Asymptotic Properties in Rational l^2-approximation

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Abstract: This paper is concerned with the problem of best rational approximation of given order n in the Hardy space H_2. We show that, generically, all critical points converge to the function in H_2 as n increases to infinity. This property shows in turn that local maxima can appear only for a finite range of orders. This has consequences on an algorithm to find local minima previously described by some of the authors [3].

1 Introduction

Let us recall briefly the l^2-approximation problem as described, for instance, in [2]. Consider the real Hardy space $H^-_{2,\mathbf{R}}$ of functions f analytic in the complement of the closed unit disk $\bar{U}$, vanishing at infinity, that can be written $f(z) = \sum_{k=1}^{\infty} f_k z^{-k}$ with $f_k \in \mathbf{R}$ and such that the norm $\|f\|^2 = \sum_k f_k^2$ is finite. The assertions that the coefficients f_k are real and square integrable are respectively equivalent to the facts that the function f maps the real line into itself and satisfies a growth condition at the frontier T of $\bar{U}$:

$$\sup_{r>1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

As in [2], let P_n be the set of real polynomials of degree at most n, and $\mathcal{P}^1_n$ the subset of monic polynomials of degree n whose roots are in U. Moreover, let $\Sigma^1_n \subset H^-_{2,\mathbf{R}}$ consist of all rational fractions $h = p/q$, where $p \in P_{n-1}$ and $q \in \mathcal{P}^1_n$. The problem is to minimize, for arbitrary n, the criterion

$$\Gamma^n_f(h) = \|f - h\|^2$$

where h is in Σ^1_n.

The relevance to system theory of this question arises from the need of describing the input-output behaviour of a given system (i.e. its transfer function) by a finite dimensional model, whose transfer function is therefore rational. For more details, and a stochastic interpretation of the l^2 norm, we refer the reader to [1]. Of course, in this context, our problem can only appear as the prototype of an integral criterion, which can admit many variations and generalizations, putting for instance weight functions, additional constraints, and going over to the multi-input multi-output case. In the sequel, however, we shall restrict ourselves to the simple formulation given above.

Apart from a classical formulation using the so called normal equations [4], this problem has been mostly tackled using differentiation, namely a gradient algorithm [6]. In [3], an algorithm is described, which converges to a local minima by integrating numerically a differential equation. In all three cases, critical points (i.e. points where the derivative is zero) are on the saddle, because they are the ones that can be computed, and the trouble comes of course from the possibility of having several local minima.

Let us take a closer look at the criterion $\Gamma_f^n(p/q)$ that we want to minimize. As q is settled, the polynomial p is uniquely determined as the orthogonal projection of f onto the n-dimensional linear subspace V_q of $H_{2,\mathbf{R}}^-$ defined by $V_q = P_{n-1}/q$. We shall denote by $L_f^n(q)$ this projection and replace the former criterion by the following one:

$$\psi_f^n(q) = \left\| f - \frac{L_f^n(q)}{q} \right\|^2 .$$

At each order, we get a set of critical points for this criterion and the goal of this paper is to describe their asymptotic behavior. These critical points may be of two different kinds: either $L_f^n(q)$ and q are coprime, in which case the point q is said to be irreducible, either they share a common factor and we get a reducible point. When q is reducible, the variational argument (due to Ruckebush) used in [1] to show that it cannot be a minimum applies as well to show it cannot be a maximum either unless f is already rational. In the sequel, we shall suppose that the element f of $H_{2,\mathbf{R}}^-$ to be approximated is not already a rational function and we shall make the mild assumption that it is analytic on a domain strictly containing the complement of the open disk U. Then we show that, generically, critical points converge in l^2-norm to the function f as the order (i.e. the degree of the polynomials) growths to infinity. Using this fact, we deduce that critical points which are local maxima can only appear for a finite range of orders.

2 Properties of the criterion ψ_f^n and its critical points

As in [2], we introduce an other Hardy space $H_{2,\mathbf{R}}^+$, which contains analytic functions g in the unit disk U, mapping the real axis into itself and such that

$$\sup_{r<1} \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^2 d\theta < \infty.$$

The operator $f \to f^\sigma$ given by $f^\sigma(z) = \frac{1}{z}\check{f}(z)$ where $\check{f}(z) = f(\frac{1}{z})$, defines an application from $H_{2,\mathbf{R}}^-$ onto $H_{2,\mathbf{R}}^+$ and conversely. Moreover the orthogonal sum $L_{2,\mathbf{R}}(T) = H_{2,\mathbf{R}}^+ \oplus H_{2,\mathbf{R}}^-$ is a Hilbert space with the following scalar product

$$< f, g >= \frac{1}{2i\pi} \int_T f(z)\overline{g(z)} \frac{dz}{z}.$$

As $\bar{z} = z^{-1}$ on the unit circle T and the coefficients in the power series expansion of g are real, we also have

$$< f, g >= \frac{1}{2i\pi} \int_T f(z) g(\frac{1}{z}) \frac{dz}{z}.$$

This scalar product verifies the two following obvious properties which will be used in the sequel:

1) for all $k \in \mathbf{Z}$, the multiplication by z^k is an isometry of $L_{2,\mathbf{R}}(T)$, i.e. for all f, $g \in L_{2,\mathbf{R}}(T)$,

$$< z^k f, z^k g >=< f, g >; \tag{1}$$

2) for all f, g, h in $L_{2,\mathbf{R}}(T)$ such that fg and $\check{f}h$ are in $L_{2,\mathbf{R}}(T)$,

$$< fg, h >=< g, \check{f}h > . \qquad (2)$$

Let f be the function of $H^-_{2,\mathbf{R}}$ to be approximated and let $g = f^\sigma$, its image in the space $H^+_{2,\mathbf{R}}$. With the assumption made on f, there exists a real $\lambda > 1$ such that g is analytic in the open disk U_λ centered at 0 of radius λ. Let q in $\mathcal{P}^1_n$ and $\widetilde{q}$ defined as in [2] in the following way:

$$\widetilde{q}(z) = z^n q(\frac{1}{z}).$$

The Weierstrass division theorem (cf [5]) or more precisely its one dimensional version (also known as the Hadamard representation, cf [7]), applied to the function $g\widetilde{q}$ of $H^+_{2,\mathbf{R}}$ shows that there exists a unique function $v(g, q)$ analytic in U_λ and a unique polynomial $w(g, q)$ of degree $n - 1$ such that:

$$g\widetilde{q} = v(g, q)q + w(g, q).$$

It follows from [2] that if you seek in the space of rational fractions V_q for the minimum $L_f(q)/q$ of ψ^n_f, you get $L_f(q) = \widetilde{w(g, q)}$.
The quotient $v(g, q)$ of the former division which we shall simply denote by v when no confusion can arise, possesses the helpful property to give the value of the criterion ψ^n_f at the corresponding point q:

proposition 1. *Let q be a point of $\mathcal{P}^1_n$ and v the corresponding quotient,*

$$\|v\|^2 = \psi^n_f(q).$$

Proof: Using (1) and (2), the value of the criterion at q is:

$$\psi^n_f(q) =< f - \frac{L(q)}{q}, f - \frac{L(q)}{q} >=< g - \frac{\widetilde{L(q)}}{\widetilde{q}}, g - \frac{\widetilde{L(q)}}{\widetilde{q}} > .$$

Then

$$\psi^n_f(q) = \left\| g - \frac{\widetilde{L(q)}}{\widetilde{q}} \right\|^2 = \left\| \frac{qv}{\widetilde{q}} \right\|^2 = \|v\|^2,$$

the last equality coming out directly from the definition of the scalar product in $L_{2,\mathbf{R}}(T)$. □

We come now to a divisibility property at critical points which ensures the existence of many zeros in the unit disk for the associated quotients v when the points are irreducible:

proposition 2. *Let q, a point of $\mathcal{P}^1_n$, the two following assertions are equivalent:*
(i) q is a critical point of ψ^n_f.
(ii) q divides $vL(q)$.
Then, if q is an irreducible critical point, q divides v.

Proof: Let $q = z^n + q_{n-1}z^{n-1} + ... + q_0 \in \mathcal{P}^1_n$, q is a critical point iff:

$$\forall i \in \{0, ..., n-1\}, \frac{\partial \psi^n_f}{\partial q_i}(q) = 0.$$

These partial derivatives are equal to

$$\begin{aligned}\frac{\partial \psi_f^n}{\partial q_i}(q) &= -2 < f - \frac{L(q)}{q}, \frac{\partial}{\partial q_i}(\frac{L(q)}{q}) > \\ &= -2 < f - \frac{L(q)}{q}, \frac{\frac{\partial}{\partial q_i}(L(q))}{q} > +2 < f - \frac{L(q)}{q}, \frac{z^i L(q)}{q^2} > .\end{aligned}$$

As $L(q)/q$ is the orthogonal projection of f on the space V_q, the first term of the right-hand side is zero, and

$$\frac{\partial \psi_f^n}{\partial q_i}(q) = 2 < f - \frac{L(q)}{q}, \frac{z^i L(q)}{q^2} > .$$

This set of derivatives vanishes iff for all polynomials P in $\mathbf{C}^{n-1}[z]$

$$< f - \frac{L(q)}{q}, P\frac{L(q)}{q^2} >= 0.$$

From the definition of the scalar product, we get

$$\int_T (g - \frac{\widetilde{L(q)}}{\widetilde{q}})(z) P(z) \frac{L(q)}{q^2}(z) dz = 0$$

or

$$\int_T v(z) \frac{q}{\widetilde{q}}(z) P(z) \frac{L(q)}{q^2}(z) dz = \int_T v(z) \frac{L(q)}{\widetilde{q}}(z) \frac{P}{q}(z) dz = 0.$$

If α is a root of order m of the polynomial q, this implies

$$\forall l \in \{1, ..., m\}, \int_T v(z) \frac{L(q)}{\widetilde{q}}(z) \frac{dz}{(z-\alpha)^l} = 0.$$

Then, by the residue formula, the following derivatives should vanish:

$$\forall l \in \{0, ..., m-1\}, \left[\frac{vL(q)}{\widetilde{q}}\right]^{(l)} (\alpha) = 0.$$

By induction, α is a zero of order m of $vL(q)$. This proves the equivalence of the two assertions and proposition 2. $\square$

3 Asymptotic behaviour of critical points

We denote by C_n the subset of $\mathcal{P}_n^1$ containing the critical points at order n, and we put $C = \cup_n C_n$. By choosing a point in C_n for each n, we construct a sequence of quotients (v_n). In order to prove that the family of functions (v_n) is normal, we use the integral representation:

$$v(z) = \frac{1}{2i\pi} \int_{T_\mu} \frac{g(\xi)\widetilde{q}(\xi)}{q(\xi)} \frac{d\xi}{\xi - z}$$

where μ is any real such that $1 < \mu < \lambda$. Using this expression we prove the

lemma 1. *Let μ' a real number such that $1 < \mu' < \mu$, there exists on the open set $U_{\mu'}$ an uniform bound for the set of functions (v_n) which depends only on the function g.*

Proof: On the unit circle T, the quotient $\widetilde{q}/q$ is of modulus 1. Then by using the maximum principle over the complement of the unit disk U, we get

$$\forall \xi \in \mathbf{C} - U, \left|\frac{\widetilde{q}(\xi)}{q(\xi)}\right| \leq 1.$$

This inequality is true on the circle T_μ so that:

$$\forall z \in U_{\mu'}, |v(z)| \leq \frac{1}{2\pi}\left(\sup_{|z|=\mu} |g|\right)\int_{T_\mu} \frac{d\xi}{|\xi - z|}.$$

But for $z \in U_{\mu'}$, $|\xi - z|$ is greater than $\mu - \mu'$ and we get the bound in question. □
We shall first consider the case when our sequence consists of irreducible points only. Note that this is the case when it is obtained by repeatedly using a gradient algorithm for increasing values of n as described in [3].

From such a sequence, take a subsequence (w_p) which converges to a limit w_{lim} uniformly over all compact subsets of U_μ. Let μ' such that $1 < \mu' < \mu$ and suppose that the analytic function w_{lim} has no zeros on the circle $T_{\mu'}$. Then

$$\exists N, \forall n \geq N, \forall z \in T_{\mu'}, |w_n(z) - w_{lim}(z)| < |w_{lim}(z)|.$$

By Rouché's theorem, w_n and w_{lim} will have the same number of zeros in the open set $U_{\mu'}$, but using proposition 2, a quotient corresponding to an irreducible critical point of order n has at least n zeros in U. As the order of points in the subsequence (w_p) tends to infinity, w_{lim} must be equal to zero. This is a contradiction with the assumption made on the circle $T_{\mu'}$. By letting μ' vary continuously, we get a compact circular annulus containing infinitely many zeros for w_{lim} and thus this limit must vanish on the open disk U_μ. We just showed that every convergent subsequence of (v_n) converges to zero uniformly on every compact set of U_μ. Since it is normal, this is true for the sequence (v_n) itself.

By using the proposition 1, we get the l^2-convergence of any sequence of irreducible critical points to the function to be approximated as the order of points tends to infinity. In order to generalize this fact to sequences containing also reducible points, we shall have to restrict ourselves to functions with the property that C_n is finite for each n. This can be proved to be generic in various contexts. For instance, it is shown in [1] that such functions form a set of first category in the disc algebra of U_μ where $\mu > 1$. We first prove the

proposition 3. *Let $p \in \mathcal{P}_n^1$, a critical point such that the fraction $L_f(p)/p$ is irreducible. Let $r \in \mathcal{P}_m^1$ and $q = pr$. Then*
(i) $L_f(q) = rL_f(p)$ iff r divides $v(g,p)$.
(ii) if (i) is verified, we have the following equivalence: q is a critical point iff p is a critical point and r divides the quotient $v(g,p)/p$.

Proof: Apply the above mentioned division theorem to $g\widetilde{q}$ and $g\widetilde{p}$:

$$g\widetilde{q} = v(g,q)q + \widetilde{L_f(q)} \text{ and } g\widetilde{p} = v(g,p)p + \widetilde{L_f(p)}.$$

Multiply the second equation by $\widetilde{r}$:

$$g\widetilde{q} = v(g,p)\widetilde{r}p + \widetilde{\widetilde{r}L_f(p)}. \tag{3}$$

Let us denote $v(g,p)$ by v_p and divide $v_p\widetilde{r}$ by r:

$$v_p\widetilde{r} = v(v_p, r)r + \widetilde{L_{v_p^\sigma}(r)}.$$

Plugging this expression in (3), we get

$$g\widetilde{q} = v(v_p, r)q + (\widetilde{L_{v_p^\sigma}(r)}p + \widetilde{r}\widetilde{L_f(p)})$$

and the second term on the right-hand side is of degree strictly lower than that of q. Thus we have

$$\left\{\begin{array}{rcl} v_p\widetilde{r} & = & v(g,q)r + \widetilde{L_{v_p^\sigma}(r)} \\ \\ \widetilde{L_f(q)} & = & p\widetilde{L_{v_p^\sigma}(r)} + \widetilde{r}\widetilde{L_f(p)}. \end{array}\right. \tag{4}$$

In order to prove (i), suppose first that $L_f(q) = rL_f(p)$ holds. Applying to the second equation of (4) the assumption that r divides $L_f(q)$, and hence that $\tilde{r}$ divides $\widetilde{L_f(q)}$ we get (using the usual notation for division)

$$r|\widetilde{p}L_{v_p^\sigma}(r).$$

As roots of $\widetilde{p}$ lie in the complement of the unit disk, r divides $L_{v_p^\sigma}(r)$. But $L_{v_p^\sigma}(r)$ is the remainder of a division by r and then it must be zero. The previous pair of equations becomes

$$\left\{\begin{array}{rcl} L_f(q) & = & rL_f(p), \\ \\ v_p\widetilde{r} & = & v(g,q)r. \end{array}\right. \tag{5}$$

The second equation of (5) shows that r divides v_p.
Conversely, suppose that $r|v_p$. The first equation of (4) implies that $r|\widetilde{L_{v_p^\sigma}(r)}$. As the degree of $\widetilde{L_{v_p^\sigma}(r)}$ is strictly lower than that of r, it must be zero and (4) reduces as in the former case to (5).
Suppose now that the assertions of (i) are verified and let prove (ii). By proposition 2, the fact that q is a critical point means that $q|v(g,q)L_f(q)$ i.e.

$$pr|v_p(\frac{\widetilde{r}}{r})rL_f(p) \text{ or } pr|v_pL_f(p)$$

which yields that $p|v_pL_f(p)$ i.e. p is critical. This yields also that $r|(\frac{v_p}{p})L_f(p)$. Moreover as (i) is verified, $r|(\frac{v_p}{p})p$. As $L_f(p)$ and p are relatively prime, we deduce that $r|\frac{v_p}{p}$. Conversely, this last relation implies that $q|v_p$. By the second equation of (5), we get $q|v(g,q)\frac{r}{\tilde{r}}$. As roots of $\tilde{r}$ lie in the complement of the unit disk, q divides $v(g,q)r$ and in particular $v(g,q)rL_f(p)$. Then using the first equation of (5), q divides $v(g,q)L_f(q)$ and q is critical. □
If critical points are irreducible, there exists an order over which the corresponding quotients v have more than any preassigned number of zeros. To get our generalization, we prove that such an order exists even in the case of reducible points. Following proposition 3, these points are generated by adjoining to irreducible critical points q of lower order, zeros from v/q. We show that for a fixed order, the number of such zeros is bounded from above. Let I_n be the subset of C_n containing irreducible critical points of order n and let q in I_n. We

denote by $Z(v/q)$, the number of zeros of the quotient v/q in the disk U. Then $Z(v/q)$ is finite. Indeed, with the assumption made on f, the quotient v is analytic on the open set U_λ which contains the compact disk $\bar{U}$. If $Z(v/q)$ is not finite, v vanishes in U which means that the function f to approximate is already a rational fraction, but we discarded this case in the introduction. Let us set one more notation:

$$R_n = \max\{Z(v/q), q \in I_n\},$$

then R_n is finite. It is obvious when I_n itself is finite. Otherwise, let suppose that R_n is not finite, then we can select a sequence of critical points (q_l) in I_n whose corresponding quotients (v_l) have a number of zeros growing to infinity. From this sequence, we can extract as before a subsequence which tends to zero. But this means that there is a sequence of critical points of order n which converges to the function f and then f is again rational. Indeed, we have

$$g\widetilde{q_l} = v_l q_l + \widetilde{L_f(q_l)}. \tag{6}$$

The functions v_l converge uniformly to zero on $\bar{U}$ and the polynomials q_l and $\widetilde{q_l}$ are also bounded on $\bar{U}$ as their degree and their coefficients are. Then by (6), $\widetilde{L_f(q_l)}$ is bounded. We can successively extract two subsequences such that $\widetilde{L_f(q_l)}$ and $\widetilde{q_l}$ will converge respectively to some polynomials p and q, uniformly on $\bar{U}$. By taking the limit, the equation (6) becomes $gq = p$ on $\bar{U}$ and thus f is equal to $\tilde{p}/\tilde{q}$.
As a conclusion, at order $n + R_n$, quotients v corresponding to irreducible critical points as well as reducible ones which come from irreducible points of order n have all at least n zeros. At order $\max_{p\leq n}\{p + R_p\} + 1$, no critical point comes from an irreducible one of order less than or equal to n. Thus all the corresponding quotients have more than n zeros. This is the result we needed and finally, we proved the

theorem 1. *Let f, be a function of the Hardy space $H^-_{2,\mathbf{R}}$, distinct from a rational fraction, analytic on a open domain containing the complement of the unit disk U. Let (v_n) be a sequence of quotients corresponding to irreducible critical points q_n whose orders tend to infinity. Then the sequence (v_n) converges uniformly to zero on every compact subset of an open set containing the closed unit disk $\bar{U}$.Consequently, the sequence of critical points $(L_f(q_n)/q_n)$ tends to the function f, accordingly to the l^2-norm. Generically, the assumption that q_n is irreducible can be dropped.*

4 Finiteness of the number of orders where local maxima appear

Let us restate theorem 1 in the following manner: if f is non rational, then for any $\epsilon > 0$, there exists n_0 such that any irreducible critical point q of order $n > n_0$ satisfies $\Psi^n_f(q_n) < \epsilon$.

Now, let q be a critical point of order n. We shall show that it cannot be a local maximum if n is large enough. First, we can assume q is irreducible. The partial derivatives of the criterion $< f - \frac{L(q)}{q}, f - \frac{L(q)}{q} >$ at q vanish ie:

$$\forall i \in \{0, ..., n-1\}, < f - \frac{L(q)}{q}, \frac{\partial}{\partial q_i}\left(\frac{L(q)}{q}\right) >= 0$$

or

$$\forall i \in \{0,...,n-1\}, < f - \frac{L(q)}{q}, \frac{\frac{\partial}{\partial q_i}(L(q))}{q} > - < f - \frac{L(q)}{q}, \frac{z^i L(q)}{q^2} >= 0.$$

As $\frac{L(q)}{q}$ is the orthogonal projection of f on the space V_q, we know that

$$\forall k \in \{0,...,n-1\}, < f - \frac{L(q)}{q}, \frac{z^k}{q} >= 0, \tag{7}$$

so the last equality reduces to

$$\forall i \in \{0,...,n-1\}, < f - \frac{L(q)}{q}, \frac{z^i L(q)}{q^2} >= 0. \tag{8}$$

Combining (7) and (8), we get

$$< f - \frac{L(q)}{q}, \frac{r_1 L(q) + r_2 q}{q^2} >= 0$$

where r_1 and r_2 are any polynomials in P_{n-1}. But q and $L(q)$ are relatively prime and the last equality is equivalent to

$$< f - \frac{L(q)}{q}, \frac{P}{q^2} >= 0, \tag{9}$$

P being any polynomial of P_{2n-1}.
Now, we come again to equalities (7). Taking partial derivatives leads us to

$$\forall i \in \{0,...,n-1\}, - < \frac{\partial}{\partial q_i}\left(\frac{L(q)}{q}\right), \frac{z^k}{q} > - < f - \frac{L(q)}{q}, \frac{z^{k+i}}{q^2} >= 0.$$

Using (9), the second term in the left-hand side is zero and we get the orthogonality relations:

$$\forall k \in \{0,...,n-1\}, < \frac{\partial}{\partial q_i}\left(\frac{L(q)}{q}\right), \frac{z^k}{q} >= 0. \tag{10}$$

Next, we shall use the following

lemma 2. *Let h be a function of $H^-_{2,\mathbf{R}}$ orthogonal to the linear space V_p, where p is any polynomial of $\mathcal{P}^1_n$. Then $\tilde{p}$ divides h.*

Proof: Let ξ_i, be the complex roots of the polynomial p ie:

$$p(z) = \prod (z - \xi_i).$$

For each root ξ_i, the function h will be orthogonal to the quotient $1/(z-\xi_i)$. Using the definition of the scalar product, we get

$$\int_T \frac{h(\frac{1}{z})}{z - \xi_i} \frac{dz}{z} = 0.$$

From Cauchy's formula applied to the function $h(1/z)/z$, we deduce that

$$\forall i \in \{0,...,n\}, h(\frac{1}{\xi_i}) = 0$$

which means that $\tilde{p}$ divides h. $\square$
Now, apply this lemma to the functions $\frac{\partial}{\partial q_i}(\frac{L(q)}{q})$ which are orthogonal to the space V_q (cf (10)). There exist polynomials ν_i of $\mathcal{P}^1_{n-1}$ such that:

$$\forall i \in \{0, ..., n-1\}, \frac{\partial}{\partial q_i}\left(\frac{L(q)}{q}\right) = \frac{\tilde{q}}{q^2}\nu_i.$$

Moreover, the polynomials ν_i are linearly independent. Indeed, let (λ_i), a family of real numbers such that $\sum \lambda_i \nu_i = 0$. Then $\sum \lambda_i \tilde{q} \nu_i = 0$ or

$$\sum \left(\frac{\partial}{\partial q_i}\left(\frac{L(q)}{q}\right)\right)\lambda_i = 0.$$

This yields

$$q \sum \lambda_i \frac{\partial L(q)}{\partial q_i} - L(q) \sum \lambda_i z^i = 0$$

and the polynomial q must divide the sum $\sum \lambda_i z^i$ which is of degree $n-1$. As q is of degree n, we get that

$$\sum \lambda_i z^i = 0$$

and each real number λ_i is zero which proves the independency of the polynomials ν_i.
Now, we can evaluate the variation of the criterion at the critical point q, using the hessian matrix H whose entries are by definition:

$$\frac{\partial^2}{\partial q_i \partial q_j} < f - \frac{L(q)}{q}, f - \frac{L(q)}{q} >$$

or

$$-2\frac{\partial}{\partial q_i} < f - \frac{L(q)}{q}, \frac{\partial}{\partial q_j}\frac{L(q)}{q} >$$

$$= -2\left[- < \frac{\partial}{\partial q_i}\left(\frac{L(q)}{q}\right), \frac{\partial}{\partial q_j}\left(\frac{L(q)}{q}\right) > + < f - \frac{L(q)}{q}, \frac{\partial}{\partial q_i}\frac{\partial}{\partial q_j}\left(\frac{L(q)}{q}\right) >\right]$$

$$= 2\left[< \frac{\nu_i}{q}, \frac{\nu_j}{q} > - < f - \frac{L(q)}{q}, \frac{\partial}{\partial q_i}\left(\frac{\tilde{q}}{q^2}\nu_j\right) >\right]$$

$$= 2\left[< \frac{\nu_i}{q}, \frac{\nu_j}{q} > - < f - \frac{L(q)}{q}, \frac{\tilde{q}}{q^2}\frac{\partial \nu_j}{\partial q_i} > - < f - \frac{L(q)}{q}, \nu_j \frac{\partial}{\partial q_i}\left(\frac{\tilde{q}}{q^2}\right) >\right]$$

and using (9), we get

$$2\left[< \frac{\nu_i}{q}, \frac{\nu_j}{q} > - < f - \frac{L(q)}{q}, \nu_j \frac{\partial}{\partial q_i}\left(\frac{\tilde{q}}{q^2}\right) >\right].$$

But

$$\frac{\partial}{\partial q_i}\left(\frac{\tilde{q}}{q^2}\right) = \frac{1}{q^4}[q^2 z^{n-i} - 2\tilde{q} q z^i] = \frac{z^{n-i}}{q^2} - 2\tilde{q}\frac{z^i}{q^3},$$

so by (9) again, we obtain

$$\frac{\partial^2}{\partial q_i \partial q_j} < f - \frac{L(q)}{q}, f - \frac{L(q)}{q} >= 2\left[< \frac{\nu_i}{q}, \frac{\nu_j}{q} > + < f - \frac{L(q)}{q}, 2\frac{\tilde{q} z^i \nu_j}{q^3} >\right].$$

Now, the variation of the criterion in a neighbourhood of the critical point q following a direction given by the real vector $(\lambda_1, ..., \lambda_n)$ in the space $\mathcal{P}_n^1$ is

$$\Delta_q(\lambda_1, ..., \lambda_n) = (\lambda_1, ..., \lambda_n) H \begin{pmatrix} \lambda_1 \\ . \\ . \\ . \\ \lambda_n \end{pmatrix}.$$

As the family of polynomials (ν_i) is independent, we choose the numbers λ_i such that

$$\sum \lambda_i \nu_i = L(q).$$

The value of $(\frac{1}{2})\Delta_q(\lambda_1, ..., \lambda_n)$ becomes

$$\sum \lambda_i \lambda_j \left[< \frac{\nu_i}{q}, \frac{\nu_j}{q} > + < f - \frac{L(q)}{q}, 2\frac{\tilde{q} z^i \nu_j}{q^3} > \right]$$

or

$$< \frac{L(q)}{q}, \frac{L(q)}{q} > + < f - \frac{L(q)}{q}, 2\frac{\tilde{q}}{q^3} L(q) \left[\sum \lambda_i z^i \right] > .$$

On the other hand

$$\left(\frac{\partial L(q)}{\partial q_i} \right) q = z^i L(q) + \tilde{q} \nu_i.$$

Using this equality together with (9) gives the following expression for $(\frac{1}{2})\Delta_q$:

$$\left\| \frac{L(q)}{q} \right\|^2 + < f - \frac{L(q)}{q}, -2\frac{\tilde{q}^2}{q^3} \sum \lambda_i \nu_i >$$

$$= \left\| \frac{L(q)}{q} \right\|^2 - 2 < f - \frac{L(q)}{q}, \frac{\tilde{q}^2}{q^3} L(q) >$$

$$\geq \left\| \frac{L(q)}{q} \right\|^2 - 2 \left\| f - \frac{L(q)}{q} \right\| \left\| \frac{L(q)}{q} \right\|.$$

As the order of q growths, $\left\| f - \frac{L(q)}{q} \right\|$ tends to zero, then the variation following the chosen direction becomes positive which means that the critical point q may not be a maximum.

theorem 2. *Let f, be a function as in theorem 1. Then, critical points which are local maxima can only appear for a finite range of orders.*

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