Genealogy of catalytic branching models

Anita Winter, Erlangen

(with Andreas Greven & Lea Popovic)

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Motivation

Branching Models - encoding the evolution of a population:

$->$ **population size**

$->$ **family forest** (genealogy/ancestral relationships)

Scaling Limits
- many individuals, each of small mass, rapid branching

Invariance Principles - general classes of models with same limit:

$->$ population - **continuous state branching processes (CSBP)**

$->$ genealogy - **continuum random trees (CRTs)**

Example: Independent Branching

critical Galton-Watson processes whose **offspring distribution** ...

- ... has **finite variance**:
  - population $\Rightarrow$ **Feller diffusion** Feller '51
  - genealogy $\Rightarrow$ **Aldous's CRT** Aldous '93
- ... is in the domain of attraction of an α**-stable distribution**:
  - population $\Rightarrow$ α**-stable CSBP** Lamperti '67
  - genealogy $\Rightarrow$ α**-stable CRT** Duquesne-LeGall '02

Our model: catalytic branching

$->$ the **catalyst** $(\eta_t)_{t\geq 0}$ is a critical binary GW with constant branching rate $=1$

$->$ the **reactant** $(\xi_t)_{t\geq 0}$ is a critical binary GW with time-inhomogeneous branching rate **depending on** the current population size of the **catalyst**.

... and the scaling limit

In the n^{th}-rescaling step $(\tilde\eta^n, \tilde\xi^n)$ is a continuous time MC with $(\tilde\eta^n_0, \tilde\xi^n_0) = (1, 1)$ and

$$(\tilde\eta^n, \tilde\xi^n) \mapsto \begin{cases} (\tilde\eta^n \pm \frac{1}{n}, \tilde\xi^n), & \text{at rate } \frac{1}{2} n^2 \tilde\eta^n, \\ (\tilde\eta^n, \tilde\xi^n \pm \frac{1}{n}), & \text{at rate } \frac{1}{2} n^2 \tilde\eta^n \tilde\xi^n. \end{cases}$$

Fact. Greven, Klenke & Wakolbinger '99

$\left(\tilde\eta^n, \tilde\xi^n\right) \Rightarrow (X, Y)$ where

$$\mathrm{d}X_t = \sqrt{2X_t}\, \mathrm{d}W_t^X$$
$$\mathrm{d}Y_t = \sqrt{2X_tY_t}\, \mathrm{d}W_t^Y$$

A remark on extinction times

$$\mathrm{d}X_t = \sqrt{2X_t}\,\mathrm{d}W_t^X$$
$$\mathrm{d}Y_t = \sqrt{2X_tY_t}\,\mathrm{d}W_t^Y$$

and

$$\rho^0 := \inf\{t \geq 0 : X_t = 0\}, \qquad \tau^0 := \inf\{t \geq 0 : Y_t = 0\}.$$

Basic Facts.

- $\rho^0 < \infty$, almost surely.
- The reactant gets absorbed in Y_{ρ^0} at time ρ^0.
- Penssel '03

$$\mathbb{P}\{\rho^0 < \tau^0\} = \frac{1}{\sqrt{5}} \in (0,1).$$

Problems we would like to answer ...

- from a **quenched view** (given the **catalyst** mass process):
  - establish **existence** of a limit reactant genealogy
  - **analytic characterizations** of the limit
- from an **annealed view:**
  - **joint convergence** of catalyst and reactant genealogies
  - **differences** in catalyst forest from Aldous's CRF

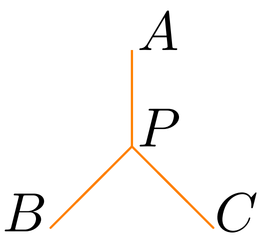
Existence of limit reactant genealogies

Rooted real trees

Dress (1984), Dress & Terhalle (1996)

A **rooted real tree** (T, r, ρ) is a

- **metric space** (T, r),
- **path-connected**,
- **0-hyperbolic**, i.e. $\forall\, A, B, C \in T \;\; \exists\, P \in [A, B]$ with $P \in [B, C] \cap [A, C]$



- with the **root** $\rho \in T$ distinguished.

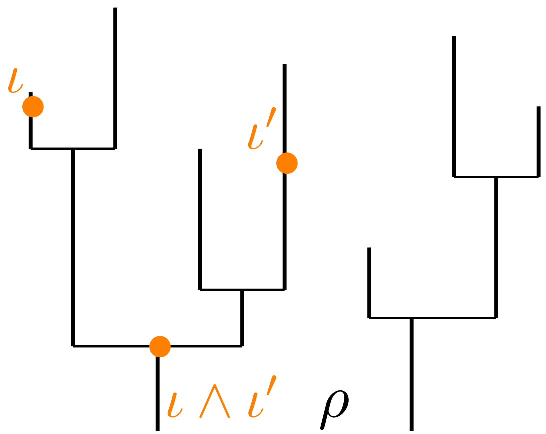
$\mathbb{T} :=$ (root invariant) isometry classes of **compact** rooted $\mathbb{R}$-trees

Coding finite forest as rooted $\mathbb{R}$-trees

$$T := \bigcup_{t\in[0,\mathrm{height}]} \text{all individuals alive time } t$$

$$d(\iota,\iota') := \text{generation of}(\iota) + \text{generation of}(\iota') - 2\cdot\text{generation of}(\iota'\wedge\iota)$$

$$\rho := \text{glueing together all the trees' ancestors}$$



Rule. At the height where the catalyst dies **cut off** the reactant forest

Hausdorff distance

Let (X, d) be a metric space.

Hausdorff distance:

For $A_1, A_2 \subseteq_{\text{closed}} X$,

$$d_{\text{H}}(A_1, A_2) := \inf\{\varepsilon > 0 : \ A_1 \subseteq A_2^{\varepsilon} \textbf{ and } A_2 \subseteq A_1^{\varepsilon}\},$$

where A^{ε} is the ε-neighborhood of A.

Gromov strong topology

$$(\mathbf{T_N}, \mathbf{d_N}, \rho_\mathbf{N}) \underset{\mathbf{N}\to\infty}{\longrightarrow} (\mathbf{T}, \mathbf{r}, \rho)$$

iff $\{(T_N, d_N);\ N \in \mathbb{N}\}$ and (T, d) can be embedded via isometries $\{\varphi_N;\ N \in \mathbb{N}\}$ and φ, resp., in one and the same compact metric space $(\mathbf{Z}, \mathbf{d_Z})$ on which

$$\varphi_\mathbf{N}(\mathbf{T_N}) \underset{\mathbf{N}\to\infty}{\longrightarrow} \varphi(\mathbf{T}), \qquad \varphi_\mathbf{N}(\rho_\mathbf{N}) \underset{\mathbf{N}\to\infty}{\longrightarrow} \varphi(\rho)$$

in the **Hausdorff topology on** $(\mathbf{Z}, \mathbf{d})$.

Existence of reactant limit forest

- let $(\tilde{\eta}^{\mathrm{for},n}, \tilde{\xi}^{\mathrm{for},n})$ be the pairs of $\mathbb{R}$-trees corresponding to the catalyst and reactant population in the n^{th}-approximation step
-

$(\cdot;\cdot) :=$ conditional law of reactant **given** a realization of the catalyst

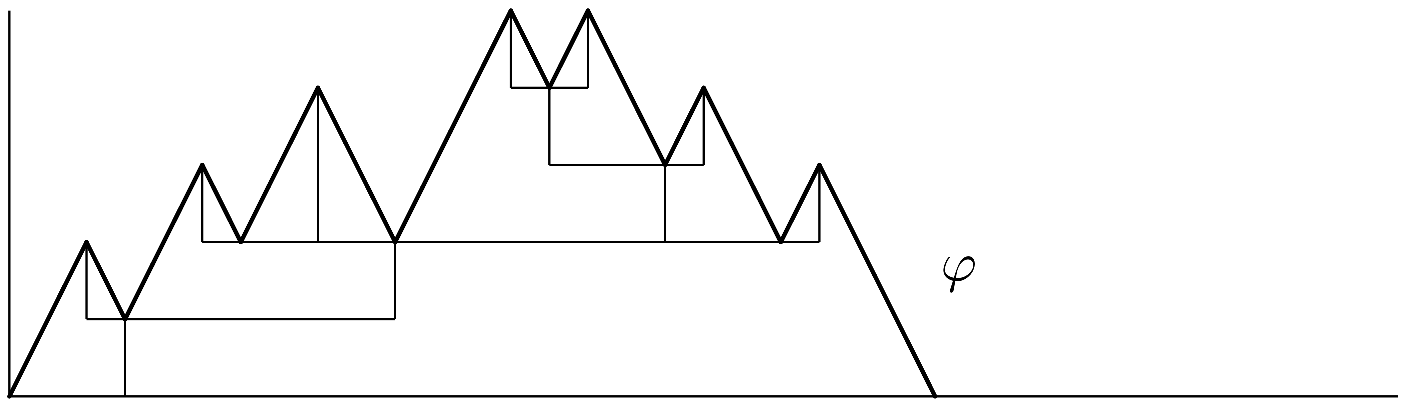
Proposition. Assume that $\sup_{t \le T} \left| \tilde{\eta}_t^n - X_t \right| \to 0$ a.s. Then the family $\{(\tilde{\xi}^{\mathrm{for},n}; \tilde{\eta}^n);\, n \in \mathbb{N}\}$ is relatively compact.

Characterization via excursions

Main example: Tree "below" an excursion

$\varphi \in C([0, L(\varphi)])\,,\ L(\varphi) \in (0, \infty)\,,\ \varphi\big|_{\{0, L(\varphi)\}} \equiv 0\,,\ \varphi\big|_{(0, L(\varphi))} > 0$

For $u, u' \in [0, L(\varphi)]$ put $\mathbf{u} \equiv_\varphi \mathbf{u}'$ iff **no minima** in between them.



$$\mathbf{T}_\varphi := \{[u]; u \in [0, L(\varphi)]\}, \quad \rho_\varphi := [0],$$

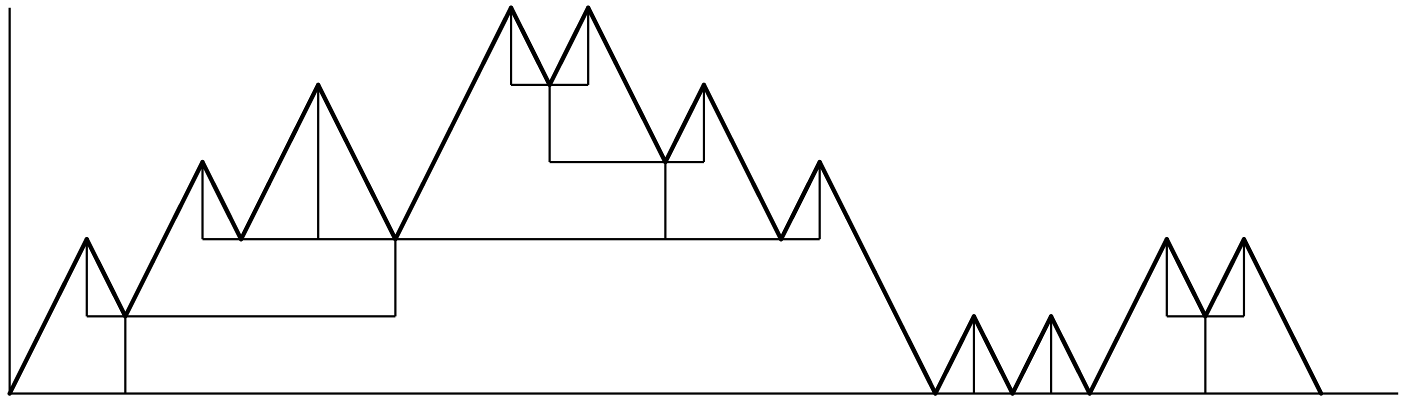
$$\mathbf{d}_\varphi([u], [u']) := \varphi(u) + \varphi(u') - 2 \inf_{s \in [u \wedge u', u \vee u']} \varphi(s).$$

Fact. $T\big|_\varphi$ is **compact** real tree.

Example. Aldous's CRT $=$ tree "below" Brownian excursion

From trees to forests

several trees in a forest = several aligned excursions



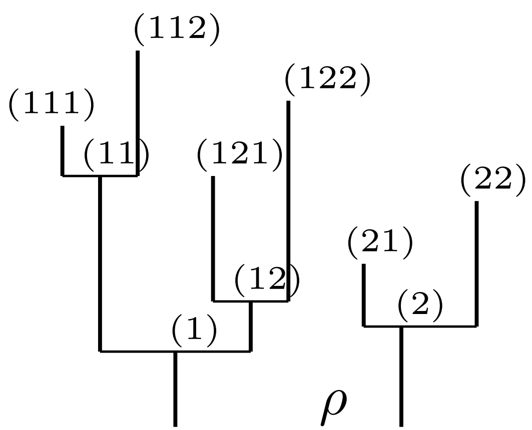
$\#$ **trees** $= \#$ **excursions** $= \#$ **of zeros** $-1.$

Planar embedding and linear order

Planar embedding requires a **linear ordering** which

- extends the partial order, and
- if x, y, x', y' are s.t. $x \leq^{\text{lin}} y$ and $x \wedge y \leq x' \wedge x$ and $x \wedge y \leq y' \wedge y$ then $x' \leq^{\text{lin}} y'$.

Example.



$+$ **lexicographic order**

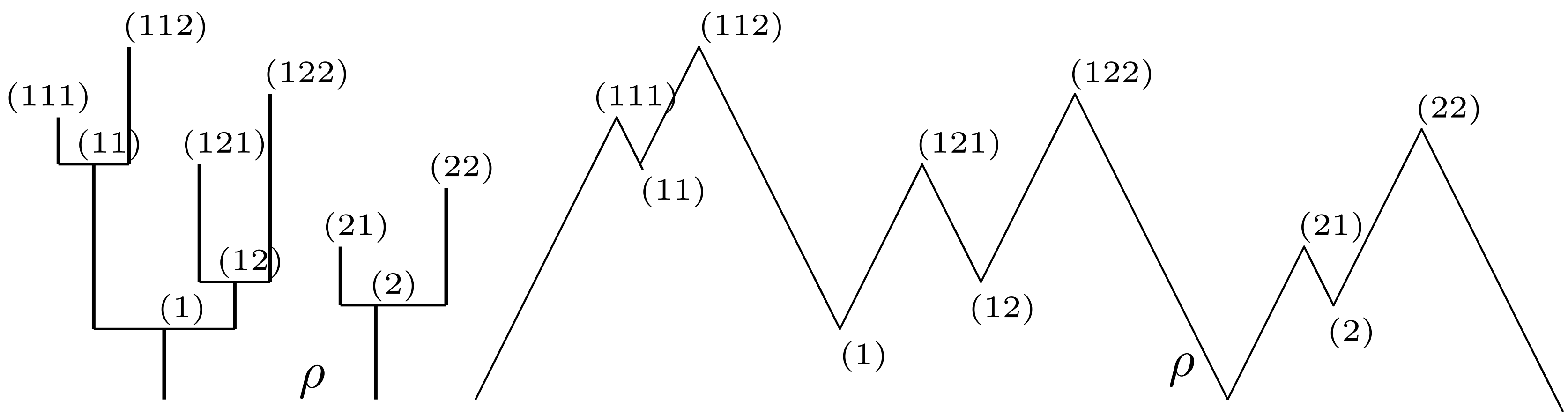
e.g., $x = (111)$, $y = (122)$, $x' = (112)$ and $y' = (12)$

The contour process of finite trees

$(\mathbf{T}, \rho)$: finite (linearly ordered) forest

σ : speed of traversal

$\mathcal{C}(T, \sigma)$: records height of the depth-first search around the forest



Lemma. In continuous time binary GW with **given** branching rate, $\mathcal{C}(T, \sigma)$ is a linear interpolation of an alternating random walk.

Re-scaling the contour

- Speeding up branching by n = rescaling **edges** by a factor $\mathbf{\frac{1}{n}}$,
- a GW-trees has height of order $\mathcal{O}(n)$ with probability $\mathcal{O}(\frac{1}{n})$

$\Rightarrow$ if we start initially with n-trees yields a **Poisson number** of trees each having height $\mathcal{O}(n)$

- given a GW-tree has height of order $\mathcal{O}(n)$ its **number of edges** is of order $\mathcal{O}(\mathbf{n^2})$

Since edges are of length $\mathcal{O}(\frac{1}{n})$, we choose in the n^{th}-approximation step

$$\sigma := \mathbf{n}.$$

The catalyst contour

Obvious. [Brownian rescaling]

If $|\beta|$ is **reflected Brownian motion**, $\ell(|\beta|)$ its **local time** process, i.e.,

$$\ell(|\beta|)_t := \lim_{\varepsilon \to 0} \varepsilon^{-1} \int_0^t \mathrm{d}u \mathbf{1}\{|\beta|_u \in [0, \varepsilon]\},$$

and $\ell(|\beta|)^{-1}$ its **inverse**, i.e.,

$$\ell(|\beta|)^{-1}(s) := \inf \left\{ t \geq 0 : \ell(|\beta|)_t = s \right\},$$

then

$$\mathcal{C}(\tilde{\eta}^{\mathrm{for},\mathbf{n}}; \mathbf{n}) \underset{\mathbf{n} \to \infty}{\Longrightarrow} \mathbf{2} \cdot |\beta|_{\cdot \wedge \ell_{\mathbf{0}}^{-\mathbf{1}}(\mathbf{1})}.$$

Random evolution and stochastic averaging (Kurtz '92)

$\delta > 0$,

$$\tilde{T}^{\delta,n} := \inf\big\{t \geq 0 : \tilde{\eta}^n_t \leq \delta\big\}.$$

Lemma.

Let $V_u := \frac{1}{2}\text{slope}(C_u)$ then $(C_u, V_u)_{u\geq 0}$ is a random evolution with

$$\tilde{\mathbf{A}}^{\delta,\mathbf{n}}\mathbf{f}(\mathbf{c},\mathbf{v}) = \mathbf{n}\mathbf{v}\frac{\partial}{\partial\mathbf{c}}\mathbf{f}(\mathbf{c},\mathbf{v}) + \mathbf{n}^{\mathbf{2}}\tilde{\eta}^{\mathbf{n}}_{\mathbf{c}}\big[\mathbf{f}(\mathbf{c},-\mathbf{v}) - \mathbf{f}(\mathbf{c},\mathbf{v})\big]$$

and domain

$$\mathcal{D}(\tilde{\mathbf{A}}^{\delta,\mathbf{n}}) = \big\{\mathbf{f} \in \mathbf{C}^{\mathbf{1},\mathbf{0}}(\mathbf{0},\tilde{\mathbf{T}}^{\delta,\mathbf{n}} \times \{-\mathbf{1},\mathbf{1}\}) : \partial_{\mathbf{c}}\mathbf{f}\big|_{\{\mathbf{0},\tilde{\mathbf{T}}^{\delta,\mathbf{n}}\}\times\{\mathbf{0},\mathbf{1}\}} \equiv \mathbf{0}\big\}.$$

Limiting contour process

For $\delta > 0$, let $\tilde{\xi}^{\mathrm{for},\delta,\mathbf{n}} :=$ reactant tree **cut off** in height $\tilde{T}^{\delta,n}$ and

$$A^{\delta} f(c) := 2\Big(\frac{1}{X_c} f'\Big)'(c)$$

on the domain

$$\mathcal{D}(A^{\delta}) := \Big\{ f \in \mathcal{C}^2([0,\tau^{\delta}]) : \frac{1}{X_{\cdot}} f' \in \mathcal{C}^1([0,\tau^{\delta}]), f'\big|_{\{0,\tau^{\delta}\}} = 0 \Big\}.$$

Theorem The $(A^{\delta}, \mathcal{D}(A^{\delta}))$-martingale problem is well-posed. If ζ^{δ} is the solution of the $(A^{\delta}, \mathcal{D}(A^{\delta}))$-martingale problem, then

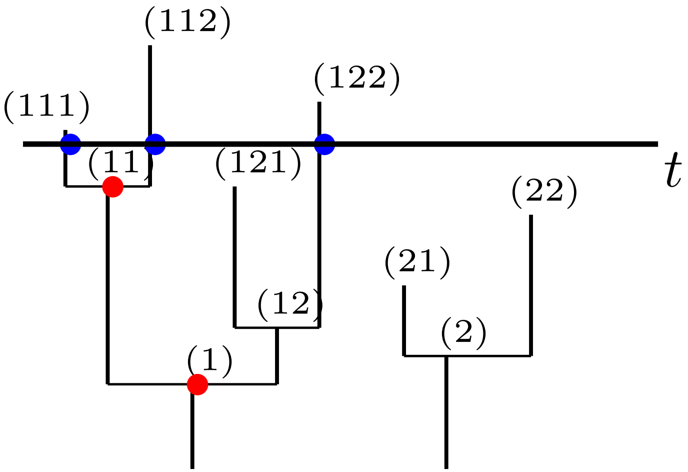
$$\Big(\mathcal{C}(\tilde{\xi}^{\mathrm{for},\delta,n}); \tilde{\eta}^{n}\Big) \underset{n\to\infty}{\Longrightarrow} \Big(\zeta^{\delta}; X\Big).$$

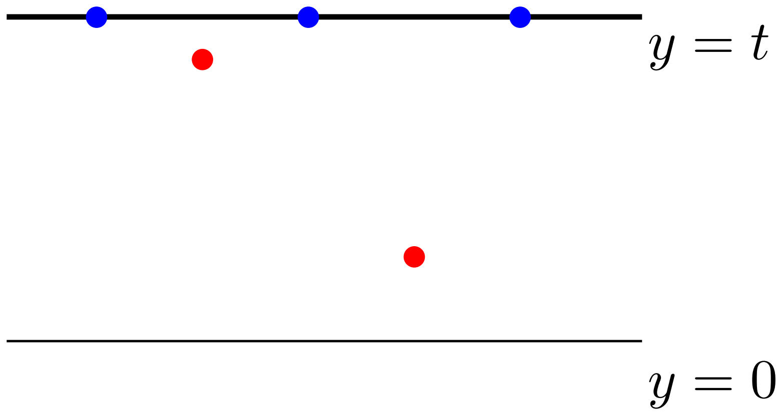
.... but what if $\delta \downarrow 0$?

Yet another useful representation

Genealogical point process (Popovic '04)

$\mathcal{P}(\mathbf{T},\zeta) = \{\pi_{\mathbf{t}}\}_{\mathbf{t}\geq\mathbf{0}}$: collection of processes indexed by t; each giving the ancestry of the population alive at time t

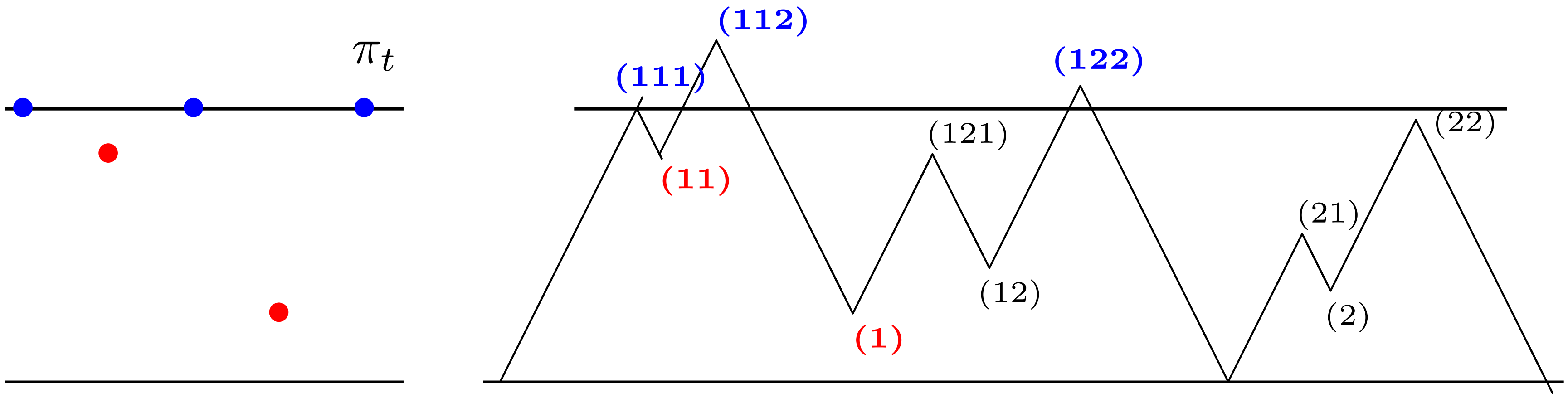




ζ := spacing

Distribution of the point process

π_t are points of maximal depths of i.i.d. excursions below t



Lemma. For all $t > 0$, given a medium η, π_t is a simple point-process, $\{\tau_i\}_{i=0,\ldots,(\xi_t-1)}$ are i.i.d. with

$$\mathbf{P}(\tau_\mathbf{i} \in \mathrm{d}\mathbf{h}) = \frac{\eta_\mathbf{h}}{(\mathbf{1} + \int_\mathbf{h}^\mathbf{t} \eta_\mathbf{s} \mathrm{d}\mathbf{s})^\mathbf{2}} \frac{\mathbf{1} + \int_\mathbf{0}^\mathbf{t} \eta_\mathbf{s} \mathrm{d}\mathbf{s}}{\int_\mathbf{0}^\mathbf{t} \eta_\mathbf{s} \mathrm{d}\mathbf{s}}, \quad \mathbf{h} \in (\mathbf{0}, \mathbf{t})$$

Rescaling the point process

Since after the rescaling of edge lengths we find at any height $t > 0$ a **Poisson number** of trees **each** having family size of order $\mathcal{O}(\mathbf{n})$, we choose in the n^{th}-approximation step

$$\zeta := \frac{\mathbf{1}}{\mathbf{n}}.$$

Limit of the reactant genealogical point-process

Theorem. For any $t > 0$,

$$\big(\mathcal{P}^{\mathbf{t}}(\tilde{\xi}^{\mathrm{for},\mathbf{n}};\frac{\mathbf{1}}{\mathbf{n}});\tilde{\eta}^{\mathbf{n}}\big) \Rightarrow \big(\pi^{\zeta,\mathbf{t}};\mathbf{X}\big),$$

where for given path of X the point process $\pi^{\zeta,t}$ consists of:

- Poisson point process on $\mathbb{R}^+ \times \mathbb{R}^+$ whose intensity measure is

$$\aleph^{\zeta,\mathbf{t}}\big[\mathrm{d}\ell \times \mathrm{d}\mathbf{h}\big] = \mathbf{1}_{[\mathbf{0},\mathbf{Y_t}]}(\ell)\mathrm{d}\ell \otimes \mathbf{1}_{(\mathbf{0},\mathbf{t}\wedge\tau^{\mathbf{0}}\wedge\rho^{\mathbf{0}})}(\mathbf{h})\frac{\mathbf{X_h}}{(\int_{\mathbf{h}}^{\mathbf{t}}\mathbf{X_s}\mathrm{d}\mathbf{s})^{\mathbf{2}}}\mathrm{d}\mathbf{h}.$$

- rate $\big(\int_{\mathbf{0}}^{\mathbf{t}}\mathbf{X_s}\mathrm{ds}\big)^{-\mathbf{1}}$ Poisson point process at height 0 whose points separate distinct trees in the forest.

Remark. Result is δ-free!!!!

Properties of the Reactant Forest

Comparison with the constant rate branching forest:

$->$ differences in tree structure due to inhomogeneity of the random environment (evolving branching rates)

- stretching of the tree metric

$->$ behavior of the forest Y^{for} at time τ^0 of extinction of the random environment (in the event the catalyst dies first)

- infinite ℓ^2-length of the tree tips!

Differences in the reactant forest due to inhomogeneous rates

Stretching Lemma.

Let $(Z^{\mathrm{for}}, d_{Z^{\mathrm{for}}}, \rho)$ be the Brownian CRF and Y^{for} the reactant forest, and let $X : [0, \tau^0) \to \mathbb{R}_+$ be a given continuous function. For a fixed $t > 0$ define an increasing function $s_t^X : [0, t] \to [0, \int_0^t X_s \mathrm{d}s]$ by

$$s_t^X(h) := \int_{t-h}^{t} X_s \mathrm{d}s,$$

and let $(s_t^X)^{-1} : [0, \int_0^t X_s \mathrm{d}s] \to [0, t]$ be its inverse. Then

$$\Big(\big(\partial Q_t(Y^{\mathrm{for}}), d_{Y^{\mathrm{for}}}, \rho\big); X\Big) \stackrel{d}{=} \big(\partial Q_{s_t^X(t)}(Z^{\mathrm{for}}), 2(s_t^X)^{-1}(\frac{1}{2} d_{Z^{\mathrm{for}}}), \rho\big)$$

Application of the stretching lemma

Theorem. [Comparing the probability to belong to different families]

Let Y^{for} be the reactant CRF and Z^{for} be Brownian CRF with the **same expected number of trees of a given height $t > 0$**. If $\mu^{t,Y}$ and $\mu^{t,Z}$ are the "uniform" distributions on $\partial Q_t(Y^{\text{for}})$ and $\partial Q_t(Z^{\text{for}})$, respectively, then

$$\mathbb{E}\Big[\int_{(\partial Q_t(Y^{\text{for}}))^2} \big(\mu^{t,Y}\big)^{\otimes 2}(\mathrm{d}u, \mathrm{d}u')\, \mathbf{1}\big\{d_{Y^{\text{for}}}\big(u, u'\big) = 2t\big\}\Big]$$
$$\leq \mathbb{E}\Big[\int_{(\partial Q_t(Z^{\text{for}}))^2} \big(\mu^{t,Z}\big)^{\otimes 2}(\mathrm{d}u, \mathrm{d}u')\, \mathbf{1}\big\{d_{Z^{\text{for}}}\big(u, u'\big) = 2t\big\}\Big];$$

Differences due to vanishing catalyst

Case. $\rho^0 < \tau^0$, i.e., reactant dies first

Proposition.

$$\lim_{\delta \downarrow \mathbf{0}} \langle \zeta^\delta, \zeta^\delta \rangle_{\ell^{-1}(\zeta^\delta)(\mathbf{1})} = \infty, \qquad \mathbb{P}\text{-a.s.}$$

Note. With positive probability there does **not exist** a limiting diffusion ζ^0 describing the contour of the full forest Y^{for} including its highest tips.

Many thanks

Compact sets

Lemma. Evans, Pitman, W. '06

A set $\mathcal{T}$ is pre-compact in $\mathbb{T}^{\text{root}}$ iff for all $\varepsilon > 0$ there is $n(\varepsilon) < \infty$ such that each $T \in \mathcal{T}$ has an ε-net with at most $n(\varepsilon)$ points.

Example.

$$A^{t}_{t-\varepsilon}(T,\rho) := \text{ ancestors of the time } t \text{ population a time } \varepsilon \text{ back}$$

- ε-net. $\varepsilon > 0$, $(T,\rho) \in \mathbb{T}^{\text{root}}$, $m_0 = \lfloor -\log_2(\varepsilon) \rfloor$

$$N(\varepsilon) := \bigcup_{k \in \mathbb{N}} A^{k \cdot 2^{-(m_0+1)}}_{(k-1)\cdot 2^{-(m_0+1)}}(T,\rho)$$

- Compact set. $\{L_m;\, m \in \mathbb{N}\}$ positive integers

$$\mathcal{T} := \bigcap_{m \in \mathbb{N}} \Big\{(T,\rho) \in \mathbb{T}^{\text{root}} : \sum_{k \geq 1} \# A^{k \cdot 2^{-(m+1)}}_{(k-1)\cdot 2^{-(m+1)}}(T,\rho) \leq L_m \Big\}.$$