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Probability Theory and Related Fields

White noise driven quasilinear SPDEs with reflection

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Summary. We study reflected solutions of the heat equation on the spatial interval $[0, 1]$ with Dirichlet boundary conditions, driven by an additive space–time white noise. Roughly speaking, at any point (x, t) where the solution $u(x, t)$ is strictly positive it obeys the equation, and at a point (x, t) where $u(x, t)$ is zero we add a force in order to prevent it from becoming negative. This can be viewed as an extension both of one-dimensional SDEs reflected at 0, and of deterministic variational inequalities. An existence and uniqueness result is proved, which relies heavily on new results for a deterministic variational inequality.

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0 Introduction

The aim of this paper is to study the existence of a pair (u, η) where u is a continuous function of $(x, t) \in Q \triangleq [0, 1] \times \mathbb{R}_+$, η is a measure over Q, which satisfy:

(i) $\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} + f(u) = \dfrac{\partial^2 W}{\partial x \partial t} + \eta$

(ii) $u(x, 0) = u_0(x)$, $u(0, t) = u(1, t) = 0$, $u(x, t) \geqq 0$

(iii) $\int_Q u d\eta = 0$

where $\{W(x, t), (x, t) \in Q\}$ is a Brownian sheet. Condition (iii) implies that the support of η is included in $\{u = 0\}$. (i) says in particular that wherever $u(x, t) > 0$, u solves the white noise driven parabolic SPDE:

$$\frac{\partial u}{\partial t}(x, t) - \frac{\partial^2 u}{\partial x^2}(x, t) + f(x, t; u(x, t)) = \frac{\partial^2 W}{\partial x \partial t}(x, t) .$$

η is there to "push u upward", so that it remains nonnegative, and (iii) says that the pushing is minimal in the sense that no pushing occurs where $u(x, t) > 0$, since u cannot become negative there.

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We shall show that the stochastic problem (i), (ii), (iii) is equivalent by translation to a deterministic problem with reflection along an irregular boundary function. Such deterministic problems with reflection are called "inequations" and have been widely studied by several authors, see in particular Bensoussan and Lions [1], the bibliography therein and Mignot and Puel [6].

Most of the present paper is devoted to the proof of an existence and uniqueness result for such an inequation. The point is that our boundary function is not smooth enough such that we might apply the usual theory of strong solutions, and we want both existence and uniqueness, which is not provided by the theory of weak solutions (see in particular [6]).

Note that a similar problem has already been considered in Haussman and Pardoux [3], in the case of a different type of driving noise and with a non constant diffusion coefficient, using quite different methods.

The paper is organized as follows. In Sect. 1 we state the results and in Sect. 2 we prove them.

1 Statement of the problem and of the main results

Our aim is to study an equation of the following type

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + f(x, t; u(x, t)) = \frac{\partial^2 W}{\partial x \partial t} + \eta , \tag{1}$$

$$t \geqq 0,\ x \in [0, 1] ,$$

with the Dirichlet boundary conditions

$$u(0, t) = u(1, t) = 0, \quad t \geqq 0 , \tag{2}$$

and the initial condition

$$u(x, 0) = u_0(x), \quad x \in [0, 1] . \tag{3}$$

We shall assume in what follows that the function f takes the following form:

$$f = [0, 1] \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$$

(A.0) $f(x, t, z) = f_1(x, t) + f_2(x, t; z) + f_3(x, t; z)$
with the properties that f_i is jointly measurable, $1 \leqq i \leqq 3$ and
(A.1) $f_1 \in \bigcap_{T>0} L^2((0, 1) \times (0, T)), f_2(x, t; 0) \equiv f_3(x, t; 0) \equiv 0;$
(A.2) $\exists c$ such that $|f_2(x, t; z) - f_2(x, t; r)| \leqq c|z - r|$, for all (x, t) in $[0, 1] \times \mathbb{R}_+$, r, z in $\mathbb{R}$;
(A.3) for any $(x, t) \in [0, 1] \times \mathbb{R}_+$, $z \to f_3(x, t; z)$ is continuous and nondecreasing, and f_3 is locally bounded.

Given $u \in C([0, 1] \times \mathbb{R}_+)$, we shall consider the mapping $(x, t) \to f(x, t; u(x, t))$. We shall often below write in short $f(u_t)$ for the mapping $x \to f(x, t; u(x, t))$.

The initial condition $u_0(x)$ will be a continuous and nonnegative function which satisfies the Dirichlet boundary conditions on $[0, 1]$. Set $Q = [0, 1] \times \mathbb{R}_+$. We will assume that $W = \{W(x, t), (x, t) \in Q\}$ is a two-parameter Wiener process defined on a complete probability space $(\Omega, \mathscr{F}, P)$. That means, W is a continuous Gaussian process with zero mean and covariance function defined by

$$E[W(x,t)W(x',t')] = (x \wedge x')(t \wedge t') .$$

We will denote by $\mathscr{F}_t$ the σ-field generated by the random variables $\{W(x,s), x \in [0,1], s \in [0,t]\}$.

The operator $-(d^2/dx^2)$ on $L^2(0,1)$, with the Dirichlet boundary conditions will be denoted by A. We will denote by $C_0([0,1])$ the set of continuous functions φ on $[0,1]$ such that $\varphi(0) = \varphi(1) = 0$, and by $C_k^\infty(D)$, D open subset of $\mathbb{R}^l$, the set of functions from D into $\mathbb{R}$ which are infinitely differentiable and whose support is a compact subset of $\mathbb{R}^l$.

The solution to the Eq. (1) will be a pair (u, η) such that $u = \{u(x,t), (x,t) \in Q\}$ is a nonnegative and continuous stochastic process which satisfies the equality (1) in a weak sense, and $\eta(\mathrm{d}x, \mathrm{d}t)$ is a random measure on Q which forces the process u to be nonnegative.

If the term η is omitted then the Eq. (1) becomes a particular case of the parabolic stochastic differential equations studied, among others, by Walsh [7], Manthey [5] and Buckdahn and Pardoux [2].

The Eq. (1) is formal and we have to give a rigorous meaning to the notion of solution. This is the purpose of the following definition (we denote here and in the sequel by $(\cdot,\cdot)$ the scalar product in $L^2(0,1)$).

Definition 1.1 A pair (u, η) is said to be a solution of Eq. (1) if:

(i) $u = \{u(x,t), (x,t) \in Q\}$ is a nonnegative, continuous and adapted process (i.e., $u(x,t)$ is $\mathscr{F}_t$-measurable $\forall t \geqq 0$, $x \in [0,1]$) with $u(0,t) = u(1,t) = 0$, $t \geqq 0$, a.s.

(ii) $\eta(\mathrm{d}x, \mathrm{d}t)$ is a random measure on $(0,1) \times \mathbb{R}_+$ such that $\eta((\varepsilon, 1-\varepsilon) \times [0,T]) < \infty$ for all $\varepsilon > 0$ and $T > 0$, and η is adapted (i.e., $\eta(B)$ is $\mathscr{F}_t$-measurable if $B \subset (0,1) \times [0,t]$).

(iii) For all $t \geqq 0$ and $\phi \in C_k^\infty((0,1))$ we have

$$(u_t, \phi) + \int_0^t (u_s, A\phi)\mathrm{d}s + \int_0^t (f(u_s), \phi)\mathrm{d}s = (u_0, \phi) + \int_0^t \int_0^1 \phi(x)\mathrm{d}W_{x,s} \tag{4}$$
$$+ \int_0^t \int_0^1 \phi(x)\eta(\mathrm{d}x, \mathrm{d}s), \quad \text{a.s.}$$

(iv) $\int_Q u \mathrm{d}\eta = 0$.

Remark 1.2 Notice that the condition (iv) is equivalent to saying that the support of the measure η is contained in the set $\{u = 0\}$.

All terms appearing in the Eq. (4) are continuous functions of $\phi \in C_k^\infty((0,1))$ with respect to the topology of the uniform convergence of ϕ, ϕ' and ϕ'' on compact subsets of $(0,1)$. For the stochastic integral with respect to the two-parameter Wiener process this continuity follows from the integration by parts formula:

$$\int_0^t \int_0^1 \phi \mathrm{d}W = -\int_0^1 W(x,t)\phi'(x)\mathrm{d}x .$$

The space $C_k^\infty((0,1))$ is separable for this topology. Consequently in property (iii) the almost sure requirement is uniform in $t \geqq 0$ and $\phi \in C_k^\infty((0,1))$. □

The main result of this paper is the following.

Theorem 1.3 *Suppose that f satisfies* (A.0), (A.1), (A.2), (A.3) *and let* $u_0 \in C_0([0,1])$ *be a non-negative function. Then there exists a unique solution* (u, η) *of Eq.* (1). *Furthermore this solution verifies*

$$\eta((0,1)\times\{t\}) = 0 \quad \text{for all } t \geqq 0, \quad \text{and} \tag{5}$$

$$\int_0^T \int_0^1 x(1-x)\eta(dx, dt) < \infty, \quad \text{for all } T > 0. \tag{6}$$

Before giving the proof of this theorem we need to introduce some notations. We denote by $G_t(x, y)$ the fundamental solution of the heat equation with Dirichlet boundary condition. That means, for any $\varphi \in C_0([0,1])$,

$$g(x,t) = \int_0^1 G_t(x,y)\varphi(y)dy$$

is the unique solution of

$$\begin{cases} \dfrac{\partial g}{\partial t} - \dfrac{\partial^2 g}{\partial x^2} = 0, & t > 0, \qquad 0 < x < 1 \\ g(x,0) = \varphi(x), & 0 \leqq x \leqq 1 \\ g(0,t) = g(1,t) = 0, & t \geqq 0. \end{cases}$$

Define the Gaussian random field

$$v(x,t) = \int_0^t \int_0^1 G_{t-s}(x,y)dW_{sy} + \int_0^1 G_t(x,y)u_0(y)dy. \tag{7}$$

It is shown in Walsh [7] that v has a version which is α-Hölder continuous for any $0 < \alpha < \frac{1}{4}$. Moreover, v satisfies $v(x,0) = u_0(x)$, $v(0,t) = v(1,t) = 0$, and v is a weak solution of the parabolic stochastic differential equation

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 W}{\partial x \partial t}. \tag{8}$$

That means, it holds that

$$(v_t, \phi) + \int_0^t (v_s, A\phi)ds = (u_0, \phi) + \int_0^t \int_0^1 \phi(x)dW_{s,x}, \tag{9}$$

for all $t \geqq 0$ and $\phi \in C_k^\infty((0,1))$, almost surely.

The basic ingredient in the proof of Theorem 1.3 is the following: making use of the change of variable $z = u - v$, Theorem 1.3 is easily seen to be a consequence of the next deterministic result:

Theorem 1.4 *Let* v *be a continuous function on* Q *such that* $v(x,0) = u_0(x)$ *and* $v(0,1) = v(1,t) = 0$, *for all* $x \in [0,1]$, $t \geqq 0$, *and suppose that f satisfies* (A.0), (A.1), (A.2), (A.3). *Then there exists a unique pair* (z, η) *such that*:

(i) *z is a continuous function on Q verifying* $z(x,0) = 0$,

$$z(0,t) = z(1,t) = 0 \quad \text{and} \quad z \geqq -v.$$

(ii) *η is a measure on* $(0, 1) \times \mathbb{R}_+$ *such that*

$$\eta((\varepsilon, 1-\varepsilon) \times [0, T]) < \infty \quad \text{for all } \varepsilon > 0 \quad \text{and} \quad T > 0$$

(iii)

$$(z_t, \phi) + \int_0^t (z_s, A\phi)\mathrm{d}s + \int_0^t (f(z_s + v_s), \phi)\mathrm{d}s = \int_0^t \int_0^1 \phi(x)\eta(\mathrm{d}x, \mathrm{d}s) \tag{10}$$

for all $t \geqq 0$, $\phi \in C_k^\infty((0, 1))$,

(iv) $\int_Q (z(x, t) + v(x, t))\eta(\mathrm{d}x, \mathrm{d}t) = 0$.

Furthermore the measure η *verifies the properties* (5) *and* (6).

Remark 1.5 Formally, (i)–(iv) can be rewritten as follows:

$$\left.\begin{aligned} \frac{\partial z_t}{\partial t} + Az_t + f(z_t + v_t) &\geqq 0 \\ z_t &\geqq -v_t \\ \left(\frac{\partial z_t}{\partial t} + Az_t + f(z_t + v_t), z_t + v_t\right) &= 0 \,. \end{aligned}\right\}$$

This is a deterministic parabolic inequation, of a type which has been largely studied in the literature (see e.g. Bensoussan and Lions [1], Mignot and Puel [6] and the bibliographies therein). Note that the non linear term in the above inequality has a very special form. However, if we define:

$$\bar{f}(x, t; r) = f(x, t; r + v_t(x))$$

and write $\bar{f}(z_t)$ for the mapping

$$x \to \bar{f}(x, t; z_t(x))$$

we can rewrite our inequality in a more general form:

$$\left.\begin{aligned} \frac{\partial z_t}{\partial t} + Az_t + \bar{f}(z_t) &\geqq 0 \\ z_t &\geqq -v_t \\ \left(\frac{\partial z_t}{\partial t} + Az_t + \bar{f}(z_t), z_t + v_t\right) &= 0 \,. \end{aligned}\right\} \tag{11}$$

(11) is a parabolic inequality with the non-smooth obstacle $-v(x, t)$.

There are several existence results in the literature concerning this kind of problem. When the obstacle is somewhat smooth, existence and uniqueness of a “strong solution” is known. In the case of a more general obstacle, existence of a minimal “weak solution” is known. Our situation is somehow intermediate between the two situations known in the literature. Our formulation will be “strong enough” so that we shall be able to prove existence and uniqueness, although the classical theory of strong solutions does not apply. The (apparently new) idea is to allow the term which we call η to be a measure.

Remark 1.6 We have not been able to decide whether $\eta((0,1)\times[0,T])$ is finite or infinite. It is clear that $\eta(\{(0,\varepsilon)\cup(1-\varepsilon,1)\}\times[0,T])$ is large (maybe infinite?). Indeed, the solution is forced to be zero on the boundary, and it is irregular in x (see Walsh [7]), therefore a lot of "pushing by η" is necessary near the boundary, in order to prevent the solution from taking negative values. Note that if we replace the zero boundary conditions by strictly positive ones (or if we replace the zero level of reflection by a negative one), then it is easy to see that $\eta((0,1)\times(0,T))<\infty$ a.s., for all $T>0$. □

2 Proof of Theorem 1.4

2.1 Reduction of the problem

In this subsection, we show that it suffices to prove the theorem under the assumptions:
(A.1′) $f(\cdot,\cdot;0)\in\bigcap_{T>0}L^2((0,1)\times(0,T))$;
(A.2′) $f-f(\cdot,\cdot;0)$ is locally bounded;
(A.3′) $z\to f(x,t;z)$ is continuous and nondecreasing, $\forall(x,t)\in[0,1]\times\mathbb{R}_+$.

Indeed, the result is equivalent to the same result with f replaced by $f_\lambda+\lambda I$, where $f_\lambda(x,t;r)=e^{-\lambda t}f(x,t;e^{\lambda t}r)$, and if f satisfies (A.1), (A.2), (A.3), and $\lambda=c$, $f_\lambda+\lambda I$ satisfies (A.1′), (A.2′), (A.3′). In order to see the equivalence, we first note that (iii) is equivalent to the following statement:

$$(1)\qquad -\int_0^\infty\left(\frac{\partial\psi_t}{\partial t},z_t\right)\mathrm{d}t+\int_0^\infty(A\psi_t,z_t)\mathrm{d}t+\int_0^\infty(f(z_t+v_t),\psi_t)\mathrm{d}t$$
$$=\int_Q\psi\,\mathrm{d}\eta,\ \forall\psi\in C_K^\infty((0,1)\times\mathbb{R}_+)\,,$$

which for any $\lambda\in\mathbb{R}$ is equivalent to

$$(1_\lambda)\quad -\int_0^\infty\left(\frac{\partial\psi_t}{\partial t},e^{-\lambda t}z_t\right)\mathrm{d}t+\int_0^\infty(A\psi_t,e^{-\lambda t}z_t)\,\mathrm{d}t+\int_0^\infty(e^{-\lambda t}[f(z_t+v_t)+\lambda z_t],\psi_t)\mathrm{d}t$$
$$=\int_Q\psi e^{-\lambda t}\mathrm{d}\eta,\ \forall\psi\in C_K^\infty((0,1)\times\mathbb{R}_+)\,.$$

Hence (z,η) solves (1) iff

$$\bar z_t=e^{-\lambda t}z_t,\ \bar\eta=e^{-\lambda t}.\eta$$

solves

$$(2)\qquad -\int_0^\infty\left(\frac{\partial\psi_t}{\partial t},\bar z_t\right)\mathrm{d}t+\int_0^\infty(A\psi_t,\bar z_t)\mathrm{d}t+\int_0^\infty(f_\lambda(\bar z_t+\bar v_t)+\lambda\bar z_t,\psi)\mathrm{d}t$$
$$=\int_Q\psi\,\mathrm{d}\bar\eta,\ \forall\psi\in C_K^\infty((0,1)\times\mathbb{R}_+)\,,$$

where $\bar v_t=e^{-\lambda t}v_t$.

2.2 Existence of a solution

Step 1. We shall construct a solution by means of the well-known penalization method. Fix $\varepsilon > 0$ and denote by z^ε the solution of the equation

$$\left.\begin{aligned} \frac{\partial z^\varepsilon}{\partial t} + Az_t^\varepsilon + f(z_t^\varepsilon + v_t) &= \frac{1}{\varepsilon}(z_t^\varepsilon + v_t)^- \\ z^\varepsilon(x, 0) &= 0 \\ z^\varepsilon(0, t) = z^\varepsilon(1, t) &= 0 \,. \end{aligned}\right\} \tag{12}$$

This equation has a unique solution $z^\varepsilon \in \bigcap_{T>0} L^2(0, T; H^2(0, 1)) \cap C(Q)$, see e.g. Lions [4]. Using the monotonicity property of f one can show the following facts:
(A) $z^\varepsilon(x, t)$ increases as ε decreases to zero.
(B) Let z^ε and $\hat{z}^\varepsilon$ be the solutions corresponding to two different functions v and $\hat{v}$. Then we have for any $T > 0$,

$$\| z^\varepsilon - \hat{z}^\varepsilon \|_\infty^T \leqq \| v - \hat{v} \|_\infty^T \,, \tag{13}$$

where

$$\| \varphi \|_\infty^T = \sup_{\substack{t \in [0, T] \\ x \in [0, 1]}} |\varphi(x, t)|, \; \varphi \in C(Q) \,.$$

Proof of (A) Define $F_1(x, t; z) = f(x, t; z) - (1/\varepsilon_1)z^-$ and $F_2(x, t; z) = f(x, t; z) - (1/\varepsilon_2)z^-$ where $\varepsilon_1 < \varepsilon_2$. Then F_1 and F_2 are nondecreasing functions such that $F_1 < F_2$. Set $\psi = z^{\varepsilon_2} - z^{\varepsilon_1}$. We want to show that $\psi \leqq 0$. We have

$$\left.\begin{aligned} \frac{\partial \psi}{\partial t} + A\psi_t + F_2(z_t^{\varepsilon_2} + v_t) - F_1(z_t^{\varepsilon_1} + v_t) &= 0 \\ \psi_0 &= 0 \,. \end{aligned}\right\}$$

Multiplying this equality by ψ_t^+ we obtain, for all $T > 0$,

$$\int_0^T \left(\frac{\partial \psi_t}{\partial t}, \psi_t^+ \right) \mathrm{d}t + \int_0^T (A\psi_t, \psi_t^+) \mathrm{d}t + \int_0^T (F_2(z_t^{\varepsilon_2} + v_t) - F_1(z_t^{\varepsilon_1} + v_t), \psi_t^+) \mathrm{d}t = 0 \,.$$

It follows from Lemma 6.1, p. 132 in Bensoussan and Lions [1] that $\psi^+ \in L^2(0, T; H^1(0, 1))$ and ($|\cdot|$ denotes the norm in $L^2(0, 1)$):

$$\int_0^T \left(\frac{\partial \psi_t}{\partial t}, \psi_t^+ \right) \mathrm{d}t = \frac{1}{2} |\psi_T^+|^2$$

$$\int_0^T (A\psi_t, \psi_t^+) \mathrm{d}t = \int_0^T \left| \frac{\partial \psi_t^+}{\partial x} \right|^2 \mathrm{d}t \,.$$

Moreover

$$(F_2(z_t^{\varepsilon_2} + v_t) - F_1(z_t^{\varepsilon_1} + v_t), \psi_t^+) = (F_2(z_t^{\varepsilon_2} + v_t) - F_2(z_t^{\varepsilon_1} + v_t), \psi_t^+) \\ + (F_2(z_t^{\varepsilon_1} + v_t) - F_1(z_t^{\varepsilon_1} + v_t), \psi_t^+) \geqq 0 .$$

Hence $|\psi_t^+|^2 = 0$, $t \geqq 0$.

Proof of (*B*). Set $k = \|v - \hat{v}\|_\infty^T$, and $F_\varepsilon(x, t; z) = f(x, t; z) - (1/\varepsilon)z^-$. We have

$$\left.\begin{aligned} &\frac{\partial z_t^\varepsilon}{\partial t} + Az_t^\varepsilon + F_\varepsilon(z_t^\varepsilon + v_t) = 0 \\ &\frac{\partial \hat{z}^\varepsilon}{\partial t} + A\hat{z}_t^\varepsilon + F_\varepsilon(\hat{z}_t^\varepsilon + \hat{v}_t) = 0 \\ &\hat{z}_0^\varepsilon = z_0^\varepsilon = 0 . \end{aligned}\right\}$$

Therefore

$$\frac{\partial}{\partial t}(z_t^\varepsilon - \hat{z}_t^\varepsilon) + A(z_t^\varepsilon - \hat{z}_t^\varepsilon) + F_\varepsilon(z_t^\varepsilon + v_t) - F_\varepsilon(\hat{z}_t^\varepsilon + \hat{v}_t) = 0 .$$

Define $w = (z^\varepsilon - \hat{z}^\varepsilon) - k$. Multiplying the above equation by w^+ yields

$$\int_0^T \left(\frac{\partial w_t}{\partial t}, w_t^+\right) dt + \int_0^T (Aw_t, w_t^+) dt + \\ + \int_0^T (F_\varepsilon(z_t^\varepsilon + v_t) - F_\varepsilon(\hat{z}_t^\varepsilon + \hat{v}_t), w_t^+) dt = 0 .$$

It holds that

$$(F_\varepsilon(z_t^\varepsilon + v_t) - F_\varepsilon(\hat{z}_t^\varepsilon + \hat{v}_t), w_t^+) \geqq 0$$

because on $\{(x, t);\ w^+(x, t) \neq 0\}$, $z^\varepsilon > \hat{z}^\varepsilon + k$ and then $z^\varepsilon + v \geqq \hat{z}^\varepsilon + \hat{v}$. Consequently the same computations as made in the proof of (A) yield $w^+ = 0$, hence $z^\varepsilon - \hat{z}^\varepsilon \leqq k$. By symmetry $\hat{z}^\varepsilon - z^\varepsilon \leqq k$.

Step 2. For any $(x, t) \in [0, 1] \times \mathbb{R}_+$, define

$$z(x, t) = \sup_{\varepsilon > 0} z^\varepsilon(x, t) .$$

We want to show that $z \in C([0, 1] \times \mathbb{R}_+)$. Let $\{v_n, n \in \mathbb{N}\} \subset C_k^\infty((0, 1) \times (\mathbb{R}_+ \setminus \{0\}))$ satisfy $v_n(x, t) \to v(x, t)$ uniformly on compact subsets of $[0, 1] \times \mathbb{R}_+$.

Let z_n^ε denote the solution of (12) where v has been replaced by v_n. From (B) above,

$$\|z^\varepsilon - z_n^\varepsilon\|_\infty^T \leqq \|v - v_n\|_\infty^T .$$

But for each fixed n, $z_n^\varepsilon \uparrow z_n$ as $\varepsilon \downarrow 0$, where z_n is the strong solution of an inequation with smooth obstacle, hence – see e.g. Corollary 2.3, p. 237 of Bensoussan and Lions [1] – $z_n \in C([0, 1]) \times \mathbb{R}_+)$.

Letting $\varepsilon \downarrow 0$ in the above inequality yields:

$$\|z - z_n\|_\infty^T \leqq \|v - v_n\|_\infty^T, \quad T > 0 .$$

The desired result follows, by letting $n \to \infty$.

Step 3. We have to show that the function $z(x, t)$ satisfies the conditions (i)–(iv) of Theorem 1.4. Clearly $z(x, 0) = 0$ and $z(0, t) = z(1, t) = 0$ for all $t \geqq 0$.

Let $\psi \in C_k^\infty((0, 1) \times \mathbb{R}_+)$. From (12) we obtain

$$\left.\begin{aligned} &-\int_0^\infty \left(z_t^\varepsilon, \frac{\partial \psi_t}{\partial t}\right) dt + \int_0^\infty (z_t^\varepsilon, A\psi_t) dt + \\ &+ \int_0^\infty (f(z_t^\varepsilon + v_t), \psi_t) dt = \frac{1}{\varepsilon} \int_0^\infty ((z_t^\varepsilon + v_t)^-, \psi_t) dt . \end{aligned}\right\} \tag{14}$$

We denote by η_ε the measure $(1/\varepsilon)(z_t^\varepsilon + v_t)^- dx dt$ on Q. From the equality (14) we deduce that η_ε converges in the distributional sense to some distribution on the open set $(0, 1) \times (0, +\infty)$. The limiting distribution is nonnegative and, therefore, it is a measure that we denote by η. For any $\psi \in C_k^\infty((0, 1) \times [0, +\infty))$ we have

$$\left.\begin{aligned} &-\int_0^\infty \left(\frac{\partial \psi_t}{\partial t}, z_t\right) dt + \int_0^\infty (A\psi_t, z_t) dt + \int_0^\infty (f(z_t + v_t), \psi_t) dt \\ &= \int_Q \psi d\eta . \end{aligned}\right\} \tag{15}$$

Actually the above convergence holds for any infinitely differentiable function ψ on $(0, 1) \times \mathbb{R}$ with compact support included in $(0, 1) \times [0, +\infty)$. So η is a distribution on $(0, 1) \times \mathbb{R}$, and hence a measure on $(0, 1) \times [0, +\infty)$. Moreover it is clear from (15) that $\eta((\varepsilon, 1 - \varepsilon) \times [0, T]) < \infty$ for all $\varepsilon > 0$ and $T > 0$.

Multiplying the Eq. (14) by ε and letting ε tend to zero, we obtain $\int_0^\infty ((z_t + v_t)^-, \psi_t) dt = 0$ for any $\psi \in C_k^\infty((0, 1) \times \mathbb{R}_+)$. This implies $z_t + v_t \geqq 0$ a.e., and $z + v$ being a continuous function we obtain $z_t(x) \geqq v_t(x)$, $(x, t) \in [0, 1] \times \mathbb{R}_+$.

It only remains to check condition (iv). For each $\varepsilon > 0$ the support of η_ε is included in the set $\{z^\varepsilon + v \leqq 0\}$, which decreases when ε decreases. Hence the support of η is included in $\{z^\varepsilon + v \leqq 0\}$ for any $\varepsilon > 0$.

Therefore $\int_{(0,1) \times [0,T]} (z^\varepsilon + v) d\eta \leqq 0$. By the monotone convergence theorem $\int_{(0,1) \times [0,T]} (z + v) d\eta \leqq 0$. Hence $\int_{(0,1) \times [0,T]} (z + v) d\eta = 0$ for all $T > 0$. This implies (iv) because the measure η is concentrated on $(0, 1) \times \mathbb{R}_+$, by definition.

Step 4. Proof of (5) Let $t_0 > 0$. For any $0 < \delta < t_0$ we define the function

$$\psi_\delta(t) = \begin{cases} \frac{1}{\delta}(t - t_0 + \delta) & \text{if} \quad t_0 - \delta \leqq t \leqq t_0 \\ \frac{1}{\delta}(t_0 + \delta - t) & \text{if} \quad t_0 \leqq t \leqq t_0 + \delta \\ 0 & \text{otherwise} . \end{cases}$$

Let $\phi \in C_k^\infty((0, 1))$.

Choosing $\psi(x, t) = \psi_\delta(t)\phi(x)$ in (15), we obtain

$$\begin{aligned} &-\delta^{-1} \int_{t_0-\delta}^{t_0} (\phi, z_t) dt + \delta^{-1} \int_{t_0}^{t_0+\delta} (\phi, z_t) dt + \int_{t_0-\delta}^{t_0+\delta} \psi_\delta(t)(A\phi, z_t) dt \\ &+ \int_{t_0-\delta}^{t_0+\delta} \psi_\delta(t)\,(f(z_t + v_t), \phi) dt = \int_{t_0-\delta}^{t_0+\delta} \int_0^1 \psi_\delta(t)\phi(x)\eta(dx, dt) . \end{aligned}$$

Letting δ tend to zero yields $\int_0^1 \phi(x)\eta(\mathrm{d}x \times \{t_0\}) = 0$, which implies $\eta((0,1) \times \{t_0\}) = 0$, for any $t_0 > 0$. The same argument can be used if $t_0 = 0$. In this case one uses the property $z(x, 0) = 0$.

Step 5. Proof of (6) For any $\delta > 0$ we define the following function

$$\phi_\delta(x) = \begin{cases} 0 & \text{if } 0 \leqq x < \delta \text{ or } 1 - \delta < x \leqq 1 \\ \frac{1}{\delta}(x - \delta) & \text{if } \delta \leqq x < 2\delta \\ 1 & \text{if } 2\delta \leqq x \leqq 1 - 2\delta \\ \frac{1}{\delta}(1 - \delta - x) & \text{if } 1 - 2\delta < x \leqq 1 - \delta . \end{cases}$$

Set $\bar{\phi}_\delta(x) = x(1-x)\phi_\delta(x)$.

We can apply the equality (15) to $\psi(x, t) = \bar{\phi}_\delta(x)\, \mathbf{1}_{[0,T]}(t)$ and we obtain

$$(\bar{\phi}_\delta, z_T) + \int_0^T (A\bar{\phi}_\delta, z_t)\mathrm{d}t + \int_0^T (\bar{\phi}_\delta, f(z_t + v_t))\mathrm{d}t = \int_0^T \int_0^1 \bar{\phi}_\delta(x)\eta(\mathrm{d}x, \mathrm{d}t) .$$

It just remains to prove that

$$\sup_{\delta > 0} \left| \int_0^T (A\bar{\phi}_\delta, z_t)\mathrm{d}t \right| < \infty ,$$

and this follows from the equality:

$$\begin{aligned} (A\bar{\phi}_\delta, z_t) = 2(\bar{\phi}_\delta, z_t) &- 2\left(\frac{1}{\delta}\int_\delta^{2\delta} (1-2x)z_t(x)\mathrm{d}x - \frac{1}{\delta}\int_{1-2\delta}^{1-\delta} (1-2x)z_t(x)\mathrm{d}x\right) \\ &- (z_t(\delta)(1-\delta) - z_t(2\delta)2(1-2\delta) - z_t(1-2\delta)2(1-2\delta) \\ &+ z_t(1-\delta)(1-\delta)) . \end{aligned}$$

2.3 *Uniqueness of the solution*

Suppose that (z, η) and $(\bar{z}, \bar{\eta})$ are two solutions. Define $\xi = z - \bar{z}$. Then for any infinitely differentiable function ψ: $(0,1) \times \mathbb{R}_+ \to \mathbb{R}$ whose support is contained in $[\delta, 1-\delta] \times \mathbb{R}_+$ (for some $\delta > 0$), and for all $T > 0$, we have

$$\left.\begin{aligned} (\psi_T, \xi_T) - \int_0^T \left(\frac{\partial \psi_t}{\partial t}, \xi_t\right)\mathrm{d}t + \int_0^T (\xi_t, A\psi_t)\mathrm{d}t + \int_0^T (f(z_t + v_t) - f(\bar{z}_t + v_t), \psi_t)\mathrm{d}t \\ = \int_0^T \int_0^1 \psi(x,t)\eta(\mathrm{d}x, \mathrm{d}t) - \int_0^T \int_0^1 \psi(x,t)\bar{\eta}(\mathrm{d}x, \mathrm{d}t) . \end{aligned}\right\} \quad (16)$$

Fix $\delta > 0$ and let φ: $[0,1] \to \mathbb{R}_+$ be an infinitely differentiable function whose support is contained in $[\delta, 1-\delta]$.

Let ε: $\mathbb{R} \to \mathbb{R}_+$ be an infinitely differentiable function, which is symmetric (i.e., $\varepsilon(x) = \varepsilon(-x)$), its support is contained in $[-1, +1]$, $\int_{-1}^{+1} \varepsilon(x)\mathrm{d}x = 1$ and it is nonnegative definite (i.e., $\sum_{i=1}^n \sum_{j=1}^n \varepsilon(x_i - x_j)y_i y_j \geqq 0$ for any $n \geqq 1$, $\{x_1, \ldots, x_n\} \subset \mathbb{R}$ and $\{y_1, \ldots, y_n\} \subset \mathbb{R}$).

By means of the function $\varepsilon(x)$ we introduce the approximation of the identity $\varepsilon_n(x) = n\varepsilon(nx)$.

Define $\varepsilon_{n,m}(x, t) = \varepsilon_n(t)\varepsilon_m(x)$ and

$$\psi_{n,m} = [(\xi\varphi) * \varepsilon_{n,m}]\varphi \ .$$

That means,

$$\psi_{n,m}(x, t) = \left(\int_{(t-(1/n))^+}^{t+(1/n)} \int_0^1 \xi(y, s)\varphi(y)\varepsilon_n(t - s)\varepsilon_m(x - y)\mathrm{d}y\,\mathrm{d}s \right) \varphi(x) \ , \qquad (17)$$

where we assume $(1/n) < \delta$.

We can choose $\psi = \psi_{n,m}$ in (16). We are going to study the asymptotic behaviour of each term as n and m tend to infinity.

(a)

$$\lim_{n,m\to\infty} (\psi_{n,m}(T), \xi(T)) = \| \xi(T)\varphi \|^2 \qquad (18)$$

(b)

$$\lim_{n,m\to\infty} \int_0^T \left(\frac{\partial \psi_{n,m}(t)}{\partial t}, \xi_t \right) \mathrm{d}t = 0 \ . \qquad (19)$$

In fact, we have

$$\int_0^T \left(\frac{\partial \psi_{n,m}(t)}{\partial t}, \xi_t \right) \mathrm{d}t$$

$$= \int_0^T \int_{(t-(1/n))^+}^{t+(1/n)} \varepsilon_n'(t - s) \left(\int_{[0,1]^2} \xi(y, s)\varphi(y)\varepsilon_m(x - y)\varphi(x)\xi(x, t)\mathrm{d}x\mathrm{d}y \right) \mathrm{d}s\mathrm{d}t \ .$$

The function $\Gamma_m(s, t) = \int_{[0,1]^2} \xi(y, s)\varphi(y)\varepsilon_m(x - y)\varphi(x)\xi(x, t)\mathrm{d}x\mathrm{d}y$ is symmetric (namely, $\Gamma_m(s, t) = \Gamma_m(t, s)$). Therefore, the integral $\int_0^T \int_{(t-(1/n))^+}^{(t+(1/n))\wedge T} \varepsilon_n'(t - s)$ $\Gamma_m(s, t)\mathrm{d}s\mathrm{d}t$ vanishes due to the property $\varepsilon'(s) = -\varepsilon'(-s)$.
Hence

$$\left| \int_0^T \left(\frac{\partial \psi_{n,m}(t)}{\partial t}, \xi_t \right) \mathrm{d}t \right| = \left| \int_{T-(1/n)}^{T} \int_T^{T+(1/n)} \varepsilon_n'(t - s)\Gamma_m(s, t)\mathrm{d}s\mathrm{d}t \right|$$

$$\leqq \frac{C}{n} \ .$$

(c)

$$\lim_{n,m\to\infty} \left(\int_0^T \int_0^1 \psi_{n,m}(x, t)\eta(\mathrm{d}x, \mathrm{d}t) - \int_0^T \int_0^1 \psi_{n,m}(x, t)\bar{\eta}(\mathrm{d}x, \mathrm{d}t) \right)$$

$$= \int_0^T \int_0^1 \xi(x, t)\varphi(x)^2\eta(\mathrm{d}x, \mathrm{d}t) - \int_0^T \int_0^1 \xi(x, t)\varphi(x)^2\bar{\eta}(\mathrm{d}x, \mathrm{d}t)$$

$$= -\int_0^T \int_0^1 \bar{z}(x, t)\varphi(x)^2\eta(\mathrm{d}x, \mathrm{d}t) - \int_0^T \int_0^1 z(x, t)\varphi(x)^2\bar{\eta}(\mathrm{d}x, \mathrm{d}t) \leqq 0 \qquad (20)$$

where the second equality follows from the properties $\int_Q (z+v)\mathrm{d}\eta = 0$ and $\int_Q (\bar{z}+v)\mathrm{d}\bar{\eta} = 0$.

(d)

$$\lim_{n,m\to\infty} \int_0^T (f(z_t+v_t) - f(\bar{z}_t+v_t), \psi_{n,m}(t))\mathrm{d}t$$

$$= \int_0^T (f(z_t+v_t) - f(\bar{z}_t+v_t), (z_t-\bar{z}_t)\varphi^2)\mathrm{d}t \geqq 0 \tag{21}$$

because f is nondecreasing.

(e) It holds that

$$\liminf_{m\to\infty} \lim_{n\to\infty} \int_0^T (A\psi_{n,m}(s), \xi_s)\mathrm{d}s \geqq -\frac{1}{2}\int_0^T\int_0^1 \xi(x,t)^2(\varphi^2)''(x)\mathrm{d}x\,\mathrm{d}t\,. \tag{22}$$

In fact, first notice that

$$\lim_{n\to\infty} \int_0^T (A\psi_{n,m}(s), \xi_s)\mathrm{d}s = \int_0^T (A\psi_m(s), \xi_s)\mathrm{d}s\,,$$

where

$$\psi_m(x,t) = \left(\int_0^1 \xi(y,t)\varphi(y)\varepsilon_m(x-y)\mathrm{d}y\right)\varphi(x)\,.$$

Suppose first that ξ is a smooth function. In this case, integrating by parts and using the fact that ε is nonnegative definite we obtain

$$\begin{aligned}
(A\psi_m(t), \xi_t) &= (A\{[(\xi_t\varphi) * \varepsilon_m]\varphi\}, \xi_t) = (((\xi_t\varphi)' * \varepsilon_m)\varphi, \xi_t') + (((\xi_t\varphi) * \varepsilon_m)\varphi', \xi_t') \\
&= ((\xi_t'\varphi) * \varepsilon_m, \varphi\xi_t') + ((\xi_t\varphi') * \varepsilon_m, \varphi\xi_t') + ((\xi_t\varphi) * \varepsilon_m, \varphi'\xi_t') \\
&\geqq ((\xi_t\varphi') * \varepsilon_m, \varphi\xi_t') + ((\xi_t\varphi) * \varepsilon_m, \varphi'\xi_t') \\
&= ((\xi_t\varphi') * \varepsilon_m, \varphi\xi_t') - ((\xi_t\varphi) * \varepsilon_m, \varphi''\xi_t) \\
&\quad - ((\xi_t'\varphi) * \varepsilon_m, \varphi'\xi_t) - ((\xi_t\varphi') * \varepsilon_m, \varphi'\xi_t) \\
&= -((\xi_t\varphi) * \varepsilon_m, \varphi''\xi_t) - ((\xi_t\varphi') * \varepsilon_m, \varphi'\xi_t)\,.
\end{aligned}$$

Approximating ξ by smooth functions we obtain the inequality

$$(A\psi_m(t), \xi_t) \geqq -((\xi_t\varphi) * \varepsilon_m, \varphi''\xi_t) - ((\xi_t\varphi') * \varepsilon_m, \varphi'\xi_t)\,,$$

and if we let m tend to infinity we get (22).

Consequently, from the relations (a), (b), (c), (d) and (e) we deduce

$$\int_0^1 \xi(x,T)^2\varphi(x)^2\mathrm{d}x \leqq \frac{1}{2}\int_0^T \mathrm{d}t \int_0^1 \xi(x,t)^2(\varphi^2)''(x)\mathrm{d}x\,, \tag{23}$$

for all $T > 0$ and any function $\varphi \in C_k^\infty((0, 1))$. This inequality still holds if φ^2 is a function with compact support such that $(\varphi^2)''$ is a measure. Suppose that

$$\varphi^2(x) = \begin{cases} \frac{1}{\varepsilon}(x-a) & \text{if } a < x \leqq a+\varepsilon \\ 1 & \text{if } a+\varepsilon < x \leqq b \\ \frac{1}{\varepsilon}(b+\varepsilon-x) & \text{if } b < x \leqq b+\varepsilon \\ 0 & \text{if } x \in [a, b+\varepsilon]^c, \end{cases}$$

where $0 < a < a + \varepsilon \leqq b < b + \varepsilon < 1$. From (23) we have

$$\varepsilon \int_0^1 \xi(x, T)^2 \varphi(x)^2 \mathrm{d}x \leqq \int_0^T \mathrm{d}t \{\xi(a, t)^2 - \xi(a+\varepsilon, t)^2 - \xi(b, t)^2 + \xi(b+\varepsilon, t)^2\}$$

Set $\beta(x) = \int_0^T \xi(x, t)^2 \mathrm{d}t$. Then

$$\beta(a) - \beta(a+\varepsilon) - \beta(b) + \beta(b+\varepsilon) \geqq 0,$$

that means,

$$\beta(b+\varepsilon) - \beta(a+\varepsilon) \geqq \beta(b) - \beta(a). \tag{24}$$

By taking $\varepsilon = b - a$ we get

$$\beta(a+2\varepsilon) - \beta(a+\varepsilon) \geqq \beta(a+\varepsilon) - \beta(a).$$

So, $\beta\left(\frac{k+1}{n}\right) - \beta\left(\frac{k}{n}\right) \geqq 0$ for $k = 0, 1, \ldots, n-1$, because $\beta\left(\frac{1}{n}\right) - \beta(0) = \beta\left(\frac{1}{n}\right) \geqq 0$. On the other hand $\beta(1) = 0$. Therefore $\beta\left(\frac{k+1}{n}\right) - \beta\left(\frac{k}{n}\right) = 0$ for all k and n and this implies $\beta \equiv 0$. So $\xi \equiv 0$, and from (16) we deduce $\eta = \bar{\eta}$ which completes the proof of uniqueness.

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