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Non-uniqueness of rational best approximants

L. Baratchart[a], Herbert Stahl[b,*,1], F. Wielonsky[a,1]

[a] INRIA 2004, route des Lucioles B.P. 93 06902, Sophia, Antipolis, Cedex, France

[b] TFH-Berlin/FB2, Luxemburger Starasse 10, 13353 Berlin, Germany

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Dedicated to Professor Haakon Waadeland on the occasion of his 70th birthday

Abstract

Let f be a Markov function with defining measure μ supported on $(-1,1)$, i.e., $f(z)=\int(t-z)^{-1}\,\mathrm{d}\mu(t)$, $\mu\geqslant 0$, and $\operatorname{supp}(\mu)\subseteq(-1,1)$. The uniqueness of rational best approximants to the function f in the norm of the real Hardy space $H^2(V)$, $V:=\bar{\mathbb{C}}\backslash\bar{\mathbb{D}}=\{z\in\bar{\mathbb{C}}\mid |z|>1\}$, is investigated. It is shown that there exist Markov functions f with rational best approximants that are not unique for infinitely many numerator and denominator degrees $n-1$ and n, respectively. In the counterexamples, which have been constructed, the defining measures μ are rather rough. But there also exist Markov functions f with smooth defining measures μ such that the rational best approximants to f are not unique for odd denominator degrees up to a given one.

MSC: 41A20; 41A50

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1. Introduction and main results

We consider rational best approximants to functions of the real Hardy space $H^2(V)$ with $V:=\bar{\mathbb{C}}\backslash\bar{\mathbb{D}}=\{z\in\bar{\mathbb{C}}\mid |z|>1\}$. This type of approximants is interesting in control theory, and they have been the object of several studies (cf. [1–4,6–9]). In the present paper, we are concerned with the uniqueness of such approximants. We consider approximants to Markov functions, i.e., functions of the form

$$f(z)=f(\mu;z):=\int\frac{\mathrm{d}\mu(t)}{t-z} \tag{1}$$

* Corresponding author.

E-mail addresses: baratcha@sophia.inria.fr (L. Baratchart); stahl@p-soft.de (H. Stahl)

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with μ a positive measure of compact support on $\mathbb{R}$. In order that f is an element of $H^2(V)$, it is assumed that

$$\operatorname{supp}(\mu) \subseteq (-1, 1). \tag{2}$$

The problem of uniqueness has practical importance, for instance, in model fitting or in algorithmic considerations. Markov functions are interesting since in their case uniqueness has been proved for certain subclasses of functions (cf. [4,5,8]). It is a natural question, how much the subclass of Markov functions in $H^2(V)$ with at least asymptotic uniqueness can be extended, and how close the results in [5] come to best possible ones. We shall prove two new theorems (Theorems 4 and 5) that will shed light on these questions. To set the stage we (very shortly) summarize relevant results from [8,5] in two theorems. The summary is also used to introduce necessary notations.

The set of all real polynomials of degree at most n is denoted by $\mathscr{P}_n$, the set of all real rational functions of numerator and denominator degree at most m and n, respectively, by $\mathscr{R}_{m,n}$, and $\mathscr{R}^1_{m,n} \subseteq \mathscr{R}_{m,n}$ denotes the set of rational functions that have all their poles in the open unit disc $\mathbb{D}$ (hence, $\mathscr{R}^1_{m,n} = \mathscr{R}_{m,n} \cap H^2(V)$). By $\|\cdot\| = \|\cdot\|_{H^2(V)}$ we denote the norm in $H^2(V)$, i.e.,

$$\|g\| := \lim_{r\to 1-} \left[\frac{1}{2\pi}\int_0^{2\pi} |g(re^{it})|^2 \, dt\right]^{1/2}, \quad g \in H^2(V), \tag{3}$$

and $R_n = R_n(f;\cdot) \in \mathscr{R}_{n-1,n}$ denotes the rational best approximant to $f \in H^2(V)$ in the norm of $H^2(V)$, i.e., $R_n \in \mathscr{R}^1_{n-1,n}$ and

$$\|f - R_n\| = \inf_{r \in \mathscr{R}^1_{n-1,n}} \|f - r\|. \tag{4}$$

For each pair of degrees $(n-1, n)$, at least one best approximant exists, but in general it is not unique (cf. [8], Section 1). In case of nonuniqueness we assume that the symbol $R_n = R_n(f;\cdot)$ denotes one of the possible functions. If necessary, different functions $R_n^{(1)}, R_n^{(2)}$, etc., are distinguished by superindices.

In [8] the following theorem has been proved:

Theorem 1 (Baratchart and Wielonsky [8], Theorem 3). *If the defining measure* μ *in* (1) *satisfies one of the three conditions*: (i) $\operatorname{supp}(\mu) \subseteq [-\lambda_0, \lambda_0]$, $\lambda_0 = \sqrt{2-\sqrt{3}} = 0.5176\ldots$, (ii) $\operatorname{supp}(\mu) \subseteq [0, \lambda_1]$, *or* (iii) $\operatorname{supp}(\mu) \subseteq [-\lambda_1, 0]$, $\lambda_1 = \sqrt{\frac{1}{2}}$, *then all* $R_n(f(\mu;\cdot);\cdot)$, $n = 0, 1, \ldots$, *are unique.*

This result is not the best possible, and it has been conjectured that it should hold true for larger supports. However, Lemma 1 in the next section (or a remark in the introduction of [8]) shows that some restrictions have to be satisfied by μ or by $\operatorname{supp}(\mu)$ if one wants to have uniqueness for all best approximants $R_n = R_n(f(\mu;\cdot);\cdot)$, $n = 1, 2, \ldots$.

A different, but practically not less interesting question concerns asymptotic uniqueness. *Asymptotic uniqueness* means that there exists $n_0 \in \mathbb{N}$ such that rational best approximants $R_n(f;\cdot)$ are unique for all $n \geq n_0$.

In a forthcoming paper [5] it has been shown that for defining measures μ in (1), that belong to the Szegö class, asymptotic uniqueness holds true.

Definition 2. A positive measure μ on $\mathbb{R}$ belongs to the *Szegö class* if
(i) $\operatorname{supp}(\mu) \subseteq \mathbb{R}$ is a compact interval $[a,b]$, and
(ii) if in the Lebesgue decomposition

$$d\mu(x) = d\mu''(x) + \frac{\mu'(x)\,dx}{\sqrt{(b-x)(x-a)}}, \quad x \in [a,b], \tag{5}$$

with μ'' a totally singular measure, and the density function μ' satisfies

$$\int_a^b \frac{\log \mu'(x)}{\sqrt{(b-x)(x-a)}}\,dx > -\infty. \tag{6}$$

The following result has been proved in [5]:

Theorem 3 (Baratchart et al. [5], Theorem 1.3). *If the defining measure μ in (1) belongs to the Szegö class and satisfies (2), then the rational best approximants $R_n = R_n(f(\mu;\cdot);\cdot)$, $n = 0,1,\ldots,$ are asymptotically unique, i.e., there exists $n_0 \in \mathbb{N}$ such that R_n is unique for all $n \geqslant n_0$.*

Naturally, the question arises whether the assumption of Theorem 3 is really necessary. The next theorem gives an answer to this question, it shows that the defining measure μ in (1) has to satisfy some conditions beyond (2) in order that asymptotic uniqueness holds true.

Theorem 4. *There exist positive Borel measures μ with $\operatorname{supp}(\mu) \subseteq (-1,1)$ such that each second rational best approximant $R_n(f(\mu;\cdot);\cdot)$ is not unique. More precisely: there exist measures μ such that for each odd index $n = 1,3,5,\ldots$ there exist at least two different rational best approximants $R_n^{(1)}(f(\mu;\cdot);\cdot)$ and $R_n^{(2)}(f(\mu;\cdot);\cdot)$.*

Theorem 4 will be proved by constructing measures μ with the stated property. These measures are not smooth. Actually, they are rather rough. For instance, each one is carried by a denumerable set. In the light of Theorem 3 it seems that it is rather difficult to construct a measure μ with smooth and positive density such that the best rational approximants are not at least asymptotically unique. However, a nonuniqueness result for smooth measures can rather easily be deduced from Theorem 4, but nonuniqueness can be guaranteed only for a finite number of approximants.

Theorem 5. *For any $n_0 \in \mathbb{N}$ there exists a positive Borel measure μ with $\operatorname{supp}(\mu)$ a closed interval in $(-1,1)$, the measure μ has a positive and smooth density function on $\operatorname{supp}(\mu)$, and for each index $n = 1,3,\ldots,n_0$ (let n_0 be chosen to be odd) there exist at least two different rational best approximants $R_n^{(1)}(f(\mu;\cdot);\cdot)$ and $R_n^{(2)}(f(\mu;\cdot);\cdot)$.*

Remark. The measure μ of Theorem 5 belongs to the Szegö class. It follows therefore from Theorem 3 that in Theorem 5 the rational best approximants $R_n(f;\cdot)$, $n \in \mathbb{N}$, are asymptotically unique.

2. Proofs

The proof of Theorems 4 and 5 will be prepared by three lemmas, of which the second one is the most important and also the one with the most involved proof. We start with some notations and some results from the theory of rational best approximants $R_n = R_n(f;\cdot) \in \mathscr{R}_{n-1,n}$ in the H^2-norm. Let R_n be represented as

$$R_n = \frac{p}{q} = \frac{p_n}{q_n}, \tag{7}$$

where $q_n \in \mathscr{P}_n$ is assumed to be monic, and $p_n \in \mathscr{P}_{n-1}$. In case of nonuniqueness different denominators and numerator polynomials q_n and p_n are denoted by $q_n^{(1)}, q_n^{(2)}, \ldots,$ and $p_n^{(1)}, p_n^{(2)}, \ldots,$ respectively. By $\tilde{q}_n$ we denote the reversed polynomial $\tilde{q}_n(z) := z^n q_n(1/z)$ of the polynomial q_n. We note that this operation assumes a given degree n, which is usually understood from the context. It is well known (cf. [8], Proposition 5) that the denominator q_n of rational best approximants R_n is exactly of degree n, has only simple zeros, and all n zeros are contained in the smallest interval I containing supp(μ). The best approximants R_n interpolate the function f with order 2 in the reciprocal of each zero of the polynomial q_n, i.e., each R_n interpolates f in the $2n$ zeros of the polynomial $\tilde{q}_n^2$. If $z = 0$ is a zero of q_n, then $f - R_n$ has a zero of order 3 at infinity (cf. [8], Proposition 5). As a consequence of the interpolation property, it is possible to derive a characterization of the polynomial q_n by an orthogonality relation. We have

$$\int t^k q_n(t) \frac{\mathrm{d}\mu(t)}{\tilde{q}_n(t)^2} = 0, \quad k = 0, \ldots, n-1, \tag{8}$$

(cf. [10, Lemma 6.1.2]). Because of the polynomial $\tilde{q}_n^2$ in relation (8), this relation is no longer linear in q_n, which is a remarkable difference to the usual orthogonality relations, and also explains why q_n is in general not uniquely determined by relation (8). It has been shown in [10, Lemma 6.1.2] that any monic polynomial q_n that satisfies relation (8) is the denominator of a rational function that interpolates f in the $2n$ point of $\tilde{q}_n^2$. If there exist different rational best approximants $R_n^{(j)} = p_n^{(j)}/q_n^{(j)}$, $j = 1, 2, \ldots,$ then each denominator polynomial $q_n^{(j)}$ satisfies relation (8). We note that in this later case orthogonality relation (8) is different for each j since the polynomials $\tilde{q}_n^{(j)}$ are different.

For the interpolation error we have the representation

$$(f - R_n)(z) = \frac{\tilde{q}_n(z)^2}{q_n(z)^2} \int \frac{q_n(t)^2}{\tilde{q}_n(t)^2} \frac{\mathrm{d}\mu(t)}{t - z} \tag{9}$$

(cf. [10, Lemma 6.1.2]). Relation (8) and formula (9) will be important tools in the proofs of the next two lemmas.

Lemma 6. *Let* $z_0 \in \left(\sqrt{\frac{1}{2}}, 1\right)$, $\mu_0 := (\delta_{-z_0} + \delta_{z_0})/2$, *and*

$$f_0(z) := \int \frac{\mathrm{d}\mu_0(t)}{t - z}. \tag{10}$$

Then there exist exactly two different rational best approximants $R_1^{(j)}(f_0;\cdot)$, $j=1,2$, to the function f_0. They are given by

$$R_1^{(j)}(f_0;\cdot)=\frac{a_j^2/2-1}{z-a_j} \qquad \textit{with } a_j:=(-1)^j\sqrt{2-z_0^{-2}}, \quad j=1,2 \tag{11}$$

and we have

$$\| f_0-R_1^{(j)}(f_0;\cdot)\| =\frac{1}{2z_0}\sqrt{\frac{3z_0^2-1}{1-z_0^4}}, \qquad j=1,2. \tag{12}$$

Remark. None of the two rational best approximants $R_1^{(j)}$, $j=1,2$, is symmetric or antisymmetric with respect to the origin, while this is the case with f_0. We have $f_0(z)=-f_0(-z)$. But the two best approximants are connected by reflection on the origin, we have $R_1^{(1)}(z)=-R_1^{(2)}(-z)$. From the proof of Lemma 6 it can be deduced that the symmetric rational best approximant $R_1^{\rm sym}$ to f_0 is given by

$$R_1^{\rm sym}(z)=R_1^{\rm sym}(f_0;z)=-\frac{1}{z}, \tag{13}$$

and the norm of the approximation error is given by

$$\| f_0-R_1^{\rm sym}\| =\frac{z_0^2}{\sqrt{1-z_0^4}}. \tag{14}$$

Proof. Let R_1 be represented by p/q as in (7). Both polynomials q and $\tilde{q}$ are of degree 1. Let $x_1\in(-z_0,z_0)$ be the only zero of q. We have $q(z)=z-x_1$, $\tilde{q}(z)=1-x_1z$, and because of the special form of μ_0, relation (8) reduces to the single equation

$$\int q(t)\frac{\mathrm{d}\mu_0(t)}{\tilde{q}(t)^2}=\frac{z_0-x_1}{2(1-z_0x_1)^2}-\frac{z_0+x_1}{2(1+z_0x_1)^2}=0, \tag{15}$$

which is equivalent to the equation

$$x_1(1-2z_0^2+z_0^2x_1^2)=0. \tag{16}$$

This equation has the three solutions $x_1^{(0)}=0$ and $x_1^{(1,2)}=\pm\sqrt{2-z_0^{-2}}$, and they are the only ones. Each solution leads to a different denominator polynomial $q^{(0)}$, $q^{(1)}$, and $q^{(2)}$, and consequently also to three different approximants $R_1^{(0)}$, $R_1^{(1)}$, and $R_1^{(2)}$. The constant $c^{(j)}$ in $R_1^{(j)}(z)=c^{(j)}/(z-x_1^{(j)})$, $j=0,1,2$, can be determined by the interpolation property of $R_1^{(j)}$. For $j=1,2$ we have interpolation of f_0 in $1/x_1^{(j)}$, which leads to

$$f_0\left(\frac{1}{x_1^{(j)}}\right)-R_1^{(j)}\left(f_0;\frac{1}{x_1^{(j)}}\right)=\frac{1}{2\left(z_0-1/x_1^{(j)}\right)}+\frac{1}{2\left(-z_0-1/x_1^{(j)}\right)}-\frac{c^{(j)}}{1/x_1^{(j)}-x_1^{(j)}}$$

$$=\frac{x_1^{(j)}}{2(z_0x_1^{(j)}-1)}-\frac{x_1^{(j)}}{2(z_0x_1^{(j)}-1)}-\frac{c^{(j)}x_1^{(j)}}{1-(x_1^{(j)})^2}=0, \quad j=1,2. \tag{17}$$

From (17) we deduce after some calculations that

$$c^{(j)} = \frac{\left(x_1^{(j)}\right)^2}{2} - 1, \quad j = 1,2. \tag{18}$$

In case of $j = 0$, i.e., $x_1^{(0)} = 0$, we have interpolation at infinity, which leads to $c^{(0)} = 1$.

For each of the three cases $j = 0,1,2$, we calculate the norm $\| f_0 - R_1^{(j)} \|$. Since $|(\tilde{q}/q)(z)| = 1$ for all $|z| = 1$, it follows from (9) that

$$\| f_0 - R_1^{(j)} \| = \left\| \frac{\tilde{q}(\cdot)^2}{q(\cdot)^2} \int \frac{q(t)^2}{\tilde{q}(t)^2} \frac{\mathrm{d}\mu_0(t)}{t - \cdot} \right\| = \left\| \int \frac{q(t)^2}{\tilde{q}(t)^2} \frac{\mathrm{d}\mu_0(t)}{t - \cdot} \right\|$$

$$= \frac{1}{2} \left\| \frac{(q(z_0)/\tilde{q}(z_0))^2}{z_0 - \cdot} - \frac{(q(-z_0)/\tilde{q}(-z_0))^2}{z_0 + \cdot} \right\|, \quad q = q^{(j)}, \; j = 0,1,2. \tag{19}$$

After some lengthy calculations it follows from (19) that

$$\| f_0 - R_1^{(0)} \|^2 = \frac{z_0^4}{1 - z_0^4}, \qquad \| f_0 - R_1^{(j)} \|^2 = \frac{3z_0^2 - 1}{4z_0^2(1 - z_0^4)}, \quad j = 1,2. \tag{20}$$

Since $z_0 > \sqrt{\frac{1}{2}}$, we have $\| f_0 - R_1^{(j)} \| < \| f_0 - R_1^{(0)} \|$ for $j = 1,2$, which shows that only the two solutions $x_1^{(1)}$ and $x_1^{(2)}$ of Eq. (16) lead to rational best approximants. With (20) the proof of Lemma 6 is complete. □

For later use we have a second look at (15). We consider the zero x of the polynomial $q(z) = z - x$ as an independent variable, and define

$$g_1(x) := \int q(t) \frac{\mathrm{d}\mu_0(t)}{\tilde{q}(t)^2} = \int \frac{t - x}{(1 - tx)^2} \mathrm{d}\mu_0(t) = \frac{z_0 - x}{2(1 - z_0 x)^2} - \frac{z_0 + x}{2(1 + z_0 x)^2}. \tag{21}$$

From (16) we know that $g_1(x_1^{(j)}) = 0$ for $j = 0,1,2$. Since $g_1(x)$ is a rational function of degree (3, 4), all three zeros of g_1 are simple and we have

$$g_1'(x_1^{(j)}) \neq 0 \quad \text{for } j = 0,1,2. \tag{22}$$

Let $\mathcal{M}_m$ denote the set of all Markov functions $f = f(\mu; \cdot)$ of type (1) with a defining measure μ that has a support of exactly $m \in \mathbb{N}$ points in $(-1,1)$. Thus, each $f \in \mathcal{M}_m$ is a rational function with m poles in $(-1,1)$, and all residua are negative. Since rational best approximants $R_m(f; \cdot)$ are rational interpolants, it follows from [10, Ch. 6.1], that if the function f is of type (1) satisfying (2), then $R_m(f; \cdot) \in \mathcal{M}_m$. Let $\mathcal{M}_m^{\text{sym}} \subseteq \mathcal{M}_m$ denote the subset of Markov functions $f \in \mathcal{M}_m$ that satisfy $f(-z) = -f(z)$, i.e., the defining measure μ of f is symmetric with respect to the origin. We define

$$\operatorname{dist}(f, \mathcal{M}_m) := \inf_{r \in \mathcal{M}_m} \| f - r \|, \qquad \operatorname{dist}(f, \mathcal{M}_m^{\text{sym}}) := \inf_{r \in \mathcal{M}_m^{\text{sym}}} \| f - r \|. \tag{23}$$

From (4) and the fact that $R_m(f; \cdot) \in \mathcal{M}_m$ we deduce that $\operatorname{dist}(f, \mathcal{M}_m) = \| f - R_m(f; \cdot) \|$ for all functions of type (1) satisfying (2).

Lemma 7. *Let μ_{2m} be a positive, symmetric (with respect to the origin) Borel measure on $(-1,1)$ with $\operatorname{supp}(\mu_{2m})$ consisting of $2m$ points. Let further $z_0 \in \left(\sqrt{\frac{1}{2}},1\right)\setminus\operatorname{supp}(\mu_{2m})$ be chosen in such a way that $\sqrt{2-z_0^{-2}} \notin \operatorname{supp}(\mu_{2m})$, and define $\mu_0 := (\delta_{-z_0}+\delta_{z_0})/2$. Then there exists $\delta_0>0$ such that for any δ with $0\leqslant\delta\leqslant\delta_0$ the Markov function*

$$f_1(z)=f_1(\delta;z):=\int\frac{\mathrm{d}(\mu_{2m}+\delta\mu_0)(t)}{t-z}, \tag{24}$$

has exactly two different rational best approximants $R_{2m+1}^{(j)}(f_1;\cdot)$, $j=1,2$, and we have

$$\operatorname{dist}(f,\mathscr{M}_m^{\text{sym}}) > \operatorname{dist}(f,\mathscr{M}_m) := \|f_1 - R_{2m+1}(f_1;\cdot)\|,\quad j=1,2, \tag{25}$$

for all $0<\delta\leqslant\delta_0$.

Proof. Let the $2m$ points of $\operatorname{supp}(\mu_{2m})$ be denoted by $z_1,\ldots,z_{2m}\in(-1,1)$. As in the proof of Lemma 1 we assume that the rational best approximants $R_{2m+1}(f_1;\cdot)$ are represented by the quotients $p/q=p_n/q_n$ with $q\in\mathscr{P}_{2m+1}$ monic polynomials, and $p\in\mathscr{P}_{2m}$. We know that as a consequence of orthogonality (8) the polynomial q has exactly $2m+1$ simple zeros $x_j\in I\subseteq(-1,1)$, $j=1,\ldots,2m+1$, with I denoting the smallest closed interval containing $\operatorname{supp}(\mu_{2m})\cup\{-z_0,z_0\}$. The zeros $x_j=x_j(\delta)$, $j=1,\ldots,2m+1$, as well as the polynomial $q=q(\delta;\cdot)$ itself depend on the parameter δ introduced in definition (24) of the function f_1.

In the first step of the proof we shall show that

$$\lim_{\delta\to 0} x_j(\delta)=z_j \quad \text{for } j=1,\ldots,2m. \tag{26}$$

Indeed, by taking subsequences if necessary, we can assume that for a given sequence $\delta_l\to 0$, $l\to\infty$, the limits $\hat z_j := \lim_{l\to\infty} x_j(\delta_l)$, $j=1,\ldots,2m+1$, exist. As a consequence the monic polynomials $q=q(\delta_l;\cdot)$ converge to $\hat q(z)=\prod_{j=1}^{2m+1}(z-\hat z_j)\in\mathscr{P}_{2m-1}$ uniformly in $\overline{\mathbb{D}}$. Since the $R_{2m+1}(f_1;\cdot)$ are best approximants, it follows that the denominator and numerator polynomials $q=q(\delta_l;\cdot)$ and $p=p(\delta_l;\cdot)$ converge to polynomials $\hat q\in\mathscr{P}_{2m+1}$ and $\hat p\in\mathscr{P}_{2m}$ that satisfy $f_1(0;\cdot)=f(\mu_{2m};\cdot)=\hat p/\hat q$, which implies that $\hat z_j=z_j$ for $j=1,\ldots,2m$, and consequently also (26). Note that the behavior of the last zero $x_{2m+1}(\delta)$ for $\delta\to 0$ is at this stage not clear, we can only conclude that the linear factor $(z-x_{2m-1})$ in q cancels out in the limit with a corresponding factor in the numerator polynomial p.

Next, we study the behavior of $x_{2m+1}(\delta)$ as $\delta\to 0$. We define

$$\eta=\eta(\delta):=\max_{j=1,\ldots,2m}|x_j(\delta)-z_j|, \tag{27}$$

$$z_{2m+1}^{(l)} := \begin{cases} 0 & \text{for } l=0,\\ -\sqrt{2-z_0^{-2}} & \text{for } l=1,\\ +\sqrt{2-z_0^{-2}} & \text{for } l=2, \end{cases} \tag{28}$$

and show that

$$\operatorname{dist}\left(x_{2m-1}(\delta),\left\{z_{2m+1}^{(0)},z_{2m+1}^{(1)},z_{2m-1}^{(2)}\right\}\right)=\mathrm{O}(\eta) \tag{29}$$

as $\delta\to 0$. Definition (28) has been motivated by Lemma 6. If (29) is proved, then it shows that the three points introduced in (28) are the only possible cluster points of the sequence $\{x_{2m+1}(\delta_k)\}_{k\in\mathbb{N}}$ for any sequence $\delta_k\to 0$.

From (26) and (27) it follows that $\lim_{\delta\to 0}\eta(\delta)=0$. Set

$$p_{2m}(z):=\prod_{j=1}^{2m}(z-z_j). \tag{30}$$

It follows from (27) that

$$q(z)=(z-x_{2m+1})[p_{2m}(z)+O(\eta)] \tag{31}$$

and

$$\tilde{q}(z)=(1-zx_{2m+1})[\widetilde{p_{2m}}(z)+O(\eta)] \tag{32}$$

for $\delta\to 0$. The Landau symbol $O(\eta)$ in (31) and (32) holds uniformly on $\overline{\mathbb{D}}$. Since $\operatorname{supp}(\mu_{2m})$ has been assumed to be symmetric with respect to the origin, we have $p_{2m}(-z)=p_{2m}(z)$. The denominator polynomial q satisfies the orthogonality relation (8), which with (24), (31), (32) and the definition of μ_0 yields

$$\begin{aligned}0&=\int p_{2m}(t)\frac{q(t)}{\tilde{q}(t)^2}\,d(\mu_{2m}+\delta\mu_0)(t)=\delta\int p_{2m}(t)\frac{q(t)}{\tilde{q}(t)^2}\,d\mu_0(t)\\&=\delta\left(\frac{p_{2m}(z_0)}{\widetilde{p_{2m}}(z_0)}\right)^2\int\frac{t-x_{2m+1}}{(1-tx_{2m+1})^2}\,d\mu_0(t)+\delta\,O(\eta)\quad\text{as }\delta\to 0.\end{aligned} \tag{33}$$

The integral in the last line of (33) is identical with the integral in (21) if we replace in (21) x by x_{2m+1}. From the assumption of Lemma 7 it follows that $p_{2m}(z_0)/\widetilde{p_{2m}}(z_0)\neq 0$. From (21) and (22) we know that the integral in the last line of (33) has exactly three simple zeros if we consider this integral as a function of x_{2m+1}. The three zeros are identical with the numbers $z_{2m+1}^{(l)}$, $l=0,1,2$, defined in (28). Assertion (29) then follows from (33) and (22).

Next we prove an estimate for $\eta(\delta)$ as $\delta\to 0$. As a byproduct we get a sharper version of (29). Set

$$p_{2m,j}(z)=p_{2m,j}^{(l)}(z):=p_{2m}(z)(z-z_{2m+1}^{(l)})/(z-z_j)\in\mathscr{P}_{2m},\quad j=1,\dots,2m,\quad l=0,1,2. \tag{34}$$

From (27) and (29) we deduce that

$$q(z)=(z-x_j)[p_{2m,j}^{(l_q)}(z)+O(\eta)],\quad j=1,\dots,2m. \tag{35}$$

Like in (31) and (32) $O(\eta)$ holds uniformly on $\overline{\mathbb{D}}$. The superindex $l_q\in\{0,1,2\}$ in (35) has to be chosen in such a way that $|z_{2m+1}(\delta)-z_{2m+1}^{(l_q)}|$ is small. Since the denominator polynomial q satisfies the orthogonality relation (8), we deduce with (35) that

$$\begin{aligned}0&=\int p_{2m,j}(t)\frac{q(t)}{\tilde{q}(t)^2}\,d(\mu_{2m}+\delta\mu_0)(t)=\int p_{2m,j}(t)\frac{q(t)}{\tilde{q}(t)^2}\,d\mu_{2m}(t)+O(\delta)\\&=(z_j-x_j)\left[\left(\frac{p_{2m,j}(z_j)}{\tilde{q}(z_j)}\right)^2+O(\eta)\right]+O(\delta)\end{aligned} \tag{36}$$

for $\delta\to 0$ and $j=1,\dots,2m$. Since $p_{2m,j}(z_j)\neq 0$, we deduce from (36) that

$$x_j(\delta)-z_j=O(\delta)\quad\text{for }\delta\to 0,\ j=1,\dots,2m. \tag{37}$$

With (27) and (29) the last estimate implies that

$$\eta(\delta) = \max_{j=1,\dots,2m} |x_j(\delta) - z_j| = O(\delta),$$
$$\operatorname{dist}\left(x_{2m+1}(\delta), \left\{z_{2m+1}^{(0)}, z_{2m+1}^{(1)}, z_{2m+1}^{(2)}\right\}\right) = O(\delta) \quad \text{for } \delta \to 0. \tag{38}$$

Up to now we have only derived necessary conditions for the asymptotic location of the $2m+1$ zeros of the denominator polynomials q as $\delta \to 0$. It has been shown that the vector $x = (x_1,\dots,x_{2m+1})$ of the $2m+1$ zeros has to lie in a neighborhood of one of the three points

$$z^{(l)} := (z_1,\dots,z_{2m}, z_{2m+1}^{(l)}), \quad l = 0,1,2, \tag{39}$$

for $\delta > 0$ sufficiently small. For each $\delta > 0$ the orthogonality relations (8), i.e.,

$$\int t^k \frac{q(t)}{\tilde{q}(t)^2} \, \mathrm{d}(\mu_{2m} + \delta\mu_0)(t) = 0, \quad k = 0,\dots,2m \tag{40}$$

define a system of $2m+1$ equations for the $2m+1$ components of the vector $x = (x_1,\dots,x_{2m+1}) \in \mathbb{R}^{2m+1}$ of zeros of q. We shall now show that if we consider the x_j, $j = 1,\dots,2m+1$, as variables, then for $\delta > 0$ sufficiently small the system of equations (40) has exactly three solutions, and each of the three solutions is lying in a small neighborhood of one of the three points (39).

For each $l = 0,1,2$ the set $\{p_{2m,1}^{(l)},\dots,p_{2m,2m}^{(l)}, p_{2m}\}$ of polynomials defined in (39) and (30) is a basis in $\mathscr{P}_{2m}$. Therefore, these polynomials can be used in (40) instead of the $2m+1$ powers z^k. With these polynomials we define three maps $F^{(l)} : \mathbb{R}^{2m+1} \to \mathbb{R}^{2m+1}$, $l = 0,1,2$, by

$$x = (x_1,\dots,x_{2m+1}) \mapsto (F_1^{(l)}(x),\dots,F_{2m+1}^{(l)}(x))$$

$$F_j^{(l)}(x) := \begin{cases} \displaystyle\int p_{2m,j}^{(l)}(t) \frac{q(t)}{\tilde{q}(t)^2} \, \mathrm{d}(\mu_{2m} + \delta\mu_0)(t) & \text{for } j = 1,\dots,2m, \\ \displaystyle\frac{1}{\delta}\int p_{2m}(t) \frac{q(t)}{\tilde{q}(t)^2} \, \mathrm{d}(\mu_{2m} + \delta\mu_0)(t) & \\ \displaystyle\quad = \int p_{2m}(t) \frac{q(t)}{\tilde{q}(t)^2} \, \mathrm{d}\mu_0(t) & \text{for } j = 2m+1. \end{cases} \tag{41}$$

In (41) the $2m+1$ zeros $x_1,\dots,x_{2m+1}$ of q are considered as variables. If $x = (x_1,\dots,x_{2m+1}) = z^{(l)}$, then we have $q(z) = p_{2m}(z)(z - z_{2m+1}^{(l)})$ and $q(z) = p_{2m,j}^{(l)}(z)(z - z_j) = p_{2m,j}(z)(z - z_j)$ for $j = 1,\dots,2m$. Considering the expressions used in (36) it is rather immediate to see that from (35) it follows that

$$\frac{\partial}{\partial x_k} F_j^{(l)}(z^{(l)}) = \begin{cases} -\left(\dfrac{p_{2m,j}(z_j)}{\tilde{q}(z_j)}\right)^2 + O(\delta) & \text{for } k = j, \\ O(\delta) & \text{for } k = 1,\dots,2m+1,\ j \neq k, \end{cases} \tag{42}$$

$j = 1,\dots,2m$, $\delta \to 0$, and $l = 0,1,2$. Using the expression in (33) in a similar analysis, one gets

$$\frac{\partial}{\partial x_k} F_{2m+1}^{(l)}(z^{(l)}) = \begin{cases} \left(\dfrac{p_{2m}(z_0)}{\widetilde{p_{2m}}(z_0)}\right)^2 g'\left(z_{2m+1}^{(l)}\right) & \text{for } k = 2m+1, \\ O(\delta) & \text{for } k = 1,\dots,2m+1, \end{cases} \tag{43}$$

$\delta \to 0$, and $l = 0, 1, 2$. The function g_1 in (43) has been defined in (21). For the components of the function value $F^{(l)}(z^{(l)})$ we have the following estimates, respective value:

$$F_j^{(l)}(z^{(l)}) = \int p_{2m,j}(t) \frac{p_{2m}(t)(t - z_{2m+1}^{(l)})}{(\widetilde{p_{2m}}(t)(1 - tz_{2m+1}^{(l)}))^2} \, \mathrm{d}(\mu_{2m} + \delta\mu_0)(t)$$

$$= \delta \left(\frac{p_{2m}(z_0)}{\widetilde{p_{2m}}(z_0)} \right)^2 \int \left(\frac{t - z_{2m+1}^{(l)}}{1 - tz_{2m+1}^{(l)}} \right)^2 \frac{\mathrm{d}\mu_0(t)}{t - z_j} = \mathrm{O}(\delta), \quad j = 1, \ldots, 2m,$$

$$F_{2m-1}^{(l)}(z^{(l)}) = \frac{1}{\delta} \int p_{2m}(t) \frac{p_{2m}(t)(t - z_{2m+1}^{(l)})}{(\widetilde{p_{2m}}(t)(1 - tz_{2m+1}^{(l)}))^2} \, \mathrm{d}(\mu_{2m} + \delta\mu_0)(t)$$

$$= \left(\frac{p_{2m}(z_0)}{\widetilde{p_{2m}}(z_0)} \right)^2 \int \frac{t - z_{2m+1}^{(l)}}{(1 - tz_{2m+1}^{(l)})^2} \, \mathrm{d}\mu_0(t) = 0, \tag{44}$$

$l = 0, 1, 2$. In the last equation, definition (21) and Eq. (15) has been used.

In (42) and (43) we have seen that the functional matrices of the three maps $F^{(l)}$, $l = 0, 1, 2$, have a dominant diagonal, and consequently $F^{(l)}$ is invertible in small neighborhoods of the three points $z^{(l)}$, $l = 0, 1, 2$. Hence, there exists $\delta_0 > 0$ such that in each ball $D^{(l)} := \{(x_1, \ldots, x_{2m+1}) \in \mathbb{R}^{2m+1} \mid |x_j - z_j| \leqslant \delta_0,\ j = 1, \ldots, 2m,\ |x_{2m+1} - z_{2m+1}^{(l)}| \leqslant \delta_0\}$, $l = 0, 1, 2$, the map $F^{(l)}$ is injective. From (42)–(44) it follows that $0 = (0, \ldots, 0) \in F^{(l)}(B^{(l)})$ for $\delta > 0$ sufficiently small. Therefore, for each $l = 0, 1, 2$, there exists exactly one solution

$$x^{(l)} = \left(x_1^{(l)}, \ldots, x_{2m+1}^{(l)} \right) := F^{(l)-1}(0) \in D^{(l)}. \tag{45}$$

Let $q^{(l)}$ be the polynomial $\prod_{j=1}^{2m+1}(z - x_j^{(l)})$, $l = 0, 1, 2$. Each of these three polynomials satisfies the orthogonality relation (40) (or equivalently (8)), and together with the first part of the proof we know that for $\delta > 0$ sufficiently small, these are the only polynomials having this property.

It is known (cf. [10, Lemma 6.1.2]) that if $q^{(l)} \in \mathscr{P}_{2m+1}$ satisfies the orthogonality relation (40), then it is the denominator of a rational interpolant $R_{2m+1}^{(l)} = p^{(l)}/q^{(l)}$ that interpolates f_1 in all zeros of $(\widetilde{q^{(l)}})^2$, taking account of multiplicities. We know from [8, Proposition 5], that a rational interpolant $R \in \mathscr{R}_{2m,2m+1}^1$ is a stationary point of the error norm $\| f_1 - R \|$ if, and only if. R interpolates f_1 at the reciprocal of each pole of R with order 2. Thus, each $R_{2m+1}^{(l)}$, $l = 0, 1, 2$, is a stationary point of the error norm $\| f_1 - R_{2m+1}^{(l)} \|$, and even more, the three interpolants $R_{2m+1}^{(l)}$, $l = 0, 1, 2$, are the only rational functions having this property, and therefore the only candidates for rational best approximants to f_1 with degree $(2m, 2m+1)$. Before we determine the error norm for these functions, we make the following observation: Since $\mathrm{supp}(\mu_{2m})$ is symmetric with respect to the origin, it follows from (28) and (39) that the two vectors $z^{(1)} \in \mathbb{R}^{2m+1}$ and $-z^{(2)} \in \mathbb{R}^{2m+1}$ are identical up to permutations of its components, and the same is true for the two vectors $z^{(0)} \in \mathbb{R}^{2m+1}$ and $-z^{(0)} \in \mathbb{R}^{2m+1}$. It is not difficult to conclude from (40) that the same also holds true for the two pairs of vectors $x^{(1)} \in \mathbb{R}^{2m+1}$ and $-x^{(2)} \in \mathbb{R}^{2m-1}$ as well as $x^{(0)} \in \mathbb{R}^{2m+1}$ and $-x^{(0)} \in \mathbb{R}^{2m+1}$, which implies that

$$q^{(1)}(-z) = -q^{(2)}(z) \quad \text{and} \quad q^{(0)}(-z) = -q^{(0)}(z). \tag{46}$$

The calculation of the norm of the approximation error $f_1 - R^{(l)}_{2m+1}$ is based on formula (9). In this formula we have $|\widetilde{q^{(l)}}(z)/q^{(l)}(z)| = 1$ for all $|z| = 1$ and $l = 0, 1, 2$. Hence, it follows that

$$\| f_1 - R^{(l)}_{2m+1} \| = \left\| \int \left(\frac{q^{(l)}(t)}{\widetilde{q^{(l)}}(t)} \right)^2 \frac{\mathrm{d}(\mu_{2m} + \delta\mu_0)(t)}{t - \cdot} \right\|, \quad l = 0, 1, 2. \tag{47}$$

From (38), (30)–(32), and the same arguments as used in (33) it follows that

$$\begin{aligned} \int \left(\frac{q^{(l)}(t)}{\widetilde{q^{(l)}}(t)} \right)^2 \frac{\mathrm{d}(\mu_{2m} + \delta\mu_0)(t)}{t - z} &= \int \left(\frac{q^{(l)}(t)}{\widetilde{q^{(l)}}(t)} \right)^2 \frac{\mathrm{d}\mu_2(t)}{t - z} + \delta \int \left(\frac{q^{(l)}(t)}{\widetilde{q^{(l)}}(t)} \right)^2 \frac{\mathrm{d}\mu_0(t)}{t - z} \\ &= \mathrm{O}(\delta^2) + \delta \left[\left(\frac{p_{2m}(z_0)}{\widetilde{p_{2m}}(z_0)} \right)^2 \int \left(\frac{t - z^{(l)}_{2m+1}}{1 - t z^{(l)}_{2m+1}} \right) \frac{\mathrm{d}\mu_0(t)}{t - z} + \mathrm{O}(\delta) \right], \end{aligned} \tag{48}$$

or

$$\begin{aligned} \| f_1 - R^{(l)}_{2m+1} \| &= \delta \left(\frac{p_{2m}(z_0)}{\widetilde{p_{2m}}(z_0)} \right)^2 \| f_0 - R^{(l)}_1 \| + \mathrm{O}(\delta^2) \\ &= \mathrm{O}(\delta^2) + \delta \left(\frac{p_{2m}(z_0)}{\widetilde{p_{2m}}(z_0)} \right)^2 \begin{cases} \sqrt{\dfrac{3 - z_0^{-2}}{4(1 - z_0^4)}} & \text{for } l = 1, 2, \\ \dfrac{z_0^2}{\sqrt{1 - z_0^4}} & \text{for } l = 0 \end{cases} \end{aligned} \tag{49}$$

as $\delta \to 0$. In (49) the formulae (19) and (20) from the proof of Lemma 6 have been used.

Since $q^{(0)}$ is an old function (cf. (46)), it follows from (9) and properties of f_1 that the rational best approximant $R^{(0)}_{2m+1}$ is also an odd function. We have already earlier discussed that all residua of $R^{(0)}_{2m+1}$ are negative. Hence, we conclude that $R^{(0)}_{2m+1} \in \mathscr{M}^{\mathrm{sym}}_{2m+1}$. From $q^{(1)}(-z) = -q^{(2)}(z)$ we deduce in a similar way that

$$R^{(1)}_{2m+1}(f_1; -z) = -R^{(2)}_{2m+1}(f_1; z). \tag{50}$$

Since $z_0 > \sqrt{\frac{1}{2}}$ has been assumed, it follows from (49) and (50) that

$$\| f_1 - R^{(1)}_{2m+1} \| = \| f_1 - R^{(2)}_{2m+1} \| < \| f_1 - R^{(0)}_{2m+1} \| \tag{51}$$

for $\delta > 0$ sufficiently small. Thus, it has been proved that for $\delta > 0$ sufficiently small f_1 has exactly two different rational best approximants of degree $(2m, 2m+1)$.

The proof of the lemma is completed if we have shown that

$$\mathrm{dist}(f_1, \mathscr{M}^{\mathrm{sym}}_{2m+1}) = \| f_1 - R^{(0)}_{2m+1} \| . \tag{52}$$

In [8, Proposition 5] it has been shown that if one considers a given denominator polynomial $q \in \mathscr{P}_{2m+1}$, then the rational best approximant p/q with this polynomial as denominator is uniquely determined, and so is also the error norm $\| f_1 - p/q \| =: \psi(q)$. The functional ψ has a stationary point if, and only if, the orthogonality relation (40) (or equivalently (8)) is satisfied by q. Let us now assume that $R_{2m+1} = p/q \in \mathcal{M}^{\rm sym}_{2m+1}$ is a minimal element in $\mathcal{M}^{\rm sym}_{2m+1}$ with respect to the norm $\| f_1 - R_{2m+1} \|$. We shall consider small variations $\hat{q} \in \mathcal{P}_{2m+1}$ of the polynomial q. If $\hat{q}(-z) = -\hat{q}(z)$, then the corresponding rational approximant $\hat{R}_{2m+1} = \hat{p}/\hat{q} \in \mathcal{M}^{\rm sym}_{2m+1}$. But if $\hat{q}(-z) \not\equiv -\hat{q}(z)$, then $\hat{q}$ and the polynomial $\hat{\hat{q}}(z) := -\hat{q}(-z)$ are different, and therefore define also two different rational approximants $\hat{R}_{2m-1}$ and $\hat{\hat{R}}_{2m+1}$. Both approximants have the same error norm $\| f_1 - \hat{R}_{2m+1} \| = \| f_1 - \hat{\hat{R}}_{2m+1} \|$. The argumentation is the same here as used for (50). As a consequence we see that if $R_{2m+1} = p/q$ is minimal in the subset $\mathcal{M}^{\rm sym}_{2m+1}$, then this rational function is also a stationary point of the functional ψ in the nonrestricted case. We have seen that the only stationary point of ψ in $\mathcal{M}^{\rm sym}_{2m+1}$ is the approximant $R^{\rm sym}_{2m+1}$, which implies that (52) holds true, and the proof of Lemma 7 is completed. □

We come to the last preparatory result.

Lemma 8. *Assume that* $f_m \in \mathcal{M}^{\rm sym}_{2m}$, $m \in \mathbb{N}$, *and*

$$\rho_m := \operatorname{dist}(f_m, \mathcal{M}^{\rm sym}_{2m-1}) - \operatorname{dist}(f_m, \mathcal{M}_{2m-1}) > 0, \tag{53}$$

then for all Markov functions $f = f(\mu; \cdot)$ *of type* (1) *with symmetric defining measure* μ *satisfying* (2) *and*

$$\| f - f_m \| \leqslant \tfrac{1}{3}\rho_m \tag{54}$$

there exist at least two different rational best approximants $R^{(l)}_{2m-1}(f; \cdot)$, $l = 1, 2$.

Proof. There always exists at least one rational best approximant $R_{2m-1} = R_{2m-1}(f; \cdot)$. We have

$$\begin{aligned}
\| f - R_{2m-1} \| &\leqslant \| f - R_{2m-1}(f_m; \cdot) \| \leqslant \| f - f_m \| + \| f_m - R_{2m-1}(f_m; \cdot) \| \\
&\leqslant \tfrac{1}{3}\rho_m + \operatorname{dist}(f_m, \mathcal{M}_{2m+1}) = \operatorname{dist}(f_m, \mathcal{M}^{\rm sym}_{2m-1}) - \tfrac{2}{3}\rho_m \\
&\leqslant \| f - f_m \| + \operatorname{dist}(f, \mathcal{M}^{\rm sym}_{2m+1}) - \tfrac{2}{3}\rho_m \leqslant \operatorname{dist}(f, \mathcal{M}^{\rm sym}_{2m-1}) - \rho_m.
\end{aligned} \tag{55}$$

Thus, $R_{2m-1} \notin \mathcal{M}^{\rm sym}_{2m-1}$. Set $R^{(1)}_{2m-1}(f; \cdot) := R_{2m-1}$ and $R^{(2)}_{2m-1}(f; z) := -R_{2m-1}(-z)$. Since $R_{2m-1} \notin \mathcal{M}^{\rm sym}_{2m-1}$, we have $R^{(1)}_{2m-1}(f; \cdot) \not\equiv R^{(2)}_{2m-1}(f; \cdot)$. □

Proof of Theorem 4. Let $I = [-a, a]$ be an interval satisfying $a \in (\sqrt{\tfrac{1}{2}}, 1)$. Recursively, we shall select numbers $\delta_j > 0$ and measures $\mu_{0j} := \tfrac{1}{2}(\delta_{-z_{0j}} + \delta_{z_{0j}})$, $z_{0j} \in \left(\sqrt{\tfrac{1}{2}}, 1\right)$, $j = 1, 2, \ldots$, in such a way that the measure

$$\mu := \sum_{j=1}^{\infty} \delta_j \mu_{0j} \tag{56}$$

has the properties stated in Theorem 4.

Set $c_1 := \max_{t \in I} \| (t - \cdot)^{-1} \|$, and assume that δ_j and μ_{0j} have already been fixed for $j = 1, \ldots, m$ with the following three properties:

(i) We have

$$\delta_1 = 1, \quad 0 < \delta_j \leqslant \tfrac{1}{2}\delta_{j-1}, \quad j = 2, \ldots, m. \tag{57}$$

(ii) The measures

$$\mu_{2j} := \sum_{k=1}^{j} \delta_k \mu_{0k}, \quad j = 1, \dots, m, \tag{58}$$

satisfy

$$\operatorname{supp}(\mu_{2j}) \quad \text{contains } 2j \text{ different points}, \tag{59}$$

$$\rho_j := \operatorname{dist}(f(\mu_{2j};\cdot), \mathscr{M}_{2j-1}^{\mathrm{sym}}) - \operatorname{dist}(f(\mu_{2j};\cdot), \mathscr{M}_{2j-1}) > 0, \tag{60}$$

and

$$\delta_j \leqslant \frac{1}{6c_1}\rho_{j-1} \tag{61}$$

for $j = 1, \dots, m$.

(iii) Each Markov function $f = f(\mu_{2j};\cdot)$ has at least two rational best approximants $R_{2j-1}^{(l)}(f;\cdot)$, $l = 1, 2$, $j = 1, \dots, m$.

Lemma 6 shows that the three properties can be satisfied for $m = 1$. From Lemma 7 it follows that if the three properties are satisfied for some $m \geqslant 1$, then it is possible to select δ_{m+1} and $\mu_{0,m+1}$ in such a way that the properties (i)–(iii) are satisfied for $m+1$, which implies that they are satisfied for all $m \in \mathbb{N}$.

Because of (56), (57), and (61) we conclude that for any $m \in \mathbb{N}$ we have

$$\begin{aligned} \| f(\mu;\cdot) - f(\mu_{2m};\cdot) \| &= \| f(\mu - \mu_{2m};\cdot) \| \leqslant \sum_{j=m+1}^{\infty} \delta_j \left\| \frac{1}{z_{0j} - \cdot} \right\| \\ &\leqslant c_1 \sum_{j=m+1}^{\infty} \delta_j \leqslant c_1 2 \delta_{m+1} \leqslant \frac{c_1 2}{6c_1} \rho_m = \frac{1}{3} \rho_m. \end{aligned} \tag{62}$$

From Lemma 8 and property (iii) it then follows that the Markov function $f = f(\mu;\cdot)$ has at least two rational best approximants for each odd degree $2m-1$, $m = 1, 2, \dots$.

Proof of Theorem 5. Let $n_0 = 2m-1$ be chosen arbitrarily and assume that $\tilde{\mu}$ denotes the measure constructed in (56) in the proof of Theorem 4. Let further $I \subseteq (-1,1)$ be an interval that contains $\operatorname{supp}(\tilde{\mu})$. It follows from (62) and Lemma 8 that variations μ of the measure $\tilde{\mu}$ can be chosen so small that the Markov function $f = f(\mu;\cdot)$ has at least two different rational best approximants $R_n^{(l)}(f;\cdot)$, $l = 1, 2$, for each odd index $n = 1, 3, \dots, n_0$. This proves Theorem 5. □

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