathbb{E}\int_0^t ds \int_{\mathbb{R}^d} dx \left(\varphi''(u)\langle \overline{a}\nabla u, \nabla u\rangle\right)(s,x) \\
&+ \mathbb{E}\int_0^t ds \int_{\mathbb{R}^d} dx \varphi'(u)[f(u^+) + \sum_i b_i \frac{\partial u^+}{\partial x_i}](s,x) \\
&+ \sum_k \mathbb{E}\int_0^t ds \int_{\mathbb{R}^d} dx \varphi''(u) h_k(u^+)[\frac{1}{2}h_k(u^+) + g_{kj}\frac{\partial u}{\partial x_j}](s,x) \\
&\leq 0
\end{aligned}$$

where we have used the

Lemma 2.4.2. *Whenever $u \in H^1(\mathbb{R}^d)$, $u^+ \in H^1(\mathbb{R}^d)$, and moreover*

$$\frac{\partial u^+}{\partial x_i}(x)\mathbf{1}_{\{u<0\}}(x) = 0, \;\; dx \;\; a. \;\; e. \;\; , \forall 1 \leq i \leq d.$$

If we admit the Lemma for a moment, we note that we have proved that for any $t \geq 0$, $\mathbb{E}\Phi(u(t)) = 0$, i. e. $\Phi(u(t)) = 0$ a. s., and in fact $u(t,x) \geq 0$, dx a. e., a. s., $\forall t$. It remains to proceed to the

PROOF OF LEMMA 2.4.2: We define a sequence of approximations of the function $r \to r^+$ of class C^1:

$$\varphi_n(r) = \begin{cases} 0, & \text{if } r < 0; \\ nr^2/2, & \text{if } 0 < r < 1/n; \\ r - 1/2n, & \text{if } r > 1/n. \end{cases}$$

Clearly, $\varphi_n(r) \to r^+$, and $\varphi_n'(r) \to \mathbf{1}_{\{r>0\}}$, as $n \to \infty$. For $u \in H^1(\mathbb{R}^d)$, let $u_n(x) = \varphi_n(u(x))$. Then $u_n \in H^1(\mathbb{R}^d)$, and

$$\frac{\partial u_n}{\partial x_i} = \varphi_n'(u_n)\frac{\partial u}{\partial x_i}.$$

It is easily seen that the two following convergences hold in $L^2(\mathbb{R}^d)$:

$$u_n \to u^+ \quad \frac{\partial u_n}{\partial x_i} \to \mathbf{1}_{\{u>0\}}\frac{\partial u}{\partial x_i}.$$

This proves the Lemma. $\square$

With a similar argument, one can also prove a comparison theorem. Let v be the solution of a slightly different SPDE

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = \dfrac{1}{2}\displaystyle\sum_{ij} \frac{\partial}{\partial x_j}\left(a_{ij}(t,x)\frac{\partial v}{\partial x_i}\right)(t,x) + \sum_i b_i(t,x)\frac{\partial v}{\partial x_i}(t,x) \\ \qquad\qquad + F(t,x;v(t,x)) \\ \qquad\qquad + \displaystyle\sum_k \left(\sum_i g_{ki}(t,x)\frac{\partial v}{\partial x_i}(t,x) + h_k(t,x;v(t,x))\right)\frac{dW^k}{dt}(t) \\ v(0,x) = v_0(x) \end{cases}$$

Theorem 2.4.3. *Assume that $u_0(x) \le v_0(x)$, x a. e., that $f(t,x;r) \le F(t,x;r)$, t, x a. e., and moreover one of the two pairs $(f,(h_k))$ or $(F,(h_k))$ satisfies the above conditions for existence–uniqueness. Then $u(t,x) \le v(t,x)$ x a. e., $\mathbb{P}$ a. s., for all $t \ge 0$.*

Sketch of the proof of Theorem 2.4.3: The proof is similar to that of the Theorem 2.4.1, so we just sketch it. We first replace v by $u \vee v$ in the last equation, in the three palces where we changed u into u^+ in the proof of the previous Theorem. The fact that

$$u, v \in H^1(\mathbb{R}^d) \Rightarrow u \vee v \in H^1(\mathbb{R}^d)$$

follows from Lemma 2.4.2 and the simple identity $u \vee v = u + (v-u)^+$. If v denotes the solution of that new equation, we show (with the same functional Φ as in the proof of Theorem 2.4.1) that $\mathbb{E}\Phi(v(t) - u(t)) \le 0$, which implies that $u(t,x) \le v(t,x)$, x a. e., $\mathbb{P}$ a. s., for all $t \ge 0$. Consequently v solves the original equation, and the result is established.

Chapter 3

SPDEs driven by space–time white noise

3.1 Restriction to one–dimensional space variable

Let us consider the following linear parabolic SPDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\Delta u(t,x) + \dot{W}(t,x), \ t \geq 0, x \in \mathbb{R}^d \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^d. \end{cases}$$

The driving noise in this equation is the so called "space–time white noise", that is $\dot{W}$ is a generalized centered Gaussian field, with covariance given by

$$\mathbb{E}[\dot{W}(h)\,\dot{W}(k)] = \int_0^\infty \int_{\mathbb{R}^d} h(t,x)k(t,x)dxdt, \ \forall h,k \in L^2(\mathbb{R}_+ \times \mathbb{R}^d).$$

Since the equation is linear, that is the mapping

$$\dot{W} \to u$$

is affine, it always has a solution as a distribution, the driving noise being a random distribution. But we want to know when that solution is a standard stochastic process $\{u(t,x), \ t \geq 0, x \in \mathbb{R}^d\}$. Let

$$p(t,x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right).$$

The solution of the above equation is given by

$$u(t,x) = \int_{\mathbb{R}^d} p(t, x-y)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} p(t-s, x-y)W(ds, dy),$$

at least if the second integral makes sense. Since it is a Wiener integral, it is a centered Gaussian random varibale, and we just have to check that its variance is finite. But that variance equals

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^d} p^2(t-s, x-y)dyds &= \frac{1}{(2\pi)^d}\int_0^t \frac{ds}{(t-s)^d}\int_{\mathbb{R}^d}\exp\left(-\frac{|x-y|^2}{t-s}\right)dy \\ &= \frac{1}{2^d\pi^{d/2}}\int_0^t \frac{ds}{(t-s)^{d/2}} < \infty\end{aligned}$$

if and only if $d = 1$! When $d \geq 2$, the solution is a generalized stochastic process, given by

$$\begin{aligned}(u(t), \varphi) = &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)p(t, x-y)u_0(y)dxdy \\ &+ \int_0^t \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\varphi(x)p(t-s, x-y)dx\right)W(ds, dy),\ t \geq 0, \varphi \in C_C^\infty(\mathbb{R}^d).\end{aligned}$$

Here the second integral is well defined. Indeed, let us assume that supp$\varphi \subset \overline{B}(0, r)$. Then

$$\int_{\mathbb{R}^d}\varphi(x)p(t-s, x-y)dx = \mathbb{E}_y\varphi(B_{t-s}),$$

where $\{B_t,\ t \geq 0\}$ is a standard $\mathbb{R}^d$–valued Brownian motion. For $|y| > r$,

$$\begin{aligned}|\mathbb{E}_y\varphi(B_{t-s})| &= |\mathbb{E}_y\left[\varphi(B_{t-s})\mathbf{1}_{|B_{t-s}|\leq r}\right]| \\ &\leq \|\varphi\|_\infty \mathbb{P}_y(B_{t-s}| \geq |y| - r) \\ &\leq \|\varphi\|_\infty \frac{\mathbb{E}(|B_{t-s}|^p)}{(|y|-r)^p}\end{aligned}$$

Choosing $2p > d$, we conclude that

$$\int_0^t \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\varphi(x)p(t-s, x-y)dx\right)^2 dsdy < \infty.$$

We note that our goal is to solve nonlinear equations of the type

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\Delta u(t,x) + f(u(t,x)) + g(u(t,x))\stackrel{\cdot}{W}(t,x),\ t \geq 0, x \in \mathbb{R}^d \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^d, \end{cases}$$

whose solution might not be more regular than that of the linear equation we considered above. Since we do not want to define the image by a nonlinear mapping of a distribution (which is essentially impossible, if we want to have some reasonable continuity properties, which is crucial when studying SPDEs), we have to restrict ourselves to the case $d = 1$!

3.2 A general existence–uniqueness result

Let us consider specifically the following SPDE with homogeneous Dirichlet boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + f(t,x;u(t,x)) + g(t,x;u(t,x))\stackrel{\cdot}{W}(t,x),\ t \geq 0, 0 \leq x \leq 1; \\ u(t,0) = u(t,1) = 0,\ t \geq 0; \\ u(0,x) = u_0(x),\ 0 \leq x \leq 1. \end{cases} \tag{3.1}$$

The equation turns out not to have a classical solution. So we first introduce a *weak formulation* of (3.1), namely

$$\begin{cases} \displaystyle\int_0^1 u(t,x)\varphi(x)dx = \int_0^1 u_0(x)\varphi(x)dx + \int_0^t\int_0^1 u(s,x)\varphi''(x)dxds \\ \displaystyle + \int_0^t\int_0^1 f(s,x;u(s,x))\varphi(x)dxds + \int_0^t\int_0^1 g(s,x;u(s,x))\varphi(x)W(ds,dx) \\ \mathbb{P} \text{ a. s.,} \quad \forall\varphi \in C^2(0,1)\cap C_0([0,1]), \end{cases} \tag{3.2}$$

where $C_0([0,1])$ stands for the set of continuous functions from $[0,1]$ into $\mathbb{R}$, which are 0 at 0 and at 1. We need to define the stochastic integral which appears in (3.2). From now on, $W(ds,dx)$ will be considered as a random Gaussian measure on $\mathbb{R}_+ \times [0,1]$. More precisely, we define the collection

$$\left\{ W(A) = \int_A W(ds,dx),\ A \text{ Borel subset of } \mathbb{R}_+ \times [0,1] \right\}$$

as a centered Gaussian random field with covariance given by

$$\mathbb{E}[W(A)W(B)] = \lambda(A \cap B),$$

where λ denotes the Lebesgus measure on $\mathbb{R}_+ \times [0,1]$.

We define for each $t > 0$ the σ–algebra

$$\mathcal{F}_t = \sigma\{W(A),\ A \text{ Borel subset of } [0,t] \times [0,1]\},$$

and the associated σ–algebra of predictable sets defined as

$$\mathcal{P} = \sigma\{(s,t] \times \Lambda \subset \mathbb{R}_+ \times \Omega :\ 0 \le s \le t, \Lambda \in \mathcal{F}_s\}.$$

The class of processes which we intend to integrate with respect to the above measure is the set of functions

$$\psi : \mathbb{R}_+ \times [0,1] \times \Omega \to \mathbb{R},$$

which are $\mathcal{P} \otimes \mathcal{B}([0,1])$–measurable and such that

$$\int_0^t \int_0^1 \psi^2(s,x) dx ds < \infty \quad \mathbb{P} \text{ a. s. } \forall t \ge 0.$$

for such ψ's, the stochastic integral

$$\int_0^t \int_0^1 \psi(s,x) W(ds,dx), \quad t \ge 0$$

can be constructed as the limit in probability of the sequence of approximations

$$\sum_{i=1}^{\infty} \sum_{j=0}^{n-1} (\psi, \mathbf{1}_{A^n_{i-1,j}})_{L^2(\mathbb{R}_+ \times (0,1))} W\left(A^n_{i,j} \cap ([0,t] \times [0,1])\right),$$

where

$$A^n_{i,j} = \left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right].$$

That stochastic integral is a local martingale, with associated increasing process

$$\int_0^t \int_0^1 \psi^2(s,x) dx ds, \quad t \ge 0.$$

If moreover

$$\mathbb{E}\int_0^t\int_0^1 \psi^2(s,x)dxds, \quad \forall t\geq 0,$$

then the stochastic integral process is a square integrable martingale, the above convergence holds in $L^2(\Omega)$, and we have the isometry

$$\mathbb{E}\left[\left(\int_0^t\int_0^1 \psi(s,x)W(ds,dx)\right)^2\right] = \mathbb{E}\int_0^t\int_0^1 \psi^2(s,x)dxds,\ \forall t\geq 0.$$

We introduce another formulation of our white–noise driven SPDE, namely the integral formulation, which is the following

$$\begin{cases} u(t,x) = \displaystyle\int_0^1 p(t;x,y)u_0(y)dy + \int_0^t\int_0^1 p(t-s;x,y)f(s,y;u(s,y))dyds \\ \qquad\qquad + \displaystyle\int_0^t\int_0^1 p(t-s;x,y)g(s,y;u(s,y))W(ds,dy),\ \mathbb{P}\text{ a. s. },t\geq 0, 0\leq x\leq 1; \end{cases} \tag{3.3}$$

where $p(t;x,y)$ is the fundamental solution of the heat equation with Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x);\ t\geq 0,\ 0<x<1; \\ u(t,0)=u(t,1)=0, \quad t\geq 0; \end{cases}$$

and $u_0\in C_0([0,1])$. We shall admit the following Lemma (see Walsh [27])

Lemma 3.2.1. *The above kernel is given explicitly by the formula*

$$p(t;x,y) = \frac{1}{\sqrt{4\pi t}}\sum_{n\in\mathbf{Z}}\left[\exp\left(-\frac{(2n+y-x)^2}{4t}\right) - \exp\left(-\frac{(2n+y+x)^2}{4t}\right)\right],$$

and for all $T>0$, there exists C_T such that

$$|p(t;x,y)| \leq \frac{C_T}{\sqrt{t}}\exp\left(-\frac{|x-y|^2}{4t}\right), \quad 0\leq t\leq T,\ 0\leq x,y\leq 1.$$

We now state two assumptions on the coeffcients

$$(H1) \quad \int_0^t\int_0^1 (f^2(s,x;0)+g^2(s,x;0))dsdx < \infty,\ t\geq 0.$$

There exists a locally bounded function $\delta : \mathbb{R} \to \mathbb{R}_+$ such that

$(H2)\quad |f(s,x;r)-f(s,x,0)|+|g(s,x;r)-g(s,x,0)| \le \delta(r),\ \forall t \ge 0, 1 \le x \le 1, r \in \mathbb{R}.$

We can now establish the

Proposition 3.2.2. *Under the assumptions* $(H1)$ *and* $(H2)$*, a continuous* $\mathcal{P} \otimes \mathcal{B}([0,1])$*–measurable function* u *satisfies* (3.2) *if and only if it satisfies* (3.3)*.*

PROOF: Let first u be a solution of (3.2), and $\lambda \in C^1(\mathbb{R}_+)$. Then by integration by parts (we use $(\cdot,\cdot)$ to denote the scalar product in $L^2(0,1)$)

$$\begin{cases} \lambda(t)(u(t),\varphi) = \lambda(0)(u(0),\varphi) + \displaystyle\int_0^t (u(s), \lambda(s)\varphi'' + \lambda'(s)\varphi)ds \\ \ + \displaystyle\int_0^t \lambda(s)(f(s,\cdot;u(s,\cdot)),\varphi)ds + \int_0^t\int_0^1 \lambda(s)g(s,x;u(s,x))\varphi(x)W(ds,dx). \end{cases}$$

But any $\phi \in C^{1,2}(\mathbb{R}_+ \times (0,1)) \cap C(\mathbb{R}_+ \times [0,1])$ such that $\phi(t,0) = \phi(t,1) = 0$ is a limit of finite sums of the form $\sum_{i=1}^n \lambda_i(t)\varphi_i(x)$. Hence we get that for all ϕ as above and all $t \ge 0$,

$$\begin{cases} (u(t),\phi(t,\cdot)) = (u(0),\phi(0,\cdot)) + \displaystyle\int_0^t (u(s), \frac{\partial^2\phi}{\partial x^2}(s,\cdot) + \frac{\partial \phi}{\partial s}(s,\cdot))ds \\ \ + \displaystyle\int_0^t (f(s,\cdot;u(s,\cdot)),\phi(s,\cdot))ds + \int_0^t\int_0^1 \phi(s,x)g(s,x;u(s,x))W(ds,dx). \end{cases}$$

Now, t being fixed, we choose for $0 \le s \le t$, $0 \le x \le 1$,

$$\phi(s,x) = \int_0^1 p(t-s;y,x)\varphi(y)dy = p(t-s;\varphi,x),$$

where $\varphi \in C_0^\infty([0,1])$. We deduce that

$$\begin{cases} (u(t),\varphi) = (u(0),p(t;\varphi,\cdot)) + \displaystyle\int_0^t (f(s,\cdot;u(s,\cdot)),p(t-s;\varphi,\cdot))ds \\ \qquad\qquad + \displaystyle\int_0^t\int_0^1 p(t-s;\varphi,y)g(s,y;u(s,y))W(ds,dy). \end{cases}$$

If we now let φ tend to δ_x, we obtain (3.3).

Let now u be a solution of (3.3). Then for all $\varphi \in C^2(0,1) \cap C_0([0,1])$, $t \geq 0$, we have, for all $0 \leq s \leq t$,

$$\begin{cases} (u(t), \varphi) = (u(s), p(t-s, \varphi, \cdot)) + \int_s^t (f(r, \cdot; u(r, \cdot)), p(t-r; \varphi, \cdot))ds \\ \qquad\qquad + \int_s^t \int_0^1 p(t-r; \varphi, y) g(r, y; u(r, y)) W(dr, dy). \end{cases}$$

We next define $t_i = it/n$, for $0 \leq i \leq n$, and $\Delta t = t/n$.

$$\begin{aligned}
&(u(t), \varphi) - (u_0, \varphi) = \sum_{i=0}^{n-1} [(u(t_{i+1}), \theta) - (u(t_i), \varphi)] \\
&= \sum_{i=0}^{n-1} [(u(t_{i+1}), \theta) - (u(t_i), p(\Delta t, \varphi, \cdot)) + (u(t_i), p(\Delta t, \varphi, \cdot)) - (u(t_i), \varphi)] \\
&= \sum_{i=0}^{n-1} \left[\int_{t_i}^{t_{i+1}} \int_0^1 p(t_{i+1} - s, \varphi, y) f(s, y; u(s, y)) dy ds \right. \\
&+ \int_{t_i}^{t_{i+1}} \int_0^1 p(t_{i+1} - s, \varphi, y) g(s, y; u(s, y)) W(dy, ds) \\
&\left. + \int_{t_i}^{t_{i+1}} \int_0^1 u(t_i, y) \frac{\partial^2 p}{\partial y^2}(s - t_i, \varphi, y) dy ds \right]
\end{aligned}$$

If we exploit the fact that u is a. s. continuous and adapted, we obtain that as $n \to \infty$, the last expression tends to

$$\begin{aligned}
\int_0^t \int_0^1 \varphi(y) f(s, y; u(s, y)) dy ds + \int_0^t \int_0^1 \varphi(y) g(s, y; u(s, y)) W(dy, ds) \\
+ \int_0^t \int_0^1 u(s, y) \varphi''(y) dy ds.
\end{aligned}$$

□

In order to prove existence and uniquenes of a solution, we need to replace the assumption $(H2)$ by the stronger assumption

$$(H3) \quad |f(t, x, r) - f(t, x, r')| + |g(t, x, r) - g(t, x, r')| \leq k|r - r'|.$$

We have the

Theorem 3.2.3. *Under the assumptions* $(H1)$ *and* $(H3)$*, if* $u_0 \in C_0([0,1])$*, there exists a unique continuous* $\mathcal{P} \otimes \mathcal{B}([0,1])$*–measurable solution* u *of equation* (3.3)*. Moreover* $\sup_{0\le x\le 1,\ 0\le t\le T} \mathbb{E}[|u(t,x)|^p] < \infty$*, for all* $p \ge 1$.

PROOF: UNIQUENESS Let u and v be two solutions. Then the difference $\overline{u} = u - v$ satisfies

$$\begin{aligned}\overline{u}(t,x) &= \int_0^t \int_0^1 p(t-s;x,y)[f(s,y;u(s,y)) - f(s,y;v(s,y))]dsdy \\ &\quad + \int_0^t \int_0^1 p(t-s;x,y)[g(s,y;u(s,y)) - g(s,y;v(s,y))]W(ds,dy).\end{aligned}$$

Using successively the inequality $(a+b)^2 \le 2(a^2+b^2)$, Cauchy–Schwarz, the isometry property of the stochastic integral, and $(H3)$, we obtain

$$\mathbb{E}[\overline{u}^2(t,x)] \le 2(t+1)k^2 \int_0^t \int_0^x p^2(t-s;x,y)\mathbb{E}[\overline{u}^2(s,y)]dyds$$

Let $H(t) = \sup_{0\le x\le 1} \mathbb{E}[\overline{u}^2(t,x)]$. We deduce from the last inequality

$$H(t) \le 2(t+1) \int_0^t \left[\sup_{0\le x\le 1} \int_0^1 p^2(t-s;x,y)dy \right] H(s)ds.$$

From the above estimate upon p, we deduce that

$$\sup_{0\le x\le 1} \int_0^1 p^2(t-s;x,y)dy \le \frac{C_T^2}{t-s} \int_{\mathbb{R}} \exp\left(-\frac{|x-y|^2}{2(t-s)}\right) dy \le \frac{C'}{\sqrt{t-s}},$$

and iterating twice the estimate thus obtained for H, we deduce that

$$H(t) \le C'' \int_0^t H(s)ds,$$

hence $H(t) = 0$ from Gronwall's Lemma.
EXISTENCE We use the well known Picard iteration procedure

$$\begin{aligned}u^0(t,x) &= 0 \\ u^{n+1}(t,x) &= \int_0^1 p(t;x,y)u_0(y)dy + \int_0^t \int_0^1 p(t-s;x,y)f(s,y;u^n(s,y))dyds \\ &\quad + \int_0^t \int_0^1 p(t-s;x,y)g(s,y;u^n(s,y))W(dy,ds).\end{aligned}$$

Let $H_n(t) = \sup_{0\le x\le 1} \mathbb{E}[|u^{n+1}(t,x) - u^n(t,x)|^2]$. Then, as in the proof of uniqueness, we have that for $0 \le t \le T$,

$$H_n(t) \le C_T \int_0^t H_{n-2}(s)ds.$$

Iterating this inequality k times, we get

$$\begin{aligned} H_n(t) &\le C_T^k \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{k-1}} H_{n-2k}(s_k)ds_k \\ &\le \frac{C_T^k t^{k-1}}{(k-1)!} \int_0^t ds H_{n-2k}(s). \end{aligned}$$

But

$$\begin{aligned} H_0(t) = \sup_{0\le x\le 1} \mathbb{E}\Bigg(\Bigg| &\int_0^1 p(t;x,y)u_0(y)dy + \int_0^t \int_0^1 p(t-s;x,y)f(s,y;0)dyds \\ &+ \int_0^t \int_0^1 p(t-s;x,y)g(s,y;0)W(dy,ds)\Bigg|^2\Bigg) < \infty, \end{aligned}$$

thanks to assumption $(H1)$. Hence the sequence $\{u^n\}$ is Cauchy in $L^\infty((0,T)\times(0,1);L^2(\Omega))$; its limit u is $\mathcal{P}\times\mathcal{B}([0,1])$–measurable and satisfies (3.3). We could have done all the argument with the exponent 2 replaced by p, hence the p–th moment estimate. It remains to show that it can be taken to be continuous, which we will do in the next Theorem.

Theorem 3.2.4. *The solution u of equation* (3.3) *has a modification which is a. s. Hölder continuous in (t,x), with the exponent $1/4-\varepsilon$, $\forall\varepsilon > 0$.*

Proof: It suffices to show that each term in the right hand side of (3.3) has the required property. We shall only consider the stochastic integral term, which is the hardest. Consider

$$v(t,x) = \int_0^t \int_0^1 p(t-s;s,y)g(s,y;u(s,y))W(ds,dy).$$

We shall use the following well known Kolmogorov Lemma

Lemma 3.2.5. *Is $\{X_\alpha,\ \alpha\in D\subset \mathbb{R}^d\}$ is a random field such that for some k, p and $\beta > 0$, for all $\alpha,\alpha'\in D$,*

$$\mathbb{E}\left(|X_\alpha - X_{\alpha'}|^p\right) \le k|\alpha-\alpha'|^{d+\beta},$$

then there exists a modification of the process $\{X_\alpha\}$ which is a. s. Hölder continuous with the exponent $\beta/p-\varepsilon$, for all $\varepsilon > 0$.

Proof of Theorem 3.2.4 Now

$$\begin{aligned}\mathbb{E}[|(v(t+k,x+h)-v(t,x)|^p]^{1/p} \leq \mathbb{E}[|(v(t+k,x+h)-v(t+k,x)|^p]^{1/p} \\ + \mathbb{E}[|(v(t+k,x)-v(t,x)|^p]^{1/p}.\end{aligned}$$

We estimate first the first term (for simplicity of notations, we replace $t+k$ by t). From Burholder and Hölder,

$$\begin{aligned}&\mathbb{E}[|(v(t,x+h)-v(t,x)|^p] \\ &\quad \leq c\mathbb{E}\left(\left|\int_0^t\int_0^1 g^2(u;s,y)[p(t-s;x+h,y)-p(t-s;x,y)]^2 dyds\right|^{4p/2}\right) \\ &\quad \leq c_p\left(\int_0^t\int_0^1 |p(s;x,z)-p(s;x+h,z|^{2p/(p-2)}dzds\right)^{(p-2)/p}\end{aligned}$$

But we have (with $\beta<3$, i. e. $p>6$)

$$\int_0^t\int_0^1 |p(s;x,z)-p(s;y,z)|^\beta dzds \leq |x-y|^\beta\int_0^\infty\int_{\mathbb{R}} |p(\eta;\frac{x}{x-y},\xi)-p(\eta;\frac{y}{x-y},\xi)|^\beta d\xi d\eta,$$

hence

$$\mathbb{E}[|(v(t,x+h)-v(t,x)|^p] \leq C|y-x|^{(p-6)/2},$$

and $x\to v(t,x)$ is Hölder with any exponent $<1/2$.

Analogously

$$\begin{aligned}&\mathbb{E}[|(v(t+k,x)-v(t,x)|^p] \\ &\quad \leq c\mathbb{E}\left(\left|\int_0^t\int_0^1 g^2(u;s,y)[p(t+k-s;x,y)-p(t-s;x,y)]^2 dyds\right|^{p/2}\right) \\ &\quad + c_p\left(\int_t^{t+k}\int_0^1 g^2(u;s,y)p^2(t+k-s;x,y)dyds\right)^{p/2} \\ &\quad \leq C\left\{\left|\int_0^t\int_0^1 |p(t+k-s;x,y)-p(t-s;x,y)|^{2p/(p-2)}dyds\right|^{(p-2)/2}\right. \\ &\quad \left. + \left|\int_0^k\int_0^1 p^{2p/(p-2)}(s;x,y)dyds\right|^{(p-2)/2}\right\} \\ &\quad \leq C\left[|t-s|^{(p-6)/4}+|t-s|^{(p-6)/2(p-2)}\right],\end{aligned}$$

hence $t\to v(t,x)$ is a. s. Hölder with any exponent $<1/4$.

3.3 More general existence and uniqueness result

One can generalize the existence–uniqueness result to coefficients satifying the following assumptions (see Zangeneh [28] and Gyöngy, P. [9])

$$(A1)\begin{cases} \forall T,R, \quad \exists K(T,R) \text{ such that } \forall 0\le x\le 1, t\le T, |r|,|r'|\le R \\ (r-r')[f(t,x;r)-f(t,x;r')]+|g(t,x;r)-g(t,x;r')|^2 \le K(T,R)|r-r'|^2 \end{cases}$$

$$(A2)\begin{cases} \exists C \text{ such that } \forall t\ge 0, r\in \mathbb{R}, 0\le x\le 1, \\ rf(t,x;r)+|g(t,x;r)|^2\le C(1+|r|^2) \end{cases}$$

$$(A3) \quad \forall t\ge 0,, 0\le x\le 1,\ r\to f(t,x;r) \text{ is continuous.}$$

Moreover, whithout the assumption $(A2)$, the solution exists and is unique up to some (possibly infinite) stopping time.

If one suppresses the above condition $(A1)$, and adds the condition that

$$\forall t\ge 0,\ 0\le x\le 1, \quad r\to g(t,x;r) \text{ is continuous,}$$

then one can show the existence of a weak solution (i. e. a solution of the associated martingale problem).

3.4 Positivity of the solution

Let us state the

Theorem 3.4.1. *Let u and v be the two solutions of the two white–noise driven SPDEs*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + f(t,x;u(t,x)) + g(t,x;u(t,x))\,\dot{W}\,(t,x),\ t\ge 0, 0\le x\le 1; \\ u(t,0)=u(t,1)=0,\ t\ge 0; \\ u(0,x)=u_0(x),\ 0\le x\le 1. \end{cases} \tag{3.4}$$

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = \dfrac{\partial^2 v}{\partial x^2}(t,x) + F(t,x;v(t,x)) + g(t,x;v(t,x))\,\dot{W}\,(t,x),\ t\ge 0, 0\le x\le 1; \\ v(t,0)=v(t,1)=0,\ t\ge 0; \\ v(0,x)=v_0(x),\ 0\le x\le 1. \end{cases} \tag{3.5}$$

Assume that u_0, $v_0 \in C_0([0,1])$ and one of the two pairs (f,g) or (F,g) satisfies the conditions for strong existence and uniqueness. Then if $u_0(x) \le v_0(x)$ $\forall x$ and $f \le F$, $u(t,x) \le v(t,x)$ $\forall t \ge 0$, $0 \le x \le 1$, $\mathbb{P}$ a. s.

SKETCH OF THE PROOF: Let $\{e_k,\ k \ge 1\}$ be an orthonormal basis of $L^2(0,1)$. Formally,

$$\dot{W}(t,x) = \sum_{k=1}^{\infty} \dot{W}^k(t) e_k(x).$$

For each $N \ge 1$, let

$$\dot{W}_N(t,x) = \sum_{k=1}^{N} \dot{W}^k(t) e_k(x),$$

and u_N (resp. v_N) be the solution of (3.4) (resp. (3.5)), where $\dot{W}$ has been replaced by $\dot{W}_N$. Then one can show (see Lemma 2.1 in [5]) that $\forall p \ge 1$, $T \ge 0$,

$$\lim_{N\to\infty} \sup_{0\le t\le T,\ 0\le x\le 1} \mathbb{E}[|(u(t,x) - u_N(t,x)|^p] = 0,$$

and the same is true for the difference $v - v_N$. It is then easy to deduce the result from Theorem 2.4.3.

Corollary 3.4.2. *Let $u_0(x) \ge 0$, assume (f,g) satisfies the conditions for strong existence–uniqueness of a solution u to equation* (3.3). *If moreover*

$$f(t,x;0) \ge 0, \quad g(t,x;0) = 0, \quad \forall t \ge 0, 0 \le x \le 1,$$

then $u(t,x) \ge 0$, $\forall t \ge 0$, $0 \le x \le 1$, $\mathbb{P}$ a. s.

PROOF: Let $v_0 \equiv 0 \le u_0(x)$, $F(t,x;r) = f(t,x;r) - f(t,x;0) \le f(t,x;r)$. Then $v \equiv 0$ solves (3.5), and the result follows from the comparison theorem (reversing the orders).

3.5 Applications of Malliavin calculus to SPDEs

We consider again equation (3.3). Our assumptions in this section are the following

$$(M1)\begin{cases} \forall 0 \le x \le 1, t \ge 0, \quad r \to (f(t,x;r), g(t,x;r)) \text{ is of class } C^1 \\ \text{and the derivatives are locally bounded, uniformly in } t \text{ and } x. \end{cases}$$

$$(M2)\begin{cases}\exists C \text{ such that } \forall t \geq 0, r \in \mathbb{R}, 0 \leq x \leq 1,\\ rf(t,x;r) + |g(t,x;r)|^2 \leq C(1+|r|^2)\end{cases}$$

$$(M3) \quad (t,x,r) \to g(t,x;r) \text{ is continuous.}$$

$$(M4) \quad \exists y \in (0,1) \text{ such that } g(0,y;u_0(y)) \neq 0.$$

The aim of this section is to show the following result from [23]

Theorem 3.5.1. *Under conditions* $(M1)$, $(M2)$ *and* $(M3)$, *for any* $t > 0$, $0 < x < 1$, *the law of the random variable* $u(t,x)$ *is absolutely continuous with respect to Lebesgue measure on* $\mathbb{R}$.

Let us first state and prove one Corollary to this result

Corollary 3.5.2. *Under the conditions of Theorem 3.5.1, if moreover* $u_0(x) \geq 0$, $u_0 \not\equiv 0$, $f(t,x;0) \geq 0$, $g(t,x;0) = 0$, *then* $u(t,x) > 0$, $\forall t > 0$, x *a. e.,* $\mathbb{P}$ *a. s.*

PROOF: From Corollary 3.4.2, we already know that $u(t,x) \geq 0$ for all t, x, $\mathbb{P}$ a. s. Moreover $\mathbb{P}(u(t,x) = 0) = 0$, hence for each fixed (t,x), $u(t,x) > 0$ $\mathbb{P}$ a. s. The result follows from the continuity of u. □

Let us recall the basic ideas of Malliavin calculus, adapted to our situation. We consider functionals of the Gaussian random measure W. We first consider the so–called *simple* random variables, which are of the following form :

$$F = f(W(k_1), \ldots, W(k_n)),$$

where $f \in C_b^\infty(\mathbb{R}^n)$, $k_1, \ldots, k_n \in H = L^2(\mathbb{R}_+ \times (0,1))$. For any $h \in H$, we define the Malliavin derivative of F in the direction h as

$$\begin{aligned} D_h F &= \frac{d}{d\varepsilon} f(W(k_1) + \varepsilon(h,k_1), \ldots, W(k_n) + \varepsilon(h,k_n))|_{\varepsilon=0} \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(k_1), \ldots, W(k_n))(h,k_i), \end{aligned}$$

and the first order Malliavin derivative of F as the random element of H $v(t,x) = D_{tx}F$ given as

$$D_{tx}F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(k_1), \ldots, W(k_n))k_i(t,x).$$

We next define the $\|\cdot\|_{1,2}$ norm of a simple random variable as follows

$$\|F\|_{1,2}^2 = \mathbb{E}(F^2) + \mathbb{E}(|DF|_H^2).$$

Now the Sobolev space $\mathbb{D}^{1,2}$ is defined as the closure of the set of simple random variables with respect to the $\|\cdot\|_{1,2}$ norm. Both the directional derivative D_h and the derivation D are closed operators, which can be extended to elements of the space $\mathbb{D}^{1,2}$. It can even be extended to elements of $\mathbb{D}^{1,2}_{\text{loc}}$, which is defined as follows. $X \in \mathbb{D}^{1,2}_{\text{loc}}$ whenever there exists a sequence $\{X_n,\ n \geq 1\}$ of elements of $\mathbb{D}^{1,2}$, which are such that the sequence $\Omega_n = \{X = X_n\}$ is increasing, and $\mathbb{P}(\Omega \backslash \cup_n \Omega_n) = 0$. We note that for $X \in \mathbb{D}^{1,2}_{\text{loc}}$, which is $\mathcal{F}_t$ measurable, $D_{sy}X = 0$ whenever $s > t$. One should think intuitively of the operator D_{sy} as the derivation of a function of $\dot{W}$ with respect to $\dot{W}(s, y)$, the white noise at point (s, y).

We shall also use the space $\mathbb{D}_h$, which is the closure of the set of simple random variables with respect to the norm whose square is defined as

$$\|X\|_h^2 = \mathbb{E}(F^2 + |D_hF|^2).$$

A simple consequence of a well–known result of Bouleau and Hirsch is the

Proposition 3.5.3. *Let $X \in \mathbb{D}^{1,2}_{loc}$. If $\|DX\|_H > 0$ a. s., then the law of the random variable X is absolutely continuous with respect to Lebesgue's measure.*

Proof (taken from Nualart [19]): It suffices to treat the case where $X \in \mathbb{D}^{1,2}$ and $|X| < 1$ a. s. It now suffices to show that whenever $g : (0, 1) \to [0, 1]$ is measurable,

$$\int_{-1}^{1} g(y)dy = 0 \Rightarrow \mathbb{E}g(X) = 0.$$

There exists a sequence $\{g_n\} \subset C_b^1((-1, 1); [0, 1])$ which converges to g a. e. both with respect to the law of X and with respect to Lebesgue's measure. Define

$$\psi_n(x) = \int_{-1}^{x} g_n(y)dy, \quad \psi(x) = \int_{-1}^{x} g(y)dy.$$

Now $\psi_n(X) \in \mathbb{D}^{1,2}$ and $D[\psi_n(X)] = g_n(X)DX$. Now, $\psi_n(X) \to \psi(X)$ in $\mathbb{D}^{1,2}$. We observe that $\psi(X) = 0$, and $D[\psi(X)] = g(X)DX$. Now from the assumption of the Proposition follows that $g(X) = 0$ a. s. $\square$

We shall prove that for fixed (t, x), $u(t, x) \in \mathbb{D}^{1,2}_{\text{loc}}$, then compute the directional Malliavin derivative $D_h u(t, x)$, and finally prove that $\|Du(t, x)\|_H > 0$ a. s.

Proof of Theorem 3.5.1: First step. By the localization argument, it suffices to prove that whenever f, f'_r, g and g'_r are bounded, $u(t, x) \in \mathbb{D}^{1,2}$. We first show that a directional derivative exists in any direction of the form $h(t, x) = \rho(t)e_\ell(x)$, where $\rho \in L^2(\mathbb{R}_+)$, and e_ℓ is an element of an orthonormal basis of $L^2(0, 1)$. This is done by approximating (3.3) by a sequence of finite dimensional SDEs indexed by n, driven by a finite dimensional Wiener process. The derivative of the approximate SDE is known to solve a linearized equation, which converges as $n \to \infty$ to the solution $v(t, x)$ of the linearized equation

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial x^2} + f'(u)v + g'(u)v\,\dot{W} + g(u)h \\ v(0, x) = 0, \end{cases} \tag{3.6}$$

and the fact that D_h is closed (that means that if $\{X_n\} \subset \mathbb{D}_h$, $X_n \to X$ in $L^2(\Omega)$, $D_h X_n \to Y$ in $L^2(\Omega; H)$, then $X \in \mathbb{D}_h$ and $Y = D_h X$), allows us to deduce that $u \in \mathbb{D}_h$, and $v = D_h u$. The fact that $u \in \mathbb{D}^{1,2}$ is proved by showing that, whenever $\{h_n,\ n \geq 1\}$ is an orthonormal basis of $H = L^2(\mathbb{R}_+ \times (0, 1))$,

$$\mathbb{E}\left(\|Du(t, x)\|_H^2\right) = \sum_n \mathbb{E}\left(|D_{h_n} u(t, x)|^2\right),$$

which can be shown to be finite using classical estimates of the kernel of the heat equation.

Step 2 Let y be such that $g(0, y; u_0(y)) \neq 0$, and suppose for example that $g(0, y; u_0(y)) > 0$. Then there exists $\varepsilon > 0$ and a stopping time τ such that $0 < \tau \leq t$, such that

$$g(s, z; u(s, z) > 0, \quad \forall z \in [y - \varepsilon, y + \varepsilon],\ 0 \leq s \leq \tau,$$

and we have

$$\begin{aligned} \|Du(t, x)\|_H > 0 &\Leftrightarrow \int_0^t \int_0^1 |D_{s,z} u(t, x)| dz ds > 0 \\ &\Leftarrow \int_0^\tau \int_{y-\varepsilon}^{y+\varepsilon} |D_{s,z} u(t, x)| dz ds > 0. \end{aligned}$$

But, $\forall h \in L^2(\Omega \times \mathbb{R}_+ \times [0,1], \mathcal{P} \otimes \mathcal{B}([0,1]), \mathbb{P} \times \lambda)$ such that $h \geq 0$ and $\mathrm{supp} h \subset \{(s,y);\; g(s,y;u(s,y)) \geq 0\}$, $D_h u(t,x) \geq 0$, as a consequence of Corollary 3.4.2, applied to (3.6). Hence a sufficient condition for $\|Du(t,x)\|_H$ to be positive is that

$$\int_0^\tau \int_{y-\varepsilon}^{y+\varepsilon} D_{s,z}u(t,x)dzds = \int_0^\tau v(s;t,x) > 0,$$

where we have defined $v(s;t,x) = \int_{y-\varepsilon}^{y+\varepsilon} D_{s,z}u(t,x)dz$. Let us just show that $v(t,x) = v(0;t,x) > 0$. v solves the linearized SPDE

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial x^2} + f'(u)v + g'(u)v\,\dot{W} \\ v(0,x) = g(0,x;u_0(x))\mathbf{1}_{[y-\varepsilon,y+\varepsilon]}(x). \end{cases} \tag{3.7}$$

Now $\exists \beta > 0$ such that $g(0,x,u_0(x)) \geq \beta$, for $x \in [y-\varepsilon, y+\varepsilon]$, then by the comparison theorem it suffices to prove our result with the initial condition of (3.7) replaced by $\beta \mathbf{1}_{[y-\varepsilon,y+\varepsilon]}(x)$, and by linearity it suffices to treat the case $\beta = 1$. In order to simplify the notations, we let $a = y-\varepsilon$, and $b = y+\varepsilon$. Since $\overline{v} = e^{ct}v$ satisfies the same equation as v, with $f'(u)$ replaced by $f'(u)+c$, it suffices, again by the comparison theorem, to treat the case $f'(u) \equiv 0$. Finaly we need to examine the random variable

$$\begin{aligned} v(t,x) &= v_1(t,x) + v_2(t,x) \\ &= \int_a^b p(t;x,z)dz + \int_0^t \int_0^1 p(t-s;x,z)g'(u)(s,z)v(s,z)W(ds,dz). \end{aligned}$$

Assume that $x \geq a$ (if this is not the case, then we have $x \leq b$, and we can adapt the argument correspondingly). Let d be such that $x \leq b+d < 1$, and define

$$\alpha = \frac{1}{2} \inf_{1\leq k\leq m} \inf_{a\leq y\leq b+dk/m} \int_a^{b+d(k-1)/m} p(\frac{t}{m};y,z)dz.$$

We have that $\alpha > 0$. We now define, for $1 \leq k \leq m$, the event

$$E_k = \left\{ v\left(\frac{kt}{m}, \cdot\right) \geq \alpha^k \mathbf{1}_{[a,b+kd/m]}(\cdot) \right\}.$$

Let us admit for a moment the

Lemma 3.5.4. *There exists $\delta_0, m_0 > 0$ such that for all $0 < \delta \leq \delta_0$, and $m \geq m_0$,*

$$\mathbb{P}(E_{k+1}^c | E_1 \cap \cdots \cap E_k) \leq \frac{\delta}{m}, \quad 0 \leq k \leq m-1.$$

Now

$$\mathbb{P}(v(t,x) > 0) \geq \lim_{m\to\infty} \mathbb{P}(E_1 \cap \cdots \cap E_m) \geq \lim_m (1 - \frac{\delta}{m})^m = e^{-\delta},$$

hence the result, since we can let $\delta \to 0$.

Proof of Lemma 3.5.4: Proving the Lemma amounts to prove that $\mathbb{P}(E_1^c) \leq \delta/m$. By the very definition of α,

$$v_1\left(\frac{t}{m}, \cdot\right) \geq 2\alpha \mathbf{1}_{[a,b+d/m]}(\cdot).$$

hence it suffices to show that for some $\delta_0 > 0$, and all $0 < \delta \leq \delta_0$, and m large enough,

$$\mathbb{P}\left(\sup_{0\leq y\leq 1}\left|v_2\left(\frac{t}{m}, y\right)\right| > \alpha\right) \leq \frac{\delta}{m}.$$

For this to be true, it suffices that there exists n, $p > 1$ and $c > 0$ such that

$$\mathbb{E}\left(\sup_{0\leq y\leq 1} |v_2(t,y)|^n\right) \leq ct^p.$$

But

$$\begin{aligned}\mathbb{E}(|v_2(t,y)|^n) &\leq c\left(\int_0^t \int_0^1 p^2(t-s;y,z)dzds\right)^{n/2} \\ &\leq c\left(\int_0^t \int_0^1 p^r(t-s;y,z)dzds\right)^{n/r} t^{n/q},\end{aligned}$$

if $\frac{2}{r} + \frac{2}{q} = 1$. Since we need $r < 3$ for the first factor to be finite, we get that for $q > 6$,

$$\mathbb{E}(|v_2(t,y)|^n) \leq ct^{n/q}.$$

Moreover, from Walsh's computations (see the prof of Theorem 3.2.4)

$$\mathbb{E}\left(|v_2(t,x) - v_2(t,y)|^n\right) \leq c|x-y|^{\frac{n}{2}-1} t^{n/q}.$$

This allows us to conclude, if we choose $n > q > 6$. $\square$

In the case where g does not vanish, and the coefficients are smooth, for any $0 < x_1 \cdots < x_n < 1$, the law of the random vector

$$(u(t, x_1, u(t, x_2), \dots, u(t, x_n))$$

has a density with respect to Lebesgue measure on $\mathbb{R}^n$, which is everywhere strictly positive. It is an open problem to show the same result under a condition similar to that of Theorem 3.5.1.

In the case of the $2D$ Navier–Stokes equation driven by certain low dimensional white noises, Mattingly and P. [13] have shown that for any $t > 0$, the projection of $u(t, \cdot)$ on any finite dimensional subspace has a density with respect to Lebesgue measure, which under appropriate conditions is smooth and everywhere positive.

3.6 SPDEs and the super Brownian motion

In this section, we want to study the SPDE

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + |u|^\gamma \dot{W}, \ t \geq 0, \ x \in \mathbb{R} \\ u(0, x) = u_0(x), \end{cases} \tag{3.8}$$

where $u_0(x) \geq 0$. We expect the solution to be non negative, so that we can replace $|u|^\gamma$ by u^γ. The behavior of the solution to this equation, which has been the object of intense study, depends very much upon the value of the positive parameter γ. The case $\gamma = 1$ is easy and has already been considered in these notes. If $\gamma > 1$, then the mapping $r \to r^\gamma$ is locally Lipschitz, and there exists a unique strong solution, up possibly to an explosion time. C. Mueller has shown that the solution is strictly positive, in the sense that

$$u_0 \not\equiv 0 \Rightarrow u(t, x) > 0, \ \forall t > 0, x \in \mathbb{R}, \ \mathbb{P} \text{ a. s.}$$

We shall consider here the case $\gamma < 1$.

3.6.1 The case $\gamma = 1/2$

In that case, the SPDE (3.8) is related to the super Brownian motion, which we now define. For a more complete introduction to superprocesses and for all the references to this subject, we refer the reader to [6]. Let $\mathcal{M}_d$ denote the space of finite measures on $\mathbb{R}^d$, and C^d_{c+} the space of C^2 functions from $\mathbb{R}^d$ into $\mathbb{R}_+$, with compact support. We shall use $\langle \cdot, \cdot \rangle$ to denote the pairing between measures and functions from C^d_{c+}.

Definition 3.6.1. *The super Brownian motion is a Markov process $\{X_t,\ t \geq 0\}$ with values in $\mathcal{M}_d$ which is such that $t \to \langle X_t, \varphi\rangle$ is right continuous for all $\varphi \in C^d_{c+}$, and whose transition probability is caracterized as follows through its Laplace transform*

$$\mathbb{E}_\mu[\exp(-\langle X_t, \varphi\rangle)] = \exp(\langle \mu, V_t(\varphi)\rangle), \quad \varphi \in C^d_{c+},$$

where $\mu \in \mathcal{M}_d$ and $V_t(\varphi)$ is the function which is the value at time t of the solution of the nonlinear PDE

$$\begin{cases} \dfrac{\partial V}{\partial t} = \dfrac{1}{2}(\Delta V - V^2) \\ V(0) = \varphi. \end{cases}$$

Let us compute the infinitesimal generator of this diffusion.

$$\begin{aligned} \text{If} \quad F(\mu) &= e^{-\langle \mu, \varphi\rangle}, \\ \lim_{t\to 0} \frac{1}{t}\left(\mathbb{E}_\mu F(X_t) - F(\mu)\right) &= \lim_{t\to 0} \frac{1}{t}\left(e^{-\langle \mu, V_t(\varphi)\rangle} - e^{-\langle \mu, \varphi\rangle}\right) \\ &= -e^{-\langle \mu, \varphi\rangle} \lim_{t\to 0} \langle \mu, \frac{V_t(\varphi) - \varphi}{t}\rangle \\ &= -\frac{1}{2} e^{-\langle \mu, \varphi\rangle} \langle \mu, \Delta\varphi - \varphi^2\rangle \\ &= \mathcal{G}F(\mu). \end{aligned}$$

From this we deduce that if F has the form $F(X_t) = f(\langle X_t, \varphi\rangle)$, then

$$\mathcal{G}F(\mu) = \frac{1}{2} f'(\langle \mu, \varphi\rangle)\langle \mu, \Delta\varphi\rangle + \frac{1}{2} f''(\langle \mu, \varphi\rangle)\langle \mu, \varphi^2\rangle.$$

Consequently, the process defined for $\varphi \in C^d_{c+}$ as

$$M^\varphi_t = \langle X_t, \varphi\rangle - \langle X_0, \varphi\rangle - \frac{1}{2}\int_0^t \langle X_s, \Delta\varphi\rangle ds$$

is a continuous martingale with associated increasing process

$$\langle M^\varphi\rangle_t = \int_0^t \langle X_s, \varphi^2\rangle ds.$$

We just formulated the martingale problem which the super Brownian motion solves. Let us show how this follows from our previous computations. We have that whenever $F(X_t) = f(\langle X_t, \varphi \rangle)$,

$$F(X_t) = F(X_0) + \int_0^t \mathcal{G}F(X_s)ds \quad \text{is a martingale.}$$

If we choose $f(x) = x$, we get that the following is a martingale

$$M_t^\varphi = \langle X_t, \varphi \rangle - \langle X_0, \varphi \rangle - \frac{1}{2}\int_0^t \langle X_s, \varphi \rangle ds.$$

If we choose now $f(x) = x^2$, we get another martingale

$$\begin{aligned} N_t = \langle X_t, \varphi \rangle^2 - \langle X_0, \varphi \rangle^2 - \int_0^t \langle X_s, \varphi \rangle \langle X_s, \Delta\varphi \rangle ds \\ - \int_0^t \langle X_s, \varphi^2 \rangle ds. \end{aligned}$$

Now applying Itô's formula to the first of the two above formulas yields

$$\begin{aligned} \langle X_t, \varphi \rangle^2 = \langle X_0, \varphi \rangle^2 + \int_0^t \langle X_s, \varphi \rangle \langle X_s, \Delta\varphi \rangle ds \\ + \langle M^\varphi \rangle_t + \text{ martingale.} \end{aligned}$$

Comparing the two last formulas gives

$$\langle M^\varphi \rangle_t = \int_0^t \langle X_s, \varphi^2 \rangle ds.$$

Existence of a density and SBM–related SPDE If $d \geq 2$, one can show that the measure X_t is a. s. singular with respect to Lebesgue measure. On the contrary, if $d = 1$, $X_t << \lambda$. Define $u(t, \cdot)$ as the density of X_t. The formula for $\langle M^x \rangle_t$ implies that there exists a Gaussian random measure on $\mathbb{R}_+ \times \mathbb{R}$ such that

$$M_t^\varphi = \int_0^t \int_{\mathbb{R}} \sqrt{u(s,x)}\varphi(x)W(ds,dx),$$

hence $u(t, x)$ is a (weak) positive solution of the SPDE

$$\begin{cases} \quad \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + \sqrt{u}\dot{W}, \ t \geq 0, \ x \in \mathbb{R} \\ u(0,x) = u_0(x). \end{cases} \tag{3.9}$$

Uniqueness in law We now show that the law of the super Brownian motion is uniquely determined, which implies uniqueness in law for the SPDE (3.9).

We note that from the Markov property, and the semigroup property of $\{V_t\}$,

$$\begin{aligned}\mathbb{E}_\mu\left(e^{-\langle X_t, V_{T-t}(\varphi)\rangle}|\mathcal{F}_s\right) &= \mathbb{E}_{X_s}\left(e^{-\langle X_{t-s}, V_{T-t}(\varphi)\rangle}\right)\\ &= e^{-\langle X_s, V_{t-s}(V_{T-t}(\varphi))\rangle}\\ &= e^{-\langle X_s, V_{T-s}(\varphi)\rangle}.\end{aligned}$$

We have just proved that $\{e^{-\langle X_t, V_{T-t}(\varphi)\rangle},\ 0 \le t \le T\}$ is a martingale. Hence in particular

$$\mathbb{E}_\mu e^{-\langle X_T, \varphi\rangle} = e^{-\langle \mu, V_T(\varphi)\rangle},$$

which caracterizes the transition probability of $\{X_t\}$, hence its law.

A construction of the SBM We start with an approximation by a branching process.

- At time 0, let N particles have i. i. d. locations in $\mathbb{R}^d$, with the common law μ.
- At each time k/N, $k \in \mathbb{N}$, each particle dies with probability 1/2 and gives birth to 2 descendants with probability 1/2.
- On each interval $[k/N, (k+1)/N]$, the living particles follow mutually independent Brownian motions.

Denote by $N(t)$ the number of particles alive at time t, and Y_t^i the position of the i–th particle ($1 \le i \le N(t)$). Let $\{X_t^N\}$ denote the $\mathcal{M}_d$–valued process

$$X_t^N = \frac{1}{N}\sum_{i=1}^{N(t)} \delta_{Y_t^i}, \quad \langle X_t^N, \varphi\rangle = \frac{1}{N}\sum_{i=1}^{N(t)} \varphi(Y_t^i).$$

Theorem 3.6.2. *$X^N \Rightarrow X$, as $N \to \infty$, where X is a SBM with initial law μ.*

Corollary 3.6.3. *There exists a stopping time τ, with $\tau < \infty$ a. s., such that $X_\tau = 0$.*

PROOF: The extinction time T of a branching process as described above satisfies, from a result due to Kolmogorov,

$$\mathbb{P}(T > t) = \mathbb{P}(NT > Nt) \simeq \frac{C}{Nt}.$$

Now with N independent such processes

$$\mathbb{P}(\sup_{i \le i \le N} T_i \le t) = \prod_{i=1}^{N} \mathbb{P}(T_i \le t) \simeq (1 - \frac{C}{Nt})^N \to e^{-C/t},$$

as $N \to \infty$. In other words, $\mathbb{P}(\tau > t) \simeq 1 - e^{-C/t}$. □

We will now show that whenever u_0 has compact support, the same is true with $u(t, \cdot)$, $\forall t > 0$. This follows from the

Theorem 3.6.4. *Let $\mu \in \mathcal{M}_d$ be such that supp$\mu \subset B(0, R_0)$. Then, for all $R > R_0$,*

$$\mathbb{P}(X_t(B(0,R)^c) = 0, \ \forall t \ge 0) = \exp\left(-\frac{\langle \mu, u(R^{-1} \cdot)\rangle}{R^2}\right),$$

where u is the positive solution of the PDE

$$\begin{cases} \Delta u = u^2, & |x| < 1; \\ u(x) \to \infty, & x \to \pm 1. \end{cases}$$

Corollary 3.6.5. *Under the assumptions of the theorem,*

$$\mathbb{P}_\mu \left(\cup_{t \ge 0} suppX_t \text{ is bounded}\right) = 1.$$

PROOF: We have

$$\begin{aligned}
&\mathbb{P}_\mu \left(\cup_{t \ge 0} \text{supp} X_t \ \text{ is bounded}\right) \\
&\quad = \mathbb{P}_\mu \left(\cup_{r > R_0} \{X_t(B(0,r)^c) = 0, \ \forall t \ge 0\}\right) \\
&\quad = \lim_{r \to \infty} \exp\left(-\frac{\langle \mu, u(r^{-1} \cdot)\rangle}{r^2}\right) \\
&\quad \ge \lim_{r \to \infty} \exp\left(-\frac{1}{r^2}[\sup_{|y| \le R_0/r} u(y)]\mu(\mathbb{R}^d)\right) \\
&\quad = 1,
\end{aligned}$$

where we have used the Theorem for the second equality. □

Before we prove the Theorem, we need one more Lemma.

Lemma 3.6.6. *$\forall t \geq 0$, $\varphi \in C^d_{c+}$, we have*

$$\mathbb{E}_\mu \exp\left(-\int_0^t \langle X_s, \varphi\rangle ds\right) = \exp\left(-\langle \mu, u_t(\varphi)\rangle\right),$$

where $\{u_t(\varphi),\ t \geq 0\}$ is the positive solution of the nonlinear parabolic PDE

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}(\Delta u - u^2) + \varphi, \quad t \geq 0; \\ u(0) = 0. \end{cases}$$

Proof: Let $n \in \mathbb{N}$, $h = t/n$, $t_i = ih$.

$$\exp\left(-\int_0^t \langle X_s, \varphi\rangle ds\right) = \lim_n \exp\left(-\sum_{i=1}^n \langle X_{t_i}, h\varphi\rangle\right).$$

Now

$$\begin{aligned} \mathbb{E}_\mu\left(e^{-\langle X_{t_n}, h\varphi\rangle} | \mathcal{F}_{t_{n-1}}\right) &= e^{-\langle X_{t_{n-1}}, V_h(h\varphi)\rangle}, \\ \mathbb{E}_\mu\left(e^{-\langle X_{t_n}, h\varphi\rangle - \langle X_{t_{n-1}}, h\varphi\rangle} | \mathcal{F}_{t_{n-2}}\right) &= \mathbb{E}_\mu\left(e^{-\langle X_{t_{n-1}}, V_h(h\varphi) + h\varphi\rangle} | \mathcal{F}_{t_{n-2}}\right) \\ &= e^{-\langle X_{t_{n-2}}, V_h(V_h(h\varphi)+h\varphi)\rangle}, \end{aligned}$$

and iterating this argument, we find that

$$\mathbb{E}_\mu \exp\left(-\sum_{i=1}^n \langle X_{t_i}, h\varphi\rangle\right) = \exp\left(-\langle \mu, v_h(t)\rangle\right),$$

where v_h solves the parabolic PDE

$$\begin{cases} \dfrac{\partial v_h}{\partial t} = \dfrac{1}{2}(\Delta v_h - v_h^2), \quad ih < t < (i+1)h; \\ v_h(ih) = v_h(ih^-) + h\varphi \\ v_h(0) = 0. \end{cases}$$

In other words (here $P(t)$ stands for the semigroup generated by $\frac{1}{2}\Delta$)

$$\begin{aligned} v_h(t) &= -\frac{1}{2}\int_0^t P(t-s) v_h^2(s) ds + h \sum_{0 \leq i:\ ih \leq t} P(t-ih)\varphi \\ &\rightarrow, \quad \text{as } n \text{ tends to } +\infty \\ u(t) &= -\frac{1}{2}\int_0^t P(t-s) u^2(s) ds + \int_0^t P(t-s)\varphi ds. \end{aligned}$$

PROOF OF THEOREM 3.6.4: Approximating the indicator function of the closed ball $B(0,R)$ by regular functions φ, and exploiting the fact that $t \to \langle X_t, \varphi\rangle$ is a. s. right continuous, as well as the monotone convergence theorem, we get that

$$\begin{aligned}
\mathbb{P}_\mu(X_t(B(0,R)^c)=0,\ \forall t\geq 0) &= \mathbb{P}_\mu\left(\int_0^\infty X_t(B(0,R)^c)dt=0\right)\\
&= \lim_{\theta\to\infty}\mathbb{E}_\mu\left(\exp\left[-\theta\int_0^\infty X_t(B(0,R)^c)dt\right]\right)\\
&= \lim_{\theta\to\infty}\lim_{n\to\infty}\lim_{m\to\infty}\lim_{T\to\infty}\mathbb{E}_\mu\left(\exp\left[-\int_0^T \langle X_t,\theta\varphi_{R,n,m}\rangle dt\right]\right)\\
&= \lim_{\theta\to\infty}\lim_{n\to\infty}\lim_{m\to\infty}\lim_{T\to\infty}\exp\left[-\langle \mu, u_{nm}(T,\cdot;R,\theta)\rangle\right],
\end{aligned}$$

where $\varphi_{R,n,m}$ is zero outside $[-m-1,-R]\cup[R,m+1]$, 1 on the interval $[-m,-R-1/n]\cup[R+1/n,m]$, increases and decreases linearly between 0 and 1; and $u_{nm}(t,\cdot,R,\theta)$, by the preceding Lemma, solves the parabolic PDE

$$\begin{cases}
\dfrac{\partial v}{\partial t} = \dfrac{1}{2}(\Delta v - v^2) + \theta\varphi_{R,n,m},\ 0\leq t\leq T,\\
v(0)=0.
\end{cases}$$

Now as $T\to\infty$, $u_{nm}(T,\cdot,R,\theta)\to u_{nm}(\cdot,R,\theta)$, which solves the PDE

$$-\Delta u_{nm} + u_{nm}^2 = 2\theta\varphi_{R,n,m},$$

and as $n,m\to\infty$, $u_{nm}(\cdot,R,\theta)\to u(\cdot,R,\theta)$, solution of

$$-\Delta u + u^2 = 2\theta\mathbf{1}_{|x|>R},$$

hence as $\theta\to\infty$, $u(\cdot,R,\theta)\to u(\cdot,R)$, solution of

$$\begin{cases}
-\Delta u + u^2 = 0, & |x|<R;\\
\quad u(x)\to\infty, & x\to\pm R.
\end{cases}$$

Since $u(x,R)=\frac{1}{R^2}u(\frac{x}{R})$, we finally get that

$$\mathbb{P}(X_t(B(0,R)^c)=0,\ \forall t\geq 0) = \exp\left(-\frac{\langle \mu, u(R^{-1}\cdot)\rangle}{R^2}\right).$$

3.6.2 Other values of $\gamma < 1$

We now prove that uniqueness in law still holds for $1/2 < \gamma < 1$, following Mytnik [18].

Theorem 3.6.7. *Let $u_0 \in C(\mathbb{R}; \mathbb{R}_+)$ be such that*

$$\sup_{x \in \mathbb{R}} e^{p|x|} u_0(x) < \infty, \quad \forall p > 0.$$

Then the martingale problem : $\forall \Phi \in D(\Delta)$,

$$Z_t(\Phi) := \langle u_t, \Phi \rangle - \langle u_0, \Phi \rangle - \frac{1}{2} \int_0^t \langle u_s, \Delta\Phi \rangle ds \tag{3.10}$$

is a continuous martingale with the associated increasing process

$$\langle Z(\Phi) \rangle_t = \int_0^t \langle u_s^{2\gamma}, \Phi^2 \rangle ds$$

possesses a unique solution.

Proof Existence of a positive solution follows from weak existence theory, usinf tightness of an approximating sequence. We now prove uniqueness. By an argument very similar to that in the proof of Theorem 4.2 in Ethier–Kurtz [7] (that theorem does not apply here), it suffices to show that for any $t > 0$, the law of $u(t, \cdot)$ is unique. We have assumed that the initial condition $u(0, \cdot)$ belongs to the space

$$C^+_{\text{rap}} = \{u \in C(\mathbb{R}; \mathbb{R}_+), \ \sup_{x \in \mathbb{R}} e^{p|x|} u(x) < \infty, \ \forall p > 0\}.$$

We first note that $u(t, \cdot) \in C^+_{\text{rap}}$, $\forall t > 0$, and that for all $T > 0$, $p \geq 1$,

$$\sup_{0 \leq t \leq T, \ x \in \mathbb{R}} \mathbb{E}\left[u^p(t, x)\right] < \infty.$$

Suppose we can find a space E, a bounded measurable mapping $f : C^+_{\text{rap}} \times E \to \mathbb{R}$ such that

- (i) the class of functions $\{f(\cdot, y); \ y \in E\}$ separates the probabilities on C^+_{rap};

- (ii) $\forall y \in E$, there exists an E–valued process $\{Y_s,\ 0 \le s \le t\}$ which is independent from $\{u(s,\cdot),\ 0 \le s \le t\}$ and satisfies

$$\mathbb{E}\left[f(u(t,\cdot),y)\right] = \mathbb{E}\left[f(u(0,\cdot),Y_t)\right],$$
$$Y_0 = y.$$

Clearly the above conditions imply that the law of $u(t,\cdot)$ is unique.

Such a process $\{Y_s,\ 0 \le s \le t\}$ is called *dual* to the process $\{u(s,\cdot),\ 0 \le s \le t\}$.

The standard method for finding a dual process is to choose both a function f and a process $\{Y_s,\ 0 \le s \le t\}$ such that, together with appropriate integrability conditions, the following are true for a given function g :

$$f(u(t,\cdot),y) - \int_0^t g(u(s,\cdot),y)ds \quad \text{is a martingale, } \forall y \in E; \tag{3.11}$$

$$f(u,Y_t) - \int_0^t g(u,Y_s)ds \quad \text{is a martingale, } \forall u \in C^+_{\text{rap}}. \tag{3.12}$$

Indeed, we deduce from those identities that for all $0 \le s,r \le t$, $u \in C^+_{\text{rap}}$, $y \in E$,

$$\frac{d}{ds}\mathbb{E}f(u(s,\cdot),y) = \mathbb{E}g(u(s,\cdot),y), \quad \frac{d}{dr}\mathbb{E}f(u,Y_r) = \mathbb{E}g(u,Y_r).$$

Since the two processes $\{u(s,\cdot),\ 0 \le s \le t\}$ and $\{Y_s,\ 0 \le s \le t\}$ are mutually independent, the last identities imply that

$$\frac{\partial}{\partial s}\mathbb{E}f(u(s,\cdot),Y_r) = \mathbb{E}g(u(s,\cdot),Y_r) = \frac{\partial}{\partial r}\mathbb{E}f(u(s,\cdot),Y_r).$$

In other words, if we define

$$h(s,r) := \mathbb{E}f(u(s,\cdot),Y_r),\ h_1(s,r) = \frac{\partial h}{\partial s}(s,t),\ h_2(s,r) = \frac{\partial h}{\partial r}(s,t),$$

we have shown that $h_1 \equiv h_2$ on $[0,t]^2$. But we have the

Lemma 3.6.8. *If h_1 and h_2 are $ds \times dr$–integrable on $[0,t]^2$, and $h_1 \equiv h_2$, then*

$$h(t,0) = h(0,t).$$

Proof of Lemma 3.6.8 It follows from an elementary change of variable in the integral that

$$\int_0^t h(t-s,s)ds = \int_0^t h(s,t-s)ds.$$

Consequently

$$\begin{aligned}\int_0^t [h(s,0)-h(0,s)]ds &= \int_0^t [h(t-s,s)-h(0,s)]ds - \int_0^t [h(s,t-s)-h(s,0)]ds\\ &= \int_0^t \int_0^{t-s} h_1(u,s)duds - \int_0^t \int_0^{t-s} h_2(s,u)duds\\ &= 0,\end{aligned}$$

since $h_1 \equiv h_2$. The result follows from differentiating the identity which we have shown. □

We choose $E = L^1_+(\mathbb{R})$, and $f(\Psi,\Phi) = \exp[-\langle\Psi,\Phi\rangle]$. We know that for any $\Phi \in D(\Delta)$,

$$e^{-\langle u(t,\cdot),\Phi\rangle} - \frac{1}{2}\int_0^t e^{-\langle u(s,\cdot),\Phi\rangle}\left[-\langle u(s,\cdot),\Delta\Phi\rangle + \langle u^{2\gamma}(s,\cdot),\Phi^2\rangle\right] ds$$

is a martingale. Hence we wish to construct an $L^1_+(\mathbb{R})$–valued process $\{Y_t,\ t\geq 0\}$ such that $\forall\Psi \in C^+_{\text{rap}}$,

$$e^{-\langle\Psi,Y_t\rangle} - \frac{1}{2}\int_0^t e^{-\langle\Psi,Y_s\rangle}\left[-\langle\Psi,\Delta Y_s\rangle + \langle\Psi^{2\gamma},Y_s^2\rangle\right] ds$$

is a martingale.

Formally, we would like to find a solution $\{Y_t,\ t\geq 0\}$ of the SPDE

$$\frac{\partial Y_t}{\partial t}(t,x) = \frac{1}{2}\Delta Y_t(t,x) + Y_t^{1/\gamma}(t,x)\dot{L}(t,x),$$

where $\{L_t,\ t\geq 0\}$ would be a stable process on $\mathbb{R}_+\times\mathbb{R}$ with non–negative jumps, whose Laplace transform would be given for $\Phi \in L^{2\gamma}_+(\mathbb{R}_+\times\mathbb{R})$ by

$$\mathbb{E}\exp\left[-\int_0^t\int_{\mathbb{R}}\Phi(s,x)L(ds,dx)\right] = \exp\left[\int_0^t\int_{\mathbb{R}}\Phi^{2\gamma}(s,x)dxds\right].$$

Remark 3.6.9. *In the case $\gamma = 1/2$, the natural choice is $\dot{L}(t,x) = -1$, i. e. L is deterministic. In that case the solution of the PDE*

$$\frac{\partial Y_t}{\partial t}(t,x) = \frac{1}{2}\Delta Y_t(t,x) - Y_t^2(t,x)$$

provides a deterministic dual. This is one way to interpret the uniqueness proof in the previous subsection.

Unfortunately, we do not know how to solve the above SPDE for the Y process. Hence we shall replace it by a sequence of approximating SPDEs, corresponding to the following martingale problems. Given an arbitrary initial condition $\Phi \in L^1_+(\mathbb{R})$, for any $\Psi \in D(\Delta)_+$,

$$\begin{aligned} Z_t^n(\Psi) &:= \exp\left[-\langle Y_t^n, \Psi\rangle\right] - \exp\left[-\langle \Phi, \Psi\rangle\right] \\ &- \frac{1}{2}\int_0^t \exp\left[-\langle Y_s^n, \Psi\rangle\right]\left(-\langle Y_s^n, \Delta\Psi\rangle + \langle [Y_s^n]^2, \eta\int_{1/n}^{\infty}[e^{-\lambda\Psi} - 1 + \lambda\Psi]\frac{d\lambda}{\lambda^{1+2\gamma}}\rangle\right) ds \end{aligned}$$

is a local martingale satisfying $Z_0^n(\Psi) = 0$, where $\eta = 2\gamma(2\gamma - 1)/\Gamma(2 - 2\gamma)$. Note that

$$\int_0^\infty (e^{-u} - 1 + u)\frac{du}{u^{1+2\gamma}} = \eta^{-1},$$

hence

$$\eta\int_0^\infty (e^{-\lambda y} - 1 + \lambda y)\frac{d\lambda}{\lambda^{1+2\gamma}} = y^{2\gamma}.$$

Our uniqueness result follows from

Proposition 3.6.10. *For all $\eta \in C^+_{rap}$, $\Phi \in L^1_+(\mathbb{R})$, there exists a sequence of processes $\{Y_t^n,\ t \geq 0\}_{n\geq 1}$ with values in the set $\mathcal{M}(\mathbb{R})$ of finite measures on $\mathbb{R}$, such that $Y_0^n = \Phi$, and for any $t \geq 0$, any solution $\{u(s,\cdot),\ s \geq 0\}$ of the martingale problem* (3.10) *corresponding to the initial condition $u(0,\cdot) = \eta$, which is independent of the processes $\{Y_t^n,\ t \geq 0\}_{n\geq 1}$, we have*

$$\mathbb{E}\exp\left[-\langle u(t,\cdot), \Phi\rangle\right] = \lim_{n\to\infty}\mathbb{E}\exp\left[-\langle \eta, Y_t^n\rangle\right]$$

We shall need the following technical Lemma

Lemma 3.6.11. *Let $\{u(t,\cdot),\ t \geq 0\}$ denote a solution of the martingale problem* (3.10)*, and $\Psi \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ such that*

$$\int_0^T\int_{\mathbb{R}} \Psi^2(t,x)dxdt < \infty, \quad \forall T > 0.$$

Then the process

$$\exp[-\langle u(t,\cdot),\Psi(t,\cdot)\rangle]$$
$$-\int_0^t e^{-\langle u(s,\cdot),\Psi(s,\cdot)\rangle}\left[-\langle u(s,\cdot),\frac{1}{2}\Delta\Psi(s,\cdot)+\frac{\partial\Psi}{\partial s}(s,\cdot)\rangle+\frac{1}{2}\langle u^{2\gamma}(s,\cdot),\Psi^2(s,\cdot)\rangle\right]ds$$

is a martingale.

Let us now describe the construction of the sequence $\{Y_t^n,\ t\geq 0\}_{n\geq 1}$.

To each $n\in\mathbb{N}$, $m\in\mathcal{M}(\mathbb{R})$, we associate the positive mild solution $\{v_t^n(m),\ t\geq 0\}$ of the PDE

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x)=\dfrac{1}{2}\dfrac{\partial^2 v}{\partial x^2}(t,x)-\dfrac{1}{2}b_n v^2(t,x);\\ \quad v(0,\cdot)=m; \end{cases}$$

where $b_n=2\gamma n^{2\gamma-1}/\Gamma(2-2\gamma)$. We have

$$\begin{aligned}\|v_t^n(m)\|_1&=m(1)-\frac{b_n}{2}\int_0^t\|v_s^n(m)\|_2^2ds\\ &\to 0,\quad \text{as } t\to\infty,\end{aligned}$$
$$\int_0^\infty\|v_t^n(m)\|_2dt=2\frac{m(1)}{b_n}.$$

Let for each $n\geq 1$ $\{S_{n,i},\ i\geq 1\}$ be i. i. d. exponential random variables with parameter $K_n=(2\gamma-1)n^{2\gamma}/\Gamma(2-\gamma)$, and $\{V_{n,i},\ i\geq 1\}$ be i. i. d. random variables, globally independent from the $S_{n,i}$'s, whose common density is given as $c(\gamma,n)x^{-(1+2\gamma)}\mathbf{1}_{\{x\geq 1/n\}}$. We let

$$\begin{aligned}T_{n,i}&=\sum_{k=1}^i S_{n,k},\quad i\geq 1;\\ A_t^n&=\langle\Phi,1\rangle+\sum_{i\geq 1}V_{n,i}\mathbf{1}_{[T_{n,i},+\infty)}(t);\\ \gamma_n(t)&=\inf\{s;\ \frac{1}{2}\int_0^s\|v_r^n(\Phi)\|_2^2dr>t\},\ 0\leq t\leq T_{n,1}.\end{aligned}$$

We then define

$$\begin{aligned}Y_t^n&=v_t^n(\Phi),\ 0\leq t<\gamma_n(T_{n,1});\\ Y_{\gamma_n(T_{n,1})}^n&=Y_{\gamma_n(T_{n,1})^-}^n+V_{n,1}\delta_{U_{n,1}},\ \ \text{if } \gamma_n(T_{n,1})<\infty;\end{aligned}$$

where

$$\mathbb{P}(U_{n,1} \in \cdot | Z_{n,1}, T_{n,1}) = G(Y^n_{\gamma_n(T_{n,1})^-}, \cdot),$$
$$G(f, A) = \left(\int_{\mathbb{R}} f^2(x) dx \right)^{-1} \int_A f^2(x) dx.$$

More generally, for $k \geq 1$, let

$$\gamma_n(t) = \inf\{s;\ \gamma_n(T_{n,k}) + \frac{1}{2} \int_0^{s-\gamma_n(T_{n,k})} \|v^n_r(Y^n_{\gamma_n(T_{n,k})}\|^2_2 dr > t\},$$
$$T_{n,k} \leq t \leq T_{n,k+1};$$
$$Y^n_t = v^n_{t-\gamma_n(T_{n,k})}(Y^n_{\gamma_n(T_{n,k})}), \quad T_{n,k} \leq t < T_{n,k+1};$$
$$Y^n_{\gamma_n(T_{n,k+1})} = Y^n_{\gamma_n(T_{n,k+1})^-} + V_{n,k+1}\delta_{U_{n,k+1}},$$

where

$$\mathbb{P}(U_{n,k+1} \in \cdot | \mathcal{F}^n_{k+1}) = G(Y^n_{\gamma_n(T_{n,k+1})^-}, \cdot).$$

Clearly, we have that for all $t \geq 0$,

$$\gamma_n(t) = \inf\{s;\ \int_0^s \|Y^n_{r^-}\|^2_2 dr > t\}.$$

Let $T^*_n = \inf\{t;\ \gamma(t) = \infty\}$. Then

$$Y^n_{\gamma_n(T^*_n)}(1) = 0, \quad \text{while } Y^n_{\gamma_n(t)}(1) = A^n_t - b_n t,$$

hence

$$T^*_n = \int \{t;\ A^n_t - b_n t = 0\}, \quad \text{and } \mathbb{P}(T^*_n < \infty) = 1.$$

We finally let $\tilde{\gamma}_n(t) = \gamma_n(\log n) \wedge t$, and we finally conclude that

$$\mathbb{E}\left[\exp\left(-\langle u(t,\cdot), \Phi\rangle\right)\right] = \lim_{n\to\infty} \mathbb{E}\left[\exp\left(-\langle u(0,\cdot), Y^n_{\tilde{\gamma}_n(t)}\rangle\right)\right].$$

□

Remark 3.6.12. *Mueller and Perkins [17] have proved that the compact support property is still true if $1/2 < \gamma < 1$. Note that that same property holds also in the case where $\gamma < 1/2$, see Shiga [25], while no uniqueness result is known to hold in that case.*

3.7 SPDEs with singular drift, and reflected SPDEs

If we consider an SPDE of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dot{W}, \\ u(0,x) = u_0(x) \geq 0, \quad u(t,0) = u(t,1) = 0, \end{cases}$$

clearly the solution is not going to remain non negative, due to the additive white noise. One way to try to keep the solution positive is to add a drift which blows up, as $u \to 0$, namely to consider, on the set $\{(t,x),\ 0 \leq x \leq 1,\ 0 \leq t \leq \tau_x\}$, where $\tau_x = \inf\{t \geq 0,\ u(t,x) = 0\}$, the SPDE

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{c}{u^\alpha} + \dot{W}, \\ u(0,x) = u_0(x), \quad u(t,0) = u(t,1) = 0. \end{cases}$$

It has been shown, see [15], [16] that the solution of such an equation remains strictly positive if $\alpha > 3$, and has positive probability of hitting 0 if $\alpha < 3$. The case $\alpha = 3$ is the most interesting, since the solution might touch zero at isolated points, but one can define the solution for all time.

In the case $\alpha \leq 3$, one can think of reflecting the solution at 0, i. e. trying the SPDE

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{c}{u^\alpha} + \eta + \dot{W}, \\ u(0,x) = u_0(x) \geq 0, \quad u(t,0) = u(t,1) = 0, \\ u \geq 0,\ \eta \geq 0,\ \displaystyle\int_0^\infty \int_0^1 u(t,x)\eta(dt,dx) = 0. \end{cases}$$

In the next subsection, we shall consider that problem with $c = 0$, following Nualart, P. [20]. Then we shall consider the case $\alpha = 3$, $c > 0$ folllowing Zambotti [29] and [30], see also [3], i. e. we shall consider the SPDE (in that case, it turns out that no reflection term η is necessary) :

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{c}{u^3} + \dot{W}, \\ u(0,x) = u_0(x), \quad u(t,0) = u(t,1) = 0. \end{cases}$$

Moreover, the reflected SPDE whithout singular drift appears as the limit as $c \to 0$ of that SPDE with a critical singular drift.

3.7.1 Reflected SPDE

In this subsection, we want first to study the following SPDE with additive white noise and reflection

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \eta + \dot{W}, \\ u(0,x) = u_0(x), \quad u(t,0) = u(t,1) = 0, \\ u \geq 0, \ \eta \geq 0, \ \displaystyle\int_0^\infty \int_0^1 u(t,x)\eta(dt,dx) = 0, \end{cases} \tag{3.13}$$

where $u_0 \in C_0([0,1];\mathbb{R}_+)$. Whithout the measure η, the sign of the solution would oscillate randomly. The measure η is there in order to prevent the solution u from crossing 0, by "pushing" the solution upwards. The last condition says the pushing is minimal, in the sense that the support of η is included in the set where u is zero. We formulate a precise

Definition 3.7.1. *A pair (u,η) is said to be a solution of equation* (3.13) *whenever the following conditions are met:*

1. *$\{u(t,x),\ t \geq 0,\ 0 \leq x \leq 1\}$ is a non negative continuous and adapted process, such that $u(t,0) = u(t,1) = 0$, $\forall t \geq 0$.*
2. *$\eta(dt,dx)$ is an adapted random measure on $\mathbb{R}_+ \times [0,1]$.*
3. *For any $t > 0$, any $\varphi \in C_C^\infty([0,1])$, we have*

$$\begin{aligned}(u(t),\varphi) = (u_0,\varphi) + \int_0^t (u(s),\varphi'')ds + \int_0^t \int_0^1 \varphi(x) W(ds,dx) \\ + \int_0^t \int_0^1 \varphi(x)\eta(ds,dx).\end{aligned}$$

We have the

Theorem 3.7.2. *If $u_0 \in C_0([0,1];\mathbb{R}_+)$, equation* (3.13) *has a unique solution.*

Proof: Step 1 We first reformulate the problem. Let v denote the solution of the heat equation with additive white noise, but whithout the reflection, i. e. v solves

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial x^2} + \dot{W}, \\ v(0,x) = u_0(x), \quad v(t,0) = v(t,1) = 0. \end{cases}$$

Defining $z = u - v$, we see that the pair (u, η) solves equation (3.13) if and only if z solves

$$\begin{cases} \quad \dfrac{\partial z}{\partial t} = \dfrac{\partial^2 z}{\partial x^2} + \eta, \\ z(0,x) = 0, \quad z(t,0) = z(t,1) = 0, \\ \qquad z \geq -v, \ \eta \geq 0, \ \displaystyle\int_0^\infty \int_0^1 (z+v)(t,x)\eta(dt,dx) = 0, \end{cases} \tag{3.14}$$

This is an obstacle problem, which can be solved path by path.

Step 2 We construct a solution by means of the penalization method. For each $\varepsilon > 0$ let z_ε solve the penalized PDE

$$\begin{cases} \quad \dfrac{\partial z_\varepsilon}{\partial t} = \dfrac{\partial^2 z_\varepsilon}{\partial x^2} + \dfrac{1}{\varepsilon}(z_\varepsilon + v)^-, \\ z(0,x) = 0, \quad z(t,0) = z(t,1) = 0. \end{cases}$$

It is easily seen that this equation has a unique solution in $L^2_{\text{loc}}(\mathbb{R}_+; H^2(0,1)) \cap C(\mathbb{R}_+ \times [0,1])$. Moreover, it is easily seen that z_ε increases, when ε decreases to 0. If z_ε and $\hat{z}_\varepsilon$ are solution to the same equation, corresponding to v and $\hat{v}$ respectively, it is easy to show that

$$\sup_{0\leq t\leq T, 0\leq x\leq 1} |z_\varepsilon(t,x) - \hat{z}_\varepsilon(t,x)| \leq \sup_{0\leq t\leq T, 0\leq x\leq 1} |v(t,x) - \hat{v}(t,x)| \tag{3.15}$$

Let us show that

$$w = z_\varepsilon - \hat{z}_\varepsilon - \|v - \hat{v}\|_{\infty,T} \leq 0,$$

the other inequality being proved analogously. w solves

$$\begin{cases} \quad \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{1}{\varepsilon}\left[(z_\varepsilon + v)^- - (\hat{z}_\varepsilon + \hat{v})^-\right], \\ w(0,x) = -k, \quad w(t,0) = w(t,1) = -k, \end{cases}$$

where $k = \|v - \hat{v}\|_{\infty,T}$. If w reaches 0, it means that $z_\varepsilon \geq \hat{z}_\varepsilon + k$, hence $z_\varepsilon + v \geq \hat{z}_\varepsilon + \hat{v}$ and $(z_\varepsilon + v)^- \leq (\hat{z}_\varepsilon + \hat{v})^-$. In that case, the drift in the equation pushes w downwards, i. e. w remains negative between $t = 0$ and $t = T$. This intuitive argument can be justified by standard methods.

Step 3 We let $z = \lim_{\varepsilon\to 0} z_\varepsilon$. We want to prove that z is continuous. If we replace v by a smooth obstacle v_n, then the difference between z_ε and $z_{n,\varepsilon}$ is dominated by $\|v - v_n\|_{\infty,T}$, and in the limit as $\varepsilon \to 0$,

$$\|z - z_n\|_{\infty,T} \leq \|v - v_n\|_{\infty,T}.$$

But it is known that when the obstacle v_n is smooth, z_n is continuous. Consequently z is the uniform limit of continuous functions, hence it is continuous.

Define

$$\eta_\varepsilon(dt,dx) = \varepsilon^{-1}(z_\varepsilon + v)^-(t,x)dtdx.$$

For any smooth function ψ of (t,x) which is zero whenever $x=0$ or $x=1$,

$$\langle \eta_\varepsilon, \psi\rangle = \int_0^\infty (z_\varepsilon, \frac{\partial \psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2})dt,$$

hence $\eta_\varepsilon \to \eta$ in the sense of distributions, as $\varepsilon \to 0$. The limit distribution is non negative, hence it is a measure, which satisfies

$$\langle \eta, \psi\rangle = \int_0^\infty (z, \frac{\partial \psi}{\partial t} + \frac{\partial^2\psi}{\partial x^2})dt.$$

Now the support of η_ε is included in the set $\{z_\varepsilon + v \le 0\}$ which decreases as $\varepsilon \to 0$. Hence the support of η is included in $\{z_\varepsilon + v \le 0\}$ for all $\varepsilon > 0$. Consequently for all $T>0$,

$$\int_0^T \int_0^1 (z_\varepsilon + v)d\eta \le 0.$$

The same is true with z_ε replaced by z by monotone convergence. Hence

$$\int_0^T \int_0^1 (z + v)d\eta = 0.$$

STEP 5 If the solution would be in $L^2_{\text{loc}}(\mathbb{R}_+; H^1(0,1))$, then the uniqueness proof would follow a very standard argument, since if (z,η) and $(\overline{z},\overline{\eta})$ are two solutions,

$$\int_0^T \int_0^1 (z - \overline{z})d(\eta - \overline{\eta}) \le 0.$$

Since the above regularity does not hold, one needs to implement a delicate regularization procedure, which we will not present here. □

3.7.2 SPDE with critical singular drift

Now, consider the SPDE with singular drift (for reasons which will become apparent below, we choose to write $c>0$ as $c=(\delta-1)(\delta-3)/8$, $\delta>3$).

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{(\delta-1)(\delta-3)}{8u^3} + \dot{W}, \\ u(0,x) = u_0(x), \quad u(t,0) = u(t,1) = 0, \end{cases}$$

where $\delta > 3$. It can be shown that the solution of this SPDE converges to the above reflected SPDE, as $\delta \to 3$. L. Zambotti has shown that the solution to those equations are ergodic, and computed explicitly their invariant measure (including in the case $\delta = 3$), with respect to which the process is reversible. It is the law of the δ Bessel bridge, i. e. that of the δ Bessel process, conditionned to be at 0 at time 1. The δ Bessel process is the solution of the one dimensional SDE

$$dX_\delta(t) = \frac{\delta - 1}{2X_\delta(t)}dt + dW(t), \;\; X_\delta(0) = 0.$$

In the case where δ is an integer, it is the law of the norm of the δ–dimensional Brownian motion. In particular, the invariant measure of the solution of the reflected SPDE studied in the previous subsection is the law of the norm of 3–dimensional Brownian motion, conditionned to be 0 at time $t = 1$.

Moreover, Dalang, Mueller and Zambotti [3] have given precise indications concerning the set of points where the solution hits zero. This set is decreasing in δ. For $\delta = 3$, with positive probability there exists three points of the form $(t, x_1), (t, x_2), (t, x_3)$ where u is zero, and the probability that there exists 5 points of the same form where u hits zero is zero. For $4 < \delta \leq 5$, there exists one such point with positive probability, and two such points with zero probability. For $\delta > 6$, the probability that there exists one point where u hits zero is zero.

Bibliography

[1] V. Bally, E. Pardoux, Malliavin calculus for white noise driven parabolic SPDEs, *Potential Analysis* **9**, 27–64, 1998.

[2] L. Bertini, G. Giacomin, Stochastic Burgers and ZPZ equations for particle systems, *Comm. Math. Phys.* **183**, 571–607, 1997.

[3] R. Dalang, C. Mueller, L. Zambotti, Hitting properties of parabolic spde's with reflection, *Ann. Probab.* **34**, 1423–1450, 2006.

[4] G. Da Prato, Zabczyk, *Stochastic equations in infinite dimension*, Cambridge Univ. Press 1995.

[5] C. Donati–Martin, E. Pardoux, White noise driven SPDEs with reflection, *Prob Theory and Rel. Fields* **95**, 1–24, 1997.

[6] A. Etheridge, Introduction to superprocesses, AMS 2000.

[7] S. N. Ethier, T. G. Kurtz, *Markov processes, characterization and convergence*, John Wiley, 1986.

[8] T. Funaki, S. Olla, Fluctuation for the $\nabla\phi$ interface model on a wall, *Stoch. Proc. and Applic.* **94**, 1–27, 2001.

[9] I. Gyöngy, E. Pardoux, Weak and strong solutions of white–noise driven parabolic SPDEs, LATP Prépublication 92–22.

[10] N. V. Krylov, B. L. Rozovskii, Stochastic partial differential equations and diffusion processes, *Russian Math. Surveys* **37**, 81–105, 1982.

[11] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires et applications*, Dunod, 1969.

[12] P. L. Lions, T. Souganidis, Notes aux CRAS, t. 326, Ser. I, pp. 1085–1092, 1998; t. 327, Ser. I, pp. 735–741; t. 331, Ser. I, pp. 617–624, 2000; t. 331, Ser. I, p. 783–790, 2000.

[13] J. Mattingly, E. Pardoux, Malliavin calculus for the stochastic $2D$ Navier–Stokes equation, *Comm. in Pure and Appl. Math* **59**, 1742–1790, 2006.

[14] M. Métivier, *Semimartingales*, de Gruyter 1982.

[15] C. Mueller, Long–time existence for for signed solutions of the heat equation with a noise term, *Probab. Theory Related Fields* **110**, 51–68, 1998.

[16] C. Mueller, E. Pardoux, The critical exponent for a stochastic PDE to hit zero, *Stochastic analysis, control, optimization and applications*, 325–338, Birkhäuser 1999.

[17] C. Mueller, E. Perkins, The compact support property for solutions to the heat equation with noise, *Probab. Theory and Rel. Fields* **93**, 325–358, 1992.

[18] L. Mytnik, Weak uniqueness for the heat equation with noise, *Ann. Probab.* **26**, 968–984, 1998.

[19] D. Nualart, *The Malliavin calculus and related topics*, Springer 1995.

[20] D. Nualart, E. Pardoux, White noise driven quasilinear SPDEs with reflection, *Probab. Theory and Rel. Fields* **93**, 77–89, 1992.

[21] E. Pardoux, Equations aux dérivées partielles stochastiques monotones, Thèse, Univ. Paris–Sud, 1975.

[22] E. Pardoux, Filtrage nonlinéaire et équations aux dérivées partielles stochastiques associées, in *Ecole d'été de Probabilités de Saint Flour XIX*, Lecture Notes in Math. **1464**, 67–163, Springer 1991.

[23] E. Pardoux, T. Zhang, Absolute continuity of the law of the solution of a parabolic SPDE, *J. of Funct. Anal.* **112**, 447–458, 1993.

[24] B. L. Rozovskii, *Evolution stochastic systems*, D. Reidel 1990.

[25] T. Shiga, Two contrasting properties of solutions of one–dimensional stochastic partial differential equations, 1990.

[26] M. Viot, Thèse, Univ. Paris 6, 1976.

[27] J. Walsh, An introduction to stochastic partial differential equations, *Ecole d'été de Probabilités de Saint Flour XIV*, Lecture Notes in Math. **1180**, 265–439, Springer 1986.

[28] B. Z. Zangeneh, Semilinear stochastic evolution equations with monotone nonlinearities, *Stochastics* **53**, 129–174, 1995.

[29] L. Zambotti, Integration by parts on convex sets of paths and applications to SPDEs with reflection, *Probab. Theory and Rel. Fields* **123**, 579–600, 2002.

[30] L. Zambotti, Integration by parts on δ–Bessel bridges, $\delta > 3$, and related SPDEs, *Ann. Probab.* **31**, 323–348, 2003.