Journal of Number Theory **74**, 230–249 (1999)
Article ID jnth.1998.2334, available online at http://www.idealibrary.com on IDEAL®

Hermite–Padé Approximants to Exponential Functions and an Inequality of Mahler

Franck Wielonsky

INRIA, 2004, Route des Lucioles, B.P. 93, 06902 Sophia Antipolis Cedex, France

Communicated by M. Waldschmidt

Received November 13, 1997; June 10, 1998

We improve Mahler's inequality

$$|e^g - a| > g^{-33g}, \qquad a \in \mathbb{N},$$

where g is any sufficiently large positive integer by decreasing the constant 33 to 19.183. This we do by computing precise asymptotics for a set of approximants to the exponential which is slightly different from the classical Hermite–Padé approximants. These approximants are related to the Legendre-type polynomials studied by Hata, which allows us to use his results about the arithmetic of the coefficients. © 1999 Academic Press

1. INTRODUCTION

In a series of papers [8, 9, 10], Mahler studied the Hermite–Padé system of approximants (of so-called Latin type or type I) to exponential or logarithms functions, from which he deduced numerous results such as a transcendence measure for e improving those of Borel and Popken, an irrationality measure for π, lower bounds for the approximation of logarithms of algebraic numbers or powers of e.

As an auxiliary result, Mahler obtained in [9] a lower bound for the distance of integral exponents of e to the nearest integer. This lower bound was improved in [10] and the following assertion was established. For any sufficiently large positive integer g,

$$|e^g - a| > g^{-33g}, \qquad a \in \mathbb{N}. \tag{1.1}$$

This elementary inequality may be seen as a particular case of more general inequalities involving algebraic numbers instead of integers and from which transcendence measures for numbers related to the exponential function can be derived [4, 5, 12, 14, 15].

The aim of this paper is to improve (1.1). Namely, we shall prove

0022-314X/99 $30.00

Copyright © 1999 by Academic Press
All rights of reproduction in any form reserved.

THEOREM 1.1. *Let g be any sufficiently large positive integer. Then*

$$|e^g - a| > g^{-19.183g}, \qquad a \in \mathbb{N}.$$

Also,

$$|\log g - a| > g^{-19.183 \log\log g}, \qquad a \in \mathbb{N}.$$

As in the original proof, the method consists in computing the asymptotics of Hermite–Padé approximants to the exponential function along with a study of the arithmetic of the coefficients (Sections 4 and 5, respectively). The asymptotics will be obtained by means of the saddle point method as applied to the integral expressions of the Hermite–Padé approximants. Actually, the approximants that we choose differ from the usual Hermite–Padé approximants and are akin to the Legendre-type polynomials studied by Hata [6]. His results concerning the arithmetic of the coefficients together with our asymptotics yield Theorem 1.1.

In [11], Mignotte displays a proof of (1.1) with a constant 17.7 instead of 33. However, the proof contains errors which rule out the assertion. Nevertheless, one may check that Mignotte's argument leads to an inequality similar to the one that can be obtained by our method when applied to ordinary Hermite–Padé approximants (Section 3).

Finally, we mention that the convergence in the complex plane of Hermite–Padé approximants of the exponential is studied in [16].

2. THE INEQUALITY OF MAHLER: METHOD OF PROOF

Hermite–Padé approximants to the exponential function and its powers are defined as a set of polynomials $A_0, ..., A_m$, not all identically zero, of *degree less than* n and such that

$$R(z) := \sum_{p=0}^{m} A_p(z)\, e^{pz} \tag{2.1}$$

vanishes at the origin to the order $(m+1)\,n-1$. The polynomials $A_0, ..., A_m$ are obtained by solving a linear system of $(m+1)\,n-1$ homogeneous equations with $(m+1)\,n$ unknown coefficients. Thus, non-trivial solutions to (2.1) always exist. Moreover, such non-trivial solutions $A_0, ..., A_m$ can be given by explicit expressions, the coefficients of which are rational numbers. Multiplying (2.1) by the least common multiple of the denominators of the coefficients of the A_p, $0 \leqslant p \leqslant m$, we get

$$\tilde{R}(z) = \sum_{p=0}^{m} \tilde{A}_p(z)\, e^{pz}, \tag{2.2}$$

where all polynomials $\tilde{A}_p$, $0 \leqslant p \leqslant m$, have integral coefficients. In fact, the sharp value of this least common multiple is difficult to compute. Usually, one uses instead a common multiple which comes out from simple derivations (see [9]). We shall evaluate (2.2) at some large positive integer $g \in \mathbb{N}$. Actually the parameters m and n will be functions of g, both tending to infinity as g does. Namely it appears appropriate to choose

$$n = [g/\beta], \qquad m = [\alpha \log n], \tag{2.3}$$

where $[x]$ denotes the integral part of x and α and β are positive real numbers to be determined in some optimal way. We write $e^g = a + \delta$, a the closest integer to e^g, $|\delta| < 1/2$. Taking into account the relations (2.3) between g, n, and m, one can easily check that, for $1 \leqslant p \leqslant m$,

$$(a+\delta)^p = a^p + p\,\delta a^{p-1}(1+o(1)) \qquad \text{as} \quad g \to +\infty,$$

where the convergence of the $o(1)$-term to zero is uniform with respect to p. Thus equation (2.2) becomes

$$\tilde{R}(g) = \sum_{p=0}^{m} \tilde{A}_p(g)\, a^p + \sum_{p=1}^{m} \tilde{A}_p(g)\, p\,\delta a^{p-1}(1+o(1)). \tag{2.4}$$

If α and β satisfy some condition, $\tilde{R}(g)$ tends to 0 as g becomes large. Assume this condition fulfilled and choose g so large that $|\tilde{R}(g)| \leqslant 1/2$. As the first sum in the right-hand side of (2.4) is an integer, it implies that

$$\begin{aligned} |\tilde{R}(g)| &\leqslant \left| \sum_{p=1}^{m} \tilde{A}_p(g)\, p\,\delta a^{p-1}(1+o(1)) \right| \\ &\leqslant (1+o(1))\,|\delta| \sum_{p=1}^{m} |pa^{p-1}\tilde{A}_p(g)|. \end{aligned} \tag{2.5}$$

In order to get the desired lower bound on $|\delta|$, we need estimates both from below and above of $|\tilde{R}(g)|$ (the second one to ensure the condition $|\tilde{R}(g)| \leqslant 1/2$) and an estimate from above of the $|\tilde{A}_p(g)|$, $1 \leqslant p \leqslant m$, allowing us to obtain an upper bound for the second sum in (2.5).

3. HERMITE-PADÉ APPROXIMANTS TO EXPONENTIALS

The estimates just alluded to above can be derived from precise asymptotics of the Hermite–Padé approximants (2.1). To achieve this, we need explicit expressions that are, for example, given in [9]. Let us denote by C_0 and C_∞ two circles both centered at the origin and of radius less than 1 and greater than m respectively. The following expressions

$$A_p(z)=\frac{1}{2i\pi}\int_{C_0}\frac{e^{\zeta z}\,d\zeta}{\prod_{l=0}^{m}(\zeta+p-l)^n},\qquad 0\leqslant p\leqslant m, \tag{3.1}$$

$$R(z)=\frac{1}{2i\pi}\int_{C_\infty}\frac{e^{\zeta z}\,d\zeta}{\prod_{l=0}^{m}(\zeta-l)^n}, \tag{3.2}$$

satisfy (2.1) as can be seen by performing a change of variable in the A_p's and using homotopy invariance of Cauchy integral. Clearly, they also satisfy the prescribed hypothesis about the degrees of the A_p's and the order of vanishing of R at the origin. Applying the saddle point method to the contour integrals in (3.2) and (3.1) leads to exact asymptotics as $n\to\infty$ and $m=[\alpha\log n]$ of the remainder term $R(\beta n)$ and of the approximants $A_p(\beta n)$, $0\leqslant p\leqslant m$. This gives the first ingredient in order to derive Mahler's inequality. The second ingredient consists of an estimation of a common multiple of the denominators of the coefficients of the A_p's, the rationality of which is clear from (3.1). Such an estimate can be found, for example, in [9, p. 203]. Using these two ingredients and following the method of proof described in Section 2, it may be shown that for g, a large positive integer,

$$|e^g-a|>g^{-21.012g},\qquad a\in\mathbb{N}. \tag{3.3}$$

This yields a first improvement of (1.1) which, actually, can also be obtained from Mignotte's computations in [11]. We take this opportunity to point out that inequality (i) on p. 127 of [11] is false. It is a consequence of the inequality derived in the appendix II on p. 130 of this same paper but the factor x^n in it is missing in (i) p. 127. Keeping track of this correction in Mignotte's proof leads to the same inequality as (3.3).

4. ASYMPTOTICS OF NON-DIAGONAL HERMITE–PADÉ APPROXIMANTS

In the following, we study a set of approximants different from (3.1). Namely, we consider the following polynomials,

$$B_p(z)=\frac{1}{2i\pi}\int_{C_0}\frac{e^{\zeta z}\,d\zeta}{\prod_{l=0}^{m}(\zeta+p-l)^n\prod_{l=q}^{m-q}(\zeta+p-l)^{sn}},\qquad 0\leqslant p\leqslant m, \tag{4.1}$$

where s and $q = [\mu m]$ are positive integers with $0 \leqslant \mu \leqslant 1/2$, a fixed real number. Note that the case $\mu = 0$ corresponds to the classical approximants (3.1) with n replaced by $(s+1)\,n$.

By analogy with the terminology used in the theory of Padé approximants, we may term the set of polynomials B_p non-diagonal Hermite–Padé approximants, meaning that their degree may change with p. Indeed, the B_p's are of degree $n-1$ when $0 \leqslant p < q$ or $m-q < p \leqslant m$ and of degree $(s+1)\,n-1$ when $q \leqslant p \leqslant m-q$. The polynomials B_p are also related to the Legendre type polynomials studied by Hata [6]. In Section 5, we shall use his results concerning arithmetical properties of their coefficients.

The B_p's satisfy an equality

$$S(z) = \sum_{p=0}^{m} B_p(z)\, e^{pz}, \tag{4.2}$$

where S vanishes at the origin to the order $n(m+1)(s+1) - 2nqs - 1$. Clearly, S admits the following integral representation obtained after plugging (4.1) into (4.2) and using homotopy invariance of Cauchy integrals,

$$S(z) = \frac{1}{2i\pi} \int_{C_\infty} \frac{e^{\zeta z}\, d\zeta}{\prod_{l=0}^{m} (\zeta - l)^n \prod_{l=q}^{m-q} (\zeta - l)^{sn}}. \tag{4.3}$$

The family of polynomials (4.1) will prove to be of some interest since they allow us to improve inequality (3.3), which is the goal of the ensuing sections.

We study the asymptotics as $n \to \infty$ and $m = [\alpha \log n]$ of the remainder term $S(\beta n)$ and of the approximants $B_p(\beta n)$, $0 \leqslant p \leqslant m$ by applying the saddle point method to the contour integrals given in (4.3) and (4.1) respectively. We recall the main result that we shall need in a form convenient for our purpose (cf. [3, 13]).

THEOREM (Saddle Point Method). *Let h and g be analytic functions in a simply connected open set Δ and assume g has no zeros in Δ. Let Γ be a smooth oriented path with a finite length and endpoints a and b, lying in Δ. Moreover, let*

$$I_n = \int_\Gamma h(\zeta)/g(\zeta)^n\, d\zeta.$$

(i) *Assume that* $\min_{\zeta \in \Gamma} |g(\zeta)|$ *is attained at the endpoint* a *only and* $g'(a) \neq 0$. *Then, if* $h(a) \neq 0$,

$$I_n = \frac{h(a)}{ng'(a)\ g(a)^{n-1}} \left(1 + O\left(\frac{1}{n}\right)\right) \qquad as \quad n \to \infty.$$

(ii) *Assume that a point* ζ_0 *of* Γ, *different from an endpoint, is a non-degenerate critical point of* g (*i.e.*, $g'(\zeta_0) = 0$, $g''(\zeta_0) \neq 0$) *and let* ω *be the phase corresponding to the direction of the tangent to the oriented path at* ζ_0. *Suppose further that* $\min_{\zeta \in \Gamma} |g(\zeta)|$ *is attained at the point* ζ_0 *only. Then, if* $h(\zeta_0) \neq 0$,

$$I_n = \sqrt{2\pi g(\zeta_0)/g''(\zeta_0)}\ \frac{h(\zeta_0)}{\sqrt{n} g(\zeta_0)^n} \left(1 + O\left(\frac{1}{n}\right)\right), \tag{4.4}$$

as $n \to \infty$, *where the phase* ω_0 *of* $g''(\zeta_0)/g(\zeta_0)$ *is chosen to satisfy* $|\omega_0 + 2\omega| \leqslant \pi/2$. *Since* $g(\zeta) - g(\zeta_0) \sim (g''(\zeta_0))/2)(\zeta - \zeta_0)^2$ *as* $\zeta \to \zeta_0$ *along* Γ, *and* $|g(\zeta)/g(\zeta_0)| \geqslant 1$, *it is always possible to choose* ω_0 *uniquely in this way.*
Before applying this theorem, we introduce the usual Pochhammer notation; we set

$$(z)_m = z(z+1) \cdots (z+m-1), \qquad m > 0.$$

4.1. *Asymptotics of the remainder term* $S(\beta n)$

THEOREM 4.1. *As* n, m *and* q *tend to infinity, with* $m = [\alpha \log n]$ *and* $q = [\mu m]$, $0 \leqslant \mu \leqslant 1/2$, *one has the following estimate*

$$S(\beta n) = e^{-n(1 + \beta'/2 + s(1 + \beta''/2))} e^{nqs\beta''} \left(\frac{e}{(a_0 - 1)m}\right)^{n(m+1)}$$
$$\times \left(\frac{e}{(a_0 - 1)m + q}\right)^{ns(m - 2q + 1)} e^{O(n/m)}, \tag{4.5}$$

where a_0 *is the unique real root lying in* $(1, \infty)$ *of*

$$e^{\beta}(a-1)(a+\mu-1)^s = a(a-\mu)^s \tag{4.6}$$

and

$$\beta' = \log\left(\frac{a_0}{a_0 - 1}\right), \qquad \beta'' = \log\left(\frac{a_0 m - q}{(a_0 - 1)m + q}\right). \tag{4.7}$$

Remark. We may replace $S(\beta n)$ by $S(g)$ in (4.5). Indeed, by the relation $n = [g/\beta]$, this is equivalent to modify the parameter β by a quantity which is less than β/n in modulus. It is easily checked that such a perturbation does not change the given estimate.

Proof. We use the saddle point method as applied to the integral expression (4.3) of $S(\beta n)$. With the same notations, we have here

$$g(\zeta)=e^{-\beta\zeta}(\zeta-m)_{m+1}(\zeta-m+q)^{s}_{m-2q+1},$$

which depends on the parameter n since m does. We first compute the expression (4.4) corresponding to $S(\beta n)$ and then, briefly explain why the dependence of m with respect to n does not modify the obtained asymptotics. The critical points of g satisfy

$$\begin{aligned}\beta &= \sum_{k=0}^{m} \frac{1}{\zeta-m+k}+\sum_{k=0}^{m-2q} \frac{s}{\zeta-m+q+k}\\ &=\psi(\zeta+1)-\psi(\zeta-m)+s\psi(\zeta-q+1)-s\psi(\zeta-m+q)\end{aligned} \tag{4.8}$$

where ψ denotes the logarithmic derivative of Γ or digamma function (cf. [13] for a definition and some properties). This equation has $m+1$ real roots, m of which lie in the segment $(0, m)$ whereas the last one, say ζ_0 is larger than m. As the contour C_∞ has to encompass all points 0, ..., m, we choose ζ_0 as the critical point that C_∞ should go through. The function ψ admits the expansion

$$\psi(z)=\log(z)-\frac{1}{2z}+O\left(\frac{1}{z^2}\right), \qquad z\notin(-\infty, 0]. \tag{4.9}$$

Using the approximation $\psi(z)\sim\log(z)$, we get from (4.8) that, as $m\to\infty$, ζ_0 is equivalent to am where a satisfies

$$\beta=\log\left(\frac{a}{a-1}\right)+s\log\left(\frac{a-\mu}{a+\mu-1}\right), \tag{4.10}$$

or equivalently

$$e^{\beta}(a-1)(a+\mu-1)^s=a(a-\mu)^s.$$

This equation has a unique real root a_0, larger than 1 and when s is odd, another real root exists, lying in (0, 1). Hence, the unique critical point of g larger than m satisfies $\zeta_0\sim a_0 m$ as $m\to\infty$. The contour C_∞ should be such that the modulus of g there attains its minimum at ζ_0 only. To achieve this, we consider the level curves of $e^{-\beta\zeta}$ and $(\zeta-m)_{m+1}(\zeta-m+q)^s_{m-2q+1}$ at ζ_0. The first one is a line $\mathscr{D}$ of direction i while the second one is a lemniscate $\mathscr{L}$, of degree $m+1+s(m-2q+1)$ tangent to $\mathscr{D}$ at ζ_0, contained in the left half plane delimited by $\mathscr{D}$. Then, a convenient contour C_∞ is any curve contained in this left half plane, surrounding all points 0, ..., m as well as the lemniscate $\mathscr{L}$. The saddle point method applies and we need an estimation of $g(\zeta_0)=e^{-\beta\zeta_0}(\zeta_0-m)_{m+1}(\zeta_0-m+q)^s_{m-2q+1}$. From the recurrence formula for the gamma function Γ, $\Gamma(z+1)=z\Gamma(z)$, we know that

$$g(\zeta_0)=e^{-\beta\zeta_0}\frac{\Gamma(\zeta_0+1)}{\Gamma(\zeta_0-m)}\frac{\Gamma^s(\zeta_0-q+1)}{\Gamma^s(\zeta_0-m+q)}. \tag{4.11}$$

Using the well-known Stirling's formula for Γ (see [13, p. 294]) and substituting ζ_0 by its approximation a_0m, one can check after some computations that

$$g(\zeta_0)=e^{s(1+(1/2-q)\beta'')+1+\beta'/2}\left(\frac{(a_0-1)m}{e}\right)^{m+1} \times\left(\frac{a_0m+q}{e}\right)^{(m-2q+1)s}(1+O(1/m)), \tag{4.12}$$

where β' and β'' are defined by (4.7). Next, we compute $g''(\zeta_0)/g(\zeta_0)$. Differentiating (4.11), we get

$$g''(\zeta_0)/g(\zeta_0)=\psi'(\zeta_0+1)-\psi'(\zeta_0-m)+s\psi'(\zeta_0-q+1)-s\psi'(\zeta_0-m+q).$$

Thus, with the equivalence $\zeta_0\sim a_0m$ and the expansion (4.9), we obtain

$$g''(\zeta_0)/g(\zeta_0)\sim\frac{1}{m}\left(\frac{(2\mu-1)s}{(a_0-\mu)(a_0-1+\mu)}-\frac{1}{a_0(a_0-1)}\right). \tag{4.13}$$

Making use of (4.12) and (4.13) to evaluate expression (4.4) leads to (4.5). It should be noted that this expression does not take into account the actual value of $g''(\zeta_0)/g(\zeta_0)$ since the logarithm of its modulus is of an order less than $O(n/m)$.

As the point ζ_0, the contour C_∞ and the function g depend on m, this last parameter increasing with n like $\alpha\log n$, a justification of (4.5) is in order. This entails screening the proof of the Laplace's method for contour integral (cf. [13]), checking that the arguments remain valid. This will not be detailed. We only mention two facts that are relevant here. The first one is that the increasing length of C_∞, which has to surround $\mathscr{L}$, as well as all of the points 0, ..., m is of order the degree of $\mathscr{L}$ (cf. [1] for an estimate giving the correct rate of growth of the arc length of a lemniscate). As this order is less than $e^{\varepsilon n}$, $\varepsilon>0$, this means that the main contribution in integral (4.3) comes as usual from its restriction to some arc in the neighborhood of ζ_0. Performing locally the change of variable $v = \log g(\zeta) - \log g(\zeta_0)$, the integral under study becomes

$$e^{-n \log g(\zeta_0)} \int e^{-nv} f(v)\, dv \qquad \text{with} \qquad f(v) = \frac{d\zeta}{dv} = \frac{g(\zeta)}{g'(\zeta)}.$$

For small $|v|$, we check that $f(v)$ has an expansion with respect to $\zeta - \zeta_0$ and m such that $f(v) = O(v^{-1/2})$. This is the second important fact here, allowing us to estimate the integral of $S(\beta n)$ by the same method as in [13, Chapter 4, Section 6], leading eventually to (4.5). ∎

4.2. *Ray asymptotics of the numbers $B_p(\beta n)$*

THEOREM 4.2. *Let p be an integer depending on m such that, as $n \to \infty$, $p/m \to \lambda$ for some fixed $\lambda \in (0, 1)$ distinct from μ and $1-\mu$. For example, choose $p = [\lambda m]$. Then, $p/m = \lambda + O(1/m)$. As $m \to \infty$, let $\mathscr{B}_\lambda$ denote the following expansion,*

$$\begin{aligned}\mathscr{B}_\lambda = {} & (1 + s(1-2\mu))(\log m - 1) + \lambda \log \lambda + (1-\lambda) \log(1-\lambda) \\ & + s(\lambda - \mu) \log |\lambda - \mu| + s(1 - \mu - \lambda) \log |1 - \mu - \lambda| + O(\log m/m).\end{aligned}$$

As $n \to \infty$, $m = [\alpha \log n]$ and $q = [\mu m]$, we have

$$|B_p(\beta n)| = e^{-nm\mathscr{B}_\lambda}. \tag{4.14}$$

Moreover, in (4.14), *the O-term of expansion $\mathscr{B}_\lambda$ is uniformly dominated by* $\log m/m$ *with respect to λ.*

Remark. It is possible to determine the sign of $B_p(\beta n)$ as p varies. Since the knowledge of this sign is not needed in the paper, it will not be given here.

Proof. To obtain the estimate of $B_p(\beta n)$, we again use the saddle point method, now with the integral representation (4.1). Here, we have

$$g(\zeta) = e^{-\beta\zeta} (\zeta + p - m)_{m+1} (\zeta + p - m + q)^s_{m-2q+1}. \tag{4.15}$$

We deform the circle of integration C_0 to a rectangle R included in the region $\{-1 < Re(z) < 1\}$ with vertices $(-a', -r)$, $(a, -r)$, (a, r), $(-a', r)$ where r is a positive real number increasing with m like $2\sqrt{m}$ and a and a' lie in $(0, 1)$. We shall choose $-a'$ and a as the two critical points of g in $(-1, 1)$. Then, the minimum of $|(\zeta + p - m)_{m+1} (\zeta + p - m + q)^s_{m-2q+1}|$ on the vertical segment joining $(-a', r)$ to $(-a', -r)$ (resp. $(a, -r)$ to (a, r)) is attained at the point $-a'$ (resp. a) only. With the assumption made on r, the value of $|(\zeta + p - m)_{m+1} (\zeta + p - m + q)^s_{m-2q+1}|$ on the two remaining horizontal segments is larger than both values at a and $-a'$. Indeed, this will be true if for any $0 \leqslant p \leqslant m$, $(p+1)^2 \leqslant r^2 + (p-1)^2$. In particular, it is satisfied when the inequality holds for $p = m$ i.e. $4m \leqslant r^2$. The exponential factor $e^{-\beta z}$ is of constant modulus on the two vertical segments and introduces only a constant independent of n on the horizontal segments. Observe also that the length of the contour R which is $O(\sqrt{m})$ will not affect the geometric estimate in n of the integral near the critical points. These critical points are among the solutions of

$$\begin{aligned}\beta &= \frac{(\zeta + p - m)'_{m+1}}{(\zeta + p - m)_{m+1}} + s\,\frac{(\zeta + p - m + q)'_{m-2q+1}}{(\zeta + p - m + q)_{m-2q+1}} \\ &= \sum_{k=0}^{m} \frac{1}{\zeta + p - m + k} + s \sum_{k=q}^{m-q} \frac{1}{\zeta + p - m + k}.\end{aligned}$$

We rewrite this equation as

$$\beta = \psi(\zeta + p + 1) - \psi(\zeta + p - m) + s\psi(\zeta + p - q + 1) - s\psi(\zeta + p - m + q). \tag{4.16}$$

Using the reflection formula for ψ, $\psi(z) = \psi(1 - z) - \pi \cot \pi z$, we get

$$\begin{aligned}\beta &= \psi(\zeta + p + 1) - \psi(1 - \zeta - p + m) + \pi \cot \pi\zeta + s\psi(-\zeta - p + q) \\ &\quad - s\psi(1 - \zeta - p + m - q) \qquad \text{if} \quad p < q, \\ \beta &= \psi(\zeta + p + 1) - \psi(1 - \zeta - p + m) + s\psi(\zeta + p - q + 1) \\ &\quad - s\psi(1 - \zeta - p + m - q) + 2\pi \cot \pi\zeta \qquad \text{if} \quad q \leqslant p \leqslant m - q,\end{aligned}$$

and

$$\begin{aligned}\beta &= \psi(\zeta + p + 1) - \psi(1 - \zeta - p + m) + \pi \cot \pi\zeta + s\psi(\zeta + p - q + 1) \\ &\quad - s\psi(\zeta + p - m + q) \qquad \text{if} \quad m - q < p.\end{aligned}$$

Using the approximation $\psi(\zeta) \sim \log \zeta$ for ζ large, one can check that the three previous equations lead for any $0 \leqslant p \leqslant m$ to

$$\tan \pi\zeta = \frac{\pi}{\beta + \log\left(\dfrac{1-\lambda}{\lambda}\left|\dfrac{1-\mu-\lambda}{\mu-\lambda}\right|^{s}\right)} + O(1/\log n). \tag{4.17}$$

Denote by f_0 the fraction in the right-hand side of (4.17). Since we choose $\lambda \in (0, 1)$ different from μ and $1-\mu$, f_0 lies in $[-\infty, \infty]-\{0\}$. Let arctan denotes the inverse of the tan function, whose image is the segment $[0, \pi)$ with the convention that $\arctan(\pm\infty)=\pi/2$. Then, there are two critical points in $(-1, 1)$, x_1 and x_2, lying in $(-1, 0)$ and $(0, 1)$ respectively,

$$x_1 = x_2 - 1 + O(1/\log n), \qquad x_2 = \frac{1}{\pi} \arctan f_0 + O(1/\log n).$$

We choose $-a' = x_1$, $a = x_2$ and determine which one of x_1 and x_2 gives the smallest modulus for g by computing the quotient

$$\frac{g(x_1)}{g(x_2)} = e^{\beta + O(1/\log n)} \frac{x_1 + p - m}{x_2 + p} \left(\frac{x_1 + p - m + q}{x_2 + p - q}\right)^s$$
$$\times \prod_{k=p-m}^{p-1} \frac{x_1 + k + 1}{x_2 + k} \prod_{k=p-m+q}^{p-q-1} \left(\frac{x_1 + k + 1}{x_2 + k}\right)^s. \tag{4.18}$$

To analyze the two products over k in the right-hand side of (4.18), we write an expression of the difference $x_1 - x_2$ at order $O(1/\log n)$:

$$x_1 - x_2 = -1 + b_0/\log n + O(1/\log^2 n),$$

where the actual value of b_0 can be computed from (4.16). Then, the first product, say, equals

$$\prod_{k=p-m}^{p-1} \left(1 + \frac{b_0}{(x_2 + k)\log n} + O(1/\log^2 n)\right),$$

which tends to 1 as n tends to infinity. Indeed, this follows from the magnitude of the bounds of summation $p-1$ and $p-m$ which is $O(\log n)$ and the convergence to 1 of the more elementary sequence whose term of order j is $\prod_{k=1}^{j} (1 + a/(kj))$, where a is some fixed real number. Since the same conclusion applies to the second product over k in (4.18), we obtain, as $n \to \infty$,

$$\frac{g(x_1)}{g(x_2)} \to e^{\beta} \frac{\lambda - 1}{\lambda} \left(\frac{\lambda + \mu - 1}{\lambda - \mu}\right)^s. \tag{4.19}$$

It may be seen that the limit in (4.19), as a function of λ, has modulus 1 one, two or three times in $(0, 1)$ according to the values of $0 \leqslant \beta$, $0 \leqslant \mu \leqslant 1/2$ and $s \in \mathbb{N}$. Note that a value of λ such that the limit in (4.19) has modulus 1, makes the denominator of f_0 vanish, i.e., we have $x_2 = 1/2 + O(1/\log n)$ and $x_1 = -1/2 + O(1/\log n)$ in this case.

Now, let us compute an expansion of $g(x_2)$, in terms of m. From (4.15) and the recurrence formula for the gamma function Γ, $\Gamma(z+1) = z\Gamma(z)$, we have

$$g(x_2) = e^{-\beta x_2} \frac{\Gamma(x_2 + p + 1)}{\Gamma(x_2 + p - m)} \frac{\Gamma^s(x_2 + p - q + 1)}{\Gamma^s(x_2 + p - m + q)}.$$

Using the reflection formula, $\Gamma(z)\,\Gamma(1-z) = \pi/\sin \pi z$, $z \notin \mathbb{Z}$, together with the expansion

$$\log \Gamma(z) = (z - 1/2) \log z - z + O(1), \qquad z \notin (-\infty, 0],$$

one can check that, as $m \to \infty$,

$$\begin{aligned} |g(x_2)| = \exp m((1 + s(1-2\mu)) \log m + (\lambda \log \lambda + (1-\lambda) \log(1-\lambda) \\ + s(\lambda - \mu) \log |\lambda - \mu| + s(1 - \mu - \lambda) \log |1 - \mu - \lambda| \\ + s(2\mu - 1) - 1) + O(\log m/m)), \end{aligned} \tag{4.20}$$

where the O-term is uniformly dominated by $\log m/m$ with respect to the parameter $\lambda \in (0, 1)$. From (4.19), we see that $|g(x_1)|$ admits the same expansion. We now apply assertion (ii) of the saddle point method with x_1 or x_2 as critical point ζ_0 according to which one of the numbers $g(x_1)$ and $g(x_2)$ has the largest modulus. To compute an expansion of order $e^{O(n)}$ or larger, the only relevant factor in (4.4) is the exponent $g(\zeta_0)^n$. Hence, taking the nth power of (4.20), we eventually get (4.14). ∎

5. ARITHMETIC OF THE COEFFICIENTS

As already mentioned, the computation of a common multiple to the denominators of the rational coefficients of the classical polynomials A_p, $0 \leqslant p \leqslant m$, may be found in [9, p. 203]. In this section, we perform such a computation for the polynomials B_p. The precise results obtained by Hata concerning the asymptotics of the coefficients of Legendre type polynomials (cf. [6, 7]) will be an important ingredient in the analysis. Set

$$T_m = \operatorname*{lcm}_{\substack{p, l = 0, 1, \ldots, m \\ p \neq l}} (p - l) = \operatorname{lcm}(1, \ldots, m),$$

where lcm denotes the least common multiple and

$$M_l = \prod_{\substack{j=0 \\ j \neq l}}^{m} (l-j) = (-1)^{m-l}\, l!(m-l)! = (-1)^{m-l}\, m! \binom{m}{l}^{-1}, \qquad 0 \leqslant l \leqslant m,$$

$$M_{l,q} = \prod_{\substack{j=q \\ j \neq l}}^{m-q} (l-j) = (-1)^{m-q-l} (l-q)!\, (m-q-l)!$$

$$= (-1)^{m-q-l} (m-2q)! \binom{m-2q}{l-q}^{-1}, \qquad q \leqslant l \leqslant m-q,$$

so that $\mathrm{lcm} M_l = m!$ and $\mathrm{lcm} M_{l,q} = (m-2q)!$. We rewrite the integral expression (4.3) of $S(z)$ as

$$S(z) = \frac{1}{2i\pi} \int_{C_\infty} \left(\sum_{l=0}^{m} \frac{1}{M_l(\zeta - l)} \right)^n \left(\sum_{l=q}^{m-q} \frac{1}{M_{l,q}(\zeta - l)} \right)^{sn} e^{\zeta z}\, d\zeta. \tag{5.1}$$

On the other hand, we can write

$$\left(\sum_{l=0}^{m} \frac{m!}{M_l(\zeta - l)} \right) \left(\sum_{l=q}^{m-q} \frac{(m-2q)!}{M_{l,q}(\zeta - l)} \right) = \sum_{l=0}^{m} \frac{c_{m,l}}{\zeta - l} + \sum_{l=q}^{m-q} \frac{d_{m,l}}{(\zeta - l)^2}$$

with rational coefficients $c_{m,l}$ and $d_{m,l}$. Now, for any subset $X \subset \mathbb{N}$, put

$$J(X) = \prod_{l \in X} l.$$

Then, from [7, Lemma 5.1], we know that

$$\begin{aligned} \{d_{m,l},\, l = q, ..., m-q\} &\subset J(X_{m-q,q})\, \mathbb{Z}, \\ \{c_{m,l},\, l = 0, ..., m\} &\subset \frac{J(X_{m-q,q})}{T_m}\, \mathbb{Z}, \end{aligned} \tag{5.2}$$

where $X_{m-q,q}$ is the set of prime numbers $p > \sqrt{m}$ satisfying

$$\left\{ \frac{m-q}{p} \right\} + \left\{ \frac{q}{p} \right\} > 1 \qquad \text{and} \qquad \left\{ \frac{m-q}{p} \right\} > \left\{ \frac{q}{p} \right\},$$

$\{x\}$ denotes the fractional part of x. Moreover, the asymptotic behavior of $J(X_{m-q,q})$ as $m \to \infty$ has been determined by Hata. For any subset $E \subset (0,1) \times (0,1)$ put $E(v) = \{x > 0;\, (\{x\}, \{vx\}) \in E\}$. Put also

$$Y = \{(x, y) \in (0,1) \times (0,1); \quad x + y > 1 \text{ and } y > x\}.$$

Then, from [6, Lemma 2.5], we have

$$\rho=\rho(v):=\lim_{m\to\infty}\frac{1}{m-q}\log J(X_{m-q,q})=\frac{1}{v}\int_{Y(v)}\frac{dy}{y^2}, \tag{5.3}$$

where

$$v=\lim_{m\to\infty}\frac{m-q}{q}=\frac{1-\mu}{\mu}.$$

Assume v is a rational number $v=\sigma/\theta$, σ, $\theta\in\mathbb{N}$. Then, one may compute the integral in (5.3) by the same way as in [7, p. 64]. It consists in the following. Let R_j be the affine mapping defined by $R_j:(x,y)\to((x+j)/\theta.y)$ and put $\tilde{Y}=\bigcup_{j=0}^{\theta-1}R_j(Y)$. Then $(\{\theta x\},y)\in Y$ if and only if $(\{x\},y)\in\tilde{Y}$. Hence, substituting $y=\theta x$ in (5.3), we get

$$\rho=\frac{1}{\sigma}\int_{\tilde{Y}(\sigma)}\frac{dx}{x^2}=\frac{1}{\sigma}\int_Z d\psi(x), \tag{5.4}$$

where $Z=\tilde{Y}(\sigma)\cap(0,1)$. As explained in [7], the set Z is a finite disjoint union of open intervals, which coincides with the projected set onto x-axis of the intersection of $\tilde{Y}$ and the line segments $y=\{\sigma x\}$. Therefore ρ can be expressed as a finite sum of the values of $\psi(x)$ at rational points.

Let us return to the study of (5.1). Expanding the nth power of

$$\sum_{l=0}^{m}\frac{m!}{M_l(\zeta-l)}$$

and using the elementary identities

$$\frac{1}{\zeta-l}\,\frac{1}{\zeta-j}=\frac{1}{l-j}\left(\frac{1}{\zeta-l}-\frac{1}{\zeta-j}\right),\qquad l\neq j,$$

show that the quantity

$$T_m^{n-1}\left(\sum_{l=0}^{m}\frac{m!}{M_l(\zeta-l)}\right)^n$$

admits a partial fraction decomposition with integral coefficients. Multiplying this decomposition by

$$\sum_{l=q}^{m-q}\frac{(m-2q)!}{M_{l,q}(\zeta-l)}$$

(sn) times and applying Hata's result (5.2) for each of these products, we deduce that

$$D_{n,m} := m!^n (m-2q)!^{sn} \, T_m^{sn+n-1} / J^{sn}(X_{m-q,q})$$

is a common multiple to the denominators of the rational coefficients in the partial fraction decomposition of

$$\left(\sum_{l=0}^{m} \frac{1}{M_l(\zeta - l)} \right)^n \left(\sum_{l=q}^{m-q} \frac{1}{M_{l,q}(\zeta - l)} \right)^{sn}.$$

Plugging this partial fraction decomposition in (5.1) and evaluating the integral by means of the identities

$$\frac{1}{2i\pi} \int_{C_\infty} \frac{e^{\zeta z} \, d\zeta}{(\zeta - l)^\lambda} = \frac{z^{\lambda - 1} e^{lz}}{(\lambda - 1)!}, \qquad \lambda = 1, ..., n,$$

we obtain that $\tilde{S} := (sn + n - 1)! \, D_{n,m} S$ is an exponential polynomial with integral coefficients. Equivalently, the

$$\tilde{B}_p := (sn + n - 1)! \, D_{n,m} B_p, \qquad 0 \leqslant p \leqslant m,$$

are polynomials with integral coefficients. Moreover, we have the well-known upper bound

$$T_m \leqslant e^{m(1 + o(1))}, \tag{5.5}$$

as m tends to infinity. Indeed,

$$\log T_m = \sum_{\substack{p \text{ prime} \\ p^l \leqslant m}} l \log p \leqslant \sum_{\substack{p \text{ prime} \\ p \leqslant m}} \log m = m(1 + o(1)),$$

where the last equality comes from the prime number theorem. From Stirling's formula for the factorial, inequality (5.5) and the definition of ρ in (5.3), one checks that an upper bound on the common multiple $(sn + n - 1)! \, D_{n,m}$ of the denominators of the coefficients of the B_p, $0 \leqslant p \leqslant m$, is given by

$$n^{(s+1)n} m^{mn(1 + s(1 - 2\mu))} (1 - 2\mu)^{mns(1 - 2\mu)} e^{2mn\mu s} e^{-s(1 - \mu)\rho mn} e^{o(mn)}. \tag{5.6}$$

6. APPLICATION TO THE INEQUALITY OF MAHLER

Proof of Theorem 1.1. We follow the scheme of proof described in Section 2. We first consider the condition

$$|\tilde{S}(g)| \leqslant 1/2. \tag{6.1}$$

From Theorem 4.1 and the upper bound (5.6), we have as $g \to \infty$ with $n = [g/\beta]$, $m = [\alpha \log n]$ and $q = [\mu m]$ that

$$|\tilde{S}(g)| < \left(\frac{e^{(1+s)(1+1/\alpha) - s(1-\mu)\rho}}{a_0 - 1} \left(\frac{a_0 - \mu}{a_0 - 1 + \mu} \right)^{s\mu} \left(\frac{1 - 2\mu}{a_0 - 1 + \mu} \right)^{s(1-2\mu)} \right)^{mn(1+o(1))}.$$

Hence the condition (6.1) is met for g large if

$$\frac{e^{(1+s)(1+1/\alpha) - s(1-\mu)\rho}}{a_0 - 1} \left(\frac{a_0 - \mu}{a_0 - 1 + \mu} \right)^{s\mu} \left(\frac{1 - 2\mu}{a_0 - 1 + \mu} \right)^{s(1-2\mu)} < 1. \tag{6.2}$$

This is the first constraint. Let us now proceed to the analog of inequality (2.5), when replacing the usual Hermite–Padé approximants A_p with our B_p's. As the normalization $B_p \to \tilde{B}_p$ does not play any role here, it is sufficient to consider the inequality

$$|S(g)| \leqslant (1 + o(1)) \, |\delta| \sum_{p=1}^{m} |pa^{p-1} B_p(g)|. \tag{6.3}$$

We establish an upper bound for the sum in the right-hand side. From Theorem 4.2, we have as $n \to \infty$,

$$\sum_{p=1}^{m} |pa^{p-1} B_p(g)| = e^{-nm(1+s(1-2\mu))(\log m - 1) + O(n \log m)} \sum_{p=1}^{m} e^{nmf_\mu(\lambda_p)}, \tag{6.4}$$

where $\lambda_p = p/m$, $1 \leqslant p \leqslant m$ and

$$\begin{aligned} f_\mu(\lambda) = {} & \beta\lambda - \lambda \log \lambda - (1-\lambda) \log(1-\lambda) - s(\lambda - \mu) \log |\lambda - \mu| \\ & - s(1 - \mu - \lambda) \log |1 - \mu - \lambda|, \qquad \lambda \in [0, 1]. \end{aligned}$$

About this equality, observe the following. First, the factor $\exp(O(n \log m))$ can be pulled out of the sum because the O-term is uniformly dominated with respect to the parameter λ_p. Second, the factor pa^{-1} disappears as it is of order less than $\exp(O(n \log m))$ and third, replacing a with e^g is possible as it does not affect the estimate.

Differentiating f_μ with respect to λ, we get

$$f'_\mu(\lambda) = \beta - \log \lambda + \log(1-\lambda) - s \log |\lambda - \mu| + s \log |1 - \mu - \lambda|.$$

Thus, critical points of f_μ are solutions of

$$e^\beta = \frac{\lambda\,|\lambda-\mu|^s}{(1-\lambda)\,|1-\mu-\lambda|^s}. \tag{6.5}$$

One can check that this equation, already met when studying the modulus of the limit in (4.19), has 1, 2 or 3 solutions in (0, 1) according to the values of $\beta>0$, $0\leqslant\mu\leqslant 1/2$ and $s\in\mathbb{N}$. One of them lies in $(\mu, 1-\mu)$ while the two other possible roots lie in $(1-\mu, 1)$. When s is odd and $\mu<\lambda<1-\mu$, (6.5) is identical to (4.6). In all other cases, (6.5) is rewritten as

$$e^\beta(1-\lambda)(\lambda+\mu-1)^s=\lambda(\lambda-\mu)^s.$$

The function f_μ increases from the value $s\mu\log\mu-s(1-\mu)\log(1-\mu)$ taken at the origin to a maximum met when λ equals the first solution of (6.5), lying in $(\mu, 1-\mu)$. Then, if (6.5) has 1 or 2 more solutions in $(1-\mu, 1)$, the function f_μ still admits a minimum, then a maximum before decreasing to the value $\beta+s\mu\log\mu-s(1-\mu)\log(1-\mu)$ met as λ equals 1. Assume first that (6.5) admits only one solution in (0, 1) and let Λ_μ be the value of λ which corresponds to the maximum of f_μ. Then, an equivalent for the sum in the right-hand side of (6.4),

$$\sum_{p=1}^{m} e^{nmf_\mu(\lambda_p)} \tag{6.6}$$

is $\exp(nmf_\mu(\Lambda_\mu))$. Indeed, denote by p_0/m the largest rational number among the p/m, $1\leqslant p\leqslant m$, less than or equal to Λ_μ. Then, divide the sum by $\exp(nmf_\mu(\Lambda_\mu))$ and split it into two parts

$$\sum_{p=1}^{p_0} e^{nm(f_\mu(\lambda_p)-f_\mu(\Lambda_\mu))} \quad \text{and} \quad \sum_{p=p_0+1}^{m} e^{nm(f_\mu(\lambda_p)-f_\mu(\Lambda_\mu))}.$$

The quotient (resp. inverse of the quotient) of a term by the previous one in the second (resp. first) sum is $\exp(-nm\,|f_\mu(\lambda_p)-f_\mu(\lambda_{p+1})|)$. This quotient is less than

$$\exp(-nm\min(f_\mu(\Lambda_\mu)-f_\mu(\Lambda_\mu-1/m),\ f_\mu(\Lambda_\mu)-f_\mu(\Lambda_\mu+1/m))). \tag{6.7}$$

The two differences $f_\mu(\Lambda_\mu)-f_\mu(\Lambda_\mu-1/m)$ and $f_\mu(\Lambda_\mu)-f_\mu(\Lambda_\mu+1/m)$ have the same expansion

$$\frac{C}{2m^2}+O\left(\frac{1}{m^3}\right), \tag{6.8}$$

where C is some constant independent from m. This shows that (6.7) tends to zero as n tends to infinity and $m=[\alpha\log n]$. Therefore, the two previous sums are majorized by any geometric series, hence bounded. Actually, they even tend to zero (except the first one if $p_0/m=\Lambda_\mu$) and decrease at most like (6.7). In regards of (6.8), this does not modify the asymptotic estimate $\exp(nmf_\mu(\Lambda_\mu))$ since we compute an expansion up to order $\exp(O(n\log m))$ only (see (6.4)).

Now, if f_μ admits more than one critical point, it suffices to split the sum (6.6) on each interval between two critical points and to repeat the previous argument on each partial sum. Comparing their contribution, we obtain that an equivalent for (6.6) is $\exp(nmf_\mu(\Lambda_\mu))$, where Λ_μ is a value of λ which corresponds to the maximum of f_μ in (0, 1).

From (6.3) along with Theorem 4.1 and the previous estimate of the sum (6.6) appearing in (6.4), we derive that

$$(a_0-1)^{-nm}\,(a_0-1+\mu)^{-nms(1-\mu)}\,(a_0-\mu)^{nms\mu}\leqslant|\delta|\;e^{O(n\log m)}e^{nmf_\mu(\Lambda_\mu)}. \tag{6.9}$$

Since $n=[g/\beta]$ and $m=[\alpha\log n]$, (6.9) is rewritten as

$$((a_0-1)(a_0-1+\mu)^{s(1-\mu)}\,(a_0-\mu)^{-s\mu}\;e^{f_\mu(\Lambda_\mu)})^{-(\alpha/\beta)\,g\,\log g(1+o(1))}\leqslant|\delta|. \tag{6.10}$$

Considering the limit case in which both sides of (6.2) are equal, we obtain that

$$\alpha=-\frac{1+s}{\left[\begin{aligned}&1+s-s(1-\mu)\rho-\log(a_0-1)\\&\qquad+s\log((1-2\mu)^{(1-2\mu)}\,(a_0-\mu)^{\mu}\,(a_0-1+\mu)^{-(1-\mu)})\end{aligned}\right]}. \tag{6.11}$$

In view of (6.10) and (6.11), we want to minimize

$$\frac{\left[\begin{aligned}&-(1+s)(f_\mu(\Lambda_\mu)+\log(a_0-1)-s\mu\log(a_0-\mu)\\&\qquad+s(1-\mu)\log(a_0-1+\mu))\end{aligned}\right]}{\left[\begin{aligned}&\beta(1+s-s(1-\mu)\rho-\log(a_0-1)\\&\qquad+s\log((1-2\mu)^{(1-2\mu)}\,(a_0-\mu)^{\mu}\,(a_0-1+\mu)^{-(1-\mu)}))\end{aligned}\right]} \tag{6.12}$$

over the set of parameters β, s and μ. Performing some numerical experiments, one can see that a large value for s should be chosen. We shall take

$$s=50,\qquad \beta=8.45,\qquad \text{and}\qquad \mu=0.07.$$

This gives

$$\alpha=1.47778...,\qquad a_0=5.71902...,\quad \Lambda_\mu=0.53563...,\quad f_\mu(\Lambda_\mu)=41.35960...\,.$$

We estimate $\rho=\rho(93/7)$ by computing the second integral in (5.4). Setting

$$I=\{(i, j)\in\mathbb{N}^2;\ 0\leqslant i\leqslant 92,\ 0\leqslant j\leqslant 6,\ -7<7i-93j\leqslant 43\}$$

and

$$J=\{(i, j)\in\mathbb{N}^2;\ 0\leqslant i\leqslant 92,\ 0\leqslant j\leqslant 6,\ 43<7i-93j<86\},$$

one can easily check that ρ is given by

$$\rho=\frac{1}{93}\left(\sum_{i=1}^{92}\psi\left(\frac{i}{93}\right)-\sum_{(i, j)\in I}\psi\left(\frac{1+i+j}{100}\right)-\sum_{(i, j)\in J}\psi\left(\frac{i-j}{86}\right)\right).$$

Numerically, this yields

$$\rho=0.23011... .$$

Finally, we obtain for (6.12) the value of 19.18235..., which completes the proof of the first inequality in Theorem 1.1. By the same way as in [9], this inequality is easily seen to imply for any $\varepsilon>0$,

$$|\log g-a|>g^{-(19.183+\varepsilon)\log\log g}, \qquad a\in\mathbb{N},$$

for any sufficiently large positive integer g. Since the first inequality actually holds with the constant 19.1824, one has

$$|\log g-a|>g^{-19.183\log\log g}, \qquad a\in\mathbb{N},$$

for any sufficiently large positive integer g, which gives the second inequality in Theorem 1.1. ∎

REFERENCES

1. P. B. Borwein, The arc length of the lemniscate $|p(z)|=1$, *Proc. Amer. Math. Soc.* **123** (1995), 797–799.
2. G. V. Chudnovsky, Hermite–Padé approximations to exponential functions and elementary estimates of the measure of irrationality of π, *in* "Lecture Notes in Math.," Vol. 925, pp. 299–322, Springer-Verlag, New York, 1982.
3. N. G. De Bruijn, "Asymptotic Methods in Analysis," 2nd ed., Wiley, New York, 1961.
4. G. Diaz, Une nouvelle minoration de $|\log\alpha-\beta|$, $|\alpha-\exp\beta|$, α et β algébriques, *Acta Arith.* **64** (1993), 43–57.
5. A. I. Galochkin, On the diophantine approximation of values of the exponential function and of solutions of some transcendental equations, *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* **3** (1972), 16–23.
6. M. Hata, Legendre type polynomials and irrationality measures, *J. Reine Angew. Math.* **407** (1990), 99–125.

7. M. Hata, A lower bound for rational approximations to π, *J. Number Theory* **43** (1993), 51–67.
8. K. Mahler, Zur Approximation der Exponentialfunktion und des Logarithmus, *J. Reine Angew. Math.* **166** (1932), 118–150.
9. K. Mahler, On the approximation of logarithms of algebraic numbers, *Philos. Trans. Roy. Soc. London* **245** (1953), 371–398.
10. K. Mahler, Applications of some formulas by Hermite to the approximation of exponentials and logarithms, *Math. Ann.* **168** (1967), 200–227.
11. M. Mignotte, Approximations rationnelles de π et quelques autres nombres, *Bull. Soc. Math. France* **37** (1974), 121–132.
12. Y. Nesterenko and M. Waldschmidt, On the approximation of the values of exponential function and logarithm by algebraic numbers (in Russian), *in* "Diophantine Approximations, Proceedings of Papers Dedicated to the Memory of Prof. N. I. Feldman" (Y. Nesterenko, Ed.), Mat. Zapiski, Vol. 2, pp. 23–42, Moscow, 1996.
13. F. Olver, "Asymptotics and Special Functions," Academic Press, San Diego, 1974.
14. M. Waldschmidt, Transcendence measures for exponentials and logarithms, *J. Austral. Math. Soc.* **25** (1978), 445–465.
15. M. Waldschmidt, A lower bound for linear forms in logarithms, *Acta Arith.* **37** (1980), 257–283.
16. F. Wielonsky, Asymptotics of diagonal Hermite–Padé approximants to e^z, *J. Approx. Theory* **90** (1997), 283–298.