June 3, 2019

POSITIVITY OF THE RATE FONCTION OF THE QUENCHED LDP FOR THE BROWNIAN MOTION IN BOUNDED RANDOM SCENERY.

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In [1], we stated in Theorem 2.3 a quenched Large Deviations Principle for the Brownian motion in random scenery, without checking the positivity of the rate function. We are really sorry for this!! Here is a short note which proves this positivity. The proof follows the same arguments as the ones used in [2]. I use the same notation as in the paper [1]. J is the quenched rate function defined by (21), and $\mathcal{J}_1$ is the first lower bound defined by (57). Setting $d\mu = f^2dx$, one has

$$\mathcal{J}_1(y) = \inf_{f\in L^2(\mathbb{R}^d), \int f^2(x)\,dx=1}\ \inf_{u\in\mathcal{B}_1(\mathbb{R}^d)} \left\{ \frac{1}{2}\int \|\nabla f(x)\|^2\,dx : \int f^2(x)u(x)\,dx = y,\ I(u) < d \right\}.$$

The key argument is the following lemma:

Lemma 1. *There exists $C > 0$ such that for any $y \in \mathbb{R}$,*

$$\mathcal{J}_1(y) \geq C\,|y|^{\frac{4}{d}\vee 1}.$$

Since it has been proven in Lemma 4.4 that $\mathcal{J}_1^{**}(y) = J(y)$, J is the greatest convex minorant of $\mathcal{J}_1$, and it follows from the lemma that there exists $C > 0$ such that for any $y \in \mathbb{R}$,

$$J(y) \geq C\,|y|^{\frac{4}{d}\vee 1}.$$

In order to prove the lemma, it is useful to give another expression of $\mathcal{J}_1$. For any $f \in L^2(\mathbb{R}^d)$ and $u \in \mathcal{B}_1(\mathbb{R}^d)$, for any $\lambda > 0$, let us define

$$f_\lambda(\cdot) = \lambda^{d/2} f(\lambda\cdot)\,,\ \text{and}\ u_\lambda(\cdot) = u(\lambda\cdot)\,,$$

so that $\|f_\lambda\|_2 = \|f\|_2$, $\|\nabla f_\lambda\|_2 = \lambda\,\|\nabla f\|_2$, $\|u_\lambda\|_\infty = \|u\|_\infty$, $I(u_\lambda) = \lambda^{-d}I(u)$, and $\int f_\lambda^2(x)u_\lambda(x)\,dx = \int f^2(x)u(x)\,dx$. Hence,

$$\begin{aligned}\mathcal{J}_1(y) &= \inf_{f,\|f\|_2=1}\ \inf_{u\in\mathcal{B}_1(\mathbb{R}^d)}\ \inf_{\lambda>0} \left\{ \frac{\lambda^2}{2}\|\nabla f\|_2^2\,dx : \int f^2(x)u(x),dx = y\,, I(u) < \lambda^d d \right\}\\ &= \frac{1}{2d^{2/d}} \inf_{f,\|f\|_2=1}\ \inf_{u\in\mathcal{B}_1(\mathbb{R}^d)} \left\{ I(u)^{2/d}\|\nabla f\|_2^2\,dx : \int f^2(x)u(x) = y \right\}\end{aligned}$$

Now, let us note that in the context we are, the function H defined by (4) is infinite outside $[-1;1]$ and is equivalent to $\frac{\sigma^2}{2}x^2$ in the neighborhood of 0 (where σ^2 is the variance of $\xi(0)$). Hence there exists a constant $C > 0$ such that $H(x) \geq Cx^2$, and $I(u) \geq C\,\|u\|_2^2$. Hence

$$\mathcal{J}_1(y) \geq C \inf_{f,\|f\|_2=1}\ \inf_{u\in\mathcal{B}_1(\mathbb{R}^d)} \left\{ \|u\|_2^{4/d}\|\nabla f\|_2^2\,dx : \int f^2(x)u(x) = y \right\}.$$

Recall now the Sobolev-Gagliardo-Nirenberg inequalities:

$$\|f\|_{p^*} \leq C\,\|\nabla f\|_2\,,\quad \text{valid for}\quad d > 2\,,\ \frac{1}{p*} = \frac{1}{2} - \frac{1}{d}; \tag{1}$$

$$\|f\|_{2p}^p \leq p\,\|f\|_{2(p-1)}^{p-1}\,\|\nabla f\|_2\,,\qquad d = 2\,,\ p \geq 1; \tag{2}$$

$$\|f\|_\infty^2 \leq 2\,\|f\|_2\,\|\nabla f\|_2\,,\qquad d = 1\,. \tag{3}$$

Case $d \leq 4$: In that case, for $d > 2$, $p^* \geq 4$. Using the interpolation inequality, for any f such that $\|f\|_2 = 1$,

$$\|f\|_4 \leq \|f\|_2^{1-d/4}\,\|f\|_{p^*}^{d/4} \leq C\,\|\nabla f\|_2^{d/4}\,.$$

This inequality remains true for $d = 1, 2$ using (2) and (3). Hence, for any (f, u) such that $\|f\|_2 = 1$, $\|u\|_\infty \leq 1$, $\int f^2(x)u(x)\,dx = y$, one has

$$|y| \leq \|f\|_4^2 \|u\|_2 \leq C \|u\|_2 \|\nabla f\|_2^{d/2}$$
$$\iff \quad |y|^{4/d} \leq C \|u\|_2^{4/d} \|\nabla f\|_2^2 \,.$$

This gives the result of the lemma for $d \leq 4$.

Case $d \geq 5$: In that case, since $p^* \geq 2$, for any (f, u) such that $\|f\|_2 = 1$, $\|u\|_\infty \leq 1$, $\int f^2(x)u(x)\,dx = y$, one can write

$$|y| \leq \|f\|_{p^*}^2 \|u\|_{d/2} \leq C \|\nabla f\|_2^2 \|u\|_2^{4/d} \|u\|_\infty^{1-4/d} \leq C \|\nabla f\|_2^2 \|u\|_2^{4/d} \,.$$

This ends the proof of the lemma

References

[1] Asselah, A; Castell, F. *Large deviations for Brownian motion in a random scenery.* Prob. Th. and Rel. Fields, Vol 126 (2003), 497–527 .

[2] Castell, F. *Moderate deviations for diffusions in a random Gaussian shear flow drift.* Ann. Inst. H. Poincaré Probab. Statist. **40** (2004), no. 3, 337–366.

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