Asymptotic upper bounds for the entropy of orthogonal polynomials in the Szegő class

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We give an asymptotic upper bound as $n \to \infty$ for the entropy integral, $E_n(w) = -\int p_n^2(x) \log(p_n^2(x)) w(x) dx$, where p_n is the nth degree orthonormal polynomial with respect to a weight $w(x)$ on $[-1,1]$ which belongs to the Szegő class. We also study two functionals closely related to the entropy integral. First, their asymptotic behavior is completely described for weights w in the Bernstein class. Then, as for the entropy, we obtain asymptotic upper bounds for these two functionals when $w(x)$ belongs to the Szegő class. In each case, we give conditions for these upper bounds to be attained. © *2004 American Institute of Physics.*
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I. INTRODUCTION

In the framework of the density functional theory (see, e.g., Refs. 6 and 11), the physical and chemical properties of fermionic systems are described by means of the single-particle probability densities. If $\Psi(\vec{r})$ is the wave function of a single-particle system in a (D-dimensional) position space, and $\hat{\Psi}(\vec{p})$ is the corresponding wave function in momentum space [that is, the Fourier transform of $\Psi(\vec{r})$], then the position and momentum densities of the system are given by

$$\rho(\vec{r}) = |\Psi(\vec{r})|^2, \quad \gamma(\vec{p}) = |\hat{\Psi}(\vec{p})|^2,$$

respectively. It is known that the Boltzmann–Gibbs–Shannon position–space entropy,

$$S(\rho) = -\int \rho(\vec{r}) \log \rho(\vec{r}) d\vec{r},$$

measures the uncertainty in the localization of the particle in space (lower entropy indicates a more concentrated wave function, with the associated higher accuracy in predicting the localization of the particle). The similar is true for the momentum–space entropy,

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$$S(\gamma) = -\int \gamma(\vec{p}) \log \gamma(\vec{p}) d\vec{p}.$$

These quantities have importance in the study of the structure and dynamics of atomic and molecular systems; we refer the reader to the survey[5] and to references therein. Both $S(\rho)$ and $S(\gamma)$ also play a role in a generalization of the Heisenberg uncertainty relation: it has been established[3] that for any pair of densities $\rho(\vec{r})$ and $\gamma(\vec{p})$ in D-dimensional space, we have the sharp inequality

$$S(\rho) + S(\gamma) \geqslant D(1 + \log \pi), \tag{1}$$

which expresses quantitatively the impossibility of the simultaneous localization of a pair of observables with no common eigenstates.

It is well known that the wave function of many important systems, such as D-dimensional harmonic oscillator and hydrogen atom, are expressible in terms of families of orthogonal polynomials. It is not surprising that, as it has been shown in Refs. 4 and 13, the computation of the entropies $S(\rho)$ and $S(\gamma)$ usually can be reduced to integrals involving these polynomials.

Let ν be a positive unit Borel measure on $\Delta := [-1,1]$ and let

$$p_n(x) = \gamma_n \prod_{j=1}^{n} (x - \zeta_j^{(n)}), \quad \gamma_n > 0, \quad n \in \mathbb{N},$$

denote the corresponding sequence of *orthonormal* polynomials such that

$$\int p_n(x) p_m(x) d\nu(x) = \delta_{mn}, \quad m,n \in \mathbb{N}.$$

We define the *information entropy* of the polynomials $p_n(x)$ as

$$E_n = E_n(\nu) = -\int p_n^2(x) \log(p_n^2(x)) d\nu(x). \tag{2}$$

Throughout the paper, we will assume that the orthogonality measure ν is absolutely continuous with respect to the Lebesgue measure λ on Δ with the Radon–Nikodym derivative

$$d\nu/d\lambda = \nu'(x) = w(x), \quad w \in L^1(\Delta).$$

For normalization purposes, we will always assume that the weight w is unitary, i.e.,

$$\int_\Delta w(x) dx = 1. \tag{3}$$

The information entropy will be indistinctly denoted by $E_n(\nu)$ and $E_n(w)$. We follow this convention below for other notations.

The asymptotic behavior of E_n as $n \to \infty$ has a special interest in the study of the so-called Rydberg states of quantum-mechanical systems. Besides physical motivations, there are some fascinating aspects of this problem because of a certain universal behavior of related integrals, and because of a close connection of the entropy E_n with important functionals of the normalized zero counting measures of the polynomials p_n,

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{\zeta_j^{(n)}}, \quad n > 0,$$

and of the following probability measures ν_n:

$$d\nu_n(x) = p_n^2(x)d\nu(x), \quad n \geqslant 0$$

(note that $\nu_0 = \nu$). Both measures are standard objects of study in the analytic theory of orthogonal polynomials. For instance, the normalized zero counting measure μ_n is closely connected with the nth root asymptotics of p_n, and as was shown in Ref. 12, ν_n is associated with the behavior of the ratio p_{n+1}/p_n as $n \to \infty$.

If μ and ν are positive Borel measures on $\mathbb{C}$, then their *mutual entropy* is defined as

$$S(u,v) = \begin{cases} -\infty & \text{if } \mu \text{ is not } \nu\text{-absolutely continuous}, \\ -\int \log\left(\dfrac{d\mu}{d\nu}\right)d\mu & \text{if } \mu \text{ is } \nu\text{-absolutely continuous}, \end{cases}$$

and their *mutual logarithmic energy* as

$$I(\nu,\mu) = -\int\int \log|z-t|d\nu(t)d\mu(z).$$

With these notations the entropy (2) is equivalently rewritten as

$$E_n(\nu) = S(\nu_n,\nu) = -2 \log \gamma_n + 2n\, I(\mu_n,\nu_n). \tag{4}$$

In particular, from a classical Jensen's inequality for integrals, it follows immediately that if both μ and ν are unit measures on Δ, then $S(\mu,\nu) \leqslant 0$, with equality if and only if $\mu = \nu$. Hence,

$$E_n(\nu) \leqslant 0,$$

with equality if and only if $n = 0$.

Aptekarev *et al.*[1] considered two subfamilies of the usual Szegő class of weights on Δ, namely the Jacobi weights and the Bernstein-Szegő class (weights being bounded above, bounded away from zero, and satisfying a Dini–Lipschitz condition). In this last case it is known that the asymptotic formula for the orthogonal polynomials p_n holds uniformly in Δ, as n tends to infinity. With these assumptions it has been proved in Ref. 1 that

$$\lim_{n\to\infty} E_n(w) = S(\rho,w) + \log(2) - 1, \tag{5}$$

where

$$\rho(x) = 1/(\pi\sqrt{1-x^2})$$

denotes the Chebyshev unit weight on Δ. We are concerned here with the problem of whether a weaker form of this equality holds in the Szegő class of weights. We will show that the right-hand side of (5) is actually an asymptotic upper bound for the entropy $E_n(w)$ when the weight w satisfies the Szegő condition [see assumption (6) below]. Furthermore, the expression (2) for the entropy can be naturally split into two functionals, which have simple asymptotic behaviors when w belongs to the Bernstein class (see Proposition 1). The situation with the Bernstein class is in a sense optimal: the corresponding limits provide asymptotic upper bounds for w in the whole Szegő class. We also give conditions for the entropy and the two functionals to tend to their upper bounds as the degree n becomes large.

Finally, we must mention that in the case of an unbounded support of the weight of orthogonality interesting results concerning the asymptotics of the E_n and related functionals have been obtained recently in Ref. 9.

II. STATEMENTS OF RESULTS

The weighted L^p norm of a function f with respect to a weight k on Δ will be denoted by

$$\|f\|_{L^p(k)} = \left(\int_\Delta |f(x)|^p k(x) dx \right)^{1/p}, \quad 1 \leqslant p \leqslant \infty.$$

We will simply write L^p when $k \equiv 1$ on Δ.

Though our main interest in this paper lies in the Szegő class of weights, some other classes appear at different places. We recall the definitions of these classes now.

The *Erdős–Turan class* $\mathcal{ET}$ consists of weights $w \in L^1$ such that $w > 0$ almost everywhere on Δ.

The *Szegő class* $\mathcal{S}$ consists of weights $w \in L^1$ such that

$$\log(w_0) \in L^1(\rho), \tag{6}$$

where

$$w_0(x) := w(x)/\rho(x) = \pi\sqrt{1-x^2}w(x)$$

denotes the trigonometric weight corresponding to w. The fact that $w_0 \in L^1(\rho)$ implies $\log^+(w_0) \in L^1(\rho)$, where, as usual, we denote

$$\log^+(x) = \max\{\log(x), 0\}, \quad x > 0.$$

Hence, condition (6) is actually equivalent to

$$S(\rho, w) = \int_\Delta \log(w_0(x))\rho(x) dx > -\infty. \tag{7}$$

Note that (6) and (7) can equivalently be rewritten as $\log(w) \in L^1(\rho)$ and

$$\int_\Delta \log(w(x))\rho(x) dx > -\infty,$$

respectively.

Finally, the *Bernstein class* $\mathcal{B}$ consists of weights w such that w_0 is given by the reciprocal of a positive polynomial on Δ. As it is well-known, the class $\mathcal{B}$ is an important class useful for establishing asymptotic properties in the Szegő theory of orthogonal polynomials. Obviously, one has the following inclusions $\mathcal{B} \subset \mathcal{S} \subset \mathcal{ET}$.

We will also use the notations

$$f_n(x) := p_n(x)\sqrt{w_0(x)}, \tag{8}$$

and for $M > 0$,

$$\Delta_n(M) := \{x \in \Delta : |f_n(x)| \geqslant M\}. \tag{9}$$

One of the main results of the paper is the following theorem.

Theorem 1: *Assume that the weight w belongs to the Szegő class $\mathcal{S}$. Then, for all $M > \sqrt{2}$,*

$$E_n(w) = S(\rho, w) + \log(2) - 1 - \int_{\Delta_n(M)} p_n^2(x)\log^+(p_n^2(x))w(x) dx + o(1), \quad n \to \infty. \tag{10}$$

As a simple consequence of the above formula, we obtain the following asymptotic upper bound together with necessary and sufficient conditions for equality.

Corollary 1: Assume that the weight w belongs to the Szegő class $\mathcal{S}$. Then the following asymptotic upper bound for the entropy holds:

$$\limsup_{n\to\infty} E_n(w) \leqslant S(\rho, w) + \log(2) - 1. \tag{11}$$

Moreover, for a subsequence $n \in \Lambda \subset \mathbb{N}$,

$$\lim_{n\in\Lambda} E_n(w) = S(\rho,w) + \log(2) - 1, \tag{12}$$

if and only if there exists a constant $M > \sqrt{2}$, *such that*

$$\lim_{n\in\Lambda} \int_{\Delta_n(M)} p_n^2(x) \log^+(p_n^2(x)) w(x) dx = 0. \tag{13}$$

In this case (13) is valid for all $M > \sqrt{2}$.

Furthermore, (13) holds if there exists $\varepsilon > 0$ *such that either*

$$\sup_{n\in\Lambda} \int_{\Delta} (\log^+(p_n^2(x)))^{1+\varepsilon} p_n^2(x) w(x) dx < \infty \quad \text{or} \quad \sup_{n\in\Lambda} \int_{\Delta} (p_n^2(x))^{1+\varepsilon} w(x) dx < \infty. \tag{14}$$

Remark 1: Notice that the findings of Ref. 1 on Bernstein–Szegő polynomials are included in Corollary 1 since for $w \in \mathcal{B}$, $\log(w_0)$ is bounded and the f_n are uniformly bounded in $[-1,1]$. In contrast, the case of Jacobi polynomials requires some extra considerations. One knows that for the orthonormal Jacobi polynomials there exists a constant c such that for $n \geqslant 0$ and $x \in [-1,1]$,

$$|P_n^{(\alpha,\beta)}(x)| \cdot \left(\sqrt{1-x} + \frac{1}{n}\right)^{\alpha+1/2} \left(\sqrt{1+x} + \frac{1}{n}\right)^{\beta+1/2} \leqslant c/\sqrt{\pi}.$$

Taking into account that here $w_0(x) = \pi(1-x)^{\alpha+1/2}(1+x)^{\beta+1/2}$, we find that for $p_n = P_n^{(\alpha,\beta)}$,

$$(p_n(x))^{2+\varepsilon} w_0(x) \leqslant c\sqrt{\pi} \left(\frac{1-x}{(\sqrt{1-x}+1/n)^{2+\varepsilon}}\right)^{\alpha+1/2} \left(\frac{1+x}{(\sqrt{1+x}+1/n)^{2+\varepsilon}}\right)^{\beta+1/2},$$

and the second condition in (14) is satisfied.

Remark 2: An inequality weaker than (11) is a straightforward consequence of the asymptotic behavior of the measures ν_n. Indeed, if $w \in \mathcal{ET}$, we know from Rakhmanov's Theorem[12] that $d\nu_n(x) \to \rho(x)dx$ as $n \to \infty$ in the weak-* topology. It follows from the weak upper semicontinuity of the mutual entropy (Ref. 7, Corollary 5.3) that $\limsup E_n(w) = \limsup S(\nu_n, w) \leqslant S(\rho, w)$. In particular, it shows that if the weight w is in $\mathcal{ET} \setminus \mathcal{S}$,

$$\lim_{n\to\infty} E_n(w) = -\infty.$$

Nevertheless, it seems that a semicontinuity argument for the entropy does not allow us to explain the additional term $\log(2)-1$ occurring on the right-hand side of (12).

The information entropy for Chebyshev polynomials orthonormal with respect to ρ has been computed in Refs. 4,14:

$$E_n(\rho) = \log(2) - 1, \quad \text{for} \quad n \geqslant 1. \tag{15}$$

Intuitively, Chebyshev polynomials are the most "uniformly" distributed polynomials, both for each n and asymptotically as $n \to \infty$. This fact is formally set in the next corollary.

Corollary 2: If

$$\limsup_{n\to\infty} E_n(w) \geqslant \log(2) - 1, \tag{16}$$

then $w = \rho$ *and* $E_n(w) = \log(2) - 1$, $n \geqslant 1$.

The proof is a simple consequence of inequality (11). Indeed, from this inequality, we see that (16) can only happen if $S(\rho, w) = 0$ that is $\rho = w$.

Now we exploit the connection between the entropy $E_n(w)$ and the mutual energy $I(\mu_n,\nu_n)$ given in (4). It is well known that in the class $\mathcal{ET}$ both μ_n and ν_n tend (as $n\to\infty$) to the Chebyshev (equilibrium) distribution given by the weight ρ on Δ. In particular, from the convexity properties of the mutual energy it follows that

$$\lim_{n\to\infty} I(\mu_n,\nu_n) = I(\rho,\rho) = \log(2).$$

What is more surprising is that the next term of the asymptotic expansion of $I(\mu_n,\nu_n)$ also exhibits a "universal" behavior, in the sense that it does not depend on the choice of the weight w. Namely, if the entropy $E_n(w)$ satisfies (12), then the following result is a direct consequence of (4) and the well known asymptotic behavior of the leading coefficient of p_n [see (29)].

Corollary 3: Assume w is a weight in the Szegő class $\mathcal{S}$ and condition (13) is satisfied. Then the mutual energy $I(\mu_n,\nu_n)$ has the following asymptotic expansion:

$$I(\mu_n,\nu_n) = \log(2) - \frac{1}{2n} + o\left(\frac{1}{n}\right), \quad n \in \Lambda, \quad n\to\infty.$$

This remarkable fact certainly deserves further study.

Another aim of the paper is to study two related functionals F_n and G_n, whose sum equals the entropy,

$$E_n(w) = F_n(w) + G_n(w),$$

and which are defined by

$$F_n(w) = -\int_\Delta \log(p_n^2(x)w_0(x))p_n^2(x)w(x)dx = S(f_n^2\rho,\rho), \tag{17}$$

and

$$G_n(w) = \int_\Delta \log(w_0(x))p_n^2(x)w(x)dx = -S(p_n^2 w, p_n^2\rho). \tag{18}$$

We will see that the functional F_n also exhibits a "universal" behavior, while G_n is sensitive to a particular choice of the weight w, and is related naturally with the mutual entropy $S(\rho,w)$. Functionals F_n and G_n have a particularly nice behavior for w in the Bernstein class $\mathcal{B}$:

Proposition 1: Let S be a polynomial of degree $2N$ $(N\geqslant 0)$ such that $S(x)>0$ for $x\in\Delta$, and assume that the orthogonality weight satisfies

$$w_0(x) = \frac{1}{S(x)}, \quad x\in\Delta.$$

Then

$$F_n(w) = \log(2) - 1, \quad \textit{for} \quad n > N. \tag{19}$$

Moreover,

$$\lim_{n\to\infty} G_n(w) = S(\rho,w), \tag{20}$$

and this limit takes place with a geometric rate. Consequently, the same holds true for the limit in (5).

The conjecture that constant entropy $E_n(w)$ is a (yet another) characterization of Chebyshev polynomials [cf. (15)] belongs to Golinsky. We were able to prove it in the Bernstein class $\mathcal{B}$.

Proposition 2: Let $w\in\mathcal{B}$ such that $E_n(w)$ is constant for all sufficiently large n. Then $w=\rho$.

Since Bernstein weights are suitable as approximation tool for the whole Szegő class, we could expect the asymptotic behavior from Proposition 1 to hold in a more general setting. Nevertheless, the behavior of the entropy, as well as the behavior of the two functionals F_n and G_n is extremely sensitive to the growth of $p_n^2 w$, which may affect convergence. In general, the following expression for the first functional F_n holds true:

Theorem 2: *Assume the weight w belongs to the Szegő class $\mathcal{S}$. Then, for all $M>\sqrt{2}$,*

$$F_n(w) = \log(2) - 1 - \int_{\Delta_n(M)} \log(f_n^2(x)) f_n^2(x) \rho(x) dx + o(1), \quad n \to \infty. \tag{21}$$

Again, as a simple consequence of the above formula, we get the following corollary.

Corollary 4: Assume the weight w belongs to the Szegő class $\mathcal{S}$. Then, the following asymptotic upper bound for F_n holds:

$$\limsup_{n\to\infty} F_n(w) \leqslant \log(2) - 1. \tag{22}$$

Moreover, for a subsequence $n \in \Lambda \subset \mathbb{N}$,

$$\lim_{n\in\Lambda} F_n(w) = \log(2) - 1, \tag{23}$$

if and only if there exists a constant $M>\sqrt{2}$, such that

$$\lim_{n\in\Lambda} \int_{\Delta_n(M)} f_n^2(x) \log(f_n^2(x)) \rho(x) dx = 0, \tag{24}$$

for f_n and $\Delta_n(M)$ defined in (8) and (9), respectively. In this case, (24) is valid for every $M>\sqrt{2}$.

Furthermore, (24) holds if there exists an $\varepsilon>0$ such that either

$$\sup_{n\in\Lambda} \int_{\Delta} (\log^+(f_n^2(x)))^{1+\varepsilon} f_n^2(x) \rho(x) dx < \infty \quad \textit{or} \quad \sup_{n\in\Lambda} \int_{\Delta} (f_n^2(x))^{1+\varepsilon} \rho(x) dx < \infty. \tag{25}$$

Remark 3: The method of proof of Theorem 2 can be applied to larger classes of weights. In fact, we only need an L^2 asymptotics of the polynomials p_n on the support Δ of the measure ν, and that has been extended beyond the Szegő class. For instance, using our technique we can prove that (11) is valid for weights $w \in \mathcal{F}(dini)$, introduced in Ref. 8.

Remark 4: Apparently, a necessary condition for (25) is that $w_0 \log(w_0) \in L^1(\rho)$ [cf. with (6)]. If $\log(w_0) \in L^\infty$ then there is equivalence between conditions (13) and (24), and between (14) and (25), respectively.

Concerning the second functional G_n, we use a result from Ref. 10 to deduce the following proposition.

Proposition 3: Assume the weight w belongs to the Szegő class $\mathcal{S}$ and $\log^+(w_0) \in L^\infty$; *then*

$$\limsup_{n\to\infty} G_n(w) \leqslant S(\rho,w) = \int_{\Delta} \log(w_0(x)) \rho(x) dx. \tag{26}$$

Similarly, assume that $\log^-(w_0) \in L^\infty$; then

$$\liminf_{n\to\infty} G_n(w) \geqslant S(\rho,w) = \int_{\Delta} \log(w_0(x)) \rho(x) dx. \tag{27}$$

Hence, if $\log(w_0) \in L^\infty$, *then*

$$\lim_{n\to\infty} G_n(w) = S(\rho,w).$$

Furthermore, if the weight w belongs to the set $\mathcal{ET}\backslash\mathcal{S}$, *the assumption* $\log^{+}(w_0)\in L^{\infty}$ *still implies inequality (26). In this case, (26) simplifies to* $\lim_{n\to\infty}G_n(w)=-\infty$.

III. PROOFS OF THEOREMS 1 AND 2, COROLLARIES 1 AND 4

Before entering the proofs of our results, let us state two preliminary lemmas. The first one is borrowed from Ref. 1.

Lemma 1 (Ref. 1, Lemma 2.1): Let g be a continuous function on $\mathbb{R}$, $g(\theta+\pi)=g(\theta)$, $f\in L^1([0,\pi])$, *and let* $\gamma(\theta)$ *be a function that is measurable and almost everywhere finite on* $[0,\pi]$. *Then, as* $n\to\infty$,

$$\int_0^{\pi} g(n\theta+\gamma(\theta))f(\theta)d\theta \to \frac{1}{\pi}\int_0^{\pi} g(\theta)d\theta\int_0^{\pi} f(\theta)d\theta.$$

As remarked in Ref. 1, when $\gamma(\theta)=0$ and $g\in L^{\infty}[0,\pi]$, the statement of the lemma becomes a well-known result of Fejer; cf. Ref. 2, Chap. I, Sec. 20.

As the second main ingredient in our proofs let us recall the Szegő asymptotics for $f_n(x)=\sqrt{w_0(x)}p_n(x)$: if

$$g_n(x)=\sqrt{2}\cos(n\arccos x+\gamma(x)),$$

where

$$\gamma(x)=\frac{1}{2\pi}\int_{\Delta}\frac{\log w_0(x)-\log w_0(t)}{x-t}\sqrt{\frac{1-x^2}{1-t^2}}dt$$

is the harmonic conjugate function to log w_0; then in the Szegő class $\mathcal{S}$, one has

$$\lim_{n\to\infty}\|f_n-g_n\|_{L^2(\rho)}=0, \tag{28}$$

and

$$\lim_{n\to\infty}\log\left(\frac{\gamma_n}{2^n}\right)=-\frac{1}{2}(\log(2)+S(\rho,w)). \tag{29}$$

The mutual entropy on the right-hand side of (29) is known as the Szegő constant for the weight w. Since the entropy integral is very sensitive to the growth of $f_n^2=p_n^2w_0$, the following lemma will be useful; roughly speaking, it shows that the subsets $\Delta_n(M)$, defined in (9), have no influence on the L^2 asymptotics (28):

Lemma 2: For $w\in\mathcal{S}$,

$$\lim_{n\to\infty}\int_{\Delta_n(M)}\rho(x)dx=0, \tag{30}$$

for every $M>\sqrt{2}$. *Furthermore, let* $\tilde{f}_n$, $n\geqslant 0$, *be the sequence of truncated functions,*

$$\tilde{f}_n(x):=\begin{cases} f_n(x), & \text{for}\quad x\in\Delta\setminus\Delta_n(M),\\ 1, & \text{for}\quad x\in\Delta_n(M).\end{cases} \tag{31}$$

Then

$$\lim_{n\to\infty}\|\tilde{f}_n-g_n\|_{L^2(\rho)}=0. \tag{32}$$

Proof: Observe first that by the Cauchy–Schwarz inequality,

$$\int_\Delta |f_n^2(x) - g_n^2(x)|\rho(x)dx \leqslant \|f_n + g_n\|_{L^2(\rho)} \cdot \|f_n - g_n\|_{L^2(\rho)} \leqslant (\|f_n\|_{L^2(\rho)} + \|g_n\|_{L^2(\rho)}) \cdot \|f_n - g_n\|_{L^2(\rho)},$$

so that

$$\int_\Delta |f_n^2(x) - g_n^2(x)|\rho(x)dx \leqslant (1+\sqrt{2})\|f_n - g_n\|_{L^2(\rho)}. \tag{33}$$

Now we can show that the Chebyshev (and hence, Lebesgue) measure of $\Delta_n(M)$ is asymptotically vanishing: by (33),

$$\begin{aligned}(M^2-2)\int_{\Delta_n(M)} \rho(x)dx &\leqslant \int_{\Delta_n(M)} (f_n^2(x)-2)\rho(x)dx \\ &\leqslant \int_{\Delta_n(M)} |f_n^2(x) - g_n^2(x)|\rho(x)dx \leqslant (1+\sqrt{2})\|f_n - g_n\|_{L^2(\rho)},\end{aligned}$$

the right-hand side tending to zero as $n\to\infty$ by (28); this proves (30). Moreover, since $|\tilde{f}_n(x)| = 1$ and $|g_n(x)| \leqslant \sqrt{2}$ for $x \in \Delta_n(M)$, we have by (33),

$$\begin{aligned}\|\tilde{f}_n - g_n\|_{L^2(\rho)}^2 &= \int_{\Delta\setminus\Delta_n(M)} |f_n(x) - g_n(x)|^2\rho(x)dx + \int_{\Delta_n(M)} |\tilde{f}_n(x) - g_n(x)|^2\rho(x)dx \\ &\leqslant (1+\sqrt{2})\|f_n - g_n\|_{L^2(\rho)} + 3\int_{\Delta_n(M)} \rho(x)dx.\end{aligned}$$

It remains to use (28) and (30) to see that (32) is satisfied. □

A. Proof of Theorem 1

Fix arbitrary $M > \sqrt{2}$ and let $\Delta_n(M)$ and $\tilde{f}$ be as defined in (9) and (31), respectively. We write the entropy as

$$E_n(w) = S(f_n^2\rho, w) = S(g_n^2\rho, w) + [S(\tilde{f}_n^2\rho, w) - S(g_n^2\rho, w)] + [S(f_n^2\rho, w) - S(\tilde{f}_n^2\rho, w)]. \tag{34}$$

In three steps let us prove that the first term on the right has as a limit the first three terms on the right-hand side of (1), the second term tends to 0, and the third term is asymptotically negative and related to the integral in (1).

Let

$$\mathcal{R}(y) = y^2 \log(y^2), \quad y \in \mathbb{R}.$$

From Lemma 1 we get

$$\begin{aligned}\lim_{n\to\infty} S(g_n^2\rho, w) &= -\lim_{n\to\infty}\int_0^\pi \mathcal{R}(g_n(\cos(\theta)))\frac{d\theta}{\pi} + \lim_{n\to\infty}\int_0^\pi \log(w_0(\cos(\theta)))g_n^2(\cos(\theta))\frac{d\theta}{\pi} \\ &= -\int_0^\pi \mathcal{R}(\sqrt{2}\cos(\theta))\frac{d\theta}{\pi} + \int_0^\pi 2\cos^2(\theta)\frac{d\theta}{\pi}\int_0^\pi \log(w_0(\cos(\theta)))\frac{d\theta}{\pi} \\ &= E_1(\rho) + S(\rho, w) = \log(2) - 1 + S(\rho, w).\end{aligned} \tag{35}$$

Hence the first term on the right-hand side of (34) has the required limit. The second term in (34) can be written as

$$S(\tilde{f}_n^2\rho,w) - S(g_n^2\rho,w) = \int_\Delta \left[\mathcal{R}\left(\frac{\tilde{f}_n(x)}{\sqrt{w_0(x)}}\right) - \mathcal{R}\left(\frac{g_n(x)}{\sqrt{w_0(x)}}\right)\right] w(x)dx. \tag{36}$$

Recall that both $\tilde{f}_n$ and g_n are uniformly bounded on Δ by M, and hence for $x \in \Delta$,

$$\left|\mathcal{R}\left(\frac{\tilde{f}_n(x)}{\sqrt{w_0(x)}}\right)\right| w_0(x) \leqslant |\mathcal{R}(\tilde{f}_n(x))| + |\log(w_0(x))|\tilde{f}_n^2(x) \leqslant M^2 \log M^2 + M^2|\log(w_0(x))| = :h(x),$$

where $h \in L^1(\rho)$ by assumption (6). Similarly,

$$\left|\mathcal{R}\left(\frac{g_n(x)}{\sqrt{w_0(x)}}\right)\right| w_0(x) \leqslant h(x), \quad x \in \Delta.$$

The integral in (36) will be split into two parts depending on whether w_0 is small or large. Fix an arbitrary $0<\varepsilon<1$; by the monotone convergence theorem there exists a constant $C=C(\varepsilon)$ such that

$$0 \leqslant \int_{h(x)>C} h(x)\rho(x)dx = \int_\Delta h(x)\rho(x)dx - \int_{h(x)\leqslant C} h(x)\rho(x)dx < \varepsilon.$$

Defining $\tau:=M^2 \exp(-C/M^2)$ we see that $w_0(x)<\tau$ implies that $h(x)>C$, and hence

$$\left|\int_{w_0(x)<\tau}\left[\mathcal{R}\left(\frac{\tilde{f}_n(x)}{\sqrt{w_0(x)}}\right) - \mathcal{R}\left(\frac{g_n(x)}{\sqrt{w_0(x)}}\right)\right] w(x)dx\right| \leqslant 2\int_{w_0(x)<\tau} h(x)\rho(x)dx \leqslant 2\varepsilon.$$

On the other hand, if $w_0(x)\geqslant\tau$, then

$$\left|\frac{\tilde{f}_n(x)}{\sqrt{w_0(x)}}\right| \leqslant \frac{M}{\sqrt{w_0(x)}} \leqslant \frac{M}{\sqrt{\tau}} = e^{C/(2M^2)} = :C_1,$$

and the same inequality is valid for $g_n/\sqrt{w_0}$. Taking into account that $\mathcal{R}$ is smooth,

$$\begin{aligned}
\left|\mathcal{R}\left(\frac{\tilde{f}_n(x)}{\sqrt{w_0(x)}}\right) - \mathcal{R}\left(\frac{g_n(x)}{\sqrt{w_0(x)}}\right)\right| &\leqslant \max_{|y|\leqslant C_1}|\mathcal{R}'(y)| \left|\frac{\tilde{f}_n(x)}{\sqrt{w_0(x)}} - \frac{g_n(x)}{\sqrt{w_0(x)}}\right| \\
&\leqslant \max_{|y|\leqslant C_1}|2y(1+\log(y^2))| \left|\frac{\tilde{f}_n(x)}{\sqrt{w_0(x)}} - \frac{g_n(x)}{\sqrt{w_0(x)}}\right| \\
&\leqslant C_2 \left|\frac{\tilde{f}_n(x)}{\sqrt{w_0(x)}} - \frac{g_n(x)}{\sqrt{w_0(x)}}\right|,
\end{aligned}$$

with $C_2:=\max\{4e^{-3/2}, 2C_1(1+\log(C_1^2))\}$. Hence, using the Cauchy–Schwarz inequality,

$$\left|\int_{w_0(x)\geqslant\tau}\left[\mathcal{R}\left(\frac{\tilde{f}_n(x)}{\sqrt{w_0(x)}}\right) - \mathcal{R}\left(\frac{g_n(x)}{\sqrt{w_0(x)}}\right)\right] w(x)dx\right| \leqslant C_2\|(\tilde{f}_n - g_n)\sqrt{w_0(x)}\|_{L^1(\rho)} \leqslant C_2\|\tilde{f}_n - g_n\|_{L^2(\rho)},$$

which by (32) tends to 0 as $n\to\infty$. Taking into account that $\varepsilon \in (0,1)$ was chosen arbitrarily, we conclude that

$$S(\tilde{f}_n^2\rho,w) - S(g_n^2\rho,w) \to 0, \quad n\to\infty. \tag{37}$$

Thus, for establishing the expression for the entropy in Theorem 1, it only remains to examine the last bracket on the right-hand side of (34). Notice that since $\tilde{f}_n=f_n$ on $\Delta\setminus\Delta_n(M)$,

$$
\begin{aligned}
S(f_n^2\rho,w)-S(\tilde{f}_n^2\rho,w) &= -\int_{\Delta_n(M)} p_n^2(x)\log(p_n^2(x))w(x)dx+\int_{\Delta_n(M)}\log\left(\frac{1}{w_0(x)}\right)\rho(x)dx \\
&= -\int_{\Delta_n(M)} p_n^2(x)\log^+(p_n^2(x))w(x)dx+\int_{\tilde{\Delta}_n(M)} p_n^2(x)|\log(p_n^2(x))|w(x)dx \\
&\quad -\int_{\Delta_n(M)}\log(w_0(x))\rho(x)dx,
\end{aligned}
$$

where

$$\tilde{\Delta}_n(M)=\{x\in\Delta_n(M):p_n^2(x)<1\}\subset\Delta_n(M).$$

Observing that, for $p_n^2(x)\leqslant 1$, we have $0\leqslant p_n^2(x)|\log(p_n^2(x))|\leqslant 1$, we obtain

$$0\leqslant\int_{\tilde{\Delta}_n(M)} p_n^2(x)|\log(p_n^2(x))|w(x)dx\leqslant\int_{\Delta_n(M)} w(x)dx=\int_{\Delta_n(M)} w_0(x)\rho(x)dx.$$

Since $w_0\in L^1(\rho)$, $\log(w_0)\in L^1(\rho)$, by the absolute continuity of the Lebesgue integral, relation (30) implies that

$$\lim_{n\to\infty}\int_{\Delta_n(M)} w_0(x)\rho(x)dx=0,\quad\text{and}\quad\lim_{n\to\infty}\int_{\Delta_n(M)}\log(w_0(x))\rho(x)dx=0, \tag{38}$$

showing that

$$S(f_n^2\rho,w)-S(\tilde{f}_n^2\rho,w)=-\int_{\Delta_n(M)} p_n^2(x)\log^+(p_n^2(x))w(x)dx+o(1),\quad n\to\infty. \tag{39}$$

Hence, gathering (35), (37), and (39) in (34), we get (10). □

B. Proof of Corollary 1

Since

$$\int_{\Delta_n(M)} p_n^2(x)\log^+(p_n^2(x))w(x)dx\geqslant 0,$$

relation (11) is a trivial consequence of Theorem 1. Suppose now that (13) holds for some $M>\sqrt{2}$, then (12) follows immediately from (10). Conversely, if (12) is true then it follows from Theorem 1 that (13) holds for all $M>\sqrt{2}$.

In order to prove that (14) is sufficient for (13), notice that, by Hölder's inequality,

$$
\begin{aligned}
\int_{\Delta_n(M)} p_n^2(x)\log^+(p_n^2(x))w(x)dx &\leqslant\left(\int_{\Delta_n(M)} p_n^2(x)(\log^+(p_n^2(x)))^{1+\varepsilon}w(x)dx\right)^{1/(1+\varepsilon)} \\
&\quad\times\left(\int_{\Delta_n(M)} p_n^2(x)w(x)dx\right)^{1-1/(1+\varepsilon)}.
\end{aligned} \tag{40}
$$

Furthermore,

$$\begin{aligned}\int_{\Delta_n(M)} p_n^2(x)w(x)dx &\leqslant \int_{\Delta_n(M)} [f_n^2(x)-g_n^2(x)]\rho(x)dx + \int_{\Delta_n(M)} g_n^2(x)\rho(x)dx\\ &\leqslant \int_{\Delta_n(M)} [f_n^2(x)-g_n^2(x)]\rho(x)dx + 2\int_{\Delta_n(M)} \rho(x)dx\\ &\leqslant (1+\sqrt{2})\|f_n-g_n\|_{L^2(\rho)} + 2\int_{\Delta_n} \rho(x)dx = o(1), \quad n\to\infty,\end{aligned}$$

where we have used (28), (30), and (33).

If we assume that the first condition in (14) holds, then the first factor on the right-hand side of (40) is uniformly bounded in n, and (13) follows.

Finally, notice that the second condition in (14) implies the first one since $\log^+(z)\leqslant z$ for $z\geqslant 0$, and hence

$$(\log^+(y))^{1+\varepsilon} = \left(\frac{1+\varepsilon}{\varepsilon}\right)^{1+\varepsilon}(\log^+(y^{\varepsilon/(1+\varepsilon)}))^{1+\varepsilon} \leqslant \left(\frac{1+\varepsilon}{\varepsilon}\right)^{1+\varepsilon} y^{\varepsilon}, \quad y\geqslant 0.$$

□

C. Proof of Theorem 2

Our proof for Theorem 2 follows closely the arguments of the proof of Theorem 1, but some parts simplify. As before let $\mathcal{R}(y)=y^2\log(y^2)$, $y\in\mathbb{R}$, and fix $M>\sqrt{2}$. We write the functional as follows:

$$\begin{aligned}F_n(w) = \int_{\Delta}[-\mathcal{R}(g_n(x))]\rho(x)dx + \int_{\Delta}[\mathcal{R}(g_n(x))-\mathcal{R}(\tilde{f}_n(x))]\rho(x)dx + \int_{\Delta}[\mathcal{R}(\tilde{f}_n(x))\\ -\mathcal{R}(f_n(x))]\rho(x)dx.\end{aligned} \tag{41}$$

Here the first integral on the right-hand side of (41) has the limit $E_1(\rho)=\log(2)-1$ by Lemma 1. The last one can be written as

$$\int_{\Delta}[\mathcal{R}(\tilde{f}_n(x))-\mathcal{R}(f_n(x))]\rho(x)dx = -\int_{\Delta_n(M)} \log(f_n^2(x))f_n^2(x)\rho(x)dx \leqslant 0,$$

the right-hand side coinciding with the integral in (21). Thus Theorem 2 follows by showing that the second integral on the right-hand side of (41) is asymptotically vanishing. Recalling that $|\tilde{f}_n(x)|$ and $|g_n(x)|$ are uniformly bounded by M for all $n\geqslant 0$ and $x\in\Delta$, we obtain

$$\begin{aligned}&\left|\int_{\Delta}[\mathcal{R}(g_n(x))-\mathcal{R}(\tilde{f}_n(x))]\rho(x)dx\right|\\ &\quad\leqslant \max_{y\in[-M,M]}|\mathcal{R}'(y)|\int_{\Delta}|g_n(x)-\tilde{f}_n(x)|\rho(x)dx\\ &\quad\leqslant M^2(1+\log M^2)\|g_n-\tilde{f}_n\|_{L^1(\rho)} \leqslant M^2(1+\log M^2)\|g_n-\tilde{f}_n\|_{L^2(\rho)},\end{aligned}$$

the term on the right tending to zero as $n\to\infty$ by (32). □

D. Proof of Corollary 4

Since

$$\int_{\Delta_n(M)} f_n^2(x)\log(f_n^2(x))\rho(x)dx \geqslant 0,$$

relation (22) is a trivial consequence of Theorem 2. Suppose now that (24) holds for some $M>\sqrt{2}$; then (23) follows immediately from (21). Conversely, if (23) is true then it follows from Theorem 2 that (24) holds for all $M>\sqrt{2}$.

In order to prove that the first condition in (25) (which clearly is weaker than the second one) is sufficient for (24), notice that, by Hölder's inequality,

$$\int_{\Delta_n} f_n^2(x)\log^+(f_n^2(x))\rho(x)dx \leqslant \left(\int_{\Delta_n} f_n^2(x)(\log^+(f_n^2(x)))^{1+\varepsilon}\rho(x)dx\right)^{1/(1+\varepsilon)}\left(\int_{\Delta_n} f_n^2(x)\rho(x)dx\right)^{1-1/(1+\varepsilon)},$$

and we may conclude as in the proof of Corollary 1 that the second factor on the right-hand side tends to zero. □

IV. PROOFS OF PROPOSITIONS 1, 2, AND 3

A. Proof of Proposition 1

Let us make the change of variables $x=(z+1/z)/2$. It is well known that since $S(x)>0$ on Δ we may write S as

$$S(x) = |q(z)|^2 = q(z)q(1/z), \tag{42}$$

with q a polynomial of degree $2N$ with real coefficients having all its zeros outside the disk and $q(0)>0$. Moreover,

$$p_n(x) = \frac{1}{\sqrt{2}}(z^n q(z^{-1}) + z^{-n}q(z)) \tag{43}$$

is the orthonormal polynomial of degree $n>N$ with respect to the Bernstein weight ρ/S. Introducing the Blaschke product,

$$B_n(z) = z^{2n}q(1/z)/q(z), \quad n \geqslant N, \tag{44}$$

we find that

$$p_n^2(x)w_0(x) = \tfrac{1}{2}|1 + B_n(z)|^2 = 1 + \tfrac{1}{2}(B_n(z) + B_n(1/z)), \quad |z| = 1.$$

Since, for $n>N$, $B_n(0)=0$, and B_n is analytic in the disk, we have

$$\begin{aligned}\log(2) - F_n(w) &= \log(2) + \int \log(p_n^2(x)w_0(x))p_n^2(x)w_0(x)\rho(x)dx\\ &= \frac{1}{2\pi}\int_{|z|=1} \log(|1 + B_n(z)|^2)\left[1 + \frac{1}{2}(B_n(z) + B_n(1/z))\right]|dz|\\ &= \mathrm{Re}\left(\frac{1}{2\pi i}\int_{|z|=1} \log(1 + B_n(z))[2 + B_n(z) + B_n(1/z)]\frac{dz}{z}\right).\end{aligned}$$

Since $|B_n(z)|<1$ for $|z|<1$, the function $\log(1+B_n)[2+B_n]$ is holomorphic inside the disk and vanishes at the origin. Thus,

$$\log(2) - F_n(w) = \mathrm{Re}\left(\frac{1}{2\pi i}\int_{|z|=1} \log(1 + B_n(z))B_n(1/z)\frac{dz}{z}\right) \tag{45}$$

$$=\mathrm{Re}\left(\frac{1}{2\pi i}\int_{|z|=1}\frac{\log(1+B_n(z))}{B_n(z)}\frac{dz}{z}\right), \tag{46}$$

where we have used that $B_n(1/z)=1/B_n(z)$. Observe that the last integrand is analytic in a neighborhood of the unit circle, and we can integrate along a smaller circle $|z|=r<1$, where $|B_n(z)|<1$. Replacing the log by its uniformly convergent Taylor expansion we get finally that this integral equals 1, which proves (19).

On the other hand, by a similar reasoning we have

$$\begin{aligned} G_n(w) &= \int_{-1}^{1}\log(w_0(x))p_n(x)^2w(x)dx \\ &= -2\mathrm{Re}\left(\frac{1}{2\pi i}\int_{|z|=1}\log(q(z))\left[1+\frac{1}{2}(B_n(z)+B_n(1/z))\right]|dz|\right) \\ &= -2\log(q(0)) - \frac{1}{2\pi i}\int_{|z|=1}\log(q(z))B_n(1/z)\frac{dz}{z}. \end{aligned} \tag{47}$$

Note that in the last expression of (47), taking the real part is not necessary since q and B_n are real functions. Integrating now along $|z|=R>1$, we observe that $|B_n(1/z)|$ becomes geometrically small, there which yields a geometric rate of convergence for

$$\lim_{n\to\infty}G_n(w) = -2\log(q(0)) = -\mathrm{Re}\frac{1}{\pi}\int_{|z|=1}\log(q(z))\frac{dz}{z} = S(\rho,w),$$

which proves (20). □

B. Proof of Proposition 2

From the computations of $F_n(w)$ and $G_n(w)$ in the proof of Proposition 1, see (45) and (47), we know that $E_n(w)$ is constant for n large, say $n>N_0>N$, if and only if

$$\frac{1}{2\pi i}\int_{|z|=1}\frac{B_N(1/z)\log(q(z))}{z^{2n-2N}}\frac{dz}{z}=0, \quad n>N_0, \tag{48}$$

where the polynomial q and the Blaschke product B_N are defined by (42) and (44), respectively. Since $\log(q(z))$ is analytic in some neighborhood $\mathcal{U}$ of the unit disk, we may conclude that $\log(q(z))B_N(1/z)$ is meromorphic in $\mathcal{U}$, and thus can be written as

$$B_N(1/z)\log(q(z)) = r(z)+f(z), \quad z\in\mathcal{U}, \tag{49}$$

where r is a rational function such that $z^{2N}q(1/z)r(z)$ is a polynomial of degree at most $2N-1$, and f is analytic in $\mathcal{U}$. Since r is analytic outside the unit disk and grows like at most $1/z$ at infinity, we deduce

$$\frac{1}{2\pi i}\int_{|z|=1}\frac{r(z)}{z^{2n-2N}}\frac{dz}{z}=0, \quad n>N_0,$$

which implies, together with (48) and (49), that

$$\frac{1}{2\pi i}\int_{|z|=1}\frac{f(z)}{z^{2n-2N}}\frac{dz}{z}=0, \quad n>N_0.$$

Hence, all sufficiently high even Taylor coefficients of f vanish. As a consequence, $f(z)+f(-z)=P(z)$ is a polynomial, and

$$B_N(1/z)\log(q(z)) + B_N(-1/z)\log(q(-z)) = r(z) + r(-z) + P(z), \quad |z| \leqslant 1. \tag{50}$$

Since the right-hand side of (50) is a rational function, the principle of analytic continuation applies, showing that (50) actually holds everywhere in $\mathbb{C}$. First, assume that the polynomial q is even, that is $q(z)=q(-z)$, $z\in\mathbb{C}$. Then, it follows from (50) that $\log(q(z))$ is a rational function so that q can only be a constant, namely 1 by the normalization (3) of the weight w. Second, assume that the polynomial q is not even (hence different from a constant). It implies the existence of some root $\alpha\in\mathbb{C}$ of q such that either $q(-\alpha)\neq 0$ or $-\alpha$ is a root of q of different multiplicity than that of α. Note that $\alpha\neq 0$ since, by assumption, $q(0)>0$. Then we get a contradiction. Indeed, in view of the definition (44) of B_N, we readily observe that the left-hand side of (50) has a branch point at α while the right-hand side has not. Hence, $q(z)$ is constant, equal to 1, and the proof of Proposition 2 is finished. □

C. Proof of Proposition 3

Choosing $p=2$ and $g=|\log(w_0)w|^{1/2}\in L^1$ in Theorem 2 of Ref. 10 shows that

$$\liminf_{n\to\infty}\int_\Delta |\log(w_0(x))|p_n^2(x)w(x)dx \geqslant \int_\Delta |\log(w_0(x))|\rho(x)dx, \tag{51}$$

for any weight w in the Erdős–Turan class $\mathcal{ET}$. If $\log^+(w_0)\in L^\infty$, there exists a constant $C>1$ such that $w_0(x)\leqslant C$, $x\in\Delta$. Hence $|\log(w_0/C)|=-\log(w_0/C)$ and substracting $\log(C)$ to both sides of (51), we get (26) since

$$\int_\Delta \rho(x)dx = \int_\Delta p_n^2(x)w(x)dx = 1.$$

A similar reasoning shows (27) when $\log^-(w_0)\in L^\infty$. Since this argument applies for any weight in the Erdős–Turan class, the last assertion in the proposition also follows. □

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