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Rational Approximation to the Exponential Function with Complex Conjugate Interpolation Points[1]

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In this paper, we study asymptotic properties of rational functions that interpolate the exponential function. The interpolation is performed with respect to a triangular scheme of complex conjugate points lying in bounded rectangular domains included in the horizontal strip $|\mathrm{Im}\, z| < 2\pi$. Moreover, the height of these domains cannot exceed some upper bound which depends on the type of rational functions. We obtain different convergence results and precise estimates for the error function in compact sets of $\mathbb{C}$ that generalize the classical properties of Padé approximants to the exponential function. The proofs rely on, among others, Walsh's theorem on the location of the zeros of linear combinations of derivatives of a polynomial and on Rolle's theorem for real exponential polynomials in the complex domain. © 2001 Academic Press

Key Words: rational interpolation in the complex plane; exponential function; error estimates.

1. INTRODUCTION

The rational interpolation problem, also known as the Cauchy–Jacobi interpolation problem, consists in finding a rational function P/Q of type (m, n) (i.e., $\deg P = m$, $\deg Q = n$) say, which takes prescribed values at $m+n+1$ given points of the complex plane, for example the values of some function f. It may happen that no solution to this problem exists, due to interpolation defects, but its linearized version, which consists in finding P and Q such that $Qf - P$ vanishes at the above $m+n+1$ points, always admits a non-trivial solution. From a computational point of view, the seeked rational functions can be represented in many different ways, for example in terms of Loewner determinants or Thiele continued fractions, and are explicitly determined by solving a structured linear system of

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equations. There has been many investigations about how to perform these computations via efficient and reliable algorithms, see e.g. [9] and the bibliography therein.

In this paper, we shall be concerned with the analytic aspect of this problem, namely the convergence behavior of the interpolants as $m+n \to \infty$. We review some of the results concerning convergence of rational interpolants, and in particular of Padé approximants that, as well-known, correspond to interpolation points all lying at the origin. The reader may also consult [26] which contains many results in the same connection.

To begin with, families of approximants simpler than rational interpolants to analytic functions, such as polynomial interpolants or rational interpolants with prescribed poles, have been studied in great detail in [33]. Their convergence can be directly derived from the well-known Hermite representation formula. For rational interpolants with free poles, the situation is more intricate. Indeed, the possible occurrence of spurious poles that do not reflect the singularities of the underlying function may locally destroy convergence. Hence, usually, only results weaker than uniform convergence can be obtained. Convergences in measure or in capacity are the appropriate notions that are used e.g. in the Nuttall–Pommerenke theorem. Roughly speaking, this theorem asserts that spurious poles only disrupt convergence in sets of small capacity. It was first proved for Padé approximants to meromorphic functions [19], then to any single-valued analytic functions with singular sets of capacity zero [22], and also to functions of fast rational approximability [7]. It was subsequently generalized to rational interpolants by Karlsson and Wallin [11, 32]. Examples in [13] and [23] show that, when considering arbitrary single-valued analytic functions, the assumption of singular sets of zero capacity in the Nuttall–Pommerenke theorem is essential.

Concerning uniform convergence, the famous conjecture of Baker–Gammel–Wills has been proposed:

Let f be meromorphic in $|z|<1$, and analytic at 0. Then there exists an infinite sequence $\mathcal{I}$ of positive integers such that

$$\lim_{\substack{n\to\infty\\ n\in\mathcal{I}}} [n/n](z) = f(z),$$

locally uniformly in $|z|<1$, omitting poles of f, where $[n/n]$ denotes the Padé approximant to f of type (n, n).

In [31], an example of an entire function is constructed which shows that, in general, the convergence cannot hold for the whole sequence $([n/n])_{n\in\mathbb{N}}$. The above conjecture has been proved for a class of entire functions with fast decreasing Taylor coefficients, hence for most entire functions in the sense of category, since the above class is dense in the space of entire functions [14]. If rational interpolation rather than Padé approximation is allowed, then by a result of Levin [12], for any function f analytic in $|z|<1$, there exists an infinite triangular scheme A of points in $|z|<1$ such that

$$\lim_{n\to\infty} R_{n,n}(f, A, z) = f(z),$$

locally uniformly in $|z|<1$, where $R_{n,n}$ denotes the rational interpolant of type (n, n) to f with respect to the points of the scheme A. For entire functions that admit sufficiently rapid and regular decay of errors of best approximation, Theorem 7.4 in [15] asserts locally uniform convergence in $\mathbb{C}$ of rational interpolants with respect to *any* scheme of points $(a_{k,j})_{j=1}^{k}$ in a compact set, satisfying for some $\rho>0$,

$$\lim_{k\to\infty} \prod_{j=1}^{k} |z-a_{k,j}|^{1/k} = |z|, \qquad |z| \geqslant \rho.$$

There are essentially only two special classes of functions which are known to admit nice properties with respect to Padé approximation, or more generally with respect to rational interpolation. These two classes consist of Markov functions on the one hand and of Polya frequency series on the other hand.

The convergence of Padé approximants to Markov functions, i.e., to functions of the form

$$\int \frac{d\mu(t)}{t-z},$$

where μ is a positive measure with compact support on $\mathbb{R}$, goes back to Markov himself [17] and was formulated in terms of Chebyshev continued fractions. The convergence has been extended to rational interpolants in [8] and this last result was subsequently refined and generalized in [28]. Recently, sharp asymptotic error estimates for rational interpolants to Markov functions whose measures are sufficiently regular, i.e. satisfy Szegő condition, have been derived from the corresponding strong asymptotics for orthonormal polynomials with varying weights, cf. [29] and [27].

Let us now turn to the case of the exponential function. The Padé approximants to e^z were first studied by Hermite in connection with his proof of the transcendence of e. Then, Padé, a student of Hermite, took up again the investigation of these approximants and proved their convergence to the exponential function. One may consult [20] where a proof of the separated convergence of the numerator and the denominator is also given.

These results have been generalized in [1] to the class of Polya frequency series, that is functions of the type

$$Cz^{\lambda}e^{\gamma z}\prod_{i=1}^{\infty}(1+\alpha_i z)\Big/\prod_{i=1}^{\infty}(1-\beta_i z),$$

where $C\geqslant 0$, $\lambda\in\mathbb{N}$, $\gamma\geqslant 0$, $\alpha_i\geqslant 0$, $\beta_i\geqslant 0$, $\sum(\alpha_i+\beta_i)<\infty$.

From a practical point of view, rational approximations to the exponential function are relevant to the design and analysis of numerical methods for stiff ordinary differential equations. In this connection, the theory of order stars, that is sets $\{z\in\mathbb{C}:|R(z)|>|f(z)|\}$ where R denotes a rational approximation to $f(z)$ (which may be the exponential function or some more general function), was started in [34] and studied in a series of papers, see [10] for a detailed account of the theory. The geometry of an order star gives, via the argument principle, knowledge about structural properties of the approximating function in an easy and natural way, but quantitative informations such as the magnitude of the error of approximation cannot be derived from this approach. Besides of these results, precise estimates on the location of the zeros and poles of the approximants, which also have applications to the numerical analysis of differential equations, were obtained in a well-known series of papers (cf. [24, 25], and the bibliography therein).

Rational interpolants to the exponential function with interpolation points all lying in some compact interval of the real axis were studied in [4, 5] in relation with the Meinardus conjecture. The above mentioned classical properties of Padé approximants to the exponential function were generalized to the case of real interpolation points in [2]. The problem of further generalizing the result of convergence as well as the asymptotic error estimates to the case of complex interpolation points was contributed by the authors of [2] in the open problems session at the 1994 conference on computational methods and function theory held at the university Sains Malaysia, Penang. In the present paper, this problem is partially solved, since the interpolation points are conjugate and lie in horizontal strips of arbitrary length, but with height less than 4π. Moreover, the height of these strips is bounded above by some number depending on the asymptotic behavior of the degrees of the interpolants (see condition (2.3), respectively condition (2.7) when the sequence of interpolants is ray). The method of proof is based on ideas which were developed in [2] and [3, Section 5]. The new ingredient consists of an analog of Rolle's theorem that holds for real exponential polynomials in the complex plane [35]. Then, it becomes possible to apply the method used in [2], though, here, Newman's method relating complex and real interpolants to e^z (cf. [18, Lecture V]) has also to be added as a supplementary component of the proof. This analysis, which is performed in Section 4, leads to a convergence theorem and estimates of the error function for interpolants with complex points generalizing those obtained in [2] for real interpolation points.

2. NOTATIONS AND MAIN RESULTS

Let ρ, p and l be three positive real numbers. Throughout, T_ρ and $D(0, \rho)$ will respectively denote the circle of radius ρ and the closed disk of radius ρ, both centered at the origin. Also we denote by $\mathscr{E}_{\rho, p}$ the ellipse such that

$$\mathscr{E}_{\rho, p} := \left\{ x + iy : \frac{x^2}{p} + y^2 = \rho^2 \right\}$$

and $\mathscr{F}_{\rho, p}$ the closed interior of this ellipse; that is

$$\mathscr{F}_{\rho, p} := \left\{ x + iy : \frac{x^2}{p} + y^2 \leqslant \rho^2 \right\}.$$

Hence, in particular, $\mathscr{F}_{\rho, 1} = D(0, \rho)$. Moreover, we set

$$\mathscr{L}_{l, \rho, p} := \{ x + iy : -l \leqslant x \leqslant l, \ -\rho \leqslant y \leqslant \rho \} \cup (\mathscr{F}_{\rho, p} - l) \cup (\mathscr{F}_{\rho, p} + l),$$

so that $\mathscr{L}_{l, \rho, p}$ is the bounded strip consisting of the interior of a rectangle of dimensions $2l \times 2\rho$, centered at the origin, whose left and right sides have been replaced with semi-ellipses of half-axis ρ and $\sqrt{p}\, \rho$.

THEOREM 2.1. *Let $B^{(m+n)} := \{ z_k^{(m+n)} \}_{k=0}^{m+n}$, $m = m_\nu$, $n = n_\nu$ be a triangular sequence of not necessarily distinct complex interpolation points possessing the symmetry property $\overline{B^{(m+n)}} = B^{(m+n)}$ (i.e. non-real points in $B^{(m+n)}$ only appear in conjugated pairs) contained in the domain $\mathscr{L}_{l, \rho, 1}$, $0 < \rho < 2\pi$, and such that*

$$\lim_{\nu \to \infty} m_\nu + n_\nu = \infty. \tag{2.1}$$

Denote by $R_{m, n} = P_{m, n} / Q_{m, n}$ the multipoint Padé approximant or linearized rational interpolant of type (m, n) to e^z in $B^{(m+n)}$ such that

$$Q_{m, n}(z)\, e^z - P_{m, n}(z) = O(\omega_{m+n+1}(z)), \tag{2.2}$$

where

$$\omega_{m+n+1}(z) = \prod_{k=0}^{m+n} (z - z_k^{(m+n)}).$$

Assume that the sequence (m_ν, n_ν) *satisfies*

$$4\left(1 - \frac{\rho}{2}\cot\frac{\rho}{2}\right) < \inf_{\nu \geqslant \nu_0} \max\left(\frac{n_\nu + 1}{m_\nu + 2}, \frac{m_\nu + 1}{n_\nu + 2}\right), \tag{2.3}$$

where ν_0 *is some positive integer. Then, the following four assertions hold true:*

(i) *For* ν *large, the polynomials* $P_{m,n}$ *and* $Q_{m,n}$ *such that* (2.2) *holds are of exact degree m and n respectively.*

(ii) *All the zeros and poles of* $R_{m,n}$ *tend to infinity, as* ν *becomes large. In particular, for* ν *large, no poles of* $R_{m,n}$ *lie in the bounded strip* $\mathcal{L}_{l,\rho,1}$ *where interpolation takes place, hence, dividing* (2.2) *by* $Q_{m,n}$, *we get that* $R_{m,n}$ *is a true rational interpolant to* e^z *in* $B^{(m+n)}$ *satisfying*

$$e^z - R_{m,n}(z) = O(\omega_{m+n+1}(z)).$$

(iii) *As* $\nu \to \infty$,

$$R_{m_\nu, n_\nu}(z) \to e^z, \tag{2.4}$$

locally uniformly in $\mathbb{C}$.

(iv) *If, in addition,* (m, n) *is a ray sequence,*

$$\lim_{\nu \to \infty} \frac{m_\nu}{n_\nu} = \lambda \qquad (0 \leqslant \lambda \leqslant +\infty), \tag{2.5}$$

then the numerator and denominator converge separately, that is, as $\nu \to \infty$,

$$P_{m_\nu, n_\nu}(z) \to e^{\lambda z/(1+\lambda)} \qquad \text{and} \qquad Q_{m_\nu, n_\nu}(z) \to e^{-z/(1+\lambda)} \tag{2.6}$$

locally uniformly in $\mathbb{C}$, *where* Q_{m_ν, n_ν} *is normalized so that* $Q_{m_\nu, n_\nu}(0) = 1$.

Note that if (2.5) *holds, then condition* (2.3) *reduces to*

$$4\left(1 - \frac{\rho}{2}\cot\frac{\rho}{2}\right) < \max(\lambda, 1/\lambda). \tag{2.7}$$

Remarks. First, assertion (i) is equivalent to the fact that, for ν large, the exponential polynomial $Q_{m,n}(z)\,e^z - P_{m,n}(z)$ cannot have more than $m+n+1$ zeros in $\mathscr{L}_{l,\rho,1}$. In the terminology of rational approximation theory, this property would usually be rephrased by saying that the exponential function is *normal* in the domain $\mathscr{L}_{l,\rho,1}$. Second, note that the conditions (2.3) and (2.7) give a constraint on the height ρ but not on the length l of the domain $\mathscr{L}_{l,\rho,1}$ where interpolation takes place. Third, in the case corresponding to the classical Montessus de Ballore theorem (i.e. the degree n is fixed while m tends to infinity, or conversely m is fixed and n tends to infinity), condition (2.7) is void. Hence, in this case, all the assertions of Theorem 2.1 hold with respect to interpolation in any domain $\mathscr{L}_{l,\rho,1}$, $\rho < 2\pi$. On the opposite, diagonal ray sequences (m, n) such that $m/n \to 1$ correspond to the most stringent condition on ρ, namely

$$4\left(1-\frac{\rho}{2}\cot\frac{\rho}{2}\right) < 1,$$

that is $\rho < 1.689....$

For simplicity, we shall usually omit the subscript ν in the sequel, writing m instead of m_ν and n instead of n_ν. In the next proposition, we present estimates on the zeros and poles of $R_{m,n}$.

Proposition 2.2. *Let $\tilde{\rho} = \rho + l$ so that $\tilde{\rho}$ equals half the diameter of the domain $\mathscr{L}_{l,\rho,1}$. Then, with the same assumptions as in Theorem* 2.1, *all zeros $\alpha_k^{(n)}$ of $Q_{m,n}$ satisfy*

$$m+1-2\tilde{\rho}+O(1/(m+n)) \leqslant |\alpha_k^{(n)}|, \qquad k=1, ..., n.$$

Symmetrically, all zeros $\beta_j^{(m)}$ of $P_{m,n}$ satisfy

$$n+1-2\tilde{\rho}+O(1/(m+n)) \leqslant |\beta_j^{(m)}|, \qquad m \geqslant 1, \quad j=1, ..., m.$$

Moreover, all zeros $\alpha_k^{(n)}$ and $\beta_j^{(m)}$ lie within a corona

$$(m+n)\,\gamma - 2\tilde{\rho} + O(1/(m+n)) \leqslant |z| \leqslant m+n+4/3+2\tilde{\rho}+O(1/(m+n)), \qquad m \geqslant 1,$$

where $\gamma \simeq 0.278$ is the unique positive root of $\gamma e^{1+\gamma} = 1$, and respectively remain within $2\tilde{\rho}$ units distance from the zeros of the Padé denominator and numerator to the exponential function.

The convergence asserted in Theorem 2.1 can be further estimated as follows.

THEOREM 2.3. *Let* $B^{(m+n)}$, $m=m_\nu$, $n=n_\nu$ *and* $R_{m,n}$ *be as in Theorem* 2.1, *where it is assumed that* (2.1), (2.2), (2.5), *and* (2.7) *hold. For* $K\subset\mathbb{C}$ *a compact set, define*

$$c_0=\min_{z\in K}|e^z|, \qquad c_1=\max_{z\in K}|e^z|,$$

and put

$$C_0=e^{-2\tilde{\rho}}c_0^{2/(1+\lambda)}, \qquad C_1=e^{2\tilde{\rho}}c_1^{2/(1+\lambda)},$$

where $\tilde{\rho}=\rho+l$. *Then, for any positive real number* $\alpha<1$, *there exists a positive integer* L *such that the rational interpolants* $R_{m,n}(z)$ *to* e^z *satisfy for all* $z\in K$ *and* $m+n\geqslant L$,

$$\alpha C_0\prod_{k=0}^{m+n}|z-z_k^{(m+n)}|\leqslant\delta_{m,n}^{-1}|e^z-R_{m,n}(z)|$$

$$\leqslant\frac{C_1}{\alpha}\prod_{k=0}^{m+n}|z-z_k^{(m+n)}|, \tag{2.8}$$

where

$$\delta_{m,n}:=\frac{m!\,n!}{(m+n)!\,(m+n+1)!}. \tag{2.9}$$

From Theorem 2.3 we can deduce absolute error bounds which do not depend on a particular ray sequence, namely:

THEOREM 2.4. *Let* $B^{(m+n)}$, $m=m_\nu$, $n=n_\nu$ *and* $R_{m,n}$ *be as in Theorem* 2.1, *where it is assumed that* (2.1), (2.2), *and* (2.3) *hold. For* $K\subset\mathbb{C}$ *a compact set, let* c_0 *and* c_1 *be as in Theorem* 2.3. *Define* m_0 *to be* c_0^2 *if* $c_0\leqslant 1$ *and to be* 1 *otherwise. In a symmetric manner, let* m_1 *be* c_1^2 *if* $c_1\geqslant 1$ *and* 1 *otherwise. Then, for any positive real number* $\alpha<1$, *there exists a positive integer* L *depending only on* ρ, K, *and* α *such that any rational interpolant* $R_{m,n}(z)$ *of type* (m,n) *to* e^z *in* $m+n+1$ *points of* $\mathscr{L}_{l,\rho,1}$ *satisfies for all* $z\in K$ *and* $m+n\geqslant L$,

$$\alpha e^{-2\tilde{\rho}}m_0\prod_{k=0}^{m+n}|z-z_k^{(m+n)}|\leqslant\delta_{m,n}^{-1}|e^z-R_{m,n}(z)|$$

$$\leqslant\frac{e^{2\tilde{\rho}}m_1}{\alpha}\prod_{k=0}^{m+n}|z-z_k^{(m+n)}|, \tag{2.10}$$

where $\tilde{\rho}=\rho+l$ *and* $\delta_{m,n}$ *is defined by* (2.9).

Remark. The previous bounds are sharp except for the factors $e^{-2\tilde{\rho}}$ and $e^{2\tilde{\rho}}$ in the left and right hand sides of (2.10). Nevertheless, the sharp estimates cannot include factors larger than $e^{-\tilde{\rho}}$ nor smaller than $e^{\tilde{\rho}}$ in place of $e^{-2\tilde{\rho}}$ and $e^{2\tilde{\rho}}$ respectively. This can be seen from considering the exact asymptotics for the classical Padé approximants in a ray sequence satisfying (2.5) at some point $z=\sigma+it$, namely

$$|e^z - R_{m,n}(z)| = \frac{m!\,n!}{(m+n)!\,(m+n+1)!}\,|z|^{m+n+1}\,e^{2\sigma/(1+\lambda)}(1+o(1)), \quad (2.11)$$

see [6, eq. (5.5) p. 138], or take $\tilde{\rho}=0$ in (2.8). Replacing z by $z-\xi$, ξ some complex number, in (2.11) gives asymptotics at z for the Padé approximant to e^z shifted at ξ. Letting the point ξ and the parameter λ vary in $[-\tilde{\rho}, \tilde{\rho}]$ and in $[0, +\infty]$ respectively leads to the above observations about the sharpness of (2.10). We refer to the discussion after Theorem 2.3 in [2] for some more details.

The results of Theorems 2.1, 2.3 and 2.4 are similar to those of [2] which considers the case of real interpolation points. They are also similar to those of [3, Theorem 5.1] that are established for a special class of rational interpolants to the exponential function in the unit disk, namely H^2 rational approximants of type $(n-1, n)$, $n \geqslant 1$.

3. PRELIMINARIES

To prove our results, several ingredients, borrowed from the literature, will be necessary. We list them now along with some references. The first one is a theorem by Walsh about the location of roots of certain combinations of polynomials and their derivatives.

THEOREM 3.1 (cf. [16, Theorem 18.1]). *Let*

$$f(z) = \sum_{j=0}^{n} a_j z^j, \qquad g(z) = \sum_{j=0}^{n} b_j z^j = b_n \prod_{j=1}^{n} (z-\beta_j),$$

and

$$h(z) = \sum_{j=0}^{n} (n-j)!\, b_{n-j} f^{(j)}(z).$$

If all the zeros of $f(z)$ lie in a circular region A, then all the zeros of $h(z)$ lie in the point set C consisting of n circular regions obtained by translating A in the amount and direction of the vectors β_j.

The second result by Trefethen concerns the asymptotic rate of the error in uniform best rational approximation to e^z on a disk. It was obtained by applying a method of Braess.

THEOREM 3.2 (cf. [30]). *Let $m, n \geqslant 0$ be integers, and let $E_{m,n}(e^z, R)$ denote the error in rational best uniform approximation of type (m, n) to e^z on the disk $|z| \leqslant R$. Then, as $m+n \to \infty$,*

$$E_{m,n}(e^z, R) = \delta_{m,n} R^{m+n+1}(1+o(1)),$$

where $\delta_{m,n}$ is defined as in Theorem 2.3.

The next result connects rational interpolants of the exponential function on a disk and on a segment, and was derived in [18]. One may also find a proof in [3], [6], or [21].

THEOREM 3.3 (Technique of Newman). *Let $R > 0$ be a fixed real number, P/Q a rational function of type (m, n) with real coefficients, and define*

$$\hat{P}(x, R) = |P(R\zeta)|^2, \quad \hat{Q}(x, R) = |Q(R\zeta)|^2, \quad |\zeta| = 1, \quad x = \operatorname{Re}(\zeta). \tag{3.1}$$

Then $\hat{P}(x, R)$ and $\hat{Q}(x, R)$ are polynomials in x and $\hat{P}(x, R)/\hat{Q}(x, R)$ is again of type (m, n). Assume the following three assertions hold:

(i) *The polynomial $Q(z)$ has no zeros on $\{|z| \leqslant R\}$.*

(ii) *For any complex number z of modulus R, we have*

$$|e^z - P/Q(z)| < 2\,|e^z|.$$

(iii) *P/Q interpolates e^z in k points (counting multiplicities) in $\{|z| \leqslant R\}$.*

Then, the rational function $\hat{P}(x, R)/\hat{Q}(x, R)$ interpolates e^{2Rx} in at least k points of $[-1, 1]$, counting multiplicities.

The fourth reminder concerns various results about rational interpolation of the exponential function on the real line.

THEOREM 3.4 (cf. [2, Theorem 2.1]). *Let $B^{(m+n)} := \{x_k^{(m+n)}\}_{k=0}^{m+n+1}$, $m = m_\nu$, $n = n_\nu$ be a triangular sequence of (not necessarily distinct) real interpolation points contained in the interval $[-R, R]$ such that*

$$\lim_{\nu \to \infty} m_\nu + n_\nu = \infty, \tag{3.2}$$

and denote by $\hat{R}_{m,n} = \hat{P}_{m,n}/\hat{Q}_{m,n}$ *the rational function of type* (m, n) *that interpolates* e^z *in* $B^{(m+n)}$. *Then*

$$\lim_{\nu \to \infty} \hat{R}_{m_\nu, n_\nu}(z) = e^z \tag{3.3}$$

locally uniformly in $\mathbb{C}$. *Furthermore, if* (m, n) *is a ray sequence,*

$$\lim_{\nu \to \infty} \frac{m_\nu}{n_\nu} = \lambda \qquad (0 \leqslant \lambda \leqslant +\infty), \tag{3.4}$$

then we also have, as $\nu \to \infty$,

$$\hat{P}_{m_\nu, n_\nu}(z) \to e^{\lambda z/(1+\lambda)} \qquad \text{and} \qquad \hat{Q}_{m_\nu, n_\nu}(z) \to e^{-z/(1+\lambda)} \tag{3.5}$$

locally uniformly in $\mathbb{C}$, *where* $\hat{Q}_{m_\nu, n_\nu}$ *is normalized so that* $\hat{Q}_{m_\nu, n_\nu}(0) = 1$.

Estimates on the zeros and poles of $\hat{R}_{m,n}$ are reminded in the next proposition.

PROPOSITION 3.5 (cf. [2, Proposition 2.8]). *With the same assumptions as in Theorem* 3.4, *all zeros of* $\hat{Q}_{m,n}$, *say* $a_k^{(n)}$, *satisfy*

$$m + 1 - R \leqslant |a_k^{(n)}|, \qquad k = 1, ..., n. \tag{3.6}$$

Symmetrically, all zeros of $\hat{P}_{m,n}$, *say* $b_j^{(m)}$, *satisfy*

$$n + 1 - R \leqslant |b_j^{(m)}|, \qquad m \geqslant 1, \quad j = 1, ..., m. \tag{3.7}$$

Moreover, all zeros $a_k^{(n)}$ *and* $b_j^{(m)}$ *lie within the corona*

$$(m+n)\,\gamma - R \leqslant |z| \leqslant m + n + R + 4/3, \qquad m \geqslant 1, \tag{3.8}$$

where $\gamma \simeq 0.278$ *is the unique positive root of* $\gamma e^{1+\gamma} = 1$, *and respectively remain within R units distance from the zeros of the Padé denominator and numerator to the exponential function.*

Proof. These estimates follow from those corresponding to the Padé approximants $P^0_{m,n}/Q^0_{m,n}$ where $R = 0$ (cf. [24, Theorem 22]) and Walsh's theorem 3.1, see the proof of [2, Lemma 2.4(i)]. Actually, we shall use the same proof in Step 1 of the forthcoming derivation of Theorem 2.1. ∎

We also need to consider the polynomials $\tilde{P}_{m,n}$ and $\tilde{Q}_{m,n}$ which are obtained upon dividing $\hat{P}_{m,n}$ and $\hat{Q}_{m,n}$ by the leading coefficient of the latter. Hence, $\tilde{Q}_{m,n}$ is monic. In the next lemma, which display estimates on the leading and constant coefficients of $\tilde{P}_{m,n}$ and $\tilde{Q}_{m,n}$, we need to keep track of the interpolation scheme. We use a superscript for this purpose. The scheme $-B$ refers to the negatives of the points of B.

LEMMA 3.6 (cf. [2, Lemma 2.5]). *Let $m=m_\nu$, $n=n_\nu$, satisfy (3.2) and (3.4). Define $\tilde{p}^B_{m,n}$ to be the leading coefficient of $\tilde{P}^B_{m,n}$. For any real number $0<\eta<1$, we have*

$$\eta e^{-R/(1+\lambda)} \leqslant (-1)^n \frac{m!}{(m+n)!} \tilde{Q}^B_{m,n}(0) \leqslant \frac{1}{\eta} e^{R/(1+\lambda)} \tag{3.9}$$

as soon as ν is large enough. Similarly, for any $0<\eta'<1$, we have

$$\eta' \frac{(-1)^n \tilde{Q}^B_{m,n}(0)}{(-1)^m \tilde{Q}^{-B}_{n,m}(0)} \leqslant (-1)^n \tilde{p}^B_{m,n} \leqslant \frac{1}{\eta'} \frac{(-1)^n \tilde{Q}^B_{m,n}(0)}{(-1)^m \tilde{Q}^{-B}_{n,m}(0)}, \tag{3.10}$$

as soon as ν is large enough.

Finally, we shall also use the following version of Rolle's theorem in the complex domain. Here, the result is stated for expressions of the type $Q(z)\,e^z - P(z)$, but a similar assertion holds more generally for any real exponential polynomials (see [35]).

THEOREM 3.7 (cf. [35, Theorem 3.2]). *Let ε, l and $\rho<2\pi$ be three real positive numbers. Let $(g_\nu)_{\nu\in\mathbb{N}}$, be a sequence of real exponential polynomials such that*

$$g_\nu(z) = Q_{m_\nu,n_\nu}(z)\,e^z - P_{m_\nu,n_\nu}(z), \quad \deg P_{m_\nu,n_\nu} = m_\nu, \quad \deg Q_{m_\nu,n_\nu} = n_\nu$$

and

$$\lim_{\nu\to\infty} m_\nu + n_\nu = \infty.$$

Furthermore, let r_ν be a sequence of integers with $1\leqslant r_\nu \leqslant m_\nu+1$, satisfying

$$4\left(1-\frac{\rho}{2}\cot\frac{\rho}{2}\right) < \inf_{\nu\in\mathbb{N}} \frac{m_\nu+n_\nu+2-r_\nu}{r_\nu+1}. \tag{3.11}$$

For each ν, assume that g_ν has $L_\nu \geqslant m_\nu+n_\nu+1$ zeros in the domain $\mathscr{L}_{l,\rho,1}$. Then, there exists a positive integer C depending on ε, l and ρ such that the r_ν-th derivative $g_\nu^{(r_\nu)}$ of g_ν has at least $L_\nu - r_\nu$ zeros in the domain $\mathscr{L}_{l,\rho+\varepsilon,r_\nu+1}$, as soon as the degree of g_ν is larger than C.

Moreover, if

$$\lim_{\nu\to\infty} r_\nu = \infty \qquad \text{and} \quad \lim_{\nu\to\infty} \frac{m_\nu + n_\nu}{r_\nu} = \mu \qquad (1 \leqslant \mu \leqslant \infty), \tag{3.12}$$

assumption (3.11) *in the previous assertion may be replaced with*

$$4\left(1 - \frac{\rho}{2}\cot\frac{\rho}{2}\right) < \mu - 1. \tag{3.13}$$

4. PROOFS

We are now in a position to demonstrate the results stated in Section 2.

Proof of Theorem 2.1. We perform the proof in three steps.

Step 1: *On the location of the zeros and poles of the interpolants.* Recall that the polynomials $P_{m,n}$ and $Q_{m,n}$ satisfy (2.2), where the roots of ω_{m+n+1} all lie in the bounded strip $\mathscr{L}_{l,\rho,1}$, $\rho < 2\pi$, and that the degrees $m = m_\nu$, $n = n_\nu$ satisfy condition (2.3). We split the sequence (m_ν, n_ν) into two subsequences with indices $\nu \in I_1$ or $\nu \in I_2$, $I_1 \cup I_2 = \mathbb{N}$, in the following way. The first subsequence corresponds to indices $\nu \in I_1$ such that the maximum in the right-hand side of (2.3) is taken by the ratio $n_\nu + 1/m_\nu + 2$, while the second one corresponds to indices $\nu \in I_2$ such that the maximum is taken by the ratio $m_\nu + 1/n_\nu + 2$. We consider these two subsequences separately.

For the first subsequence, Theorem 3.7 applies with $g_\nu(z) = Q_{m_\nu, n_\nu}(z)\, e^z - P_{m_\nu, n_\nu}(z)$ and $r_\nu = \deg P_{m_\nu, n_\nu} + 1$. It shows that the derivative

$$g_\nu^{(\deg P_{m_\nu, n_\nu} + 1)} = ((I + D)^{\deg P_{m_\nu, n_\nu} + 1} Q_{m_\nu, n_\nu})\, e^z,$$

where D denotes differentiation, has $m_\nu + n_\nu - \deg P_{m_\nu, n_\nu}$ zeros in the domain $\mathscr{L}_{l,\rho+\varepsilon,\deg P_{m_\nu, n_\nu}+2}$, $\varepsilon > 0$, and $\nu \in I_1$ large. Since

$$s_{n_\nu} := (I + D)^{\deg P_{m_\nu, n_\nu} + 1} Q_{m_\nu, n_\nu}$$

is a polynomial of degree equal to the degree of Q_{m_ν, n_ν}, we deduce that

$$\deg Q_{m_\nu, n_\nu} \geqslant m_\nu + n_\nu - \deg P_{m_\nu, n_\nu}.$$

Hence P_{m_ν, n_ν} and Q_{m_ν, n_ν} cannot be of degree less than m_ν and n_ν respectively. This proves assertion (i) for $\nu \in I_1$. Moreover, since all the zeros of the polynomial s_{n_ν} lie in the domain $\mathscr{L}_{l,\rho+\varepsilon, m_\nu+2}$, $\nu \in I_1$ large, they all lie *a* *fortiori* interior to the closed disk $D(0, (\rho+\varepsilon)\sqrt{m_\nu+2+l})$, whose radius equals half the diameter of $\mathscr{L}_{l, \rho+\varepsilon, m_\nu+2}$. Now, writing the Taylor expansion

$$(1+x)^{-(m_\nu+1)}=\sum_{j=0}^{\infty}(-1)^j\binom{m_\nu+j}{m_\nu}x^j,$$

we get

$$Q_{m_\nu, n_\nu}=\sum_{j=0}^{n_\nu}(-1)^j\binom{m_\nu+j}{m_\nu}s_{n_\nu}^{(j)},$$

where we have used the fact that s_{n_ν} is a polynomial of degree n_ν. Then, one checks easily that Theorem 3.1 applies with $f(z)=s_{n_\nu}(z)$, $g(z)=Q^0_{m_\nu, n_\nu}(z)/n_\nu!$ and $h(z)=Q_{m_\nu, n_\nu}(z)$, where $Q^0_{m_\nu, n_\nu}$ denotes the monic Padé denominator. From (3.8) where we take $\rho=0$, we know that the zeros of $Q^0_{m_\nu, n_\nu}$ have modulus larger than or equal to $(m_\nu+n_\nu)\gamma$, thus, by Walsh's theorem, all the zeros of Q_{m_ν, n_ν} have modulus larger than or equal to $(m_\nu+n_\nu)\gamma-(\rho+\varepsilon)\sqrt{m_\nu+2-l}$. Hence, in view of (2.1), all the zeros of Q_{m_ν, n_ν} tend to infinity as $\nu\in I_1$ becomes large.

For the second subsequence, Theorem 3.7 applies with $Q_{m_\nu, n_\nu}(-z)-e^zP_{m_\nu, n_\nu}(-z)$ and $r_\nu=\deg Q_{m_\nu, n_\nu}+1$. Reasoning as in the previous case shows that $\deg P_{m_\nu, n_\nu}=m_\nu$, $\deg Q_{m_\nu, n_\nu}=n_\nu$, so that assertion (i) of Theorem 2.1 is proved for any $\nu\in\mathbb{N}$ large. It also shows that all zeros of P_{m_ν, n_ν} have modulus larger than or equal to $(m_\nu+n_\nu)\gamma-(\rho+\varepsilon)\sqrt{n_\nu+2-l}$ for $\nu\in I_2$ large. In view of (2.1), we deduce this time that all the zeros of P_{m_ν, n_ν} tend to infinity as $\nu\in I_2$ becomes large.

Step 2: *Locally uniform convergence of the rational interpolants.* By assumption, all interpolation points lie in the disk $D(0, \tilde{\rho})$ with $\tilde{\rho}=l+\rho$. We show that the rational interpolants $R_{m,n}$ are *near-best approximants* i.e. satisfy, for any $R\geqslant\tilde{\rho}$, and any given $0<\delta<1$,

$$\|Q_{m,n}e^z-P_{m,n}\|_R\leqslant\|Q_{m,n}\|_R\,E_{m,n}(e^z, R)^{1-\delta}, \tag{4.1}$$

$m+n$ large enough. This is essentially well known (cf. [11, 14, 32]) for any function admitting faster than geometric rational approximation or, equivalently, functions belonging to the *Gonchar–Walsh class*, but we prefer to include a proof for completeness. We denote by $R^*_{m,n}=Q^*_{m,n}/P^*_{m,n}$, a rational best approximant of type (m, n) to e^z on the closed disk $D(0, R+\eta)$, where $\eta>0$. We have

$$\begin{aligned} &Q^*_{m,n}(Q_{m,n}e^z-P_{m,n})\\ &\qquad=Q^*_{m,n}Q_{m,n}(e^z-R^*_{m,n})+(Q_{m,n}P^*_{m,n}-Q^*_{m,n}P_{m,n}),\end{aligned}$$

where the last term in the right-hand side is a polynomial of degree at most $m+n$. Hence, dividing this equality by the polynomial of interpolation ω_{m+n+1}, and applying Cauchy's formula on the circle $T_{R+\eta}$, we get for z inside $T_{R+\eta}$,

$$Q^*_{m,n}(z)\,\frac{Q_{m,n}e^z-P_{m,n}}{\omega_{m+n+1}}(z)=\frac{1}{2i\pi}\int_{T_{R+\eta}}\frac{Q^*_{m,n}Q_{m,n}(e^z-R^*_{m,n})(t)}{\omega_{m+n+1}(t)(t-z)}\,dt.$$

Since the roots of ω_{m+n+1} remain in the disk $D(0,\tilde{\rho})$, there exists a constant C_1 such that

$$\|\omega_{m+n+1}\|_R/\min_{t\in T_{R+\eta}}|\omega_{m+n+1}(t)|\leqslant C_1^{m+n+1}.$$

Moreover, as all zeros of $Q^*_{m,n}$ lie outside the disk $D(0,R+\eta)$, there exists a constant C_2 such that

$$\|Q^*_{m,n}\|_{R+\eta}/\min_{z\in T_R}|Q^*_{m,n}(z)|\leqslant C_2^n.$$

Thus, we obtain

$$\begin{aligned}\|Q_{m,n}e^z-P_{m,n}\|_R&\leqslant C_3C_1^{m+n+1}C_2^n\,\|Q_{m,n}\|_{R+\eta}\,E_{m,n}(e^z,R+\eta)\\&\leqslant\|Q_{m,n}\|_R\,E_{m,n}(e^z,R)^{1-\delta}.\end{aligned}$$

In the second inequality, we have used Bernstein inequality for comparing the norm of the polynomial $Q_{m,n}$ on the circles of radius R and $R+\eta$, and the fact that (cf. Theorem 3.2)

$$\lim_{m+n\to\infty}E_{m,n}(e^z,R)^{1/(m+n)}=R\lim_{m+n\to\infty}\delta_{m,n}^{1/(m+n)}=0,$$

where the last equality is easily checked from the definition (2.9) of $\delta_{m,n}$. This proves (4.1). Let us observe that the inequality

$$\|P_{m,n}e^{-z}-Q_{m,n}\|_R\leqslant\|P_{m,n}\|_R\,E_{m,n}(e^z,R)^{1-\delta},\tag{4.2}$$

$m+n$ large enough, can be obtained in the same way we proved (4.1), since $Q_{m,n}/P_{m,n}(-z)$ is a linearized rational interpolant to e^z in the points opposite to those of $B^{(m+n)}$.

Now, as in Step 1, we split the sequence (m_ν,n_ν) into the two subsequences with indices $\nu\in I_1$ and $\nu\in I_2$ respectively. Let us first consider the subsequence with indices $\nu \in I_1$. For any disk $D(0, R)$, $R > \tilde{\rho}$, we deduce from (4.1) that

$$\|e^z - P_{m,n}/Q_{m,n}\|_R \leqslant \frac{\|Q_{m,n}\|_R}{\min_{z \in T_R} |Q_{m,n}(z)|} E_{m,n}(e^z, R)^{1-\delta}.$$

For $\nu \in I_1$ large, the right-hand side tends to zero. Indeed, by the result of Step 1, we know that the roots of $Q_{m,n}$ become of modulus larger than $R' > R$, hence there exits a constant C such that the ratio in the right-hand side is less than C^n. This implies the uniform convergence of $R_{m,n}$ to e^z in any compact set of $\mathbb{C}$, hence proves (2.4) when $\nu \in I_1$. As a consequence of this convergence, we deduce also that all the zeros of $P_{m,n}$ tend to infinity for $\nu \in I_1$ large.

For the second subsequence with indices $\nu \in I_2$, we deduce from (4.2) that

$$\|e^{-z} - Q_{m,n}/P_{m,n}\|_R \leqslant \frac{\|P_{m,n}\|_R}{\min_{z \in T_R} |P_{m,n}(z)|} E_{m,n}(e^z, R)^{1-\delta},$$

for any disk $D(0, R)$, $R > \tilde{\rho}$. For $\nu \in I_2$ large, the right-hand side tends to zero since we know by the result of Step 1 that the roots of $P_{m,n}$ become of modulus larger than $R' > R$. This implies the uniform convergence of $1/R_{m,n}$ to e^{-z} in any compact set of $\mathbb{C}$. Consequently, all the zeros of $Q_{m,n}$ tend to infinity for $\nu \in I_2$ large, and $R_{m,n}$ converges locally uniformly to e^z in $\mathbb{C}$. So far, we have proved assertions (i), (ii), and (iii) of Theorem 2.1. If (2.5) holds, it remains to show the separated convergence of $P_{m,n}$ and $Q_{m,n}$. This is the goal of the next step.

Step 3: *Applying Newman's technique*. By the uniform convergence of the interpolants $R_{m,n}$ established in Step 2, condition (i) of Theorem 3.3 is satisfied in the disk $D(0, \tilde{\rho})$, $\tilde{\rho} = l + \rho$, for ν large. It also implies that condition (ii) is met on the circle $T_{\tilde{\rho}}$, for ν large. Moreover, by assumption, condition (iii) is satisfied in $D(0, \tilde{\rho})$ as well. We set

$$\hat{P}_m(x) = |P_{m,n}(\tilde{\rho}\zeta)|^2, \quad \hat{Q}_n(x) = |Q_{m,n}(\tilde{\rho}\zeta)|^2, \quad |\zeta| = 1, \quad x \in \mathrm{Re}(\zeta), \tag{4.3}$$

and apply Theorem 3.3 on $D(0, \tilde{\rho})$: the rational function $\hat{R}_{m,n}(z) = \hat{P}_m/\hat{Q}_n(z)$ interpolates $e^{2\tilde{\rho}z}$ at $m+n+1$ points of $[-1, 1]$. Thus, $\hat{P}_m/\hat{Q}_n(z/2\tilde{\rho})$ interpolates e^z at $m+n+1$ points of $[-2\tilde{\rho}, 2\tilde{\rho}]$, and we deduce from Theorem 3.4 that

$$\hat{R}_{m,n}(z) \to e^{2\tilde{\rho}z},$$

locally uniformly in $\mathbb{C}$, and moreover that

$$\hat{P}_m(z)/\hat{Q}_n(0) \to e^{2\tilde{\rho}\lambda z/1+\lambda} \qquad \text{and} \qquad \hat{Q}_n(z)/\hat{Q}_n(0) \to e^{-2\tilde{\rho}z/1+\lambda} \tag{4.4}$$

locally uniformly in $\mathbb{C}$, since we are assuming that (2.5) holds. Denoting by $a_k^{(n)}$, the zeros of $\hat{Q}_n(z)$, $k=1, ..., n$, we see from the first inequality in (3.8) that

$$(m+n)\,\gamma - 2\tilde{\rho} \leqslant |2\tilde{\rho} a_k^{(n)}|, \qquad k=1, ..., n. \tag{4.5}$$

Moreover, from (4.3), we know the relation between the zeros $\alpha_k^{(n)}$ of $Q_{m,n}$ and the zeros of $\hat{Q}_n$,

$$2a_k^{(n)} = \frac{\alpha_k^{(n)}}{\tilde{\rho}} + \frac{\tilde{\rho}}{\alpha_k^{(n)}}, \qquad k=1, ..., n. \tag{4.6}$$

The inequality (4.5), the relations (4.6), and the fact that the polynomial $Q_{m,n}$ has no zeros near the origin implies that, for v large,

$$(m+n)\,\gamma/\alpha \leqslant |\alpha_k^{(n)}|, \qquad k=1, ..., n, \tag{4.7}$$

where α is any real number $\alpha > 1$. We prove that $(Q_{m,n})$ is a normal family. Indeed, when v is large enough,

$$|Q_{m,n}(z)| = \prod_{k=1}^{n} |1 - z/\alpha_k^{(n)}| \leqslant (1 + \alpha\,|z|/(m+n)\,\gamma)^n \leqslant e^{\alpha\,|z|/\gamma}.$$

Note that $Q_{m,n}$ is normalized so that $Q_{m,n}(0)=1$, which is possible as, by assumption, $Q_{m,n}$ has no zeros in a neighborhood of the origin, v large. In addition, from the definition (4.3) of $\hat{Q}_n$, we get that

$$\hat{Q}_n(0) = |Q_{m,n}(i\tilde{\rho})|^2 = \prod_{k=1}^{n} (1 + \tilde{\rho}^2/\alpha_k^{(n)^2})$$

is bounded from below by some positive constant, thanks to (4.7). Hence

$$h_n = Q_{m,n}/\sqrt{\hat{Q}_n(0)}$$

again defines a normal family of functions. Let $h = \lim_{k\to\infty} h_{n_k}$ be a limit function of this family, and notice that, on $T_{\tilde{\rho}}$,

$$|h(\zeta)|^2 = \lim_{k\to\infty} \hat{Q}_{n_k}(\mathrm{Re}(\zeta)/\tilde{\rho})/\hat{Q}_{n_k}(0) = |e^{-2\zeta/1+\lambda}|$$

by (4.4). This entails that h does not vanish identically, and as h_n is zero-free in $D(0, (m+n)\,\gamma/\alpha)$ for n large by (4.7), we derive from Hurwitz's theorem that h is zero-free in $\mathbb{C}$. Therefore $h = e^{-z/1+\lambda}$, because these two functions share the same modulus on $T_{\tilde{\rho}}$, have no zeros in $D(0, \tilde{\rho})$, and $h(0) > 0$. Thus, h_n actually converges to $e^{-z/1+\lambda}$ since this is the only possible limit function. As $Q_{m,n}(0) = 1$ for all indices ν, we now deduce that $\hat{Q}_n(0) \to 1$ so that $Q_{m,n}(z) \to e^{-z/1+\lambda}$ as $\nu \to \infty$, locally uniformly in $\mathbb{C}$. The uniform convergence of $P_{m,n}$ to $e^{\lambda z/1+\lambda}$ in any compact set of $\mathbb{C}$ follows from the above convergence of $Q_{m,n}$ and the locally uniform convergence of $R_{m,n}$ to e^z in $\mathbb{C}$. This proves (2.6) and finishes the proof of Theorem 2.1. ▌

Proof of Proposition 2.2. We apply Theorem 3.3 on the disk $D(0, \tilde{\rho})$ as we did in Step 3 of the proof of Theorem 2.1. Recall the definition (4.3) of the polynomial $\hat{Q}_n(z)$. Since $\hat{P}_m/\hat{Q}_n(z/2\tilde{\rho})$ interpolates e^z at $m+n+1$ points of the segment $[-2\tilde{\rho}, 2\tilde{\rho}]$, we know that the zeros of the polynomial $\hat{Q}_n(z/2\tilde{\rho})$ satisfy the assertions of Proposition 3.5 with R replaced with $2\tilde{\rho}$. Multiplying by $\tilde{\rho}$ the relation (4.6) between the zeros $\alpha_k^{(n)}$ of $Q_{m,n}$ and the zeros $a_k^{(n)}$ of $\hat{Q}_n$, we get

$$2\tilde{\rho} a_k^{(n)} = \alpha_k^{(n)} + \frac{\tilde{\rho}^2}{\alpha_k^{(n)}}, \qquad k = 1, ..., n.$$

On the other hand, from the estimates (3.8), there exists two constants C_1 and C_2 such that

$$C_1(m+n) \leqslant |a_k^{(n)}| \leqslant C_2(m+n), \qquad k = 1, ..., n,$$

and since $Q_{m,n}$ does not vanish near the origin similar inequalities hold for its zeros. Thus,

$$2\tilde{\rho} a_k^{(n)} = \alpha_k^{(n)} + O(1/(m+n)), \qquad k = 1, ..., n,$$

whence the asserted inequalities and assertion for the zeros of $Q_{m,n}$ since the left-hand side of the previous relations are equal to the zeros of $\hat{Q}_n(z/2\tilde{\rho})$. The corresponding estimates for the zeros of the polynomial $P_{m,n}$ follow in a similar way from the remark that $Q_{m,n}/P_{m,n}$ interpolates e^z at the negative of the points of $B^{(m+n)}$. ▌

Proof of Theorem 2.3. The technique is borrowed from [5] and was also used in [2]. As a first step, we will treat the diagonal case and write $R_j = P_j/Q_j$ for the multipoint interpolant of type (j, j). Moreover, for j large, *we will assume that Q_j is normalized to be monic*. Note that, by assertion (i) of Theorem 2.1, this is always possible. Since several interpolation schemes enter into the proof, we shall keep track of them by using a superscript as in (3.9) and (3.10). As in [5] and [2], we consider for each positive n the triangular interpolation scheme C_n whose $(2n+2k)$th row is obtained by adding to the set $B^{(2n)} = \{z_i^{(2n)}\}_{i=0}^{2n}$ the point zero with multiplicity $2k$ for $k \geqslant 0$, while the first $2n-1$ rows can be chosen arbitrarily in $D(0, \tilde{\rho})$. This defines a family of interpolation schemes indexed by n, and it is important to notice that $B^{(2n)} = C_n^{(2n)}$ for each n. With the scheme C_n, we consider the rational interpolants $R_{n+k}^{C_n}$, $k \geqslant 0$, of type $(n+k, n+k)$ which still define a ray sequence with $\lambda = 1$. By assertion (iii) of Theorem 2.1 as applied to C_n, we have that for any $z \in \mathbb{C}$,

$$e^z - R_n^B(z) = \sum_{k=0}^{\infty} [R_{n+k+1}^{C_n}(z) - R_{n+k}^{C_n}(z)]. \tag{4.8}$$

From the interpolation conditions and upon checking degrees, we get the following factorization:

$$D_k(z) := R_{n+k+1}^{C_n}(z) - R_{n+k}^{C_n}(z) = \frac{\beta_{k+1} z^{2k} \prod_{i=0}^{2n} (z - z_i^{(2n)})}{Q_{n+k}^{C_n}(z)\, Q_{n+k+1}^{C_n}(z)}, \tag{4.9}$$

where β_{k+1} is the leading coefficient of $P_{n+k+1}^{C_n} Q_{n+k}^{C_n} - P_{n+k}^{C_n} Q_{n+k+1}^{C_n}$. As the polynomials $Q_{n+k}^{C_n}$ and $Q_{n+k+1}^{C_n}$ are monic, we have

$$\beta_{k+1} = p_{n+k+1}^{C_n} - p_{n+k}^{C_n}, \tag{4.10}$$

where $p_{n+k}^{C_n}$ and $p_{n+k+1}^{C_n}$ denote the leading coefficients of $P_{n+k}^{C_n}$ and $P_{n+k+1}^{C_n}$, respectively.

We derive first the upper estimate in (2.8). Since C_n is a scheme satisfying the assumptions of Theorem 2.1 and $R_j^{C_n}$ is a ray sequence with $\lambda = 1$, we obtain from the limit (2.4) and the second limit in (2.6) that for any $0 < \delta < 1$ and j large enough

$$\frac{P_j^{C_n}(0)}{Q_j^{C_n}(0)} \geqslant \delta, \qquad \frac{|Q_j^{C_n}(0)|}{|Q_j^{C_n}(i\tilde{\rho})|} \geqslant \delta, \tag{4.11}$$

and

$$|Q_j^{C_n}(z)| \geqslant \frac{\delta\, |Q_j^{C_n}(0)|}{\sqrt{c_1}}, \qquad z \in K. \tag{4.12}$$

Moreover, it is clear from Theorem 2.1 that the three conditions of Theorem 3.3 are met with the interpolant $R_j^{C_n}$, j large enough. Hence, when n is fixed and $j = n+k$ is large enough, (4.11), (4.12) hold true and Theorem 3.3 applies. *We claim there exists n_0 such that the previous assertion holds true for any scheme C_n and all $k \geqslant 0$, as soon as n is larger than n_0*. Indeed, assume the contrary. Then, we can find a sequence $n' + k'$ with $n' \to \infty$ and $k' \geqslant 0$ such that the interpolant $R_{n'+k'}^{C_{n'}}$ constantly violates the assertion. But the scheme obtained by selecting for each pair of indices (n', k'), the row $C_{n'}^{(2n'+2k')}$ is again a ray sequence of interpolation points in $D(0, \tilde{\rho})$ to which our analysis can be applied. *This proves the claim by contradiction.*

Now, let us denote by $\hat{R}_j^{\hat{C}_n} = \hat{P}_j^{\hat{C}_n} / \hat{Q}_j^{\hat{C}_n}$, the rational function of type (j, j) such that

$$\hat{P}_j^{\hat{C}_n}(2\tilde{\rho}x) = |P_j^{C_n}(\tilde{\rho}\zeta)|^2, \quad \hat{Q}_j^{\hat{C}_n}(2\tilde{\rho}x) = |Q_j^{C_n}(\tilde{\rho}\zeta)|^2, \quad |\zeta| = 1, \quad x = \mathrm{Re}(\zeta),$$

which, from Theorem 3.3, interpolates e^z in $2j+1$ points of $[-2\tilde{\rho}, 2\tilde{\rho}]$, as soon as $n > n_0$ and $k \geqslant 0$. Observe that, by definition, the scheme $\hat{C}_n$ has its $(2j)$-th row, $j \geqslant n$, consisting of these $2j+1$ points. Let $\hat{p}_j^{\hat{C}_n}$ and $\hat{q}_j^{\hat{C}_n}$ denote the leading coefficients $\hat{P}_j^{\hat{C}_n}$ and $\hat{Q}_j^{\hat{C}_n}$. From the previous relations, we check that

$$\hat{p}_j^{\hat{C}_n} = P_j^{C_n}(0)\, p_j^{C_n}, \qquad \hat{q}_j^{\hat{C}_n} = Q_j^{C_n}(0), \tag{4.13}$$

recalling that $Q_j^{C_n}$ has been chosen to be monic. As in Theorem 3.4, we denote by $\tilde{P}_j^{\hat{C}_n}$ and $\tilde{Q}_j^{\hat{C}_n}$ the polynomials which are obtained upon dividing $\hat{P}_j^{\hat{C}_n}$ and $\hat{Q}_j^{\hat{C}_n}$ by the leading coefficient of the latter, so that

$$\tilde{p}_j^{\hat{C}_n} = p_j^{C_n} P_j^{C_n}(0) / Q_j^{C_n}(0), \tag{4.14}$$

where $\tilde{p}_j^{\hat{C}_n}$ denotes the leading coefficient of $\tilde{P}_j^{\hat{C}_n}$. Moreover, by the second inequality in (3.10), we know that for any $0 < \eta' < 1$ we have

$$(-1)^l\, \tilde{p}_j^{\hat{C}_n} \leqslant \frac{1}{\eta'} \frac{\tilde{Q}_j^{\hat{C}_n}(0)}{\tilde{Q}_j^{-\hat{C}_n}(0)} \tag{4.15}$$

for j large enough. Using the lower estimate in (3.9) for the real schemes $\hat{C}_n$ and $-\hat{C}_n$ in $[-2\tilde{\rho}, 2\tilde{\rho}]$, we further obtain that for any $0 < \eta < 1$

$$\eta e^{-\tilde{\rho}} \leqslant (-1)^j \frac{j!}{(2j)!} \tilde{Q}_j^{\hat{C}_n}(0), \qquad \eta e^{-\tilde{\rho}} \leqslant (-1)^j \frac{j!}{(2j)!} \tilde{Q}_j^{-\hat{C}_n}(0), \tag{4.16}$$

for j large. Reasoning by contradiction as we did after (4.12), we get that *there exists n_1 such that the inequalities* (4.15) *and* (4.16) *hold true for any scheme $\hat{C}_n$ and all $j = n + k \geqslant n$, as soon as n is larger than n_1.*

Now, from the inequalities (4.11), (4.12), (4.15), and the relation (4.14) where we substitute $n+k$ and then $n+k+1$ for j, we get in view of (4.9)

$$|D_k(z)| \leqslant \frac{c_1}{\eta' \delta^3} \left[\left| \frac{\tilde{Q}_{n+k+1}^{\hat{C}_n}(0)}{\tilde{Q}_{n+k+1}^{-\hat{C}_n}(0)} \right| + \left| \frac{\tilde{Q}_{n+k}^{\hat{C}_n}(0)}{\tilde{Q}_{n+k}^{-\hat{C}_n}(0)} \right| \right] \left| \frac{z^{2k} \prod_{i=0}^{2n} (z - z_i^{(2n)})}{Q_{n+k}^{C_n}(0)\, Q_{n+k+1}^{C_n}(0)} \right|, \tag{4.17}$$

as soon as $n > \max(n_0, n_1)$ and $k \geqslant 0$. On the other hand, from the second inequality in (4.11), the definition of $\hat{Q}_j^{\hat{C}_n}$, $\tilde{Q}_j^{\hat{C}_n}$, and the second equality in (4.13), we obtain successively

$$\begin{aligned}(Q_j^{C_n}(0))^2 &\geqslant \delta^2\,|Q_j^{C_n}(i\tilde{\rho})|^2 = \delta^2 \hat{Q}_j^{\hat{C}_n}(0) = \delta^2 \tilde{Q}_j^{\hat{C}_n}(0)\,\hat{q}_j^{\hat{C}_n}\\ &= \delta^2 \tilde{Q}_j^{\hat{C}_n}(0)\,Q_j^{C_n}(0).\end{aligned} \tag{4.18}$$

Dividing by the absolute value of $Q_j^{C_n}(0)$, we get

$$|Q_j^{C_n}(0)| \geqslant \delta^2\,|\tilde{Q}_j^{\hat{C}_n}(0)|.$$

Making use of this inequality in (4.17), we deduce

$$\begin{aligned}|D_k(z)| \leqslant \frac{c_1}{\eta'\delta^7}&\left[\left|\frac{1}{\tilde{Q}_{n+k}^{\hat{C}_n}(0)\,\tilde{Q}_{n+k+1}^{-\hat{C}_n}(0)}\right| + \left|\frac{1}{\tilde{Q}_{n+k+1}^{\hat{C}_n}(0)\,\tilde{Q}_{n+k}^{-\hat{C}_n}(0)}\right|\right]\\ &\times |z|^{2k} \prod_{i=0}^{2n} |z - z_i^{(2n)}|\end{aligned}$$

as soon as $n > \max(n_0, n_1)$ and $k \geqslant 0$. Making use of (4.16), we now obtain

$$|D_k(z)| \leqslant 2\,\frac{e^{2\tilde{\rho}}c_1}{\eta'\delta^7\eta^2}\,\frac{(n+k+1)!\,(n+k)!}{(2n+2k+2)!\,(2n+2k)!}\,|z|^{2k} \prod_{i=0}^{2n} |z - z_i^{(2n)}|,$$

as soon as n is large enough and for all k. Note that

$$\begin{aligned}&\frac{(n+k+1)!\,(n+k+2)!}{(2n+2k+2)!\,(2n+2k+4)!}\,|z|^{2k+2}\Big/\frac{(n+k)!\,(n+k+1)!}{(2n+2k)!\,(2n+2k+2)!}\,|z|^{2k}\\ &\quad\leqslant \frac{|z|^2}{16(n+k)^2} \leqslant \frac{M^2}{16(n+k)^2},\end{aligned}$$

where $M = \max_{z\in K} |z|$. Suppose that n is so large that $M^2/16n^2 \leqslant 1$. Then, with $\alpha' = \eta'\eta^2\delta^7 < 1$,

$$\begin{aligned}|e^z - R_n^B(z)| &\leqslant \sum_{k=0}^{\infty} |D_k(z)|\\ &\leqslant \frac{2e^{2\tilde{\rho}}c_1}{\alpha'}\,\frac{n!(n+1)!}{(2n)!\,(2n+2)!}\left[1 + \sum_{k=1}^{\infty} \frac{M^2}{16(n+k)^2}\right] \times \prod_{i=0}^{2n} |z - z_i^{(2n)}|.\end{aligned} \tag{4.19}$$

As $\sum_k 1/(n+k)^2$ is the tail of a convergent series, the quantity

$$\left[1+\sum_{k=1}^{\infty} \frac{M^2}{16(n+k)^2}\right] \Big/ \alpha'$$

can be made arbitrarily close to 1 by choosing η, η' and δ close enough to 1 and n sufficiently large. Hence, we get that for any $\alpha < 1$ and n large enough

$$|e^z - R_n^B(z)| \leqslant \frac{c_1 e^{2\tilde{\rho}}}{\alpha} \frac{n!\, n!}{(2n)!\,(2n+1)!} \prod_{i=0}^{2n} |z - z_i^{(2n)}|,$$

which is our claimed upper estimate for $\lambda = 1$.

Let us now proceed with the lower estimate in (2.8). We have

$$|e^z - R_n^B(z)| \geqslant |D_0(z)| - \sum_{k=1}^{\infty} |D_k(z)|.$$

By Theorem 2.1, for any $0 < \delta < 1$, j large, we have

$$\frac{P_j^{C_n}(0)}{Q_j^{C_n}(0)} \leqslant \frac{1}{\delta}, \qquad \frac{|Q_j^{C_n}(0)|}{|Q_j^{C_n}(i\tilde{\rho})|} \leqslant \frac{1}{\delta}, \tag{4.20}$$

and

$$|Q_j^{C_n}(z)| \leqslant \frac{1}{\delta\sqrt{c_0}} |Q_j^{C_n}(0)| \qquad z \in K.$$

Note that from the second inequality in (4.20) and the chain of equality in (4.18), we deduce that

$$|Q_j^{C_n}(0)| \leqslant |\tilde{Q}_j^{\hat{C}_n}(0)|/\delta^2.$$

From (3.9) we observe that for any $0 < \eta < 1$, j large,

$$\left|\frac{j!}{(2j)!}\, \tilde{Q}_j^{\hat{C}_n}(0)\right| \leqslant \frac{e^{\tilde{\rho}}}{\eta}, \qquad \left|\frac{j!}{(2j)!}\, \tilde{Q}_j^{-\hat{C}_n}(0)\right| \leqslant \frac{e^{\tilde{\rho}}}{\eta}.$$

Moreover, from (3.10), we get for any $0 < \eta' < 1$ and $j = n+k$ large

$$\eta' \frac{\tilde{Q}_j^{\hat{C}_n}(0)}{\tilde{Q}_j^{-\hat{C}_n}(0)} \leqslant (-1)^j\, \tilde{p}_j^{C_n}.$$

Reasoning as before, these estimates can be made uniform with respect to n when the latter is sufficiently large and yield together with relation (4.14)

$$|D_0(z)| \geqslant 2\eta'\eta^2\delta^7 c_0 e^{-2\tilde{\rho}} \frac{n!(n+1)!}{(2n)!\,(2n+2)!} \prod_{i=0}^{2n} |z - z_i^{(2n)}|$$

$$= \eta'\eta^2\delta^7 c_0 e^{-2\tilde{\rho}} \frac{n!\,n!}{(2n)!\,(2n+1)} \prod_{i=0}^{2n} |z - z_i^{(2n)}|. \tag{4.21}$$

On the other hand, we know that for large n (cf. (4.19))

$$\sum_{k=1}^{\infty} |D_k(z)| \leqslant \frac{c_1 e^{2\tilde{\rho}}}{\alpha'} \frac{n!n!}{(2n)!\,(2n+1)!} \left[1 + \sum_{k=1}^{\infty} \frac{M^2}{16(n+k)^2}\right] \prod_{i=0}^{2n} |z - z_i^{(2n)}|.$$

As the term in (4.21) is dominant compared to the above summation, we get that for any $\alpha < 1$ and n large enough

$$|e^z - R_n^B(z)| \geqslant \alpha c_0 e^{-2\tilde{\rho}} \frac{n!\,n!}{(2n)!\,(2n+1)!} \prod_{i=0}^{2n} |z - z_i^{(2n)}|,$$

thereby establishing (2.8) in the diagonal case.

The general case where the rational function is of type (m, n) with $m = m_\nu$, $n = n_\nu$ satisfying (2.1), (2.5) and (2.7) can be handled in a similar way. It consists in decomposing, as in equality (4.8), the error $e^z - R_{m,n}^B(z)$ as a sum of differences by introducing schemes $C_{m,n}$ whose $(m+k_1)+(n+k_2)$-rows are obtained by adding the point zero with multiplicity $k_1 + k_2$ to the set $B^{(m+n)}$. As in the diagonal case, we also use Theorem 3.3 in order to exhibit rational interpolants on the segment $[-2\tilde{\rho}, 2\tilde{\rho}]$ to which the estimates in Lemma 3.6 can be applied. For more details, the interested reader may consult the proof of Theorem 2.2 in [2], where the same technique is applied for general ray sequences with respect to rational interpolants on finite segments of the real axis. ∎

Proof of Theorem 2.4. It can be done by contradiction and is identical to that of [2, Theorem 2.3]. Hence, we refer to [2] for the full proof.

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