A CRITERION FOR UNIQUENESS OF A CRITICAL POINT IN H_2 RATIONAL APPROXIMATION

By

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Abstract. This paper presents a criterion for uniqueness of a critical point in $H_{2,\mathbf{R}}$ rational approximation of type (m,n), with $m \geq n-1$. This criterion is differential-topological in nature, and turns out to be connected with corona equations and classical interpolation theory. We illustrate its use with three examples, namely best approximation of fixed type on small circles, a de Montessus de Ballore type theorem, and diagonal approximation to the exponential function of large degree.

Notations

T, U, V	unit circle, open unit disk, complement in $\overline{\mathbf{C}}$ of the closed unit disk
T_r, U_r, V_r	circle of radius r, open disk of radius r, complement in $\overline{\mathbf{C}}$ of the closed disk of radius r (with centers at the origin)
$\mathcal{P}_n$	space of real polynomials of degree at most n ; regarding the coefficients as coordinates, we endow $\mathcal{P}_n$ with the Euclidean topology of $\mathbf{R}^{n+1}$
$\mathcal{M}_n$	monic real polynomials of degree n
$\widetilde{\mathcal{M}}_n$	real polynomials of degree at most n with constant coefficient equal to 1
$\mathcal{M}_n^r$	monic real polynomials of degree n having all their roots in U_r
$\widetilde{\mathcal{M}}_n^r$	real polynomials of degree at most n with constant coefficient equal to 1 having all their roots in V_r
Δ_n	real monic polynomials of degree n having all their roots in $\overline{U}$; alternatively, closure of $\mathcal{M}_n^1$ with respect to the Euclidean topology of $\mathcal{P}_n$
$\widetilde{\Delta}_n$	real polynomials of degree at most n with constant coefficient equal to 1 having all their roots in $\overline{V}$; alternatively, closure of $\widetilde{\mathcal{M}}_n^1$ with respect to the Euclidean topology of $\mathcal{P}_n$
$\Vert\cdot\Vert_\infty$, $\Vert\cdot\Vert_2$	norms in $L_\infty(T)$, and in $L_2(T)$, respectively
$<\cdot,\cdot>$	scalar product in $L_2(T)$
$L_{2,\mathbf{R}}(T)$	real subspace of $L_2(T)$ consisting of functions with real Fourier coefficients

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$H_{2,\mathbf{R}}(U)$	real Hardy space of exponent 2 of the unit disk consisting of functions in $L_{2,\mathbf{R}}(T)$ whose Fourier coefficients with negative index vanish
$H^0_{2,\mathbf{R}}(V)$	real Hardy space of exponent 2 of the complement of the closed unit disk restricted to those functions vanishing at infinity; alternatively, orthogonal complement of $H_{2,\mathbf{R}}(U)$ in $L_{2,\mathbf{R}}(T)$
$\mathbf{P}_+, \mathbf{P}_-$	orthogonal projections $L_{2,\mathbf{R}}(T) \to H_{2,\mathbf{R}}(U)$ and $L_{2,\mathbf{R}}(T) \to H^0_{2,\mathbf{R}}(V)$, respectively
$H_{\infty,\mathbf{R}}(U)$	real subspace of $H_{2,\mathbf{R}}(U)$ consisting of essentially bounded functions
$\mathcal{R}^0_{m,n}(V)$	subset of $H^0_{2,\mathbf{R}}(V)$ consisting of rational functions $p/z^{m-n+1}q$ with $p \in \mathcal{P}_m$ and $q \in \mathcal{M}^1_n$
$\widetilde{\mathcal{R}}_{m,n}(U)$	subset of $H_{2,\mathbf{R}}(U)$ consisting of rational functions $\widetilde{p}/\widetilde{q}$ with $\widetilde{p} \in \mathcal{P}_m$ and $\widetilde{q} \in \widetilde{\mathcal{M}}^1_n$

1. Introduction

We consider the following two rational approximation problems:

$\mathbf{Pb}(V,m,n)$: *Given $f \in H^0_{2,\mathbf{R}}(V)$ and positive integers m, n with $m \geq n-1$, minimize*

$$\left\| f - \frac{p}{z^{m-n+1}q} \right\|_2$$

as $p/z^{m-n+1}q$ ranges over $\mathcal{R}^0_{m,n}(V)$.

$\mathbf{Pb}(U,m,n)$: *Given $g \in H_{2,\mathbf{R}}(U)$ and positive integers m, n with $m \geq n-1$, minimize*

$$\left\| g - \frac{\widetilde{p}}{\widetilde{q}} \right\|_2$$

as $\widetilde{p}/\widetilde{q}$ ranges over $\widetilde{\mathcal{R}}_{m,n}(U)$.

It will be convenient to say that a rational function is of type (m,n) if it can be written as the quotient of a member of $\mathcal{P}_m$ by a member of $\mathcal{P}_n$. Now, in the statement of $\mathbf{Pb}(U,m,n)$, the requirement that $\widetilde{p}/\widetilde{q}$ be analytic in $\overline{U}$ is, in fact, redundant because any rational function of type (m,n) with $m \geq n-1$ that minimizes the distance to g in $L_2(T)$ has to belong to $H_{2,\mathbf{R}}(U)$ when g does; this follows by partial fraction expansion from the orthogonality of $H_{2,\mathbf{R}}(U)$ and $H^0_{2,\mathbf{R}}(V)$.

The two approximation problems above are, in fact, equivalent: if we let

$$h^{\sigma}(z) := \frac{1}{z}h\left(\frac{1}{z}\right),$$

the map $h \to h^{\sigma}$ is an involutive isometry of $L_{2,\mathbf{R}}(T)$ interchanging $H_{2,\mathbf{R}}(U)$ (resp. $\widetilde{\mathcal{R}}_{m,n}(U)$) and $H^0_{2,\mathbf{R}}(V)$ (resp. $\mathcal{R}^0_{m,n}(V)$), so that $\mathbf{Pb}(U,m,n)$ is the transform of $\mathbf{Pb}(V,m,n)$ under this map.

Problem $\mathbf{Pb}(U,m,n)$ is classical in approximation theory (see e.g. [3, 6, 9, 11, 12, 14, 26]) and of importance in applied sciences, notably in control [10], and in signal processing [8, 15, 20]. Two noteworthy features of the present formulation are the restrictions to real coefficients and to the super-diagonal case $m \geq n-1$. The first restriction reflects the authors's interest in the applications and is not essential; the technique developed in the paper can be carried over to the case of complex coefficients.* The second restriction is more serious: whereas the super-diagonal case is an easy extension of the diagonal one, the sub-diagonal case involves additional difficulties that leave it uncovered.

The basic question of existence of a minimizer in $\mathbf{Pb}(U,m,n)$ or $\mathbf{Pb}(V,m,n)$ reduces to the case $m = n-1$ by Lemma 2.2 and is settled in the above references. The aim of the present paper is the analysis of the more delicate issue of *local minima* that arises here as in many nonlinear approximation questions in the complex domain, where uniqueness results are rare. In spite of its intriguing character and its practical relevance, since the occurrence of local *minima* is the major obstacle to numerical approaches, the only positive answers seem to be in [23], for the elementary case where f is rational of type (m,n), and in [6], for f a Stieltjes function whose supporting measure lies within some interval. When f is analytic in $\overline{V}$, we derive in Section 2 a general criterion for uniqueness that rests on the differential approach of [6], and consists in computing the index and then checking the signature of the second derivatives at the critical points. It will turn out that this signature depends on a corona-type equation with norm constraints expressing the degree of coprimeness of the numerator and the denominator of such points, and that one way to construct solutions to this equation is related to classical interpolation theory, and more specifically to the decay of the error in multipoint Padé approximation. Subsequently, in the remaining three sections, we apply the technique to three specific problems. The first of them deals with best $H_{2,\mathbf{R}}$ rational approximation of fixed type on small circles. It was selected, firstly because our method applies easily, secondly because it may prove practically useful by allowing to initialize continuation methods, and thirdly because it may, by a heuristic duality principle, support the conjecture that an entire function of finite

* The complex version, however, does not subsume the real one because the best H_2 rational approximation of given degree to a function with real Fourier coefficients need not itself be real.

order that is normal with respect to rational interpolation of type (m, n) in the disk has a unique critical point for n large. The second result is an $H_{2,\mathbf{R}}$ analog of the de Montessus de Ballore theorem, and is included because the subject is classical, and also because it is a nice instance of a problem whose linear part eventually dominates the nonlinear one. The third problem is the asymptotic uniqueness in $H_{2,\mathbf{R}}$ of best rational approximation to the exponential function in the diagonal case, when the degree becomes large. The exponential is the primary example for which the above-mentioned conjecture should be checked, and also turns out to be the prototype of the functions to which our criterion should apply, since the error is nearly circular and decreases rapidly. To establish these last properties, however, requires a somewhat detailed analysis that will provide us with precise asymptotics, both on the error and on the location of the poles of the approximants.

2. Critical points and a criterion for uniqueness

We develop in this section the differential theory of $\mathbf{Pb}(V, m, n)$. We assume that $m \geq n-1$ throughout, and we shall reduce to the case where $m = n-1$ for which the properties given below already appear in the literature except for Proposition 2.5, and for Proposition 2.8 when q has roots of unit modulus (cf. [4, 6] and the bibliographies therein).

Differentiating under the integral sign, we see that the map $\mathcal{P}_m \times \mathcal{M}_n^1 \to \mathbf{R}$ given by

$$(p, q) \to \left\| f - \frac{p}{z^{m-n+1}q} \right\|_2^2 \tag{2.1}$$

is smooth. Any pair (p_c, q_c) where the derivative vanishes will be termed *critical.* A minimizer of $\mathbf{Pb}(V, m, n)$, and more generally any local minimizer, is a critical pair; but there may also be others like saddles or local maximizers. Now a critical pair interpolates f maximally in V, as we shall shortly see, and the main result of this section may be summarized as follows (compare Theorems 2.9 and 2.10):

If each critical pair (p_c, q_c) is such that $p_c/z^{m-n+1}q_c$ interpolates f in no more than $m+n+1$ points in $\overline{V}$, and if in addition the corona equation

$$bp_c + cq_c = 1, \quad b, c \in H_{\infty,\mathbf{R}}(U)$$

is solvable with

$$\|(fz^{m-n+1}q_c - p_c)b\|_\infty < 1/2,$$

then the critical pair is unique.

Subsequently, we give a criterion based on the interpolation error to construct solutions to the above equation, and we translate it to $\mathbf{Pb}(U, m, n)$ (cf. Theorems 2.12 and 2.13).

We begin our study of the critical pairs by equating to zero the partial derivative of (2.1) with respect to the numerator. We obtain

$$\left\langle f - \frac{p_c}{z^{m-n+1}q_c}, \frac{\mathcal{P}_m}{z^{m-n+1}q_c} \right\rangle = 0;$$

if for $q \in \mathcal{M}_n^1$ we denote by $L_m^f(q) \in \mathcal{P}_m$ the numerator of the orthogonal projection of f onto the subspace $\mathcal{P}_m/z^{m-n+1}q \subset H_{2,\mathbf{R}}^0(V)$, the above equation means that a critical pair is necessarily of the form $(L_m^f(q_c), q_c)$. Hence, determining critical pairs reduces to finding their denominators, and these in turn arise as the critical points of the map $\mathcal{M}_n^1 \to \mathbf{R}$ given by

$$\psi_{m,n}^f(q) := \left\| f - L_m^f(q)/z^{m-n+1}q \right\|_2^2 = \|f\|_2^2 - \|L_m^f(q)/z^{m-n+1}q\|_2^2. \tag{2.2}$$

Here, the smoothness of $\psi_{m,n}^f$ as a function of q depends on the following formula (cf. [6]):

$$\widetilde{L_m^f}(q)(z) = \frac{1}{2i\pi}\int_T \frac{f^\sigma(\xi)\widetilde{q}(\xi)}{\xi^{m-n+1}q(\xi)} \left[\frac{\xi^{m-n+1}q(\xi) - z^{m-n+1}q(z)}{\xi - z}\right] d\xi, \tag{2.3}$$

where

$$\widetilde{p}(z) := z^k p(1/z)$$

defines the reciprocal polynomial of $p \in \mathcal{P}_k$. We offer a word of warning about this notation: if $k' > k$ and $p \in \mathcal{P}_k$ is considered as a member of $\mathcal{P}_{k'}$ whose leading coefficients do vanish, the two definitions of $\widetilde{p}$ may be inconsistent. For this reason, *we shall always specify the value of k under consideration*; in (2.3) for instance, it is understood that $L_m^f(q) \in \mathcal{P}_m$ and $q \in \mathcal{P}_n$. Formula (2.3) merely rephrases, using Hermite integral representation, a nice characterization of $L_m^f(q)$ essentially due to Walsh ([26], cf. also [3]):

Lemma 2.1 *$L_m^f(q)$ is the reciprocal of the remainder of the division of $f^\sigma\widetilde{q}$ by $z^{m-n+1}q$.*

We just saw that critical pairs are those pairs $(L_m^f(q_c), q_c)$ where q_c is critical for $\psi_{m,n}^f$. We shall say that $(L_m^f(q_c), q_c)$ is an irreducible critical pair, or also that q_c is an irreducible critical point of $\psi_{m,n}^f$, if $L_m^f(q_c)$ and q_c are coprime; otherwise, we call them reducible. The critical points of $\psi_{n-1,n}^f$ were studied in [2, 3, 4, 6], and we shall carry the corresponding results over to the case where $m \geq n-1$ using the following lemma:

Lemma 2.2 *Let $f \in H^0_{2,\mathbf{R}}(V)$ and $q \in \mathcal{M}^1_n$. Let further*

$$f^\sigma \widetilde{q} = v_q q z^{m-n+1} + \widetilde{L}^f_m(q), \qquad L^f_m(q) \in \mathcal{P}_m, \qquad v_q \in H_{2,\mathbf{R}}(U), \tag{2.4}$$

be the division of $f^\sigma \widetilde{q}$ by qz^{m-n+1}. Set $f_1 = \mathbf{P}_+(z^{m-n+1}f)$ and $f_2 = \mathbf{P}_-(z^{m-n+1}f)$. Then $L^{f_2}_{n-1}(q)$ and f_1 are the remainder and the quotient respectively of the division of $L^f_m(q)$ by q:

$$L^f_m(q) = qf_1 + L^{f_2}_{n-1}(q), \tag{2.5}$$

and

$$f_2^\sigma \widetilde{q} = v_q q + \widetilde{L}^{f_2}_{n-1}(q), \qquad L^{f_2}_{n-1}(q) \in \mathcal{P}_{n-1}, \tag{2.6}$$

is the division of $f_2^\sigma \widetilde{q}$ by q, namely the quotient is again v_q. Moreover, we have that

$$\psi^f_{m,n} = \psi^{f_2}_{n-1,n}. \tag{2.7}$$

In particular, q is an irreducible (resp. reducible) critical point of $\psi^f_{m,n}$ iff it is an irreducible (resp. reducible) critical point of $\psi^{f_2}_{n-1,n}$.

Proof It is easily checked that $f_1^\sigma = \mathbf{P}_-(z^{-(m-n+1)}f^\sigma)$ and $f_2^\sigma = \mathbf{P}_+(z^{-(m-n+1)}f^\sigma)$. Thus, upon dividing (2.4) by z^{m-n+1} and observing that $\mathbf{P}_+ + \mathbf{P}_-$ is the identity $L_{2,\mathbf{R}}(T) \to L_{2,\mathbf{R}}(T)$, we obtain

$$f_1^\sigma \widetilde{q} + f_2^\sigma \widetilde{q} = v_q q + \frac{\widetilde{L}^f_m(q)}{z^{m-n+1}}.$$

Since $f_1 \in \mathcal{P}_{m-n}$, we have $f_1^\sigma = \widetilde{f}_1 / z^{m-n+1}$ so that

$$f_2^\sigma \widetilde{q} = v_q q + \left[\frac{\widetilde{L}^f_m(q) - \widetilde{q}\widetilde{f}_1}{z^{m-n+1}} \right]. \tag{2.8}$$

Now, $f_2^\sigma \widetilde{q}$ and $v_q q$ belong to $H_{2,\mathbf{R}}(U)$ whence also the bracketed term does. This term is therefore a polynomial of degree at most $n-1$ and consequently (2.8) is the division of $f_2^\sigma \widetilde{q}$ by q. It follows now from Lemma 2.1 that

$$\widetilde{L}^{f_2}_{n-1}(q) = \left[\frac{\widetilde{L}^f_m(q) - \widetilde{q}\widetilde{f}_1}{z^{m-n+1}} \right], \tag{2.9}$$

and changing z into $1/z$ yields (2.5); furthermore, (2.9) and (2.8) together imply (2.6). Substituting (2.5) in (2.2) and using that multiplication by z is an isometry gives

$$\psi^f_{m,n}(q) = \left\| f - f_1/z^{m-n+1} - L^{f_2}_{n-1}(q)/z^{m-n+1}q \right\|_2^2 = \left\| f_2 - L^{f_2}_{n-1}(q)/q \right\|_2^2 = \psi^{f_2}_{n-1,n}(q).$$

□

Now, the characterization of the critical points of $\psi^f_{m,n}$ runs as follows:

Proposition 2.3 *Let q belong to $\mathcal{M}^1_n$, and let again*

$$f^\sigma\tilde{q} = v_q q z^{m-n+1} + \tilde{L}^f_m(q), \qquad L^f_m(q) \in \mathcal{P}_m, \qquad v_q \in H_{2,\mathbf{R}}(U),$$

be the division of $f^\sigma\tilde{q}$ by qz^{m-n+1}. Let $d \in \mathcal{M}_k$ be the monic g.c.d. of $L^f_m(q)$ and q, whose degree k may be positive or zero. Then:

(i) $L^f_m(q)/d = L^f_{m-k}(q/d)$.

(ii) *q is a critical point of $\psi^f_{m,n}$ iff q divides $v_q L^f_m(q)$. In particular, if q is an irreducible critical point, then q divides v_q.*

(iii) *q is a critical point of $\psi^f_{m,n}$ iff $q_1 = q/d$ is an irreducible critical point of $\psi^f_{m-k,n-k}$ and the polynomial d divides v_{q_1}/q_1, where v_{q_1} is the quotient of the division of $f^\sigma\tilde{q}_1$ by $q_1 z^{m-n+1}$ (note that v_{q_1}/q_1 actually lies in $H_{2,\mathbf{R}}(U)$ by* (ii) *as applied to q_1).*

Proof This is an immediate consequence of Lemma 2.2 and [6, Thm. 1, Prop. 2]. □

From now on, we assume that f not only belongs to $H^0_{2,\mathbf{R}}(V)$ but is in fact holomorphic on $\overline{V}$:

(H) there exists $\eta > 0$ such that $f(z)$ is analytic for $|z| > 1 - \eta$, $f(\bar{z}) = \overline{f(z)}$ and $f(\infty) = 0$.

Hypothesis **(H)** is a technical one, allowing us to extend the domain of definition of $\psi^f_{m,n}$ from $\mathcal{M}^1_n$ to Δ_n (Proposition 2.4). This is important because our criterion for uniqueness (Theorems 2.12 and 2.13), based on the Index Theorem (Theorem 2.9), is differential-topological in nature, whereas in differential topology functions have to be defined over compact sets to exhibit homotopy invariants (e.g. the degree).

Proposition 2.4 *When* **(H)** *is satisfied, the maps $L^f_m : \mathcal{M}^1_n \to \mathcal{P}_m$ and $\psi^f_{m,n} : \mathcal{M}^1_n \to \mathbf{R}$ extend smoothly to a neighborhood of Δ_n in $\mathcal{M}_n$.*

Proof Clearly, f_2 satisfies **(H)** if f does. Now apply Lemma 2.2 and [3, Prop. 2]. □

Denote again by $\psi^f_{m,n}$ and L^f_m the extended functions. When **(H)** is satisfied, Proposition 2.4 will allow us to speak of a critical point q_c or a critical pair $(L^f_m(q_c), q_c)$ when q_c lies on the boundary $\partial\Delta_n$ of Δ_n; this boundary consists of monic polynomials of degree n whose roots are bounded by 1 in modulus, and such that at least one of them has modulus 1. Now, we need to handle the critical points that $\psi^f_{m,n}$ may have on $\partial\Delta_n$. Such critical points are reducible since $L^f_m(q)/q$ cannot have a pole on T as $\psi^f_{m,n}$ is bounded by $\|f\|_2^2$; for such points, Proposition 2.3 must be supplemented as follows.

Proposition 2.5 *Assume* **(H)** *holds and let* $q \in \Delta_n$. *Let further*

$$f^\sigma \widetilde{q} = v_q q z^{m-n+1} + \widetilde{L}^f_m(q)$$

be the division of $f^\sigma \widetilde{q}$ *by* qz^{m-n+1}. *Write*

$$q = q_1 \nu_1^{\alpha_1} \cdots \nu_l^{\alpha_l},$$

where $q_1 \in \mathcal{M}^1_{n-k}$ *and the* ν_j*'s are distinct irreducible factors over* $\mathbf{R}[z]$ *having roots of modulus* 1. *Designate by* d_j *the degree of* ν_j *(either* 1 *or* 2*) and by*

$$f^\sigma \widetilde{q}_1 = v_{q_1} q_1 z^{m-n+1} + \widetilde{L}^f_{m-k}(q_1) \tag{2.10}$$

the division of $f^\sigma \widetilde{q}_1$ *by* $q_1 z^{m-n+1}$. *Then*

$$v_q = \pm v_{q_1} \quad \text{and} \quad L^f_m(q) = \nu_1^{\alpha_1} \cdots \nu_l^{\alpha_l} L^f_{m-k}(q_1). \tag{2.11}$$

Moreover, the following are equivalent:

(i) q *is a critical point of* $\psi^f_{m,n}$.

(ii) q_1 *is a critical point of* $\psi^f_{m-k,n-k}$ *and* $\nu_j^{[(\alpha_j+1)/2]}$ *divides* v_{q_1} *for* $j \in \{1, \ldots, l\}$, *where the bracket denotes the integer part.*

For the proof, we need two lemmas.

Lemma 2.6 *For* $q \in \Delta_n$, *we have*

$$\psi^f_{m,n}(q) = \|v_q\|_2^2.$$

Proof By continuity, we can assume $q \in \mathcal{M}_n^1$. Then

$$\psi_{m,n}^f(q) = \|f - L_m^f(q)/z^{m-n-1}q\|_2^2 = \|f^\sigma - \widetilde{L}_m^f(q)/\widetilde{q}\|_2^2 = \|v_q q z^{m-n+1}/\widetilde{q}\|_2^2 = \|v_q\|_2^2,$$

where the last equality is due to the fact that $|qz^{m-n-1}/\widetilde{q}| = 1$ on T. □

Lemma 2.7 *For* $q = q_1q_2 \in \Delta_n$ *with* $q_1 \in \Delta_{n-k}$, $q_2 \in \Delta_k$, *and* v_{q_1} *as in* (2.10), *we have*

$$\psi_{m,n}^f(q_1q_2) = \psi_{k-1,k}^{v_{q_1}^\sigma}(q_2).$$

Proof First observe that $v_{q_1}^\sigma$ satisfies hypothesis **(H)** when f does, so that the statement makes sense, i.e. we may evaluate $\psi_{k-1,k}^{v_{q_1}^\sigma}$ on $\partial\Delta_k$. By continuity, we can assume that $q_1 \in \mathcal{M}_{n-k}^1$ and $q_2 \in \mathcal{M}_k^1$. Perform the division of $v_{q_1}\widetilde{q}_2$ by q_2:

$$v_{q_1}\widetilde{q}_2 = aq_2 + \widetilde{L}_{k-1}^{v_{q_1}^\sigma}(q_2), \quad a \in H_{2,\mathbf{R}}(U);$$

multiplying (2.10) by $\widetilde{q}_2$ and substituting for $v_{q_1}\widetilde{q}_2$ yields

$$f^\sigma\widetilde{q}_1\widetilde{q}_2 = a(q_1q_2)z^{m-n+1} + \left[\widetilde{L}_{k-1}^{v_{q_1}^\sigma}(q_2)q_1z^{m-n+1} + \widetilde{q}_2\widetilde{L}_{m-k}^f(q_1)\right],$$

which is nothing but the division of $f^\sigma\widetilde{q}_1\widetilde{q}_2$ by $q_1q_2z^{m-n+1}$. Thus, we deduce from Lemma 2.1 that

$$\text{(2.12)} \qquad \frac{L_m^f(q_1q_2)}{z^{m-n+1}q_1q_2} = \frac{\widetilde{q}_1L_{k-1}^{v_{q_1}^\sigma}(q_2)}{z^{m-n+1}q_1q_2} + \frac{L_{m-k}^f(q_1)}{z^{m-n+1}q_1}.$$

It is to be observed that the two terms in the right-hand side of (2.12) are mutually orthogonal in $H_{2,\mathbf{R}}(V)$ because multiplying by $q_1z^{m-n+1}/\widetilde{q}_1$ is an isometry of $L_2(T)$ sending the first of these terms into $H_{2,\mathbf{R}}(V)$ and the second into $H_{2,\mathbf{R}}(U)$. By (2.2) and Pythagoras' rule, we get successively

$$\psi_{m,n}^f(q_1q_2) = \|f\|_2^2 - \left\|\frac{L_m^f(q_1q_2)}{z^{m-n+1}q_1q_2}\right\|_2^2 = \|f\|_2^2 - \left\|\frac{L_{m-k}^f(q_1)}{z^{m-n+1}q_1}\right\|_2^2 - \left\|\frac{\widetilde{q}_1L_{k-1}^{v_{q_1}^\sigma}(q_2)}{z^{m-n+1}q_1q_2}\right\|_2^2$$

$$= \psi_{m-k,n-k}^f(q_1) - \left\|\frac{L_{k-1}^{v_{q_1}^\sigma}(q_2)}{q_2}\right\|_2^2.$$

By Lemma 2.6 and since $h \to h^\sigma$ is an isometry of $L_{2,\mathbf{R}}(T)$, this may be rewritten as

$$\|v_{q_1}^\sigma\|_2^2 - \left\|\frac{L_{k-1}^{v_{q_1}^\sigma}(q_2)}{q_2}\right\|_2^2 = \psi_{k-1,k}^{v_{q_1}^\sigma}(q_2). \qquad \square$$

Proof of Proposition 2.5 Set $\mu = \nu_1^{\alpha_1} \cdots \nu_l^{\alpha_l} \in \mathcal{P}_k$. Since $\widetilde{\mu} = \pm\mu$ according to whether the multiplicity of the root 1 is even or odd, (2.11) follows immediately upon multiplying (2.10) by $\widetilde{\mu}$.

We turn to the equivalence of (i) and (ii). Let $\mathcal{Q}_1 \subset \mathcal{M}_{n-k}$ and $\mathcal{N}_j \subset \mathcal{M}_{\alpha_j d_j}$ be neighborhoods of q_1 and $\nu_j^{\alpha_j}$ respectively. Due to the pairwise coprimeness of q_1, $\nu_1, \ldots, \nu_l$, the components of $(\chi_1, \theta_1, \ldots, \theta_l) \in \mathcal{Q}_1 \times \mathcal{N}_1 \times \cdots \times \mathcal{N}_l$ are coordinates around $q \in \mathcal{M}_n$, so that q is critical iff the $l+1$ partial maps

$$\begin{aligned} \Xi_1 : \quad & \chi_1 \to \psi^f_{m,n}(\chi_1 \nu_1^{\alpha_1} \cdots \nu_l^{\alpha_l}), \\ \Theta_j : \quad & \theta_j \to \psi^f_{m,n}(q_1 \nu_1^{\alpha_1} \cdots \nu_{j-1}^{\alpha_{j-1}}\, \theta_j\, \nu_{j+1}^{\alpha_{j+1}} \cdots \nu_l^{\alpha_l}), \quad j = 1, \ldots, l, \end{aligned}$$

have vanishing derivatives at q_1 and $\nu_j^{\alpha_j}$ respectively. Since by a previous remark the roots of modulus 1 of any $\chi \in \Delta_n$ cancel in $L^f_m(\chi)/\chi$, we have for all $j \in \{1, \ldots, n\}$ that

$$\Xi_1(\chi_1) = \psi^f_{m-k,n-k}(\chi_1), \tag{2.13}$$

$$\Theta_j(\theta_j) = \psi^f_{m-k+\alpha_j d_j, n-k+\alpha_j d_j}(q_1 \theta_j). \tag{2.14}$$

Applying Lemma 2.7 to (2.14) with $n - k + \alpha_j d_j$, $m - k + \alpha_j d_j$ and $\alpha_j d_j$ in place of n, m and k respectively, we see that

$$\Theta_j = \psi^{v^\sigma_{q_1}}_{\alpha_j d_j - 1, \alpha_j d_j}, \quad j = 1, \ldots, l. \tag{2.15}$$

In view of (2.13) and (2.15), we see that q is a critical point of $\psi^f_{m,n}$ iff q_1 is critical for $\psi^f_{m-k,n-k}$ and $\nu_j^{\alpha_j}$ is critical for $\psi^{v^\sigma_{q_1}}_{\alpha_j d_j - 1, \alpha_j d_j}$. It remains to prove that $\nu_j^{\alpha_j}$ is critical for $\psi^{v^\sigma_{q_1}}_{\alpha_j d_j - 1, \alpha_j d_j}$ iff $\nu_j^{[(\alpha_j+1)/2]}$ divides v_{q_1}. Considering this assertion for each j separately, we may as well drop the index j and rename $v^\sigma_{q_1}$ as f. In other words, we are back to the case where $k = n$, $m = n - 1$, and $l = 1$, namely *we have to prove, for ν an irreducible factor over $\mathbf{R}[z]$ of degree d (either 1 or 2) having roots of modulus 1, that $q = \nu^\alpha$ is a critical point of $\psi^f_{\alpha d-1,\alpha d}$ iff $\nu^{[(\alpha+1)/2]}$ divides f^σ.*

By the Hermite integral formula (cf. e.g. [26]), the function v_q can be represented as

$$v_q(\xi) = \frac{1}{2i\pi} \int_{T_{1+\epsilon}} f^\sigma(\gamma) \frac{\widetilde{q}(\gamma)}{q(\gamma)} \frac{d\gamma}{\gamma - \xi}, \quad \xi \in U_{1+\epsilon}, \tag{2.16}$$

where $\epsilon > 0$ is chosen so that $(1+\epsilon)^{-1} > 1 - \eta$. This formula shows in particular that $q \to v_q$ is smooth $\mathcal{M}_n^{1+\epsilon} \to H_{2,\mathbf{R}}(U)$ and, since $\psi^f_{\alpha d-1,\alpha d}(q) = \|v_q\|_2^2$ by Lemma 2.6, differentiating with respect to the coefficients of $q(z) = z^{\alpha d} + a_{\alpha d-1} z^{\alpha d-1} + \cdots + a_0$ yields for $k \in \{0, \ldots, \alpha d - 1\}$:

$$\frac{\partial \psi^f_{\alpha d-1,\alpha d}(q)}{\partial a_k} = 2 < \frac{\partial v_q}{\partial a_k}, v_q >

$$= \frac{1}{\pi} \int_0^{2\pi} \frac{\partial v_q}{\partial a_k}(e^{i\theta}) \overline{v_q(e^{i\theta})}\, d\theta = \frac{1}{i\pi} \int_T \frac{\partial v_q}{\partial a_k}(\xi) v_q^\sigma(\xi) d\xi, \tag{2.17}$$

where we have used the identity $v_q^\sigma(\xi) = \overline{v_q(\xi)}/\xi$ on T (remembering that the Fourier coefficients are real). Plugging (2.16) into (2.17) and differentiating under the integral sign, we obtain

$$\frac{\partial \psi^f_{\alpha d-1,\alpha d}(q)}{\partial a_k} = \frac{1}{i\pi} \int_T v_q^\sigma(\xi) \left[\frac{1}{2i\pi} \int_{T_{1+\epsilon}} f^\sigma(\gamma) \frac{\gamma^{\alpha d-k} q(\gamma) - \gamma^k \widetilde{q}(\gamma)}{q^2(\gamma)} \frac{d\gamma}{\gamma - \xi} \right] d\xi$$

$$= \frac{1}{i\pi} \int_{T_{1+\epsilon}} f^\sigma(\gamma) \frac{\gamma^{\alpha d-k} q(\gamma) - \gamma^k \widetilde{q}(\gamma)}{q^2(\gamma)} \left[\frac{1}{2i\pi} \int_T v_q^\sigma(\xi) \frac{d\xi}{\gamma - \xi} \right] d\gamma, \tag{2.18}$$

where the second equality uses Fubini's theorem. On the other hand, by the residue formula as applied to the function v_q^σ (which is analytic in $V_{1-\eta}$ and vanishes at infinity),

$$\frac{1}{2i\pi} \int_T v_q^\sigma(\xi) \frac{d\xi}{\gamma - \xi} = v_q^\sigma(\gamma), \quad \gamma \in T_{1+\epsilon},$$

whence (2.18) becomes

$$\frac{1}{2} \frac{\partial \psi^f_{\alpha d-1,\alpha d}(q)}{\partial a_k} = \frac{1}{2i\pi} \int_{T_{1+\epsilon}} v_q^\sigma(\gamma) f^\sigma(\gamma) \frac{\gamma^{\alpha d-k} q(\gamma) - \gamma^k \widetilde{q}(\gamma)}{q^2(\gamma)} d\gamma.$$

As $q = \nu^\alpha$ is a real polynomial with roots of modulus 1 only, either $\widetilde{q} = q$ or $\widetilde{q} = -q$ depending upon whether the multiplicity of the root 1 is even or odd; accordingly, either $v_q = f^\sigma$ or $v_q = -f^\sigma$.

Assume first that $\widetilde{q} = q$. We get

$$\frac{1}{2} \frac{\partial \psi^f_{\alpha d-1,\alpha d}(q)}{\partial a_k} = \frac{1}{2i\pi} \int_{T_{1+\epsilon}} f(\gamma) f^\sigma(\gamma) \frac{\gamma^{\alpha d-k} - \gamma^k}{q(\gamma)} d\gamma,$$

whence q is a critical point of $\psi^f_{\alpha d-1,\alpha d}$ iff

$$\frac{1}{2i\pi} \int_{T_{1+\epsilon}} f(\gamma) f^\sigma(\gamma) \frac{\gamma p(\gamma) - \widetilde{p}(\gamma)}{q(\gamma)} d\gamma = 0, \quad \forall p \in \mathcal{P}_{\alpha d-1}. \tag{2.19}$$

The image of the map $\phi : \mathcal{P}_{\alpha d-1} \to \mathcal{P}_{\alpha d}$ sending $p(\gamma)$ to $\gamma p(\gamma) - \widetilde{p}(\gamma)$ is the set of anti-reciprocal polynomials of degree at most αd, namely

$$\mathbf{Im}\phi = \{\chi \in \mathcal{P}_{\alpha d} \,;\, \widetilde{\chi} = -\chi\}.$$

Indeed, one checks easily that the image of ϕ is included in this set. Conversely, let

$$\chi(\gamma) = \chi_{\alpha d}\gamma^{\alpha d} + \chi_{\alpha d-1}\gamma^{\alpha d-1} + \cdots - \chi_{\alpha d-1}\gamma - \chi_{\alpha d}$$

be anti-reciprocal; then

$$p(\gamma) = \chi_{\alpha d}\gamma^{\alpha d-1} + \cdots + \chi_{[\alpha d/2]+1}\gamma^{[\alpha d/2]} \in \mathcal{P}_{\alpha d-1}$$

satisfies $\phi(p) = \chi$. Therefore, q is a critical point of $\psi^f_{\alpha d-1,\alpha d}$ iff

$$\frac{1}{2i\pi}\int_{T_{1+\epsilon}} f(\gamma)f^\sigma(\gamma)\frac{p(\gamma)}{q(\gamma)}d\gamma = 0, \quad \forall p \in \mathcal{P}_{\alpha d}, \quad p = -\widetilde{p}. \tag{2.20}$$

In another connection, changing γ into $1/\gamma$ in the integral below yields the identity

$$\frac{1}{2i\pi}\int_{T_{1+\epsilon}} f(\gamma)f^\sigma(\gamma)\frac{p(\gamma)}{q(\gamma)}d\gamma = \frac{1}{2i\pi}\int_{T_{1/(1+\epsilon)}} f(\gamma)f^\sigma(\gamma)\frac{\widetilde{p}(\gamma)}{q(\gamma)}\,d\gamma, \quad p \in \mathcal{P}_{\alpha d}, \tag{2.21}$$

whence

$$\frac{1}{2i\pi}\int_{T_{1+\epsilon}-T_{1/(1+\epsilon)}} f(\gamma)f^\sigma(\gamma)\frac{p(\gamma)}{q(\gamma)}\,d\gamma = \frac{1}{2i\pi}\int_{T_{1+\epsilon}} f(\gamma)f^\sigma(\gamma)\frac{p-\widetilde{p}}{q}(\gamma)\,d\gamma, \quad p \in \mathcal{P}_{\alpha d}.$$

In view of (2.20), we see now that $q = \nu^\alpha$ is a critical point of $\psi^f_{\alpha d-1,\alpha d}$ iff

$$\frac{1}{2i\pi}\int_{T_{1+\epsilon}-T_{1/(1+\epsilon)}} f(\gamma)f^\sigma(\gamma)\frac{p(\gamma)}{\nu^\alpha(\gamma)}\,d\gamma = 0, \quad p \in \mathcal{P}_{\alpha d}.$$

Thanks to the residue formula, and since all the roots of ν^α have modulus 1 hence lie within the contour, this is equivalent to asserting that ν^α divides ff^σ or also that $\nu^{[(\alpha+1)/2]}$ divides f because f and f^σ share the same roots of modulus 1. Finally, if $\widetilde{q} = -q$, (2.19) has to be replaced by

$$\frac{1}{2i\pi}\int_{T_{1+\epsilon}} f(\gamma)f^\sigma(\gamma)\frac{\gamma p(\gamma) + \widetilde{p}(\gamma)}{q(\gamma)}d\gamma = 0, \quad \forall p \in \mathcal{P}_{\alpha d-1},$$

and one can check in the same manner that $\nu^{[(\alpha+1)/2]}$ again divides f in this case. □

Having characterized the critical points of $\psi^f_{m,n}$ in terms of the zeros of v_q and v_{q_1} in Propositions 2.3 and 2.5, we now recognize by applying σ to (2.4) that $(L^f_m(q), q)$ is an irreducible critical pair iff $L^f_m(q)/q$ is a multipoint Padé approximant to f of a particular type (the interpolation takes place at infinity with order $m - n + 1$ and at the reciprocals of the roots of q with order 2), and that a reducible critical pair is generated by a lack of normality in $\overline{V}$. Since there are no spurious poles for such approximants, and since we assume that **(H)** holds, it is natural to conjecture that *any* sequence of critical points such that m goes to infinity actually converges to f uniformly on $\overline{V}$. This is the content of the next proposition, which will be used in the proof of asymptotic uniqueness for the exponential function and may be of interest in its own right.

Proposition 2.8 *Assume f is analytic in a simply-connected domain Ω containing $\overline{V}$. Assume also that f vanishes at infinity and satisfies $f(\bar{z}) = \overline{f(z)}$. Define $\check{\Omega} = \{1/z; z \in \Omega\}$. For $q \in \Delta_n$, let*

$$f^\sigma \widetilde{q} = v_q q z^{m-n+1} + \widetilde{L}^f_m(q)$$

be the division of $f^\sigma \widetilde{q}$ by qz^{m-n+1}. Then the collection

$$\{v_q;\ \ q \in \Delta_n,\ n \in \mathbf{N},\ m \in \mathbf{N},\ m \geq n-1\}$$

is a normal family of functions in $\check{\Omega}$. If (m_k, n_k) is a sequence of pairs of non-negative integers such that $m_k \geq n_k - 1$ and $\lim_{k\to\infty} m_k = \infty$, and if, for each k, we let $q_{n_k} \in \Delta_{n_k}$ be a critical point of $\psi^f_{m_k,n_k}$, then

$$\lim_{k\to\infty} \frac{L^f_{m_k}(q_{n_k})}{q_{n_k}} = f \tag{2.22}$$

uniformly on $\overline{V}$.

Proof Let $K \subset \check{\Omega}$ be a compact set containing $\overline{U}$ and $\Gamma \subset \check{\Omega}$ a contour surrounding K. By the Hermite formula, the function v_q can be represented as

$$v_q(\xi) = \frac{1}{2i\pi} \int_\Gamma f^\sigma(\gamma) \frac{\widetilde{q}(\gamma)}{\gamma^{m-n+1} q(\gamma)} \frac{d\gamma}{\gamma - \xi}, \quad \xi \in K. \tag{2.23}$$

On the unit circle, the function $\widetilde{q}/\gamma^{m-n+1}q$ has modulus 1, so by the maximum principle

$$\left| \frac{\widetilde{q}(\gamma)}{\gamma^{m-n+1} q(\gamma)} \right| \leq 1 \qquad \forall \gamma \in V.$$

This inequality is in particular true on the circle Γ, so that (2.23) implies

$$|v_q(\xi)| \leq \frac{1}{2\pi} \left(\sup_\Gamma |f^\sigma| \right) \int_\Gamma \frac{d\gamma}{|\gamma - \xi|} \qquad \forall \xi \in K. \tag{2.24}$$

Because the distance from K to Γ is positive, we see from (2.24) that the family (v_q) is uniformly bounded over K, thereby establishing that it is normal.

To prove (2.22), observe as in the proof of Lemma 2.6 that

$$|f - L_m^f(q)/z^{m-n+1}q| = |v_q| \quad \text{on } T,$$

so we must prove that $\|v_{q_{n_k}}\|_\infty \to 0$ when $k \to \infty$.

Assume first that n_k remains bounded. Then $m_k - n_k + 1$ goes to infinity and

$$f_{k,2} = \mathbf{P}_-(z^{m_k-n_k+1}f)$$

converges to zero uniformly on $\overline{V}$ because the Taylor expansion of f at infinity is normally convergent there. According to Lemma 2.2, let

$$f_{k,2}^\sigma \widetilde{q}_{n_k} = v_{q_{n_k}} q_{n_k} + \widetilde{L}_{n_k-1}^{f_{k,2}}(q_{n_k})$$

be the division of $f_{k,2}^\sigma \widetilde{q}_{n_k}$ by q_{n_k}. From (2.24) applied with $f_{k,2}$ instead of f, we deduce that

$$|v_{q_{n_k}}(\xi)| \le \frac{1}{2\pi}\left(\sup_\Gamma |f_{k,2}^\sigma|\right)\int_\Gamma \frac{d\gamma}{|\gamma-\xi|} \qquad \forall \xi \in \overline{U},$$

which goes to zero uniformly as k goes to infinity. We thus get the desired conclusion when n_k is bounded. Assume now that n_k goes to infinity. In accordance with Propositions 2.3 and 2.5, we decompose q_{n_k} as $\nu_k d_k \chi_k$ where ν_k is a polynomial of degree α_k having only roots of modulus 1, $d_k \in \mathcal{M}_{\beta_k}^1$ is a common divisor of $L_{m_k}^f(q_{n_k})$ and q_{n_k}, and $\chi_k \in \mathcal{M}_{\delta_k}^1$ is an irreducible critical point of $\psi_{m_k-n_k+\delta_k,\delta_k}^f$. Let v_{χ_k} be the quotient of the division of $f^\sigma \widetilde{\chi}_k$ by $z^{m_k-n_k+1}\chi_k$. Cancelling the common roots between $L_{m_k}^f(q_{n_k})$ and q_{n_k} gives

$$\|f - L_{m_k}^f(q_{n_k})/z^{m_k-n_k+1}q_{n_k}\|_\infty = \|f - L_{m_k-n_k+\delta_k}^f(\chi_k)/z^{m_k-n_k+1}\chi_k\|_\infty = \|v_{\chi_k}\|_\infty,$$

and since (v_{χ_k}) is a normal family on $\check{\Omega} \supset \overline{U}$ by the first part of the proof, it is enough to show that the number of zeros of v_{χ_k} in $\overline{U}$ goes to infinity with k. By Proposition 2.3, we know that $d_k\chi_k$ divides v_{χ_k}. In addition, it is easily checked that

$$v_{\chi_k} = \pm v_{q_{n_k}} d_k/\widetilde{d}_k,$$

where the sign depends upon whether $\nu_k = \widetilde{\nu}_k$ or $\nu_k = -\widetilde{\nu}_k$, so that v_{χ_k} has at least $[\alpha_k/2]$ zeros of modulus 1 since $v_{q_{n_k}}$ does by Proposition 2.5. Hence v_{χ_k} has at least

$$\beta_k + \delta_k + [\alpha_k/2] \ge n_k/2$$

zeros in $\overline{U}$, and this completes the proof. □

We turn to a result which is central to our approach since it will allow us to pass from the local to the global when analysing uniqueness of a critical point. Recall that a critical point is said to be nondegenerate if the second derivative is a nondegenerate quadratic form. In this case, the number of negative eigenvalues is called the Morse index of the critical point and is invariant under change of coordinates. The theorem below links the Morse indices of the critical points of $\psi^f_{m,n}$ together when they are nondegenerate and may be viewed as an analogue of the Poincaré–Hopf theorem granted that Δ_n is a topological n-ball [3].

Theorem 2.9 (The Index Theorem) *Assume* **(H)** *holds and* $\psi^f_{m,n}$ *has only nondegenerate critical points in* Δ_n*, none of which lies on* $\partial\Delta_n$*. Let* C *be the collection of these critical points and* $\varepsilon(q)$ *designate the Morse index of* $q \in C$*. Then*

$$\sum_{q\in C}(-1)^{\varepsilon(q)} = 1.$$

Proof The case $m = n - 1$ is established in [3], so we appeal to Lemma 2.2. □

Remarks The nondegeneracy of the critical points is generic in $H_{2,\mathbf{R}}(U_r)$ for $r > 1$ [2]. One can in fact prove that critical points on $\partial\Delta_n$ are degenerate, so that the hypotheses we gave are somewhat redundant, but this will not be a concern for us.

The criterion for uniqueness of a critical point that we seek rests on ensuring that each critical point is a nondegenerate local minimum, hence has index 0, and then applying the index theorem. Therefore, what we really need now is a sufficient condition for a critical point to be a local minimum. While it is not difficult to see that a reducible point is never a local minimum, unless f is rational of type $(m-1, n-1)$, because the problem is normal [3], the forthcoming theorem asserts that a critical q is a local minimum provided $L^f_m(q)$ and q are "sufficiently coprime".

Theorem 2.10 *Let* $f \in H^0_{2,\mathbf{R}}(V)$ *and* $q \in \mathcal{M}^1_n$ *be an irreducible critical point of* $\psi^f_{m,n}$*. Assume there exists a corona relation*

$$bL^f_m(q) + cq = 1, \quad b, c \in H_{\infty,\mathbf{R}}(U), \tag{2.25}$$

such that

$$\|(fq - L^f_m(q)/z^{m-n+1})b\|_\infty < 1/2. \tag{2.26}$$

Then, q *is a nondegenerate local minimum of* $\psi^f_{m,n}$*.*

Proof Set $f_1 = \mathbf{P}_+(z^{m-n+1}f)$ and $f_2 = \mathbf{P}_-(z^{m-n+1}f)$. Then $\psi^f_{m,n} = \psi^{f_2}_{n-1,n}$ by Lemma 2.2, so that q will be a nondegenerate local minimum of $\psi^f_{m,n}$ iff it is a nondegenerate local minimum of $\psi^{f_2}_{n-1,n}$. Now, (2.5) and (2.25) give us a corona relation between q and $L^{f_2}_{n-1}(q)$, where the coefficient of the latter is again b:

$$bL^{f_2}_{n-1}(q) + (c + bf_1)q = 1.$$

Moreover, it is straightforward to check that

$$\|(f_2 q - L^{f_2}_{n-1}(q))b\|_\infty = \|(fq - L^f_m(q)/z^{m-n+1})b\|_\infty,$$

whence it is enough to prove the theorem when $m = n - 1$. In this case, it is shown in [6, eqn. (22)] that the second derivative of $\psi^f_{n-1,n}$ at q can be expressed in suitable coordinates as a quadratic form on the space $\mathcal{P}_{n-1}$ by the formula

$$H(p,p) = \left\|\frac{p}{q}\right\|^2 - \left\langle f - \frac{L^f_{n-1}(q)}{q}, 2\frac{\tilde{q}^2 bp^2}{q^3}\right\rangle, \qquad p \in \mathcal{P}_{n-1},$$

which depends on b modulo q only, by the critical point property (cf. [6]). From the inequality

$$\left|\left\langle f - \frac{L^f_{n-1}(q)}{q}, 2\frac{\tilde{q}^2 bp^2}{q^3}\right\rangle\right| \leq 2\left\|(fq - L^f_{n-1}(q))b\right\|_\infty \left\|\frac{p}{q}\right\|_2^2,$$

which is obvious if one writes the scalar product in integral form, we see that (2.26) implies the positivity of H. □

In order to complete our construction, it remains for us to find a way of manufacturing b and c satisfying (2.25) and (2.26). The next lemma provides us with a means of doing this when precise estimates of the error in the multipoint Padé approximation are available. It will be convenient to use the notation $\mathrm{Ord}_\infty(h)$ to designate the order at ∞ of a meromorphic function h, i.e., the finite integer ν such that

$$h(z) = O(z^\nu) \qquad \text{as } |z| \to \infty.$$

We also denote by $Z_V(h)$ (resp. $Z_U(h)$) the number of finite zeros of h in V (resp. U) counting multiplicities.

Lemma 2.11 *Assume that* **(H)** *holds and let* $f_1 = \mathbf{P}_+(z^{m-n+1}f)$ *and* $f_2 = \mathbf{P}_-(z^{m-n+1}f)$. *Let also* $q \in \mathcal{M}^1_n$ *be prime to* $L^f_m(q)$. *If* B/A *is a rational function in irreducible form with real coefficients such that*

$$[Z_V - \mathrm{Ord}_\infty](z^{m-n+1}Af - B) \geq n, \tag{2.27}$$

and if in addition

$$|f - L^f_m(q)/z^{m-n+1}q| < |f - B/z^{m-n-1}A| \quad \text{on } T, \tag{2.28}$$

then the polynomial $AL^f_m(q) - Bq$ *has no zeros in* $\overline{U}$ *and we may set in* (2.25)

$$b = \frac{A}{AL^f_m(q) - Bq}, \qquad c = \frac{-B}{AL^f_m(q) - Bq}.$$

If (2.28) *is replaced by the stronger inequality*

$$3|f - L^f_m(q)/z^{m-n+1}q| < |f - B/z^{m-n+1}A| \quad \text{on } T, \tag{2.29}$$

then in addition (2.26) *holds.*

Remark The proof will actually show that equality necessarily holds in (2.27) under the stated hypothesis. If B/A has poles on T, the right-hand sides of (2.28) and (2.29) have to be interpreted as $+\infty$ at those points.

Proof The difference $B/A - L^f_m(q)/q$ has no zeros on T since by (2.28)

$$|B/z^{m-n+1}A - L^f_m(q)/z^{m-n+1}q| \geq |f - B/z^{m-n-1}A| - |f - L^f_m(q)/z^{m-n-1}q| > 0 \quad \text{on } T.$$

Then $Bq - AL^f_m(q)$ has no zero on T either; hence the winding number $W(Bq - AL^f_m(q))$ of the curve $(Bq - AL^f_m(q))(T)$ around the origin is well defined. Assume first that A has no zero on T. By the argument principle,

$$W\left(Bq - AL^f_m(q)\right) = W\left(B/z^{m-n+1}A - L^f_m(q)/z^{m-n+1}q\right) + m + 1 + Z_U(A),$$

so that (2.28) and Rouché's theorem together imply

$$W\left(Bq - AL^f_m(q)\right) = W(f - B/z^{m-n-1}A) + m + 1 + Z_U(A)$$

$$= W(z^{m-n+1}Af - B) + n = -Z_V(z^{m-n+1}Af - B) + \mathrm{Ord}_\infty(z^{m-n+1}Af - B) + n \leq 0,$$

where the last inequality uses (2.27). As $Bq - AL^f_m(q)$ is analytic, this winding number equals the number of zeros it has in $\overline{U}$, so this number is zero and the first assertion of the lemma is proved. Assuming (2.29), one obtains on T

$$|(fq - L^f_m(q)/z^{m-n+1})b| = \left|\frac{f - L^f_m(q)/z^{m-n+1}q}{L^f_m(q)/z^{m-n+1}q - B/z^{m-n+1}A}\right|$$

$$\leq \frac{|f - L_m^f(q)/z^{m-n+1}q|}{|f - B/z^{m-n+1}A| - |f - L_m^f(q)/z^{m-n+1}q|} < 1/2,$$

which proves (2.26).

If A happens to have zeros on T, it cannot have a zero on $T_{1+\epsilon}$ for $0 < \epsilon < \epsilon_0$, say. Since (2.27) and (2.28) will remain true on $V_{1+\epsilon}$ and $T_{1+\epsilon}$ respectively when ϵ is small enough, we obtain by the same reasoning as before that $Bq - AL_m^f(q)$ has no zero in $\overline{U}_{1+\epsilon}$. If (2.29) happens to hold, we first replace 3 by $3+\delta$ for some $\delta > 0$ which is small enough, and then argue that this stronger inequality also remains true on $T_{1+\epsilon}$. We conclude as in the first part of the proof that

$$|(fq - L_m^f(q)/z^{m-n+1})b| < \frac{1}{2+\delta} \quad \text{on } T_{1+\epsilon}.$$

Letting ϵ go to zero yields the desired conclusion. □

We are in a position now to state our criterion for uniqueness of a critical point.

Theorem 2.12 (Criterion for uniqueness in $\mathbf{Pb}(V, m, n)$) *Assume that* **(H)** *holds, and that any critical point q of $\psi_{m,n}^f$ satisfies*

(i) *q is irreducible,*

(ii) *there exists a rational function B/A in irreducible form with real coefficients such that*

$$3|f - L_m^f(q)/z^{m-n+1}q| < |f - B/z^{m-n+1}A| \quad \text{on } T, \tag{2.30}$$

and such that $B/z^{m-n+1}A$ interpolates f in V to yield

$$[Z_V - \mathrm{Ord}_\infty](z^{m-n+1}Af - B) \geq n. \tag{2.31}$$

Then, $\psi_{m,n}^f$ has a unique critical point $q^* \in \mathcal{M}_n^1$ and $L_m^f(q^*)/q^*$ is the unique minimizer of $\mathbf{Pb}(V, m, n)$.

Proof The proof is immediate from Theorem 2.10, Lemma 2.11, and Theorem 2.9. □

Theorem 2.12 can also be translated to $\mathbf{Pb}(U, m, n)$ using the equivalence of the two problems under σ. To emphasize symmetry, we shall denote by $\widetilde{p}/\widetilde{q}$ a typical element of $\widetilde{\mathcal{R}}_{m,n}(U)$, observing that indeed any member of $\mathcal{P}_m$ is the reciprocal of some unique $p \in \mathcal{P}_m$ and that any member of $\widetilde{\Delta}_n$ is the reciprocal of some unique $q \in \Delta_n$. We further define $\widetilde{\psi}_{m,n}^g : \widetilde{\Delta}_n \to \mathbf{R}$ by

$$\widetilde{\psi}_{m,n}^g(\widetilde{q}) := \min_{\widetilde{p} \in \mathcal{P}_m} \|g - \widetilde{p}/\widetilde{q}\|_2^2 = \|g - \widetilde{L}_m^{g^\sigma}(q)/\widetilde{q}\|_2^2 = \psi_{m,n}^{g^\sigma}(q),$$

and we extend in a natural way to $\mathbf{Pb}(U,m,n)$ and $\widetilde{\psi}^g_{m,n}$ the notions of reducible or irreducible critical pair and critical point respectively: $\widetilde{q}$ is a critical point of $\widetilde{\psi}^g_{m,n}$ iff q is a critical point of $\psi^{g^\sigma}_{m,n}$; and $(\widetilde{p},\widetilde{q})$ is a critical pair for $\mathbf{Pb}(U,m,n)$ iff it is of the form $(\widetilde{L}^{g^\sigma}_m(q),\widetilde{q})$ where $\widetilde{q}$ is a critical point. Note that the status of being critical for a pair $(\widetilde{p},\widetilde{q})$ depends on the interpolation properties of $\widetilde{p}/\widetilde{q}$ to g in the disk, namely, it depends on whether the quotient v_q of the division

$$g\widetilde{q} = v_q q z^{m-n+1} + \widetilde{p}$$

meets the requirements of Propositions 2.3 and 2.5.

Theorem 2.13 (Criterion for uniqueness in $\mathbf{Pb}(U,m,n)$) *Let g be analytic in the closed disk $\overline{U}$ and $g(\overline{z}) = \overline{g(z)}$. Assume that any critical point $\widetilde{q}$ of $\widetilde{\psi}^g_{m,n}$ satisfies*

(i) *$\widetilde{q}$ is irreducible,*

(ii) *there exists a rational function B/A in irreducible form, with $B \in \mathcal{P}_{k'}$, $A \in \mathcal{P}_k$, such that*

$$3|g - \widetilde{L}^{g^\sigma}_m(q)/\widetilde{q}| < |g - B/A| \quad \text{on } T, \tag{2.32}$$

and such that B/A interpolates g in U to yield

$$Z_U(Ag - B) \geq \max(m+k, n+k'). \tag{2.33}$$

Then $\widetilde{\psi}^g_{m,n}$ has a unique critical point $\widetilde{q}^* \in \widetilde{\mathcal{M}}^1_n$, and $\widetilde{L}^{g^\sigma}_m(q^*)/\widetilde{q}^*$ is the unique minimizer of $\mathbf{Pb}(U,m,n)$.

Remark A typical B/A will be a multipoint Padé approximant to g of type $(m-1,n-1)$. More generally, (2.33) holds for any such approximant of type (k',k) with $k' \geq m-1$ and $k \geq n-1$.

Proof Set $f = g^\sigma$; upon applying σ, inequality (2.32) is equivalent to

$$3|f - L^f_m/z^{m-n+1}q| < |f - (B/A)^\sigma| \quad \text{on } T.$$

Assume first that $m-n \geq k'-k$. Then

$$\begin{aligned}
&[Z_V - \mathrm{Ord}_\infty]\left(z^{m-n+1}\widetilde{A}f - z^{m-n+k-k'}\widetilde{B}\right) \\
&= [Z_V - \mathrm{Ord}_\infty]\left(z^{m-n+k}(A(1/z)g(1/z) - B(1/z))\right) \\
&= [Z_V - \mathrm{Ord}_\infty](A(1/z)g(1/z) - B(1/z)) - m + n - k \\
&= Z_U(Ag - B) - m + n - k \geq n,
\end{aligned}$$

where the last inequality uses (2.33). Since

$$(B/A)^\sigma = \frac{z^{m-n+k-k'}\widetilde{B}}{z^{m-n+1}\widetilde{A}},$$

we may apply Theorem 2.12 with B replaced by $z^{m-n+k-k'}\widetilde{B}$ and A replaced by $\widetilde{A}$ to conclude that $\psi^f_{m,n}$ has a unique critical point in $q^* \in \mathcal{M}^1_n$, whence $\widetilde{\psi}^g_{m,n}$ has a unique critical point in $\widetilde{q}^* \in \widetilde{\mathcal{M}}^1_n$.

If $m - n < k' - k$, then

$$\begin{aligned}[Z_V - \mathrm{Ord}_\infty]\left(z^{k'-k+1}\widetilde{A}f - \widetilde{B}\right) &= [Z_V - \mathrm{Ord}_\infty]\left(z^{k'}(A(1/z)g(1/z) - B(1/z))\right) \\ &= Z_U(Ag - B) - k' \geq n.\end{aligned}$$

As

$$(B/A)^\sigma = \frac{\widetilde{B}}{z^{m-n+1}[z^{k'-k-m+n}\widetilde{A}]},$$

we apply Theorem 2.12, this time with B replaced by $\widetilde{B}$ and A replaced by $z^{k'-k-m+n}\widetilde{A}$. □

3. Approximation of fixed type on shrinking disks

In this section we consider the following rational approximation problem:

Pb(U_r, m, n): *for g analytic in U_{r_0} such that $g(\bar{z}) = \overline{g(z)}$, and given $0 < r < r_0$ and positive integers m, n, $m \geq n - 1$, minimize*

$$\frac{1}{2\pi}\int_0^{2\pi}\left|g - \frac{\widetilde{p}}{\widetilde{q}}\right|^2 (re^{i\theta})d\theta$$

as $\widetilde{p}$ ranges over $\mathcal{P}_m$ and $\widetilde{q}$ ranges over $\widetilde{\mathcal{M}}^r_n$.

If we set $f_r(z) := f(rz)$, for any function f and any positive real number r, the relation

$$\int_0^{2\pi}\left|g - \frac{\widetilde{p}}{\widetilde{q}}\right|^2 (re^{i\theta})d\theta = \int_0^{2\pi}\left|g_r - \frac{\widetilde{p}_r}{\widetilde{q}_r}\right|^2 (e^{i\theta})d\theta$$

shows **Pb**(U_r, m, n) for g to be equivalent to **Pb**(U, m, n) for g_r. This allows us to carry over to the first problem the terminology introduced for the second, and in particular to define the notion of a critical pair: $(\widetilde{p}, \widetilde{q}) \in \mathcal{P}_m \times \widetilde{\Delta}_n$ is critical for **Pb**(U_r, m, n) with g iff $(\widetilde{p}_r, \widetilde{q}_r)$ is critical for **Pb**(U, m, n) with g_r, and we have that

$$\widetilde{p}_r = \widetilde{L}^{g_r^\sigma}_m(\widetilde{\widetilde{q}_r}) = \widetilde{L}^{g_r^\sigma}_m(r^n q_{1/r}). \tag{3.1}$$

Using the theory developed in the previous section, we shall establish the following result:

Theorem 3.1 *Let $g(z) = \sum_{l=0}^{\infty} g_l z^l$, with $g_l \in \mathbf{R}$, and define*

$$\mathcal{G}_{l,k} := \begin{vmatrix} g_l & g_{l-1} & \cdots & g_{l-k+1} \\ \vdots & \vdots & \vdots & \vdots \\ g_{l+k-1} & g_{l+k-2} & \cdots & g_l \end{vmatrix}$$

with the convention that $g_s = 0$ for $s < 0$. Let us assume that

$$\mathcal{G}_{k+m-n,k} \neq 0, \quad 1 \leq k \leq n. \tag{3.2}$$

Then $\mathbf{Pb}(U_r, m, n)$ has a unique critical pair, whence a unique solution, when r is small enough. Moreover, if $(\widetilde{p}^*(z,r), \widetilde{q}^*(z,r))$ denotes this pair and if P_m^0/Q_n^0, $Q_n^0(0) = 1$ denotes the Padé approximant of type (m,n) to g, we have

$$\widetilde{p}^*(z,r) \to P_m^0(z) \quad in\ \mathcal{P}_m, \qquad \widetilde{q}^*(z,r) \to Q_n^0(z) \quad in\ \widetilde{\mathcal{M}}_n \quad as\ r \to 0. \tag{3.3}$$

Remark This theorem applies in particular to totally positive functions, as condition (3.2) is satisfied for all $m \geq n-1$ in this case.

Proof First we shall prove (3.3) with $(\widetilde{p}^*(z,r), \widetilde{q}^*(z,r))$ replaced by any irreducible critical pair $(\widetilde{p}, \widetilde{q}) = (\widetilde{p}(z,r), \widetilde{q}(z,r))$ of $\mathbf{Pb}(U_r, m, n)$. Thus $\widetilde{q}_r$ is an irreducible critical point of $\widetilde{\psi}_{m,n}^{g_r}$. By the equivalence of $\mathbf{Pb}(U, m, n)$ and $\mathbf{Pb}(V, m, n)$, Proposition 2.3 and (3.1) tell us that

$$g_r \widetilde{q}_r - \widetilde{p}_r = O(z^{m-n+1} q_{1/r}^2),$$

or equivalently

$$g\widetilde{q} - \widetilde{p} = O(z^{m-n+1} q_{1/r^2}^2). \tag{3.4}$$

Note that all the roots of q_{1/r^2} have modulus less than r.

Equation (3.4) means that $\widetilde{p}$ is the Lagrange interpolant to $g\widetilde{q}$ at the $m+n+1$ zeros of $z^{m-n+1} q_{1/r^2}^2$. We denote this by

$$\widetilde{p}(s) = \mathcal{L}(s, g\widetilde{q}, z^{m-n+1} q_{1/r^2}^2).$$

Setting $\widetilde{q}(z) = \sum_{k=0}^{n} b_k z^k$, $b_0 = 1$, we get by linearity of the Lagrange operator

$$\widetilde{p}(s) = \sum_{k=0}^{n} b_k \mathcal{L}(s, gz^k, z^{m-n+1} q_{1/r^2}^2).$$

As $\widetilde{p} \in \mathcal{P}_m$, expressing that the coefficients of $s^{m+1}, \ldots, s^{m+n}$ in the right-hand side vanish yields the linear system of equations satisfied by b_k, $k = 1, \ldots, n$. Let $K \subset \mathbf{C}$ be a compact set. We have by the Hermite formula that for $s \in K$,

$$\mathcal{L}(s, gz^k, z^{m-n+1}q_{1/r^2}^2) = \frac{1}{2i\pi}\int_C \left[1 - s^{m-n+1}q_{1/r^2}^2(s)/t^{m-n+1}q_{1/r^2}^2(t)\right]\frac{t^k g(t)}{t-s}dt,$$

where C denotes a contour surrounding K and the $m + n + 1$ roots of $z^{m-n+1}q_{1/r^2}^2$. Since s remains in a compact set and t belongs to C, the quotient $s^{m-n+1}q_{1/r^2}^2(s)/t^{m-n+1}q_{1/r^2}^2(t)$ tends uniformly to s^{m+n+1}/t^{m+n+1} as r tends to zero. Thus,

$$\lim_{r\to 0}\mathcal{L}(s, gz^k, z^{m-n+1}q_{1/r^2}^2) = \mathcal{L}(s, gz^k, z^{m+n+1}) = g_0 s^k + \cdots + g_{m+n-k}s^{m+n} \quad \text{in } \mathcal{P}_{m+n}$$

whence the coefficients of the linear system of equations defining the b_k's converge as r tends to zero to those of the linear system defining the Padé denominator Q_n^0, which is

$$\left\{\begin{array}{rcl} g_{m+1} + x_1 g_m + \cdots + x_n g_{m-n+1} & = & 0 \\ \vdots & \vdots & \vdots \\ g_{m+n} + x_1 g_{m+n-1} + \cdots + x_n g_m & = & 0. \end{array}\right.$$

Because the determinant of this system is $\mathcal{G}_{m,n} \neq 0$, it follows that the limit of the solution is the solution of the limiting system, namely, $\widetilde{q}(z, r)$ converges to $Q_n^0(z)$ in $\widetilde{\mathcal{M}}_n$ as $r \to 0$. As to the numerator $\widetilde{p}$, the Hermite formula gives

$$\widetilde{p}(s) = \frac{1}{2i\pi}\int_C \left[1 - s^{m-n+1}q_{1/r^2}^2(s)/t^{m-n+1}q_{1/r^2}^2(t)\right]\frac{g(t)\widetilde{q}(t)}{t-s}dt.$$

Since $\widetilde{q}(t, r)$ tends to $Q_n^0(t)$ uniformly on C, we get in turn the convergence of $\widetilde{p}(s, r)$ to the Padé numerator $P_m^0(s)$ in $\mathcal{P}_m$ as $r \to 0$.

Secondly, we establish that all critical pairs are irreducible when r is small enough. Indeed, assume to the contrary that there exists a sequence of reducible pairs as $r \to 0$. In view of Propositions 2.3 and 2.5, one obtains a sequence of irreducible critical pairs of type $(l + m - n, l)$, for some fixed $l < n$, whose associated rational functions interpolate g in more than $2l + m - n + 1$ points on $\overline{U}_r$. By the first part of the proof, which can be applied with n replaced by l and m replaced by $l + m - n$ because of (3.2), these rational functions converge to the Padé approximant P_{l+m-n}^0/Q_l^0 to g of type $(l + m - n, l)$. This gives a contradiction because P_{l+m-n}^0/Q_l^0 vanishes at the origin with multiplicity exactly $2l + m - n + 1$, as can be seen from (3.2) and the equality (cf. [17])

$$(3.5) \quad (gQ_l^0 - P_{l+m-n}^0)(z) = (-1)^l\frac{\mathcal{G}_{l+m-n+1,l+1}}{\mathcal{G}_{l+m-n,l}}z^{2l+m-n+1} + O(z^{2l+m-n+2}).$$

Thirdly, we estimate the error at a critical pair when r becomes small. By what precedes, we may assume that all critical pairs are irreducible; we may also require that $r < r_1 < 1$, where r_1 is chosen so small that $\overline{U}_{r_1}$ does not contain any zero of Q_n^0. Since $\widetilde{p}\widetilde{q} \in \mathcal{P}_{m+n}$ interpolates $g\widetilde{q}^2$ at the roots of $z^{m-n+1}q_{1/r^2}^2$ by (3.4), the Hermite formula gives

$$(3.6) \qquad (g\widetilde{q}^2 - \widetilde{p}\widetilde{q})(s) = \frac{1}{2i\pi}\int_T \frac{s^{m-n+1}q_{1/r^2}^2(s)}{t^{m-n+1}q_{1/r^2}^2(t)}\frac{g\widetilde{q}^2(t)}{t-s}dt, \qquad s \in U.$$

As the roots of q_{1/r^2} lie in U_r, we obtain for $t \in T$

$$\left|\frac{s^{m-n+1}q_{1/r^2}^2(s)}{t^{m-n+1}q_{1/r^2}^2(t)}\right| \leq \left(\frac{2}{1-r_1}\right)^{2n} r^{m+n+1}, \qquad |s| = r.$$

Moreover, $\widetilde{q}(z,r)$ tends uniformly to $Q_n^0(z)$ on the closed unit disk so that, by our choice of r_1, we get upon dividing (3.6) by $\widetilde{q}^2(s)$ and taking absolute values

$$(3.7) \qquad |g - \widetilde{p}/\widetilde{q}|\,(s) \leq Cr^{m+n+1}, \qquad |s| = r,$$

where C is some constant independent of r.

Finally, we prove the uniqueness part of the theorem by applying Theorem 2.13 to $\widetilde{\psi}_{m,n}^{g_r}$ with $B = (P_{m-1}^0)_r$ and $A = (Q_{n-1}^0)_r$, whence $k' = m-1$ and $k = n-1$. Indeed, it follows from (3.2) and (3.5) that there exists a constant $C_1 > 0$ such that for r small enough

$$(3.8) \qquad C_1 r^{m+n-1} \leq |g - P_{m-1}^0/Q_{n-1}^0|(s) \qquad \text{for } |s| = r.$$

It follows from the definition of the Padé approximant that the interpolation condition (2.33) is met while (3.7) and (3.8) together imply that (2.32) is satisfied as soon as $r < \sqrt{C_1/3C}$. □

4. Approximation of fixed denominator degree to meromorphic functions

The goal of this section is to establish the following $H_{2,\mathbf{R}}$-version of the de Montessus de Ballore theorem.

Theorem 4.1 *Fix a nonnegative integer n and a real number $R > 1$. Let*

$$Q_n(z) = \prod_{j=1}^{n}(z - x_j), \quad 1 < |x_j| < \rho < R$$

be a real polynomial, and $g = G/Q_n$ where G is analytic in U_R, and satisfies $G(\bar{z}) = \overline{G(z)}$. Moreover, assume $G(x_j) \neq 0$ for all j. Then the map $\widetilde{\psi}^g_{m,n}$ has a unique critical point $\widetilde{q}^*_n \in \widetilde{\Delta}_n$ for m large enough; in particular, problem $\mathbf{Pb}(U,m,n)$ has a unique solution $\widetilde{p}^*_m/\widetilde{q}^*_n$. In addition, still for m large, $\widetilde{p}^*_m/\widetilde{q}^*_n$ has exactly n poles in $\mathbf{C}$ and

$$\widetilde{q}^*_n \to Q_n/Q_n(0) \qquad \text{in } \mathcal{P}_n \qquad \text{as } m \to \infty. \tag{4.1}$$

Furthermore,

$$\widetilde{p}^*_m/\widetilde{q}^*_n \to g \qquad \text{as } m \to \infty, \tag{4.2}$$

locally uniformly in $U'_R := U_R \setminus \bigcup_{j=1}^k \{x_j\}$. More precisely, if $K \subset U'_R$ is a compact set,

$$\limsup_{m\to\infty} \|g - \widetilde{p}^*_m/\widetilde{q}^*_n\|_K^{1/m} \leq \max\{|z|, z \in K\}/R, \tag{4.3}$$

where $\|\cdot\|_K$ denotes the supremum norm on K.

First, we prove a lemma which gives a lower bound on the rate of convergence of certain Padé approximants to meromorphic functions.

Lemma 4.2 *For g as in Theorem 4.1, consider the Padé approximant $P^0_{m,n-1}/Q^0_{m,n-1}$ to g of type $(m, n-1)$ and let $\epsilon > 0$ be such that $\rho + \epsilon < R$. Then for m large,*

$$|g - P^0_{m,n-1}/Q^0_{m,n-1}| \geq (\rho+\epsilon)^{-m} \quad \text{on } T. \tag{4.4}$$

Proof Assume there exists a sequence of points $z_m \in T$ such that

$$|(g - P^0_{m,n-1}/Q^0_{m,n-1})(z_m)| < (\rho+\epsilon)^{-m}, \tag{4.5}$$

for infinitely many m. Combining the two relations

$$(Q^0_{m,n-1}g - P^0_{m,n-1})(z) = O(z^{m+n}), \qquad (Q^0_{m,n}g - P^0_{m,n})(z) = O(z^{m+n+1}),$$

where $P^0_{m,n}/Q^0_{m,n}$ denotes the Padé approximant to g of type (m,n), yields

$$(Q^0_{m,n}P^0_{m,n-1} - Q^0_{m,n-1}P^0_{m,n})(z) = O(z^{m+n}). \tag{4.6}$$

From de Montessus de Ballore's theorem (cf. [1]), the rational function $P^0_{m,n}/Q^0_{m,n}$ has precisely n finite poles for m large. Hence, the left-hand side of (4.6) is a nonzero polynomial of degree at most $m+n$, and there exists a nonzero constant $c_{m,n}$ such that

$$(4.7)\qquad (Q^0_{m,n}P^0_{m,n-1} - Q^0_{m,n-1}P^0_{m,n})(z) = c_{m,n}z^{m+n}.$$

In another connection, the convergence of $Q^0_{m,n}$ to Q_n asserted by the cited theorem implies that $\|g - P^0_{m,n}/Q^0_{m,n}\|_\infty$ is majorized, up to a constant, by the linearized error $\|gQ^0_{m,n} - P^0_{m,n}\|_\infty$ for m large. Since the latter decreases like the truncation of the Taylor series, we get

$$\limsup_{m\to\infty} \|g - P^0_{m,n}/Q^0_{m,n}\|^{1/m}_\infty \le 1/R,$$

implying by (4.5) for m large,

$$(4.8)\qquad |(P^0_{m,n-1}/Q^0_{m,n-1} - P^0_{m,n}/Q^0_{m,n})(z_m)| < 2(\rho+\epsilon)^{-m}.$$

We normalize $P^0_{m,n-1}$ and $Q^0_{m,n-1}$ so that the coefficient of largest modulus of $Q^0_{m,n-1}$ is 1 (in case there is more than one largest coefficient we choose the one with smallest subscript), and we normalize $Q^0_{m,n}$ so that $Q^0_{m,n}(0) = 1$. Now, de Montessus de Ballore's theorem asserts that $Q^0_{m,n}$ converges to $Q_n/Q_n(0)$ as $m\to\infty$ and hence is uniformly bounded in a neighborhood of the unit circle. Therefore, multiplying (4.8) by $Q^0_{m,n-1}(z_m)Q^0_{m,n}(z_m)$, we get for m large

$$(4.9)\qquad |(Q^0_{m,n}P^0_{m,n-1} - Q^0_{m,n-1}P^0_{m,n})(z_m)| < C(\rho+\epsilon)^{-m},$$

where C is a positive constant independent of m. In view of (4.7), we obtain $|c_{m,n}| \le C(\rho+\epsilon)^{-m}$, from which we deduce that (4.9) is actually satisfied for *all* points of T. Applying the Bernstein–Walsh lemma [26], we get for m large

$$(4.10)\quad |(Q^0_{m,n}P^0_{m,n-1} - Q^0_{m,n-1}P^0_{m,n})(z)| < C(\rho+\epsilon/2)^{m+n}(\rho+\epsilon)^{-m}, \quad z\in T_{\rho+\epsilon/2}.$$

Using the fact that $P^0_{m,n}$ (resp. $Q^0_{m,n}$) converges locally uniformly to $G/Q_n(0)$ (resp. to $Q_n/Q_n(0)$) on $T_{\rho+\epsilon/2}$ and evaluating the left-hand side of (4.10) at a zero of $Q^0_{m,n}$, we see, by the maximum principle and the fact that $G(x_j)\neq 0$ for all j, that any limit function of $Q^0_{m,n-1}$ as $m\to\infty$ vanishes at each x_j. We obtain a contradiction since this limit function can only be a polynomial of degree at most $n-1$, which is nonzero since the largest coefficient has modulus 1. □

Proof of Theorem 4.1 Let $(\widetilde{p}_m, \widetilde{q}_n)$, for $m\ge 1$, be an arbitrary sequence of irreducible critical pairs (where $\widetilde{q}_n$ depends on m). We know from Proposition 2.3

that $\widetilde{p}_m/\widetilde{q}_n$ interpolates g in U at the zeros of $z^{m-n+1}q_n^2$. Since the coefficients of q_n are bounded, we have

$$\lim_{m\to\infty} |z^{m-n+1}q_n^2(z)|^{1/(m+n)} = |z|$$

uniformly on closed subsets of V. By the extension of de Montessus de Ballore's theorem to interpolating rational functions (cf. [21, Thm. 2]), we now obtain (4.1), (4.2), and (4.3), not only for the optimal sequence $(\widetilde{p}_m^*, \widetilde{q}_n^*)$ but more generally for any sequence $(\widetilde{p}_m, \widetilde{q}_n)$ of irreducible critical pairs.

Let us next prove by contradiction that any critical pair is irreducible for m large enough. Indeed, an infinite sequence of reducible critical pairs $(\widetilde{p}_m, \widetilde{q}_n)$ would provide us, after reduction, with an infinite sequence of irreducible critical pairs $(\widetilde{p}_{m-k}, \widetilde{q}_{n-k})$ for some $0 < k \leq n$, such that $\widetilde{p}_{m-k}/\widetilde{q}_{n-k}$ interpolates g in U at the zeros of $z^{m-n+1}q_{n-k}^2$:

$$(g\widetilde{q}_{n-k} - \widetilde{p}_{m-k})(z) = O(z^{m-n+1}q_{n-k}^2(z)).$$

Upon multiplying by Q_n, we get

$$(G\widetilde{q}_{n-k} - Q_n\widetilde{p}_{m-k})(z) = O(z^{m-n+1}Q_n(z)q_{n-k}^2(z)).$$

Since the degree of $Q_n\widetilde{p}_{m-k}$ is less than $m + 2n - 2k + 1$, it is the interpolating polynomial of $G\widetilde{q}_{n-k}$ at the zeros of $z^{m-n+1}Q_n(z)q_{n-k}^2(z)$. From the Hermite formula, one has for z in U_ρ

$$(4.11)\quad (G\widetilde{q}_{n-k} - Q_n\widetilde{p}_{m-k})(z) = \frac{1}{2i\pi}\int_{T_\sigma} \frac{z^{m-n+1}Q_n(z)q_{n-k}^2(z)}{t^{m-n+1}Q_n(t)q_{n-k}^2(t)}\frac{G\widetilde{q}_{n-k}(t)}{t-z}dt,$$

where $\rho < \sigma < R$. For z in U_ρ and t in T_σ, $|t - z|$ is larger than $\sigma - \rho$ so that

$$\left|\frac{z^{m-n+1}Q_n(z)q_{n-k}^2(z)}{t^{m-n+1}Q_n(t)q_{n-k}^2(t)}\right| \leq \left(\frac{\rho}{\sigma}\right)^{m-n+1}\left(\frac{2\rho}{\sigma-\rho}\right)^{3n-2k}$$

Furthermore, $G\widetilde{q}_{n-k}$ is bounded on T_σ as G and $\widetilde{q}_{n-k}$ are (recall $\widetilde{q}_{n-k}(0) = 1$ and the roots are in V). Hence taking absolute values in (4.11) shows the locally uniform convergence of $G\widetilde{q}_{n-k} - Q_n\widetilde{p}_{m-k}$ to zero in U_ρ as $m \to \infty$. Thus, any limit function of the sequence $\widetilde{q}_{n-k}$, which is a nonzero polynomial of degree at most $n-k$, should vanish at the zeros of Q_n, a contradiction.

To conclude the uniqueness part of the theorem, it remains only to apply Theorem 2.13 with $A = Q_{m,n-1}^0$ and $B = P_{m,n-1}^0$, whence $k = n - 1$ and $k' = m$. It follows from the definition of the Padé approximant that (2.33) is met, while (2.32) follows from (4.3) applied with $K = \overline{U}$, which is valid for any sequence of irreducible critical pairs as we have seen, and from (4.4), which is valid for m large. □

5. Diagonal $H_{2,\mathbf{R}}$-approximation of the exponential function

This section is devoted to the proof of the following theorem.

Theorem 5.1 *When $g = e^z$, there exists, for n large, a unique critical pair $(\widetilde{p}^*_{n-1}, \widetilde{q}^*_n)$ (hence a unique local and global minimum) for* $\mathbf{Pb}(U, n-1, n)$.

If we set $R^*_n = \widetilde{p}^*_{n-1}/\widetilde{q}^*_n$, we have

$$\lim_{n\to\infty} R^*_n(z) = e^z$$

locally uniformly in $\mathbf{C}$. *More precisely, for $K \subset \mathbf{C}$ a compact set, there exist two constants $\widetilde{C}_1 = \widetilde{C}_1(K)$ and $\widetilde{C}_2 = \widetilde{C}_2(K)$ such that for n large and $z \in K$,*

$$\widetilde{C}_1 |q^*_n(z)|^2 \le \delta_n^{-1} |e^z - R^*_n(z)| \le \widetilde{C}_2 |q^*_n(z)|^2,$$

where

$$\delta_n := \frac{n!(n-1)!}{(2n)!(2n-1)!}.$$

Moreover, as $n \to \infty$,

$$\widetilde{p}^*_{n-1}(z) \to e^{z/2} \quad \textit{and} \quad \widetilde{q}^*_n(z) \to e^{-z/2}$$

locally uniformly in $\mathbf{C}$; *and for any constant $\kappa > 3$ the zeros of $\widetilde{q}^*_n$ and the zeros of $\widetilde{p}^*_{n-1}$ eventually lie within κ/n of the zeros of Q^0_n and P^0_{n-1} respectively, where P^0_{n-1}/Q^0_n denotes the Padé approximant of type $(n-1, n)$ to e^z.*

Remark An analogous theorem holds for the approximation problem $\mathbf{Pb}(V, n-1, n)$ to the function $(1/z)e^{1/z}$.

The proof of Theorem 5.1 would be quite easy if we knew that the estimates of the interpolation error that are available for real nodes (Theorem 5.4) extend to complex nodes as well. This question, however, is still open, and the proof of Theorem 5.1 will mainly consist of deriving such estimates in the case of $H_{2,\mathbf{R}}$ critical pairs.

We first recall independent results from the literature that we shall need in the sequel. The first is a result by Trefethen obtained by applying a method of Braess. It concerns the asymptotic rate of the error in best uniform rational approximation to e^z on a disk.

Theorem 5.2 (cf. [24]) *Let $m, n \ge 0$ be integers, and let $E_{m,n}$ denote the error in best uniform rational approximation of type (m, n) to e^z on the disk $|z| \le \rho$. Then*

$$E_{m,n} = \frac{m!n!\rho^{m+n+1}}{(m+n)!(m+n+1)!}(1 + o(1)) \tag{5.1}$$

as $m + n \to \infty$.

The next two results concern rational interpolation of the exponential function. The first one connects rational interpolants on a disk and on a segment.

Theorem 5.3 (Technique of Newman) *Let $R > 0$ be a fixed real number, P/Q a rational function of type (m,n), and define*

$$p(x,R) = |P(R\zeta)|^2, \qquad q(x,R) = |Q(R\zeta)|^2, \qquad |\zeta| = 1, \qquad x = \mathrm{Re}(\zeta).$$

Then $p(x,R)$ and $q(x,R)$ are polynomials in x and $p(x,R)/q(x,R)$ is again of type (m,n). Assume that the following three assertions hold:
(i) *The polynomial $Q(z)$ has no zeros on $\{|z| \le R\}$.*
(ii) *For any complex number z of modulus R, we have*

$$\left| e^z - \frac{P}{Q}(z) \right| < 2|e^z|.$$

(iii) *P/Q interpolates e^z in k points (counting multiplicities) in $\{|z| \le R\}$.*
Then the rational function $p(x,R)/q(x,R)$ interpolates e^{2Rx} in at least k points of $[-1,1]$, counting multiplicities.

As this result is a key ingredient in establishing Theorem 5.1, we provide a proof along the lines of [7] or [18] (see also Newman [16]).

Proof We may assume $\deg P = m$, $\deg Q = n$. Let $P(z) = a\prod_{i=1}^{m}(z-\xi_i)$, $a \in \mathbf{R}$. As P is a real polynomial and $|\zeta| = 1$, we have

$$(5.2) \qquad |P(R\zeta)|^2 = a^2 \prod_{i=1}^{m}(R\zeta - \xi_i)(R\overline{\zeta} - \xi_i) = a^2 \prod_{i=1}^{m}(R^2 - 2xR\xi_i + \xi_i^2).$$

This shows that $p(x,R)$ is a real polynomial in x of degree at most m, and similarly $q(x,R)$ is a real polynomial of degree at most n.

Let us now prove under the stated assumptions that $p(x,R)/q(x,R)$ interpolates e^{2Rx} in k points of $[-1,1]$. If α and β are complex numbers, note that

$$\alpha\overline{\alpha} - \beta\overline{\beta} = 2\mathrm{Re}\{\overline{\alpha}(\alpha - \beta)\} - |\alpha - \beta|^2;$$

applying this equality with $\alpha = e^z$, $\beta = P(z)/Q(z)$, and $z = R\zeta$ yields

$$(5.3) \qquad e^{2Rx} - \frac{p(x,R)}{q(x,R)} = 2\mathrm{Re}\left\{ e^{\overline{z}} \left(e^z - \frac{P(z)}{Q(z)} \right) \right\} - \left| e^z - \frac{P(z)}{Q(z)} \right|^2.$$

Let us define

$$h(z) = \frac{e^{-z}}{Q(z)}(Q(z)e^z - P(z)).$$

By assumptions (i) and (iii), h is analytic and has k zeros in $|z| \leq R$.

Assume first that P/Q does not interpolate e^z on the circle of radius R. Then h has winding number k on this circle. Hence, when an entire circuit has been completed on T_R, the argument of the complex number $h(z)$ has increased by $2k\pi$; and since h is a real function, the argument is increased by $k\pi$ as z traverses the upper half of the circle. Thus, h assumes real values in at least $k+1$ points $z_l = R(x_l + iy_l)$,

$$1 = x_0 > \cdots > x_k = -1, \qquad y_l \geq 0, \qquad l = 0, \ldots, k,$$

such that $h(x_l)$ and $h(x_{l-1})$ have opposite signs for $1 < l \leq k$. The same is true of $e^{\bar{z}}(e^z - \frac{P}{Q}(z))$, because it has the same argument as h. Then, by (5.3), there exists $\epsilon \in \{-1, 1\}$ such that

$$e^{2Rx_l} - \frac{p(x_l,R)}{q(x_l,R)} = 2\epsilon(-1)^l \left| e^{\bar{z}_l}\left(e^{z_l} - \frac{P}{Q}(z_l)\right)\right| - \left| e^{z_l} - \frac{P}{Q}(z_l)\right|^2, \qquad l = 0, \ldots, k.$$

Assumption (ii) shows that the sign of the right-hand side alternates with l, so that $p(x,R)/q(x,R)$ interpolates e^{2Rx} in at least k points of $[-1,1]$.

Assume now that P/Q does interpolate e^z at some point on the circle of radius R. We consider a sequence of radii $R_n > R$ such that $\lim_{n\to\infty} R_n = R$. Assumptions (i), (ii), and (iii) are satisfied on these circles as soon as R_n is sufficiently close to R. Now, P/Q does not interpolate e^z on T_{R_n} for n large, and we can apply the first part of the proof. This gives a sequence of analytic functions

$$f_n(z) = q(z, R_n)e^{2R_n z} - p(z, R_n)$$

having at least k zeros on $[-1,1]$. Moreover, this sequence converges uniformly on compact sets to the limit function $q(z,R)e^{2Rz} - p(z,R)$. By a classical theorem of Hurwitz, this function has at least k zeros on $[-1,1]$. □

Theorem 5.4 (cf. [5]) *Let* $B^{(n)} := \{x_k^{(n)}\}_{k=1}^{2n}$, $n = n_\nu$, *be a triangular sequence of (not necessarily distinct) real interpolation points contained in the interval* $[-\rho, \rho]$ *such that* $\lim_{\nu\to\infty} n_\nu = \infty$, *and denote by* $\widehat{R}_n = \widehat{p}_{n-1}/\widehat{q}_n$ *the rational function of type* $(n-1, n)$ *that interpolates* e^z *in* $B^{(n)}$. *Then*

$$\lim_{\nu\to\infty} \widehat{R}_{n_\nu}(z) = e^z \tag{5.4}$$

locally uniformly in **C**. *Furthermore, the numerator and denominator converge separately, that is, as* $\nu \to \infty$

$$\widehat{p}_{n_\nu - 1}(z) \to e^{z/2} \quad \text{and} \quad \widehat{q}_{n_\nu}(z) \to e^{-z/2} \tag{5.5}$$

locally uniformly in $\mathbf{C}$, *where* $\widehat{q}_{n_\nu}$ *is normalized so that* $\widehat{q}_{n_\nu}(0) = 1$. *For* $K \subset \mathbf{C}$ *a compact set, there exist two constants* $C_1 = C_1(K, \rho)$ *and* $C_2 = C_2(K, \rho)$ *such that for* ν *large and* $z \in K$,

$$C_1 \prod_{k=1}^{2n} |z - x_k^{(n)}| \leq \delta_n^{-1} |e^z - \widehat{R}_n(z)| \leq C_2 \prod_{k=1}^{2n} |z - x_k^{(n)}|, \tag{5.6}$$

where δ_n *is as in Theorem* 5.1. *Moreover, all zeros of* $\widehat{q}_n$, *say* $z_k^{(n)}$, *satisfy*

$$n - \rho \leq |z_k^{(n)}| \leq 2n + \rho + 1/3, \qquad k = 1, \ldots, n, \tag{5.7}$$

and remain within distance ρ *from the zeros of the Padé denominator* Q_n^0. *Symmetrically, all zeros of* $\widehat{p}_{n-1}$, *say* $y_l^{(n)}$, *satisfy*

$$n + 1 - \rho \leq |y_l^{(n)}| \leq 2n + \rho + 1/3, \qquad n \geq 2, \qquad l = 1, \ldots, n-1, \tag{5.8}$$

and remain within distance ρ *from the zeros of the Padé numerator* P_{n-1}^0.

Proof The limits in (5.4), (5.5) and the estimates (5.6) are particular cases of [5, Thms. 2.1, 2.2]. That the $z_k^{(n)}$'s and the $y_l^{(n)}$'s remain within distance ρ of the roots of Q_n^0 and P_{n-1}^0 respectively follows from the proof of [5, Lemma 2.4 (i)] and from the remark that $\widehat{q}_n/\widehat{p}_{n-1}$ again interpolates e^z, this time at the points $-x_j^{(n)}$ for $1 \leq j \leq 2n$. Keeping this in mind, we see that the lower bounds in (5.7) and (5.8) are consequences of the first assertion of [5, Prop. 2.8]. As to the upper bounds, we rely on the following result (cf. the upper bound in [22, Thm. 2.2 p.198]):

For any $m \geq 1$ *and* $n \geq 0$, *all the zeros of the Padé approximant* $P_{m,n}^0(z)/Q_{m,n}^0(z)$ *of type* (m, n) *to the exponential function lie in* $\{|z| \leq m + n + 4/3\}$.

Because of the (unnormalized) identity $Q_{m,n}^0(z) = P_{n,m}^0(-z)$, the previous inequality also holds for the zeros of $Q_{m,n}^0(z)$ when $m \geq 0$ and $n \geq 1$. □

After this reminder of known results, we proceed to a series of lemmas on critical pairs that will eventually lead us to the proof of Theorem 5.1. We fix $g(z) = e^z$ in $\mathbf{Pb}(U, n-1, n)$, and any critical pair $(\widetilde{p}_{n-1}, \widetilde{q}_n)$ relates to this problem.

Lemma 5.5 *Any critical pair* $(\widetilde{p}_{n-1}, \widetilde{q}_n)$ *is irreducible.*

Proof If $(\widetilde{p}_{n-1}, \widetilde{q}_n)$ is reducible, we obtain after reduction an irreducible critical pair $(\widetilde{\nu}_{n'-1}, \widetilde{\chi}_{n'})$ with $n' < n$. If we write the division

$$e^z \widetilde{\chi}_{n'} = v_{\chi_{n'}} \chi_{n'} + \widetilde{\nu}_{n'-1},$$

we see from Propositions 2.3 and 2.5 that $v_{\chi_{n'}}$ has at least $n' + [(n - n' + 1)/2]$ zeros in $\overline{U}$, contradicting the normality of the exponential function in the horizontal strip $\{z : \pi \leq \mathrm{Im}(z) < \pi\}$ (cf. [19, Pb. 206.2]). □

Lemma 5.6 *Let $(\widetilde{p}_{n-1}, \widetilde{q}_n)_{n\in\mathbf{N}}$ be a sequence of critical pairs. Then:*

(i) *The zeros of $\widetilde{q}_n$, say $\alpha_1^{(n)}, \ldots, \alpha_n^{(n)}$ counting multiplicities, satisfy for any $\alpha > 1$ and n large enough,*

$$n/\alpha \leq |\alpha_k^{(n)}| \leq 2\alpha n, \qquad k = 1, \ldots, n, \tag{5.9}$$

and the same inequality also holds for the zeros $\beta_1^{(n)}, \ldots, \beta_{n-1}^{(n)}$ of $\widetilde{p}_{n-1}$.

(ii) *As $n \to \infty$,*

$$\widetilde{p}_{n-1}(z) \to e^{z/2} \quad \text{and} \quad \widetilde{q}_n(z) \to e^{-z/2} \tag{5.10}$$

locally uniformly in $\mathbf{C}$.

(iii) *There exists a constant C such that for n large,*

$$\max_{z\in T} |e^z - \widetilde{p}_{n-1}/\widetilde{q}_n(z)| \leq nC\delta_n 4^{2n}. \tag{5.11}$$

(iv) *There exists a constant C_0 such that for n large,*

$$\min_{z\in T} |e^z - \widetilde{p}_{n-1}/\widetilde{q}_n(z)| \leq C_0\delta_n. \tag{5.12}$$

Proof We know by Lemma 5.5 that $\widetilde{q}_n$ is irreducible and by Proposition 2.8 that

$$|e^z - \widetilde{p}_{n-1}/\widetilde{q}_n| < 2\,|e^z|, \qquad z \in T,$$

is eventually satisfied for n large. To prove (i), we set

$$\widehat{p}_{n-1}(x) = |\widetilde{p}_{n-1}(\zeta)|^2, \quad \widehat{q}_n(x) = |\widetilde{q}_n(\zeta)|^2, \quad |\zeta| = 1, \quad x = \mathrm{Re}(\zeta), \tag{5.13}$$

and apply Theorem 5.3 with $R = 1$: the rational function $\widehat{R}_n(z) = \widehat{p}_{n-1}/\widehat{q}_n(z)$, which is of type $(n-1, n)$, interpolates e^{2z} at $2n$ points of $[-1, 1]$. Thus $\widehat{p}_{n-1}/\widehat{q}_n(z/2)$ interpolates e^z at $2n$ points of $[-2, 2]$, and we deduce from Theorem 5.4 that

$$\widehat{p}_{n-1}(z)/\widehat{q}_n(0) \to e^z \quad \text{and} \quad \widehat{q}_n(z)/\widehat{q}_n(0) \to e^{-z} \tag{5.14}$$

locally uniformly in $\mathbf{C}$. Moreover, denoting by $a_k^{(n)}$ the zeros of $\widehat{q}_n(z)$, $k = 1, \ldots, n$, we see from (5.7) that

$$n - 2 \leq \left|2a_k^{(n)}\right| \leq 2n + 7/3. \tag{5.15}$$

From (5.2), we have the relation between zeros of $\widetilde{q}_n$ and zeros of $\widehat{q}_n$:

$$2a_k^{(n)} = \alpha_k^{(n)} + \frac{1}{\alpha_k^{(n)}}, \qquad k = 1, \ldots, n, \tag{5.16}$$

whence $\alpha_k^{(n)} = O(n)$ since $|\alpha_k^{(n)}| > 1$ and $2a_k^{(n)} = O(n)$ by (5.15). But then, (5.9) follows from (5.16) and (5.15). The reasoning leading to the same inequalities with $\alpha_k^{(n)}$ replaced by $\beta_l^{(n)}$ is similar using (5.8), except that we do not know beforehand that $|\beta_l^{(n)}| \geq 1$. This, however, follows for n large from the uniform convergence of $\widetilde{p}_{n-1}/\widetilde{q}_n$ to e^z on $\overline{U}$ asserted in Proposition 2.8.

To prove (ii), we first observe that $(\widetilde{q}_n)$ is a normal family; indeed, when n is large enough, we have by (5.9)

$$|\widetilde{q}_n(z)| = \left|\prod_{k=1}^{n}(1 - z/\alpha_k^{(n)})\right| \leq (1 + \alpha|z|/n)^n \leq e^{\alpha|z|}.$$

In addition, recalling the definition of $\widehat{q}_n$ from (5.13), we find that

$$\widehat{q}_n(0) = \prod_{k=1}^{n}\left(1 + 1/\alpha_k^{(n)2}\right)$$

is bounded from below by some positive constant, thanks to (5.9). Hence

$$h_n = \widetilde{q}_n/\sqrt{\widehat{q}_n(0)}$$

again defines a normal family of functions. Let $h = \lim_{k\to\infty} h_{n_k}$ be a limit function of this family, and notice that, on T,

$$|h(\zeta)|^2 = \lim_{k\to\infty} \widehat{q}_{n_k}(\mathrm{Re}(\zeta))/\widehat{q}_{n_k}(0) = \left|e^{-\zeta}\right|$$

by virtue of (5.14). Thus h does not vanish identically, and as h_n is zero-free in $U_{n/\alpha}$ for n large by (5.9), Hurwitz's theorem implies that h is zero-free in $\mathbf{C}$. Therefore $h(z) = e^{-z/2}$, because these two functions share the same modulus on T and have no zeros in $\overline{U}$. Thus, h_n actually converges to $e^{-z/2}$ since this is the only possible limit function. As $\widetilde{q}_n(0) = 1$ for all n, we now deduce that $\widehat{q}_n(0) \to 1$ so that $\widetilde{q}_n(z) \to e^{-z/2}$ as $n \to \infty$, locally uniformly in $\mathbf{C}$. This gives the right half of (5.10). To get the other half, observe that $\widetilde{p}_{n-1}(0) \to 1$ when $n \to \infty$ because

$e^z - \widetilde{p}_{n-1}/\widetilde{q}_n \to 0$ on $\overline{U}$ by Proposition 2.8. Reasoning analogous to the first part of the proof now shows that $\widetilde{p}_{n-1}$ is a normal family of functions on $\mathbf{C}$. As $e^z\widetilde{q}_n$ converges locally uniformly to $e^{z/2}$ in $\mathbf{C}$ and since

$$\widetilde{q}_n e^z - \widetilde{p}_{n-1} = \widetilde{q}_n(e^z - \widetilde{p}_{n-1}/\widetilde{q}_n)$$

tends to zero on $\overline{U}$, we again conclude that $e^{z/2}$ is the only possible limit function of the family $(\widetilde{p}_{n-1})$.

We now prove (iii). By (ii), the function $e^{-z}\widetilde{p}_{n-1}/\widetilde{q}_n(z)$ is analytic and has no zeros in $\overline{U}_2$ for n large. Set

$$F(z) = \log\left(\frac{e^{-z}\widetilde{p}_{n-1}(z)}{\widetilde{q}_n(z)}\right),$$

where log designates the principal branch of the logarithm. For z in $\overline{U}_2$, we have

$$\operatorname{Re}(F(z)) = \log\left|\frac{\widetilde{p}_{n-1}/\widetilde{q}_n(z)}{e^z}\right| = \frac{1}{2}\log\left|\frac{\widetilde{p}_{n-1}/\widetilde{q}_n(z)}{e^z}\right|^2$$

$$= O\left(1 - \left|\frac{\widetilde{p}_{n-1}/\widetilde{q}_n(z)}{e^z}\right|^2\right) = O\left(|e^z|^2 - |\widetilde{p}_{n-1}/\widetilde{q}_n(z)|^2\right), \tag{5.17}$$

where we have used the fact that $\widetilde{p}_{n-1}/e^z\widetilde{q}_n(z)$ is uniformly close to 1 on $\overline{U}_2$ when n is large enough.

Let $1 < R \leq 2$ and notice that $|e^z - \widetilde{p}_{n-1}/\widetilde{q}_n(z)| < 2|e^z|$ certainly holds on T_R when n is larger than some integer independent of R. We apply Newman's technique on the circle of radius R by putting

$$\widehat{p}_{n-1,R}(x) = |\widetilde{p}_{n-1}(R\zeta)|^2, \qquad \widehat{q}_{n,R}(x) = |\widetilde{q}_n(R\zeta)|^2, \qquad |\zeta| = 1, \quad x = \operatorname{Re}(\zeta),$$

and see from Theorem 5.3 that $\widehat{p}_{n-1,R}/\widehat{q}_{n,R}(x)$ interpolates e^{2Rx} at $2n$ points of $[-1,1]$. Thus $\widehat{p}_{n-1,R}/\widehat{q}_{n,R}(t/2R)$ interpolates e^t at $2n$ points $\widehat{t}_k^{(n)}$, $k = 1,\ldots,2n$, of $[-2R, 2R]$. By the upper estimate (5.6) of Theorem 5.4 applied with $K = [-2R, 2R]$, we have for n large

$$\delta_n^{-1}|e^t - \widehat{p}_{n-1,R}/\widehat{q}_{n,R}(t/2R)| \leq C_2 \prod_{k=1}^{2n} |t - \widehat{t}_k^{(n)}|, \qquad t \in [-2R, 2R],$$

where C_2 is independent of $1 < R \leq 2$. Upon substituting back $x = t/2R$, we get

$$|e^{2Rx} - \widehat{p}_{n-1,R}/\widehat{q}_{n,R}(x)| \leq C_2\delta_n \prod_{k=1}^{2n} |2Rx - \widehat{t}_k^{(n)}| \leq C_2\delta_n(4R)^{2n}$$

or, equivalently,

$$||e^z|^2 - |\widetilde{p}_{n-1}/\widetilde{q}_n(z)|^2| \le C_2\delta_n(4R)^{2n}, \qquad |z| = R.$$

Plugging this into (5.17) shows that there exists a constant C_3 independent of $R \in (1, 2]$ such that for n large and z in $\overline{U}_R$,

$$|\mathrm{Re}(F(z))| \le C_3\delta_n(4R)^{2n}.$$

We now use the Borel–Carathéodory inequality (cf. e.g. [13, Thm. 5.1 p. 238]):

$$\max_{|z|=1} |F(z)| \le \frac{2}{R-1} \max_{|z|=R} |\mathrm{Re}(F(z))| + \frac{R+1}{R-1}|F(0)| \le \frac{R+3}{R-1} C_3\delta_n(4R)^{2n},$$

where we have used that $F(0)$ is real. By choosing a circle of radius $R = 1 + 1/n$, we get for n large and some absolute constant C_4

$$\max_{|z|=1} |F(z)| \le nC_4\delta_n 4^{2n}.$$

Writing now

$$\left|e^z - \frac{\widetilde{p}_{n-1}}{\widetilde{q}_n}(z)\right| = |e^z| \left|1 - \frac{\widetilde{p}_{n-1}/\widetilde{q}_n(z)}{e^z}\right|,$$

we see that

$$e^z - \widetilde{p}_{n-1}/\widetilde{q}_n(z) = O(F(z)), \qquad z \in \overline{U}_2.$$

Thus, there exists a constant C such that for n large

$$|e^z - \widetilde{p}_{n-1}/\widetilde{q}_n(z)| \le nC\delta_n 4^{2n}, \qquad z \in T,$$

which proves (iii).

To establish (iv), we appeal again to Theorem 5.3: defining $\widehat{p}_{n-1}$ and $\widehat{q}_n$ as in (5.13), we know from the proof of this theorem that there exist $2n + 1$ points $z_l = x_l + iy_l$ on T,

$$1 = x_0 > \cdots > x_{2n} = -1, \qquad y_l \ge 0, \qquad l = 0, \ldots, 2n,$$

such that

$$e^{2x_l} - \frac{\widehat{p}_{n-1}}{\widehat{q}_n}(x_l) = 2\epsilon(-1)^l \left|e^{\overline{z_l}}\left(e^{z_l} - \frac{\widetilde{p}_{n-1}}{\widetilde{q}_n}(z_l)\right)\right| - \left|e^{z_l} - \frac{\widetilde{p}_{n-1}}{\widetilde{q}_n}(z_l)\right|^2, \qquad l = 0, \ldots, 2n, \tag{5.18}$$

where $\epsilon = \pm 1$ and the sign of this expression alternates with l. In other words, the set $\{x_l\}_{l=0}^{2n}$ is a maximal alternation set for $e^{2x} - \widehat{p}_{n-1}/\widehat{q}_n(x)$, and consequently $\{2x_l\}_{l=0}^{2n}$ is a maximal alternation set in $[-2,2]$ for $e^x - \widehat{p}_{n-1}/\widehat{q}_n(x/2)$. From de La Vallée–Poussin's theorem for rational functions (cf. [18, Thm. 2.3]), we obtain

$$\min_l \left| e^{2x_l} - \frac{\widehat{p}_{n-1}}{\widehat{q}_n}(x_l) \right| \leq E_{n-1,n}(e^x, [-2,2]), \tag{5.19}$$

where the right-hand side of (5.19) denotes the error in uniform best rational approximation of type $(n-1,n)$ to e^x on $[-2,2]$. Letting C_n be the familiar Chebyshev polynomial and observing that the monic polynomial of least deviation to zero of degree n on $[-2,2]$ is $2^n C_n(x/2)$, and therefore has norm 2, we can take the corresponding nodes as $x_k^{(n)}$'s in (5.6) to obtain a constant C_5 such that, for n large,

$$E_{n-1,n}(e^x, [-2,2]) \leq C_5 \delta_n.$$

Hence, we deduce from (5.18) and (5.19) that for n large,

$$\min_l \left| e^{z_l} - \frac{\widetilde{p}_{n-1}}{\widetilde{q}_n}(z_l) \right| \, |2|e^{z_l}| - |e^{z_l} - \widetilde{p}_{n-1}/\widetilde{q}_n(z_l)|| \leq C_5 \delta_n.$$

But the second modulus in the left-hand side of the previous inequality is uniformly bounded away from 0 as $n \to \infty$, since $|e^z - \widetilde{p}_{n-1}/\widetilde{q}_n(z)|$ tends uniformly to 0 on T. This gives (5.12). □

Lemma 5.7 *There exists an integer N_0 and a real number $\beta > 0$ such that for $n \geq N_0$, the function $(e^z - \widetilde{p}_{n-1}/\widetilde{q}_n(z))/q_n^2(z)$ has no zeros in $\{|z| \leq \beta n\}$, where $(\widetilde{p}_{n-1}, \widetilde{q}_n)$ is any critical pair of type $(n-1,n)$.*

Proof We first prove that there exists a real number $\beta > 0$ such that for n large,

$$|e^z - \widetilde{p}_{n-1}/\widetilde{q}_n(z)| < 2|e^z|, \qquad |z| = \beta n. \tag{5.20}$$

To derive (5.20), we use the well-known formula for the Padé approximant (cf. [17] p. 436)

$$Q_n^0(z)e^z - P_{n-1}^0(z) = (-1)^n \frac{z^{2n}}{(2n-1)!} \int_0^1 e^{tz} t^n (1-t)^{n-1} dt,$$

from which we deduce, thanks to the value of the beta integral $B(n+1,n)$, that

$$|Q_n^0(z)e^z - P_{n-1}^0(z)| \leq e\delta_n, \qquad |z| = 1 \tag{5.21}$$

and

$$|Q_n^0(z)e^z - P_{n-1}^0(z)| \le e^{\beta n}(\beta n)^{2n}\delta_n, \qquad |z| = \beta n, \tag{5.22}$$

for any $\beta > 0$. Choose $0 < \beta < 1/2$. From (5.7), we know that all zeros of Q_n^0 have modulus larger than or equal to n. As $Q_n^0(0) = 1$ by our normalization, we get

$$1/4 \le \left(1 - \frac{1}{n}\right)^n \le |Q_n^0(z)|, \qquad |z| = 1, \qquad n \ge 2,$$

and

$$(1-\beta)^n \le |Q_n^0(z)|, \qquad |z| = \beta n. \tag{5.23}$$

Thus, (5.21) and (5.22) imply respectively that

$$\left|e^z - P_{n-1}^0/Q_n^0(z)\right| \le 4e\delta_n, \qquad |z| = 1, \tag{5.24}$$

and

$$\left|e^z - P_{n-1}^0/Q_n^0(z)\right| \le (\beta n)^{2n}e^{\beta n}\delta_n/(1-\beta)^n, \qquad |z| = \beta n. \tag{5.25}$$

Making use of Lemma 5.6 (iii), together with (5.24), we get for any $C_6 > C$ and n large

$$\left|\frac{P_{n-1}^0}{Q_n^0}(z) - \frac{\widetilde{p}_{n-1}}{\widetilde{q}_n}(z)\right| \le \left|\frac{P_{n-1}^0}{Q_n^0}(z) - e^z\right| + \left|e^z - \frac{\widetilde{p}_{n-1}}{\widetilde{q}_n}(z)\right| \le nC_6\delta_n 4^{2n}, \qquad |z| = 1.$$

Consequently, as $Q_n^0(z)$ and $\widetilde{q}_n(z)$ both converge to $e^{-z/2}$ on $\overline{U}$, there exists a constant C_7 such that for n large,

$$|P_{n-1}^0(z)\widetilde{q}_n(z) - Q_n^0(z)\widetilde{p}_{n-1}(z)| \le nC_7\delta_n 4^{2n}, \qquad |z| = 1,$$

and from the Bernstein–Walsh lemma we deduce, still for n large,

$$|P_{n-1}^0(z)\widetilde{q}_n(z) - Q_n^0(z)\widetilde{p}_{n-1}(z)| \le nC_7 4^{2n}(\beta n)^{2n-1}\delta_n, \qquad |z| = \beta n. \tag{5.26}$$

Moreover, choosing $\alpha = 1/2\beta$ in (5.9), we get, again for n large,

$$(1/2)^n \le |\widetilde{q}_n(z)|, \qquad |z| = \beta n.$$

As Q_n^0 satisfies (5.23), whence *a fortiori* the previous inequality, we deduce upon dividing (5.26) by $\widetilde{q}_n Q_n^0$ that for n large,

$$\left|\frac{P_{n-1}^0}{Q_n^0}(z) - \frac{\widetilde{p}_{n-1}}{\widetilde{q}_n}(z)\right| \leq nC_7 4^{2n}(\beta n)^{2n-1}\delta_n(2)^{2n}, \qquad |z| = \beta n$$

and we obtain by (5.25), still for $|z| = \beta n$ and n large,

$$(5.27)\quad \left|e^z - \frac{\widetilde{p}_{n-1}}{\widetilde{q}_n}(z)\right| \leq \left[\beta(1/2)^n e^{\beta n} + C_7 4^{2n}\right]\frac{(2\beta n)^{2n}\delta_n}{\beta} \leq (C_8)^n \frac{(2\beta n)^{2n}\delta_n}{\beta}$$

where $C_8 = e/2 + 16C_7$. Stirling's formula yields

$$\delta_n \simeq \frac{e^{2n}}{2^{4n}n^{2n}}.$$

Together with (5.27), this implies for $\epsilon > 0$, $|z| = \beta n$, and n large that

$$\left|e^z - \frac{\widetilde{p}_{n-1}}{\widetilde{q}_n}(z)\right| \leq \frac{(1+\epsilon)}{\beta}\frac{(2\beta e)^{2n}(C_8)^n}{2^{4n}}.$$

As C_8 is independent of β, we may have chosen β so small that

$$\frac{C_8 e^2 \beta^2}{4} < 1$$

which implies (5.20) for n large. Now, because $\widetilde{q}_n$ has no zeros in the closed disk of radius βn for n large by (5.9), Theorem 5.3 applies to a circle of radius βn: if $e^z - \widetilde{p}_{n-1}/\widetilde{q}_n(z)$ had more than $2n$ zeros in $\{|z| \leq \beta n\}$ we would get a rational function of type $(n-1, n)$ which interpolates e^x at more than $2n$ points of the real axis, contradicting the normality of the exponential function. □

Lemma 5.8 *Let β and N_0 be as in Lemma 5.7. Pick, for each $n \geq N_0$, a critical pair $(\widetilde{p}_{n-1}, \widetilde{q}_n)$ and define two sequences of functions as follows:*

$$w_n(z) = \frac{e^z - \widetilde{p}_{n-1}/\widetilde{q}_n(z)}{\lambda_n \delta_n q_n^2(z)},$$

where λ_n denotes the sign of $(1 - \widetilde{p}_{n-1}/\widetilde{q}_n(0))/q_n^2(0)$, and

$$u_n(z) = w_n(z)^{1/2n}, \qquad u_n(0) \geq 0, \qquad |z| \leq \beta n.$$

(Notice that by Lemma 5.7, λ_n is unambiguously defined and u_n is a well-defined analytic function on $\{|z| \leq \beta n\}$.) The following three assertions hold:

(i) *The sequence* (u_n) *is bounded, uniformly with respect to n, on* $\{|z| \le \beta n\}$.

(ii) *As* $n \to \infty$,

$$u_n(z) \to 1$$

locally uniformly in $\mathbf{C}$.

(iii) *The sequence* (w_n) *is uniformly bounded from above and below on any compact set K of the complex plane for n large, that is, there exist two constants* $\widetilde{C}_1 = \widetilde{C}_1(K)$ *and* $\widetilde{C}_2 = \widetilde{C}_2(K)$ *such that*

$$\widetilde{C}_1 \le |w_n(z)| \le \widetilde{C}_2, \qquad z \in K, \qquad n \ge n(K).$$

Proof Fix $\alpha > 1$ such that $\alpha\beta < 1$. As the roots of q_n are the reciprocals of those of $\widetilde{q}_n$, we have from (5.9)

$$|q_n(z)| \ge \left(\beta n - \frac{\alpha}{n}\right)^n \ge (\beta' n)^n, \qquad |z| = \beta n,$$

for any $0 < \beta' < \beta$ and n large enough. Together with (5.27), this implies that for n large,

$$|w_n(z)| \le \frac{(C_8)^n (2\beta n)^{2n}}{\beta(\beta' n)^{2n}}, \qquad |z| = \beta n,$$

and by taking $2n$-th roots,

$$|u_n(z)| \le \frac{2C_9\beta}{\beta'}, \qquad |z| = \beta n, \tag{5.28}$$

where C_9 is any constant larger than $\sqrt{C_8}$. This proves (i).

Let g be the limit function of a subsequence (u_{n_k}). Letting n_k tend to ∞ in (5.28), we see that g is a bounded entire function in the complex plane, hence, by Liouville's theorem, a constant a. Remark that a is a nonnegative real number as $u_n(0) \ge 0$ for all n. Next, we show that $a = 1$. To this effect, let $a' > a$ so that for n large,

$$|u_{n_k}(z)| \le a', \qquad |z| \le 1.$$

From the definition of u_n, we infer that

$$|e^z - \widetilde{p}_{n_k - 1}/\widetilde{q}_{n_k}(z)| \le (a')^{2n_k} \delta_{n_k} |q_{n_k}(z)|^2, \qquad |z| \le 1. \tag{5.29}$$

Since the modulus of the roots of q_n is less than or equal to α/n, we get as soon as $n \ge 2\alpha$,

$$(1/2)^{2\alpha} \le (1 - \alpha/n)^n \le |q_n(z)| \le (1 + \alpha/n)^n \le e^\alpha, \qquad |z| = 1; \tag{5.30}$$

plugging the upper estimate into (5.29) shows that

$$|e^z - \widetilde{p}_{n_k-1}/\widetilde{q}_{n_k}(z)| \leq (a')^{2n_k}\delta_{n_k}e^{2\alpha}, \qquad |z| \leq 1.$$

If $a < 1$, we can choose $a' < 1$ as well, but this violates the optimal rate of convergence given by Theorem 5.2. Hence $a \geq 1$, and if we let $0 < a'' < a$, we obtain for k large

$$\max_{z\in T} |u_{n_k}(z)| / \min_{z\in T} |u_{n_k}(z)| \leq a'/a''.$$

Taking $2n$-th roots in (5.12) and using (5.30), we get

$$\min_{z\in T} |u_n(z)| \leq C_0^{1/2n} \max_{z\in T} |q_n(z)|^{1/n} \leq C_0^{1/2n} e^{\alpha/n},$$

and this implies

$$\max_{z\in T} |u_{n_k}(z)| \leq C_0^{1/2n_k} e^{\alpha/n_k} a'/a'' \tag{5.31}$$

as soon as k is large enough. Now, we may require in this relation that the ratio a'/a'' be arbitrarily close to 1, and since $a \geq 1$ we see from the definition and from (5.31) that $a = 1$ is the only possibility. Therefore all convergent subsequences of (u_n) have the same limit, namely the constant function 1, so that (u_n) itself converges locally uniformly in **C**, which proves (ii).

We now turn to the proof of (iii). It is enough to consider $K = \overline{U}_\rho$. In the sequel we choose n so large that $\rho < \beta n$. The Cauchy formula implies that

$$u_n'(z) = \frac{1}{2i\pi} \int_{T_{\beta n}} \frac{u_n(t)}{(z-t)^2} dt, \qquad |z| < \beta n.$$

From (ii) and the above integral representation, we deduce for n large enough that

$$|u_n'(z)| \leq \frac{\alpha\beta n}{(\beta n - \rho)^2} \leq \frac{\alpha_1}{n}, \qquad |z| = \rho,$$

where we have used that $\alpha\beta < 1$, and where α_1 is any real number larger than β^{-2}. The relation between u_n and w_n yields $w_n'/w_n = 2nu_n'/u_n$. As u_n tends uniformly to 1 on $\{|z| \leq \rho\}$, we obtain

$$|w_n'/w_n(z)| \leq \alpha_2, \qquad |z| \leq \rho,$$

for any $\alpha_2 > 2\alpha_1$ and n large. To obtain uniform bounds for w_n on $\{|z| \leq \rho\}$, we introduce its logarithm

$$\log(w_n(z)) = \log(w_n(z_n)) + \int_{z_n}^{z} \frac{w_n'}{w_n}(t)dt,$$

where $z_n \in \{|z| \leq \rho\}$ will be chosen below. We have

$$\left|\log\left|\frac{w_n(z)}{w_n(z_n)}\right|\right| \leq \left|\log\frac{w_n(z)}{w_n(z_n)}\right| \leq 2\rho\alpha_2, \qquad |z| \leq \rho,$$

and thus

$$|\log|w_n(z)|| \leq |\log|w_n(z_n)|| + 2\rho\alpha_2, \qquad |z| \leq \rho. \tag{5.32}$$

Besides, the upper estimate (5.12), along with the lower bound in (5.30), imply that

$$\min_{z\in T}|w_n(z)| \leq C_0 16^{\alpha}. \tag{5.33}$$

Let $\widehat{z}_n$ be a point on T where the above minimum is attained and assume that the maximum modulus of w_n in $\{|z| \leq \rho\}$ is larger than 1. If $|w_n(\widehat{z}_n)| \leq 1$, we choose for z_n a point where $|w_n(z_n)| = 1$, which leads to

$$|\log|w_n(z)|| \leq 2\rho\alpha_2 \quad \text{whence } \exp(-2\rho\alpha_2) \leq |w_n(z)| \leq \exp(2\rho\alpha_2), \qquad |z| \leq \rho.$$

If $|w_n(\widehat{z}_n)| > 1$, we choose $z_n = \widehat{z}_n$ and we get, in view of (5.33),

$$|\log|w_n(z)|| \leq \log(C_0 16^{\alpha}) + 2\rho\alpha_2, \qquad |z| \leq \rho,$$

and so

$$C_0^{-1}16^{-\alpha}\exp(-2\rho\alpha_2) \leq |w_n(z)| \leq C_0 16^{\alpha}\exp(2\rho\alpha_2), \qquad |z| \leq \rho.$$

If $|w_n(z)| \leq 1$ in $\{|z| \leq \rho\}$, we may take $\widetilde{C}_2 = 1$, and all we have to establish in order to get $\widetilde{C}_1$ is, in view of (5.32), that there exists a sequence of points (z_n) such that $|w_n(z_n)|$ is bounded away from zero. However, w_n cannot tend to zero uniformly in $\overline{U}_\rho$ without contradicting the optimal rate of convergence given by Theorem 5.2, because of the very definition of w_n and of the upper bound in (5.30). □

Lemma 5.9 *Let P_{n-2}^0/Q_{n-1}^0 be the Padé approximant of type $(n-2, n-1)$ to e^z and $(\widetilde{p}_{n-1}, \widetilde{q}_n)$ be any critical pair. We have, for n large enough,*

$$3|e^z - \widetilde{p}_{n-1}/\widetilde{q}_n(z)| < |e^z - P_{n-2}^0/Q_{n-1}^0(z)|, \qquad |z| = 1.$$

Proof On the one hand, we know from Lemma 5.8 (iii) that there exists a constant $\widetilde{C}_2$ such that for n large,

$$|e^z - \widetilde{p}_{n-1}/\widetilde{q}_n(z)| \leq \widetilde{C}_2\delta_n|q_n(z)|^2 \leq \widetilde{C}_2\delta_n e^{2\alpha}, \qquad |z| = 1,$$

where the last inequality uses the upper bound in (5.30). On the other hand, we know from Theorem 5.4 that there exists a constant C_1 such that, for n large,

$$C_1\delta_{n-1} \leq |e^z - P_{n-2}^0/Q_{n-1}^0(z)|, \qquad |z| = 1.$$

Consequently, in order to prove the lemma, it suffices to show that, for n large, $3\widetilde{C}_2\delta_n e^{2\alpha} < C_1\delta_{n-1}$. But, after reduction, this is equivalent to

$$3\widetilde{C}_2 e^{2\alpha} < 4C_1(2n-1)^2,$$

which is evident for n large. □

Proof of Theorem 5.1 For n large, uniqueness of a critical pair $(\widetilde{p}_{n-1}^*, \widetilde{q}_n^*)$ for the approximation problem $\mathbf{Pb}(U, n-1, n)$ to e^z follows from Lemma 5.5, Lemma 5.9 and Theorem 2.13 with $A = Q_{n-1}^0$ and $B = P_{n-2}^0$. The separated convergence of $\widetilde{p}_{n-1}^*$ and $\widetilde{q}_n^*$ was proved in Lemma 5.6 (ii). The lower and upper estimates for the error $e^z - \widetilde{p}_{n-1}^*/\widetilde{q}_n^*(z)$ are given by Lemma 5.8 (iii) (see the definition of w_n in this lemma). We finally prove the estimates relating the zeros of $\widetilde{p}_{n-1}^*$ and $\widetilde{q}_n^*$ to those of P_{n-1}^0 and Q_n^0 respectively. By (5.9), we know that for any $\alpha > 1$ and n large the $2n$ interpolation points of $\widetilde{p}_{n-1}/\widetilde{q}_n$ to e^z in U lie in $\overline{U}_{\alpha/n}$. Thus, Theorem 5.3 eventually applies on the circle of radius α/n, and Theorem 5.4 shows that twice the poles of the rational interpolant with real nodes thus obtained lie within $2\alpha/n$ of the zeros of Q_n^0. From (5.16) and the lower estimate in (5.9), we deduce that the zeros of $\widetilde{q}_n$ lie within $3\alpha/n$ from the zeros of Q_n^0, for n large. The corresponding assertion about the zeros of $\widetilde{p}_{n-1}$ and P_{n-1}^0 is derived in the same way. □

REFERENCES

[1] G. A. Baker, Jr., *Essentials of Padé Approximants*, Academic Press, New York, 1975.

[2] L. Baratchart, *Sur l'approximation rationnelle l^2 pour les systèmes dynamiques linéaires*, Thèse de doctorat d'état, Université de Nice, 1987.

[3] L. Baratchart and M. Olivi, *Index of critical points in l^2-approximation*, Systems & Control Letters **10** (1988), 167–174.

[4] L. Baratchart, M. Olivi and F. Wielonsky, *On a rational approximation problem in the real Hardy space H_2*, Theoretical Computer Science **94** (1992), 175–197.

[5] L. Baratchart, E. B. Saff and F. Wielonsky, *Rational interpolation of the exponential function*, Canad. J. Math. **47** (1995), 1121–1147.

[6] L. Baratchart and F. Wielonsky, *Rational approximation in the real Hardy space H_2 and Stieltjes integrals: a uniqueness theorem*, Constr. Approx. **9** (1993), 1–21.

[7] D. Braess, *Nonlinear Approximation Theory*, Springer Series in Computational Mathematics, Vol. 7, Springer-Verlag, Berlin, 1986.

[8] P. E. Caines, *Linear Stochastic Systems, Probability and Mathematical Statistics*, Wiley, New York, 1988.

[9] J. Della Dora, *Contribution à l'approximation de fonctions de la variable complexe au sens de Hermite–Padé et de Hardy*, Thèse d'état, Univ. Scient. et Medicale de Grenoble, 1980.

[10] J. C.Doyle, B. A. Francis and A. R. Tannenbaum, *Feedback Control Theory*, Macmillan, New York, 1992.

[11] M. Duc-Jacquet, *Approximation des fonctionelles linéaires sur les espaces Hilbertiens à noyaux reproduisants*, Thèse d'état, Univ. Scient. et Medicale de Grenoble, 1973.

[12] V. D. Erohin, *On the best approximation of analytic functions by rational functions*, Doklady Akad. Nauk SSSR **128** (1959), 29–32 (in Russian).

[13] S. Lang, *Complex Analysis*, Addison-Wesley, Reading, MA, 1977.

[14] A. L. Levin, *The distribution of poles of rational functions of best approximation and related questions*, Math. USSR Sbornik **9**, No. 2 (1969), 267–274.

[15] L. Ljung, *System Identification: Theory for the User*, Prentice-Hall, New Jersey, 1987.

[16] D. J. Newman, *Approximation with Rational Functions*, Regional Conf. Series 41, AMS, Providence, Rhode Island, 1979.

[17] O. Perron, *Die Lehre von den Kettenbrüchen*, 3rd ed., Vol. 2, B. G. Teubner, Stuttgart, 1957.

[18] P. P. Petrushev and V. A. Popov, *Rational Approximation of Real Functions*, in *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 1987.

[19] G. Pólya and G. Szegö, *Problems and Theorems in Analysis, I*, Springer-Verlag, Berlin, 1972.

[20] P. A. Regalia, *Adaptative IIR Filtering in Signal Processing and Control*, Dekker, New York, 1995.

[21] E. B. Saff, *An extension of Montessus de Ballore's theorem on the convergence of interpolating rational functions*, J. Approx. Theory **6** (1972), 63–67.

[22] E. B. Saff and R. S. Varga, *On the zeros and poles of Padé approximants to e^z. II*, in *Padé and Rational Approximations: Theory and Applications* (E. B. Saff and R. S. Varga, eds.), Academic Press, New York, 1977, pp. 195–213.

[23] T. Söderström, *On the uniqueness of maximum likelihood identification*, Automatica **11** (1975), 193–197.

[24] L. N. Trefethen, *The asymptotic accuracy of rational approximations to e^z on a disk*, J. Approx. Theory **40** (1984), 380–383.

[25] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo, 1959.

[26] J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, AMS Colloq. Publ. XX, 1969.

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