Optimal superluminal systems

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We demonstrate that significant effects in the "superluminal propagation" of light pulses cannot be observed without involving systems whose gain explodes outside the pulse spectrum. We explicitly determine the minimum norm of the gain to attain given superluminal effects and the transfer function of the corresponding optimal system. The gain norms, which would be required with the *most efficient* systems considered up to now (dispersive media, photonic barriers) to attain the same effects, are shown to exceed the minimum by several orders of magnitude. We finally estimate the largest superluminal advances which could be attained in a realistic experiment.

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The apparently superluminal propagation of light pulses has been observed with various systems, mainly with systems involving anomalously dispersive media or photonic barriers. For reviews, see, e.g., [1–5]. In these experiments, the envelope of the pulse having covered some distance L is nearly identical to that of the incident pulse and in advance of that of a pulse which has covered the same distance L at the velocity c of light in vacuum. This surprising behavior is not at odds with the relativistic causality. Indeed the signal received at some time t is not the consequence of the signal emitted at a well-defined time but of all the signals anterior to t by more than L/c. Otherwise said, there is no cause-and-effect relation between the homologous points of the envelopes of the incident and transmitted pulses, and the widespread statement that *the pulse maximum leaves the system before it even enters it* is somewhat misleading. The phenomenon is, however, quite puzzling and keeps it the subject of an intense theoretical and experimental activity.

In fact, Mother Nature resists to a violation of her principles even when this violation is only apparent and convincing experiments of superluminal transmission are very difficult to achieve. By convincing experiments, we mean experiments where (i) the envelopes or the intensity profiles of the pulses are detected in real time and true shape; (ii) the transmitted pulse is directly compared to the pulse having propagated at the velocity c; (iii) the superluminal advance a is large compared to the optical period; (iv) it is significant with respect to the pulse duration, say, larger than 10% of the full width at half maximum (FWHM) of the intensity profile τ_p; and (v) the pulse distortion (including noise and parasitic signals) is small compared to the relative advance a/τ_p. Note that (iii) is a consequence of (i) since the real-time detection of the envelope requires a time constant large compared to the optical period. There are few experiments meeting, even approximately, the previous conditions [6–13]. Though all-optical experiments are possible, only hybrid systems have been used up to now. They combine an optical part, responsible for the superluminal effects, and a wide-band electronic device whose function is to normalize the amplitude of the transmitted pulse. In most experiments, the transmission of the optical part, usually a resonantly absorbing medium [6,9–12] or a photonic barrier [7,8,14], is low and the electronic device is an amplifier. One experiment [13] has evidenced significant superluminal effects with an active optical part (amplifying medium). The normalization is then achieved by a suitable attenuation. In the following, we naturally include the normalization device (amplifier or attenuator) in the system under consideration.

As already noted in previous papers dealing with particular arrangements (see, e.g., [5]), large superluminal effects are only attained with systems whose gain explodes outside the pulse spectrum. We will show that this is true for any physically realizable system and determine the lower limit to the gain norm required to observe given superluminal effects. This result is of special importance since in a real experiment the gain norm should be limited to avoid problems of noise (no matter its origin), of instability, and of hypersensitivity to parasitic signal and to localized defects in the incident pulse profile [6]. Conversely restricting the gain to realistic values determines the upper limit to the actual observable effects.

The problem is studied in the frame of the linear systems theory [15]. We denote by $e(t)$ and $s(t)$ the envelopes of the incident and transmitted pulses, and by $E(\omega) = \int_{-\infty}^{\infty} e(t)\exp(-i\omega t)dt$ and $S(\omega)$ their Fourier transforms. The envelopes are assumed to be slowly varying at the scale of the optical period. Their Fourier transforms are then concentrated around 0 in a region of width small compared to the optical frequency. In all the sequel, t designates the local time, equal to the real time in $e(t)$ and retarded by the luminal transit time L/c in $s(t)$. The system is characterized by its impulse response $h(t)$ or its transfer function $H(\omega)$, such that $s(t)=h(t)\otimes e(t)$ and $S(\omega)=H(\omega)E(\omega)$. We assume that $E(\omega)$ and $H(\omega)$ have a finite energy and that $H(\omega)$, the Fourier transform of $h(t)$, has a continuation $H(z)$ in the complex plane $(z=x+iy=\rho e^{i\theta})$. In our local time picture, the relativistic causality imposes that $h(t<0)=0$. Otherwise said, $H(z)$ belongs to $L^2(\mathbb{R})$, the Hilbert space of functions $F(z)$ square summable on the real line $\mathbb{R}$ endowed with the norm $\|F\|_{\mathbb{R}}$

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such that $\|F\|_{\mathbb{R}}^2 = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$ and, more precisely, to the Hardy space $H^2(\Pi_-)$ of functions F analytic in the lower half plane Π_- ($y < 0$) which are the Fourier transforms of causal functions $f \in L^2(0, \infty)$ [16].

We want $s(t)$ to be as close as possible to $e(t+a)$ where a is the superluminal advance ($a > 0$). In L^2 norm, the distortion is defined by

$$D = \frac{\|e(t+a) - s(t)\|_{\mathbb{R}}}{\|e(t)\|_{\mathbb{R}}} = \frac{\|(H_a - H)E\|_{\mathbb{R}}}{\|E\|_{\mathbb{R}}}, \tag{1}$$

where $H_a = e^{i\omega a}$ is the transfer function of the noncausal system perfectly achieving the advance a without any distortion. With a real (causal) system, the distortion will be low if $H(\omega) \approx H_a(\omega)$ in the region around $\omega = 0$ where $|E(\omega)|$ is concentrated.

To keep tractable calculations, we consider the case $E(\omega) = E_0$ for $|\omega| < \omega_c$ and 0 elsewhere. By taking $E_0 = \pi$ and $\omega_c = 1$, this amounts to take as reference a pulse of intensity profile $|e(t)|^2 = \sin^2 t / t^2$ (FWHM $\tau_p = 2.78$). The distortion then reads $D = \|H_a - H\|_I / \sqrt{2}$ where $\|F\|_I$ denotes the norm L^2 of F restricted to $I = [-1, 1]$. In the situations of physical interest $D \ll 1$ and $\|H\|_{\mathbb{R}}^2 = \|H\|_I^2 + \|H\|_J^2 \approx 2 + \|H\|_J^2$ where $J = [-\infty, -1] \cup [1, \infty]$. In this model, the problem may then be stated: *given $a > 0$ and $D > 0$, minimize $Q = \|H\|_J$ under the constraints $H \in H^2(\Pi_-)$ and $\|H_a - H\|_I \leq D\sqrt{2}$.*

Based upon a conformal map that sends the unit disk $\mathbb{D}$ ($\rho = 1$) onto the lower half plane, we introduce the map $\tilde{F} = \Psi(F)$ defined by

$$\tilde{F}(z) = \Psi(F)(z) = \frac{\sqrt{2\pi}}{1-z} F\left(\frac{1+z}{2i(1-z)}\right). \tag{2}$$

It is an isometry from $L^2(\mathbb{R})$ to the Hilbert space $L^2(\mathbb{T})$ of the unit circle $\mathbb{T}$ endowed with the norm $\|F\|_{\mathbb{T}}$ such that $\|F\|_{\mathbb{T}}^2 = \int_0^{2\pi} |F(e^{i\theta})|^2 d\theta / 2\pi$. It sends the subspace $H^2(\Pi_-)$ onto the corresponding Hardy space $H^2(\mathbb{D})$ of the unit disk $\mathbb{D}$. We denote by $\tilde{I}$ and $\tilde{J}$ the subarcs of $\mathbb{T}$, transforms of I and J by the map Ψ. Then this map allows one to restate the problem in the unit disk $\mathbb{D}$ instead of the lower half plane: *given $a > 0$ and $D > 0$, minimize $Q = \|\tilde{H}\|_{\tilde{J}}$ under the constraints $\tilde{H} \in H^2(\mathbb{D})$ and $\|\tilde{H}_a - \tilde{H}\|_{\tilde{I}} \leq D\sqrt{2}$.*

Stated with a general function $\tilde{K} \in L^2(\tilde{I})$ instead of the particular $\tilde{H}_a$, this question has been originally considered in [16] and more recently in [17], with important extensions. The solution $\tilde{H}_{opt}$ of the problem exists and is unique. Note that, in our case ($\tilde{K} = \tilde{H}_a$), the constraint $\|H_a - H\|_I \leq D\sqrt{2}$ is saturated, i.e., $\|H_a - H\|_I = D\sqrt{2}$. The solution $\tilde{H}_{opt}$ can formally be written under the analytic form [17]

$$\tilde{H}_{opt} = (1 + \lambda\Phi)^{-1} P_{H^2}(\hat{H}_a). \tag{3}$$

In this expression $\hat{H}_a$ is defined as $\tilde{H}_a$ on $\tilde{I}$ and 0 on $\tilde{J}$, P_{H^2} denotes the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{D})$, and Φ is the so-called Toeplitz operator [17] acting on $H^2(\mathbb{D})$. It is such that $\Phi(\tilde{F}) = P_{H^2}(\check{F})$ where $\check{F}$ is defined as $\tilde{F}$ on $\tilde{J}$ and 0 on $\tilde{I}$. Finally, $\lambda \in [-1, \infty]$ is an implicit parameter.
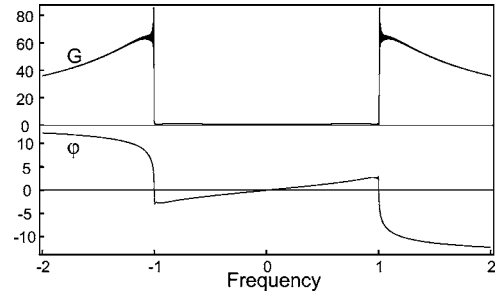

FIG. 1. Amplitude gain G and phase φ (radian) of the optimal system as functions of the frequency. Parameters: $a = \tau_p$ and $D = 15\%$.

It is the unique real number such that $\|H_a - H\|_I = D\sqrt{2}$.

From a computational viewpoint, it appears natural to consider Q and D as functions of λ [17]. It follows from Eq. (3) that Q and D, respectively, increases and decreases as λ decreases. As $\lambda \to -1$, $Q \to \infty$ while $D \to 0$. In physical terms, this confirms that a low distortion will always be paid at the price of a large gain norm. We have then $\|H_{opt}\|_{\mathbb{R}} = \|\tilde{H}_{opt}\|_{\mathbb{T}} \approx Q$.

Given a and D, the previous analysis leads to the following algorithm for the computation of the minimum gain norm Q and the corresponding function $\tilde{H}_{opt}$: (i) Choose $\lambda < -1$ and compute $\tilde{H}_{opt}$ given by Eq. (3). (ii) Compute D. If it is too large (resp. small), decrease (resp. increase) λ. Go to (i). Such a dichotomy algorithm has been implemented in the software package HYPERION developed at Institut National de Recherche en Informatique et Automatique (INRIA) by the APICS team [18]. See, also, [19] for a closely related algorithm. Equation (3), which is infinite dimensional, is approached by truncating the expansions of the involved functions so as to consider only their Fourier coefficients of indices $-N \leq j \leq N$. The optimal transfer function $H_{opt}(\omega)$ is finally obtained by inverting Eq. (2)

$$H_{opt}(\omega) = \frac{\sqrt{2/\pi}}{2i\omega + 1} \tilde{H}_{opt}\left(\frac{2i\omega - 1}{2i\omega + 1}\right). \tag{4}$$

Note that $H_{opt}(\omega)$ behaves as $1/i\omega$ for $|\omega| \to \infty$. This behavior is that of a first-order filter as used in every detection chain. Any further filtering of the high frequencies will obviously damage the performances of the system. To close this short presentation of our minimization procedure, we remark that it mainly lies on the separation of the spectral domains where the distortion and the gain norm are computed. We have chosen the pulse profile leading to the simplest calculations but the procedure might be adapted to any pulse provided that its Fourier transform has a compact support.

Calculations of the minimum gain norm Q, of the corresponding transfer function $H_{opt}(\omega)$ and of the transmitted signal $s(t)$, have been made for a/τ_p (resp. D) ranging from 0.36 to 2.2 (resp. 2% to 30%). Satisfactorily enough, the optimal system would allow one to conciliate significant advance, moderate distortion, and reasonable gain. For instance, $a = \tau_p$ with $D = 15\%$ would be obtained for $Q = 100$. Figure 1 shows the overall frequency dependence of the am-
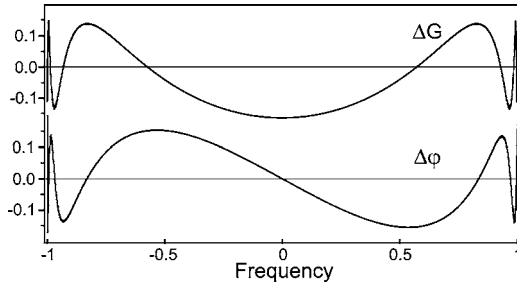
FIG. 2. Frequency dependence of $\Delta G = G-1$ and of $\Delta\varphi = \varphi - \omega a$ in the useful band. The group advance a_g deviates from a by $\Delta a = d(\Delta\varphi)/d\omega|_{\omega=0}$, that is $\Delta a \approx -0.40$ and $\Delta a/a \approx -14\%$. The parameters are the same as in Fig. 1.

FIG. 4. Relation between the peak gain M and the relative advance a/τ_p for a given distortion ($D = 15\%$). ($\bigcirc$) stands for the optimal system while (a), (b), (c), and (d), respectively, relate to the so-designated systems (see text).

plitude gain $G(\omega) = |H_{opt}(\omega)|$ and of the phase $\varphi(\omega) = \arg[H_{opt}(\omega)]$ in this reference case. As expected, the gain reaches its peak value near the frontiers of the "stop band" (in fact, the useful band for superluminal systems). The short ringing close to these frontiers originates from the finite number of Fourier coefficients used in the calculations ($N = 2000$). The asymptotic values of the phase are $\varphi = \pm 9\pi/2$ for $\omega = \mp\infty$, in agreement with Eq. (4). The extra phase rotation of 8π entails that $H_{opt}(z)$ has four zeros in the half plane $y < 0$ and, consequently, that $H_{opt}(\omega)$ is not minimum phase [15]. The differences $\Delta G = G-1$ and $\Delta\varphi = \varphi - \omega a$ for $-1 < \omega < 1$ (Fig. 2) illustrate how $H_{opt}(\omega)$ deviates from the ideal transfer function $H_a = e^{i\omega a}$ in the useful band. We remark that the group advance $a_g = d\varphi/d\omega|_{\omega=0}$ differs from the effective advance a by an amount approximately equal to the distortion (in our local time picture $a_g = L/c - L/v_g$ where v_g is the group velocity). Finally, the envelope $s(t)$, the inverse Fourier transform of $H_{opt}(\omega)E(\omega)$, and the intensity profile $|s(t)|^2$ of the transmitted pulse are displayed Fig. 3.

The efficiency of a superluminal system may be characterized by its ability to achieve given effects with gains as small as possible. As above noticed, the gain of all the optimal systems has the same asymptotic behavior ($G \propto 1/\omega$) and reaches its peak value M near $\omega = \pm 1$. Consequently, Q and M are roughly proportional and can indifferently characterize the system gain. The peak gain M, independent of the frequency scaling, is retained in the sequel. This choice
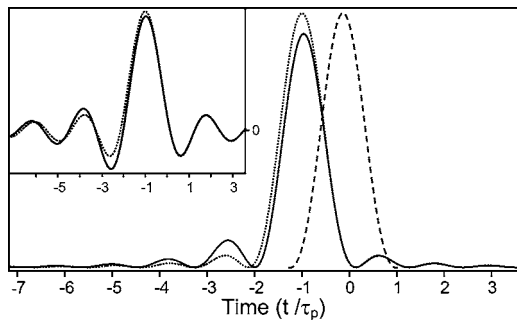

FIG. 3. Intensity profile of the pulse transmitted by the optimal system (full line). The profiles of the incident pulse advanced by $a = \tau_p$ (dotted line) and the main lobe of the incident pulse (dashed line) are given for reference. Inset: Envelopes $s(t)$ (full line) and $e(t+a)$ (dotted line). The parameters are the same as in Fig. 1.
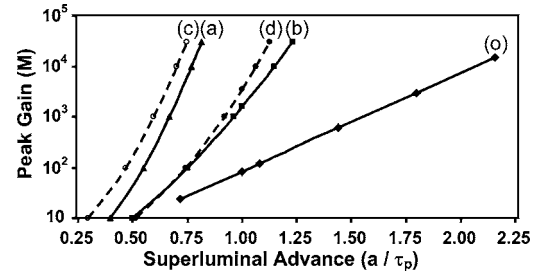
facilitates the comparison of the optimal systems with the most efficient systems used or proposed up to now. Since high optical gains exaggerate the problems of instability and noise (amplified spontaneous emission), and are difficult to achieve with the suitable frequency profile [13], we restrict ourselves to systems whose optical element, responsible for the superluminal effects, is passive. More specifically, we consider a dilute medium involving (a) an isolated absorption line [6,9–12] or (b) a doublet of absorption line [5], and a uniform Bragg grating written (c) on a classical optical fiber [8] or (d) on a hollow fiber. Since all these elements are almost transparent outside the low-transmission region (the useful band), the peak gain M is nothing but the gain of the amplifier used to normalize the amplitude of the transmitted pulse. The transfer functions are optimized by adjusting the system parameters with a genetic algorithm. A rapid convergence is obtained by starting the calculations with initial values such that $H(0) = 1$ and $a_g = a$. For the doublet (b), the initial value of the line splitting is chosen such that the second-order distortion cancels [5]. Figure 4 shows the results obtained for a reference distortion $D = 15\%$ and M ranging from 10 to 3×10^4. There is no need of a lens to see that the optimal system is much more efficient that the systems (a), (b), (c), and (d) to attain large superluminal advances. For instance, a peak gain $M = 84$ theoretically suffices to observe an advance $a = \tau_p$ with $D = 15\%$ (Fig. 1) but values as large as 1600, 3400, 6.4×10^6, and 4.9×10^7 would be required with the systems (b), (d), (a), and (c), respectively [20]. The latter dramatically increase if a lower distortion is required. Again for $a = \tau_p$ but with $D = 7\%$ they raise to 7.9 $\times 10^4$, 2.1×10^7, 2.3×10^{14}, and 4.9×10^{15} while M only reaches 174 for the optimal system. By comparison, we stress that achieving experiments with systems whose peak amplitude gain exceeds 10^4 is absolutely unrealistic.

The situation is much less catastrophic when one examines the superluminal effects that can be attained for a fixed peak gain. Taking $M = 1000$ (realizable in a careful experiment) and $D = 15\%$ as reference values, Fig. 4 shows that the relative advance a/τ_p attained with the simplest arrangement (medium with an isolated absorption line) is only 2.4 times below the theoretical limit (1.6) and that the ratio falls to 1.7 by involving a line doublet. Using nonuniform fiber-Bragg gratings could further reduce this ratio. Indeed, at least in principle, these elements allow one to synthesize any transfer function in transmission as long as it is minimum phase [21].

This restriction entails that the optimal transfer function (not minimum phase), and thus the upper limit to the advance, could be approached but not equaled with these systems. The same remark applies to the dispersive media whose transfer function is the exponential of a causal function and is thus also minimum phase [5]. Anyway, whatever the system is, superluminal advances exceeding two times the full width at half maximum of the pulse intensity profile are unattainable.

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