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A ROLLE'S THEOREM FOR REAL EXPONENTIAL POLYNOMIALS IN THE COMPLEX DOMAIN

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ABSTRACT. – We present a version of Rolle's theorem for real exponential polynomials having a number L sufficiently large of zeros in a compact set $\mathcal{K}$ of the complex plane. We show that the derivative of the exponential polynomials have at least $L-1$ zeros in a region slightly larger than $\mathcal{K}$. The method of proof is elementary and similar to that of the classical Jensen's theorem about the location of the zeros of the derivative of a real polynomial. The proof also relies on known results concerning the distribution of the zeros of real exponential polynomials. Besides, we display a Rolle's theorem for higher-order derivatives and as a conclusion make a few comments about the maximal number of zeros a real exponential polynomials may have in a given compact set of $\mathbb{C}$. © 2001 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Nous présentons un analogue du théorème de Rolle pour les polynômes exponentiels réels admettant un nombre L suffisamment grand de zéros dans un ensemble compact $\mathcal{K}$ du plan complexe. Nous montrons que la dérivée de ces polynômes exponentiels possède au moins $L-1$ zéros dans une région légèrement plus grande que $\mathcal{K}$. La méthode de démonstration est élémentaire et s'inspire de celle du théorème classique de Jensen sur la distribution des zéros de la dérivée d'un polynôme réel. La démonstration utilise en outre des résultats classiques sur la distribution des zéros des polynômes exponentiels réels. Nous donnons également une version du théorème de Rolle pour les dérivées d'ordre supérieur et en conclusion, quelques remarques sur le nombre maximal de zéros d'un polynôme exponentiel réel dans un ensemble compact de $\mathbb{C}$. © 2001 Éditions scientifiques et médicales Elsevier SAS

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1. Introduction

Exponential polynomials are entire functions g of the form

$$g(z)=\sum_{j=1}^{n} q_j(z)\,\mathrm{e}^{\omega_j z},$$

where the ω_j are complex numbers, usually called the frequencies of g, and the coefficients q_j are complex polynomials. We assume that the ω_j are distinct and the polynomials q_j not zero. We set:

$$m_j=\deg q_j,\quad j=1,\dots,n.$$

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Since the coefficients q_j are uniquely determined by the function g, we can define the *degree* of g as

$$\deg g := n - 1 + \sum_{j=1}^{n} m_j.$$

Properties of exponential polynomials, which are solutions of linear homogeneous differential equations with constant coefficients, are of interest in analysis, number theory or in applications of the type occurring in control theory. The zeros sets and ideals generated by exponential polynomials have been the subject of many studies. We refer to [2,3] and the bibliography therein for an exposition of this subject.

In this paper, we will be interested in studying the zeros of *real exponential polynomials*,

$$g(z) = \sum_{j=1}^{n} p_j(z)\,\mathrm{e}^{\alpha_j z}, \tag{1.1}$$

with *real frequencies*

$$0 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_n,$$

and *real polynomial coefficients* p_j.

Apparently, there does not seem to exist easy generalizations of the classical Rolle's theorem for functions of a complex variable. A few references for this topic are, e.g., [4,7,10]. Here, we derive an analog of Rolle's theorem for exponential polynomials g having $L \geqslant \deg g$ zeros in a compact region $\mathcal{K}$ around the origin. When the degree of g is sufficiently large, we show that at least $L - 1$ zeros of the derivative g' lie in a region $\mathcal{K}'$ slightly larger than the original region $\mathcal{K}$ (for example, if $\mathcal{K}$ is the disk of radius ρ, then $\mathcal{K}'$ consists of the interior of an ellipse whose semi axis have lengths $\sqrt{2}\rho$ and ρ).

Note that the above assertion bears some resemblance with Lucas's theorem, which says that the zeros of the derivative of a polynomial p lie in the convex hull of the ($\deg p$) zeros of p. The proof of our result is elementary. It is adapted from the method used in proving the Jensen theorem for polynomials [6, Theorem 7.1, p. 26], asserting that every non real critical point of a real polynomial p lies in at least one of the circles whose diameters are the line-segments joining the pairs of conjugate zeros of p, the so-called Jensen circles. Applying the argument principle to the logarithmic derivative of our exponential polynomials on a convenient contour and comparing the contributions of the zeros that are interior with those that are exterior to this contour leads to our result. Here, the distribution of the zeros of exponential polynomials plays an important role. In particular, the number of these zeros is known to be bounded above in any given horizontal strips (see Proposition 2.4). The proof of this Rolle's theorem will be the content of Section 2. In Section 3, we show how to derive Rolle's theorem for higher order derivatives and also for a natural analog of them. In Section 4, we make a few comments about the results and suggest an open problem.

2. A complex Rolle's theorem for real exponential polynomials

We will need a few notations. Let ρ, p and l be three positive real numbers. Throughout, T_ρ and $D(0,\rho)$ will respectively denote the circle of radius ρ and the closed disk of radius ρ, both centered at the origin. Also we denote by $\mathcal{E}_{\rho,p}$ the ellipse such that

$$\mathcal{E}_{\rho,p} := \left\{x + \mathrm{i}y\colon\ \frac{x^2}{p} + y^2 = \rho^2\right\},$$

and $\mathcal{F}_{\rho,p}$ the closed interior of this ellipse, that is

$$\mathcal{F}_{\rho,p} := \left\{x + \mathrm{i}y\colon\ \frac{x^2}{p} + y^2 \leqslant \rho^2\right\}.$$

Hence, in particular, $\mathcal{F}_{\rho,1} = D(0,\rho)$. Moreover, we set:

$$\mathcal{L}_{l,\rho,p} := \{x + \mathrm{i}y\colon\ -l \leqslant x \leqslant l,\ \ -\rho \leqslant y \leqslant \rho\} \cup (\mathcal{F}_{\rho,p} - l) \cup (\mathcal{F}_{\rho,p} + l),$$

so that $\mathcal{L}_{l,\rho,p}$ is the bounded strip consisting of the interior of a rectangle of dimensions $2l \times 2\rho$, centered at the origin, whose left and right sides have been replaced with semi-ellipses of half-axis ρ and $\sqrt{p}\rho$. We denote by $\mathcal{K}_{l,\rho,p}$ the closed curve which consists of the boundary of the domain $\mathcal{L}_{l,\rho,p}$.

Moreover, we will use the following notation. For M and N two points in the complex plane, we set:

$$Q(M,N) := \frac{d^2(N,M_0) - d^2(M,M_0)}{d^2(N,M)d^2(N,\overline{M})}, \tag{2.1}$$

where M_0 denotes the projection of M on the real axis $\mathcal{O}x$, $\overline{M}$, the conjugate point of M, i.e. the point symmetric to M with respect to $\mathcal{O}x$ and $d(\cdot,\cdot)$, the usual Euclidean distance between points in the plane.

Finally, we define the *diameter* of a real exponential polynomial (1.1) to be the positive number $\alpha_n - \alpha_1$.

The aim of this section will be to prove the next two theorems:

THEOREM 2.1 (A complex Rolle's theorem). – *Let g be any real exponential polynomial of diameter less than or equal to some fixed positive real number α and let l, ρ and ρ' be three positive real numbers such that $0 < \rho < \rho' < 2\pi/\alpha$. Finally, let $p \geqslant 1$ be some integer. Assume that g has $L \geqslant \deg g$ zeros in the domain $\mathcal{L}_{l,\rho,p}$. Then the derivative of g has at least $L-1$ zeros in the domain $\mathcal{L}_{l,\rho',p+1}$ as soon as*

$$\deg g \geqslant \frac{1}{\rho'^2}\left(1 - \frac{\alpha\rho'}{2}\cot\frac{\alpha\rho'}{2}\right) \max_{\substack{M\in\mathcal{L}_{l,\rho,p}\\ N\in\mathcal{K}_{l,\rho',p+1}}} Q(M,N)^{-1}. \tag{2.2}$$

Remark 1. – From the upper bound in the forthcoming Proposition 2.4 and the assumption $\rho < 2\pi/\alpha$, we see that the integer L can only assume the values $\deg g$ or $\deg g + 1$.

Remark 2. – The assumption $\rho' < 2\pi/\alpha$ has been given because the function

$$\rho' \mapsto 1 - (\alpha\rho'/2)\cot(\alpha\rho'/2)$$

which appears in (2.2) has a singularity at $2\pi/\alpha$. Obviously, the conclusion of Rolle's theorem with respect to a domain $\mathcal{L}_{l,\rho',p+1}$, $\rho' \geqslant 2\pi/\alpha$, follows if one can apply the theorem on a smaller domain with $\rho' < 2\pi/\alpha$.

In order to make more explicit the maximum in the right-hand side of (2.2), we state the following proposition:

PROPOSITION 2.2. – *Let p be a positive integer and let l, ρ, ρ' be three positive real numbers with $\rho < \rho'$. Then, for any $M \in \mathcal{L}_{l,\rho,p}$ and any $N \in \mathcal{K}_{l,\rho',p+1}$, we have*:

$$Q(M,N) \geqslant \min\left(\frac{\rho'^2-\rho^2}{(\rho'^2+\rho^2)^2}, \frac{(\rho'+\sqrt{2}l)^2-\rho^2}{(2(\rho'+\sqrt{2}l)^2-\rho^2)^2}\right), \quad \textit{if } p=1, \tag{2.3}$$

and

$$Q(M,N) \geqslant \min\left(\frac{\rho'^2-\rho^2}{(\rho'^2+\rho^2)^2}, \frac{1}{(\sqrt{p+1}\rho'+\sqrt{p}\rho+2l)^2}\right), \quad \textit{if } p\geqslant 2. \tag{2.4}$$

Remark 1. – Actually, a weaker condition than that in (2.2) may be sufficient in order to apply Rolle's theorem. Indeed, denote by $S(\rho,\rho')$ the expression in the right-hand side of (2.2) where we replace the maximum of the $Q(M,N)^{-1}$'s with the upper bound deduced from Lemma 2.2. Note that $S(\rho,\rho')$ tends to infinity when ρ' tends to ρ and to $2\pi/\alpha$. In the segment $(\rho, 2\pi/\alpha)$, it meets a minimum S for some particular value ρ'_0 of ρ' and then increases up to infinity at $2\pi/\alpha$. Since the conclusion of Rolle's theorem holds in $\mathcal{L}_{l,\rho'_0,p+1}$ as soon as $\deg g \geqslant S$, it holds *a fortiori* in $\mathcal{L}_{l,\rho',p+1}$, $\rho' \geqslant \rho'_0$, as soon as $\deg g \geqslant S$. Consequently, one can replace the expression $S(\rho,\rho')$ in the right-hand side of (2.2) with $\inf_{u'\in(\rho,\rho']} S(\rho,u')$.

Remark 2. – The minimum in (2.3) is smaller than or equal to the minimum in (2.4), when $p=1$. In particular, with the proof to be given here, inequality (2.4) does not extend to the case $p=1$.

Remark 3. – When $\varepsilon = \rho'-\rho$ tends to 0, the degree of g needed in order to apply Theorem 2.1 becomes of order C/ε, where C is a constant depending only on ρ and α.

Examples. – Assume g is a real exponential polynomial of diameter 1 having $\deg g \geqslant 2$ zeros in the circle T_1, so that with the notations of Theorem 2.1 $\rho=1$, $l=0$, $\alpha=1$. One checks that, when replacing the maximum in the expression $S(1,\rho')$ in the right-hand side of (2.2) by the upper bound deduced from (2.3), one obtains a quantity which is decreasing when ρ' increases from 1 to $\sqrt{2}$ and then increasing up to infinity at 2π. The minimum value at $\sqrt{2}$ is approximatively equal to 0.776. Moreover, the expression $S(1,\rho')$ has value 2 for $\rho'=\rho'_0 \simeq 1.098$. Hence, Rolle's theorem holds for the polynomial g in the domain $\mathcal{L}_{0,\rho'_0,2}$ (and a fortiori in any domain $\mathcal{L}_{0,\rho',2}$, $\rho' \geqslant \rho'_0$) without any assumption on its degree. If one wants to apply Rolle's theorem in smaller domains $\mathcal{L}_{0,\rho',2}$, e.g. $\rho'=1.05$ or $\rho'=1.01$, condition (2.2) then translates to $\deg g \geqslant 4$ or $\deg g \geqslant 18$, respectively.

In Fig. 1, we have graphed the zeros (denoted with circles "o") in the strip $-\mathrm{i}\pi \leqslant y \leqslant \mathrm{i}\pi$ of the ninth derivatives of the exponential polynomials $g_1(z) = \mathrm{e}^{5z} - P_{19}(z)$, $\deg P_{19} = 19$, and $g_2(z) = Q_9(z)\,\mathrm{e}^{5z} - P_{10}(z)$, $\deg Q_9 = 9$, $\deg P_{10} = 10$, Q_9 monic, such that

$$g_1^{(k)}(\mathrm{i}\pi) = g_1^{(k)}(-\mathrm{i}\pi) = g_2^{(k)}(\mathrm{i}\pi) = g_2^{(k)}(-\mathrm{i}\pi) = 0, \quad k=0,\dots,9. \tag{2.5}$$

Hence, g_1 and g_2, which are of degree 20, are uniquely determined by (2.5). Moreover, the derivatives $g_1^{(9)}$ and $g_2^{(9)}$ both admit a simple zero at $\mathrm{i}\pi$ and $-\mathrm{i}\pi$. In Figure 2, we have also graphed the zeros in the strip $-\mathrm{i}\pi \leqslant y \leqslant \mathrm{i}\pi$ of the ninth derivative of the exponential polynomial $g_3(z) = Q_{19}(z)\,\mathrm{e}^{5z} - 1$, $\deg Q_{19} = 19$ such that

$$g_3^{(k)}(\mathrm{i}\pi) = g_3^{(k)}(-\mathrm{i}\pi) = 0, \quad k=0,\dots,9.$$

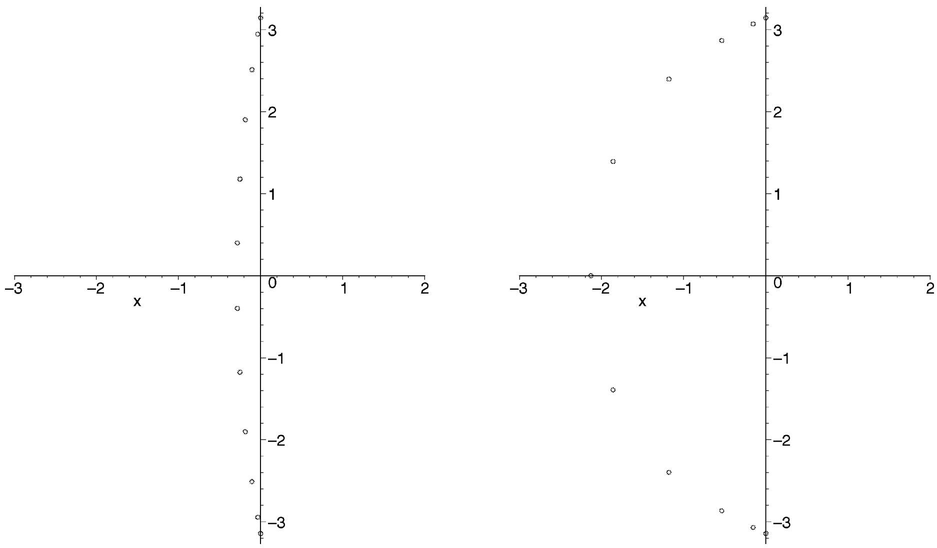

Fig. 1. The zeros of the ninth derivatives of g_1 and g_2 in the strip $-\pi \leqslant y \leqslant \pi$.

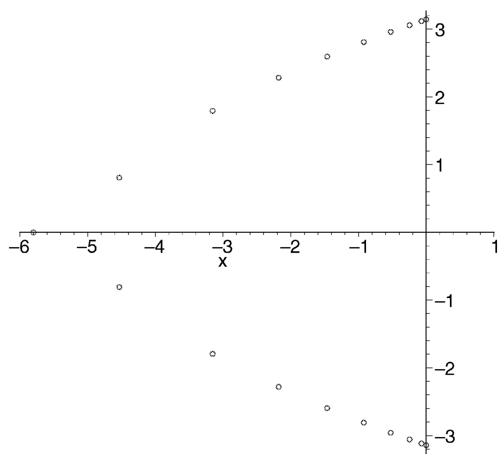

Fig. 2. The zeros of the ninth derivative of g_3 in the strip $-\pi \leqslant y \leqslant \pi$.

On these three examples, we observe that the zeros of the derivatives deviate from the segment $[-\mathrm{i}\pi, \mathrm{i}\pi]$ and on the last one, also leave the circle of radius π.

Theorem 2.1 may be generalized to a domain $\mathcal{L}_{l,\rho,p}$ of arbitrary height ρ. Indeed, we have:

THEOREM 2.3. – *Let α, l and $\rho < \rho'$ be four real positive numbers, and let $a \in \mathbb{N}$ and $p \geqslant 1$ be integers. There exists an integer $C(l, \rho, \rho', a, p, \alpha)$ such that for any real exponential polynomial g of diameter less than or equal to α having at least L, $L \geqslant \deg g - a$, zeros in the domain $\mathcal{L}_{l,\rho,p}$, the derivative of g has at least $L - 1$ zeros in the domain $\mathcal{L}_{l,\rho',p+1}$, as soon as the degree of g is larger than $C(l, \rho, \rho', a, p, \alpha)$.*

Remark. – Theorem 2.3 is more general than Theorem 2.1 because here we allow a domain $\mathcal{L}_{l,\rho,p}$ of arbitrary height ρ but also because the number L of zeros of the exponential polynomial

g has in $\mathcal{L}_{l,\rho,p}$ is allowed to be less than the degree of g, up to some fixed constant. Of course, since we do not compute an explicit expression for the integer $C(l,\rho,\rho',a,p,\alpha)$, it is also less precise than Theorem 2.1.

Before proving the above theorems, let us begin with displaying some known results concerning the zeros of exponential polynomials. The following proposition was alluded to in the introduction. It gives estimates on the number of zeros an exponential polynomial may have in horizontal strips (cf. [9, Problem 206.2]).

PROPOSITION 2.4. – *Let g be a real exponential polynomial of diameter α and let $N(g,a,b)$ the number of zeros of g that are contained in the horizontal strip $a \leqslant \operatorname{Im} z \leqslant b$, we have*:

$$\alpha\frac{b-a}{2\pi} - \deg g \leqslant N(g,a,b) \leqslant \alpha\frac{b-a}{2\pi} + \deg g. \tag{2.6}$$

Idea of proof. – The derivation of these inequalities consists in applying the argument principle on a rectangle $\{z\colon a \leqslant \operatorname{Im} z \leqslant b, -c \leqslant \operatorname{Re} z \leqslant c\}$, and then let c tend to ∞. □

We proceed with recalling the Polya–Dickson theorem (cf. [2,3]), restricting ourselves to the real frequencies case. Following [2], we set a few notations. Let P_j, $j = 0,\dots,n$, be the points in the plane with coordinates (α_j, m_j) where α_j and m_j have been defined in Section 1 and let L be the upper convex envelope of these points. In other words, L is the polygonal line which joins P_0 to P_n, has vertices only at points of the set $\{P_j\}$, no points P_j lie above it and the domain below L is convex. Let the successive segments of L be denoted by $L_1,\dots,L_k$, and let the slope of L_r be denoted by μ_r (from the definition of L, the slope μ_r is a decreasing function of r). For $c > 0$, we consider the curvilinear strips V_r defined by:

$$V_r = \left\{z \in \mathbb{C}\colon \left|\operatorname{Re}(z + \mu_r \log z)\right| \leqslant c\right\}, \quad r = 1,\dots,k.$$

Note that the strips are disjoint, for large $|z|$, and that V_{r+1} lies to the right of V_r.

PROPOSITION 2.5. – *Let g be an exponential polynomial as in* (1.1), *possibly with complex coefficients. Outside a certain disk $\{z\colon |z| \leqslant c_2\}$, the following assertions hold true*:

(i) *All zeros of g are contained in the union of the regions V_r.*

(ii) *In any region R,*

$$\left|\operatorname{Re}(z + \mu_r \log z)\right| \leqslant c, \qquad \left|\operatorname{Im}(z + \mu_r \log z) - a\right| \leqslant b,$$

with no zeros of g on the boundary, the number of zeros in R satisfies

$$(b/\pi)\alpha + 1 - n_r \leqslant n(R) \leqslant (b/\pi)\alpha - 1 + n_r, \tag{2.7}$$

where n_r is the number of frequencies lying in the segment L_r and α is the difference in values of the frequencies at the end-points of L_r.

(iii) *In any strip V_r with $\mu_r \neq 0$, the zeros of g lie asymptotically along a finite number of curves $|z^{\mu_r} \mathrm{e}^z| = |w^{\mu_r}|$, w in the set of roots of an algebraic equation determined by the points on L_r. Clearly, these curves are symmetric with respect to the real axis and if $z = x + \mathrm{i}y$ is a point lying on one of these curves, then $|y/x| \to +\infty$ as $|z| \to +\infty$. Moreover, the zeros $z = x + \mathrm{i}y$ of large modulus accumulating near $|z^{\mu_r} \mathrm{e}^z| = |w^{\mu_r}|$ are described by the following asymptotic formulas. Let l be any large integer such that*

$l\mu_r > 0$. *Then, for z lying in the upper-half plane, we have* $z = z_l = x_l + \mathrm{i}y_l$, *with*

$$x_l = \mu_r\bigl(\log|w| - \log|\mu_r \arg w + 2l\mu_r\pi - \mu_r\pi/2|\bigr) + \mathrm{o}(1)$$

and

$$y_l = \mu_r(\arg w + 2l\pi - \pi/2) + \mathrm{o}(1),$$

whereas, for z lying in the lower-half plane, we have $z = z_{-l} = x_{-l} + \mathrm{i}y_{-l}$, *with*

$$x_{-l} = \mu_r\bigl(\log|w| - \log|\mu_r \arg w - 2l\mu_r\pi + \mu_r\pi/2|\bigr) + \mathrm{o}(1)$$

and

$$y_{-l} = \mu_r(\arg w - 2l\pi + \pi/2) + \mathrm{o}(1).$$

Remark. – When the exponential polynomial has real coefficients, the algebraic equation alluded to in assertion (iii) is real as well, so that the points z_l and z_{-l} corresponding to the roots w and $\overline{w}$ respectively, are conjugate roots of g.

From these results we may derive a precise form of the Hadamard factorization of a real exponential polynomial.

PROPOSITION 2.6. – *Assume that $g(z)$ is a real exponential polynomial vanishing at the origin with a multiplicity n_0, $n_0 \geqslant 0$, and let Z denote the set of zeros of the entire function $g(z)/z^{n_0}$. Let N be a fixed large positive integer, we decompose Z into an union of a finite subsets $Z_0 = \{z_i\}$ of roots of small modulus and an infinite subsets of roots $(z_l)_{l=\pm N,\ldots}$, given by asymptotic expressions as in Proposition* 2.5, *assertion* (iii). *Then, $g(z)$ admits the following representation*:

$$g(z) = cz^{n_0}\,\mathrm{e}^{az} \prod_{z_i \in Z_0}\left(1 - \frac{z}{z_i}\right) \prod_{l \geqslant N}\left(1 - \frac{z}{z_l}\right)\left(1 - \frac{z}{z_{-l}}\right), \tag{2.8}$$

where the infinite product converges and where a and c are some real constants.

Proof. – Since g is an entire function of order 1, the Hadamard factorization theorem tells us that

$$g(z) = z^{n_0}\,\mathrm{e}^{h(z)} \prod_{z_i \in Z_0}\left(1 - \frac{z}{z_i}\right) \prod_{l=\pm N,\ldots}\left(1 - \frac{z}{z_l}\right)\mathrm{e}^{z/z_l}, \tag{2.9}$$

where $h(z)$ is a polynomial of degree less than or equal to 1. In the last product, we bracket together the factors corresponding to indices l and $-l$ and show that the series

$$S = \sum_{l \geqslant N}\left(\frac{1}{z_l} + \frac{1}{z_{-l}}\right)$$

converges. Indeed, when z_l, l large, lies in a strip V_r with $\mu_r \neq 0$, we know from the expressions in assertion (iii) of Proposition 2.5, that:

$$z_l = -\mu_r \log|2l\mu_r\pi| + 2\mathrm{i}l\mu_r\pi + \mathrm{O}(1),$$

$$z_{-l} = -\mu_r \log|2l\mu_r\pi| - 2\mathrm{i}l\mu_r\pi + \mathrm{O}(1).$$

Thus, the general term of S satisfies

$$\frac{1}{z_l}+\frac{1}{z_{-l}}=\frac{-2\mu_r\log|2l\mu_r\pi|}{(2l\mu_r\pi)^2}\bigl(1+\mathrm{o}(1)\bigr),$$

which implies the convergence of the series. When $\mu_r=0$, the result remains true since the zeros are in $|\mathrm{Re}(z)|\leqslant c$ and by the estimates (2.7), they lie along the imaginary axis with a constant density, equal to $\alpha/2\pi$.

Hence, the exponential factors e^{z/z_l} are not necessary to the convergence of the canonical product in the right-hand side of (2.9). Taking them out of this product and collecting them together with the polynomial h leads to the asserted expression for g. Finally, the constants a and c are real since g is. □

Remark. – The representation (2.8) of g can also be deduced from the fact that g is a function of completely regular growth, see [5, Chapter III].

In the sequel, we shall need the following simple geometric lemma:

LEMMA 2.7. – *The circles whose diameters are the vertical chords of the ellipse $\mathcal{E}_{\rho,p}$ lie in the closed interior of the ellipse $\mathcal{E}_{\rho,p+1}$ and have this ellipse as their envelope. Evidently, this implies that the circles whose diameters are the vertical chords of the domain $\mathcal{L}_{l,\rho,p}$ lie in the domain $\mathcal{L}_{l,\rho,p+1}$ and have $\mathcal{K}_{l,\rho,p+1}$ as their envelope.*

Proof. – Easily verified by elementary calculus. □

Proof of Proposition 2.2. – Let (x,y) and (u,v) be the coordinates in the complex plane of N and M respectively. Then, from the definition (2.1) of $Q(M,N)$, we deduce

$$Q(M,N)=\frac{[(x-u)^2+y^2-v^2]}{[(x-u)^2+(y-v)^2][(x-u)^2+(y+v)^2]}. \tag{2.10}$$

We have

$$\bigl[(x-u)^2+(y-v)^2\bigr]\bigl[(x-u)^2+(y+v)^2\bigr]\leqslant\bigl[(x-u)^2+y^2+v^2\bigr]^2.$$

Hence, setting

$$X:=(x-u)^2+y^2,\qquad Y:=v^2, \tag{2.11}$$

we get

$$Q(M,N)\geqslant\frac{X-Y}{(X+Y)^2}.$$

Let $F(X,Y):=(X-Y)/(X+Y)^2$. Since

$$\frac{\partial F}{\partial X}=\frac{3Y-X}{(X+Y)^3},\qquad \frac{\partial F}{\partial Y}=\frac{Y-3X}{(X+Y)^3}, \tag{2.12}$$

F has no local minimum and the magnitude of F must be studied on the boundary of the domain in $\mathbb{R}^2$ consisting of those points (X,Y) such that (2.11) holds with $(u,v)\in\mathcal{L}_{l,\rho,p}$ and $(x,y)\in\mathcal{K}_{l,\rho',p+1}$. Assume Y has a fixed value v^2. Let M be the point on the boundary $\mathcal{K}_{l,\rho,p}$ of coordinates (u,v) with $u\geqslant 0$ (so that $v=l$ when $0\leqslant u\leqslant l$ and $(u-l)^2/p=\rho^2-v^2$ when $u\geqslant l$), and M_0 its projection on the real axis. Since N lies on the curve $\mathcal{K}_{l,\rho',p+1}$, X can only range from the square of the minimal distance between M_0 and $\mathcal{K}_{l,\rho',p+1}$ to the square of their maximal distance. This maximal distance evidently equals $\sqrt{p+1}\rho'+l+u$ and it is also straightforward to check that the minimal distance equals ρ' if $u\leqslant l$, $\sqrt{\rho'^2-(u-l)^2/p}$ if $l\leqslant u\leqslant l+p\rho'/\sqrt{p+1}$ and $\sqrt{p+1}\rho'+l-u$ otherwise. We thus have to study the minimum of the following three values of F:

$$F\big((\sqrt{p+1}\rho'+l+u)^2,\rho^2-(u-l)^2/p\big),\qquad F\big(\rho'^2-(u-l)^2/p,\rho^2-(u-l)^2/p\big),$$

and

$$F\big((\sqrt{p+1}\rho'+l-u)^2,\rho^2-(u-l)^2/p\big),$$

as u ranges from l to $l+\sqrt{p}\rho$ on one hand, and the minimum of the following two values of F:

$$F\big((\sqrt{p+1}\rho'+l+u)^2,\rho^2\big)\quad\text{and}\quad F\big(\rho'^2,\rho^2\big),$$

as u ranges from 0 to l, on the other hand. Since

$$(\sqrt{p+1}\rho'+l-u)^2-\big(\rho'^2-(u-l)^2/p\big)=\big(\sqrt{p+1}(u-l)-p\rho'\big)^2/p\geqslant 0,$$

we have

$$\rho'^2-(u-l)^2/p\leqslant(\sqrt{p+1}\rho'+l-u)^2\leqslant(\sqrt{p+1}\rho'+l+u)^2,$$

as u ranges from l to $l+\sqrt{p}\rho$, and also

$$\rho'^2<(\sqrt{p+1}\rho'+l)^2\leqslant(\sqrt{p+1}\rho'+l+u)^2\leqslant(\sqrt{p+1}\rho'+2l)^2,$$

as u ranges from 0 to l. Since F increases then decreases as a function of X, we only have to look for the minimums of

$$F\big((\sqrt{p+1}\rho'+l+u)^2,\rho^2-(u-l)^2/p\big)=p\frac{p(\sqrt{p+1}\rho'+l+u)^2+(u-l)^2-p\rho^2}{(p(\sqrt{p+1}\rho'+l+u)^2-(u-l)^2+p\rho^2)^2}\tag{2.13}$$

and

$$F\big(\rho'^2-(u-l)^2/p,\rho^2-(u-l)^2/p\big)=\frac{\rho'^2-\rho^2}{(\rho'^2-2(u-l)^2/p+\rho^2)^2},\tag{2.14}$$

as u ranges from l to $l+\sqrt{p}\rho$. One can check that if p is distinct from 1 or if $p=1$ and $\rho'+\sqrt{2}l\geqslant\sqrt{2}\rho$ then (2.13) decreases as u increases. Thus, assuming $\rho'+\sqrt{2}l\geqslant\sqrt{2}\rho$ when $p=1$, the minimum of (2.13) is

$$(\sqrt{p+1}\rho'+\sqrt{p}\rho+2l)^{-2}.$$

When $p=1$ and $\rho'+\sqrt{2}l<\sqrt{2}\rho$, the minimum of (2.13) as $u\in[l,l+\sqrt{p}\rho]$ is either

$$\frac{2(\rho'+\sqrt{2}l)^2-\rho^2}{(2(\rho'+\sqrt{2}l)^2+\rho^2)^2}\quad\text{or}\quad\frac{(\rho'+\sqrt{2}l)^2-\rho^2}{(2(\rho'+\sqrt{2}l)^2-\rho^2)^2},$$

depending whether $\rho' + \sqrt{2}l \leqslant \sqrt{3/2}\rho$ or $\rho' + \sqrt{2}l \geqslant \sqrt{3/2}\rho$. Since the last value is the smallest one among the three, we shall consider this one when $p = 1$.

As for (2.14), it is clear that its minimum equals

$$\frac{\rho'^2 - \rho^2}{(\rho'^2 + \rho^2)^2},$$

as u ranges from l to $l + \sqrt{p}\rho$, and this finishes the proof of Proposition 2.2. □

Proof of Theorem 2.1. – To establish this result, we shall adapt the classical proof of Jensen's theorem (see, e.g., [6, Theorem 7.1]). This theorem asserts that the zeros of the derivative of a real polynomial lie into certain circles, usually called Jensen circles, that are determined by the roots of the polynomial. Denoting by Z the set of roots z_j of g, we know from Proposition 2.6 that

$$g(z) = cz^{n_0}\,\mathrm{e}^{az} \prod_{\substack{z_j \in Z \cap \mathcal{L}_{l,\rho,p} \\ z_j \neq 0}} \left(1 - \frac{z}{z_j}\right) \prod_{z_j \in Z \setminus Z \cap \mathcal{L}_{l,\rho,p}} \left(1 - \frac{z}{z_j}\right), \tag{2.15}$$

where a and c are some real constants and the roots of g are symmetric with respect to the real axis since g is real. Moreover, by Proposition 2.4 and the assumption that g is real and has $\deg g$ zeros in the domain $\mathcal{L}_{l,\rho,p}$, $0 < \rho < 2\pi/\alpha$, we know that the other zeros of g actually lie outside the horizontal strip $-2\pi/\alpha < \operatorname{Im} z < 2\pi/\alpha$, except for one possible extra real zero that we shall denote by z_0. From the factorization (2.15), the logarithmic derivative of g is equal to

$$g'(z)/g(z) = a + \delta/(z - z_0) + \sum_{z_j \in \mathcal{L}_{l,\rho,p}} 1/(z - z_j) + \sum_{|\operatorname{Im} z_j| \geqslant 2\pi/\alpha} 1/(z - z_j), \tag{2.16}$$

where δ equals 0 or 1, depending whether z_0 does exist or not. The real zeros of g all lie in $\mathcal{L}_{l,\rho,p}$, except the possible extra zero z_0. The term $1/(x + \mathrm{i}y - x_j)$ in (2.16) corresponding to a real zero $z_j = x_j$ and $z = x + \mathrm{i}y$ has the imaginary part

$$-y/\big[(x - x_j)^2 + y^2\big]. \tag{2.17}$$

Remark that the sign of (2.17) is always opposite to the sign of y.

The sum of terms $1/(x + \mathrm{i}y - x_j - \mathrm{i}y_j)$ and $1/(x + \mathrm{i}y - x_j + \mathrm{i}y_j)$ corresponding to the pair of conjugate zeros $z_j = x_j + \mathrm{i}y_j$ and $\bar{z}_j = x_j - \mathrm{i}y_j$ has the imaginary part

$$\frac{-2y[(x - x_j)^2 + y^2 - y_j^2]}{[(x - x_j)^2 + (y - y_j)^2][(x - x_j)^2 + (y + y_j)^2]}. \tag{2.18}$$

Denote by $\mathcal{C}_j$ the Jensen circle of the pair of zeros z_j and $\bar{z}_j$, that is the circle whose diameter is the segment joining z_j to $\bar{z}_j$. If the point $z = x + \mathrm{i}y$ lies outside $\mathcal{C}_j$, the sign of (2.18) and the sign of y are opposite, whereas if z lies inside $\mathcal{C}_j$, the signs are equal. Now, consider some bounded contour $\mathcal{C}$ that encloses all Jensen circles corresponding to roots of g that are located into the domain $\mathcal{L}_{l,\rho,p}$. Then, for z on $\mathcal{C}$, the sign of expression (2.18) corresponding to roots into $\mathcal{L}_{l,\rho,p}$ is opposite to the sign of y. When (2.18) corresponds to roots lying outside the strip $|\operatorname{Im} z| < 2\pi/\alpha$, its sign may be equal or opposite to the sign of y, depending whether the chosen point on $\mathcal{C}$ lies inside or outside the Jensen circle. Though this will not be needed in the sequel, note that, from Proposition 2.5, the zeros of large modulus of g asymptotically lie near the imaginary axis. Hence, their Jensen circle completely encloses the contour $\mathcal{C}$ which means that, in this case, expression (2.18) has the sign of y everywhere on $\mathcal{C}$.

Next, we shall prove that, when summing up terms in (2.16), the contribution from the first sum (taken over zeros $z_j \in \mathcal{L}_{l,\rho,p}$) to the imaginary part of $g'(z)/g(z)$ becomes, uniformly on $\mathcal{C}$, larger than the contribution of the second sum (taken over zeros with $|\mathrm{Im}\, z_j| \geqslant 2\pi/\alpha$) as the degree of g exceeds some explicitly given bound. Here, note that the possible extra fraction $\delta/(z-z_0)$ can be neglected, since its contribution only adds to the first sum.

From Lemma 2.7, we know that the convex hull of the Jensen circles whose diameters are the vertical chords of the domain $\mathcal{L}_{l,\rho,p}$ is the curve $\mathcal{K}_{l,\rho,p+1}$. Thus, one can choose as a contour $\mathcal{C}$ the curve $\mathcal{K}_{l,\rho',p+1}$, $\rho<\rho'<2\pi/\alpha$, surrounding $\mathcal{K}_{l,\rho,p+1}$ and contained in the strip $|\mathrm{Im}\, z| < 2\pi/\alpha$. For z a point on $\mathcal{K}_{l,\rho',p+1}$, we give an upper bound for the modulus of

$$\begin{aligned} I &:= \sum_{|\mathrm{Im}\, z_j| \geqslant 2\pi/\alpha} \mathrm{Im}\big(1/(z-z_j)\big) y^{-1} \\ &= -\sum_{y_j \geqslant 2\pi/\alpha} \frac{2[(x-x_j)^2+y^2-y_j^2]}{[(x-x_j)^2+(y-y_j)^2][(x-x_j)^2+(y+y_j)^2]} \end{aligned} \tag{2.19}$$

and a lower bound for the modulus of

$$\begin{aligned} J &:= \sum_{z_j \in \mathcal{L}_{l,\rho,p}} \mathrm{Im}\big(1/(z-z_j)\big) y^{-1} \\ &= -\sum_{0 \leqslant y_j \leqslant \rho} \frac{\delta_{z_j}[(x-x_j)^2+y^2-y_j^2]}{[(x-x_j)^2+(y-y_j)^2][(x-x_j)^2+(y+y_j)^2]}, \end{aligned} \tag{2.20}$$

where δ_{z_j} equals 1 or 2 depending whether z_j is real or complex. First, let us consider I. We have

$$\big[(x-x_j)^2+y^2-y_j^2\big]^2 \leqslant \big[(x-x_j)^2+(y-y_j)^2\big]\big[(x-x_j)^2+(y+y_j)^2\big].$$

Thus

$$|I| \leqslant \sum_{y_j \geqslant 2\pi/\alpha} \frac{2}{[(x-x_j)^2+(y-y_j)^2]^{1/2}[(x-x_j)^2+(y+y_j)^2]^{1/2}} \leqslant \sum_{y_j \geqslant 2\pi/\alpha} \frac{2}{y_j^2-y^2}. \tag{2.21}$$

From Proposition 2.4, we know that roots of g cannot accumulate in any bounded horizontal strip. More precisely, from the upper bound in (2.6), roots z_j of g with $|y_j| \geqslant 2\pi/\alpha$ are spread along the imaginary axis with a density at most $\alpha/2\pi$. Hence, the last sum in (2.21) cannot be larger than

$$2\left(\frac{1}{(2\pi/\alpha)^2-\rho'^2}+\frac{1}{(4\pi/\alpha)^2-\rho'^2}+\cdots\right) = \frac{\alpha}{2\pi\rho'}\left[\psi\left(\frac{\alpha\rho'}{2\pi}\right)-\psi\left(-\frac{\alpha\rho'}{2\pi}\right)\right]+\frac{2}{\rho'^2}, \tag{2.22}$$

where ψ denotes the logarithmic derivative of the Gamma function Γ, or Digamma function:

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{z+n}\right),$$

and γ denotes the usual Euler constant. Using the recurrence relationship and the reflection formula for ψ,

$$\psi(1+z)=\psi(z)+1/z, \qquad \psi(1-z)=\psi(z)+\pi\cot\pi z,$$

respectively, we get for (2.22) the expression

$$\frac{1}{\rho'^2}-\frac{\alpha}{2\rho'}\cot\left(\frac{\alpha\rho'}{2}\right),$$

which implies

$$|I|\leqslant\frac{1}{\rho'^2}\left(1-\frac{\alpha\rho'}{2}\cot\frac{\alpha\rho'}{2}\right). \tag{2.23}$$

Second, we give a lower bound for $|J|$. From the definitions (2.1) of $Q(M,N)$ and (2.20) of J, we get

$$|J|\geqslant\min_{\substack{M\in\mathcal{L}_{l,\rho,p}\\ N\in\mathcal{K}_{l,\rho',p+1}}}Q(M,N)\deg g. \tag{2.24}$$

From (2.23) together with (2.24), we see that a necessary condition to ensure $|J|>|I|$ is

$$\min_{\substack{M\in\mathcal{L}_{l,\rho,p}\\ N\in\mathcal{K}_{l,\rho',p+1}}}Q(M,N)\deg g\geqslant\frac{1}{\rho'^2}\left(1-\frac{\alpha\rho'}{2}\cot\frac{\alpha\rho'}{2}\right). \tag{2.25}$$

Assume that the degree of g satisfies the previous inequality. Then, from (2.16), the discussion after (2.18) and the definitions of I and J, we know that the imaginary part of $g'(z)/g(z)$ is negative as z describes the upper-half of $\mathcal{K}_{l,\rho',p+1}$ and positive as z describes the lower-half of $\mathcal{K}_{l,\rho',p+1}$. Hence, $\mathcal{K}_{l,\rho',p+1}$ is mapped by the function $g'(z)/g(z)$ into a curve which encircles the origin at most once. Thus, by the argument principle, the number of zeros of $g'(z)$ within $\mathcal{K}_{l,\rho',p+1}$ differs by at most one from the number of zeros of $g(z)$ in $\mathcal{K}_{l,\rho',p+1}$. This implies the assertion of Theorem 2.1. □

Proof of Theorem 2.3. – We shall apply the same idea as in the proof of Theorem 2.1 but, now, we have to be careful about the extra zeros that g may have near the curve $\mathcal{K}_{l,\rho',p+1}$. We determine some contour $\mathcal{C}=\mathcal{K}_{l,\rho_0,p+1}$, $\rho+\varepsilon\leqslant\rho_0\leqslant\rho'$, $0<\varepsilon<\rho'-\rho$, surrounding the curve $\mathcal{K}_{l,\rho,p+1}$, with $\rho_0-\rho$ larger than some fixed constant ε, in order that a lower bound for $|J|/L$ exists. Also, we need that the distance from $\mathcal{K}_{l,\rho_0,p+1}$ to zeros of g that do not belong to $\mathcal{L}_{l,\rho,p}$ remains bounded away from zero in order that an upper bound for $|I|$ exists. This can be achieved as follows. From Proposition 2.4, the horizontal strip

$$\mathcal{H}:=\{z\colon\ |\operatorname{Im}z|\leqslant l\pi/\alpha\},\quad l=[\rho'\alpha/\pi]+1$$

($[x]$ denoting the integral part of x), contains at most $\deg g+l$ zeros of g, that is, $\mathcal{H}$ contains at most $l+\deg g-L$ extra zeros in addition to the $L\geqslant\deg g-a$ zeros of g in $\mathcal{L}_{l,\rho,p}$. One can always determine some ρ_0, $\rho+\varepsilon\leqslant\rho_0\leqslant\rho'$, such that the curve $\mathcal{K}_{l,\rho_0,p+1}$ is at least at a distance $(\rho'-\rho-\varepsilon)/2(l+\deg g-L)$ from the $l+\deg g-L$ possible extra zeros of g in $\mathcal{L}_{l,\rho',p+1}\setminus\mathcal{L}_{l,\rho,p}$. Since this distance is bounded below and the number of extra zeros is bounded above independently from the degree of g, the quantity $|I|$ still admits an absolute upper bound depending only on l, ρ, ρ' and α. Thus, assuming $\deg g$ large enough, the argument principle can be applied to the function $g'(z)/g(z)$ as in the last part of the proof of Theorem 2.1, eventually showing that g' admits at least $L-1$ zeros in $\mathcal{L}_{l,\rho_0,p+1}$, hence in $\mathcal{L}_{l,\rho',p+1}$. □

3. A complex Rolle's theorem for higher-order derivatives

In this section we shall prove results similar to those established in Theorems 2.1, 2.3, now considering a real exponential polynomial g and derivatives whose order can possibly grow up to the degree of g. Actually, we shall consider derivatives of g and also slight modifications of them, which consist in taking the derivative of the quotient of $g(z)$ by $\exp(\alpha z)$, where α is the smallest frequency of g. We shall denote by $g^{\{1\}}$, the exponential polynomial, image of g by this transformation, and similarly $g^{\{k\}}$, k integer, for the iterates of this transformation. From the definition, if zero is a frequency of g, then $g^{\{1\}}$ equals the usual derivative $g^{(1)}$ of g. Note that, contrary to the derivative, the degree of $g^{\{1\}}$ is always one less than the degree of g. With this definition at hand, we are in a position to state our result. As a first step, we only consider derivatives of a fixed order, while the degree of the exponential polynomial g goes large.

THEOREM 3.1. – *Let α, l and $\rho < \rho'$ be four real positive numbers, and let r be a positive integer. There exists an integer $C(l,\rho,\rho',\alpha,r)$ such that for any real exponential polynomial g of diameter less than or equal to α, of degree $\deg g \geqslant r+1$, and having L zeros in the domain $\mathcal{L}_{l,\rho,1}$ with $L \geqslant \deg g$, the exponential polynomials $g^{(r)}$ and $g^{\{r\}}$ have at least $L-r$ zeros interior to the curve $\mathcal{K}_{l,\rho',r+1}$, as soon as the degree of g is larger than $C(l,\rho,\rho',\alpha,r)$.*

Proof. – First, remark that Theorem 2.3 applies in the same way, when the derivative of g is replaced with the function $g^{\{1\}}$ defined above. Then, it suffices to apply Theorem 2.3 r times, successively with the polynomials $g, g^{(1)}, \ldots, g^{(r-1)}$ or the polynomials $g, g^{\{1\}}, \ldots, g^{\{r-1\}}$ on two consecutive curves taken among the set of $(r+1)$ concentric curves

$$\mathcal{K}_{l,\rho+k(\rho'-\rho)/r,k+1}, \quad k=0,\ldots,r, \tag{3.1}$$

leading to the conclusion that, for $\deg g$ large, $g^{(r)}$ or $g^{\{r\}}$ have $L-r$ zeros interior to the curve $\mathcal{K}_{l,\rho',r+1}$. The integer C in the statement of the theorem exists and may be chosen as the maximum of the r constants

$$C\big(l,\rho+k(\rho'-\rho)/r,\rho+(k+1)(\rho'-\rho)/r,k,k+1,\alpha\big)+k, \quad k=0,\ldots,r-1,$$

where these constants refer to those introduced in Theorem 2.3. Observe that if we deal with the sequence of derivatives of type $g^{\{k\}}$, then the fourth argument in the above constants can actually be equal to zero since $\deg g^{\{k\}} = \deg g - k$, $k=0,\ldots,r-1$. □

Now, we shall consider a sequence of exponential polynomials g_ν whose degrees tend to infinity, and derivatives of type $g_\nu^{\{r_\nu\}}$ whose order r_ν may possibly grow to infinity with the degree of g_ν. We restrict ourselves to bounded domains included in the horizontal strip $|\operatorname{Im} z| < 2\pi/\alpha$, where α is an upper bound for the diameter of the g_ν's.

THEOREM 3.2. – *Let ε, α, l and ρ be four real positive numbers such that $\rho < 2\pi/\alpha$. Let $(g_\nu)_{\nu\in\mathbb{N}}$, be a sequence of real exponential polynomials of diameter less than or equal to α such that*

$$\lim_{\nu\to\infty} \deg g_\nu = \infty.$$

Let r_ν be a sequence of integers with $1 \leqslant r_\nu \leqslant \deg g_\nu - 1$, satisfying:

$$4\left(1-\frac{\alpha\rho}{2}\cot\frac{\alpha\rho}{2}\right) < \inf_{\nu\in\mathbb{N}} \frac{\deg g_\nu + 1 - r_\nu}{r_\nu + 1}. \tag{3.2}$$

For each ν, assume that g_ν has $L_\nu \geqslant \deg g_\nu$ zeros in the domain $\mathcal{L}_{l,\rho,1}$. Then, there exists a positive integer C depending on ε, α, l and ρ such that the exponential polynomial $g_\nu^{\{r_\nu\}}$ has at least $L_\nu - r_\nu$ zeros interior to the curve $\mathcal{K}_{l,\rho+\varepsilon,r_\nu+1}$, as soon as the degree of g_ν is larger than C.

Moreover, if

$$\lim_{\nu\to\infty} r_\nu = \infty \quad \textit{and} \quad \lim_{\nu\to\infty} \frac{\deg g_\nu}{r_\nu} = \mu \quad (1 \leqslant \mu \leqslant \infty), \tag{3.3}$$

assumption (3.2) *in the previous assertion may be replaced by*

$$4\left(1-\frac{\alpha\rho}{2}\cot\frac{\alpha\rho}{2}\right) < \mu - 1. \tag{3.4}$$

Remark 1. – First, as in Theorem 2.1, we know from the upper bound in Proposition 2.4 and the assumption $\rho < 2\pi/\alpha$ that the integer L_ν can only assume the two values $\deg g_\nu$ or $\deg g_\nu + 1$. Second, observe that the conditions (3.2) and (3.4) are independent from the length l of the strip $\mathcal{L}_{l,\rho,1}$ containing all the zeros of the g_ν, $\nu \in \mathbb{N}$.

Remark 2. – Theorem 3.2 improves asymptotically the upper bound in (2.6) for certain exponential polynomials in the strip $|\operatorname{Im} z| < \rho$, $\pi/\alpha \leqslant \rho < 2\pi/\alpha$. Let us give an example. Consider a sequence of real exponential polynomials $Q_{n_\nu}\mathrm{e}^z - P_{m_\nu}$ of diameter $\alpha = 1$, with $\deg P_{m_\nu} = m_\nu$, $\deg Q_{n_\nu} = n_\nu$, $m_\nu + n_\nu \to \infty$, and satisfying

$$4\left(1-\frac{\rho}{2}\cot\frac{\rho}{2}\right) < \inf_{\nu\in\mathbb{N}} \frac{n_\nu + 1}{m_\nu + 2}. \tag{3.5}$$

From Proposition 2.4, we know that $Q_{n_\nu}\mathrm{e}^z - P_{m_\nu}$ cannot have more than $m_\nu + n_\nu + 2$ zeros in the strip $|\operatorname{Im} z| < \rho$, $\pi \leqslant \rho < 2\pi$. Assume it has exactly $m_\nu + n_\nu + 2$ zeros there. Denote by D the differentiation operator. In view of (3.5), for ν large, we can differentiate $m_\nu + 1$ times $Q_{n_\nu}\mathrm{e}^z - P_{m_\nu}$ and get that $(I + D)^{m_\nu+1}Q_{n_\nu}$, which is a polynomial of degree n_ν, has $n_\nu + 1$ zeros in $\mathcal{L}_{l,\rho+\varepsilon,m_\nu+2}$, a contradiction. Consequently, for ν large, $Q_{n_\nu}\mathrm{e}^z - P_{m_\nu}$ has no more than $m_\nu + n_\nu + 1$ zeros in the strip $|\operatorname{Im} z| < \rho$, $\pi \leqslant \rho < 2\pi$, which improves in this example the upper bound in (2.6) by 1.

Remark 3. – The domain containing the zeros of $g_\nu^{\{r_\nu\}}$, that is the domain $\mathcal{L}_{l,\rho+\varepsilon,r_\nu+1}$ remains bounded along the imaginary axis and has a length along the real axis which is of order $\sqrt{r_\nu}$, if r_ν tends to infinity as ν tends to infinity. The precise magnitude of this length prove to be important since it allows one to use Theorem 3.2 in order to obtain convergence properties in the problem of rational interpolation to the exponential function with complex conjugate interpolation points (see [14]).

In the previous section, we made use of Proposition 2.2, which gives a lower bound for an expression involving distances between points located interior to the domain $\mathcal{L}_{l,\rho,p}$ on the one hand and on the surrounding curve $\mathcal{K}_{l,\rho',p+1}$ on the other hand. Here we shall need the order of this expression when the radius ρ' tends to ρ, first when the parameter p is fixed, and second also when p tends to infinity. This is the content of the next two lemmas:

LEMMA 3.3. – *With the same notations and assumptions as in Proposition* 2.2, *we have*:

$$\min_{\substack{M\in\mathcal{L}_{l,\rho,p}\\ N\in\mathcal{K}_{l,\rho',p+1}}} Q(M,N)=\frac{2p}{\rho^3}(\rho'-\rho)-p\frac{7p+5}{2\rho^4}(\rho'-\rho)^2+\mathrm{O}\big((\rho'-\rho)^3\big), \tag{3.6}$$

as p is a fixed integer and ρ' tends to ρ.

Remarks. – As ρ' tends to ρ, the minimums in (2.3) and (2.4) are both given by the ratio $(\rho'^2-\rho^2)/(\rho'^2+\rho^2)^2$, which is of order $(\rho'-\rho)/2\rho^3$. Hence, the order $2p(\rho'-\rho)/\rho^3$ in the right-hand side of (3.6) improves the previous one, as ρ' tends to ρ. In particular, it takes the parameter p into account. Remark also that the estimate in (3.6) is independent from the parameter l.

Proof. – Let us first consider the case $l=0$, i.e. $M\in\mathcal{F}_{\rho,p}$, and $N\in\mathcal{E}_{\rho',p+1}$. In the limit case $\rho'=\rho$, we know from Lemma 2.7 that the minimum in (3.6) actually equals 0. It is easily checked that for any $M\in\mathcal{E}_{\rho,p}$, of coordinates (u,v) satisfying

$$|u|\leqslant (p/\sqrt{p+1})\rho,\quad u\neq 0, \tag{3.7}$$

this minimum vanishes when $N\in\mathcal{E}_{\rho,p+1}$ has coordinates (x,y) such that

$$x=\frac{p+1}{p}u,\qquad y^2=v^2-\frac{u^2}{p^2}. \tag{3.8}$$

If inequality (3.7) is not met, then the minimum is distinct from 0.

Assume now ρ fixed. Since $Q(M,N)$ is minimized only when M lies on the boundary of $\mathcal{F}_{\rho,p}$, that is on $\mathcal{E}_{\rho,p}$, this expression can be seen as a function of three parameters, namely the two arguments of M and N and the ratio $\eta:=\rho'/\rho$. Plugging the parameterizations

$$u=\rho\sqrt{p}\cos\alpha,\qquad v=\rho\sin\alpha,\qquad x=\rho'\sqrt{p+1}\cos\beta,\qquad y=\rho'\sin\beta \tag{3.9}$$

in expression (2.10) of $Q(M,N)$, then differentiating with respect to η and evaluating this derivative at $\eta=1$ and arguments α and β corresponding to points M and N such that (3.8) holds, leads to the following simple expression:

$$\frac{1}{2\rho^2}\frac{p}{\cos^2\alpha(1-\cos^2\alpha)}.$$

Now, it only remains to take the minimum of this ratio as $\cos^2\alpha$ ranges from 0 to $p/(p+1)$ (see (3.7) and the first equation in (3.9)). Obviously, this minimum is met as $\cos^2\alpha=1/2$, which is always possible since $p\geqslant 1$ entails $1/2\leqslant p/(p+1)$. It is thus equal to $2p/\rho^2$. Consequently, considering an expansion of $Q(M,N)$ in a neighborhood of $\eta=1$, $M\in\mathcal{E}_{\rho,p}$ and $N\in\mathcal{E}_{\rho',p+1}$ such that $\cos^2\alpha=1/2$ and (3.8) holds, we obtain that the minimum in the left-hand side of (3.6) is of order $2p(\rho'-\rho)/\rho^3$ as ρ' tends to ρ. The second term in the expansion is obtained by evaluating the second-order derivative of $Q(M,N)$ at the above points. This finishes the proof of (3.6) when $l=0$.

To obtain the same result for the general case $l>0$, it is sufficient to remark that when $M\in\mathcal{L}_{l,\rho,p}$ has coordinates (u,ρ) with $|u|\leqslant l$, the minimum of $Q(M,N)$, $N\in\mathcal{K}_{l,\rho,p+1}$, does not vanish. On the other hand, when $l\leqslant |u|\leqslant l+\sqrt{p}\rho$, the analysis given in the case $l=0$

remains valid since when $l > 0$, one merely performs a shift of $-l$ (resp. l) on the left (resp. right) parts of both $\mathcal{F}_{l,\rho,p}$ and $\mathcal{E}_{l,\rho,p+1}$. □

LEMMA 3.4. – *With the same notations and assumptions as in Proposition* 2.2*, we have*:

$$\min_{\substack{M\in\mathcal{L}_{l,\rho,p}\\ N\in\mathcal{K}_{l,\rho',p+1}}} Q(M,N)\simeq \min\left(\frac{2\varepsilon}{\rho^2},\left(\sqrt{p+1}\rho'+\sqrt{p}\rho+2l\right)^{-2}\right), \tag{3.10}$$

where p tends to infinity and $\rho'/\rho = 1+\varepsilon/p$, with $\varepsilon\to 0$ as $p\to\infty$.

Remark. – When $\rho'/\rho = 1+\varepsilon/p$, the dominant term in the expansion (3.6) and the first expression in the minimum of (3.10) coincide.

Proof. – First assume $l=0$. We consider any point $M\in\mathcal{E}_{\rho,p}$, of coordinates (u,v) satisfying (3.7) and $N\in\mathcal{E}_{\rho',p+1}$ of coordinates (x,y) such that

$$x=\frac{\rho'}{\rho}\frac{p+1}{p}u,\qquad y^2=\left(\frac{\rho'}{\rho}\right)^2\left(v^2-\frac{u^2}{p^2}\right). \tag{3.11}$$

The point N has been chosen in this way, because, as $\rho'\to\rho$, N tends to the point N_ρ of $\mathcal{E}_{\rho,p+1}$ such that $Q(M,N_0)$ vanishes. Plugging the parameterizations (3.9) and relations (3.11) in expression (2.10) of $Q(M,N)$, then using the assumption $\rho'/\rho=1+\varepsilon/\rho$, $\eta\to 0$ as $p\to\infty$, we find after some computations that the dominant term in $Q(M,N)$ equals $2\varepsilon/\rho^2$, as $p\to\infty$. The identical estimate for the general case $l>0$ follows from the same observations as in the proof of Lemma 3.3.

Now, since p tends to infinity, we need to compare the latter minimum with the other possible one which occurs when $M\in\mathcal{L}_{l,\rho,p}$ has coordinates $(l+\sqrt{p}\rho,0)$, $N\in\mathcal{L}_{l,\rho',p+1}$ has coordinates $(-l-\sqrt{p+1}\rho',0)$, whose value equals $(\sqrt{p+1}\rho'+\sqrt{p}\rho+2l)^{-2}$. This shows (3.10) and finishes the proof of the lemma. □

We are now in a position to prove Theorem 3.2.

Proof of Theorem 3.2. – For simplicity, we shall omit the subscript ν. The difficulty in applying Theorem 2.1 as in the proof of Theorem 3.1 lies in that the concentric curves (3.1) have their mutual distances tending to 0 and their lengths along the real axis tending to ∞, as r possibly goes large. Hence, in view of the Lemmas 3.3 and 3.4, the maximum in the right-hand side of (2.2) tends to ∞ and it becomes unclear whether this inequality can still be satisfied. Here, we define a sequence of concentric curves

$$\mathcal{K}_{l,\rho,1},\mathcal{K}_{l,\rho_1,2},\ldots,\mathcal{K}_{l,\rho_r,r+1},\qquad \rho=:\rho_0<\rho_1<\cdots<\rho_r, \tag{3.12}$$

distinct from the sequence (3.1): let a be some positive real number to be chosen later, and define the sequence ρ_k, $k=0,\ldots,r$, by the recurrence relations:

$$\rho_0=\rho,\qquad \rho_k=\rho_{k-1}+\frac{a\rho_{k-1}}{k(\deg g+1-k)},\quad k=1,\ldots,r. \tag{3.13}$$

Since the product

$$\prod_{k=1}^{r}\left(1+\frac{a}{k(\deg g+1-k)}\right)$$

converges to 1 as ν tends to ∞, ρ_r tends to ρ. In particular, the differences $\rho_k - \rho_{k-1}$, $k = 1, \ldots, r$, tend to zero, and moreover, from (3.13), $\rho_k/\rho_{k-1} = 1 + a/k(\deg g + 1 - k)$ with $a/(\deg g + 1 - k) \to 0$, as ν tends to ∞ (see the assumption (3.2)). Hence, in the proof of Theorem 2.1, instead of using Proposition 2.2 in order to get an explicit lower bound for $Q(M, N)$ in (2.24), we may appeal, as ν tends to ∞, to the more precise estimates established in Lemmas 3.3 and 3.4. If r remains bounded, we deduce from Lemma 3.3 that

$$\min_{\substack{M \in \mathcal{L}_{l,\rho_{k-1},k} \\ N \in \mathcal{K}_{l,\rho_k,k+1}}} Q(M, N) \simeq \frac{2a}{\rho_{k-1}^2(\deg g + 1 - k)}, \quad k = 1, \ldots, r,$$

while, if r and k tend to infinity, we deduce from Lemma 3.4 that

$$\min_{\substack{M \in \mathcal{L}_{l,\rho_{k-1},k} \\ N \in \mathcal{K}_{l,\rho_k,k+1}}} Q(M, N) \simeq \min\left(\frac{2a}{\rho_{k-1}^2(\deg g + 1 - k)}, (\sqrt{k+1}\rho_k + \sqrt{k}\rho_{k-1} + 2l)^{-2}\right).$$

Consequently, for $k = 1, 2, \ldots, r$, the condition (2.2) may be replaced with the two following ones

$$\deg g + 1 - k \geqslant \frac{1}{\rho_k^2}\left(1 - \frac{\alpha\rho_k}{2}\cot\frac{\alpha\rho_k}{2}\right)\frac{\eta}{2a}\rho_{k-1}^2(\deg g + 1 - k), \tag{3.14}$$

and

$$\deg g + 1 - k \geqslant \frac{1}{\rho_k^2}\left(1 - \frac{\alpha\rho_k}{2}\cot\frac{\alpha\rho_k}{2}\right)(\sqrt{k+1}\rho_k + \sqrt{k}\rho_{k-1} + 2l)^2, \tag{3.15}$$

for some $\eta > 1$. Observe that in our situation, Theorem 2.1 applies with $g^{\{k\}}$. Indeed, it has (contrary to $g^{(k)}$) exact degree $\deg g - k$ and thus no extra zeros in the complement of $\mathcal{L}_{l,\rho_k,k+1}$ in the strip $|\operatorname{Im} z| < 2\pi/\alpha$, except for one possible real zero whose contribution, as was seen in the proof of Theorem 2.1, can be neglected. Obviously, condition (3.14) will be fulfilled as soon as the parameter a is chosen sufficiently large, so that only condition (3.15) has to be met. Here, we may remark that, since the diameter of $g^{\{k\}}$ is only decreasing as k increases and since the function $x \mapsto 1 - x\cot x$ is increasing for $x \geqslant 0$, (3.15) is actually stronger than what is needed. Now, as its right-hand side is less than

$$4(r+1)\left(1 - \frac{\alpha\rho_r}{2}\cot\frac{\alpha\rho_r}{2}\right)\left(1 + \frac{l}{\sqrt{r+1}\rho}\right)^2,$$

a sufficient condition for (3.15), $k = 1, \ldots, r$, to hold is given by the inequality

$$4\left(1 - \frac{\alpha\rho_r}{2}\cot\frac{\alpha\rho_r}{2}\right)\left(1 + \frac{l}{\sqrt{r+1}\rho}\right)^2 < \frac{\deg g + 1 - r}{r+1}. \tag{3.16}$$

This last condition is implied by the stronger inequality

$$4\eta_1\left(1 - \frac{\alpha\rho}{2}\cot\frac{\alpha\rho}{2}\right)\left(1 + \frac{l}{\sqrt{r+1}\rho}\right)^2 < \frac{\deg g + 1 - r}{r+1}, \tag{3.17}$$

where $\eta_1 > 1$, since ρ_r tends to ρ as ν tends to ∞. Now, we consider two cases. First, if $r \leqslant \sqrt{\deg g}$, (3.17) is satisfied as soon as

$$4\eta_1\left(1-\frac{\alpha\rho}{2}\cot\frac{\alpha\rho}{2}\right)\left(1+\frac{l}{\rho}\right)^2 < \frac{\deg g + 1 - \sqrt{\deg g}}{\sqrt{\deg g}+1},$$

which will be granted as soon as $\deg g$ is larger than some constant depending only on α, ρ and l. Second, if $r > \sqrt{\deg g}$, the factor $1 + l/\sqrt{r+1}\rho$ tends to 1 as $\deg g$ tends to ∞, which shows that, for $\deg g$ large, the condition (3.17) is implied by the condition (3.2), for some $\eta_1 > 1$. Thus, if (3.2) is satisfied, and if $\deg g$ is large enough, we obtain a sequence of concentric curves (3.12) with the property that Rolle's theorem can be applied on each pair of two consecutive curves taken from this sequence. Doing so, we eventually obtain that $g^{\{r\}}$ has $L - r$ zeros interior to $\mathcal{K}_{l,\rho_r,r+1}$, hence to $\mathcal{K}_{l,\rho+\varepsilon,r+1}$, for ν large. Finally, if (3.3) holds, the right-hand side of (3.16) tends to $\mu - 1$ as ν tends to ∞. It is then clear that the factor η_1 is not necessary in the sequel of the argument, after (3.16), which means that the inequality (3.2) transforms into the inequality (3.4), as asserted. □

4. Some remarks concerning the previous results

First, the assertions in Theorem 3.2 have been applied in [14] to the problem of rational interpolation to the exponential function by means of complex conjugate interpolation points, allowing to recover in this case all the classical properties of the Padé approximants, such as separated convergences of the numerator and of the denominator, as well as error estimates (cf., e.g., [8] for these classical results and [1] for the case of real interpolation points).

Second, Theorem 3.2 may also give some hints when asking for the maximal number of zeros a real exponential polynomial can have, e.g., in a disk. Several authors, Polya, Gelfond, Turan, Mahler, have given such bounds for general exponential polynomials, i.e. allowing complex frequencies and complex coefficients. These latter bounds have been subsequently improved by Tijdeman [11], Waldschmidt [13], Voorhoeve [12], leading to the following result:

Let $N(g, z_0, r)$ denote the number of zeros of the exponential polynomial

$$g(z) = \sum_{j=1}^{n} q_j(z)\,\mathrm{e}^{\omega_j z}, \quad \omega_j \in \mathbb{C},$$

where the q_j are complex polynomials, that are contained in the closed disk of radius r, centered at z_0. Then,

$$N(g, z_0, r) \leqslant 4\Omega r/\pi + 2\deg g, \tag{4.1}$$

where

$$\Omega = \max\bigl\{|\omega_j|,\ j = 1, \ldots, n\bigr\}.$$

For real polynomials, an upper bound is easier to compute. Indeed, from Proposition 2.4, we knows that for g a real polynomial of diameter α,

$$N(g, z_0, r) \leqslant \alpha r/\pi + \deg g, \tag{4.2}$$

and this upper bound even holds true in any horizontal strips of height $2r$ (incidentally, note that the previous upper bound is half the upper bound in (4.1)). Now, we may ask about the sharpness of this upper bound and in particular what happens asymptotically that is when z_0 goes to infinity or when the degree of g grows to infinity. The Polya–Dickson theorem, which gives the asymptotic location of the zeros of large modulus, shows that outside a compact set near the origin, the right-hand side of (4.2) can be simplified to $\alpha r/\pi$. This answers the previous question when z_0 goes to infinity. Now, if z_0 is fixed while $\deg g$ grows to infinity, Theorem 3.2 would rather indicate that the right-hand side of (4.2) can be simplified to $\deg g$. Indeed, let us consider a sequence of exponential polynomials with a given number n of terms, say,

$$g_\nu(z) = \sum_{j=1}^{n} p_{j,\nu}(z)\,\mathrm{e}^{\alpha_{j,\nu} z}, \quad \deg p_{j,\nu} = m_{j,\nu},$$

and the sequence of integers

$$r_\nu = \sum_{j=2}^{n} (m_{j,\nu} + 1) = \deg g - m_{1,\nu},$$

such that Theorem 3.2 applies. Obviously, from (3.2), we see that this will be the case when $m_{1,\nu}$ is larger than r_ν. Then, if the g_ν have more than $\deg g_\nu$ zeros, we deduce that the $g_\nu^{\{r_\nu\}}$, which are polynomials of degree $m_{1,\nu}$, have more than $m_{1,\nu}$ zeros in the complex plane, a contradiction.

Based on these observations, we ask more generally the following:

Open Question. – Let be given a closed disk, centered at the origin, of radius r, and a diameter α. Does there exist an integer C depending only on r and α such that, for any real exponential polynomial g with diameter less than or equal to α, one has

$$N(g, 0, r) \leqslant \deg g,$$

as soon as the degree of g is larger than C?

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