Asymptotic uniqueness of best rational approximants of given degree to Markov functions in $\mathcal{L}^2$ of the circle.

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Abstract

We improve over a sufficient condition given in [8] for uniqueness of a nondegenerate critical point in best rational approximation of prescribed degree over the conjugate-symmetric Hardy space $\overline{\mathcal{H}}^2_{\mathbf{R}}$ of the complement of the disk. The improved condition connects to error estimates in AAK approximation, and is necessary and sufficient when the function to be approximated is of Markov type. For Markov functions whose defining measure satisfies the Szegő condition, we combine what precedes with sharp asymptotics in multipoint Padé approximation from [43, 40] in order to prove uniqueness of a critical point when the degree of the approximant goes large. This lends perspective to the uniqueness issue for more general classes of functions defined through Cauchy integrals.

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1 Introduction and statement of the results

Although rational approximation to analytic functions over compact subsets of their domain of analyticity is a classical subject that has undergone many developments in particular concerning error rates (see *e.g.* [19, 21, 22, 28, 33, 34, 39, 41]), most uniqueness issues for rational approximants remain open and intriguing in the complex domain. We aim at clarifying the situation in the special case where the function to be approximated is of Markov type:

$$f_\mu(z) = \int \frac{d\mu(t)}{z-t}, \tag{1.1}$$

with μ a positive measure compactly supported on $(-1, 1)$, when best approximation is seeked with respect to the $\mathcal{L}^2$ norm on the unit circle among all rational functions of degree at most n having real coefficients[2]; here, the degree of p/q, where p and q are coprime polynomials, is defined to be $\max\{\deg(p), \deg(q)\}$. Hopefully, the present paper also lays ground for generalization to broader classes of functions.

Let $\overline{\mathcal{H}}^2_{\mathbf{R}}$ denote the Hardy class of square summable functions on the unit circle whose Fourier coefficients are real and do vanish for non-negative index. For functions belonging to $\overline{\mathcal{H}}^2_{\mathbf{R}}$ like f_μ, it follows by partial fraction expansion and the Parseval relation that a best approximant, in the above sense, again lies in $\overline{\mathcal{H}}^2_{\mathbf{R}}$. For such functions, best $\mathcal{L}^2$ rational approximation of prescribed degree on the circle is thus equivalent to best rational approximation of prescribed degree in $\overline{\mathcal{H}}^2_{\mathbf{R}}$: a best approximant has poles in the open unit disk only and is of type $(n-1, n)$; when saying that a rational function is of type (m, n), we mean it can be written as a quotient of two polynomials of degree m and n, respectively.

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[2]For Markov functions, it can be shown that such an approximant with real coefficients is still optimal out of rational functions of degree at most n with *complex* coefficients [4].

Although we shall be working entirely on the circle, it is worth mentioning at this point that our $\mathcal{L}^2$ approximation problem may equivalently be stated on the imaginary axis, because if we set

$$\varphi(s) = \frac{s+1}{s-1},$$

then

$$f(z) \to F(s) = \frac{1}{\sqrt{\pi}} \frac{f \circ \varphi(s)}{s-1} \tag{1.2}$$

is an isometry from $\overline{\mathcal{H}}^2_{\mathbf{R}}$ onto the Hardy space of exponent 2 of the right half-plane that preserves rationality and the degree. Moreover, this isometry establishes a one-to-one correspondence between Markov functions that are analytic in a neighborhood of $\{|z| \geq 1\}$ and Markov functions that are analytic in a neighborhood of $\{\mathbf{Re}\, s \geq 0\}$, the image of f_μ defined by (1.1) being f_ν where the positive measure ν is now compactly supported on $(-\infty, 0)$, and given by the formula:

$$d\nu(x) = (1-x)\frac{d\widetilde{\mu}(x)}{2\sqrt{\pi}},$$

where $\widetilde{\mu}$ denotes the image of μ under φ. Therefore our results on uniqueness of best rational $\mathcal{L}^2$ approximants to Markov functions on the circle translate to the imaginary axis, where they acquire some significance in identification and control since Markov functions over the right half-plane are transfer-functions of so-called relaxation systems [45][15][11]. Specifically, Theorem 1.2 and Theorem 1.3 remain valid for measures supported in $(-\infty, 0)$ when approximation is taken in the Hardy space of the right half-plane, while Theorem 1.1 still holds provided a and b are replaced by $\varphi(a)$ and $\varphi(b)$ in condition (1.4).

Generally speaking, several kinds of uniqueness may be envisaged. First comes uniqueness of a best approximant, which is conceptually important to define a best approximation projection. This type of uniqueness holds generically within most function spaces by a theorem of Stechkin [13], but the proof relies on the Baire category theorem and does not allow one to decide whether it applies in specific cases. Second comes uniqueness of a *local* best approximant (which is then also the global one), a stronger property which is much more interesting from the constructive viewpoint as it ensures the convergence of a numerical search. This kind of uniqueness is not generic and accounts a good deal for the computational difficulty of best rational approximation. A third notion is uniqueness of a *critical point*, in other words uniqueness of a rational function at which the derivatives of the error with respect to all the parameters at hand do vanish. This third type of uniqueness lends itself better to analysis, and is the one we shall consider; it subsumes the previous two: uniqueness of a critical point implies uniqueness both of a local and a global best approximant.

To state the results, we need a few definitions. Let T be the unit circle and U the open unit disk. We denote by $\mathcal{P}_k$ the space of real polynomials of degree at most k, and by $\mathcal{M}_k$ the subset of monic polynomials of degree k. Let also $\mathcal{M}_k^1 \subset \mathcal{M}_k$ consist of those polynomials whose roots lie in U. The set of rational functions of degree at most n in $\overline{\mathcal{H}}^2_{\mathbf{R}}$ is indicated by $\mathcal{R}_n$; it consists of quotients p/q where $p \in \mathcal{P}_{n-1}$ and $q \in \mathcal{M}_n^1$.

Given $f \in \overline{\mathcal{H}}^2_{\mathbf{R}}$, that need not be of Markov type, a *best approximant* of degree at most n to f is any $r_n^* \in \mathcal{R}_n$ satisfying

$$\|f - r_n^*\|_2 = \inf_{r \in \mathcal{R}_n} \|f - r\|_2,$$

where $\| \ \|_2$ denotes the norm in $\mathcal{L}^2(T)$. A best approximant always exists, an early reference in the case of real coefficients being [16].

A *local best approximant* of degree at most n to f is any $r_n^\ell \in \mathcal{R}_n$ such that

$$\|f - r_n^\ell\|_2 = \inf_{r \in \mathcal{R}_n \cap \mathcal{V}_{r_n^\ell}} \|f - r\|_2$$

for some neighborhood $\mathcal{V}_{r_n^\ell}$ of r_n^ℓ in $\overline{\mathcal{H}}^2_{\mathbf{R}}$.

Besides, we attach to an arbitrary pair $(p, q) \in \mathcal{P}_{n-1} \times \mathcal{M}_n^1$ the squared approximation error:

$$E_f(p, q) := \left\| f - \frac{p}{q} \right\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} \left| f(e^{i\theta}) - \frac{p(e^{i\theta})}{q(e^{i\theta})} \right|^2 d\theta. \tag{1.3}$$

Writing

$$p(z) = p_{n-1}z^{n-1} + p_{n-2}z^{n-2} + \ldots + p_0 \quad \text{and} \quad q(z) = z^n + q_{n-1}z^{n-1} + \ldots + q_0,$$

we say that $(p^c, q^c) \in \mathcal{P}_{n-1} \times \mathcal{M}_n^1$ is a *critical pair* of degree n if the partial derivatives of E_f with respect to the variables p_j and q_k vanish at $p = p^c$ and $q = q^c$ for $j, k \in \{0, \ldots, n-1\}$.

Critical pairs turn out to be deeply connected to classical interpolation theory: (p^c, q^c) is critical if, and only if, whenever α is a root of q^c of multiplicity k giving rise to a pole of p^c/q^c of order l with $0 \le l \le k$, then p^c/q^c interpolates f in the point $1/\overline{\alpha}$ with multiplicity $k + l$. Hence p^c/q^c is a multipoint Padé approximant to f; a *multipoint Padé approximant* is, by definition, a rational function of type (m, n) interpolating f at $m + n + 1$ points, counting multiplicities. The fact that the interpolation points have to match the reflections of the poles makes for an implicit problem that stresses the nonlinear character of best approximation with free poles.

Henceforth we rule out the exceptional case where $f \in \mathcal{R}_{n-1}$, which is of little interest from the point of view of approximation anyway. This is to the effect that every local best approximant of degree at most n has *exact* degree n [6] hence can be written in a unique fashion as p^ℓ/q^ℓ with $p^\ell \in \mathcal{P}_{n-1}$ and $q^\ell \in \mathcal{M}_n^1$. Clearly (p^ℓ, q^ℓ) is a critical pair, but it is generally not true that, conversely, p^c/q^c is a local best approximant as soon as (p^c, q^c) is critical. In fact, if f is analytic in $\{|z| > 1 - \varepsilon\}$ for some $\varepsilon > 0$ and if the critical pairs meet a mild nondegeneracy condition (*e.g.* they should have nonsingular second derivative and the error functions associated to critical pairs in degree less than n should have no zero on T), then the above-mentioned converse holds *if and only if the critical pair of degree n is unique.* This follows from the index theorem, to be stated here as Theorem 3.3, that was originally proved in [3, 5].

This conditional converse forms the basis of an approach to the uniqueness issue, as it provides us with a *local* criterion:

let f be analytic in a neighborhood of T; if each critical pair of degree n is a nondegenerate local minimum of E_f, and if moreover critical pairs of degree less than n do not interpolate[3] f on T, then there is a unique critical pair of degree n.

When $f = f_\mu$ is of Markov type (1.1), critical pairs never interpolate f_μ on T by the well-known normality of multipoint Padé approximation to such functions, and the above criterion reduces to checking that each critical point is a nondegenerate local minimum[4].

The technique was first used in [10] to prove uniqueness of a critical pair, when $f = f_\mu$ is a Markov function and the support of μ does not come too close to the unit circle. We state below a slightly generalized form of this result:

[3]There is a slight abuse of language here: we say that (p^c, q^c) interpolates f at $\xi \in \overline{\mathbf{C}} \setminus U$ if p^c/q^c does.

[4]It is interesting to note that, in the context of *real* approximation on an interval, a similar approach was taken in [14] to tackle uniqueness of generalized monosplines of least $\mathcal{L}^p$ norm.

Theorem 1.1 *If μ is a positive measure supported on $[a,b] \subset (-1,1)$ where a and b satisfy*

$$b - a < \sqrt{2}(1 - \max\{a^2, b^2\}), \tag{1.4}$$

and if μ has at least n points of increase, then there is a unique critical pair in rational $\overline{\mathcal{H}}^2_{\mathbf{R}}$ approximation of degree n to f_μ defined by (1.1)*. In particular, r_n^* is unique and is the unique local best approximant to f_μ in this case.*

When $a = -b$ or $a = 0$, this is [10, thm 3] whose proof requires only a very minor modification in order to yield Theorem 1.1[5].

It was also noticed in [10] that uniqueness cannot hold without assumptions on μ. This is well illustrated by the following statement which is borrowed from [9]:

Theorem 1.2 [9, thm 3] *There exists a positive measure μ with compact support in $(-1,1)$ such that, for each odd index $n = 1, 3, 5, \ldots$, there are two distinct best rational approximants of degree n to f_μ defined by* (1.1)*.*

The measure in question is rather wild, and the question arises whether uniqueness prevails, at least for n large, when μ is sufficiently smooth but othewise supported on an arbitrary compact subinterval of $(-1,1)$. We prove in this paper that this is indeed the case, an appropriate notion of smoothness being here Szegő regularity. The latter, whose definition is recalled below, already plays an important role in the asymptotic theory of orthogonal polynomials on the unit circle and on the line [42, ch. XII], [18].

A positive measure μ on $\mathbf{R}$ is said to satisfy the *Szegő condition* if the following two assertions hold:
(i) $\mathrm{supp}(\mu)$ is a finite interval, $[a,b] \subset \mathbf{R}$.
(ii) For $d\mu(t) = \mu'(t)dt + d\mu_s(t)$ the Radon-Nykodym decomposition, with μ_s a singular measure and $\mu' \in \mathcal{L}^1[a,b]$, then

$$\int_a^b \frac{\log \mu'(t)}{\sqrt{(t-a)(b-t)}} dt > -\infty. \tag{1.5}$$

We can now state the main result of the paper as follows:

Theorem 1.3 *If μ is a positive measure on $[a,b] \subset (-1,1)$ which satisfies the Szegő condition, then there is a unique critical pair in rational $\overline{\mathcal{H}}^2_{\mathbf{R}}$ approximation of degree n to f_μ defined by* (1.1) *provided that n is large enough. In particular, as soon as $n \geq n(f_\mu)$ for some integer $n(f_\mu)$, then r_n^* is unique and is the unique local best approximant to f_μ.*

Theorem 1.3 may be seen as a natural counterpart to Theorem 1.2 since measures satisfying the Szegő condition have a density function that can only vanish on rather thin subsets of their supporting interval, while the measure constructed in the proof of Theorem 1.2 is carried by a countable set. The asymptotic nature of the conclusion in Theorem 1.3 cannot be improved: given any positive odd integer n, there exists a measure μ satisfying the hypotheses of the theorem, yet f_μ has two distinct best approximants of degree n [9, thm 4].

Although each of them implies uniqueness, the hypotheses of Theorems 1.1 and 1.3 do not look alike. Their hidden common feature is: both ensure that the numerator and the denominator of a critical pair are "coprime enough". Indeed, for any $f \in \overline{\mathcal{H}}^2_{\mathbf{R}}$ which is analytic on a neighborhood of T, it was shown in [8, thm 2.10] that a critical pair (p^c, q^c) gives rise to a nondegenerate local

[5] Apply inequality (32) of [10] to the function $(t - \frac{a+b}{2})q/\widetilde{q}$ instead of $tq/\widetilde{q}$.

minimum p_c/q_c as soon as there are bounded analytic function h_1, h_2 in U satisfying the constrained corona equation:

$$h_1 p^c + h_2 q^c = 1, \qquad \|h_1(fq^c - p^c)\|_\infty < 1/2; \tag{1.6}$$

here $\| \|_\infty$ refers to the *sup* norm on T where h_1 is to be interpreted as a nontangential limit. We shall prove that the solvability of (1.6) is not only sufficient but is in fact also *necessary* for p_c/q_c to be a nondegenerate local minimum when $f = f_\mu$ is of Markov type. Therefore it is no surprise that the proof of [10, thm 3] (that estimates in disguise the interpolation constant of the zeros of p^c and q^c) as well as the the proof of Theorem 1.2 (that ultimately rests on a de la Vallée-Poussin lower bound principle in AAK theory) may both be viewed as constructive means to solve (1.6).

The paper is organized as follows. Section 2 gives some general background on $\overline{\mathcal{H}}^2_{\mathbf{R}}$ rational approximation, and section 3 discusses critical pairs and states the index theorem, while section 4 is a brief account of Hankel operators and AAK theory. These three sections are instrumental and do not contain new results, whereas the remaining of the paper presents original material. In section 5, we express the second derivative of E_f at a critical pair in terms of the pseudo-inverse of a certain Hankel operator; using AAK theory, we then deduce a criterion for uniqueness, that extends [8, thm 2.12] and likewise requires the construction of a special rational interpolant of type $(n-2, n-1)$ associated to a critical pair of degree n. In section 6, we prove Theorem 1.3 by showing that the previous criterion asymptotically applies to the class of Markov functions whose defining measure satisfies the Szegő condition. For this we use a sharp asymptotic estimate of the error in multipoint Padé approximation to Markov functions, that was established in [43] and in a refined form in [40]. The special interpolant required by the criterion is then obtained by picking the interpolation points to be the support of a discrete measure yielding appropriate asymptotics for a certain Green potential. Finally, section 7 concludes the paper with some perspectives on more general Cauchy integrals than Markov functions, in connection with so-called complex orthogonal polynomials.

2 $\overline{\mathcal{H}}^2_{\mathbf{R}}$ rational approximation

Recall from the introduction the definitions of T, U, $\overline{\mathcal{H}}^2_{\mathbf{R}}$, $\mathcal{R}_n$, $\mathcal{P}_k$, $\mathcal{M}_k$, $\mathcal{M}^1_k$, as well as $\| \|_2$, and $\| \|_\infty$. Consider the *real* Hilbert space $\mathcal{L}^2_{\mathbf{R}}(T)$ consisting of square summable functions on the unit circle with real Fourier coefficients. The scalar product can be expressed as a line integral:

$$< f, g >= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\, d\theta \;=\; \frac{1}{2i\pi}\int_T f(z)g(1/z)\frac{dz}{z}.$$

The Hardy space $\mathcal{H}^2_{\mathbf{R}}$ of the unit disk is the closed subspace of $\mathcal{L}^2_{\mathbf{R}}(T)$ consisting of analytic functions in U of the form

$$g(z) = \sum_{k=0}^{\infty} g_k z^k,$$

where $g_k \in \mathbf{R}$ and the squared norm $\|g\|_2^2 = \sum_k g_k^2$ is finite. The orthogonal complement of $\mathcal{H}^2_{\mathbf{R}}$ in $\mathcal{L}^2_{\mathbf{R}}(T)$ is the Hardy space $\overline{\mathcal{H}}^2_{\mathbf{R}}$ of the complement of the disk already defined in the introduction: it consists of functions analytic in $\overline{\mathbf{C}} \setminus \overline{U}$ of the form

$$f(z) = \sum_{k=1}^{\infty} f_k z^{-k},$$

with $f_k \in \mathbf{R}$ and $\|f\|_2^2 = \sum_k f_k^2 < \infty$. We also introduce the space $\mathcal{L}^\infty_{\mathbf{R}}(T)$ of essentially bounded measurable functions on T with real Fourier coefficients, and the Hardy space $\mathcal{H}^\infty_{\mathbf{R}} = \mathcal{H}^2_{\mathbf{R}} \cap \mathcal{L}^\infty_{\mathbf{R}}(T)$ consisting of nontangential limits of bounded analytic functions in U. The symbol $C_{\mathbf{R}}(T)$ refers to continuous functions in $\mathcal{L}^\infty_{\mathbf{R}}(T)$. Note that we only consider functions with real Fourier coefficients; this is equivalent to let them have the conjugate symmetry $f(\overline{z}) = \overline{f(z)}$, whenever z is in the domain of definition of f.

The problem of best rational approximation of degree at most n in $\overline{\mathcal{H}}_{2,\mathbf{R}}$ can be stated as follows:

Given $f \in \overline{\mathcal{H}}^2_{\mathbf{R}}$ and $n > 0$ an integer, find a rational function $p/q \in \mathcal{R}_n$ minimizing $\|f - p/q\|_2$.

Upon changing z into $1/z$ and dividing again by z, this is easily seen to be equivalent to the corresponding problem in $\mathcal{H}^2_{\mathbf{R}}$.

We mentioned in the introduction already that a best approximant indeed exists, need not be unique, and has exact degree n unless f is itself rational of degree less than n. This maximality of the degree of a best approximant, that may be called *normality* by analogy with the terminology in Padé approximation, is in fact true for local *minima* and *maxima*. The interested reader may consult [10] for a detailed discussion of these elementary properties and further references.

For $q \in \mathcal{M}^1_n$, we define the subspace $V_q \subset \overline{\mathcal{H}}^2_{\mathbf{R}}$ of rational functions with denominator q:

$$V_q := \frac{\mathcal{P}_{n-1}}{q}. \tag{2.1}$$

When $f \in \overline{\mathcal{H}}^2_{\mathbf{R}}$ and $q \in \mathcal{M}^1_n$ is fixed, the smallest value of $\|f - p/q\|_2$ is obtained when p/q is the orthogonal projection of f onto V_q. Therefore, if we denote this projection by L_q/q where $L_q \in \mathcal{P}_{n-1}$, our approximation problem amounts to seek the minimum, with respect to $q \in \mathcal{M}^1_n$, of the function:

$$\psi_n(q) := \left\| f - \frac{L_q}{q} \right\|_2^2. \tag{2.2}$$

To characterize L_q more effectively, we need a few notations. For $p \in \mathcal{P}_k$, we define its *reciprocal* polynomial in $\mathcal{P}_k$ by

$$\widetilde{p}(z) := z^k p(1/z).$$

We offer a word of warning about this notation. If $k' > k$ and some member p of $\mathcal{P}_k$ is considered as a member of $\mathcal{P}_{k'}$ whose leading coefficients do vanish, the two definitions of $\widetilde{p}$ may be inconsistent. For that reason, we shall always specify which $\mathcal{P}_k$ is involved in this process. For any function h of a complex variable z, we set

$$\check{h}(z) = h(1/z) \quad \text{and} \quad h^\sigma(z) := \frac{1}{z} h(1/z). \tag{2.3}$$

Clearly, $h \mapsto h^\sigma$ is an involution that maps $\overline{\mathcal{H}}^2_{\mathbf{R}}$ isometrically onto $\mathcal{H}^2_{\mathbf{R}}$ and conversely. Two elementary rules of computation are

$$(h_1 h_2)^\sigma = h_1^\sigma \check{h}_2 \quad \text{and} \quad < h_1 h_2, h_3 > = < h_1, \check{h}_2 h_3 >; \tag{2.4}$$

the first is obvious by inspection, and the second holds because $\check{h} = \overline{h}$ on T if h has real Fourier coefficients. Whenever h is holomorphic in U and $q \in \mathcal{M}^1_n$, we have the well-known division formula

$$h = vq + r, \tag{2.5}$$

where the quotient v is again holomorphic in U and the remainder r is the unique polynomial in $\mathcal{P}_{n-1}$ interpolating h at the zeroes of q. Whenever $h \in \mathcal{H}^2_{\mathbf{R}}$, it is easily checked that $v \in \mathcal{H}^2_{\mathbf{R}}$ also.

We can now proceed with the announced characterization of L_q, which is just a reformulation of [44, ch.9, thm 2] in terms that do not involve the roots of q.

Proposition 2.1 [5] *For $f \in \overline{\mathcal{H}}^2_{\mathbf{R}}$ and $q \in \mathcal{M}^1_n$, let us write the division of $f^\sigma \widetilde{q}$ by q:*

$$f^\sigma \widetilde{q} = v_q q + r,$$

where $r \in \mathcal{P}_{n-1}$ and $v_q \in \mathcal{H}^2_{\mathbf{R}}$. Then $L_q = \widetilde{r}$.

Proposition 2.1 is nicely rephrased in terms of interpolation by saying that, for fixed denominator q, the best approximant interpolates f at the reflections of the roots of q across T.

In view of the Hermite formula for the remainder [44], that may be used in this context because a function in $\mathcal{H}^2_{\mathbf{R}}$ is the Cauchy integral of its boundary values on T, we obtain from Proposition 2.1 the following representation of L_q:

$$L_q(z) = \frac{1}{2\pi} \int_T \frac{f^\sigma(\xi)\widetilde{q}(\xi)}{q(\xi)} \, \frac{z^n q(\xi) - \widetilde{q}(z)}{z\xi - 1} \, d\xi, \qquad z \in U.$$

The main virtue of his formula is to show that the coefficients of L_q are smooth functions of the coefficients of q, the word *smooth* meaning troughout infinitely differentiable.

3 Critical points

For $p \in \mathcal{P}_{n-1}$ and $q \in \mathcal{M}^1_n$, we write

$$p(z) = p_{n-1}z^{n-1} + p_{n-2}z^{n-2} + \ldots + p_0, \qquad q(z) = z^n + q_{n-1}z^{n-1} + \ldots + q_0, \tag{3.1}$$

and we consider the coefficients p_j and q_k as coordinates when j and k range over $\{0, \ldots, n-1\}$.

In the introduction, a critical pair was defined as a pair $(p^c, q^c) \in \mathcal{P}_{n-1} \times \mathcal{M}^1_n$ at which all the derivatives of $E_f(p, q)$ with respect to p_j and q_k do vanish. Writing

$$E_f(p, q) = < f - p/q, f - p/q >$$

and differentiating under the integral sign, we find that E_f is smooth and that (p^c, q^c) is critical if, and only if,

$$\frac{\partial E_f}{\partial p_j}(p^c, q^c) = - < f - \frac{p^c}{q^c}, \frac{z^j}{q^c} >= 0, \qquad j \in \{0, \ldots, n-1\}, \tag{3.2}$$

$$\frac{\partial E_f}{\partial q_k}(p^c, q^c) = < f - \frac{p^c}{q^c}, \frac{p^c z^k}{(q^c)^2} >= 0, \qquad k \in \{0, \ldots, n-1\}. \tag{3.3}$$

Equation (3.2) means that $p^c = L_{q^c}$, and combining it with (3.3) yields

$$< f - \frac{L_{q^c}}{q^c}, \frac{q^c u + L_{q^c} v}{(q^c)^2} >= 0, \qquad \forall u, v \in \mathcal{P}_{n-1}. \tag{3.4}$$

Upon choosing

$$u = \frac{\partial L_q}{\partial q_k}(q^c) \quad \text{and} \quad v = -z^k, \quad k \in \{0, \ldots, n-1\},$$

we conclude that q^c is a *critical point* of ψ_n introduced in (2.2), that is, by definition, a member of $\mathcal{M}_n^1$ at which all partial derivatives $\partial\psi_n/\partial q_k$ do vanish. Conversely, every critical point q^c of ψ_n gives rise to a critical pair (L_{q^c}, q^c).

Henceforth we drop the superscript "c" for simplicity, and we say that q is critical, or a critical point, to mean that it is a critical point of ψ_n, reserving the term "critical pair" to designate the couple (L_q, q). The reader will easily check that q is a local minimum of ψ_n if, and only if, (L_q, q) is a local minimum of E_f.

A critical point q will be termed *irreducible* if L_q is prime to q and *reducible* otherwise. For irreducible critical points, the coprimeness of L_q and q allows one to put (3.4) into the more manageable form:

$$< f - \frac{L_q}{q}, \frac{p}{q^2} >= 0, \qquad \forall p \in \mathcal{P}_{2n-1}. \tag{3.5}$$

Using the division formula (2.5) with respect to q^2 rather than q, and appealing to the orthogonality of $\mathcal{H}^2_{\mathbf{R}}$ and $\overline{\mathcal{H}}^2_{\mathbf{R}}$ in $\mathcal{L}^2_{\mathbf{R}}(T)$, we see that (3.5) strengthens to

$$< f - \frac{L_q}{q}, \frac{h}{q^2} >= 0, \qquad \forall h \in \mathcal{H}^2_{\mathbf{R}}. \tag{3.6}$$

This relation is the most useful version of the irreducible critical point equation.

Critical points can be characterized by a certain divisibility property:

Proposition 3.1 [6, prop.2] *Let $f \in \overline{\mathcal{H}}^2_{\mathbf{R}}$, $q \in \mathcal{M}_n^1$, and*

$$f^\sigma \widetilde{q} = v_q q + \widetilde{L}_q \tag{3.7}$$

be the division of $f^\sigma \widetilde{q}$ by q. The following assertions are equivalent:
(i) *q is a critical point of ψ_n.*
(ii) *q divides $v_q L_q$ in $\mathcal{H}^2_{\mathbf{R}}$.*
In particular if q is an irreducible critical point, then q divides v_q.

Like Proposition 2.1, Proposition 3.1 can be rephrased in terms of interpolation. Indeed, changing z into $1/z$ transforms (3.7) into

$$f - \frac{L_q}{q} = v_q^\sigma \frac{\widetilde{q}}{q},$$

and it follows from (ii) that:

q is critical if, and only if, whenever ξ is a pole of L_q/q of order $n_1 > 0$ which is a zero of q of multiplicity $n_2 \geq n_1$, then L_q/q interpolates f at $1/\overline{\xi}$ with order $n_1 + n_2$, not withstanding the structural interpolation condition at infinity where both functions do vanish[6].

If for instance q is an irreducible critical point, this means that L_q/q interpolates f at each of the reflections of its poles across the unit circle with order twice the multiplicity of the pole (plus an additional interpolation condition at infinity in case 0 is a pole) [7]. Thus, whenever q is a critical

[6] As stated, this also characterizes critical points in $\overline{\mathcal{H}}_2$ rational approximation where the coefficients may be imaginary numbers.

[7] The corresponding property for $\mathcal{H}_2$ best rational approximants, which is easily deduced from the present one upon changing z into $1/z$, was apparently first stated in [17] when the poles are simple and in [25] generally speaking.

point, L_q/q appears to be a multipoint Padé approximant of type $(n-1,n)$ of a particular kind, having a normality defect in $\overline{C}\setminus\overline{U}$ if the critical point is reducible; by a *normality defect*, we mean that the irreducible form p_1/p_2 of L_q/q interpolates f in $\overline{C}\setminus\overline{U}$ in *more* than $\deg p_1+\deg p_2+1$ points, counting multiplicities.

We turn to differential-topological properties of critical points. Whereas everything stated so far is valid for $f\in\overline{\mathcal{H}}^2_{\mathbf{R}}$, the theory now requires the additional assumption that f is analytic in a neighborhood of $\overline{\mathbf{C}}\setminus U$. In other words, we will suppose in the sequel that f satisfies the following hypotheses:

(H) there exists $\eta>0$ such that $f(z)$ is analytic in $\{|z|>1-\eta\}$, $f(\overline{z})=\overline{f(z)}$ and $f(\infty)=0$.

Monic polynomials have been coordinatized by their coefficients except the leading one, thereby identifying $\mathcal{M}_n$ with $\mathbf{R}^n$ and $\mathcal{M}_n^1$ with an open subset. Let Δ_n denote the closure, in $\mathcal{M}_n$, of $\mathcal{M}_n^1$. It is clear that Δ_n is a compact set consisting of those monic polynomials of degree n whose roots have modulus *at most* 1. We further denote by $\partial\Delta_n$ the boundary of Δ_n in $\mathcal{M}_n$; it consists of those polynomials in Δ_n having *at least* one root of modulus 1. The importance of hypothesis (**H**) lies with the fact that it allows for us to extend the domain of definition of ψ_n as follows.

Proposition 3.2 [5, prop.2] *When* **(H)** *is satisfied, then the map $\psi_n:\mathcal{M}_n^1\to\mathbf{R}$ extends smoothly to a neighborhood of Δ_n in $\mathcal{M}_n$.*

With a slight abuse of notation, we continue to denote this extended function by ψ_n. Its actual domain depends on η appearing in (**H**), and is not important. Proposition 3.2 definitely rests on the introduction of L_q: only with this adapted numerator can we let the roots of q go to the unit circle and still get a differentiable error function $\psi_n(q)=\|f-L_q/q\|_2^2$. This is the basis for a differential-topological study, because smoothly defining ψ_n over a *compact* set, namely Δ_n, makes it possible to use degree theory. Let us first recall some classical terminology. Given a smooth map $\Phi:V\to\mathbf{R}$, where V is an open subset of $\mathbf{R}^n$, a point $x\in V$ is called *critical* if the derivative $\mathbf{D}\Phi(x):\mathbf{R}^n\to\mathbf{R}$ is the zero linear map. This definition of course agrees with the one we gave for ψ_n. A critical point is called *nondegenerate* if the second derivative at this point is a nonsingular quadratic form; it follows from the implicit function theorem that nondegenerate critical points are isolated. The *Morse index* of a nondegenerate critical point is, by definition, the number of negative eigenvalues of the second derivative. With these definitions, one has the following *global* result concerning the critical points of ψ_n which is fundamental to our approach:

Theorem 3.3 (The Index Theorem) [5, sec.5] [8] *Assume* **(H)** *holds and ψ_n has only nondegenerate critical points in Δ_n, none of which lie on $\partial\Delta_n$. Let $\mathcal{C}_f$ be the set of these critical points. Then $\mathcal{C}_f$ is finite, and denoting by $M(q)$ the Morse index of $q\in\mathcal{C}_f$, we have that*

$$\sum_{q\in\mathcal{C}_f}(-1)^{M(q)}=1. \tag{3.8}$$

In the statement of Theorem 3.3 we envisage for the first time the possibilty that ψ_n has a critical point *on* $\partial\Delta_n$ (*i.e.* having roots of modulus ≤ 1 but one or several roots of modulus precisely 1), whereas Proposition 3.1 only deals with critical points lying in $\mathcal{M}_n^1$ (*i.e.* having only roots of modulus <1). Characterizing these new critical points is a little more difficult, but can nevertheless be done as follows.

[8] With a natural definition of L_q, this theorem is in fact valid more generally in $\mathcal{L}^p_{\mathbf{R}}(T)$ rational approximation of type (m,n) when $m\geq n-1$ and $2\leq p<\infty$, including meromorphic approximation if we formally set $m=\infty$; it also extends to imaginary Fourier coefficients.

Proposition 3.4 [8, prop.2.5] *Let f satisfy hypothesis* **(H)**, *and $q \in \partial\Delta_n$. Write $q = q_T q_U$ where $q_T \in \mathcal{M}_k$ has all its roots of modulus 1 while $q_U \in \mathcal{M}^1_{n-k}$. Then, q is a critical point of ψ_n if, and only if, the following two assertions hold:*
(i) *q_U is a critical point of ψ_{n-k} in $\mathcal{M}^1_{n-k}$;*
(ii) *whenever $\xi \in T$ is a root of q_T of multiplicity n_ξ, then L_{q_U}/q_U interpolates f at ξ with order $[(n_\xi + 1)/2]$ at least, where the brackets indicate the integer part.*

In the above statement, q_U is to be interpreted as 1 and L_{q_U} as 0 in case $k = n$.

Proposition 3.4 shows that critical points lying in $\partial\Delta_n$ share one important common feature with reducible critical points lying in $\mathcal{M}^1_n$, namely if q is such a point then L_q/q is a multipoint Padé approximant with a normality defect in $\overline{C} \setminus U$. When the function f is normal with respect to multipoint Padé approximation in $\{|z| \geq 1\}$, it is therefore automatic that no critical point lies on $\partial\Delta_n$ and that each critical point in $\mathcal{M}^1_n$ is irreducible. In particular, this is the case for Markov functions whose well-known properties with respect to interpolation yield the following result:

Proposition 3.5 [10, prop.5] *Let $f = f_\mu$ be a Markov function of type* (1.1) *and assume that the support of μ contains at least n points, or equivalently that $f \notin \mathcal{R}_{n-1}$. Then the following assertions hold:*
(i) *for $q \in \Delta_n$, the quotient v_q in* (3.7) *has at most n roots in $\overline{U}$, counting multiplicities;*
(ii) *each critical point of ψ_n lies in $\mathcal{M}^1_n$ and is irreducible;*
(iii) *the roots of a critical point are simple, and lie interior to the minimal segment containing the support of μ.*

Remark: for later use, let us complement Proposition 3.5 with a result that follows readily from [10, eqn 25], namely *the critical pairs are themselves of Markov type when f is*; that is to say, the residues of L_q/q at the zeros of q are positive when q is critical.

4 Hankel operators

We need now introduce Hankel operators for which the reader may further consult [27, 29, 32, 46]. The terminology refers to those operators that can be represented, in suitable Hilbert bases, by matrices of Hankel type, in other words by infinite matrices whose entry (j, k) only depends on the sum $j + k$. Many concrete definitions meet this requirement, but we adopt one which is suitable for our purposes: given $\varphi \in \mathcal{L}^\infty_{\mathbf{R}}(T)$, we define the *Hankel operator of symbol φ* to be the bounded operator

$$\Gamma_\varphi : \overline{\mathcal{H}}^2_{\mathbf{R}} \to \overline{\mathcal{H}}^2_{\mathbf{R}}$$

given by

$$\Gamma_\varphi g = \mathbf{P}_{\overline{\mathcal{H}}^2_{\mathbf{R}}}(\varphi g^\sigma), \qquad g \in \overline{\mathcal{H}}^2_{\mathbf{R}},$$

where $\mathbf{P}_{\overline{\mathcal{H}}^2_{\mathbf{R}}}$ denotes the orthogonal projection of $\mathcal{L}^2_{\mathbf{R}}(T)$ onto $\overline{\mathcal{H}}^2_{\mathbf{R}}$. A Hankel operator is clearly bounded, and is in fact compact whenever it admits a continuous symbol (see *e.g.* [29, thm 3.14]); note that the operator only characterizes the symbol up to the addition of some $\mathcal{H}^\infty_{\mathbf{R}}$-function. Although everything in this section is valid more generally, our restriction to functions with real Fourier coefficients will confine the discussion to self-adjoint Hankel operators.

Hankel operators whose symbol is of Markov type (1.1) deserve a special mentioning because they are positive. Indeed, the matrix of Γ_{f_μ} in the basis $\{e^{-i\theta}, e^{-2i\theta}, e^{-3i\theta}, \ldots\}$ has entry (j, k)

equal to $\int t^{j+k-2}\, d\mu(t)$, the moment of order $j+k-2$ of the positive measure μ; consequently, we get by density for $g \in \overline{H}^2_{\mathbf{R}}$ that

$$< \Gamma_{f_\mu} g, g >= \int |g^\sigma(t)|^2 \, d\mu(t) \geq 0.$$

For $k = 0, 1, 2, \ldots$, the *singular values* of Γ_φ are classically defined by the formula:

$$s_k(\Gamma_\varphi) := \inf \left\{ ||\Gamma_\varphi - A||; \quad A \text{ an operator of rank } \leq k \text{ on } \overline{\mathcal{H}}^2_{\mathbf{R}} \right\},$$

where $|| \; ||$ denotes the operator norm. When $\varphi \in C_{\mathbf{R}}(T)$, the singular values are, by compactness, the square roots of the eigenvalues of $\Gamma_\varphi^* \Gamma_\varphi$ arranged in non-increasing order; a k-th *singular vector* is then any normalized eigenvector of $\Gamma_\varphi^* \Gamma_\varphi$ associated to $s_k^2(\Gamma_\varphi)$. Since Γ_φ is self-adjoint by the conjugate-symmetry of φ, the singular values in fact coincide with the *moduli* of the eigenvalues of Γ_φ, and the set of k-th singular vectors is the unit sphere of the subspace generated by the eigenvectors of Γ_φ associated to the eigenvalues $\pm s_k(\Gamma_\varphi)$.

A celebrated connexion between the spectral theory of Hankel operators and best meromorphic approximation on the unit circle is given by the Adamjan-Arov-Krein (in short: AAK) theory [1, 2]. Specifically, let us define for each $n \geq 0$ the set of *meromorphic functions with at most n poles in* $\mathcal{L}^\infty_{\mathbf{R}}(T)$*:*

$$\mathcal{H}^\infty_{\mathbf{R},n} := \{h/q; \; h \in \mathcal{H}^\infty_{\mathbf{R}}, \; q \in \mathcal{M}^1_n\},$$

thereby giving rise to a nested sequence:

$$\mathcal{H}^\infty_{\mathbf{R}} = \mathcal{H}^\infty_{\mathbf{R},0} \subset \mathcal{H}^\infty_{\mathbf{R},1} \subset \mathcal{H}^\infty_{\mathbf{R},2} \subset \ldots.$$

The main result of AAK theory asserts that

$$\inf_{g \in \mathcal{H}^\infty_{\mathbf{R},n}} \|\varphi - g\|_\infty = s_n(\Gamma_\varphi) \tag{4.1}$$

where the *infimum* is attained; moreover, if φ is continuous, then the *unique* minimizer is given by the formula

$$g^* = \varphi - \frac{\Gamma_\varphi v}{v^\sigma},$$

where v is *any* n-th singular vector of Γ_φ. The error function $\varphi - g^*$ has further remarkable properties when $\varphi \in C_{\mathbf{R}}(T) \setminus \mathcal{H}^\infty_{\mathbf{R},n}$: it has constant modulus $s_n(\Gamma_\varphi) \neq 0$ a.e. on T and, though it need not be continuous, it is quasi-continuous [31] hence has a well-defined winding number[9] $\mathbf{w}_T(f - g^*)$ over T that turns out to be at most $-2m-1$, where $m \leq n$ is the number of poles of g^* in U. These properties actually *characterize* the optimal error, and $-2m - \mathbf{w}_T(f - g^*)$ is the multiplicity of the singular value $s_n(\Gamma_\varphi) = s_m(\Gamma_\varphi)$.

Using Rouché's theorem, the peculiar properties of the error function give rise to a lower bound principle of the de la Vallée-Poussin type in AAK theory, namely any $g \in \mathcal{H}^\infty_{\mathbf{R},n}$ such that $\mathbf{w}(\varphi - g)$ is defined and not greater than $-2n-1$ has to satisfy

$$\inf_{\xi \in T} |\varphi(\xi) - g(\xi)| \leq s_n(\Gamma_\varphi).$$

[9] Quasicontinuity means that both $\varphi - g^*$ and its conjugate lie in the Douglas algebra $\mathcal{H}^\infty + C(T)$; since it has constant modulus, $\varphi - g^*$ is thus invertible in this algebra, and the winding number can then be defined as that of its Poisson integral on circles of radius $1 - \varepsilon$ for small $\varepsilon > 0$ [38].

In the present work, we need not consider refined regularity conditions to define the winding number: our use of AAK theory will be limited to the case where φ is analytic in a neighborhood of T, in which the reflexion principle shows that g^* also has this property [36, 30]. In fact, we even confine ourselves to situations where φ is meromorphic in a neighborhood of $\overline{U}$, so that $\mathbf{w}_T(\varphi - g^*)$ is equal to the number of zeros minus the number of poles of $\varphi - g^*$ in U.

To conclude this section, we discuss the structure of Γ_φ when φ belongs to $\mathcal{H}^\infty_{\mathbf{R},m}$ for some $m \geq 0$. Assume that m is the smallest non-negative integer with this property. If $m = 0$, then $\Gamma_\varphi = 0$. Otherwise, using the division (2.5), we can write

$$\varphi = \frac{p}{q} + h \qquad \text{with } h \in \mathcal{H}^\infty_{\mathbf{R}}, \quad p \in \mathcal{P}_{m-1}, \quad q \in \mathcal{M}^1_m, \quad p, q \text{ coprime.}$$

Then $\Gamma_\varphi = \Gamma_{p/q}$ has rank m [29, thm 3.11] and its range is V_q defined in (2.1); the kernel of Γ_φ is the orthogonal complement of its range in $\overline{\mathcal{H}}^2_{\mathbf{R}}$:

$$\mathbf{Ker}\Gamma_\varphi = V_q^\perp = \{\frac{\widetilde{q}}{q}h; \quad h \in \overline{\mathcal{H}}^2_{\mathbf{R}}\}, \tag{4.2}$$

whose representation as the set of multiples of $\widetilde{q}/q$ emphasizes its invariance under multiplication by z^{-1} in connection with Beurling's theorem. In particular, Γ_φ induces an isomorphism $V_q \to V_q$ and, as it will appear in a forthcoming computation, we give a name to the *inverse* of this isomorphism by calling it the *pseudo-inverse* of Γ_φ, denoted by $\Gamma^\sharp_\varphi$.

We are now through with the preliminaries, and ready to make our way towards the proof of Theorem 1.3.

5 A criterion for uniqueness

To approach the uniqueness issue along the lines indicated in the introduction, we need to produce a sufficient condition for a critical point $q \in \mathcal{M}^1_n$ to be a nondegenerate local minimum of ψ_n. First of all we know, from section 2, that this point must be irreducible, so we assume this beforehand. Then q divides v_q by Proposition 3.1, so that (3.7) can be rewritten as

$$f^\sigma \widetilde{q} = w_q q^2 + \widetilde{L}_q, \quad w_q \in \mathcal{H}^2_{\mathbf{R}}. \tag{5.1}$$

In the forthcoming discussion, w_q will always designate, like in (5.1), the quotient of the division of $f^\sigma \widetilde{q}$ by q^2. Note, because of hypothesis **(H)**, that w_q is analytic on a neighborhood of $\overline{U}$ since f^σ does.

If we differentiate with respect to $\chi \in \mathcal{M}^1_n$ the relation

$$< f - L_\chi/\chi, p/\chi >= 0 \qquad \text{for fixed } p \in \mathcal{P}_{n-1},$$

and if we evaluate at $\chi = q$ taking (3.6) into account, we obtain as in [10, eqn 17] that each partial derivative $\partial(L_q/q)/\partial q_k$ belongs to $V_q^\perp$. Because these partial derivatives are of the form p/q^2 where $p \in \mathcal{P}_{2n-1}$, we deduce in view of (4.2), like in [10, eqn 19], that there are polynomials $\nu_k \in \mathcal{P}_{n-1}$ satisfying:

$$\frac{\partial}{\partial q_k}\left(\frac{L_q}{q}\right) = \frac{\widetilde{q}}{q^2}\nu_k, \qquad k \in \{0, ..., n-1\},$$

or equivalently

$$q\frac{\partial L_q}{\partial q_k} - z^k L_q = \widetilde{q}\nu_k, \qquad k \in \{0, ..., n-1\}. \tag{5.2}$$

As $\mathcal{M}_n^1$ was coordinatized in (3.1) by its coefficients q_k, the tangent space to $\mathcal{M}_n^1$ at q identifies naturally with $\mathcal{P}_{n-1}$, a typical tangent vector being customarily written as

$$\delta q = \sum_{k=0}^{n-1} \lambda_k z^k, \quad \lambda_k \in \mathbf{R}. \tag{5.3}$$

As a consequence of the coprimeness of q and L_q, it is immediately checked from (5.2) that the polynomials $\nu_1, \ldots, \nu_n$ are linearly independent over $\mathbf{R}$ and thus form a basis of $\mathcal{P}_{n-1}$. This allows for us to define a linear isomorphism of $\mathcal{P}_{n-1}$ by associating to each polynomial of degree at most $n-1$ the linear combination of the ν_k obtained from the negative of its coefficients:

$$\delta q = \sum_{k=0}^{n-1} \lambda_k z^k \quad \longleftrightarrow \quad \nu = -\sum_{k=0}^{n-1} \lambda_k \nu_k. \tag{5.4}$$

This one-to-one correspondence plays an important role when analysing the second derivative of ψ_n. To avoid heavy notations, we do not make the dependency on δq explicit, but we rather *reserve the symbols ν and λ_k to designate the quantities associated to δq by* (5.4).

Our first step towards a computation of the second derivative is a lemma connecting δq and ν through the pseudo-inverse of a certain Hankel operator. For this, it will be convenient to assume that the error function $f - L_q/q$ has no zero on T, this generic case being sufficient for our purposes. It is of course equivalent to require that $f^\sigma - \widetilde{L_q}/\widetilde{q}$ hence also w_q has no zero on T. Hence if we number the zeros of w_q in U, counting multiplicities, as $z_1, z_2 \ldots, z_m$, we can write the inner-outer factorization $w_q = b_q \Omega_q$ where

$$b_q(z) := \prod_{l=1}^{m} \frac{z - z_l}{1 - \overline{z}_l z}$$

is the *Blaschke product* associated with the z_l normalized so that $b_q(1) = 1$, while Ω_q is the *outer factor of* w_q which is analytic and zero-free in a neighborhood of $\overline{U}$ and has the same modulus as w_q pointwise on T (if w_q has no zero in $\overline{U}$, we set of course $b_q = 1$ and $\Omega_q = w_q$). We then define

$$\Phi_q(z) = \frac{L_q}{\Omega_q q \widetilde{q}}, \tag{5.5}$$

a function that will be central to the computations in this section. Since $1/\Omega_q$ and $1/\widetilde{q}$ are analytic in a neighborhood of $\overline{U}$ by construction, Φ_q assumes the form Φ_1/q, where Φ_1 is analytic in a neighborhood of $\overline{U}$ and nonzero at every root of q by the coprimeness of q and L_q. In view of the discussion in section 4, the Hankel operator Γ_{Φ_q} maps $\overline{\mathcal{H}}^2_{\mathbf{R}}$ onto V_q, and its pseudo inverse $\Gamma^\sharp_{\Phi_q} : V_q \to V_q$ is an isomorphism.

Lemma 5.1 *Let f satisfy hypothesis* **(H)**, *and $q \in \mathcal{M}_n^1$ be an irreducible critical point of ψ_n such that $f - L_q/q$ has no zero on T. With the notations introduced above, we get for any $\delta q \in \mathcal{P}_{n-1}$ that*

$$(\Omega_q \delta q)^\sigma - \Gamma^\sharp_{\Phi_q} \frac{\nu}{q} \ \in \ V_q^\perp. \tag{5.6}$$

Proof. Let us linearly combine the equations (5.2) with the coefficients λ_k. This yields a relation of the form

$$\delta q \, L_q = \widetilde{q}\nu + qh, \qquad h \in \mathcal{H}^2_{\mathbf{R}},$$

or equivalently

$$\frac{L_q}{q}\frac{\delta q}{\widetilde{q}} = \frac{\nu}{q} + g, \qquad g = \frac{h}{\widetilde{q}} \in \mathcal{H}^2_{\mathbf{R}}. \tag{5.7}$$

Using the definition of the Hankel operator and formula (5.5), we compute:

$$\Gamma_{\Phi_q}(\Omega_q \delta q)^\sigma = \mathbf{P}_{\overline{\mathcal{H}}^2_{\mathbf{R}}}(\Phi_q \Omega_q \delta q) = \mathbf{P}_{\overline{\mathcal{H}}^2_{\mathbf{R}}}\left(\frac{L_q}{q}\frac{\delta q}{\widetilde{q}}\right) = \frac{\nu}{q},$$

where the last equality follows from (5.7). Therefore, by definition of $\Gamma^\sharp_{\Phi_q}$, we get

$$(\Omega_q \delta q)^\sigma = \Gamma^\sharp_{\Phi_q}\frac{\nu}{q} + u, \qquad u \in V_q^\perp,$$

which is the desired conclusion. ■

We turn to the second derivative of ψ_n at q which is, by definition, the quadratic form on the tangent space given by

$$H(\delta q, \delta q) := \sum_{j,k} \lambda_j \lambda_k \frac{\partial^2}{\partial q_j \partial q_k} < f - \frac{L_q}{q}, f - \frac{L_q}{q} > .$$

We shall presently compute this quadratic form using the Hankel operator introduced in the preceding lemma.

Proposition 5.2 *Let f satisfy hypothesis* **(H)**, *and $q \in \mathcal{M}^1_n$ be an irreducible critical point of ψ_n such that $f - L_q/q$ has no zero on T. Notations being as above, the second derivative of ψ_n at q can be expressed by the formula:*

$$H(\delta q, \delta q) = 2 < \left(I - 2M_{\check{b}_q}\Gamma^\sharp_{\Phi_q}\right)\frac{\nu}{q}, \frac{\nu}{q} >, \tag{5.8}$$

where I denotes the identity mapping and $M_{\check{b}_q} : \overline{\mathcal{H}}^2_{\mathbf{R}} \to \overline{\mathcal{H}}^2_{\mathbf{R}}$ the multiplication by $\check{b}_q$.

Proof. Let

$$h_1 L_q + h_2 q = 1, \qquad h_1, h_2 \in \mathcal{H}^2_{\mathbf{R}}, \tag{5.9}$$

be a Bezout relation between L_q and q. Such a relation certainly exists since L_q and q are coprime polynomials. Our starting point will be the formula

$$\frac{1}{2}H(\delta q, \delta q) = \left\|\frac{\nu}{q}\right\|^2 - 2 < f - \frac{L_q}{q}, \frac{\widetilde{q}^2 h_1 \nu^2}{q^3} >, \tag{5.10}$$

which is [10, eqn 22] and results from a straightforward differentiation under the integral sign using (3.6). In view of the latter, we remark that the numerator $\widetilde{q}^2 h_1 \nu^2$ of the second argument in the scalar product only matters up to a multiple of q in $\mathcal{H}^2_{\mathbf{R}}$, and this is why the exact representative of h_1 *modulo* $q\mathcal{H}^2_{\mathbf{R}}$ is unimportant in (5.9) to evaluate (5.10). Observing from (5.7) and (5.9) that

$$\delta q - \widetilde{q} h_1 \nu \in q\mathcal{H}^2_{\mathbf{R}}, \tag{5.11}$$

we may take advantage of the preceding remark to rewrite the scalar product in the right-hand side of (5.10) in the form

$$2 < f - \frac{L_q}{q}, \frac{\widetilde{q}}{q^3}\delta q\, \nu > . \tag{5.12}$$

In another connection, applying σ to (5.1) yields

$$f - \frac{L_q}{q} = w_q^\sigma \frac{\widetilde{q}}{q} \check{q}, \tag{5.13}$$

and using that $|\widetilde{q}/q| = 1$ pointwise on T together with (5.13) allows us to express (5.12) as

$$2 < f - \frac{L_q}{q}, \frac{\widetilde{q}}{q^3} \delta q \, \nu >= 2 < w_q^\sigma \check{q}, \frac{\delta q}{q} \frac{\nu}{q} >= 2 < (w_q \delta q)^\sigma, \frac{\nu}{q} >,$$

where the last equality follows from (2.4). Altogether, we have shown that

$$\frac{1}{2} H(\delta q, \delta q) = \left\| \frac{\nu}{q} \right\|^2 - 2 < (w_q \delta q)^\sigma, \frac{\nu}{q} > . \tag{5.14}$$

Now, it is clear from (4.2) that $V_q^\perp$ is invariant under multiplication by bounded $\overline{\mathcal{H}}^2_{\mathbf{R}}$ functions. Hence applying $M_{\check{b}_q}$ to both sides of (5.6) furnishes

$$(w_q \delta q)^\sigma = M_{\check{b}_q} \Gamma^\sharp_{\Phi_q} \frac{\nu}{q} + u_1, \qquad u_1 \in V_q^\perp, \tag{5.15}$$

and substituting (5.15) into (5.14) yields (5.8). ∎

We can now give sufficient conditions for an irreducible critical point to be a nondegenerate local minimum, that connect to the approximation error $f - L_q/q$, and improve over [8, lemma 2.11] when this error has no zero on T.

Theorem 5.3 *Let f satisfy hypothesis* **(H)** *and $q \in \mathcal{M}_n^1$ be an irreducible critical point of ψ_n. Assume that $f - L_q/q$ has no zero on T, and let Φ_q be defined as in* (5.5)*. Then, each of the following three conditions is equivalent to the other two and implies that the second derivative of ψ_n at q is positive definite.*
(i) *the Hankel operator Γ_{Φ_q} has all its nonzero singular values strictly larger than* 2*;*
(ii) *there is a function $\beta \in H^\infty_{\mathbf{R},n-1} \cap C_{\mathbf{R}}(T)$ satisfying*

$$2|f - L_q/q| < |\beta - L_q/q| \qquad \text{everywhere on} \quad T, \tag{5.16}$$

and such that the winding number $\mathbf{w}_T(f - \beta)$ of the curve $(f - \beta)(T)$ around the origin is less than or equal to $1 - 2n$.
(iii) *there exist $h_1, h_2 \in \mathcal{H}^\infty_{\mathbf{R}}$ such that*

$$h_1 L_q + h_2 q = 1 \tag{5.17}$$

and

$$\|h_1(fq - L_q)\|_\infty < 1/2. \tag{5.18}$$

Proof. We show that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i).

Assume that (i) holds, and let g^* be a best approximant to Φ_q from $H^\infty_{\mathbf{R},n-1}$. We set $\beta = \Omega_q \widetilde{q} g^*$ which is again a member of $H^\infty_{\mathbf{R},n-1}$ because $\Omega_q \widetilde{q}$ is analytic in a neighborhood of $\overline{U}$. By (i) and the elements of AAK theory reported in section 4, we know that g^* is analytic in a neighborhood of T

and that $|\Phi_q - g^*|$ is constant and strictly larger than 2 on T. Consequently β is also analytic on a neighborhood of T, and (5.16) holds because $|f - L_q/q| = |\Omega_q \widetilde{q}|$ pointwise on T in view of (5.13). Now, it follows from the very definition of β and Ω_q that

$$L_q/q - \beta = \Omega_q \widetilde{q}(\Phi_q - g^*).$$

Moreover, as $\Omega_q \widetilde{q}$ has no zero in $\overline{U}$, we get $\mathbf{w}_T(\Omega_q \widetilde{q}) = 0$ and therefore

$$\mathbf{w}_T(L_q/q - \beta) = \mathbf{w}_T(\Omega_q \widetilde{q}) + \mathbf{w}_T(\Phi_q - g^*) = \mathbf{w}_T(\Phi_q - g^*) \leq 1 - 2n, \tag{5.19}$$

where the last inequality follows from the winding number property of the error in AAK theory. In another connection, we can write

$$|f - \beta - (L_q/q - \beta)| = |f - L_q/q| < |L_q/q - \beta| \tag{5.20}$$

as this inequality is weaker than (5.16). Hence

$$\mathbf{w}_T(f - \beta) = \mathbf{w}_T(Lq/q - \beta)$$

by the Rouché theorem, and this number is at most $1 - 2n$ by (5.19) so that (ii) indeed holds.

Assume next that (ii) holds, and write $\beta = \beta_1/\chi$ where $\beta_1 \in H^\infty_{\mathbf{R}} \cap C_{\mathbf{R}}(T)$ and $\chi \in \mathcal{M}^1_{n-1}$. Set

$$h_1 = \frac{\chi}{L_q\chi - \beta_1 q} \quad \text{and} \quad h_2 = -\frac{\beta_1}{L_q\chi - \beta_1 q}.$$

The function $L_q\chi - \beta_1 q$ is analytic in U, continuous in $\overline{U}$, and it cannot vanish on T by (5.16). Its winding number is thus well-defined and may be decomposed as

$$\mathbf{w}_T(L_q\chi - \beta_1 q) = \mathbf{w}_T(L_q/q - \beta) + \mathbf{w}_T(\chi q). \tag{5.21}$$

On the one hand, we deduce as before from (5.16) *via* (5.20) that

$$\mathbf{w}_T(Lq/q - \beta) = \mathbf{w}_T(f - \beta)$$

which is at most $1 - 2n$ by assumption; on another hand, $\chi q \in \mathcal{M}^1_{2n-1}$ has winding number $2n - 1$. Therefore, it follows from (5.21) that $\mathbf{w}_T(L_q\chi - \beta_1 q)$, which is the number of zeros of the analytic function $L_q\chi - \beta_1 q$ in U, is negative or zero. Hence $L_q\chi - \beta_1 q$ has no zeros in $\overline{U}$, its inverse belongs to $H^\infty_{\mathbf{R}}$, and h_1, h_2 are certainly functions in $H^\infty_{\mathbf{R}}$ for which (5.17) holds. Moreover, we get from (5.16) and the definition of h_1 that

$$|h_1(fq - L_q)| = \frac{|f - L_q/q|}{|L_q/q - \beta|} < 1/2 \text{ on } T,$$

so that (5.18) is satisfied and (iii) holds.

Assume, finally, that (iii) holds. Picking $\nu, p \in \mathcal{P}_{n-1}$ and letting δq be associated to ν *via* the isomorphism (5.4), we deduce from Lemma 5.1 that

$$< \Gamma^\sharp_{\Phi_q} \frac{\nu}{q}, \frac{p}{q} > = < (\Omega_q \delta q)^\sigma, \frac{p}{q} > = < \Omega^\sigma_q, \delta q \frac{p}{q} >, \tag{5.22}$$

where the last equality follows from (2.4). In another connection, we get upon multiplying (5.7) by h_1 and using (5.17) that

$$\widetilde{q} h_1 \nu - \delta q \in q\mathcal{H}^2_{\mathbf{R}}.$$

Now, if δq gets computed *modulo* $q\mathcal{H}^2_{\mathbf{R}}$ only, the value of the third scalar product in (5.22) remains unchanged since $\Omega_q^\sigma \in \overline{\mathcal{H}}^2_{\mathbf{R}}$ is orthogonal to $\mathcal{H}^2_{\mathbf{R}}$ in $\mathcal{L}^2_{\mathbf{R}}(T)$. Therefore we may substitute $\widetilde{q}h_1\nu$ for δq in this expression to obtain:

$$< \Gamma^\sharp_{\Phi_q}\frac{\nu}{q}\,,\,\frac{p}{q} >=< \Omega_q^\sigma\,,\,\widetilde{q}h_1\nu\frac{p}{q} >=< (\Omega_q q\widetilde{q}h_1)^\sigma\,,\,\frac{\nu p}{q^2} >, \tag{5.23}$$

where we used (2.4) again. Applying successively the Schwartz and the Hőlder inequality to (5.23) yields

$$\left|< \Gamma^\sharp_{\Phi_q}\frac{\nu}{q}\,,\,\frac{p}{q} >\right| \le \|(\Omega_q q\widetilde{q}h_1)^\sigma\|_\infty \left\|\frac{\nu}{q}\right\|_2 \left\|\frac{p}{q}\right\|_2.$$

As $|\Omega_q| = |w_q|$ on T, and since both the σ and the check operation preserve the modulus pointwise on the unit circle, we get from (5.13) that

$$|(\Omega_q q\widetilde{q}h_1)^\sigma| = |h_1(fq - L_q)| \quad \text{on} \quad T$$

and we conclude, in view of (5.18), that

$$\left|< \Gamma^\sharp_{\Phi_q}\frac{\nu}{q}\,,\,\frac{p}{q} >\right| < \frac{1}{2} \left\|\frac{\nu}{q}\right\|_2 \left\|\frac{p}{q}\right\|_2. \tag{5.24}$$

Because ν/q and p/q were arbitrary members of V_q, equation (5.24) means that each eigenvalue of the self-adjoint operator $\Gamma^\sharp_{\Phi_q}$ is strictly less than $1/2$ in modulus. But these eigenvalues are the inverses of the nonzero eigenvalues of Γ_{Φ_q}, so it is equivalent to assert that the nonzero singular values of Γ_{Φ_q} are strictly greater than 2, thereby proving (i).

Having shown the equivalence of (i), (ii), and (iii), it remains for us to establish that one (and thus any) of these conditions entails the positive definiteness of the second derivative of ψ_n at q. But if (ii) holds, we just saw that the eigenvalues of $\Gamma^\sharp_{\Phi_q}$ are strictly less than $1/2$ in modulus and, since $M_{\check{b}_q}$ is an isometry, it is clear from (5.8) and the Schwartz inequality that H is indeed positive definite in this case. ■

Remarks. It follows from the proof that the functions β, h_1, and h_2 appearing in conditions (ii) and (iii) may as well be assumed analytic on a neighborhood of T; this regularity is of course induced by the hypothesis **(H)**. In another connection, it was noticed in [8, thm 2.10] already that (iii) implies the positive definiteness of the second derivative, and it was proved in [8, lemma 2.11] that it is implied by a stronger version of (ii) in which the constant 2 is replaced by 3. Although the present improvement may seem minor, it is actually crucial to demonstrate Theorem 1.3. Condition (iii) will not be used in what follows, and could have been omitted from the exposition since the implication (ii) $\Longrightarrow$ (i) follows directly from the de la Vallée-Poussin lower bound principle in AAK approximation. We did nevertheless include it, because it illuminates the role of coprimeness between the numerator and the denominator in $\overline{\mathcal{H}}^2_{\mathbf{R}}$ rational approximation and proposes a qualitative version of normality for this problem.

It is natural to ask how far the sufficient conditions of Theorem 5.3 are from being necessary. When f is of Markov type, these conditions turn out to be sharp:

Corollary 5.4 *Let f_μ be of Markov type* (1.1)*, and $q \in \mathcal{M}^1_n$ be a critical point of ψ_n. Then q is irreducible and $f_\mu - L_q/q$ has no zero and T. Moreover, q is a nondegenerate local minimum if, and only if, the equivalent conditions of Theorem 5.3 hold.*

Proof. By Proposition 3.5, we know that q is irreducible and that $v_q = w_q q$ has at most n zeros in $\overline{U}$, hence w_q has no zero in $\overline{U}$. This implies on the one hand that $f_\mu - L_q/q$ has no zero on T, and on the other hand that $b_q = 1$ and $\Omega_q = w_q$. Consequently $M_{\check{b}_q} = I$ and, by (5.8), the second derivative of ψ_n at q is positive definite if, and only if, the positive eigenvalues of $\Gamma^{\sharp}_{\Phi_q}$ are strictly less than 1/2. An equivalent requirement is that the strictly positive eigenvalues of Γ_{Φ_q} be strictly larger than 2. *We claim that the eigenvalues of* Γ_{Φ_q} *are all nonnegative*, and this will achieve the proof because the above requirement will become condition (i) of Theorem 5.3.

Let $[a, b] \subset (-1, 1)$ be the minimal segment containing the support of μ. By Proposition 3.5 (iii), we can write

$$q(z) = \prod_{k=1}^{n} (z - \xi_k)$$

where the ξ_k are distinct real numbers in $[a, b]$. Singling out the singular part of $\Phi_q = L_q/w_q q\widetilde{q}$ in U, we can write

$$\Phi_q(z) = \sum_{k=1}^{n} \frac{x_k}{z - \xi_k} + h, \qquad x_k \in \mathbf{R}, \quad h \in \mathcal{H}^\infty_{\mathbf{R}},$$

the x_k being given by the formula:

$$x_k = \frac{L_q(\xi_k)}{q'(\xi_k)\widetilde{q}(\xi_k)w_q(\xi_k)}, \tag{5.25}$$

where the superscript $'$ indicates the derivative. By the remark after Proposition 3.5, we know that $L_q(\xi_k)/q'(\xi_k) > 0$. In another connection it is clear that $\widetilde{q}(\xi_k) > 0$ because $\widetilde{q}$ has no root in U and assumes the value 1 at 0. The fact that $w(\xi_k) > 0$ follows from the formula:

$$w_q(\xi_k) = \frac{1}{\widetilde{q}(\xi_k)} \int \frac{q^2(t)}{\widetilde{q}^2(t)} \frac{d\mu(t)}{1 - t\xi_k},$$

which is easily obtained by applying successively the Fubini and the residue theorems to the Hermite integral representation for the quotient of a division (see [10, eqn 24]). Altogether, we see that $x_k > 0$ for all k, hence

$$\varphi = \sum_{k=1}^{n} \frac{x_k}{z - \xi_k}$$

is a rational function of Markov type (1.1) and Γ_φ is then a positive operator as was noticed in section 4. But $\Gamma_\varphi = \Gamma_{\Phi_q}$ *which proves the claim.* ∎

Let us conclude this section by gathering up what we did into a criterion for uniqueness of a critical point, that will be used in the next section to prove Theorem 1.3.

Theorem 5.5 (Uniqueness criterion) *Assume that* **(H)** *holds, and that all critical points of* ψ_n *on* Δ_n *lie in* $\mathcal{M}^1_n$ *and are irreducible. Suppose further, whenever* q *is such a critical point, that* $f - L_q/q$ *has no zeros on* T *and that there is a function* $\beta_q \in H^\infty_{\mathbf{R},n-1} \cap C_{\mathbf{R}}(T)$ *satisfying*

$$2|f - L_q/q| < |\beta_q - L_q/q| \qquad \text{everywhere on} \quad T, \tag{5.26}$$

as well as

$$\mathbf{w}_T(f - \beta_q) \leq 1 - 2n. \tag{5.27}$$

Then ψ_n *has a unique critical point* q^* *in* $\mathcal{M}^1_n$, *and* L_{q^*}/q^* *is the unique local (hence also global) best approximant to* f *from* $\mathcal{R}_n \cap \overline{\mathcal{H}}^2_{\mathbf{R}}$.

Proof. In view of Theorem 5.3, the hypotheses are to the effect that each critical point is a nondegenerate local minimum and therefore has Morse index 0. Since there are no critical points on $\partial\Delta_n$ by assumption, the conclusion follows from Theorem 3.3. ∎

6 Asymptotic uniqueness of a critical point for Markov functions

This section is devoted to the proof of Theorem 1.3. *From now on, we set accordingly $f = f_\mu$ in the definition* (2.2) *of ψ_n, where it is assumed that μ is a positive measure on $[a,b] \subset (-1,1)$ satisfying the Szegő condition* (1.5). Because the theorem we want to prove is of asymptotic nature, it will be convenient to make the dependency on n explicit and to write q_n for a critical point of ψ_n in $\mathcal{M}_n^1$. We also put for simplicity

$$I = [a,b] \quad \text{and} \quad I^{-1} := \{x \in \overline{\mathbf{R}};\ x^{-1} \in I\};$$

note that I^{-1} contains ∞ if I contains 0.

Proof of Theorem 1.3.

From Proposition 3.5 (*ii*), we know that each critical point q_n of ψ_n on Δ_n lies in $\mathcal{M}_n^1$ and is irreducible. We also know from Corollary 5.4 that $f_\mu - L_{q_n}/q_n$ has no zero on T. To each sequence $\{q_n\}$ of critical points indexed by $n \in \mathbf{N}$, we will associate a sequence $\{\beta_{q_n}\}$ of functions in $H^\infty_{\mathbf{R},n-1} \cap C_{\mathbf{R}}(T)$ satisfying (5.26) and (5.27) provided that n is large enough. In view of Theorem 5.5, this will prove Theorem 1.3.

We shall *design* β_{q_n} to be a multipoint Padé approximant to f_μ of type $(n-2, n-1)$, that interpolates this function in $n-1$ points

$$y_{n,1}, y_{n,2}, \ldots, y_{n,n-1} \in I^{-1}$$

with multiplicity 2 at each of them (so that we have indeed $2n-2$ interpolation conditions in total). By classical properties of Padé approximants to Markov functions [21, 41], such a β_{q_n} uniquely exists when the $y_{n,j}$ are given, and we can write

$$\beta_{q_n} = \frac{p_{n-2}}{\chi_{n-1}}, \qquad p_{n-2} \in \mathcal{P}_{n-2}, \chi_{n-1} \in \mathcal{M}_{n-1}^1, \qquad p_{n-2} \text{ and } \chi_{n-1}\text{coprime},$$

where χ_{n-1} has all its roots on I. In particular, we see that

$$\beta_{q_n} \in H^\infty_{\mathbf{R},n-1} \cap C_{\mathbf{R}}(T).$$

Let us number the interpolation points so that $y_{n,1}, \ldots, y_{n,k}$, say are *finite*, while $y_{n,k+1} = \ldots = y_{n,n-1} = \infty$ for some $k \in \{0 \ldots n-1\}$. Let further

$$\omega_{2k}(z) = \prod_{j=1}^{k} (z - y_{n,j})^2 \in \mathcal{M}_{2k}^1$$

designate the monic polynomial that vanishes with order 2 at each *finite* $y_{n,j}$ (if $k = 0$, we set $\omega_0 = 1$). A direct computation shows (see *e.g.* [21, eqn 9] or [41, lemma 6.1.2.]) that the error function can be expressed as

$$f_\mu(z) - \beta_{q_n}(z) = \frac{\omega_{2k}(z)}{\chi_{n-1}^2(z)} \int_I \frac{\chi_{n-1}^2(t)}{\omega_{2k}(t)} \frac{d\mu(t)}{z-t}, \qquad z \in \overline{\mathbf{C}} \setminus I,$$

hence applying the σ operation yields:

$$(f_\mu - \beta_{q_n})^\sigma(z) = \frac{z^{2(n-1-k)}\widetilde{\omega_{2k}}(z)}{\widetilde{\chi}^2_{n-1}(z)}\,\mathcal{J}(z)$$

where we have put

$$\mathcal{J}(z) = \int_I \frac{\chi^2_{n-1}(t)}{\omega_{2k}(t)}\frac{d\mu(t)}{1-zt}, \qquad z \in \overline{\mathbf{C}} \setminus I^{-1}.$$

The integral $\mathcal{J}(z)$ is zero-free in $\overline{U}$, because if it did vanish at z_0 the relation

$$\frac{\mathcal{J}(z_0) + \mathcal{J}(\overline{z_0})}{2} = 0$$

would lead to

$$\int_I \frac{\chi^2_{n-1}(t)(1 - t\mathbf{Re}z_0)}{\omega_{2k}(t)|1 - z_0 t|^2}\,d\mu(t) = 0$$

which is impossible since the integrand is strictly positive almost everywhere on I. Hence $(f_\mu - \beta_{q_n})^\sigma$, which is analytic in $\overline{\mathbf{C}} \setminus I^{-1} \supset \overline{U}$, has no zero on T and exactly $2n-2$ zeros in U counting multiplicities, namely a zero of order $2(n-1-k)$ at the origin and a double zero at each point $y^{-1}_{n,j}$ for $1 \leq j \leq k$. Its winding number on T is therefore well-defined and equal to $2n-2$, so that conjugating and dividing by $e^{i\theta}$ gives

$$\mathbf{w}_T(f - \beta_{q_n}) = 1 - 2n.$$

Thus our choice of β_{q_n} meets (5.27) no matter how the interpolation points $y_{n,j} \in I^{-1}$ are chosen, and it remains for us to choose them so that (5.26) holds for all n sufficiently large.

Let us rewrite (5.26) as

$$\left|1 - \frac{(f - \beta_{q_n})(z)}{(f - L_{q_n}/q_n)(z)}\right| > 2, \qquad \forall z \in T. \tag{6.1}$$

A key ingredient to estimate the left hand-side of (6.1) is a precise asymptotic formula for the error in rational interpolation to Markov functions whose defining measure satisfies the Szegő condition. This formula was first derived in [43] as a consequence of the asymptotic theory of orthogonal polynomials with varying weights under the additional assumption that μ is absolutely continuous, and was then generalized in [40] to measures having a singular part. Let us mention that [40] also weakens considerably the restrictions on the interpolation points by allowing them to approach the support of μ in a controlled manner, but this is not a concern here. To state the result, we need to introduce two functions. The first one is the conformal map Ψ from $\overline{\mathbf{C}} \setminus I$ onto U normalized so that $\Psi(\infty) = 0$ and $\Psi(b^+) = 1$, whose explicit form is

$$\Psi(z) = \frac{2}{b-a}\left(z - \frac{a+b}{2} - \sqrt{(z-a)(z-b)}\right) \tag{6.2}$$

where the principal branch of the square root was selected. The second function we need to introduce is the Szegő function for the measure μ, defined by

$$D_\mu(z) = \exp\left(\sqrt{(z-a)(z-b)}\frac{1}{2\pi}\int_I \frac{\log \mu'(t)}{\sqrt{(t-a)(b-t)}}\frac{dt}{z-t}\right), \qquad z \in \overline{\mathbf{C}} \setminus I. \tag{6.3}$$

The Szegő function D_μ is characterized by the following properties:
(i) D_μ is an analytic function in $\overline{\mathbf{C}} \setminus I$, and $D_\mu(z) \neq 0$ for all $z \in \overline{\mathbf{C}} \setminus I$.
(ii) $|D_\mu(t)|^2 = \mu'(t)$ almost everywhere on I, where $|D_\mu(t)|$ denotes the limit of $|D_\mu(z)|$ as z tends nontangentially to $t \in (a,b)$ ($D_\mu(t)$ itself is multiply defined as the limits of $D_\mu(z)$ from above and below the cut I will be conjugate to each other). The announced asymptotic estimate can now be formulated as follows.

Theorem 6.1 [43, Chapter 16][10][40, thm 2] *Let f_μ be a Markov function of type* (1.1) *whose defining measure μ is supported on some real interval $[a,b]$ where it satisfies the Szegő condition. For each $n \in \mathbf{N}$, let $S_n = \{z_{n,1}, \ldots, z_{n,2n}\}$ be a set of $2n$ non-necessarily distinct points in $\overline{\mathbf{C}} \setminus [a,b]$ which is invariant under conjugation, and assume that $\cup S_n$ has no accumulation point on $[a,b]$. Let further $R_{n-1,n}$ be the rational interpolant of type $(n-1,n)$ to f_μ in the points of S_n. Then the asymptotic relation*

$$f_\mu(z) - R_{n-1,n}(z) = (1+o(1))2\pi\Psi(z)D_\mu^2(z)\prod_{j=1}^{2n}\frac{\Psi(z)-\Psi(z_{n,j})}{1-\Psi(z_{n,j})\Psi(z)}, \qquad n \to \infty, \tag{6.4}$$

holds locally uniformly in $\overline{\mathbf{C}} \setminus [a,b]$.

Remark. Whenever $z_{n,j} = \infty$, the corresponding factor of the Blaschke product in the right-hand side of (6.4) reduces to $\Psi(z)$.

Theorem 6.1 allows for us to evaluate the *ratio* $(f - \beta_{q_n})/(f - L_{q_n}/q_n)$ appearing in (6.1). To this effect, we let q_n be a critical point of ψ_n and we factorize it as

$$q_n(z) = \prod_{k=1}^{n}(z - \xi_{n,k}).$$

By Proposition 3.5 and the comments we made just after Proposition 3.1, we know that the $\xi_{n,k}$ lie interior to I and that L_{q_n}/q_n is a multipoint Padé approximant of type $(n-1,n)$ to f_μ interpolating this function with multiplicity 2 in each of the points $\xi_{n,k}^{-1} \in I^{-1}$, notwithstanding the structural interpolation at infinity. Recalling that β_{q_n} was defined via double interpolation in $y_{n,1}, y_{n,2}, \ldots, y_{n,n-1} \in I^{-1}$, we conclude from (6.4) that

$$\frac{f_\mu(z) - \beta_{q_n}(z)}{f_\mu(z) - L_{q_n}/q_n(z)} = (1+o(1))\left[\prod_{j=1}^{n-1}\frac{\Psi(z)-\Psi(y_{n,j})}{1-\Psi(y_{n,j})\Psi(z)}\Big/\prod_{k=1}^{n}\frac{\Psi(z)-\Psi(\xi_{n,k}^{-1})}{1-\Psi(\xi_{n,k}^{-1})\Psi(z)}\right]^2, \tag{6.5}$$

locally uniformly in $\overline{\mathbf{C}} \setminus [a,b]$. In view of (6.1), the whole point will be to choose the $y_{n,j}$ in close connection to the $\xi_{n,k}$, so as to make the right hand-side of (6.5) converge to some function whose distance to 1 is strictly larger than 2 everywhere on T. To this end, it is convenient to first identify a sufficiently rich class of possible limit functions for this right hand-side. A clue to what such a class may be is the observation that the log-modulus of the quantity under consideration can be viewed as a Green potential, in $\overline{C} \setminus I$, of a discrete signed measure with point mass 2 at each $\xi_{n,k}^{-1}$ and -2 at each $y_{n,j}$, the mass of I^{-1} being thus equal to 2. Elaborating on this, we let $\Sigma_{I^{-1}}$

[10] By considering $R_{n-1,n}$ to be of type (n,n) with at least one interpolation condition at infinity, the reader will have no difficulty to check that the formula at the bottom of page 108 is equivalent to (6.4), up to the normalization $[a,b] = [-1,1]$ and except for a factor -2 which we found was missing in this reference.

designate the space of signed measures on I^{-1}, and we associate a function h_ν to every $\nu \in \Sigma_{I^{-1}}$ such that $\nu(I^{-1})$ is an integer by means of the formula:

$$h_\nu(z) := \exp\left\{ \int_{I^{-1}} \log \frac{1-\Psi(t)\Psi(z)}{\Psi(z)-\Psi(t)} d\nu(t) \right\}. \tag{6.6}$$

Although the logarithm in the integrand is multiply defined, the condition $\nu(I^{-1}) \in \mathbf{N}$ is to the effect that the imaginary part of the integral in (6.6) increases by an integer multiple of 2π as z encircles I^{-1}, so that

$$h_\nu : \overline{\mathbf{C}} \setminus \{I \cup I^{-1}\} \to \mathbf{C}$$

is indeed well-defined and analytic. Let us formally define

$$\mathcal{F} = \{h_\nu; \quad \nu \in \Sigma_{I^{-1}}, \quad \nu(I^{-1}) \in \mathbf{N}\},$$

out of which we single out the subclass:

$$\mathcal{F}_2 = \{h_\nu \in \mathcal{F}; \quad \nu(I^{-1}) = 2\}.$$

We claim that every member of $\mathcal{F}_2$ is the limit of the right hand-side of (6.5) *for some appropriate choice of the $y_{n,j}$, the convergence being locally uniform on $\overline{\mathbf{C}} \setminus \{I \cup I^{-1}\}$.*

The proof of this claim will occupy us for a while. Without loss of generality, let us number the zeros of q_n in increasing order so that

$$a < \xi_{n,1} < \cdots < \xi_{n,n} < b,$$

and put

$$\Delta_n := \max_{0 \le j \le n} \{\xi_{n,j+1} - \xi_{n,j}\}, \qquad \xi_{n,0} = a, \qquad \xi_{n,n+1} = b.$$

Our immediate goal is to show that $\Delta_n \to 0$ as $n \to \infty$[11]. For any algebraic polynomial p, we get on the one hand upon applying Fubini's theorem and the Cauchy formula successively that

$$\frac{1}{2i\pi} \int_T p(\xi) f_\mu(\xi)\, d\xi = \int \left[\frac{1}{2i\pi} \int_T \frac{p(\xi)}{\xi - t}\, d\xi \right] d\mu(t) = \int p(t)\, d\mu(t).$$

On the other hand, the residue theorem implies that

$$\frac{1}{2i\pi} \int_T p(\xi) \frac{L_{q_n}}{q_n}(\xi)\, d\xi = \sum_{k=1}^{n} p(\xi_{n,k}) \frac{L_{q_n}(\xi_{n,k})}{q_n'(\xi_{n,k})} \tag{6.7}$$

and, since L_{q_n}/q_n converges uniformly to f_μ on T as n tends to infinity (this is obvious from (6.4) but actually does not depend on the fact that f_μ is of Markov type, see [8, prop 2.8]), we conclude that

$$\lim_{n\to\infty} \int p(t)\, d\mu_n(t) = \int p(t)\, d\mu(t) \tag{6.8}$$

where we have set

$$\mu_n := \sum_{k=1}^{n} \frac{L_{q_n}(\xi_{n,k})}{q_n'(\xi_{n,k})} \delta_{\xi_{n,k}}$$

[11]Much more in fact can be proved, namely the measure $n^{-1}\sum_{k=1}^n \delta_{\xi_{n,k}}$ having a point mass $1/n$ at each $\xi_{n,k}$ converges weak-* on I to the equilibrium distribution of the condenser (T, I). This result will be published elsewhere.

and $\delta_{\xi_{n,k}}$ is the unit point mass at $\xi_{n,k}$. As was noticed already in the proof of Corollary 5.4, the remark after Proposition 3.5 implies that $L_q(\xi_k)/q'(\xi_k) > 0$ so that μ_n is a *positive* measure, and its total variation is bounded independently of n since

$$\sum_{k=1}^{n} \frac{L_{q_n}(\xi_{n,k})}{q_n'(\xi_{n,k})} = \frac{1}{2i\pi} \int_T \frac{L_{q_n}}{q_n}(\xi)\, d\xi \leq \left\| \frac{L_{q_n}}{q_n} \right\|_2 \leq \|f_\mu\|_2,$$

where we have used (6.7) for $p = 1$ together with the Schwarz inequality and the fact that L_{q_n}/q_n is the orthogonal projection of f_μ onto V_{q_n}. By weak-* compactness of balls in dual spaces, any subsequence of $\{\mu_n\}$ has a weak-* accumulation point in the space of complex measures on I when the latter is viewed as the dual of $C[a,b]$, the space of continuous complex-valued functions on $[a,b]$. But since polynomials are dense in $C[a,b]$, (6.8) implies that μ is the only accumulation point, so that μ_n actually converges weak-* to μ:

$$\lim_{n\to\infty} \int h(t)\, d\mu_n(t) = \int h(t)\, d\mu(t), \qquad \forall h \in C[a,b]. \tag{6.9}$$

If Δ_n did not go to zero, there would exist a nonnempty interval $J \subset I$ containing none of the $\xi_{n,k}$ for infinitely many n. Letting h be strictly positive on J and zero on $I \setminus J$, we would get from (6.9) that $\int h\, d\mu = 0$ whence μ should have zero density on J, and this contradicts the Szegő condition on μ. Therefore

$$\Delta_n \to 0 \qquad \text{as} \quad n \to \infty \tag{6.10}$$

as announced.

Now, pick any $\nu \in \mathcal{F}_2$, and set $\check{\nu}$ to be the image of ν under the map $t \to 1/t$, i.e. $\check{\nu}$ is a signed measure on I with $\check{\nu}(I) = 2$. This use of the check symbol for measures is in agreement with our notational convention (2.3) for functions. Observe also from the definition (6.2) of Ψ that $\check{\Psi}$ is the conformal mapping of $\overline{\mathbf{C}} \setminus I^{-1}$ onto U normalized by $\check{\Psi}(0) = 0$ and $\check{\Psi}'(0) > 0$, so that

$$K(z,t) := \log \left| \frac{\check{\Psi}(z) - \check{\Psi}(t)}{1 - \check{\Psi}(t)\check{\Psi}(z)} \right| \tag{6.11}$$

is the negative of the Green function in $\overline{\mathbf{C}} \setminus I^{-1}$.

For $j = 1, \ldots, n$, we consider the coefficients $c_{n,j}$ given by the formulas:

$$c_{n,1} := \check{\nu}\left([a, \frac{\xi_{n,1} + \xi_{n,2}}{2}) \right),$$

$$c_{n,j} := \check{\nu}\left([\frac{\xi_{n,j-1} + \xi_{n,j}}{2}, \frac{\xi_{n,j} + \xi_{n,j+1}}{2}) \right), \qquad j = 2, \ldots, n-1,$$

$$c_{n,n} := \check{\nu}\left([\frac{\xi_{n,n-1} + \xi_{n,n}}{2}, b] \right),$$

and we subsequently define $b_{n,0}, b_{n_1}, \ldots, b_{n,n-1}$ and $a_{n,1}, a_{n,2}, \ldots, a_{n,n}$ by means of the relations:

$$b_{n,0} := 0, \quad b_{n,j} := \sum_{k=1}^{j} c_{n,k} = \check{\nu}\left([a, \frac{\xi_{n,j} + \xi_{n,j+1}}{2}) \right), \qquad j = 1, \ldots, n-1,$$

$$a_{n,j} := 2 - b_{n,j}, \qquad j = 1, \ldots, n-1, \qquad a_{n,n} := 0.$$

It is a mechanical consequence of these definitions that

$$a_{n,j} + b_{n,j-1} + c_{n,j} = 2, \qquad j = 1, \dots, n,$$

whence

$$2\sum_{j=1}^{n} K(z,\xi_{n,j}) = \sum_{j=1}^{n-1} a_{n,j} K(z,\xi_{n,j}) + \sum_{j=1}^{n-1} b_{n,j} K(z,\xi_{n,j+1}) + \sum_{j=1}^{n} c_{n,j} K(z,\xi_{n,j}). \tag{6.12}$$

Next we introduce auxiliary points $\check{y}_{n,j}$ by setting

$$\check{y}_{n,j} := \frac{a_{n,j}\,\xi_{n,j} + b_{n,j}\,\xi_{n,j+1}}{2}, \qquad j = 1, \dots, n-1. \tag{6.13}$$

Since $a_{n,j}$ and $b_{n,j}$ add up to 2, it follows from the definition that

$$|\check{y}_{n,j} - \xi_{n,j}| = \left| \frac{b_{n,j}}{2} (\xi_{n,j+1} - \xi_{n,j}) \right| \leq \frac{\|\check{\nu}\|}{2} |\xi_{n,j+1} - \xi_{n,j}|, \tag{6.14}$$

where $\|\check{\nu}\|$ denotes the total variation of ν. Let $\mathcal{K} \subset \overline{C} \setminus \{I \cup I^{-1}\}$ be compact and $\mathcal{U} \subset \overline{C} \setminus I$ be an open set containing $I^{-1} \cup \mathcal{K}$. From (6.14) and (6.10), we see that $\check{y}_{n,j}$ remains outside $\mathcal{U}$ as soon as n is large enough, and for $z \in \mathcal{K}$ we can then write first-order Taylor expansions:

$$K(z,\xi_{n,j}) - K(z,\check{y}_{n,j}) = \frac{\partial}{\partial t} K(z,\check{y}_{n,j})(\xi_{n,j} - \check{y}_{n,j}) + O\left((\xi_{n,j} - \check{y}_{n,j})^2\right), \tag{6.15}$$

$$K(z,\xi_{n,j+1}) - K(z,\check{y}_{n,j}) = \frac{\partial}{\partial t} K(z,\check{y}_{n,j})(\xi_{n,j+1} - \check{y}_{n,j}) + O\left((\xi_{n,j+1} - \check{y}_{n,j})^2\right). \tag{6.16}$$

If we add (6.15) multiplied by $a_{n,j}$ to (6.16) multiplied by $b_{n,j}$, we obtain:

$$a_{n,j} K(z,\xi_{n,j}) + b_{n,j} K(z,\xi_{n,j+1}) - 2K(z,\check{y}_{n,j}) = O\left((\xi_{n,j} - \xi_{n,j+1})^2\right), \tag{6.17}$$

where we have taken (6.13) and (6.14) into account; of course, the same notation big "Oh" is used to designate different functions. By construction, the right hand-side of (6.17) is majorized in modulus by

$$M \, \|\check{\nu}\|^2 \, \frac{(\xi_{n,j} - \xi_{n,j+1})^2}{2},$$

where M is an upper bound for $\partial^2 K(z,t)/\partial t^2$ with respect to $t \in \overline{\mathbf{R}} \setminus \mathcal{U}$, and it follows from the smoothness of $K(z,t)$ that M can be made uniform with respect to $z \in \mathcal{K}$. Consequently, since

$$\sum_{j=1}^{n-1} (\xi_{n,j} - \xi_{n,j+1})^2 \leq \Delta_n (b-a) < 2\Delta_n,$$

we deduce from (6.17) that

$$\sum_{j=1}^{n-1} a_{n,j} K(z,\xi_{n,j}) + \sum_{j=1}^{n-1} b_{n,j} K(z,\xi_{n,j+1}) - 2\sum_{j=1}^{n-1} K(z,\check{y}_{n,j}) = O(\Delta_n), \tag{6.18}$$

where $O(\Delta_n)$ is uniform with respect to $z \in \mathcal{K}$. Moreover, we get in view of (6.12) and (6.18) that

$$2\sum_{j=1}^{n} K(z,\xi_{n,j}) - 2\sum_{j=1}^{n-1} K(z,\check{y}_{n,j}) = O(\Delta_n) + \sum_{j=1}^{n} c_{n,j} K(z,\xi_{n,j}). \tag{6.19}$$

Now, since $K(z,t)$ is jointly continuous with respect to $z \in \overline{C} \setminus \{I \cup I^{-1}\}$ and $t \in I$, it follows from (6.10) and the definition of $c_{n,j}$ that $\sum_{j=1}^{n} c_{n,j} K(z, \xi_{n,j})$ is a Riemann sum satisfying

$$\sum_{j=1}^{n} c_{n,j} K(z, \xi_{n,j}) \to \int \log \left| \frac{\check{\Psi}(z) - \check{\Psi}(t)}{1 - \check{\Psi}(t)\check{\Psi}(z)} \right| d\check{\nu}(t) = -\log |h_\nu(1/z)| \qquad \text{as} \quad n \to \infty, \tag{6.20}$$

locally uniformly with respect to $z \in \overline{C} \setminus \{I \cup I^{-1}\}$. Altogether, (6.19) and (6.20) prove that

$$2 \sum_{j=1}^{n-1} K(z, \check{y}_{n,j}) - 2 \sum_{j=1}^{n} K(z, \xi_{n,j}) \to \log |h_\nu(1/z)| \qquad \text{as} \quad n \to \infty, \tag{6.21}$$

uniformly with respect to $z \in \mathcal{K}$.

Let us define

$$g_n(z) = \left[\prod_{j=1}^{n-1} \frac{\check{\Psi}(z) - \check{\Psi}(\check{y}_{n,j})}{1 - \check{\Psi}(\check{y}_{n,j})\check{\Psi}(z)} \Big/ \prod_{j=1}^{n} \frac{\check{\Psi}(z) - \check{\Psi}(x_{n,j})}{1 - \check{\Psi}(x_{n,j})\check{\Psi}(z)} \right]^2, \tag{6.22}$$

which is analytic in $\overline{\mathbf{C}} \setminus \{I \cup I^{-1}\}$, and observe from (6.21) that $\log |g_n(z)| \to \log |h_\nu(1/z)|$ uniformly on $\mathcal{K}$. Since $\mathcal{K}$ was an arbitrary compact subset of $\overline{C} \setminus \{I \cup I^{-1}\}$, it follows that g_n is a normal family of functions whose limit g has the same modulus as $h_\nu(1/z)$ everywhere in $\overline{C} \setminus \{I \cup I^{-1}\}$. Therefore these two functions differ by a multiplicative constant of modulus 1, but the normalization of Ψ shows that $h_\nu(x) \to 1$ as $x \in \mathbf{R}$ tends to b from above, while g is obviously positive for real arguments; thus $g = h_\nu(1/z)$. Upon setting $y_{n,j} := 1/\check{y}_{n,j}$, this can be reformulated as:

$$\lim_{n \to \infty} \left[\prod_{j=1}^{n-1} \frac{\Psi(z) - \Psi(y_{n,j})}{1 - \Psi(y_{n,j})\Psi(z)} \Big/ \prod_{k=1}^{n} \frac{\Psi(z) - \Psi(\xi_{n,k}^{-1})}{1 - \Psi(\xi_{n,k}^{-1})\Psi(z)} \right]^2 = h_\nu(z) \tag{6.23}$$

locally uniformly in $\overline{C} \setminus \{I \cup I^{-1}\}$, *thereby proving our claim that the right hand-side of (*6.5*) may be construed, by an appropriate choice of the* $y_{n,j}$*, to converge locally uniformly in* $\overline{\mathbf{C}} \setminus \{I \cup I^{-1}\}$ *to an arbitrary function in* $\mathcal{F}_2$*.*

To achieve the proof of Theorem 1.3, it remains for us to find $h_\nu \in \mathcal{F}_2$ such that

$$|h_\nu(z) - 1| > 2 \qquad \forall z \in T. \tag{6.24}$$

This raises the question of characterizing in a manageable way the members of $\mathcal{F}_2$, namely those conjugate-symmetric analytic functions in $\overline{\mathbf{C}} \setminus \{I \cup I^{-1}\}$ whose log-modulus is the Green potential of a signed measure of mass 2 supported on I^{-1}.

By considering the positive and the negative part of the measure ν separately in (6.6), it follows from the Riesz representation theorem (cf. [24, ch I, thm 1.24']) and the minimum principle for superharmonic functions[12] that $h \in \mathcal{F}$ if, and only if, it is analytic $\overline{\mathbf{C}} \setminus \{I \cup I^{-1}\} \to \mathbf{C} \setminus \{0\}$, positive on $(b, 1/b)$, and if we can write $\log |h(z)| = S_1 - S_2$ where, for $i = 1, 2$, the function S_i is superharmonic in $\overline{\mathbf{C}} \setminus I$ and satisfies $S_i(z) \to 0$ when $z \to t \in I$. In the language of functional

[12] We need it over the unbounded domain $\overline{\mathbf{C}} \setminus I$ to ensure that the functions S_i defined hereafter are non-negative and therefore bounded from below; the minimum principle over unbounded domains is immediately deduced from the usual one by conformal mapping, see for instance [35] where the equivalent maximum principle for subharmonic functions is stated for general domains after corollary 2.4.3.

analysis, the last condition says that $\log|h(z)|$ tends to 0 as $z \to t \in I$ and extends to a locally integrable function on $\overline{\mathbf{C}} \setminus I$ whose Laplacian, in the sense of distributions, is a finite measure.

Having characterized members of $\mathcal{F}$, we now describe those belonging to $\mathcal{F}_2$. For this, given $h = h_\nu \in \mathcal{F}$, we appeal to Gauss's theorem [37, ch. 2, thm 1.1][13] to compute $\nu(I^{-1})$, and we see that $h \in \mathcal{F}_2$ if, and only if,

$$\frac{1}{2\pi}\int_\gamma \frac{\partial \log|h|}{\partial \mathbf{n}}\, d|\gamma| = 2, \tag{6.25}$$

where γ is any smooth Jordan contour in $\mathbf{C} \setminus \{I \cup I^{-1}\}$ surrounding I^{-1}, and where $\partial/\partial\mathbf{n}$ indicates the normal derivative while $d|\gamma|$ is the differential of the arclength. As $\log|h|$ is harmonic in $\overline{\mathbf{C}} \setminus \{I \cup I^{-1}\}$, the integral in (6.25) is the increment of its conjugate differential when z winds around I^{-1} once and, by the argument principle, we can rewrite condition (6.25) by means of a line integral:

$$\frac{1}{2i\pi}\int_\gamma \frac{h'}{h}\, d\gamma = 2. \tag{6.26}$$

We are now ready to construct $h_\nu \in \mathcal{F}_2$ satisfying (6.24). Because members of $\mathcal{F}_2$ must have real limits at a and b and conjugate limits from above and below at every point of (a, b), by the smoothness of Ψ, and since all these limits have modulus 1, it is quite natural to consider the conformal map

$$\Psi_1 : \overline{\mathbf{C}} \setminus \{I \cup I^{-1}\} \longrightarrow \{z;\ 1 < |z| < R\}$$

of $\overline{\mathbf{C}} \setminus \{I \cup I^{-1}\}$ onto an annular region whose inner circle is T, normalized by the conditions $\Psi_1(z) > 0$ for $z \in (b, 1/b)$ and $\lim_{z\to b^+} \Psi_1(z) = 1$. The log-modulus of Ψ_1 is well-known to be the Green potential with respect to $\overline{\mathbf{C}} \setminus I$ of a positive measure of unit mass supported on I^{-1}, the so-called equilibrium measure of the condenser (I, I^{-1}) (see *e.g.* [37, ch. 8, sec. 6]), and the number R is then equal to $e^{1/C}$ where C is the capacity of this condenser. Observe also that $e^{1/C}\Psi_1^{-1}(1/z) = \Psi_1(z)$ by uniqueness of the normalized conformal map, whence Ψ_1 maps T onto the circle centered at 0 of radius $e^{1/2C}$. Altogether, the map

$$\Psi_2(z) = \Psi_1^2(z)$$

belongs to $\mathcal{F}_2$ and satisfies

$$|\Psi_2(z)| = e^{1/C} \qquad z \in T.$$

However, Ψ_2 is not good enough for our purposes if $e^{1/C} < 2$, because then (6.24) will fail if we set $h_\nu = \Psi_2$.

A natural idea is then to compose Ψ_2 with a Möbius transformation that leaves T invariant: such a transformation will send the circle centered at 0 of radius $e^{1/C}$ to another circle whose distance to the point 1 will intuitively be maximum when 1 is in fact its center. Now, the unique Möbius transformation that preserves T and maps the circle centered at 0 of radius $e^{1/C}$ to a circle centered at 1 while preserving the conjugate symmetry is

$$\varphi(z) = \frac{z - e^{-2/C}}{1 - ze^{-2/C}},$$

and we will show that the map

$$\Psi_3(z) := \varphi(\Psi_2(z))$$

does indeed satisfy all our requirements.

[13]The proof given there for logarithmic potentials extends immediately to Green potentials.

First of all, we get by construction that

$$|\Psi_3(z) - 1| = e^{1/C} + e^{-1/C} > 2 \qquad z \in T,$$

so that (6.24) will be satisfied if we set $h_\nu = \Psi_3$.

Next, *we claim that* Ψ_3 *belongs to* $\mathcal{F}$. Indeed, by Carathéodory's theorem, a conformal map from a simply connected domain onto a disk extends continuously at every accessible point of the boundary of this domain, and even extends to a homeomorphism of the closures if every boundary point is accessible [35, thm 4.4.13]. Applying this to the upper and lower halves of $\overline{\mathbf{C}} \setminus \{I \cup I^{-1}\}$, that are mapped by Ψ_1 onto the upper and lower halves of $\{z;\ 1 < |z| < e^{1/C}\}$ respectively due to the conjugate symmetry and the normalization, we see that Ψ_1 has real limits at $1/a$ and $1/b$ and conjugate limits from above and below at every point of $(1/a, 1/b)$; moreover, since these limits have to lie on the circle centered at 0 of radius $e^{1/C}$, the map Ψ_1 extends analytically through $(1/a, 1/b)$ from above and below by the reflexion principle. The same is then true of Ψ_3, although this time the limits at $1/a$ and $1/b$ are ∞ because φ sends the point $e^{2/C}$ to infinity. Observe also that φ maps the annulus $\{1 < |z| < e^{2/C}\}$ onto the region

$$\left\{\mathbf{Re} z > -\frac{e^{2/C} + e^{-2/C}}{2}\right\} \setminus \overline{U},$$

and the circle $\{|z| = e^{2/C}\}$ to the line

$$\left\{\mathbf{Re} z = -\frac{e^{2/C} + e^{-2/C}}{2}\right\}. \tag{6.27}$$

For $t \in (1/a, 1/b)$, let D_t be a disk centered at t which is so small that $\mathbf{Re}\Psi_3 < 0$ on $D_t \setminus \mathbf{R}$; this is possible because the limits of Ψ_3 from above and below on $(1/a, 1/b)$ lie on the line (6.27). Denote by Ψ_3^+ and Ψ_3^- respectively the restriction to D_t of the analytic extensions of Ψ_3 from above and below across $(1/a, 1/b)$. Thus Ψ_3^+ and Ψ_3^- are analytic functions in D_t such that

$$\Psi_3(z) = \Psi_3^+(z) \qquad \text{for } z \in D_t,\ \mathbf{Im} z > 0, \tag{6.28}$$

$$\Psi_3(z) = \Psi_3^-(z) \qquad \text{for } z \in D_t,\ \mathbf{Im} z < 0. \tag{6.29}$$

Putting

$$\kappa = \frac{e^{2/C} + e^{-2/C}}{2}$$

for simplicity, the above-mentioned properties of φ imply that $\Psi_3^+(z) + \kappa$ is purely imaginary on $D_t \cap (1/a, 1/b)$. Hence

$$\Psi_3^+(z) = -2\kappa - \overline{\Psi_3^+(\overline{z})} \tag{6.30}$$

by the reflexion principle and the uniqueness of analytic continuation. Moreover, as

$$-\kappa < \mathbf{Re}\Psi_3(z) < 0 \qquad \text{for } z \in D_t,$$

it follows from (6.28) and (6.30) that

$$|\Psi_3^+(z)|^2 = \left(2\kappa + \mathbf{Re}\Psi_3^+(\overline{z})\right)^2 + \left(\mathbf{Im}\Psi_3^+(\overline{z})\right)^2 > |\Psi_3(\overline{z})|^2 \qquad \text{for } z \in D_t \cap \{\mathbf{Im} z < 0\}.$$

But since $\Psi_3(\overline{z}) = \overline{\Psi_3(z)}$ by the conjugate-symmetry, we deduce from the above that

$$|\Psi_3^+(z)| > |\Psi_3(z)| = |\Psi_3^-(z)| \qquad \text{for } z \in D_t \cap \{\mathbf{Im} z < 0\}.$$

Using (6.29), a similar reasoning shows that

$$|\Psi_3^-(z)| > |\Psi_3(z)| = |\Psi_3^+(z)| \qquad \text{for } z \in D_t \cap \{\mathbf{Im} z > 0\}.$$

Altogether, we obtain

$$\log|\Psi_3(z)| = \min\{\log|\Psi_3^+(z)|, \log|\Psi_3^-(z)|\} \qquad \text{for } z \in D_t, \ \mathbf{Im} z \neq 0,$$

thereby establishing that $\log|\Psi_3|$ extends to a superharmonic function on the whole of D_t. Since it is otherwise harmonic in $\overline{\mathbf{C}} \setminus \{I \cup I^{-1}\}$ and tends to $+\infty$ as z tends to $1/a$ or $1/b$, we now see that $\log|\Psi_3|$ extends superharmonically to $\overline{\mathbf{C}} \setminus I$. Considering that it tends to 0 as z tends to a point of I, we conclude that $\Psi_3 \in \mathcal{F}$ by a previous characterization *which proves our claim.*

To show that in fact $\Psi_3 \in \mathcal{F}_2$, we use the definition $\Psi_3 = \varphi \circ \Psi_2$ to compute

$$\begin{aligned} \frac{1}{2\pi}\int_T \frac{\Psi_3'}{\Psi_3}\,d\xi &= \frac{1}{2\pi}\int_T \frac{(1-e^{-4/C})\Psi_2'}{(\Psi_2 - e^{-2/C})(1-\Psi_2 e^{-2/C})}\,d\xi \\ &= \frac{1}{2\pi}\int_T \frac{\Psi_2'}{\Psi_2 - e^{-2/C}}\,d\xi - \frac{1}{2\pi}\int_T \frac{\Psi_2'}{\Psi_2 - e^{2/C}}\,d\xi. \end{aligned} \tag{6.31}$$

As we noticed that $\Psi_2 \in \mathcal{F}_2$, we know by (6.26) that Ψ_2 winds around 0 twice as z traverses T. Since $|\Psi_2(z)| = e^{1/C}$ for $z \in T$, we deduce that Ψ_2 also winds twice around $e^{-2/C}$ and does not wind at all around $e^{2/C}$. Hence the first integral in the right hand-side of (6.31) is equal to 2 and the second is equal to zero. By the characterization (6.26) again, this shows that $\Psi_3 \in \mathcal{F}_2$, and achieves the proof of Theorem 1.3. ∎

7 Conclusion

Theorems 1.1, 1.2, and 1.3 together yield a definite insight into uniqueness issues for $\overline{\mathcal{H}}_{2,\mathbf{R}}$ rational approximation to Markov functions: roughly speaking, uniqueness holds if the measure has small support, it prevails asymptotically if the measure is smooth enough, but it may fail for infinitely many n if the measure is irregular and charges too much the endpoints. Note that there is still room for a conjecture asserting that, in any case, a critical point is unique at least for a subsequence.

It is natural to wonder whether the analysis will extend to more general functions. The machinery around the index theorem is admitedly heavy but quite general, and the specific character of Markov functions was used in the asymptotic formula (6.4) and in the knowledge that the poles lie on the minimal segment containing the support of μ. Formula (6.4) itself is exceedingly accurate for our purposes: what we really need is that

$$(f_\mu(z) - R_{n-1,n}) \Big/ \prod_{j=1}^{2n} \frac{\Psi(z) - \Psi(z_{n,j})}{1 - \Psi(z_{n,j})\Psi(z)}$$

is a normal family of functions in $\overline{\mathbf{C}} \setminus I$ whose limit is zero-free on T. This allowed for us to asymptotically replace the error function in multipoint Padé approximation by the Blaschke product in $\overline{\mathbf{C}} \setminus I$ built on the interpolation points, and to subsequently reduce the construction of the "comparatively bad" approximant β_n of Theorem 5.5 to a Green energy problem.

Now, let us consider the problem of best rational approximation with complex coefficients in $\mathcal{L}^2(T)$ to a function defined by as Cauchy integral over an open contour $C \subset U$:

$$f(z) = \frac{1}{2i\pi} \int_C \frac{w(t)dt}{t-z},$$

The same computation as in the Markov case yields the critical point equation:

$$\int_C \frac{q_n}{\widetilde{q}_n^2} \, w(t) \, t^k \, dt = 0, \qquad 0 \leq k \leq n-1, \tag{7.32}$$

where this time the definition of $\widetilde{q}_n$ involves conjugating the coefficients as well:

$$\widetilde{q}_n(z) := z^n \overline{q_n(1/\bar{z})}.$$

From 7.32, we see that q_n appears to be a *non-Hermitian* orthogonal polynomial with respect to the complex measure $w(t)/\widetilde{q}_n^2(t)$. Let us put for definiteness

$$q_n(z) = \prod_{j=1}^{n} (z - \xi_{n,j}).$$

If one could prove, under suitable conditions on w and C, that there is a sequence of closed neighborhoods $\mathcal{N}_n$ of C, with $\cap_n \mathcal{N}_n = C$, such that $\mathcal{N}_n$ contains the $\xi_{n,j}$ except perhaps for a fixed number of them, and if he established the normality of the sequence

$$\left(f(z) - L_{q_n}/q_n\right) \Big/ \left[\prod_{j=1}^{n} \frac{\Psi_n(z) - \Psi_n(\overline{\xi}_{n,j}^{-1})}{1 - \Psi_n(\overline{\xi}_{n,j}^{-1}) \Psi_n(z)} \right]^2$$

where $\Psi_n : \overline{\mathbf{C}} \setminus \mathcal{N}_n \to U$ is a conformal map, then he may be in position to parallel the proof of Theorem 1.3. In this connection, it has been shown recently when f is a function with two branch points that the roots of any q_n satisfying (7.32) converge to the arc of circle orthogonal to T joining these points [4]. This arc minimizes the capacity of the pair (T, C) among all *continua* joining the branch points, and this fits with the results of [39] if Padé approximants are interpreted as optimal rational approximants on a circle with infinite radius. It is to be hoped that sharpened versions of these statements will lead to generalizations of Theorem 1.3 to more general Cauchy integrals than Markov functions.

In any case, a remarkably strong connection apparently arises between uniqueness issues and error estimates in $\overline{\mathcal{H}}_{2,\mathbf{R}}$ rational approximation.

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