Jacobian curves for normal complex surfaces

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ABSTRACT. Let (X, p) be a germ of complex surface, let $\psi : (X, p) \to (\mathbf{C^2}, \mathbf{0})$ be a finite holomorphic germ. The talk is dedicated to the study the contact zone of the strict transform of the jacobian curve Γ of ψ in the minimal good resolution of ψ. We define "bunches" of branches of Γ which have the same contact and we evaluate the "multiplicity" of every bunch.

1. Definitions and Notations

Let (X, p) be a germ of complex surface, let f and g be two germs of holomorphic functions on (X, p) and let $\psi = (f, g) : (X, p) \to (\mathbf{C^2}, \mathbf{0})$. We suppose that ψ is a finite morphism. Let $\Sigma(\psi)$ be the critical locus of ψ. Let Γ be the union of the irreducible components, with their multiplicities, of $\Sigma(\psi)$ which are not included in $(fg)^{-1}(0)$. We say that Γ is the **jacobian curve** and that $\Delta = \psi(\Gamma)$ is the **discriminant** of ψ. If $\psi = (f, g) : (X, p) \to (\mathbf{C^2}, \mathbf{0})$ is a generic projection Γ is the **polar curve** of ψ.

DEFINITIONS 1.4. Let R be a good resolution of (f, g) and let E_i be an irreducible component of E. A smooth germ of curve c_i which meets transversely E'_i is a **curvetta** of E_i. The quotient

$$q_i = \frac{V_{f \circ R}(c_i)}{V_{g \circ R}(c_i)}$$

is the **Hironaka number** of E_i. As $V_{f \circ R}(c_i)$ depends only of E_i we write $v_i(f) = V_{f \circ R}(c_i)$.

Let q be a Hironaka number. Let $E(q)$ be the union of the E'_i such that $q_i = q$ to which we add $E_i \cap E_j$ if $q_i = q_j = q$. Let $E^k(q), k = 1, ...n_q$, be the connected components of $E(q)$. A q**-zone** is a connected component $E^k(q)$ of $E(q)$. Let us denote by $\Gamma^k(q)$ the union (with their multiplicities) of the branches of the jacobian curve Γ whose strict transform in Y meets E in the zone $E^k(q)$. By definition $\Gamma^k(q)$ is a **q-bunch** of the jacobian curve.

A q-zone $E^k(q)$ is a **rupture** zone if there exists at least one E'_i in $E^k(q)$ such that $\chi(E'_i) < 0$.

Let $\nu : (X', p') \to (X, p)$ be the normalization of (X, p). We consider $f' = f \circ \nu$ and $g' = g \circ \nu$. We can use the Milnor fibration of f' (resp. g') to obtain the following results:

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THEOREM 4.9 (MAIN PART). *Let R be the minimal good resolution of (f, g). Then $\Gamma^k(q)$ is not empty if and only if $E^k(q)$ is a rupture zone. Moreover we have :*

$$V_f(\Gamma^k(q)) = -\Big(\sum_{E'_j \subset E^k(q)} (v_j(f)).(\chi(E'_j))\Big)$$

$$V_g(\Gamma^k(q)) = -\Big(\sum_{E'_j \subset E^k(q)} (v_j(g)).(\chi(E'_j))\Big)$$

Moreover, we prove the following result in Section 3:

THEOREM 3.4. *Let R be a good resolution of (f, g).Then:*
i) When $q_i \neq q_l$, the strict transform $\tilde{\Gamma}$ of the jacobian curve of (f, g) does not meet E at $E_i \cap E_l$.
ii) The intersection between $\tilde{\Gamma} \cap E$ and $(\tilde{F}_0 \cup \tilde{G}_0) \cap E$ is empty.

COROLLARY 4.7. *Suppose that the germ (X, p) is irreducible. The three following properties are equivalent: 1) The jacobian curve is empty. 2) M' is a thickened torus. 3) The minimal good resolution of (f, g) has the following dual graph :*

$$(f) \longleftarrow \bullet \!-\!\!-\!\!-\!\!-\!\!-\! \bullet \ \ldots \ \bullet \!-\!\!-\!\!-\!\!-\!\!-\! \bullet \longrightarrow (g)$$

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