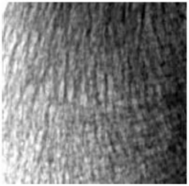
Local anisotropy of Gaussian fields–Joint work with K. Polisano, V. Perrier, L. Condat

Modeling anisotropy?

Some scientific questions

- Anisotropic textures? Images admitting different characteristics along the directions
- Suitable concept for local anisotropy?
- Done in the deterministic case using the Riesz transform!
- Extension to random fields?






A convenient framework in the deterministic case [Felsberg,2001]

The Riesz transform

- For $f \in L^2(\mathbb{R})$

$$\mathcal{R}f = \begin{pmatrix} \mathcal{R}_1 f \\ \mathcal{R}_2 f \end{pmatrix} \text{ with } \widehat{\mathcal{R}_\ell f}(\xi) = j \frac{\xi_\ell}{||\xi||} \widehat{f}(\xi)$$

- Can be viewed as a smooth version of gradient

A convenient framework in the deterministic case [Felsberg,2001]

The Riesz transform

Main properties :

- invariance by dilation, translation and steerability
- energy conservation : $||\mathcal{R}f||^2_{L^2(\mathbb{R}^2,\mathbb{R}^2)} = ||f||^2_{L^2(\mathbb{R}^2,\mathbb{R})}$

A convenient framework in the deterministic case `[Felsberg,2001]`

Amplitude and local orientation

- Amplitude : $A(x) = \sqrt{(\mathcal{R}_1 f(x))^2 + (\mathcal{R}_2 f(x))^2}$
- If $\mathcal{R}f(x) \neq 0$, one then defines the `local orientation`

$$n_f(x) = \frac{\mathcal{R}f(x)}{||\mathcal{R}f(x)||}$$

A convenient framework in the deterministic case [Felsberg,2001]

Degree of directionality

- Structure tensor

$$J_f(x) = \mathcal{R}f(x) \cdot \mathcal{R}f^T(x) = \begin{pmatrix} \mathcal{R}_1 f(x)^2 & \mathcal{R}_1 f(x)\mathcal{R}_2 f(x) \\ \mathcal{R}_1 f(x)\mathcal{R}_2 f(x) & \mathcal{R}_2 f(x)^2 \end{pmatrix}$$

  with eigenvalues $\lambda_1(x) \geq \lambda_2(x)$
- In practice, J_f replaced with $W \star J_f$ where W window
- The orientation is always an eigenvector associated to $\lambda_1(x)$
- Coherency index

$$\chi(x) = \frac{\lambda_1(x) - \lambda_2(x)}{\lambda_1(x) + \lambda_2(x)}$$

Extension to Gaussian Fields? Sample paths not in L^2 in general....

A global definition in the stationary case

Gaussian fields with spectral density

- Gaussian fields with spectral density

$$X(x) = \int (e^{jx\xi} - 1) f^{1/2}(\xi) d\widehat{W}(\xi)$$

- If X is H—self similar, that is if $X(ax) = a^H X(x)$, one has

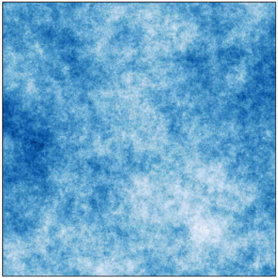
$$f(\xi) = ||\xi||^{-2H-d} S(\xi) \text{ where } S \text{ homogeneous}$$

S : anisotropy function

A global definition in the stationary case

Example 1 : Fractional Brownian Field

FBF is defined as $B_H(x) = \int \frac{e^{ix\xi}-1}{||\xi||^{H+d/2}} d\widehat{W}(\xi)$



A global definition in the stationary case

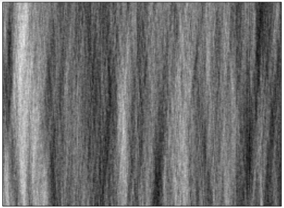
Example 2 : Anisotropic Fractional Brownian Field [Bonami-Estrade,2003]

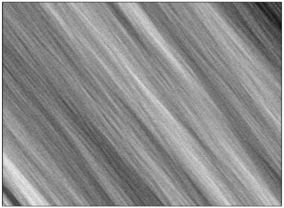
We consider a special class of AFBF is defined as

$$X_{\alpha_0,\delta}(x) = \int (e^{ix\xi} - 1) f^{1/2}(\xi) d\widehat{W}(\xi)$$

with $f(\xi) = ||\xi||^{-2H-d} S(\xi/||\xi||)$ where $S(\Theta) = (2\delta)^{-1} 1_{[-\delta,\delta]}(arg(\Theta) - \alpha_0)$

Realisations of EFBF via the turning band method of Biermé et al.





A global definition of orientation in the stationary case

Definition of the structure tensor in the self similar stationary case?

- Let W be an isotropic, zero–mean window with fast decay admitting two vanishing moments.
- We define

$$J_f^W = \begin{pmatrix} | < X, \mathcal{R}_1\psi > |^2 & < X, \mathcal{R}_1\psi >< X, \mathcal{R}_2\psi > \\ < X, \mathcal{R}_1\psi >< X, \mathcal{R}_2\psi > & | < X, \mathcal{R}_2\psi > |^2 \end{pmatrix}$$

A global definition of orientation in the stationary case

Proposition

Define $\varphi(||\xi||) = \widehat{W}(\xi)$. Then

$$\mathbb{E}[J_f^W] = \left[\int_0^\infty \frac{|\varphi(r)|^2 dr}{r^{2H+1}}\right] J(X)$$

where for any ℓ_1, ℓ_2 $J_{\ell_1,\ell_2}(X) = \int_{\mathbb{S}^1} \Theta_{\ell_1}\Theta_{\ell_2} S(\Theta) d\Theta$

A global definition of orientation in the stationary case

Definition of the structure tensor in the self similar stationary case

- Matrix $J(X)$ is called the structure tensor of X. The coherency index of the field is then

$$\chi(X) = \frac{|\lambda_1 - \lambda_2|}{\lambda_1 + \lambda_2}$$

- An orientation of X is defined as any eigenvector of $J(X)$ associated to its largest eigenvalue

A global definition of orientation in the stationary case

Example 1 : Fractional Brownian Field

One has

- $J_{1,1}(B_H) = J_{2,2}(B_H) > 0$ and $J_{1,2}(B_H) = J_{2,1}(B_H) = 0$
- Any unit vector is an eigenvector and $\chi(B_H) = 0$

A global definition of orientation in the stationary case

Example 2 : Anisotropic Fractional Brownian Field
`[Bonami-Estrade,2003]`

- One has

$$J(X) = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}\cos(2\alpha_0)\frac{\sin(2\delta)}{2\delta} & \frac{1}{2}\sin(2\alpha_0)\frac{\sin(2\delta)}{2\delta} \\ \frac{1}{2}\sin(2\alpha_0)\frac{\sin(2\delta)}{2\delta} & \frac{1}{2} - \frac{1}{2}\cos(2\alpha_0)\frac{\sin(2\delta)}{2\delta} \end{pmatrix}$$

- Hence $n = \begin{pmatrix} \cos(\alpha_0) \\ \sin(\alpha_0) \end{pmatrix}$
- Coherency index $\chi(X) = \sin(2\delta)/(2\delta)$

A global definition of orientation in the stationary case

Example 3 : Sum of two independent Anisotropic Fractional Brownian Field

- We add two independent EFBF, $X_{\alpha_0,\delta}(x) + X_{\alpha_1,\delta}(x)$
- Here $n = \begin{pmatrix} \cos((\alpha_0 + \alpha_1)/2) \\ \sin((\alpha_0 + \alpha_1)/2) \end{pmatrix}$
- $\chi(X) = \sin(2\delta)/(2\delta) \cos(\alpha_0 - \alpha_1)$

A global definition of orientation in the stationary case

Example 4 : Deformation of an Anisotropic Fractional Brownian Field

- Let L be an invertible 2×2 matrix

$$X_L(x) = X_{\alpha_0,\delta}(L^{-1}x)$$

- $n_L = (L^{-1})^T n / \|(L^{-1})^T n\|$

A global definition of orientation in the stationary case

- We have a definition of orientation in a very simple case
- Can we turn our global definition into a local one?
- What about more complex model?
- We are now considering random fields that locally behave as self similar ones

Localisable Gaussian fields with spectral density

Definition

The random field Y is H localisable at x_0 with tangent field $Y_{x_0}(x)$ if

$$\frac{Y(x_0+\rho h)}{\rho^H} \to Y_{x_0}(x)$$

(convergence in distribution)

Roughly speakig Y_{x_0} is the "local form" of Y

Localisable Gaussian fields with spectral density

If Y is localisable, all local anisotropy characteristics are defined from its tangent field

Definition

Let Y be a H localisable random field at x_0 with tangent field Y_{x_0}. Its local structure tensor (resp. local orientation, local coherency) at x_0 is the structure tensor (resp. orientation, cohenrency) of Y_{x_0}

Localisable Gaussian fields with spectral density

Example 5 : Multifractional Brownian Motion
`Benassi-Jaffard-Roux, 1995`

- MBM is defined as

$$X(x) = \int \frac{e^{ix\xi} - 1}{||\xi||^{h(x)+d/2}} d\widehat{W}(\xi)$$

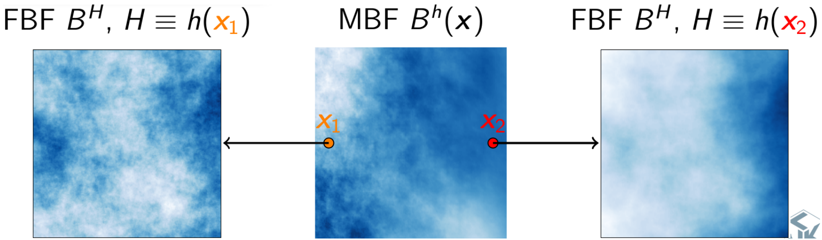
- Local form : FBF
- Then at any point, it admits any unit vector as local orientation

Localisable Gaussian fields with spectral density

- Can we synthetize textures with prescribed orientation and Holder exponent at each point?
- Could be benchmarks to test estimation procedures
- Previous works to synthetize textures with prescribed orientation using the framework of locally parallel textures : `[Peyré (2007), Aujol et al. (2010)]`
- Our trick is once more tangent fields!

Localisable Gaussian fields with spectral density

Tangent fields are also a synthesis tool!



Model 1 : Generalized Anisotropic Fractional Brownian Field [Polisano et al. (2017)]

- We define a multifractional version of EFBF

$$X_{\alpha_0,\delta}(x) = \int (e^{ix\xi} - 1)\frac{C(x,\xi)}{||\xi||^{h(x)+d/2}} d\widehat{W}(\xi)$$

  with $C(x,\xi) = 1_{[-\delta(x),\delta(x)]}(arg(\xi) - \alpha_0(x))$ and h β Holderian with $0 < a = \inf h \leq \sup h = b < \beta < 1$
- Local properties of this Gaussian field?

Model 1 : Generalized Anisotropic Fractional Brownian Field [Polisano et al. (2017)]

Properties satisfied by C

Assumptions $\mathcal{H}$

- $(x, \xi) \to C(x, \xi)$ is bounded
- $\xi \to C(x, \xi)$ is even and homogenous
- $x \to C(x, \xi)$ is continuous and for some $\eta > 0, \beta \leq \eta < 1$ s.t.

$$\sup_z ||z||^{-2\eta} \int [C(x+z, \Theta) - C(x, \Theta)]^2 d\Theta \leq A_x < \infty$$

Model 1 : Generalized Anisotropic Fractional Brownian Field [Polisano et al. (2017)]

Theorem

Since Assumptions $\mathcal{H}$ are satisfied, the GAFBF X admits at x_0 the tangent field Y_{x_0} defined as

$$Y_{x_0}(x) = \int (e^{ix\xi} - 1) \frac{C(x_0, \xi)}{||\xi||^{h(x_0)+d/2}} d\widehat{W}(\xi)$$

Model 1 : Generalized Anisotropic Fractional Brownian Field [Polisano et al. (2017)]

A two step synthesis of the GAFBF

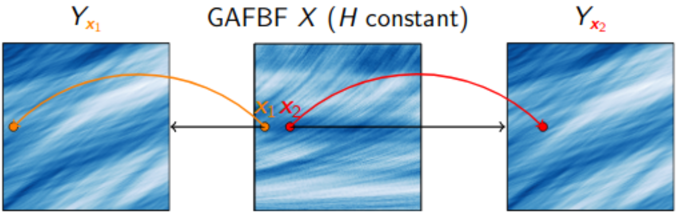
If $h \equiv H$

- Step 1 :Synthesis of the EFBF Y_{x_0} using the turning band method of Biermé et al.
- Step 2 : $X(x_0) \leftarrow Y_{x_0}(x = x_0)$

If h varies, use of kriegeage

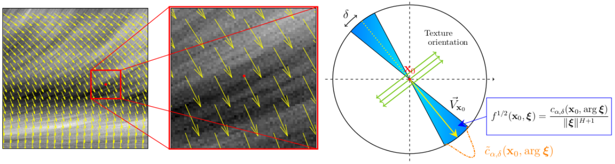
Model 1 : Generalized Anisotropic Fractional Brownian Field [Polisano et al. (2017)]

Synthesis of the GAFBF from its tangent field Y_{x_0}



Model 1 : Generalized Anisotropic Fractional Brownian Field [Polisano et al. (2017)]

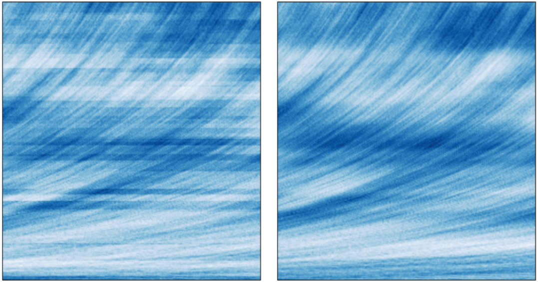
Improvement : Synthesis of the EFBF Y_{x_0} using the turning band method of Biermé et al.



(a) (b) (c)

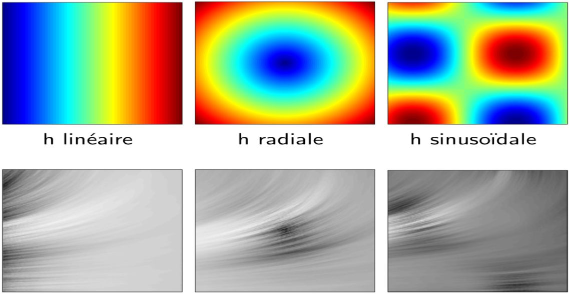
Synthesis of the GAFBF [Polisano et al. (2018)]

Case where $h \equiv H$



Synthesis of the GAFBF [Polisano et al. (2018)]

Case where h varies : use of Kriegage



Model 2 : Wrapped Anisotropic Fractional Brownian Field [Polisano et al. (2017)]

We extend previous models of Perrin et al. Guyon et al. (2000)

Definition

Let X a H-ssi random field, Φ a continuously differentiable function. The WAFBF $Z_{\Phi,X}$ is defined as

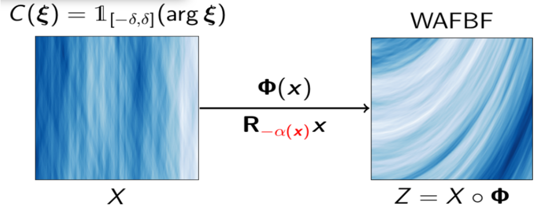
$$Z_{\Phi,X}(x) = X(\Phi(x))$$

Theorem

$Z_{\Phi,X}$ admits at any point x_0 the tangent field

$$Y_{x_0}(x) = X(D\Phi(x_0).x)$$

Model 2 : Wrapped Anisotropic Fractional Brownian Field [Polisano et al. (2017)]



$$\alpha(x_1, x_2) = -\frac{\pi}{2} + x_1$$

Model 2 : Wrapped Anisotropic Fractional Brownian Field [Polisano et al. (2017)]

Some questions

- What about control of the directionality?
- How can we choose Φ so that the local orientation is prescribed?

Model 2 : Wrapped Anisotropic Fractional Brownian Field [Polisano et al. (2017)]

A proposal to control the orientation

Let $Z_{\Phi_\alpha,X}(x)$ be $X = X_{0,\delta}$ with local orientation $e_1 = (1,0)^T$ wrapped by Φ_α defined as :

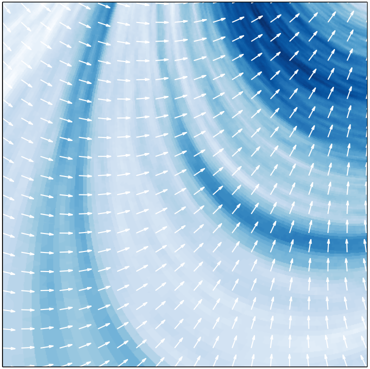
1. $\alpha : \mathbb{R}^2 \to \mathbb{R}$ is an harmonic function
2. λ such that $\Psi_\alpha = \begin{pmatrix} \lambda \\ -\alpha \end{pmatrix}$ is holomorphic,
3. Φ_α a complex primitive of $\exp(\Psi_\alpha)$.

The local orientation (up to δ^2) of $Z_{\Phi_\alpha,X}$ at x_0 is

$$n_Z(x_0) = \begin{pmatrix} \cos\alpha(x_0) \\ \sin\alpha(x_0) \end{pmatrix} = u(\alpha(x_0))$$

Model 2 : Wrapped Anisotropic Fractional Brownian Field [Polisano et al. (2017)]

WAFBF with prescribed orientation



Towards estimation : the wavelet–based method

- Wavelet isotropic tight frames `Unser et al. 2013` : $\psi_{j,k}$
- Properties of Riesz transform implies $\mathcal{R}\psi_{j,k} = (\mathcal{R}\psi)_{j,k}$
- Wavelet coefficients of X H sssi

$$c_{j,k}^{(\mathcal{R})} = \begin{pmatrix} c_{j,k}^{(1)} \\ c_{j,k}^{(2)} \end{pmatrix} = \begin{pmatrix} < X, \mathcal{R}_1\psi_{j,k} > \\ < X, \mathcal{R}_2\psi_{j,k} > \end{pmatrix}$$

Towards estimation : the wavelet–based method

The stationary case

- Covariance of wavelet coefficients related to the structure tensor of X
$$\mathbb{E}[c_{j,k}^{(\ell_1)} c_{j,k}^{(\ell_2)}] = 2^{-2i(H+d/2)} C(\psi, H) J(X)$$
- Possible strategy?
- Log–log regression of empirical covariance to estimate H and thereafter J

Towards estimation : the wavelet–based method

Perspectives?

The non stationary case

- Adaptation to the non stationary case as in `Coeurjolly, 2000`
- Test the procedures on our two models?
- Statistical results on our estimation procedures?

Thanks!

- Feel free to visit the website of Kevin : http://www.kevinpolisano.com/
- Our preprint https://arxiv.org/abs/1708.00267v1