2-D Inverse Problems for the Laplacian: a Meromorphic Approximation Approach

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Abstract

We consider a classical inverse problem: detecting an insulating crack inside a homogeneous 2-D conductor, using overdetermined boundary data. Our method involves meromorphically approximating the complexified solution to the underlying Dirichlet-Neumann problem on the outer boundary of the conductor, and relating the singularities of the approximant (*i.e.* its poles) to the singular set of the approximated function (*i.e.* the crack). This approach was introduced in [18] when the crack is a real segment embedded in the unit disk. Here we show, more generally, that the best L^2 and L^∞ meromorphic approximants to the complexified solution on the outer boundary of the conductor have poles that accumulate on the hyperbolic geodesic arc linking the endpoints of the crack if the latter is analytic and "not too far" from a geodesic. The extension of the method to the case where the crack is piecewise analytic is briefly discussed. We provide numerical examples to illustrate the technique; as the computational cost is low, the results may be used to initialize a heavier local search. The bottom line of the approach is to regard the problem of "optimally" discretizing a potential using finitely many point masses as a regularization scheme for the underlying inverse potential problem. This point of view may be valuable in higher dimension as well.

Key words: Laplacian, Inverse problems, Dirichlet-Neumann problem, Meromorphic approximation, Hankel operators.

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1 Introduction

The present paper explores a new connection between complex analysis and inverse problems with free boundary of the 2-D Laplacian. This may be viewed as one particular instance of a general issue in inverse potential theory, namely *what does the optimal discretization of a potential (with respect to some criterion) tell us about the support of its generating measure?*

Recall that, given a fundamental solution E of some elliptic operator A, the potential of a compactly supported measure μ is the convolution $p_\mu = E * \mu$. The inverse problem of potential theory is to recover information on μ from the knowledge of p_μ outside a neighborhood $\mathcal{N}$ of the support of μ [43]. Now, if one approximates p_μ outside $\mathcal{N}$, say optimally with respect to some criterion, by the potential of a *discrete* measure with n point masses, the issue that we raise is: how do these masses distribute with respect to the support of μ and how do they behave asymptotically with n?

Of course, the relevance of that issue to the inverse problem is clear only if the optimal discretization can be carried out constructively and if the distribution of the corresponding point masses can be related to μ explicitly. To the authors' knowledge, such questions have received little

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attention so far, and no general framework is presently available to study them in a systematic way. However, when A is the 2-D Laplacian so that $E = -\log|z|$, we remark that $\partial p_\mu/\partial z$ is the Cauchy integral of μ, which is a rational function if and only if μ is discrete; consequently, optimal discretization with respect to Sobolev-type norms (*i.e.* approximating the derivative) amounts to best rational approximation of Cauchy integrals. This last subject has made considerable progress in recent years, enabling one to describe the asymptotic behavior of the approximants when the degree grows large, provided that the Cauchy integral to be approximated is taken over so-called *symmetric arcs* for the logarithmic or the Green potential [36, 68, 53, 69, 37, 21, 23, 50, 17]. This is how the authors of the present paper were led to approach the problem of crack detection using tools from Approximation Theory in the complex domain. Actually, we shall deal with a situation where potentials are only known up to the addition of a harmonic function, and therefore we consider *meromorphic* rather than rational approximation. This makes hardly any difference for the application we have in mind.

Crack detection from overdetermined boundary data of diffusive phenomena is a classical inverse problem for which many approaches exist already. Some exploit the propagation dynamics while others rely on the analysis of the steady state solution satisfying Laplace's equation. In Section 2 we review briefly three methods of the latter type on simply connected planar domains, in order to put the present work into perspective. Due to ill-posedness, every method involves *a priori* assumptions on the crack, ranging from the very strong (*e.g.* it is a line segment) to mere Hölder-smoothness; as a general rule, the stronger the assumptions the more constructive the method. The one we present here requires analyticity on the crack while putting additional constraints on its shape (it should not be too far from a geodesic arc in the hyperbolic geometry; see the quantitative bounds in Theorem 7.2). This range of applicability can be enlarged considerably by appealing to finer properties of meromorphic approximants (*cf.* [24]), making it possible to deal with more general, piecewise analytic cracks; see the discussion in Section 7.2. This more extensive treatment is beyond the scope of this paper, but the account below already contains all the necessary ingredients to illustrate the method.

What we shall present is *not* a recovery algorithm, in that it can only locate the *endpoints* of the crack in general. Indeed, the outcome of the procedure at step n is a set of n points (*i.e.* the poles of the approximant) that converge *in proportion*, when n gets large, to the Green equilibrium distribution on the geodesic arc linking these endpoints. The equilibrium distribution charges the endpoints which is why they can be recovered asymptotically, whereas the crack itself will not attract the poles unless it is a geodesic arc. On the one hand, a full assessment of the method calls for estimates on the speed of convergence that are currently not available and would require further *a priori* assumptions both on the crack and on the boundary data that are used. On the other hand, numerical experiments indicate that this convergence is rather fast, and in any case the non-asymptotic bounds in Proposition 6.1 and Corollary 6.2 provide one already with some indication of how quickly the poles approach the geodesic arc. The computational cost is very low as compared to other techniques, so our endeavor of spotting the endpoints of the crack is of major interest to initialize a more accurate but "heavier" local search. Alternatively, recovering the crack from the knowledge of its endpoints can be recast analytically as yet another extremal problem, but we will not touch upon that issue here. Let us mention also that a similar approach can be taken for the recovery of pointwise monopolar or dipolar sources [15].

The complex analytic tools that we shall use are mostly standard (cf. e.g. [1, 27, 75]), except perhaps for the approximation-theoretic part. Specifically, we shall be concerned with two types of meromorphic approximation on a smooth simply connected planar domain with rectifiable boundary, namely those associated, respectively, with the L^∞ and the L^2 norms on the boundary of the domain. The first type is particularly interesting in that it is conformally invariant, and constructively solvable *via* the Adamjan–Arov–Krein (in short: AAK) theory [2, 3]. Unfortunately, it is not continuous with respect to the L^∞ norm on the data, but only with respect to stronger ones (*e.g.* Hölder or Wiener norms) [58] which makes it delicate to use in noisy situations. The L^2 meromorphic approximation is better behaved in that respect, but it cannot be computed in closed form and one has to rely on numerical search; that is why we take the precaution to stress results that are valid for *local minima* and not just the global one. As a trade-off, one could work with L^p meromorphic approximation with $2 \leq p \leq \infty$ using results from [23], but we shall not deal with such a generalization here. In all cases the asymptotic distribution of the poles is the same. In principle, one could as well use multipoint Padé interpolation [**?**] instead of meromorphic approximation. Our line of approach applies *mutatis mutandis* to this setting, replacing throughout Green potentials by logarithmic ones and hyperbolic geodesics by straight lines. Although this makes for a *linear* algorithm, it is not feasible in practice because interpolation data have to be estimated from pointwise values of the solution on the outer boundary, and such computations turn out to be unstable. This is why we do not pursue this direction.

For simplicity, the numerical experiments reported at the end of the paper have been carried out when the domain under consideration is normalized to be the unit disk. Then the analytic projection of a function is immediately deduced from its Fourier expansion, which makes for an easier computation, both of the Hankel matrix needed for AAK-theory and of the long division providing us with the L^2-criterion, see comments in Section 8. Of course, the results that we illustrate are formulated in a conformally invariant manner that does not depend on such a normalization.

2 2-D crack detection from Neumann-to-Dirichlet data

Consider a bounded simply connected domain D in the plane with oriented boundary Γ. Suppose D is filled with a homogeneous body, except for a one-dimensional crack modeled by an oriented Jordan arc $\gamma \subset D$ with distinct endpoints, say, γ_0 and γ_1. The smoothness of Γ and γ will be discussed shortly. For the time being, simply assume they have well-defined oriented normals n_Γ and $n_\gamma^\pm$ whenever needed. Here n_Γ points inward with respect to D, while n_γ^+ (resp. n_γ^-) points into the positive (resp. negative) region determined by the oriented arc γ, i.e. the region "to the left" (resp. "right") as one travers on γ. Functions defined on $D \setminus \gamma$ will generally have two-fold limits on γ; to avoid confusion, the limit will be superscripted by $+$ or $-$, depending on which side it is to be taken.

Let the conductor D be subject to some physical experiment, governed by the Laplace operator (*e.g.* it could be heated or electrified), for which the crack acts as a perfect insulator. Specifically, assume that we apply a flux Φ (of heat or current) on the outer boundary of the conductor. When the equilibrium is reached, the physical phenomenon u (the heat or the potential) is a real-valued function on $D \setminus \gamma$ subject to the following Neumann boundary value problem:

$$\left\{ \begin{array}{lcl} \Delta u & = & 0 \quad \text{in } D_\gamma = D \setminus \gamma\,, \\[2ex] \dfrac{\partial u}{\partial n_\Gamma} & = & \Phi \quad \text{on the boundary } \Gamma = \partial D\,, \\[2ex] \dfrac{\partial u^\pm}{\partial n_\gamma^\pm} & = & 0 \quad \text{on } \overset{\circ}{\gamma} = \gamma \setminus \{\gamma_0, \gamma_1\}\,, \end{array} \right. \tag{2.1}$$

where Δu is the Laplacian of u.

In order that a solution to (2.1) exists, the *compatibility condition*

$$\int_\Gamma \Phi\,|dw| = 0, \tag{2.2}$$

must hold, where $|dw|$ is the differential of arclength on Γ. The necessity of (2.2) comes from the fact that the distribution u (of temperature or electricity) can be time-independent only if the total flux (of heat or current) is zero. Also, it is clear that a solution to (2.1) is determined up to an additive constant only. To avoid such trivial non-uniqueness, we may impose the normalization:

$$\int_\Gamma u\,|dw| = 0. \tag{2.3}$$

The inverse problem under consideration is:

> *If the crack γ is unknown, how can one recover it from overdetermined measurements (u, Φ) on the boundary Γ of D?*

Note that we *assume* there is *only one crack*. The case of finitely many cracks will be commented upon in Section 7.2.

Our immediate goal is to review some of the results and techniques available in the literature to tackle this inverse problem. These were derived assuming that Γ and γ are *smooth*, say of class C^2. Using the trace theorem [39, Thm. 1.5.2.6], most of them would carry over with few modifications to a piecewise-$C^{1,1}$ setting, but we make no attempt at describing them in this greater generality. Accordingly, in the remainder of this section, we let Γ and γ be C^2-smooth.

Denote by $\mathcal{S}^1(D\setminus\gamma)$ the Sobolev space of functions in $L^2(D\setminus\gamma)$ whose distributional derivatives of the first order again lie in $L^2(D\setminus\gamma)$. Traces of $\mathcal{S}^1(D\setminus\gamma)$-functions continuously exist in $\mathcal{S}^{\frac{1}{2}}(\Gamma)$, the interpolating space of exponent 1/2 between $L^2(\Gamma)$ and $\mathcal{S}^1(\Gamma)$ [52, Ch. 1, Sec. 9-10]. Hence by the Riesz-Fisher representation of linear functionals in a Hilbert space, a *variational* solution to (2.1) uniquely exists in $\mathcal{S}^1(D\setminus\gamma)$ whenever Φ belongs to the dual space $\mathcal{S}^{-\frac{1}{2}}(\Gamma)$ of $\mathcal{S}^{\frac{1}{2}}(\Gamma)$ [52, Ch.1, Sec. 12], meaning that there is one and only one $u \in \mathcal{S}^1(D\setminus\gamma)$ satisfying (2.3) such that

$$\int_\Gamma \Phi\psi\,|dw| + \int_{D\setminus\gamma} \nabla\psi\cdot\nabla u\,dm = 0 \qquad \forall\psi\in\mathcal{S}^1(D\setminus\gamma), \tag{2.4}$$

where m is the 2-D Lebesgue measure and ∇ indicates the gradient vector field. Equation (2.4) is obtained upon formally substituting (2.1) in Green's first identity over $D\setminus\gamma$, and does not formally require that the Laplacian of u be a function.

The existence and uniqueness of a variational solution makes it possible to define the *Neumann-to-Dirichlet operator:*

$$\begin{array}{rccc} \mathcal{F}_\gamma: & \mathcal{S}^{-\frac{1}{2}}(\Gamma) & \longrightarrow & \mathcal{S}^{\frac{1}{2}}(\Gamma) \\ & \Phi & \longmapsto & u|_\Gamma, \end{array} \tag{2.5}$$

which is convenient to discuss the three basic issues facing every inverse problem, namely:

- **Identifiability.**

  Is the map $\gamma \mapsto \mathcal{F}_\gamma$ injective? This question was originally considered in [34] where it is shown that two particular fluxes on Γ are enough to characterize a single crack; one flux is not sufficient in general. We remark that this result was subsequently extended [29, 9, 46] to finitely many cracks, and that a similar statement holds for *emerging* cracks [11, 32] although these are not a concern to us here.

- **Stability.**

  Granted identifiability, let Φ_1, Φ_2 be a pair of characteristic fluxes for the crack under consideration. Then, the stability of the solution with respect to the data amounts to requiring that the map
  $$(\mathcal{F}_\gamma(\Phi_1), \mathcal{F}_\gamma(\Phi_2)) \longmapsto \gamma$$
  be continuous with respect to suitable topologies. The case of a rather general crack is considered in [6], but there only conditional stability can be proved since the problem is not well-posed. Estimates of the same kind under less restrictive regularity assumptions may be found in [62, 8]. We also remark that if there are several cracks, the Lipschitzian stability of their relative angles is established in [34] for the case when they are line segments; see also [7].

- **Identification.**

  When identifiability and stability are met, one may ask for an identification procedure; that is, a constructive means of approximating γ from the evaluation on Γ of finitely many functions $\mathcal{F}_\gamma(\Phi_j)$, $1 \leq j \leq N$.

  We shall distinguish between two types of identification methods:

  - Iterative methods, usually of the descent type, where the direct problem is solved at each step. These are usually based on minimizing a criterion (typically some distance from the measurements) with respect to a parametrized family of cracks.
  - Semi-explicit methods, where characteristic properties of γ are sought from the knowledge of $\mathcal{F}_\gamma - \mathcal{F}$, where $\mathcal{F}$ is the Neumann-to-Dirichlet operator corresponding to the "sane" domain D; *i.e.* when there is no crack. This way the direct problem need not be solved repeatedly.

  Generally speaking, semi-explicit methods are fast but not fully constructive unless strong assumptions are made on the crack. They may typically be used to initialize iterative methods that are more flexible but computationally heavier and flawed with local minima that could prevent them from converging if the initial guess is inappropriate.

  We refer the reader to [66] for a prototypical example of an iterative method on a problem which is conjugate to (2.1). The method can be used quite generally but is only proved convergent in the above reference when the crack is a line segment.

  Below we sketch two examples of semi-explicit methods that are closer in spirit to the present work. Both are based on a comparison between the Neumann-to-Dirichlet map of the "sane" domain D and that of the "cracked" domain $D \setminus \gamma$. The first one makes the very strong assumption that the crack lies on a straight line whereas the second is much more general but not fully constructive.

  - **The reciprocity gap method.**

    This method was introduced in [10]. The direct problem being modeled by (2.1), let v be harmonic in the sane domain D and set $h = u|_\Gamma$. Then, by Green's second identity:
    $$\int_\Gamma \left(\Phi v - \frac{\partial v}{\partial n_\Gamma} u\right) |dw| = \int_\gamma [u]_\gamma \frac{\partial v}{\partial n_\gamma} |d\xi|, \tag{2.6}$$

where $[u]_\gamma = u^+ - u^-$ denotes the jump of u across γ. The left-hand side of (2.6) defines the so-called reciprocity Gap operator $RG_{[\Phi,h]}(v)$. Choosing elementary harmonic polynomials for v generates linear equations for the parameters of the straight line L containing γ. The latter is then determined by localizing the set $\{t \in L;\ [u]_\gamma(t) > \epsilon\}$ using Fourier analysis. The method could be adapted to more general algebraic curves but very strong *prior* assumptions on γ have to be made anyway.

- **The factorization method.** This method was introduced in [28] after earlier work [47] on inverse scattering. The direct problem being still described by (2.1), the difference operator $\mathcal{F}_\gamma - \mathcal{F}$ turns out to be positive $L^2_\diamond(\Gamma) \to L^2_\diamond(\Gamma)$ where the subscript "$\diamond$" indicates that the mean vanishes [28, Thm. 2.2.]. For σ an open arc of class C^2 in D, let v_1 be the double-layer potential:

$$v_1(z) = \frac{1}{2\pi} \int_\sigma \varphi(\zeta) \frac{\partial}{\partial n_\sigma} \log \frac{1}{|z-\zeta|} d|\zeta|, \quad z \in D_\sigma,$$

  where φ is a smooth positive density function on σ that vanishes at the endpoints. Let further v_0 be harmonic in D with $\partial v_0/\partial n_\Gamma = \partial v_1/\partial n_\Gamma$, and c be the mean of $v_1 - v_0$. If we set $v = v_1 - v_0 - c$, then $\sigma \subset \gamma$ if, and only if, $v_{|\Gamma}$ lies in the range of $(\mathcal{F}_\gamma - \mathcal{F})^{\frac{1}{2}}$ [28, Thm. 3.1.].

  Unfortunately one cannot check constructively whether a function belongs to a non-closed subspace, nor can one exhaust the candidate-cracks σ in D. We refer the reader to [28] for a heuristic criterion of whether $v_{|\Gamma}$ is "close" to lying in the range of the compact operator $(\mathcal{F}_\gamma - \mathcal{F})^{\frac{1}{2}}$. This criterion is based on the numerical computation of a large number of pairs (u, Φ) in order to evaluate sufficiently many eigenvalues and eigenvectors of the square root operator, and on the estimation of a mean geometric decay to guess the nature of the Picard series for the inverse.

Other approaches also exist, that are of a more heuristical type. Let us quote in particular [49], where some approximation to a conformal map from an annulus to $D \setminus \gamma$ is seeked in the sense of kernel convergence, using numerical integration of conjugate differentials.

Note that the methods we just mentioned proceed by approximating, in different ways, the *solutions* to (2.1) subject to the boundary conditions provided by the measurements. In contrast, the approach below is based on approximating the *boundary conditions themselves.*

3 Overview of the results

Having reviewed in the previous section some existing methods to tackle the inverse problem of recovering γ from the knowledge of Φ and $u_{|\Gamma}$ in (2.1), let us now briefly indicate how the results of the present paper, that are function-theoretic in nature, can be used for that purpose. We assume that D is simply connected with piecewise $C^{1,\alpha}$ boundary Γ and no outward-pointing cusps; see hypotheses **H1-H2** in Section 4.

1 If γ is piecewise $C^{1,\alpha}$ without cusps (see hypotheses **H3-H4** in Section 4), and if $\Phi \in L^p(\Gamma)$ with $1 < p < 2$ while (2.2) is met, then (up to an additive constant) there is a unique solution u to (2.1) whose gradient is uniformly summable over a sequence of curves tending to $\Gamma \cup \gamma$ in $D \setminus \gamma$; moreover, u is the real part of a holomorphic function f whose derivative belongs to the Smirnov class $\mathcal{E}^p(D \setminus \gamma)$. This is the content of Theorem 4.1.

2 The function f is the sum of a holomorphic function on D which is continuous on $\overline{D}$ (in fact absolutely continuous on Γ with $L^p(\Gamma)$-derivative), and of a Cauchy integral over γ. This is the content of Theorem 4.4. In particular, it follows from characteristic properties of conjugate differentials that

$$f(\xi) = u(\xi) - i \int_{\xi_0}^{\xi} \Phi(\zeta)\,|d\zeta|, \qquad \xi \in \Gamma,$$

so that f can indeed be computed on Γ from the knowledge of u and Φ.

3 Granted f on Γ one can, for increasing values of n, compute best meromorphic approximants to f from $\mathcal{E}_n^\infty(D)$ using AAK-theory or local best approximants from $\mathcal{E}_n^2(D)$ using descent algorithms; see the details in sections 5 and 8.

4 If the crack γ is analytic in D and close enough to the hyperbolic geodesic arc $\mathcal{G}$ linking its endpoints γ_0 and γ_1, and if the flux Φ does not make γ a level line of u, then the poles of the best meromorphic approximants asymptotically distribute, as n increases, according to the Green equilibrium measure of $\mathcal{G}$ in D. This is the content of Theorems 7.2 and 7.4. In particular, since the Green equilibrium measure heavily charges γ_0 and γ_1, they can in principle be spotted as clusters of poles.

5 The non-asymptotic relations (6.6) or (6.11) give a quantitative estimate of how far the poles may lie from $\mathcal{G}$ for fixed n. This can be used to obtain geometric bounds on the location of γ_0 and γ_1, see the remarks after Proposition 6.1 and Corollary 6.2.

In item 4, the geometric restrictions on γ can be weakened by choosing the flux Φ conveniently, and the method can further handle piecewise analytic cracks whose hyperbolic convex hull is not too big. In this case, however, the hyperbolic geodesic arc must be replaced by some appropriate *symmetric contour* for the Green potential. Such generalization is beyond the scope of this paper; see Section 7.2 for a short discussion and a conjecture that the endpoints and the vertices of a piecewise smooth γ should always attract a positive proportion of poles, no matter whether it is piecewise analytic or not.

The computational cost is very low, and the technique seems suited to initialize other methods of crack recovery proper. Its limitation is of course that the accuracy on computing meromorphic approximants for large n decreases with the precision on the data: at some point one starts approximating the measurement and truncation errors, and then the behavior of the poles becomes different. This is why, in practice, one would rather iterate the steps above for several pairs (u, Φ) while keeping n within reasonable bounds. Section 8 displays a few numerical examples.

4 Cauchy integrals

Below, we shall need a stronger type of solution to (2.1) than the variational one to make contact with complex analysis. Specifically, we want to represent u as the real part of a holomorphic function f whose derivative extends in an L^p manner on $\Gamma = \partial D$ and from both sides on γ. This will both enable us to represent u as the real part of a Cauchy integral and provide us with the Hölder-continuity of f which is important to ensure continuity properties of best meromorphic approximants. These requirements lead us to choose Φ to be a true function and not merely a distribution on Γ, although we still want to allow this function to be somewhat irregular since the flux which is applied to the outer boundary of D may well be discontinuous in practice.

The situation that we face here is not completely classical: the theory of layer potentials on Lipschitz domains (see *e.g.* [44, 45]) is not directly available because the endpoints of the crack are (inward-pointing) cusps, and the regularity theory of [39] on polygonal domains is not easily adapted either as it deals with homogeneous boundary conditions and does not include the estimates that are necessary to control the conjugate function near the boundary. That is why we devise in this section a theorem which suits our purposes.

Since we must deal with two inward-pointing cusps by the very nature of the problem, we may as well handle any number of them and in addition allow for ∂D to have finitely many corners, which is often convenient in applications (outward-pointing cusps are not allowed). However, we make rather strong regularity assumptions on these singular points by requiring ∂D to be piecewise $C^{1,\alpha}$-smooth. This will enable us to localize the singularities of the conjugate function, and to prove a result which may be of independent interest, as it stands somewhat half-way between classical L^p theorems on smooth domains [4, 33] and theorems on Lipschitz domains where the range of p gets restricted [45]. We shall see in particular that the occurrence of inward-pointing cusps forbids the use of $p = 2$, in contrast with Lipschitz domains. The technique of proof consists in conformally mapping the Neumann problem over to an analytic domain, thus multiplying the boundary conditions by the derivative of the mapping function, and to solve it there using classical theorems on smooth domains. When mapping the solution back, we need to handle summability with respect to the extra-weight coming from that derivative, which is done *via* the theory of Muckenhoupt weights granted the detailed knowledge one has of the singularities of the conformal map in the piecewise $C^{1,\alpha}$ setting. Appealing to weighted norm inequalities to handle summability of boundary values in the Neumann problem is not original in itself, as it was used to design counterexamples to L^p theorems with variable coefficients [45]. But to the authors knowledge, their use to obtain existence theorems on non-Lipschitz domains is new.

First, let us fix some terminology. We say that Γ is a piecewise-$C^{1,\alpha}$ polygon, $0 < \alpha \leq 1$, if there is on some interval $[a, b]$ a continuous parameterization $w : [a, b] \to \Gamma$ with $w(a) = w(b)$ which is one-to-one $[a, b) \to \Gamma$ and satisfies, for some partition $a = s_0 < s_1 < \cdots < s_N = b$, that its restriction to each interval $[s_j, s_{j+1}]$ has a nonvanishing derivative which is Lipschitz-continuous of order α. Recall that Lipschitz-continuity of order α of w' on $[s_j, s_{j+1}]$ means that there exists a constant $M_j > 0$ such that

$$|w'(\tau) - w'(\eta)| \leq M_j |\tau - \eta|^\alpha, \quad \tau, \eta \in [s_j, s_{j+1}].$$

The points $W_j = w(s_j)$ with $0 \leq j \leq N-1$ are the vertices of Γ. We denote by $\Gamma_j = w([s_j, s_{j+1}])$ the closed oriented arc that links W_j to W_{j+1}, and we endow Γ with the orientation inherited from the Γ_j's. We let further $\sigma_j\pi$, for $1 \leq j \leq N-1$, be the oriented angle $\widehat{w_{j-1}, v_j} \in [-\pi, \pi]$, where w_{j-1} and v_j are the respective tangents to Γ_{j-1} and Γ_j at W_j; we also set $\sigma_0\pi = \widehat{w_{N-1}, v_0}$. Thus $\sigma_j\pi$ is the *oriented jump of the argument of the forward tangent to* Γ *at* W_j. Note that the jump of the argument of the tangent at each vertex *depends on the orientation*: changing the orientation changes its sign. If $\sigma_j = 0$ the tangent behaves continuously up to a reparametrization, and W_j is not a true corner. If Γ is oriented counter-clockwise and $\sigma_j = 1$ we get an outward-pointing cusp at W_j, whereas if $\sigma_j = -1$ we get an inward-pointing cusp. At every $z \in \Gamma \setminus \cup_j W_j$ the oriented unit tangent $t_\Gamma(z)$ does exist, and then the oriented normal at z is the unit vector $n_\Gamma(z)$ such that $(t_\Gamma(z), n_\Gamma(z))$ is a positively oriented orthonormal frame.

We say that γ is a piecewise-$C^{1,\alpha}$ closed arc if it obeys the same definition as given above for a polygon, except that we do not require this time $w(a) = w(b)$. The points $\gamma_0 = w(a)$, $\gamma_1 = w(b)$ are called the endpoints of γ, and we shall distinguish them from other vertices.

We now enumerate the hypotheses on the domain D and the crack γ that will be made in what follows. In the forthcoming sections, when we consider the inverse problem of locating γ, we shall strengthen the assumptions on the latter.

H1 D is a bounded Jordan domain in $\mathbb{C}$ whose boundary Γ is a piecewise-$C^{1,\alpha}$ polygon for some α satisfying $0 < \alpha \leq 1$, with vertices $W_0, \dots, W_{N-1}$ where $N \geq 0$. We orient Γ counter-clockwise.

H2 For each $j \in \{0, \dots, N-1\}$, the jump $\sigma_j \pi$ of the argument of the tangent to Γ at W_j is neither 0 nor π (*i.e.* the W_j are truly corners if $N > 0$ and there is no outward-pointing cusp).

H3 γ is a piecewise-$C^{1,\alpha}$ oriented closed Jordan arc that lies interior to D, with distinct endpoints γ_0, γ_1 and vertices $V_0, \dots, V_{M-1}$ where $M \geq 0$.

H4 For each $k \in \{0, \dots, M-1\}$, the jump $\kappa_k \pi$ of the argument of the tangent to γ at V_k is neither 0 nor $\pm\pi$ (*i.e.* the V_k are truly corners if $M > 0$ and there is no cusp).

To define the precise meaning that we assign to the boundary conditions in (2.1), we need now introduce some standard terminology. For $w \in \Gamma \setminus \cup_j \{W_j\}$ and $a > 0$, we define a nontangential region of approach to w from inside $D \setminus \gamma$ by setting

$$C(a, w) \stackrel{\Delta}{=} \{z \in D \setminus \gamma; \quad |z - w| < (1 + a)\, d(z, \Gamma \cup \gamma)\},$$

where $d(z, \Gamma \cup \gamma)$ is the Euclidean distance from z to $\Gamma \cup \gamma$; the fact that w is a smooth boundary point guarantees that $w \in \overline{C(a, w)}$. We say that a (complex or vector-valued) function v on $D \setminus \gamma$ *converges nontangentially* to v_0 at w if, for each $a > 0$, one has

$$\lim_{\substack{z \to w \\ z \in C(a,w)}} v(z) = v_0.$$

Associated with a, the *nontangential maximal function of* v on Γ is given by

$$M_a v(w) \stackrel{\Delta}{=} \sup_{z \in C(a,w)} \|v(z)\|,$$

which is well-defined except perhaps at the W_j's with values in $[0, +\infty]$.

On γ, these definitions will be modified to distinguish between nontangential approaches from each side. The easiest way is to imbed γ into a bigger $C^{1,\alpha}$-arc that cuts out D in two pieces D^+ and D^-, to which we apply the previous definitions. This gives rise to one-sided notions of nontangential limit and maximal function at each $s \in \gamma \setminus (\{\gamma_0, \gamma_1\} \cup (\cup_k V_k))$, that we distinguish by putting a superscript $\pm$, depending whether the approach is taken along $n_\gamma(\zeta)$ or its negative.

For A a disjoint union of rectifiable Jordan arcs and σ the linear measure induced by arclength, we let $L^p(A)$ denote, for $1 \leq p \leq \infty$, the familiar space of (equivalence classes of σ-a.e. coinciding) measurable functions f on A such that $\|f\|_{L^p(A)} < \infty$, where

$$\|f\|_{L^p(A)} \stackrel{\Delta}{=} \left(\int_A |f|^p \, d\sigma \right)^{1/p} \quad \text{if } p < \infty, \qquad \|f\|_{L^\infty(A)} \stackrel{\Delta}{=} \text{ess. sup. } \{|f(\zeta)|; \ \zeta \in A\}.$$

If W is a non-negative weight function on A, we write $f \in L^p(A, W)$ to mean that $fW \in L^p(A)$.

For $0 < \beta < 1$, denote by $\Lambda^\beta(A)$ the space of Hölder-continuous functions with exponent β on A, endowed with the norm

$$\|f\|_{\Lambda^\beta(A)} = \|f\|_\infty + \sup_{\substack{\xi,\zeta\in A\\ \xi\neq\zeta}} \frac{|f(\xi)-f(\zeta)|}{\sigma([\xi,\zeta])^\beta}, \tag{4.1}$$

where $[\xi,\zeta]$ is an arc of minimal length linking ξ and ζ in A.

Recall that, for $0 < p < \infty$, the *Hardy space* $H^p(\Omega)$ of a plane domain Ω consists of all functions f holomorphic in Ω such that $|f|^p$, which is subharmonic, has a harmonic majorant there. When $p \geq 1$, a family of equivalent norms is obtained by picking a point in Ω and evaluating there the least harmonic majorant of $|f|^p$ to the $1/p$; this makes $H^p(\Omega)$ into a Banach space. The space $H^\infty(\Omega)$ consists of all bounded holomorphic functions in Ω, endowed with the *sup* norm. Clearly, these definitions are conformally invariant.

Two particular cases will be of importance to us; the first is $\Omega = \mathbb{D}_r$ where $\mathbb{D}_r$ denotes the open disk centered at 0 of radius r, and the second is $\Omega = \mathcal{A}_{r_1,r_2}$ where $\mathcal{A}_{r_1,r_2}$ denotes the annulus $\mathbb{D}_{r_2} \setminus \overline{\mathbb{D}_{r_1}}$ for $0 < r_1 < r_2$. When $0 < p < \infty$ it is easy to check [35, Thm. 6.7] that

$$f \in H^p(\mathbb{D}_r) \quad \text{if and only if} \quad \sup_{0\leq\rho<r} \left(\int_0^{2\pi} |f(\rho e^{i\theta}|^p \, d\theta \right)^{1/p} < \infty, \tag{4.2}$$

and if $p \geq 1$ the above *supremum* yields an equivalent norm on $H^p(\mathbb{D}_r)$. Moreover [31, Thm. 2.6.], each $f \in H^p(\mathbb{D}_r)$ has a nontangential limit a.e. on the circle $\mathbb{T}_r$ centered at 0 of radius r, which is the $L^p(\mathbb{T}_r)$ limit of $re^{i\theta} \mapsto f(\rho e^{i\theta})$ as $\rho \to r$ from below. Subsequently, by the decomposition theorem [63], [31, Thm. 10.12], it holds that

$$f \in H^p(\mathcal{A}_{r_1,r_2}) \;\text{ if and only if } \sup_{r_1<\rho<r_2} \left(\int_0^{2\pi} |f(\rho e^{i\theta}|^p \, d\theta \right)^{1/p} < \infty, \tag{4.3}$$

and the *supremum* yields an equivalent norm on $H^p(\mathcal{A}_{r_1,r_2})$ for $p \geq 1$. Each $f \in H^p(\mathcal{A}_{r_1,r_2})$ has a nontangential limit a.e. on $\mathbb{T}_{r_\ell}$, $\ell = \{1,2\}$, which is the $L^p(\mathbb{T}_{r_\ell})$ limit of $r_\ell e^{i\theta} \mapsto f(\rho e^{i\theta})$ as $\rho \to r_\ell$.

For $1 \leq p \leq \infty$, we also introduce the so-called *Smirnov class* $\mathcal{E}^p(\Omega)$. It consists of all functions f holomorphic in Ω for which there is a sequence of domains Δ_n with $\overline{\Delta}_n \subset \Omega$, whose boundary $\partial\Delta_n$ is a finite union of rectifiable Jordan curves, such that each compact subset of Ω is eventually contained in Δ_n, and having the property that

$$\sup_{n\in\mathbb{N}} \|f\|_{L^p(\partial\Delta_n)} < \infty. \tag{4.4}$$

It follows from the definition and the maximum principle that $\mathcal{E}^\infty(\Omega) = H^\infty(\Omega)$.

It is true, although not immediately clear, that $\mathcal{E}^p(\Omega)$ is a Banach space on which the *infimum* of (4.4) over all sequences Δ_n as above defines a norm; in fact, there is a *fixed* sequence of this type such that (4.4) already yields a norm [74], [31, Sec. 10.5]. When the boundary $\partial\Omega$ of Ω consists of finitely many rectifiable Jordan curves, each $f \in \mathcal{E}^p(\Omega)$ has nontangential limits a.e. on $\partial\Omega$ with respect to arclength and the boundary function thus defined lies in $L^p(\partial\Omega)$; this boundary function characterizes f completely in that it cannot vanish on a set of positive arclength unless $f \equiv 0$, and its $L^p(\partial\Omega)$-norm is a norm on $\mathcal{E}^p(\Omega)$ thereby identifying the latter with a closed subspace of $L^p(\partial\Omega)$. Moreover, f can be recovered from its boundary function by a Cauchy integral [73, 72], [31, Sec. 10.5]. It is not difficult to see that these properties continue to hold if some components of $\partial\Omega$ are rectifiable Jordan *arcs* rather than curves: the only difference is that nontangential limits have to be taken from each side of the arcs, see [20, App. A1] for such generalizations with Hardy spaces rather than Smirnov classes.

On finitely connected domains bounded by analytic Jordan curves, $\mathcal{E}^p$ turns out to be identical with H^p, but it is not so on domains with corners like $D \setminus \gamma$ (see [73, 72, 74] and [31, Sec. 10.5]). More precisely, one has the following criterion of membership to $\mathcal{E}^p(\Omega)$ when $1 \leq p < \infty$:

CS *Let Ψ map Ω conformally onto a domain Ω_1 bounded by analytic Jordan curves, and f be holomorphic in Ω. Then, $f \in \mathcal{E}^p(\Omega)$ iff $|f \circ \Psi^{-1}|^p |(\Psi^{-1})'|$ has a harmonic majorant in Ω_1.*

Introducing the complex derivatives:

$$\partial u/\partial z = \frac{1}{2}(\partial u/\partial x - i\partial u/\partial y), \qquad \partial u/\partial \bar{z} = \frac{1}{2}(\partial u/\partial x + i\partial u/\partial y), \tag{4.5}$$

it is easily checked that $f = \partial u/\partial z$ is holomorphic when u is harmonic, and the property that $f \in \mathcal{E}^p(\Omega)$ becomes a uniform summability condition for $\|\nabla u\|^p$ on *at least one* system of curves tending to the boundary.

Theorem 4.1 *Let D and γ satisfy assumptions* **H1**–**H4**, *and p satisfy $1 < p < 2$. Assume that $\Phi \in L^p(\Gamma)$ and $\phi^\pm \in L^p(\gamma)$ are real-valued functions meeting*

$$\int_\Gamma \Phi\, |dw| = 0 \qquad \text{and} \qquad \int_\gamma \phi^+ |ds| = \int_\gamma \phi^- |ds|. \tag{4.6}$$

Then, up to an additive constant, there is a unique harmonic function u in $D \setminus \gamma$ such that:

(i) $\partial u/\partial z \in \mathcal{E}^1(D \setminus \gamma)$,

(ii) the function $z \mapsto \nabla u(z) \cdot n_\Gamma(\zeta)$ converges nontangentially to $\Phi(\zeta)$ at almost every $\zeta \in \Gamma$ and the function $z \mapsto \nabla u(z) \cdot n_\gamma^\pm(\xi)$ converges nontangentially to $\phi^\pm(\xi)$ at almost every $\xi \in \gamma$.

It holds in fact that $\partial u/\partial z \in \mathcal{E}^p(D \setminus \gamma)$. Moreover, u is the real part of a function f holomorphic in $D \setminus \gamma$ that extends continuously to Γ and to γ from both sides. The boundary maps $f_{|\Gamma}$ on Γ and $f^\pm$ on γ are absolutely continuous with derivative in $L^p(\Gamma)$ and $L^p(\gamma)$ respectively, and the map

$$(\Phi, \phi^+, \phi^-) \mapsto \left(\frac{df_{|\Gamma}}{|dw|}, \frac{df^+}{|dw|}, \frac{df^-}{|dw|} \right)$$

is continuous from $L^p(\Gamma) \times L^p(\gamma) \times L^p(\gamma)$ into itself.

Before we proceed with the proof, a couple of remarks are perhaps in order:

- We stated the theorem in slightly greater generality than needed to handle (2.1) since the latter only deals with $\phi^+ = \phi^- = 0$. This restriction, however, would have been artificial, and the present version is for instance useful to handle *via* fixed-point methods other boundary conditions like $\partial u^\pm/\partial n^\pm = a(u^\mp - u^\pm)$. There, a positive constant a would express that the crack is not perfectly insulating. We do not pursue this case below.

- Theorem 4.1 would hold for several piecewise $C^{1,\alpha}$ cracks and holes with obvious modifications. The proof is essentially the same, but the case of a single crack allows us to map conformally $D \setminus \gamma$ onto an annulus whose circular symmetry makes for an easy connection with classical Fourier analysis.

- In standard treatments on Lipschitz domains [44, 45], an important role is played by nontangential estimates of the form

$$\|M_a \nabla u\|_{L^p(\Gamma)} \leq c\,\|\Phi\|_{L^p(\Gamma)}, \qquad \|M_a \nabla u^{\pm}\|_{L^p(\gamma)} \leq c\,\|\phi^{\pm}\|_{L^p(\gamma)}$$

  where the constant c depends only on p and the geometry. In our case, such estimates are not sufficient to control ∇u at the cusps nor at the endpoints of γ, and we replaced them by the belonging to some Smirnov class.

- By Hölder's inequality, the fact that f has L^p derivative on Γ, and on γ from above and below, implies that $f_{|\Gamma}$ and $f^{\pm}$ are Hölder-continuous of exponent $1-1/p$. This, however, does not imply that f satisfies a Hölder condition on $D\setminus\gamma$. With a bit of extra-work, one can show that, if $\varepsilon>0$ is so small that $\max\{\sigma_j,|\kappa_k|\}<1/(1+\varepsilon)$, then f is Hölder-continuous of exponent $\alpha_\varepsilon=\varepsilon(p-1)/2p(1+\varepsilon)$ with respect to the Riemannian metric $\mathcal{R}$ induced by $|dz|$ on $D\setminus\gamma$ (which is equivalent to the Euclidean one away from cusps and endpoints of γ).

It will be convenient to formally define the closure of the doubly connected domain $D\setminus\gamma$ in a way that distinguishes between the positive and negative "sides" of γ. For this, we attach to every $\zeta\in\gamma\setminus\{\gamma_0,\gamma_1\}$ a real number $r(\zeta)>0$ such that, if $B(\zeta,r(\zeta))$ denotes the open disk centered at ζ of radius $r(\zeta)$, then $B(\zeta,r(\zeta))\setminus\gamma$ consists of two connected components B_ζ^+ and B_ζ^- lying respectively on the positive and negative side of γ with respect to the orientation. Subsequently we define:

$$\overline{D\setminus\gamma}^{\pm} \triangleq \left(\overline{D}\setminus \overset{\circ}{\gamma}\right)\cup \overset{\circ}{\gamma}^{+}\cup \overset{\circ}{\gamma}^{-}, \tag{4.7}$$

where $\overset{\circ}{\gamma}^{+}$ and $\overset{\circ}{\gamma}^{-}$ are disjoint copies of $\overset{\circ}{\gamma}=\gamma\setminus\{\gamma_0,\gamma_1\}$, and where a neighborhood of $\zeta\in\overset{\circ}{\gamma}^{+}$ (resp. $\overset{\circ}{\gamma}^{-}$) in $\overline{D\setminus\gamma}^{\pm}$ contains a Euclidean neighborhood of ζ in $\overline{B_\zeta^+}$ (resp. $\overline{B_\zeta^-}$). For convenience, if $\zeta\in\gamma\setminus\{\gamma_0,\gamma_1\}$, we sometimes denote by $\zeta^{\pm}$ the image of ζ in $\overset{\circ}{\gamma}^{\pm}$. It is easily seen that $\overline{D\setminus\gamma}^{\pm}$ is just the completion of $D\setminus\gamma$ under the metric $\mathcal{R}$.

The technical facts from conformal mapping that we need are gathered in the next proposition.

Proposition 4.2 *Let D and γ satisfy assumptions* **H1**–**H4**. *Then, there exists a conformal map*

$$\Psi : D\setminus\gamma \to \mathcal{A},$$

where $\mathcal{A}$ is an annulus bounded by the unit circle $\mathbb{T}$ and some circle $\mathbb{T}_R$ of radius $R>1$, having the following properties:

(i) Ψ extends to a homeomorphism $\overline{D\setminus\gamma}^{\pm}\to\overline{\mathcal{A}}$ that maps Γ onto $\mathbb{T}_R$ and $\overset{\circ}{\gamma}^{+}\cup\overset{\circ}{\gamma}^{-}\cup\{\gamma_0,\gamma_1\}$ onto $\mathbb{T}$ (cf. definition (4.7)).

(ii) The derivative $(\Psi^{-1})'$ of Ψ^{-1} is such that

$$z \longrightarrow \frac{\prod_{j=0}^{N-1}(z-\Psi(W_j))^{\sigma_j}\ \prod_{k=0}^{M-1}(z-\Psi^+(V_k))^{\kappa_k}(z-\Psi^-(V_k))^{-\kappa_k}}{(z-\Psi(\gamma_0))(z-\Psi(\gamma_1))}\ \left(\Psi^{-1}\right)'(z)$$

extends continuously to $\overline{\mathcal{A}}\to\mathbb{C}\setminus\{0\}$.

Proof: Let Ψ_1 map conformally D onto the unit disk $\mathbb{D}$. Since Γ is a Jordan curve, Ψ_1 extends to a topological homeomorphism $\overline{D} \to \overline{\mathbb{D}}$ by Carathéodory's theorem [61, Thm. 4.4.13]. Because Γ is piecewise-$C^{1,\alpha}$, the corners W_j are Dini-smooth, and their opening angles (π minus oriented angle) are strictly positive since there is no outward-pointing cusp. Hence it follows from [60, Thm. 3.9] that

$$z \longrightarrow \left(\Psi_1^{-1}\right)'(z) \prod_{j=0}^{N-1} (z - \Psi_1(W_j))^{\sigma_j} \text{ extends continuously } \overline{\mathbb{D}} \to \mathbb{C} \setminus \{0\}. \tag{4.8}$$

Next we put $\gamma' = \Psi_1(\gamma)$ which is a piecewise-$C^{1,\alpha}$ closed Jordan arc with endpoints $\gamma_0' = \Psi_1(\gamma_0)$, $\gamma_1' = \Psi_1(\gamma_1)$, and vertices $V_k' = \Psi_1(V_k)$ for $0 \le k < M$. Let Ψ_2 map conformally $\overline{\mathbb{C}} \setminus \gamma'$ onto $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ with $\Psi_2(\infty) = \infty$. Note that Ψ_2^{-1} extends continuously to $\mathbb{T}$ by the local connectedness of γ' ([60, Thm. 2.1]). From Caratheodory's prime ends theorem [60, Thm. 2.15] there are unique preimages $\Psi_2(\gamma_0')$ and $\Psi_2(\gamma_1')$ on $\mathbb{T}$ of γ_0' and γ_1', whereas each of the two closed arcs on $\mathbb{T}$ with endpoints $\Psi_2(\gamma_0')$ and $\Psi_2(\gamma_1')$ is mapped homeomorphically onto γ' by Ψ_2^{-1}. In particular, each $\zeta \in \gamma' \setminus \{\gamma_0', \gamma_1'\}$ has two preimages $\Psi_2^{\pm}(\zeta)$ on $\mathbb{T}$ (so that γ is covered twice). Therefore Ψ_2^{-1} extends to a continuous bijective map $\overline{\mathbb{C}} \setminus \mathbb{D} \to \overline{\mathbb{C} \setminus \gamma'}^{\pm}$ (compare definition (4.7)) which must be a homeomorphism by the compactness of $\overline{\mathbb{C}} \setminus \mathbb{D}$ on the Riemann sphere.

Moreover, since $\overline{\mathbb{C}} \setminus \gamma'$ is a simply connected domain having (flat) piecewise-$C^{1,\alpha}$ boundary with two Dini-smooth inward-pointing cusps at γ_0', γ_1' and $2M$ Dini-smooth corners of aperture $\pi \pm \kappa_k$ at $(V_k')^{\pm}$, similar arguments to those we gave concerning the boundary behavior of Ψ_1 will apply to Ψ_2 as well; the only difference is that, in order to formally use [60, Thm. 3.9], one must deal with a conformal map from $\mathbb{D}$ (rather than $\overline{\mathbb{C}} \setminus \mathbb{D}$) onto a subdomain of $\mathbb{C}$ (rather than $\overline{\mathbb{C}}$). To remedy this, we first change $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ into $\mathbb{D}$ by setting $z = 1/s$, and we also change $\overline{\mathbb{C}} \setminus \gamma'$ into a subdomain of $\mathbb{C}$ by performing a Möbius transformation that maps some smooth point of γ' to infinity. These minor modifications are easily unwound to yield that

$$z \longrightarrow \frac{\left(\Psi_2^{-1}\right)'(z) \prod_{k=0}^{M-1} \left(z - \Psi_2^{+}(V_k')\right)^{\kappa_k} \left(z - \Psi_2^{-}(V_k')\right)^{-\kappa_k}}{(z - \Psi_2(\gamma_0'))(z - \Psi_2(\gamma_1'))} \text{ extends continuously to } \overline{\mathbb{C}} \setminus \mathbb{D} \to \mathbb{C} \setminus \{0\}. \tag{4.9}$$

Now, $\Psi_3 = \Psi_2 \circ \Psi_1$ conformally maps $D \setminus \gamma$ onto an annular region $\mathcal{A}'$ whose outer boundary is the analytic Jordan curve $C = \Psi_2(\mathbb{T})$ and whose inner boundary is $\mathbb{T}$. Since Ψ_1 induces a homeomorphism $\overline{D} \to \overline{\mathbb{D}}$ that maps γ onto γ' and Ψ_2 a homeomorphism $\overline{\mathbb{C} \setminus \gamma'}^{\pm} \to \overline{\mathbb{C}} \setminus \mathbb{D}$ that maps $\overset{\circ}{\gamma'}^{+} \cup \overset{\circ}{\gamma'}^{-} \cup \{\gamma_0', \gamma_1'\}$ onto $\mathbb{T}$, it follows that Ψ_3 extends to a homeomorphism $\overline{D \setminus \gamma}^{\pm} \to \overline{\mathcal{A}'}$ that maps Γ onto C and $\overset{\circ}{\gamma}^{+} \cup \overset{\circ}{\gamma}^{-} \cup \{\gamma_0, \gamma_1\}$ onto $\mathbb{T}$. Finally, let Ψ_4 map conformally $\mathcal{A}'$ onto an annulus $\mathcal{A} = \{z; 1 < z < R\}$; the existence of such a map is well-known [65, Thm.VIII.6.1]. *We claim* that Ψ_4 extends to a homeomorphism $\overline{\mathcal{A}'} \to \overline{\mathcal{A}}$ and that the derivative Ψ_4' extends continuously to $\overline{\mathcal{A}'} \to \mathbb{C} \setminus \{0\}$. The fact that Ψ_4 extends continuously and injectively to $\overline{\mathcal{A}'} \to \overline{\mathcal{A}}$ can be established as in the case of a conformal map from a simply connected domain with accessible boundary points onto $\mathbb{D}$; in fact, granted that each point of $\partial\mathcal{A}'$ is accessible and that every bounded analytic function on $\mathcal{A}'$ has nontangential limits at almost every boundary point [31, Thm. 10.3 and 10.12], the proof of [64, Thm. 14.18] applies almost *mutatis mutandis*. This extension is necessarily a homeomorphism $\overline{\mathcal{A}'} \to \overline{\mathcal{A}}$ by the compactness of $\overline{\mathcal{A}'}$. We may assume up to an inversion $z \to R/z$ that Ψ_4 maps $\mathbb{T}$ onto itself and C onto $\mathbb{T}_R$. Then, Ψ_4 extends holomorphically and injectively across $\mathbb{T}$ according to the reflection rule $\Psi_4(z) = 1/\overline{\Psi_4(1/\bar{z})}$, so that Ψ_4' extends continuously to $\mathbb{T} \to \mathbb{C} \setminus \{0\}$. Similarly $\Psi_5 = \Psi_4 \circ \Psi_{2|_{\mathbb{D}\setminus\gamma'}}$ extends holomorphically and injectively across $\mathbb{T}$ *via* $\Psi_5(z) = R^2/\overline{\Psi_5(1/\bar{z})}$, and by the conformal character of Ψ_2 we deduce that $\Psi_4'(z) = \Psi_5'(\Psi_2^{-1}(z))/\Psi_2'(\Psi_2^{-1}(z))$ extends continuously to $C \to \mathbb{C}\setminus\{0\}$. *This proves the claim*

Since Ψ_1 is smooth (even holomorphic) on γ with nonvanishing derivative and similarly for Ψ_2 on $\mathbb{T}$, we get from (4.8), (4.9), the previous claim, and the chain rule, that (i) and (ii) hold.

■

We are now ready for the proof of Theorem 4.1, in which we shall have a first opportunity to deal with Hardy spaces and Smirnov classes.

Proof of Theorem 4.1: We first show that a harmonic function u satisfying (i) and (ii) is unique up to an additive constant. Let Ψ be as in Proposition 4.2, and observe by (ii) of the latter that Ψ^{-1} is conformal up to $\partial\mathcal{A}\setminus\cup_{j,k,l}\{\Psi(W_j),\Psi(V_k),\Psi(\gamma_l)\}$, where $\partial\mathcal{A} = \mathbb{T}_R\cup\mathbb{T}$ is the boundary of $\mathcal{A}$. In particular, identifying a complex number with a vector in $\mathbb{R}^2$, we have for $\zeta \in \partial\mathcal{A}\setminus\cup_{j,k,l}\{\Psi(W_j),\Psi(V_k),\Psi(\gamma_l)\}$ that $n_{\partial\mathcal{A}}(\zeta)$ (which is just $\pm\zeta/|\zeta|$) is mapped under multiplication by $(\Psi^{-1})'(\zeta)$ to a vector which is normal to $\Gamma\cup\gamma$ at $\Psi^{-1}(\zeta)$ and points inward $D\setminus\gamma$. In terms of the variables $w\in\Gamma$ and $t\in\gamma$, this means exactly:

$$n_\Gamma(w) = -\frac{\left(\Psi^{-1}\right)'\circ\Psi(w)\ \ \Psi(w)}{\left|(\Psi^{-1})'\circ\Psi(w)\right| R},\qquad w\in\Gamma\setminus\{W_0,\dots,W_{N-1}\},$$

$$n_\gamma^\pm(t) = \frac{\left(\Psi^{-1}\right)'\circ\Psi^\pm(t)\ \ \Psi^\pm(t)}{\left|(\Psi^{-1})'\circ\Psi^\pm(t)\right|},\qquad t\in\gamma\setminus\{\gamma_0,\gamma_1,V_0,\dots,V_{M-1}\}.$$

Thus upon setting $\Psi(w)=\zeta$, $\Psi^\pm(t)=\xi^\pm$, and $\Psi(z)=s$ for $z\in D\setminus\gamma$, we get from (4.5):

$$\begin{aligned}\nabla u(z)\cdot n_\Gamma(w) &= -2\mathbf{Re}\left\{\zeta\left(\Psi^{-1}\right)'(\zeta)\ \partial u/\partial z\circ\Psi^{-1}(s)\right\}\left|R\left(\Psi^{-1}\right)'(\zeta)\right|^{-1},\\ \nabla u(z)\cdot n_\gamma^\pm(t) &= 2\mathbf{Re}\left\{\xi^\pm\left(\Psi^{-1}\right)'(\xi^\pm)\ \partial u/\partial z\circ\Psi^{-1}(s)\right\}\left|\left(\Psi^{-1}\right)'(\xi^\pm)\right|^{-1}.\end{aligned}\tag{4.10}$$

Next, observe from criterion **CS**, since $\partial u/\partial z$ is holomorphic, that (i) is equivalent to:

$$h_u \triangleq \left(\partial u/\partial z\circ\Psi^{-1}\right)\left(\Psi^{-1}\right)'\in H^1(\mathcal{A}).\tag{4.11}$$

In particular, see *e.g.* [31, Thms. 10.3, 10.12] or [67, 74], h_u has nontangential limit $h_u(\xi)$ at almost every $\xi\in\partial A$. Now, since Ψ^{-1} is conformal up to $\partial\mathcal{A}\setminus\cup_{j,k,l}\{\Psi(W_j),\Psi(V_k),\Psi(\gamma_l)\}$, it is easy to check with the notations of (4.10) that $z\to w$ (resp. $z\to t$) within a region $C(a,w)$ (resp. $C^\pm(a,t)$) in $D\setminus\gamma$ if, and only if, $s\to\zeta$ (resp. $s\to\xi^\pm$) within a region $C(a',\zeta)$ (resp. $C(a',\xi^\pm)$) in $\mathcal{A}$. Therefore, letting $\mathbb{T}^\pm$ be the closed arc in $\mathbb{T}$ which is mapped homeomorphically by Ψ^{-1} onto $\overset{\circ}{\gamma}{}^\pm\cup\{\gamma_0,\gamma_1\}$, we get from (ii) and (4.10) that

$$\begin{aligned}\Phi\left(\Psi^{-1}(\zeta)\right) &= -2\mathbf{Re}\left\{\zeta h_u(\zeta)\right\}\left|R\left(\Psi^{-1}\right)'(\zeta)\right|^{-1} &&\text{for a.e. } \zeta\in\mathbb{T}_R,\\ \phi^+\left(\Psi^{-1}(\xi^+)\right) &= 2\mathbf{Re}\left\{\xi^+ h_u(\xi^+)\right\}\left|\left(\Psi^{-1}\right)'(\xi^+)\right|^{-1} &&\text{for a.e. } \xi^+\in\mathbb{T}^+,\\ \phi^-\left(\Psi^{-1}(\xi^-)\right) &= 2\mathbf{Re}\left\{\xi^- h_u(\xi^-)\right\}\left|\left(\Psi^{-1}\right)'(\xi^-)\right|^{-1} &&\text{for a.e. } \xi^-\in\mathbb{T}^-.\end{aligned}\tag{4.12}$$

If u, v are harmonic functions in $D\setminus\gamma$ satisfying (i) and (ii), and if we set $h=h_u-h_v$, then $F(z)=zh(z)$ lies in $H^1(\mathcal{A})$ since $h(z)$ does, and from (4.12) we see that $\mathbf{Re}\,F=0$ a.e. on $\partial\mathcal{A}$. By a classical reflection principle*, F extends holomorphically to the annulus $R^{-1}<|z|<R^2$ according

*The extension issue being a local one, one can restrict oneself to a subdomain $\{z=re^{i\theta};\ 1<r<R,\theta_1<\theta<\theta_2\}$ which is simply connected, and then apply [48, Ch. III, Sec. E] after conformal mapping onto the unit disk, thanks to the conformal invariance of Hardy spaces.

to the rule $F(z) = -\overline{F(R^2/\bar{z})}$ if $|z| > R$ and $F(z) = -\overline{F(1/\bar{z})}$ if $|z| < 1$. This shows *a posteriori* that F was bounded in $\mathcal{A}$; therefore its extension is also bounded and purely imaginary on $\mathbb{T}_{R^2}$ and $\mathbb{T}_{1/R}$. Iterating this extension process yields a holomorphic function in $\mathbb{C} \setminus \{0\}$ which is bounded and therefore a constant, for zero must be a removable singularity [64, Thm. 10.21] and we can use Liouville's theorem [64, Thm 10.23]. Thus $h(z) = \zeta_0/z$, where ζ_0 is purely imaginary. In another connection, since by holomorphy $\partial\Psi^{-1}/\partial z = \left(\Psi^{-1}\right)'$ and $\partial\overline{\Psi^{-1}}/\partial z = 0$, we get from (4.11) and the chain rule that $h_u = \partial\left(u \circ \Psi^{-1}\right)/\partial z$. Hence

$$\frac{\partial\left((u-v)\circ\Psi^{-1}\right)}{\partial z} = \frac{\zeta_0}{z}, \qquad z \in \mathcal{A},$$

from which it follows (recall $u - v$ is real-valued and ζ_0 is pure imaginary) that

$$(u-v)\circ\Psi^{-1} + 2\zeta_0 \log|z|$$

must be an analytic function in $\mathcal{A}$ because it vanishes when applying $\partial/\partial\bar{z}$ to it. Consequently the pure imaginary number ζ_0 is zero (since $\log z$ has no single-valued branch in $\mathcal{A}$) and so u and v must differ by a real constant as desired.

Next we prove the existence of u and f claimed in Theorem 4.1. Put

$$\begin{cases} \Phi_1(\zeta) &= \Phi\left(\Psi^{-1}(\zeta)\right)\left|\left(\Psi^{-1}\right)'(\zeta)\right|, \quad \zeta \in \mathbb{T}_R, \\ \Phi_1(\xi) &= \pm\phi^{\pm}\left(\Psi^{-1}(\xi)\right)\left|\left(\Psi^{-1}\right)'(\xi)\right|, \quad \xi \in \mathbb{T}^{\pm}, \end{cases} \tag{4.13}$$

which is well-defined a.e. on $\partial\mathcal{A}$. From (4.13) and the fact that $\Phi \in L^p(\Gamma)$, $\phi^{\pm} \in L^p(\gamma)$, we get

$$\begin{aligned} \int_{\partial\mathcal{A}} |\Phi_1|^p \left|\left(\Psi^{-1}\right)'\right|^{1-p} |d\zeta| &= \int_{\mathbb{T}_R} \left|\Phi\circ\Psi^{-1}\right|^p \left|\left(\Psi^{-1}\right)'\right| |d\zeta| + \int_{\mathbb{T}^+\cup\mathbb{T}^-} \left|\phi^{\pm}\circ\Psi^{-1}\right|^p \left|\left(\Psi^{-1}\right)'\right| |d\xi| \\ &= \int_\Gamma |\Phi|^p |dw| + \int_\gamma |\phi^+|^p |ds| + \int_\gamma |\phi^-|^p |ds| < +\infty. \end{aligned} \tag{4.14}$$

Factoring out

$$|\Phi_1(\zeta)| = \left(\frac{|\Phi_1(\zeta)|}{\left|\left(\Psi^{-1}\right)'(\zeta)\right|^{(p-1)/p}}\right)\left(\left|\left(\Psi^{-1}\right)'(\zeta)\right|^{(p-1)/p}\right),$$

we see from (4.14) that the first factor lies in $L^p(\partial\mathcal{A})$ and from Proposition 4.2 (*ii*) that the second factor lies in $L^{p(1+\varepsilon)/(p-1)}(\partial\mathcal{A})$, as soon as $\varepsilon > 0$ is so small that $\max\{\sigma_j, |\kappa_k|\} < 1/(1+\varepsilon)$. We can pick such an ε because σ_j and $|\kappa_k|$ are strictly less than 1 and then, by Hölder's inequality, we obtain

$$\|\Phi_1\|_{L^\beta(\partial\mathcal{A})} \le c_0\left(\int_\Gamma |\Phi|^p |dw| + \int_\gamma |\phi^+|^p |ds| + \int_\gamma |\phi^-|^p |ds|\right)^{1/p}, \qquad \beta \triangleq \frac{1+\varepsilon}{1+\varepsilon/p} > 1, \tag{4.15}$$

where $c_0 = \|(\Psi^{-1})'\|_{L^{1+\varepsilon}}^{(p-1)/p}$ depends only on p and the geometry. Also, by (4.6) and (4.13),

$$\int_{\mathbb{T}_R} \Phi_1(\zeta)\,|d\zeta| = \int_\Gamma \Phi_1(\Psi(w))\left|\Psi'(w)\right| |dw| = \int_\Gamma \Phi(w)\,|dw| = 0 \tag{4.16}$$

and likewise

$$\int_{\mathbb{T}} \Phi_1(\xi)\,|d\xi| = \int_\gamma \phi^+(s)\,|ds| - \int_\gamma \phi^-(s)\,|ds| = 0.$$

Hence by a well-known existence result for the Neumann problem on smooth domains (see *e.g.* [4]or [71, Chap. XVII], [33], [45, cor 2.2.14] for a more general version on C^1-domains), there is a harmonic function u_1 in $\mathcal{A}$ such that:

a) $M_a \nabla u_1$ lies in $L^\beta(\partial\mathcal{A})$ for some (and then every) $a > 0$,

b) $z \mapsto \nabla u_1(z).n_{\mathbb{T}_R}(\zeta)$ converges nontangentially to $\Phi_1(\zeta)$ at almost every $\zeta \in \mathbb{T}_R$,

c) $z \mapsto \nabla u_1(z).n_{\mathbb{T}}(\xi)$ converges nontangentially to $\Phi_1(\xi)$ at almost every $\xi \in \mathbb{T}$.

Actually (*cf.* the previous references), property *a*) can be made more precise in that

$$\|M_a \nabla u_1\|_{L^\beta(\partial\mathcal{A})} \leq c_1 \|\Phi_1\|_{L^\beta(\partial\mathcal{A})}, \tag{4.17}$$

where the constant c_1 depends only on a and R. As u_1 is harmonic, the function

$$g(z) \triangleq \partial u_1/\partial z \tag{4.18}$$

is holomorphic in $\mathcal{A}$, and since $2|g| = \|\nabla u_1\|$ it follows from *a*), (4.17) and (4.15) by an obvious majorization that

$$\sup_{1<r<R} \left(\int_0^{2\pi} |g(re^{i\theta})|^\beta \, d\theta \right)^{1/\beta} \leq c_2 \left(\int_\Gamma |\Phi|^p \, |dw| + \int_\gamma |\phi^+|^p \, |ds| + \int_\gamma |\phi^-|^p \, |ds| \right)^{1/p} \tag{4.19}$$

where c_2 depends only on p and the geometry. Taking for Δ_n a nested sequence of subannuli that exhausts $\mathcal{A}$ in (4.4), we see from (4.19) that $g \in \mathcal{E}^\beta(\mathcal{A}) = H^\beta(\mathcal{A})$; thus it has nontangential limit $g(\xi)$ at almost every $\xi \in \partial A$, and moreover [31, Thms. 10.3, 10.12], [67, 74],

$$\lim_{\substack{r\to R \\ 1<r<R}} \int_0^{2\pi} |g(Re^{i\theta}) - g(re^{i\theta})|^\beta \, d\theta = \lim_{\substack{r\to 1 \\ 1<r<R}} \int_0^{2\pi} |g(e^{i\theta}) - g(re^{i\theta})|^\beta \, d\theta = 0. \tag{4.20}$$

Let us show that

$$\int_\Upsilon g(z) \, dz = 0 \tag{4.21}$$

whenever Υ is a smooth Jordan curve winding around $\mathbb{T}$ in $\mathcal{A}$. First, this integral is pure imaginary, because u_1 is real so that

$$0 = \int_\Upsilon du_1 = \int_\Upsilon \frac{\partial u_1}{\partial z} \, dz + \int_\Upsilon \frac{\partial u_1}{\partial \bar{z}} \, d\bar{z} = \int_\Upsilon \frac{\partial u_1}{\partial z} \, dz + \overline{\int_\Upsilon \frac{\partial u_1}{\partial z} \, dz} = 2\,\mathbf{Re}\left\{ \int_\Upsilon g(z) \, dz \right\}$$

by (4.18). Second, using (4.18) again, it is straightforward to compute that

$$\mathbf{Im}\left\{ \int_\Upsilon g(z) \, dz \right\} = \frac{1}{2} \int_\Upsilon \frac{\partial u_1}{\partial x} \, dy - \frac{\partial u_1}{\partial y} \, dx = -\int_\Upsilon \nabla u_1 \cdot n_\Upsilon \, |dz|. \tag{4.22}$$

By Cauchy's theorem, we may deform Υ into $\mathbb{T}_r$ for $r \in (1, R)$ without changing the value of (4.22). Thus in view of *a*), *b*), and (4.16), we get by dominated convergence when $r \to R$ that

$$\mathbf{Im}\left\{ \int_\Upsilon g(z) \, dz \right\} = -\lim_{r\to R} \int_{\mathbb{T}_r} \nabla u_1 \cdot n_{\mathbb{T}_r} \, |dz| = -\int_{\mathbb{T}_R} \Phi_1(z) \, |dz| = 0.$$

From (4.21) it follows by elementary path integration that g has an integral in $\mathcal{A}$, *i.e.* there is an analytic function G, unique up to an additive constant, such that $G' = g$ there.

Define

$$f = 2G \circ \Psi \quad \text{and} \quad u = \mathbf{Re} f. \tag{4.23}$$

Clearly f is holomorphic in $D \setminus \gamma$ and u is harmonic there. Moreover, we get for the derivatives:

$$f' = 2(g \circ \Psi)\Psi' \quad \text{and} \quad \partial u/\partial z = f'/2. \tag{4.24}$$

For $z = \Psi^{-1}(s) \in D \setminus \gamma$ and $w = \Psi^{-1}(\zeta) \in \Gamma \setminus \cup_j \{W_j\}$, we compute as in (4.10) using (4.23)-(4.24)

$$\begin{aligned} \nabla u(z) \cdot n_\Gamma(w) &= -2\mathbf{Re}\left\{\zeta \left(\Psi^{-1}\right)'(\zeta)\ \partial u/\partial z \circ \Psi^{-1}(s)\right\} \left|\zeta \left(\Psi^{-1}\right)'(\zeta)\right|^{-1} \\ &= -2\mathbf{Re}\left\{\zeta \left(\Psi^{-1}\right)'(\zeta)\ g(s)\ \Psi' \circ \Psi^{-1}(s)\right\} \left|\zeta \left(\Psi^{-1}\right)'(\zeta)\right|^{-1} \\ &= -2\mathbf{Re}\left\{(\zeta g(s)/|\zeta|)\ \left(\left(\Psi^{-1}\right)'(\zeta)/\left(\Psi^{-1}\right)'(s)\right)\right\} \left|\left(\Psi^{-1}\right)'(\zeta)\right|^{-1}. \end{aligned} \tag{4.25}$$

As $2\mathbf{Re}\left\{\zeta g(\zeta)|\zeta|^{-1}\right\} = -\Phi_1(\zeta)$ a.e. on $\mathbb{T}_R$ by $b)$ and (4.18), we see from (4.13) and Proposition 4.2 (ii) that the last term in (4.25), when viewed as a function of s, converges nontangentially to $\Phi(w)$ at almost every $\zeta \in \mathbb{T}_R$. But since $z \to w$ nontangentially in $D \setminus \gamma$ if and only if $s \to \zeta$ nontangentially in $\mathcal{A}$, we conclude that $\nabla u \cdot n_\Gamma(w)$ converges nontangentially to $\Phi(w)$ at a.e. $w \in \Gamma$. A similar argument shows that $\nabla u^\pm \cdot n_\gamma(\xi)$ converges nontangentially to $\phi^\pm(\xi)$ at almost every $\xi \in \gamma$, so that point (ii) of Theorem 4.1 holds true.

Pick θ_1 such that g has a nontangential limit both at $Re^{i\theta_1}$ and at $e^{i\theta_1}$, and select $r_0 \in (1, R)$. Then, upon writing

$$G(re^{i\theta_2}) = G(r_0 e^{i\theta_1}) + \int_{r_0}^{r} g(\rho e^{i\theta_1})e^{i\theta_1}\, d\rho + \int_{\theta_1}^{\theta_2} g(re^{i\theta}) ire^{i\theta}\, d\theta, \tag{4.26}$$

we deduce from (4.20) that G extends continuously to $\overline{\mathcal{A}}$ and that it is absolutely continuous on $\partial\mathcal{A}$ with derivative $dG/|d\zeta| = i\zeta g(\zeta)/|\zeta|$. Hence, in view of (4.23) and Proposition 4.2, f in turn extends continuously to $\overline{D \setminus \gamma}^{\pm}$; moreover $f_{|\Gamma}$ and $f^\pm$ are absolutely continuous with respect to arclength on $\Gamma \setminus \cup_j \{W_j\}$ and $\overset{\circ}{\gamma} \setminus \cup_k \{V_k\}$ respectively, where they have derivative:

$$\frac{df_{|\Gamma}}{|dw|} = 2i\frac{\Psi}{|\Psi|}\,(g \circ \Psi)\,|\Psi'|, \qquad \frac{df^\pm}{|dw|} = 2i\frac{\Psi^\pm}{|\Psi^\pm|}\,\left(g \circ \Psi^\pm\right)\,|\left(\Psi'\right)^\pm|. \tag{4.27}$$

Using the parametrization $\theta \to \Psi^{-1}(Re^{i\theta})$ of Γ (which is absolutely continuous by Proposition 4.2 (ii) because $\sigma_j < 1$ for all j), we deduce from (4.27) that

$$\left\|\frac{df_{|\Gamma}}{|dw|}\right\|^p_{L^p(\Gamma)} = 2^p \int_{\mathbb{T}_R} |g(\zeta)|^p \left|\left(\Psi^{-1}\right)'(\zeta)\right|^{1-p} |d\zeta| \tag{4.28}$$

that we now prove is finite. Since $2\mathbf{Re}\left\{e^{i\theta} g\left(Re^{i\theta}\right)\right\} = -\Phi_1\left(Re^{i\theta}\right)$ for a.e. θ, equation (4.14) yields

$$(2/R)^p \int_{\mathbb{T}_R} |\mathbf{Re}\{\zeta g(\zeta)\}|^p \left|\left(\Psi^{-1}\right)'(\zeta)\right|^{1-p} |d\zeta| \le \int_\Gamma |\Phi|^p\,|dw| + \int_\gamma |\phi^+|^p\,|ds| + \int_\gamma |\phi^-|^p\,|ds|. \tag{4.29}$$

As a member of $\mathcal{E}^\beta(\mathcal{A})$, the function $zg(z)$ is the Cauchy integral of its boundary values [74][31, Sec. 10.5], hence we can write $zg(z) = g_1(z) + g_2(z)$ with

$$g_1(z) = \frac{1}{2i\pi}\int_{\mathbb{T}} \frac{\zeta g(\zeta)}{z - \zeta}\, d\zeta \ \text{ for } |z| > 1, \qquad g_2(z) = \frac{1}{2i\pi}\int_{\mathbb{T}_R} \frac{\zeta g(\zeta)}{\zeta - z}\, d\zeta \ \text{ for } |z| < R, \tag{4.30}$$

where we note that $g_1 \in H^\beta(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ (equivalently $g_1(1/z) \in H^\beta(\mathbb{D})$) and $g_2 \in H^\beta(\mathbb{D}_R)$ by the continuity of the Cauchy projection $L^\beta(\mathbb{T}_r) \to H^\beta(\mathbb{D}_r)$ for $1 < \beta < \infty$ (which is an immediate consequence of the M. Riesz theorem, see [35, Thm. 2.3]). Applying Hölder's inequality in (4.30), we obtain in view of (4.19) that

$$\|g_1\|_{L^\infty(\mathbb{T}_R)} < c_3 \left(\int_\Gamma |\Phi|^p \, |dw| + \int_\gamma |\phi^+|^p \, |ds| + \int_\gamma |\phi^-|^p \right)^{1/p} \tag{4.31}$$

where c_3 depends only on p and the geometry. As $1 < p < 2$ while $-1 \leq \sigma_j$, we certainly have

$$p < 1 - 1/\sigma_j \quad \text{whenever} \quad \sigma_j < 0 \tag{4.32}$$

so that $1/(\Psi^{-1})' \in L^{p-1}(\mathbb{T}_R)$ by Proposition 4.2 (*ii*). Thus by (4.29) and (4.31)

$$\int_{\mathbb{T}_R} |\mathbf{Re}\{g_2(\zeta)\}|^p \left| \left(\Psi^{-1}\right)'(\zeta) \right|^{1-p} |d\zeta| \leq c_4 \left(\int_\Gamma |\Phi|^p \, |dw| + \int_\gamma |\phi^+|^p \, |ds| + \int_\gamma |\phi^-|^p \, |ds| \right) \tag{4.33}$$

where c_4 depends only on p and the geometry. At this point, recall that a non-negative locally integrable weight W on $\mathbb{T}_R$ is said to satisfy the *condition* A_p of Muckenhoupt if, and only if† :

$$\sup_{t \in [0,2\pi)} \sup_{0<\varepsilon<\pi} \left(\frac{1}{2R\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} W^p \, d\theta \right)^{1/p} \left(\frac{1}{2R\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} W^{-q} \, d\theta \right)^{1/q} < +\infty, \qquad 1/p + 1/q = 1. \tag{4.34}$$

Moreover, since $|(\Psi^{-1})'|$ behaves like a power weight on $\mathbb{T}_R$ by Proposition 4.2 (*ii*), it follows from [30, Thm. 2.2] that $W = |(\Psi^{-1})'|^{1/p-1}$ has A_p if, and only if (4.32) holds. But it is well-known [42] (or *e.g.* [35, Thm. 6.2] for an exposition on the line rather than the circle) that A_p characterizes the boundedness of the conjugate operator from $L^p(\mathbb{T}_R, W)$ into itself. Since $g_2 \in H^\beta(\mathbb{D}_R)$ its imaginary part is conjugate to its real part on $\mathbb{T}_R$, and therefore $g_2(\zeta) \in L^p(\mathbb{T}_R, W)$; adding g_1 back, we thereby conclude from (4.31) and (4.28) that

$$\left\| \frac{df_{|\Gamma}}{|dw|} \right\|_{L^p(\Gamma)}^p \leq c_5 \left(\int_\Gamma |\Phi|^p \, |dw| + \int_\gamma |\phi^+|^p \, |ds| + \int_\gamma |\phi^-|^p \right) \tag{4.35}$$

where c_5 depends only on p and the geometry. Starting from the relation:

$$\left\| \frac{df^+}{|dw|} \right\|_{L^p(\gamma)}^p + \left\| \frac{df^-}{|dw|} \right\|_{L^p(\gamma)}^p = 2^p \int_{\mathbb{T}} |g(\zeta)|^p \left| \left(\Psi^{-1}\right)'(\zeta) \right|^{1-p} |d\zeta|,$$

a similar argument where g_1 and g_2 get swaped while (4.32) gets replaced by

$$-1 < \kappa_k < 1 \quad \text{and} \quad 1 < p < 2$$

(this ensures that W has A_p on $\mathbb{T}$ hence that $g_1 \in L^p(\mathbb{T}, W)$) leads us to

$$\left\| \frac{df^+}{|dw|} \right\|_{L^p(\gamma)}^p + \left\| \frac{df^-}{|dw|} \right\|_{L^p(\gamma)}^p \leq c_6 \left(\int_\Gamma |\Phi|^p \, |dw| + \int_\gamma |\phi^+|^p \, |ds| + \int_\gamma |\phi^-|^p \right) \tag{4.36}$$

where c_6 depends only on p and the geometry. The continuity of $(\Phi, \phi^+, \phi^-) \mapsto \left(\frac{df_{|\Gamma}}{|dw|}, \frac{df^+}{|dw|}, \frac{df^-}{|dw|} \right)$ asserted in the theorem now follows from (4.35) and (4.36).

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†This definition is not the most commonly used : many articles and textbooks would say that W satisfies A_p if $W^{1/p}$ meets (4.34) and accordingly would write $f \in L^p(A, W)$ to mean $f^p W \in L^1(A)$ instead of $f^p W^p \in L^1(A)$ which is our present convention. Our definition of A_p is as in [30] in order to conveniently quote Thm. 2.2 there.

Finally let us prove that $\partial u/\partial z \in \mathcal{E}^p(D \setminus \gamma)$. Define two functions on $\mathbb{D}_R$ and $\mathbb{D}$ by:

$$\begin{aligned} \varphi_\Gamma(z) &= \prod_{j=0}^{N-1} (z - \Psi(W_j))^{\sigma_j(1-1/p)}, \quad z \in \mathbb{D}_R, \\ \varphi_\gamma(z) &= \Big((1 - z\,\Psi(\gamma_0))\,(1 - z\,\Psi(\gamma_1)) \Big)^{1/p-1} \prod_{k=0}^{M-1} \left(\frac{1 - z\,\Psi^+(V_k)}{1 - z\,\Psi^-(V_k)} \right)^{\kappa_k(1-1/p)}, \quad z \in \mathbb{D}. \end{aligned} \tag{4.37}$$

Clearly φ_Γ and φ_γ are well-defined and holomorphic since $z - \Psi(W_j) \neq 0$ has no zero in $\mathbb{D}_R$ while neither $1 - z\Psi^{\pm}(V_j)$ nor $1 - z\Psi(\gamma_\ell)$ can have a zero in $\mathbb{D}$. Moreover, since $1 < p < 2$ while $\sigma_j > -1$ and $-1 < \kappa_k < 1$, it is easy to check that $\varphi_\Gamma \in H^p(\mathbb{D}_R)$ and $\varphi_\gamma \in H^p(\mathbb{D})$ (their $L^p(\mathbb{T}_r)$-means are uniformly bounded by dominated convergence). In another connection, we saw through (4.34)-(4.36) that g_1, g_2 defined in (4.30) are such that $g_1 \in L^p(\mathbb{T}, W)$, $g_2 \in L^p(\mathbb{T}_R, W)$ with $W = |(\Psi^{-1})'|^{1/p-1}$. Taking Proposition 4.2 (*ii*) into account, and using that $1/\zeta = \bar{\zeta}$ on $\mathbb{T}$, this may equivalently be rewritten as the following two conditions:

$$\int_{\mathbb{T}} |g_1(1/\zeta)|^p \, |\varphi_\gamma(\zeta)|^p \, |d\zeta| < +\infty, \qquad \int_{\mathbb{T}_R} |g_2(\zeta)|^p \, |\varphi_\Gamma(\zeta)|^p \, |d\zeta| < +\infty. \tag{4.38}$$

Consider the functions

$$h_1(z) = g_1(1/z)\,\varphi_\gamma(z), \quad z \in \mathbb{D} \qquad \text{and} \qquad h_2(z) = g_2(z)\,\varphi_\Gamma(z), \quad z \in \mathbb{D}_R. \tag{4.39}$$

Pick some real λ such that $0 < \lambda < \varepsilon/(1+\varepsilon)$, where ε is as in (4.15). Since $\lambda < 1$, we have that

$$|h_1(z)|^\lambda \le (1 + |g_1(1/z)|)\,|\varphi_\gamma(z)|^\lambda, \quad z \in \mathbb{D}, \qquad \text{and} \qquad |h_2(z)|^\lambda \le (1 + |g_2(z)|)\,|\varphi_\Gamma(z)|^\lambda, \quad z \in \mathbb{D}_R,$$

hence by Hölder's inequality we get upon letting $1/\beta + 1/\beta' = 1$:

$$\int_0^{2\pi} \left| h_1(re^{i\theta}) \right|^\lambda d\theta \le \left(\int_0^{2\pi} \left(1 + \left| g_1\left(e^{-i\theta}/r \right) \right| \right)^\beta d\theta \right)^{1/\beta} \left(\int_0^{2\pi} \left| \varphi_\gamma(re^{i\theta}) \right|^{\lambda\beta'} d\theta \right)^{1/\beta'} \tag{4.40}$$

$$\int_0^{2\pi} \left| h_2(re^{i\theta}) \right|^\lambda d\theta \le \left(\int_0^{2\pi} \left(1 + \left| g_2(re^{i\theta}) \right| \right)^\beta d\theta \right)^{1/\beta} \left(\int_0^{2\pi} \left| \varphi_\Gamma(re^{i\theta}) \right|^{\lambda\beta'} d\theta \right)^{1/\beta'}, \tag{4.41}$$

where it is understood that $0 \le r < 1$ in (4.40) and that $0 \le r < R$ in (4.41). The first factor in the right-hand side of (4.40) and (4.41) is bounded independently of r because $g_1(1/z) \in H^\beta(\mathbb{D})$ and $g_2(z) \in H^\beta(\mathbb{D}_R)$. Besides, it is easy to see from (4.37) that the second factor is also bounded, for $|\sigma_j|$, $|\kappa_k|$ do not exceed 1 while

$$\beta'\lambda(1 - 1/p) = \lambda \frac{1+\varepsilon}{\varepsilon} < 1.$$

As (4.40) and (4.41) are majorized independently of r, it holds that $h_1 \in H^\lambda(\mathbb{D})$ and $h_2 \in H^\lambda(\mathbb{D}_R)$ [35, Ch. II, Sec. 1]; but since their boundary functions lie in $L^p(\mathbb{T})$ and $L^p(\mathbb{T}_R)$ respectively by (4.38), we deduce that in fact $h_1 \in H^p(\mathbb{D})$ and $h_2 \in H^p(\mathbb{D}_R)$ [35, Ch. II, Cor. 4.3].

Now, according to criterion **CS**, we shall have that $\partial u/\partial z \in \mathcal{E}^p(D \setminus \gamma)$ if only

$$\left| \partial u/\partial z \circ \Psi^{-1} \right|^p \left| (\Psi^{-1})' \right| \qquad \text{has a harmonic majorant in } \mathcal{A}. \tag{4.42}$$

From (4.23), it is straightforward to check that

$$\left| \partial u/\partial z \circ \Psi^{-1} \right|^p \left| (\Psi^{-1})' \right| = |g|^p \left| (\Psi^{-1})' \right|^{1-p}$$

and therefore, granted Proposition 4.2 (*ii*) and (4.37), property (4.42) is equivalent to:

$$z \mapsto |zg(z)\, \varphi_\Gamma(z)\, \varphi_\gamma(1/z)|^p\, |z|^{2-3p} \qquad \text{has a harmonic majorant in } \mathcal{A}. \tag{4.43}$$

The factor $|z|^{2-3p}$ is bounded, so it can safely be ignored. If we write $zg(z) = g_1(z) + g_2(z)$, where g_1, g_2 are as in (4.30), and if we take into account the convexity of $x \mapsto x^p$ for $x \geq 0$ when $p > 1$, we are left to show in view of (4.39), that

$$|h_1(1/z)\, \varphi_\Gamma(z)|^p + |h_2(z)\, \varphi_\gamma(1/z)|^p \tag{4.44}$$

has a harmonic majorant in $\mathcal{A}$. We argue on each summand separately, and distinguish whether $R > |z| \geq R/2$ or $R/2 > |z| > 1$. For $R > |z| \geq R/2$, we know that $|h_1(1/z)|$ is bounded since $h_1 \in H^p(\mathbb{D})$ while $|\varphi_\Gamma(z)|^p$ has a harmonic majorant because $\varphi_\Gamma \in H^p(\mathbb{D}_R)$; for $R/2 > |z| > 1$, we observe that $|h_1(1/z)|^p$ has a harmonic majorant because $h_1(1/z) \in H^p(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ by conformal invariance of Hardy spaces, while $|\varphi_\Gamma(z)|$ remains bounded. Therefore, the first summand in (4.44) has a harmonic majorant in $\mathcal{A}$. A similar argument, using that $h_2 \in H^p(\mathbb{D}_R)$ and $\varphi_\gamma(1/z) \in H^p(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$, shows that the second summand in (4.44) also has a harmonic majorant in $\mathcal{A}$, thereby achieving the proof of the Theorem 4.1. ■

In the situation of Theorem 4.1, we say for short that u *is a solution to the Neumann problem with flux* Φ *on* Γ *and* $\phi^\pm$ *on* γ. This causes no confusion since, by the uniqueness part of the theorem, u is well-defined no matter the value of $p > 1$ for which $\Phi \in L^p(\Gamma)$ and $\phi^\pm \in L^p(\gamma)$. The next corollary is useful to normalize the geometry of the inverse problem described in Section 2.

Corollary 4.3 *Notations being as in Theorem 4.1, let* Ψ_1 *map* D *conformally onto the unit disk* $\mathbb{D}$ *and put* $\gamma' = \Psi_1(\gamma)$. *Then* Ψ_1 *extends continuously to* $\Gamma \to \mathbb{T}$, *and* $u \circ \Psi_1^{-1}$ *is a solution to the Neumann problem in* $\mathbb{D} \setminus \gamma'$ *with flux* $(\Phi \circ \Psi_1^{-1})|(\Psi_1^{-1})'|$ *on* $\mathbb{T}$ *and* $(\phi^\pm \circ \Psi_1^{-1})|(\Psi_1^{-1})'|$ *on* γ'.

Proof: In the proof of Proposition 4.2, we saw that Ψ_1 extends continuously $\Gamma \to \mathbb{T}$ and we obtained the representation $\Psi = \Psi_5 \circ \Psi_1$, where Ψ_5 conformally maps $\mathbb{D} \setminus \gamma'$ onto $\mathcal{A}$. Now, from (4.8), it follows as in (4.14)-(4.15) that $(\Phi \circ \Psi_1^{-1})|(\Psi_1^{-1})'| \in L^\beta(\mathbb{T})$ and $(\phi^\pm \circ \Psi_1^{-1})|(\Psi_1^{-1})'| \in L^\beta(\gamma')$ where $\beta > 1$. Thus, it makes sense to speak of the solution to the Neumann problem on $\mathbb{D} \setminus \gamma'$ associated with these fluxes; call this solution v. In the proof of Theorem 4.1, we can put $\Psi_1 = \mathrm{id}$ when $\Gamma = \mathbb{T}$ to obtain that $v = u_1 \circ \Psi_5$, where u_1 solves the Neumann problem on $\mathcal{A}$ with fluxes given by (4.13); but from the same proof $u = u_1 \circ \Psi = u_1 \circ \Psi_5 \circ \Psi_1$, thereby proving the corollary. ■

Theorem 4.1 also yields a Cauchy representation of the solution to the Neumann problem on $D \setminus \gamma$ which is basic to our approach of the inverse problem in Section 7:

Theorem 4.4 *Let* f *be as in Theorem 4.1. Then, we can write*

$$f(z) = H(z) - \frac{1}{2i\pi} \int_\gamma \frac{\sigma(\xi)}{z - \xi} d\xi, \qquad z \in D \setminus \gamma, \tag{4.45}$$

where H *is holomorphic in* D *and continuous on* $\overline{D}$. *The boundary map* $H_{|\Gamma}$ *is absolutely continuous with* $L^p(\Gamma)$ *derivative. The density* σ *of the Cauchy integral in (4.45) is equal to* $f^+ - f^-$, *which is absolutely continuous on* γ *with* $L^p(\gamma)$ *derivative, and it vanishes at the endpoints* γ_0, γ_1.

Proof: Fix $z \in D \setminus \gamma$ and imbed γ into a rectifiable Jordan arc that splits D into two domains D^+ and D^- in such a way that z lies in one of them, say D^+. Since f extends continuously to $\overline{D}$, we can apply the Cauchy integral formula in D^+ and Cauchy theorem in D^- to obtain

$$f(z) = \frac{1}{2i\pi} \int_\Gamma \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2i\pi} \int_\gamma \frac{f^+(\xi) - f^-(\xi)}{z - \xi} d\xi. \tag{4.46}$$

The first integral in (4.46) will serve as a definition of H, thereby establishing (4.45). Clearly H is smooth near γ while the Cauchy integral of σ on γ is smooth near Γ, so the remaining assertions on the smoothness of $H_{|\Gamma}$ and σ follow from Theorem 4.1. ■

Recall that the Wiener algebra $\mathcal{W}$ is the space of functions on $\mathbb{T}$ whose Fourier series is absolutely convergent. When $D = \mathbb{D}$ so that $\Gamma = \mathbb{T}$, the absolute continuity of $H_{|\mathbb{T}}$ asserted in Theorem 4.4 implies that $H \in \mathcal{W}$ [41, Ch. 5, Sec. 4] and $\Phi \to H$ is continuous $L^p(\Gamma) \to \mathcal{W}$. This will warrant the use of truncation in our numerical treatment of Fourier series; see comments in Section 8.

5 Meromorphic approximation

Let D and γ satisfy hypotheses **H1**–**H4** of the preceding section, and $C(\Gamma)$ be the space of complex continuous functions on Γ. We denote by $\mathcal{P}_n$ the space of algebraic polynomials of degree at most n, and by $\mathcal{M}_n^D$ the set of monic polynomials of degree n having all their roots in D.

For $1 \leq q \leq \infty$, we introduce a class of *meromorphic functions with at most n poles in D* by setting:

$$\mathcal{E}_n^q(D) = \{h/q_n; \quad h \in \mathcal{E}^q(D), \ q_n \in \mathcal{M}_n^D\}. \tag{5.1}$$

From elementary division one sees that, alternatively,

$$\mathcal{E}_n^q(D) = \{g + p_{n-1}/q_n; \quad g \in \mathcal{E}^q(D), \ q_n \in \mathcal{M}_n^D, \ p_{n-1} \in \mathcal{P}_{n-1}\}. \tag{5.2}$$

Identifying functions with their nontangential limits, $\mathcal{E}_n^q(D)$ becomes a subset of $L^q(\Gamma)$.

We shall consider two types of meromorphic approximation with at most n poles to a function F defined on Γ: the first is with respect to the $L^2(\Gamma)$-norm in which case we seek best approximants out of $\mathcal{E}_n^2(D)$, and the second is with respect to the $L^\infty(\mathbb{T})$-norm in which case we seek best approximants out of $\mathcal{E}_n^\infty(D)$. Actually, we only deal with functions of the form:

$$F(z) = G(z) - \int_\gamma \frac{d\nu(\xi)}{z - \xi}, \qquad G \in \mathcal{E}^q(D), \qquad \nu \text{ a complex measure on } \gamma, \tag{5.3}$$

where $q = 2$ or $q = \infty$ depending on which type of approximation we are considering. Later, when dealing with $q = \infty$ in Section 5.2, we will assume in addition that F (thus also G) is continuous on Γ. Note in particular that the representation (5.3) holds when F is as in (4.45).

The fact that in each case a best approximant does exist follows easily from the weak compactness of balls in $L^2(\Gamma)$, the weak-* compactness of balls in $L^\infty(\Gamma)$, and the next lemma.[‡]

Lemma 5.1 *Let D satisfy hypotheses* **H1**–**H2**. *Then $\mathcal{E}_n^q(D)$ is weakly closed in $L^q(\Gamma)$ for $1 < q < \infty$, and $\mathcal{E}_n^\infty(D)$ is weak-* closed in $L^\infty(\Gamma)$.*

[‡]This argument actually shows that a best approximant from $\mathcal{E}_n^q$ to $F \in L^q(\Gamma)$ exists for $1 < q \leq \infty$. It fails when $q = 1$, because $\mathcal{E}_n^1(D) \subset L^1(\Gamma)$ is closed but not weak-* closed when viewed as a set of measures on Γ. Still a best approximant exists in this case too, as can be proved by adapting to D the reasoning carried out on $\mathbb{D}$ in [23, p. 74].

Proof: Let Ψ_1 map conformally D onto the unit disk $\mathbb{D}$ as in the proof of Proposition 4.2, and put $\Xi = \Psi_1^{-1}$. The function Ξ' cannot vanish on $\mathbb{D}$ and therefore it has a well-defined q-th root for $1 \le q < \infty$. From criterion **CS** in Section 4, it follows that $h \in \mathcal{E}^q(D)$ if, and only if, $(h \circ \Xi)(\Xi')^{1/q} \in H^q(\mathbb{D}) = \mathcal{E}^q(\mathbb{D})$. Moreover, since $g \in \mathcal{E}_n^\infty(D)$ if, and only if, it is meromorphic with at most n poles in D and bounded outside some compact subset of D, it is elementary to check that $g \in \mathcal{E}_n^\infty(D)$ if, and only if, $g \circ \Xi \in \mathcal{E}_n^\infty(\mathbb{D})$. Consequently, as $\mathcal{E}_n^q(D) = \mathcal{E}^q(D)\,\mathcal{E}_n^\infty(D)$ by (5.1), we see upon writing $h/q_n \in \mathcal{E}_n^q$ in the form h times $1/q_n$ that

$$g \in \mathcal{E}_n^q(D) \quad \text{if and only if} \quad (g \circ \Xi)(\Xi')^{1/q} \in \mathcal{E}_n^q(\mathbb{D}), \qquad 1 \le q \le \infty. \tag{5.4}$$

From [23, lemma 5.1.], the conclusion of the lemma holds when $D = \mathbb{D}$. By (5.4), $\mathcal{E}_n^q(D)$ is the preimage of $\mathcal{E}_n^q(\mathbb{D})$ under the continuous (indeed: isometric) map $g \mapsto (g \circ \Xi)(\Xi')^{1/q}$ from $L^q(\Gamma)$ onto $L^q(\mathbb{T})$. Hence we conclude that $\mathcal{E}_n^q(D) \subset L^q(\Gamma)$ is weakly closed for $1 < q < \infty$. Also, $\mathcal{E}_n^\infty(D)$ is the preimage of $\mathcal{E}_n^\infty(\mathbb{D})$ under the transpose of the continuous map $f \mapsto (f \circ \Psi_1)\Psi_1'$ from $L^1(\mathbb{T})$ onto $L^1(\Gamma)$ (this transpose maps $L^\infty(\Gamma)$ onto $L^\infty(\mathbb{T})$). Therefore $\mathcal{E}_n^\infty(D) \subset L^\infty(\Gamma)$ is weak-* closed. ■

In this section, we state some basic properties of best approximants of either type above, and we point out a common feature to them namely that their denominators, when written in the form (5.1), satisfy certain non-Hermitian orthogonality relations. From these, information on the distribution of the poles can be obtained after the work in [17] when γ is a hyperbolic geodesic arc. Our approach consists in mapping the meromorphic approximation problem onto the unit circle, where we can quote existing results.

From a computational point of view, the two types of approximation that we consider are complementary in the following sense. On the one hand, the closed expression (5.22) for the best L^∞ approximant on the unit circle, together with the conformal invariance of such approximants, make for fast and guaranteed computations. However, these are sensitive to irregular perturbations of the *data*: the best approximation projection is *not* continuous with respect to the L^∞-norm, and only generically continuous with respect to stronger norms like C^α, Besov, or Wiener norms (all of which take into account the variation of the function) [58, 40]. On the other hand, best L^2 approximants are more robust numerically as they generically depend L^2-continuously on the *data* [13], but their computation requires a numerical search that can get trapped in *local minima* namely points where the map $h \mapsto \|F - h\|_{L^2(\Gamma)}$ has a relative minimum with respect to $h \in \mathcal{E}_n^2$. Such points are also called *local best approximants* to F from $\mathcal{E}_n^2$, and their possible occurence is the reason why, besides best approximants, we consider more generally *critical points* of the L^2 criterion, *i.e.* triples $(g, p_{n-1}, q_n) \in \mathcal{E}^2(D) \times \mathcal{P}_{n-1} \times \mathcal{M}_n^D$ in the notation of (5.2) such that the (Fréchet) derivative of $\|F - g - p_{n-1}/q_n\|^2_{L^2(\Gamma)}$ with respect to g vanishes together with its (ordinary) derivatives with respect to the coefficients of p_{n-1} and q_n. A local best approximant is the primary example of a critical point.

5.1 Meromorphic Approximation in the $L^2(\Gamma)$ norm

Let us assume first that $D = \mathbb{D}$, the unit disk, so that $\Gamma = \mathbb{T}$, the unit circle. Since $\mathcal{E}^q(\mathbb{D}) = H^q(\mathbb{D})$, it does not matter whether we use the Smirnov or the Hardy class and we shall work with the latter to match the references that we quote. For simplicity, we write H^q instead of $H^q(\mathbb{D})$ and we set and H_n^q to mean $\mathcal{E}_n^q(\mathbb{D})$ throughout.

When functions get identified with their nontangential limits on the unit circle, H^2 becomes the subspace of $L^2(\mathbb{T})$ consisting of functions with vanishing Fourier coefficients of strictly negative index [64, 17.10]. Then, if we put $H^{2,0}$ to mean the subspace of H^2 consisting of functions with vanishing mean, we get the orthogonal decomposition:

$$L^2(\mathbb{T}) = H^2 \oplus \overline{H}^{2,0} \tag{5.5}$$

where $\overline{H}^{2,0}$ indicates the complex conjugates of functions in $H^{2,0}$ (this follows from Parseval's theorem since $\overline{H}^{2,0}$ consists of $L^2(\mathbb{T})$-functions with vanishing Fourier coefficients of non-negative index). By conformal invariance of Hardy spaces, it is easily checked on considering $z \mapsto 1/z$ that $\overline{H}^{2,0}$ isometrically identifies with those functions in $H^2(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}})$ that vanish at infinity. In particular, the Cauchy integral in (5.3) defines a $\overline{H}^{2,0}$-function since it is bounded in $|z| \geq 1$ and zero at infinity. This remark yields the following fact:

Lemma 5.2 *Let F be given by (5.3) where $q = 2$ and $D = \mathbb{D}$ while γ satisfies* **H3**-**H4**. *If $F = F_1 - F_2$ is the orthogonal decomposition of F according to (5.5), then*

$$F_1 = G \quad \text{and} \quad F_2 = \int_\gamma \frac{d\nu(\xi)}{z - \xi}. \tag{5.6}$$

Proof: Obvious from what precedes. ■

On the unit circle, best meromorphic $L^2(\mathbb{T})$-approximation with at most n poles *reduces to rational approximation.* Indeed, if $F = F_1 + F_2$ according to (5.5) so that $F_1 = P_{H^2}F$ and $F_2 = P_{\overline{H}^{2,0}}F$ where P_{H^2}, $P_{\overline{H}^{2,0}}$ indicate the orthogonal projections, and if we parametrize the approximant as in (5.2), we get by orthogonality since $p_{n-1}/q_n \in \overline{H}^{2,0}$ that

$$\|F - g + p_{n-1}/q_n\|^2_{L^2(\mathbb{T})} = \|F_1 - g\|^2_{L^2(\mathbb{T})} + \|F_2 - p_{n-1}/q_n\|^2_{L^2(\mathbb{T})}. \tag{5.7}$$

From (5.7), it is apparent that $g - p_{n-1}/q_n$ is a best approximant to F from $\mathcal{E}^2_n$ if, and only if, $g = F_1$ and p_{n-1}/q_n is a best rational approximant with at most n poles to F_2 in $\overline{H}^{2,0}$. In fact, it is clear that $g - p_{n-1}/q_n$ is a *local* best approximant to F from $\mathcal{E}^2_n$ if, and only if, $g = F_1$ and p_{n-1}/q_n is a *local* best rational approximant with at most n poles to F_2 in $\overline{H}^{2,0}$ (the notion of *local best rational approximant* is defined analogously to that of local best meromorphic approximant). More generally, it follows easily from (5.7) that $g - p_{n-1}/q_n$ is critical for $\|F - g + p_{n-1}/q_n\|^2_{L^2(\mathbb{T})}$ if, and only if $g = F_1$ and the derivatives of $\|F_2 - p_{n-1}/q_n\|^2_{L^2(\mathbb{T})}$ with respect to the coefficients of p_{n-1}, q_n, do vanish; in this case we say that p_{n-1}/q_n is a *critical point in rational approximation with at most n poles to F_2*. Considering that $g = P_{H^2}F$ is determined explicitly, we are thus left with the following rational approximation problem:

$\mathbf{Pb}^2_{n-1,n}$: *Given $F_2 \in \overline{H}^{2,0}$ and some integer $n \geq 0$, minimize $\|F_2 - p_{n-1}/q_n\|_{L^2(\mathbb{T})}$ over $p_{n-1} \in \mathcal{P}_{n-1}$ and $q_n \in \mathcal{M}^{\mathbb{D}}_n$ (note that the rational function p_{n-1}/q_n belongs to $\overline{H}^{2,0}$).*

A solution to $\mathbf{Pb}^2_{n-1,n}$ has exact degree n (*i.e.* p_{n-1} and q_n are coprime) unless F_2 is rational of degree less than n; this is actually true of all *local minima* of $\|F - p_{n-1}/q_n\|^2$ [13]. A best approximant needs not be unique [51] although uniqueness is a strongly generic property (*i.e.* holding on an open dense subset of $\overline{H}^{2,0}$ [13, 12]), and in any case there may be local *minima* . This is why we stress below properties of critical points of exact degree n and not merely of best approximants, for a local *minimum* is all one can guarantee in general from a numerical search. More on the uniqueness issue can be found in [26, 22, 25, 14], and an efficient algorithm to generate local *minima* is described in [16, 38].

When F_2 is as in (5.6), the somewhat degenerate case where F_2 is rational of degree less than n happens if, and only if ν is a discrete measure consisting of less than n point masses. Indeed, the "if" part is clear and conversely, if F_2 coincides on $\mathbb{T}$ with some rational function R_{n-1} having at most $n-1$ poles, then F_2 and R_{n-1} must agree on $\overline{\mathbb{C}}\setminus\gamma$ by analytic continuation (here we use that γ does not disconnect the plane because it is an arc and not a closed curve). As ν is a complex measure F_2 is locally integrable with respect to the area measure m, and since $m(\gamma)=0$ the same is true of R_{n-1}; in particular the latter has simple poles. Now, F_2 and R_{n-1} agree a.e. as locally integrable functions on $\mathbb{C}$ hence they agree as distributions. Because $1/z$ is a fundamental solution of the $\overline{\partial}$ equation, we conclude from this and the definition of F_2 that $\partial R_{n-1}/\partial\bar z=\partial F_2/\partial\bar z=\nu$, and since $\partial R_{n-1}/\partial\bar z$ is the sum of at most $n-1$ point masses located at the poles of R_{n-1}, we get the "only if" part.

On the disk, the non-Hermitian orthogonality relations that we mentioned previously go as follows.

Theorem 5.3 *Let F be given by (5.3) where $q=2$ and $D=\mathbb{D}$ while γ satisfies* **H3**-**H4**. *Assume moreover that the support of ν comprises at least n points. If g_n is a critical point in best meromorphic approximation to F from $\mathcal{E}_n^2$ having exactly n poles, and if we write $g_n=g+p_{n-1}/q_n$ according to (5.2), then*

$$\int_\gamma \frac{q_n(\xi)}{\widetilde{q}_n^2(\xi)}\xi^k d\nu(\xi)=0, \quad k\in\{0,1,\dots,n-1\}, \tag{5.8}$$

where $\widetilde{q}_n(\xi)=\xi^n\,\overline{q_n(1/\bar\xi)}$ is the reciprocal polynomial of q_n. This holds in particular if g_n is a local best approximant to F from $\mathcal{E}_n^2$.

Proof: From Lemma 5.2 and the discussion after (5.7), we know that $g_n=g+p_{n-1}/q_n$ is a critical point in best meromorphic approximation to F from $\mathcal{E}_n^2$ having exactly n poles if, and only if $g=G$ and p_{n-1}/q_n is a critical point of degree n in problem $\mathbf{Pb}^2_{n-1,n}$. Granted the Cauchy representation (5.6) for F_2, the orthogonality relations (5.8) now follow from [23, Prop. 10.3., Eqn. (136)]; alternatively, the argument given in Section 4 of [26] when $\gamma\subset(-1,1)$ and ν is positive also applies here without modification. Finally, as the support of ν contains at least n points, we saw that F_2 cannot be rational of degree less than n and therefore, as mentioned after $\mathbf{Pb}^2_{n-1,n}$, each local best rational approximant to F_2 with at most n poles has exact degree n. ∎

The orthogonality relations (5.8) are nonlinear and difficult to solve in general, but they yield information on the behavior of the zeros of q_n as we shall see in Section 6. To give these relations a more intrinsic meaning, we need to bring in two classical definitions. The first one is the complex Green "function" of a simply connected domain. Recall that when $D\subset\mathbb{C}$ is a simply connected domain whose complement contains at least two points, the *Green function with pole at* $w\in D$ is the unique real-valued function $z\mapsto g_D(z,w)$ which is harmonic in $D\setminus\{w\}$, whose value at z is $\mathbf{O}(\log|z-w|^{-1})$, and which is such that $\lim_{z\to\xi}g_D(z,w)=0$ for every $\xi\in\partial D$ (see *e.g.* [61, Thms. 4.2.11, 4.4.2 and 4.4.9]). The *complex Green function with pole at* w, denoted by $\mathcal{G}_D(z,w)$, is then the (multi-valued) holomorphic function in $D\setminus\{w\}$ whose real part is $g_D(z,w)$. When $D=\mathbb{D}$, we have the explicit formula:

$$\mathcal{G}_{\mathbb{D}}(z,w)=\log\left(\frac{1-\overline{w}z}{z-w}\right), \qquad z,w\in\mathbb{D}.$$

The second notion that we must introduce is that of reproducing kernel: if $\mathcal{H}$ is a Hilbert space of functions, defined on a set E, for which pointwise evaluation is continuous, the *reproducing kernel*

$\xi \mapsto K_{\mathcal{H}}(\xi,\zeta)$ is, for fixed $\zeta \in E$, the unique member of $\mathcal{H}$ such that $f(\zeta) =< f, K_{\mathcal{H}}(\cdot,\zeta) >_{\mathcal{H}}$ for all $f \in \mathcal{H}$, where $< \cdot,\cdot >_{\mathcal{H}}$ designates the scalar product. In H^2, the reproducing kernel is just the familiar Cauchy kernel (see *e.g.* [59, Ch.I, Sec. 6]):

$$K_{H^2}(\xi,\zeta) = (1-\xi\overline{\zeta})^{-1}, \quad \xi,\zeta \in \mathbb{D}.$$

Now, for $q_n \in \mathcal{M}_n^{\mathbb{D}}$, let us write

$$q_n(z) = \prod_{j=1}^{d}(z-\xi_j)^{m_j}, \qquad \sum_{j=1}^{d} m_j = n, \tag{5.9}$$

where $\xi_1,\ldots,\xi_d \in \mathbb{D}$ denote the zeros of q_n with respective multiplicity m_j. Subsequently, we define

$$b_n(z) \triangleq \frac{q_n(z)}{\widetilde{q}_n(z)} = \prod_{j=1}^{d}\left(\frac{z-\xi_j}{1-z\overline{\xi}_j}\right)^{m_j} \tag{5.10}$$

which is called the (normalized) *Blaschke product* with zeros ξ_j of multiplicity m_j. Up to a multiplicative unimodular constant, b_n it is the unique rational function with the afore-mentioned zeros –and no others– whose modulus on $\mathbb{T}$ is identically 1. Note that we can rewrite b_n in the form:

$$b_n(z) = \exp\left(-\sum_{j=1}^{d} m_j\,\mathcal{G}_{\mathbb{D}}(z,\xi_j)\right), \tag{5.11}$$

where this time the exponential makes the right-hand side single-valued. More generally, on a domain D with complex Green function $\mathcal{G}_D$, the function

$$B_n(z) = \exp\left(-\sum_{j=1}^{d} m_j\,\mathcal{G}_D(z,\xi_j)\right) \tag{5.12}$$

is called the (normalized) *Blaschke product on* D with zeros ξ_j of multiplicity m_j.

With these definitions, (5.8) translates into:

$$\int_\gamma b_n(\xi)\,K_{H^2}(\xi,\xi_j)^k\,d\nu(\xi) = 0, \qquad 1 \le j \le d,\ 1 \le k \le m_j. \tag{5.13}$$

As we shall see, (5.13) –with b_n given by (5.11)– is a conformally invariant version of (5.8).

Let us now come back to best approximation from $\mathcal{E}_n^2(D)$ in $L^2(\Gamma)$ and no longer assume that $D = \mathbb{D}$. This more general case reduces to the previous one by means of the following result.

Proposition 5.4 *Let D and γ satisfy hypotheses* **H1**–**H4***, and Ψ_1 map D conformally onto the unit disk $\mathbb{D}$ with $\Xi = \Psi_1^{-1}$ the inverse map. Then, it holds that*

(i) g is a best approximant (resp. a local best approximant, a critical point in best approximation) from $\mathcal{E}_n^2(D)$ to $F \in L^2(\Gamma)$ if, and only if $(g\circ\Xi)(\Xi')^{1/2}$ is a best approximant (resp. a local best approximant, a critical point in best approximation) from H_n^2 to $(F\circ\Xi)(\Xi')^{1/2} \in L^2(\mathbb{T})$.

(ii) If F is given by (5.3) where $q = 2$, then $(F\circ\Xi)(\Xi')^{1/2}$ assumes a similar form on $\mathbb{D}$:

$$(F\circ\Xi)(\Xi')^{1/2}(z) = G_1(z) - \int_{\gamma'}\frac{d\nu_1(\xi)}{z-\xi}, \qquad G_1 \in H^2,$$

with $\gamma' = \Psi_1(\gamma)$ and $d\nu_1 = (\Xi')^{-1/2}d(\nu\circ\Xi)$.

Proof: Letting $\xi = \Xi(\zeta)$, we get from (4.8) that

$$\int_\Gamma |F(\xi) - g(\xi)|^2 \, |d\xi| = \int_{\mathbb{T}} \left| (F \circ \Xi(\zeta)) \left(\Xi'\right)^{1/2}(\zeta) - (g \circ \Xi(\zeta)) \left(\Xi'\right)^{1/2}(\zeta) \right|^2 \, |d\zeta|$$

from which (i) follows in view of (5.4).

To establish (ii), define F_2 through formula (5.6) and let $\nu_0 = \nu \circ \Xi$ be the complex measure on $\gamma' = \Psi_1(\gamma)$ given by $\nu_0(E) = \nu(\Xi(E))$ for every Borel subset $E \subset \gamma'$. Composing F_2 with Ξ and letting $\xi = \Xi(\zeta)$ in the integral, we obtain:

$$F_2 \circ \Xi(s) = \int_{\gamma'} \frac{d\nu_0(\zeta)}{\Xi(s) - \Xi(\zeta)}, \qquad s \in \mathbb{D} \setminus \gamma'.$$

Put

$$(\Xi(s) - \Xi(\zeta))^{-1} = \frac{1/\Xi'(\zeta)}{s - \zeta} + H(\zeta, s) \tag{5.14}$$

where $H(\zeta, s)$ is holomorphic in $\mathbb{D} \times \mathbb{D}$, and introduce a measure $\tilde{\nu}_0$ on γ' by $d\tilde{\nu}_0 = d\nu_0/\Xi'$. Then

$$F_2 \circ \Xi = H_1 + H_2, \qquad \text{with } \; H_1(s) = \int_{\gamma'} H(\zeta, s)\, d\nu_0(\zeta), \quad H_2(s) = \int_{\gamma'} \frac{d\tilde{\nu}_0(\zeta)}{s - \zeta}. \tag{5.15}$$

Now, by elementary properties of the Cauchy projection, it holds for any $f \in L^2(\mathbb{T})$ that

$$P_{\overline{H}^{2,0}} f(s) = \frac{1}{2i\pi} \int_{\mathbb{T}} \frac{f(\xi)}{s - \xi}\, d\xi, \qquad |s| > 1.$$

Hence by Fubini's theorem and Cauchy's formula as applied to $H_2(s)$ in (5.15):

$$P_{\overline{H}^{2,0}} \left((\Xi')^{1/2} H_2 \right)(s) = \int_{\gamma'} d\tilde{\nu}_0(\zeta) \frac{1}{2i\pi} \int_{\mathbb{T}} \frac{\left(\Xi'\right)^{1/2}(\xi)}{\xi - \zeta} \frac{d\xi}{s - \xi} = \int_{\gamma'} \frac{\left(\Xi'\right)^{1/2}(\zeta)}{s - \zeta}\, d\tilde{\nu}_0(\zeta), \qquad |s| > 1.$$

By inspection the above formula extends holomorphically to $s \in \overline{\mathbb{C}} \setminus \gamma'$, thus letting ν_1 be such that $d\nu_1 = (\Xi')^{1/2} d\tilde{\nu}_0 = (\Xi')^{-1/2} d\nu_0$, we get that

$$(\Xi')^{1/2}(s)\, H_2(s) = P_{H^2} \left((\Xi')^{1/2} H_2 \right)(s) + \int_{\gamma'} \frac{d\nu_1(\zeta)}{s - \zeta}, \qquad s \in \mathbb{D} \setminus \gamma'. \tag{5.16}$$

Next we observe that H_1 in (5.15) is holomorphic on $\mathbb{D}$ and continuous (in fact even Hölder-continuous) on $\overline{\mathbb{D}}$ because, in view of (5.14) and (4.8), so is $H(s, \zeta)$, uniformly on $\overline{\mathbb{D}} \times \gamma'$. Consequently, since $(\Xi')^{1/2} \in H^2$ as $\Xi' \in H^1$ by the rectifiability of Γ [60, Thm. 6.8], it follows that $(\Xi')^{1/2} H_1 \in H^2$. In another connection, since $G \in \mathcal{E}^2(D)$ by (5.3), we know from criterion **CS** that $(\Xi')^{1/2} G \circ \Xi \in H^2$. Thus if we let

$$G_1 = (\Xi')^{1/2}(G \circ \Xi) - (\Xi')^{1/2} H_1 - P_{H^2} \left((\Xi')^{1/2} H_2 \right),$$

then $G_1 \in H^2$ and from (5.3), (5.6), (5.15) and (5.16), we get:

$$(F \circ \Xi(s))\, (\Xi')^{1/2}(s) = G_1(s) - \int_{\gamma'} \frac{d\nu_1(\zeta)}{s - \zeta}, \qquad s \in \mathbb{D} \setminus \gamma', \tag{5.17}$$

which is the announced decomposition. ■

One consequence of Proposition 5.4 is that the poles of a best (or local best) approximant to $F \in L^2(\Gamma)$ from $\mathcal{E}_n^2(D)$ are the images under Ξ (counting multiplicities) of the poles of a best (or local best) approximant to $(F \circ \Xi)(\Xi')^{1/2} \in L^2(\mathbb{T})$ from H_n^2.

We can now state a conformally invariant version of Theorem 5.3.

Theorem 5.5 *Let D and γ satisfy hypotheses* **H1**-**H4**, *and F be given by (5.3) where $q = 2$. Assume moreover that the support of ν comprises at least n points. If g_n is a critical point in best meromorphic approximation to F from $\mathcal{E}_n^2(D)$ having exactly n poles, and if we write $g_n = g + p_{n-1}/q_n$ according to (5.2), where $q_n \in \mathcal{M}_n^D$ has zeros $\zeta_1, \ldots, \zeta_d \in D$ of respective multiplicity $m_1, \ldots, m_d$ with $\sum_j m_j = n$, then it holds that*

$$\int_\gamma B_n(\zeta)\, K_{\mathcal{E}^2(D)}(\zeta, \zeta_j)^k \, d\nu(\zeta) = 0, \qquad 1 \le j \le d,\ 1 \le k \le m_j, \tag{5.18}$$

where B_n is the Blaschke product on D defined in (5.12). This holds in particular if g_n is a local best approximant to F from $\mathcal{E}_n^2$.

Proof: Let Ψ_1 map D conformally onto the unit disk $\mathbb{D}$ with $\Xi = \Psi_1^{-1}$ the inverse map. Clearly, the points $\xi_j = \Psi_1(\zeta_j) \in \mathbb{D}$ with respective multiplicity m_j are the poles of $(g \circ \Xi)(\Xi')^{1/2} \in H_n^2$. Therefore, if we define q_n, b_n through (5.9)-(5.10), we deduce from Proposition 5.4 and Theorem 5.3 that (5.8), thus also (5.13) hold with γ replaced by $\gamma' = \Psi_1(\gamma)$ and $d\nu$ replaced by $d\nu_1 = (\Xi')^{-1/2} d(\nu \circ \Xi)$. Putting $\zeta = \Psi_1(\xi)$ in this last relation, we obtain (5.18) from (5.11) if we take into account the identity $\mathcal{G}_{\mathbb{D}}(\xi, \xi_j) = \mathcal{G}_D(\Xi(\xi), \zeta_j)$ [61, 4.4.4] and the fact that

$$K_{H^2}(\xi, \xi_j) = (\Xi'(\xi))^{1/2} (\Xi'(\xi_j))^{1/2} K_{\mathcal{E}^2(D)}(\Xi(\xi), \zeta_j)$$

which is immediate from the definition of a reproducing kernel in view of criterion **CS** and the Cauchy formula. ■

5.2 Meromorphic Approximation in the L^∞ norm

As in the preceding section, we first assume that $D = \mathbb{D}$ and consequently that $\Gamma = \mathbb{T}$. When considering best approximation to F from H_n^∞ in $L^\infty(\mathbb{T})$, we shall assume from the start that $F \in H^\infty + C(\mathbb{T})$, the space of all functions on $\mathbb{T}$ of the form $h + \varphi$ where $h \in H^\infty$ and φ is continuous; this space is in fact a closed subalgebra of $L^\infty(\mathbb{T})$ [35, Ch.IX]. The hypothesis that $F \in H^\infty + C(\mathbb{T})$ leads to a rather explicit description of best approximants and will be no restriction to us since it follows automatically from (5.3) when $q = \infty$.

The meromorphic approximation problem with at most n poles, in the uniform norm on the circle, can be stated as follows:

$\mathbf{Pb}_{\infty,n}^\infty$: *Given $F \in H^\infty + C(\mathbb{T})$ and some integer $n \ge 0$, minimize $\|F - g\|_{L^\infty(\mathbb{T})}$ over $g \in H_n^\infty$.*

The solution of problem $\mathbf{Pb}_{\infty,n}^\infty$ is connected to the spectral decomposition of Hankel operators by the celebrated AAK theory [2, 3], for which the reader may consult the textbooks [57, 59] or [55, Ch. 7]. To explain the connection, let us define the Hankel operator with symbol F:

$$\begin{aligned} A_F :\ \ & H^2 \to \overline{H}^{2,0} \\ & g \mapsto P_{\overline{H}^{2,0}}(Fg). \end{aligned} \tag{5.19}$$

Since $H^\infty H^2 \subset H^2$, the Hankel operator A_F only characterizes the symbol F up to the addition of some H^∞-function, and it turns out to be a compact operator when $F \in H^\infty + C(\mathbb{T})$ (see *e.g.* [57, Thm 3.14]). For $k = 0, 1, 2, \ldots$, let us introduce the *singular values* of A_F by the standard formula:

$$s_k(A_F) := \inf \left\{ ||A_F - A||; \quad A \text{ an operator of rank } \leq k \text{ from } H^2 \text{ into } \overline{H}^{2,0} \right\}, \tag{5.20}$$

where $|| \; ||$ denotes the operator norm. The singular values are the square roots of the eigenvalues of the compact self-adjoint operator $A_F^* A_F$, arranged in non-increasing order, where A_F^* denotes the adjoint. A k-th *singular vector* is then an eigenvector of $A_F^* A_F$ of unit $L^2(\mathbb{T})$-norm associated to $s_k^2(A_F)$. The main result of AAK theory asserts that

$$\inf_{g \in H_n^\infty} \|F - g\|_\infty = s_n(A_F) \tag{5.21}$$

where the *infimum* is attained, and that the *unique* minimizer g_n is given by the formula

$$g_n = F - \frac{A_F v_n}{v_n} = \frac{P_{H^2}(F v_n)}{v_n}, \tag{5.22}$$

where $v_n \in H^2$ is *any* n-th singular vector of A_F; moreover the error function $F - g_n$, which is equal to $A_F v_n / v_n$ by (5.22), has constant modulus $s_n(A_F)$ a.e. on $\mathbb{T}$. Thus (5.21) tells us about the value of problem $\mathbf{Pb}^\infty_{\infty,n}$ and (5.22) about its solution, in terms of the spectral decomposition of the operator $A_F^* A_F$.

Henceforth we rule out the case where $F \in H_n^\infty$, to the effect that $s_n(A_F) \neq 0$. Note that when F is given by (5.3), the discussion before Theorem 5.3 makes the requirement $F \notin H_n^\infty$ equivalent to the assumption that ν is not a discrete measure with less than n points of support.

If the singular value $s_n(A_F)$ is simple (the generic case), then v_n is unique up to multiplication by a unimodular constant, and it has exactly n zeros in $\mathbb{D}$ (counting multiplicities) that are the poles of g_n. More precisely, one can write $v_n = c b_n w_n$ where $c \in \mathbb{T}$ and $b_n = q_n / \widetilde{q}_n$ is a Blaschke product of degree n as defined in (5.10) for some $q_n \in \mathcal{M}_n^{\mathbb{D}}$, while $w_n \in H^2$ is a so-called *outer function*, meaning that $\log |w_n(e^{i\theta})| \in L^1(\mathbb{T})$ and that w_n can be represented as

$$w_n(z) = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |w_n(e^{i\theta})| \, d\theta \right\}, \qquad z \in \mathbb{D}; \tag{5.23}$$

here, we recall that any H^q function is log-modulus summable on $\mathbb{T}$ [35, Ch.II, Thm 4.1.].

When $s_n(A_F)$ has multiplicity $\delta > 1$ (δ is finite since $A_F^* A_F$ is compact and $s_n(A_F) \neq 0$ by assumption), things get more complicated: if $k_n \leq n$ is the smallest non-negative integer such that $s_{k_n}(A_F) = s_n(A_F)$, *i.e.*

$$s_{k_n}(A_F) = s_{k_n+1}(A_F) = \ldots = s_n(A_F) = \ldots = s_{k_n+\delta-1}(A_F) > s_{k_n+\delta}(A_F),$$

then a n-th singular vector v_n is no longer unique up to a multiplicative constant but they all give rise to the same g_n *via* (5.22). Each of them will have at least k_n zeros in $\mathbb{D}$ (counting multiplicities) which are the poles of g_n, plus possibly finitely many extra-zeros that will cancel out with zeros of $P_{H^2}(F v_n)$ in formula (5.22) so as to leave g_n unchanged. To be specific, let $q_{k_n} \in \mathcal{M}_{k_n}^{\mathbb{D}}$ be the monic polynomial of degree k_n whose zeros are the poles of g_n and put $b_{k_n} = q_{k_n} / \widetilde{q}_{k_n}$ for the Blaschke product having the same zeros. Then, for v_n an *arbitrary* n-th singular vector, it holds that $v_n = c b_{k_n} b_{v_n} w_{v_n}$, where $c \in \mathbb{T}$ and b_{v_n} is a Blaschke product carrying the extra-zeros of v_n (that are not poles of g_n), while w_{v_n} is an outer function. With these notations, it is a consequence of AAK theory that $v_n/b_{v_n} = b_{k_n}w_{v_n}$ is *also* a n-th singular vector (see [23, Eqn. (79)] and the discussion thereafter). Thus there *always* exists a n-th singular vector whose zeros are *precisely* the poles of g_n, and any such vector will be called *minimal*. For minimal singular vectors, the non-Hermitian orthogonality relations that we seek go as follows:

Theorem 5.6 [23, Eqns. (125)-(126)] *Let F be given by (5.3) where $q = \infty$ and $D = \mathbb{D}$ while γ satisfies* **H3**-**H4**. *Assume that the support of ν comprises at least n points. If $v_n = b_{k_n}w_n$ is a minimal n-th singular vector of A_F with $b_{k_n} = q_{k_n}/\widetilde{q}_{k_n}$ for some $q_{k_n} \in \mathcal{M}^{\mathbb{D}}_{k_n}$ while w_n an outer function, then*

$$\int_\gamma \frac{q_{k_n}(\xi)}{\widetilde{q}^2_{k_n}(\xi)}\xi^k w_n(\xi)d\nu(\xi) = 0, \quad k \in \{0, 1, \ldots, k_n - 1\}. \tag{5.24}$$

Here one should note the similarity between (5.24) and (5.8).

Let us now consider a general domain D with boundary Γ satisfying **H1**–**H2**. Using conformal mapping, the issue of best approximating $F \in \mathcal{E}^\infty(D) + C(\Gamma)$ from $\mathcal{E}^\infty_n(D)$ easily reduces to a problem of type $\mathbf{Pb}^\infty_{\infty,n}$ as follows.

Proposition 5.7 *Let D and γ satisfy hypotheses* **H1**–**H4**, *and Ψ_1 map D conformally onto the unit disk $\mathbb{D}$ with $\Xi = \Psi_1^{-1}$ the inverse map. Then, it holds that*

(i) g_n is a best approximant from $\mathcal{E}^\infty_n(D)$ to $F \in \mathcal{E}^\infty(D) + C(\Gamma)$ if, and only if $g_n \circ \Xi$ is a best approximant from H^∞_n to $F \circ \Xi \in H^\infty + C(\mathbb{T})$.

(ii) If F is given by (5.3) where $q = \infty$, then $F \circ \Xi$ assumes a similar form on $\mathbb{D}$:

$$F \circ \Xi(z) = G_1(z) - \int_{\gamma'} \frac{d\nu_1(\xi)}{z - \xi}, \qquad G_1 \in H^\infty, \tag{5.25}$$

with $\gamma' = \Psi_1(\gamma)$ and $d\nu_1 = (\Xi')^{-1}d(\nu \circ \Xi)$.

Proof: Note that indeed $F \circ \Xi \in H^\infty + C(\mathbb{T})$ by the conformal invariance of $\mathcal{E}^\infty(D) = H^\infty(D)$ and the continuity of Ξ on $\mathbb{T}$ (*cf.* Proposition 4.2 (*i*)). Statement (*i*) is now obvious from (5.4) and the conformal invariance of the *sup*-norm. Assertion (*ii*) in turn follows from (5.15) with $G_1 = G - H_1$. ∎

By Proposition 5.7 the poles of a best approximant to $F \in \mathcal{E}^\infty(D) + C(\Gamma)$ from $\mathcal{E}^\infty_n(D)$ are the images under Ξ (counting multiplicities) of the poles of a best approximant to $(F \circ \Xi) \in L^\infty(\mathbb{T})$ from H^∞_n. At this point, the question arises whether best meromorphic approximation in $L^\infty(\Gamma)$ can be carried out using an appropriate definition of a Hankel operator on D, without reference to conformal maps. This is indeed the case as the following construction shows.

On a domain D with boundary Γ satisfying **H1**–**H2**, we define the Hankel operator $\mathcal{A}_F$ with symbol $F \in \mathcal{E}^\infty(D) + C(\Gamma)$ in analogy with (5.19):

$$\begin{aligned} \mathcal{A}_F : \quad & \mathcal{E}^2(D) \to \mathcal{E}^{2,\perp}(\Gamma) \\ & g \mapsto P_{\mathcal{E}^{2,\perp}(\Gamma)}(Fg), \end{aligned} \tag{5.26}$$

where $\mathcal{E}^{2,\perp}(\Gamma)$ denotes the orthogonal complement of $\mathcal{E}^2(D)$ in $L^2(\Gamma)$. If we let Ψ_1 denote as before a conformal map from D onto $\mathbb{D}$ and $\Xi = \Psi_1^{-1}$ its inverse, we can define a unitary transformation:

$$\begin{aligned} \mathcal{J} : \quad & L^2(\Gamma) \to L^2(\mathbb{T}) \\ & g \mapsto (g \circ \Xi)(\Xi')^{1/2}, \end{aligned} \tag{5.27}$$

and it follows from criterion **CS** that $\mathcal{J}$ maps $\mathcal{E}^2(D)$ onto H^2. Using the properties of $\mathcal{J}$, it is easily checked from (5.19) and (5.26) that

$$\mathcal{A}_F = \mathcal{J}^{-1}\, A_{F\circ\Xi}\, \mathcal{J}, \tag{5.28}$$

i.e. the operators $\mathcal{A}_F$ and $A_{F\circ\Xi}$ are unitarily equivalent. Since $F\circ\Xi \in H^\infty + C(\mathbb{T})$, we see from (5.28) that $\mathcal{A}_F$ is compact because so is $A_{F\circ\Xi}$, and that its singular values $s_k(\mathcal{A}_F)$ (*i.e.* the eigenvalues of $\mathcal{A}_F^*\mathcal{A}_F$ arranged in non-increasing order) are the singular values as $A_{F\circ\Xi}$; moreover, the n-th singular vectors of $\mathcal{A}_F$ (*i.e.* the normalized eigenvectors of $\mathcal{A}_F^*\mathcal{A}_F$ associated with $s_n(\mathcal{A}_F)$) are precisely the functions $\mathcal{J}^{-1}(v_n)$ where v_n is a n-th singular vector of $A_{F\circ\Xi}$. Now, by conformal invariance, the best approximant to F from $\mathcal{E}_n^\infty(D)$ in $L^\infty(\Gamma)$ is $g_n\circ\Psi_1$, where g_n is the the best approximant to $F\circ\Xi$ from H_n^∞, and therefore we obtain from what precedes a conformally invariant version of AAK-theory:

Proposition 5.8 *Let D satisfy hypotheses* **H1**–**H2** *and $F\in\mathcal{E}^\infty(D)+C(\Gamma)$. Then, the unique best approximant to F from $\mathcal{E}_n^\infty(D)$ in $L^\infty(\Gamma)$ is given by the formula*

$$g_n = F - \frac{\mathcal{A}_F V_n}{V_n} = \frac{P_{\mathcal{E}^2(D)}(FV_n)}{V_n}, \tag{5.29}$$

where $V_n\in\mathcal{E}^2(D)$ is any *n-th singular vector of the operator $\mathcal{A}_F$ defined in (5.26). Moreover, the error function $F-g_n$ has constant modulus $s_n(\mathcal{A}_F)$ a.e. on Γ, where $s_n(\mathcal{A}_F)$ is the n-th singular value of $\mathcal{A}_F$.*

Proof: This is obvious from (5.21), (5.22), (5.27), (5.28) and what precedes. ∎

Proposition 5.8 is interesting from the constructive viewpoint, because on the one hand algebraic polynomials are dense in $\mathcal{E}^2(D)$ since D is a so-called *Smirnov domain* (this comes from the outer character of the derivative of the conformal map $\Xi:\mathbb{D}\to D$ to be checked from (4.8), see [31, Thm. 10.6]), and on the other hand polynomials in z and $\overline{z}$ are dense in $C(\Gamma)$ thus also in $L^2(\Gamma)$ by the Stone-Weierstrass theorem. Therefore, one can in principle recursively compute orthogonal bases of $\mathcal{E}^2(D)$ and $\mathcal{E}^{2,\perp}(\Gamma)$ and form the infinite matrix for $\mathcal{A}_F$, so as to estimate its singular values and vectors without recourse to conformal mapping.

As in the case of A_F, we say that a n-th singular vector V_n of $\mathcal{A}_F$ is minimal if its zeros are exactly the poles of the best approximant g_n to F from $\mathcal{E}_n^\infty$, that is to say if it has no common zeros with $P_{\mathcal{E}^2(D)}(FV_n)$. When $s_n(\mathcal{A}_F)$ is simple, a n-th singular vector is unique up to a multiplicative constant thus it is necessarily minimal, and if $s_n(\mathcal{A}_F)$ has multiplicity $\delta>1$ then the existence of a n-th minimal singular vector for $\mathcal{A}_F$ follows from the existence of such a vector for $A_{F\circ\Xi}$. In any case, denoting by $\zeta_1,\dots,\zeta_d$ the poles of g_n and by $m_1,\dots,m_d$ their respective multiplicities, any minimal n-th singular vector for $\mathcal{A}_F$ assumes the form $B_{k_n}W_n$, where B_{k_n} is the Blaschke product on D given by the right-hand side of (5.12) and where $W_n\in\mathcal{E}^2(D)$ is such that $\mathcal{J}(W_n)=(W_n\circ\Xi)(\Xi')^{1/2}$ is outer. To fix some terminology, let us recall that a holomorphic function h on D is called *outer* if $h\circ\Xi$ is outer on $\mathbb{D}$ as defined in (5.23)[§]. But from (4.8) it is easy to see that $(\Xi')^{-1/2}$ is outer on $\mathbb{D}$ (see *e.g.* [35, Ch.II, Cor. 4.7]), and it is otherwise clear that the product of two outer functions is outer, so that W_n is in fact outer on D. We call it the *outer factor* of V_n.

We can now give a conformally invariant version of the orthogonality relations (5.24):

[§]This does not depend on the particular choice of the conformal map $\Xi:\mathbb{D}\to D$ and amounts to say that h has no zeros and $\log|h|$ solves a generalized Dirichlet problem in D, namely it is the integral of its nontangential boundary values with respect to harmonic measure [65, App. A3].

Theorem 5.9 *Let D and γ satisfy* **H3**-**H4**, *and F be given by (5.3) where $q = \infty$. Assume that the support of ν comprises at least n points. If $V_n = B_{k_n} W_n$ is a minimal n-th singular vector of $\mathcal{A}_F$ with B_{k_n} a Blaschke product on D given by the right-hand side of (5.12) and W_n its outer factor, then*

$$\int_\gamma B_{k_n}(\xi)\, K_{\mathcal{E}^2(D)}(\zeta, \zeta_j)^k\, W_n(\zeta)\, d\nu(\zeta) = 0, \quad 1 \le j \le d,\ 1 \le k \le m_j. \tag{5.30}$$

Proof: If Ξ maps $\mathbb{D}$ conformally onto D, and if we let $b_{k_n} = B_{k_n} \circ \Xi$ and $w_n = (W_n \circ \Xi)(\Xi')^{1/2}$, then we know from (5.28) and the discussion thereafter that $v_n = b_{k_n} w_n$ is a minimal n-th singular vector of $A_{F\circ\Xi}$ with b_{k_n} a Blaschke product and w_n and outer function. Granted this, (5.30) follows from Proposition 5.7 and Theorem 5.6 in exactly the same way as (5.18) follows from Proposition 5.4 and Theorem 5.3. ∎

6 Behavior of the poles

On a plane domain whose complement contains at least two points, the *hyperbolic distance* is the maximal conformal Riemannian metric with curvature less than or equal to -1, see [5, Sec. 1.5.]. On the unit disk its differential element is $2|dz|/(1-|z|^2)$, so the hyperbolic distance is given by

$$\lambda(z_1, z_2) = \min_C \int_C \frac{|dz|}{1-|z|^2}, \quad z_1,\ z_2 \in \mathbb{D}, \tag{6.1}$$

where the minimum is taken over all rectifiable curves C in $\mathbb{D}$ from z_1 to z_2. The geodesic arc (*i.e.* the minimizing C in (6.1)) is simply the arc of circle between z_1 and z_2 which is orthogonal to $\mathbb{T}$ (here a radius is an arc of circle through infinity). Such an arc is the image of a real segment $[a, b] \subset (-1, 1)$ under a Möbius transformation of the form

$$z \to \xi_0 \frac{z - z_0}{1 - z\bar{z}_0}, \qquad \xi_0 \in \mathbb{T},\ z_0 \in \mathbb{D}. \tag{6.2}$$

These are precisely the conformal automorphisms of $\mathbb{D}$ so they preserve the hyperbolic distance. The latter can be explicitly computed [5, 60] so as to yield:

$$\lambda(z_1, z_2) = \operatorname{Arctanh} \left| \frac{z_1 - z_2}{1 - \overline{z}_1 z_2} \right|. \tag{6.3}$$

If Ψ_1 conformally maps D onto $\mathbb{D}$, the differential element of the hyperbolic metric on D is $2|\Psi_1'||dz|/(1-|\Psi_1|^2)$, and the geodesics are the images under Ψ_1^{-1} of the geodesics in $\mathbb{D}$.

When the arc γ in (5.3) is a hyperbolic geodesic, rather precise information can be deduced on the geometry of the poles of a best (or local best) meromorphic approximant to F. To state it, we first need to introduce some notation as follows.

If C is a Jordan arc with distinct endpoints and $c : I \to C$ an injective parametrization defined on some real interval I, we say that $x_0, \ldots, x_k \in C$ are *consecutively ordered* if $c^{-1}(x_0), \ldots, c^{-1}(x_k)$ is a monotonic sequence in I. Clearly this does not depend on the parametrization, and for φ a function defined on $S \subset C$ we denote its *variation* by

$$V(\varphi, S) = \sup\left\{ \sum\nolimits_{j=1}^k |\varphi(x_j) - \varphi(x_{j-1})|;\ k \in \mathbb{N},\ x_0, x_1, \ldots, x_k \text{ consecutively ordered in } S \right\}$$

which is a non-negative number or $+\infty$. In particular, when C is smooth and $t_C(x)$ indicates the unit tangent at x, we let the *total curvature* of C be

$$\Theta(C) \triangleq V(\arg t_C, C) \tag{6.4}$$

where $\arg t_C$ is any argument function for t_C which is continuous on C.

In this section we consider the special case where γ is a hyperbolic geodesic arc in D. Using the Radon-Nikodym theorem, we write the complex measure ν appearing in (5.3) in polar form:

$$d\nu(t) = e^{i\varphi(t)} d\mu(t), \tag{6.5}$$

where μ is a positive Borel measure supported on γ –the so-called *total variation* of ν– and φ is a real μ-measurable function which is an argument function for $d\nu/d\mu$ on γ. Of course φ is by no means unique since it is defined up to the addition of an arbitrary μ-measurable function with values in the multiples of 2π; thus when we make an assumption on φ, we mean that φ can be chosen so as to meet that assumption. Note that the support of μ is identical to the support of ν, and it is a compact subset of γ that we denote by S.

Finally, for $\zeta \in D$, we put $\theta_D(\zeta, \gamma)$ to designate the hyperbolic angle in which γ is seen at ζ, that is the angle at ζ between the two hyperbolic geodesics in D that run through ζ and the endpoints of γ. It is well-defined unless ζ is an endpoint of γ, in which case we set $\theta_D(\zeta, \gamma) = \pi$. This hyperbolic angle is conformally invariant, meaning that $\theta_D(\zeta, \gamma) = \theta_{\Psi(D)}(\Psi(\zeta), \Psi(\gamma))$ whenever Ψ is a conformal map on D. Clearly $\theta_D(\zeta, \gamma) \leq \pi$ with equality if, and only if $\zeta \in \gamma$.

Proposition 6.1 *Let D satisfy hypotheses* **H1**-**H2** *and γ be a hyperbolic geodesic arc in D. Assume that ν is a complex measure on γ whose support S contains at least n points, and let (6.5) be the polar decomposition of ν. With the above notations, if $\zeta_1, \ldots, \zeta_d \in D$ satisfy (5.18) with $m_1 + \ldots + m_d = n$, then we have that*

$$\sum_{j=1}^{d} m_j(\pi - \theta_D(\zeta_j, \gamma)) \leq 2V(\varphi, S) + \Theta(\gamma). \tag{6.6}$$

This holds in particular if the ζ_j, with respective multiplicity m_j, are the poles of a best (or local best approximant) from $\mathcal{E}_n^2(D)$ to F as in (5.3) with $q = 2$.

Proof: When $D = \mathbb{D}$, the result appears in slightly refined form as [17, lem. 6.1]. To carry it over to more general D, consider a conformal map Ψ_1 from D onto $\mathbb{D}$ and let $\Xi = \Psi_1^{-1}$ be the inverse map. Because $\Psi_1(\gamma)$ is a hyperbolic geodesic arc in $\mathbb{D}$, we may assume up to composing with a Möbius transformation that $\Psi_1(\gamma)$ is a real segment $[a, b] \subset (-1, 1)$. If we set $\xi_j = \Psi_1(\zeta_j)$, we get as in the proof of Theorem 5.5, that (5.13) holds with γ replaced by $[a, b]$ and $d\nu$ replaced by $d\nu_1 = (\Xi')^{-1/2} d(\nu \circ \Xi)$. Let $d\nu_1(s) = e^{i\varphi_1(s)} d\mu_1(s)$, be the polar decomposition of $d\nu_1$. From the known result on the disk, we get the inequality:

$$\sum_{j=1}^{d} m_j(\pi - \theta_D(\xi_j, [a, b])) \leq 2V(\varphi_1, \Psi_1(S)), \tag{6.7}$$

where we used the fact that $\Theta([a, b]) = 0$. By conformal invariance of hyperbolic angles the left-hand side of (6.7) equals that of (6.6), so it remains to prove that

$$2V(\varphi_1, \Psi_1(S)) \leq 2V(\varphi, S) + \Theta(\gamma). \tag{6.8}$$

But it is immediate from (6.5) and the definition of ν_1 that

$$d\nu_1(s) = \left(\frac{\Xi'(s)}{|\Xi'(s)|}\right)^{-1/2} e^{i\varphi(\Xi(s))} \left|\Xi'(s)\right|^{-1/2} d(\mu \circ \Xi),$$

and therefore we may choose

$$\varphi_1(s) = \varphi(\Xi(s)) - \frac{1}{2}\arg(\Xi'(s)), \qquad s \in \Psi_1(S), \tag{6.9}$$

where $\arg(\Xi')$ is a continuous argument function for Ξ' on $[a,b]$; such a function exists because Ξ' is smooth and does not vanish on $\mathbb{D}$. Now, since $\Xi(s)$ yields a smooth parametrization of γ for $s \in [a,b]$, it follows from definition (6.4) that $\Theta(\gamma) = V(\arg(\Xi'), [a,b])$ hence (6.8) follows from (6.9) and the triangle inequality. ∎

If $V(\varphi, S) < \infty$, the proposition gives a quantitative appraisal of the fact that most of the poles of best (or local best) approximants from $\mathcal{E}_n^2(D)$ to F must cluster to γ as n goes large. Indeed, the right-hand side of (6.6) is independent of n, whereas $\pi - \theta_D(\zeta_j, \gamma)$ is non-negative and may become small only if ζ_j approaches γ.

Remark: For the inverse problem considered in Section 7, it is of special interest to make (6.6) effective in order to check the hypothesized location of γ against the computation of the ζ_j as poles of a best (or local best) meromorphic approximant from $\mathcal{E}_n^2$ to F given by (5.3). This requires bounding $V(\varphi, S)$ from above, granted γ and the restriction of F to Γ. In fact, we can assume up to a conformal mapping that D is a disk of radius $R > 1$ while $\gamma = [-1, 1]$, and then the question is to majorize the variation of the argument of a complex measure ν on $[-1, 1]$, granted its moments which are just (up to a power of R) the Fourier coefficients of strictly negative index of the function $F(Re^{i\theta})$. This is an interesting but apparently open issue in general. In the particular case dealt with in Theorem 7.1, where $d\nu = \sigma(t)dt$ with σ a nonzero function which is analytic in a neighborhood of $[-1, 1]$ except for branchpoints of order 2 at -1 and 1, we can at least obtain an asymptotic estimate as follows. Put $h(t) = (1-t^2)^{1/2}\sigma(t)$ whose argument is the same as the argument $\varphi(t)$ of σ for $t \in [-1, 1]$. Then h is smooth on $[-1, 1]$ and the Jackson polynomials $Q_{2m-2}(h, t)$ (see *e.g.* [70, Ch. 5, sec. 1]) can be computed from the moments of σ. They will converge uniformly to h and their derivatives will likewise converge to those of h, from which it is easily seen that $V(\varphi, [-1, 1])$ (where it is understood that each zero of h on $(-1, 1)$ with odd multiplicity contributes to the variation by π) is subject to the inequality:

$$V(\varphi, [-1, 1]) \leq \limsup_{m\to\infty} \left(\pi \mathcal{Z}_o\left(Q_{2m-2}(h_1, .)\right) + \int_{[-1,1]} \left|\operatorname{Im} \frac{Q'_{2m-2}(h_1, t)}{Q_{2m-2}(h_1, t)}\right| dt\right), \tag{6.10}$$

where $\mathcal{Z}_o$ indicates the number of zeros of odd multiplicity on $(-1, 1)$. It is not difficult to check that the right-hand side of (6.10) is indeed finite, and exceeds the left-hand side by at most 2π in the generic case where σ has no zero on $(-1, 1)$ and a zero of *exact order* $1/2$ at -1 and 1 (*i.e.* $k_j = 1$ in (7.2)).

Corollary 6.2 *Let D, γ, and ν be as in Proposition 6.1, and F be as in (5.3) with $q = \infty$. If $\zeta_1, \ldots, \zeta_{m_n}$ are the poles of a best approximant to F from $\mathcal{E}_n^\infty$, each of them repeated with its multiplicity, and if W_n is the outer factor of a minimal n-th singular vector of the Hankel operator*

$\mathcal{A}_F$ defined in (5.26), then

$$\sum_{j=1}^{d} m_j(\pi - \theta_D(\zeta_j, \gamma)) \leq 2V(\varphi, S) + 2V(\arg(W_n), S) + \Theta(\gamma). \tag{6.11}$$

Proof: Note that $\arg(W_n)$ is well-defined on D since W_n has no zeros there. Now, in view of Theorem 5.9, we obtain the corollary from AAK-theory on applying Proposition 6.1, replacing ν by $\tilde{\nu}$ such that $d\tilde{\nu} = W_n d\nu$. ■

Remark: To make (6.11) effective in the context of section 7, we need to bound $V(\varphi, S)$ and $V(\arg(W_n), S)$ from above granted γ and F on Γ. The first majorization was already discussed in the remark after Proposition 6.1. As to the second, it is much easier because W_n is known on Γ by the very computation of the best approximant to F from $\mathcal{E}_n^\infty$ (*cf.* Section 5.2) and then it can be computed on γ as the Cauchy integral of its boundary values.

Inequality (6.11) substantially differs from (6.6) in that the right-hand side depends on n through W_n. In particular, it does not imply alone that the poles of the best approximant to F from $\mathcal{E}_n^\infty$ cluster to γ as n goes large. To see that this is indeed the case, we need to clarify the asymptotic behavior of W_n. Recall that a family of holomorphic functions on a domain Ω is *normal* if it is uniformly bounded on compact subsets of Ω. This is equivalent requiring that it is relatively compact in the space of holomorphic functions on Ω, endowed with the topology of uniform convergence on compact sets. When this is the case, the family of derivatives is also normal.

Proposition 6.3 *Assume that D satisfies hypotheses* **H1**-**H2** *and let γ be a hyperbolic geodesic arc in D. Consider a complex measure ν on γ with infinite support S, whose polar decomposition is given by (6.5). Let F be as in (5.3) with $q = \infty$, and for each positive integer n define W_n to be the outer factor of some minimal n-th singular vector of the Hankel operator $\mathcal{A}_F$. If $V(\varphi, S) < \infty$, then $\{W_n\}_{n\in\mathbb{N}}$ is a normal family on D, no limit point of which has a zero. In particular, $V(\arg(W_n), S)$ is bounded independently of n.*

Proof: If we establish the normality of $\{W_n\}$ and the fact that no limit function has a zero in $\mathbb{D}$, the boundedness of $V(\arg(W_n), S)$ independently of n will follow from the uniform boundedness of W_n'/W_n in $L^1(\gamma)$.

Let Ψ_1 map D conformally onto $\mathbb{D}$ and $\Xi = \Psi_1^{-1}$ be the inverse map. Up to further composing Ψ_1 with a Möbius transformation, we may assume that $\gamma' = \Psi_1(\gamma)$ is a real segment. From the discussion before Theorem (5.9), we know that W_n is the outer factor of a minimal n-th singular vector of $\mathcal{A}_F$ if, and only if $w_n = (W_n \circ \Xi)(\Xi')^{1/2}$ is the outer factor of a minimal n-th singular vector of $A_{F\circ\Xi}$. Since Ξ is a topological map and Ξ' does not vanish on $\mathbb{D}$, it is clear that the family $\{W_n\}$ is normal on D if, and only if $\{w_n\}$ is normal on $\mathbb{D}$. Note from Proposition 5.7(*ii*) that $F \circ \Xi$ is in turn of the form (5.3) on $\mathbb{T}$, the Cauchy integral now being taken over γ'. Moreover, in the polar decomposition $d\nu_1(s) = e^{i\varphi_1(s)}d\mu_1(s)$ of the measure ν_1 appearing in (5.25), we may choose $\varphi_1 = \varphi \circ \Xi - \arg(\Xi')$ and then we get by hypothesis and the smoothness of Ξ' that

$$V(\varphi_1, \Psi_1(S)) \leq V(\varphi, S) + V(\arg(\Xi'), \Psi_1(S)) < \infty.$$

Altogether, we see it is enough to prove the proposition when $D = \mathbb{D}$ and γ is a real segment.

In this case the normality of $\{w_n\}$ follows by essentially the same argument as in the proof of [23, Thm. 10.1], although there are minor modifications that we now indicate. First, as $V(\varphi, S) < \infty$, we can extend φ which is *a priori* defined μ-a.e. on S, into a function of bounded variation defined on all of γ. Indeed φ may first be defined *everywhere* on S without increasing the variation since it has limits from the left and from the right at every $s \in S$, and then one can extend φ linearly on every component of $\gamma \setminus S$; the latter extension does not increase the variation either. Next, we appeal to [17, lem. 3.4] and we find a polynomial T and a $\delta > 0$ such that

$$|\arg(e^{i\varphi(t)}T(t))| < \pi/2 - \delta \qquad \text{for } t \in \gamma,\ T(t) \neq 0. \tag{6.12}$$

Now, following the argument in the proof of [23, Thm. 10.1], inequality (6.12) is all we need to establish an analog to equation (144) of [23], where the functions b and P in that equation are replaced in our case by T and 0 respectively. Specifically, this analog assumes the form:

$$\int_\gamma \left|T(t)\, b_{k_n}^2(t)\, w_n(t)\right| d\mu(t) \leq C\, s_n(A_F), \tag{6.13}$$

where b_{k_n} is the Blaschke product of degree k_n such that $v_n = b_{k_n} w_n$ is a minimal n-th singular vector of A_F (see the discussion before Theorem 5.6) and where C is a constant depending only on δ, T, and the geometry. In another connection, [23, eqn. (142)] tells us (set $p = \infty$ thus $s = 2$ in that equation) that since v_n is a n-th singular vector of A_F, we have

$$s_n(A_F)\,\overline{j_n(\bar z)\, w_n(\bar z)} = \int_\gamma \frac{b_{k_n}^2(t)\, w_n(t)}{1 - zt}\, d\nu(t) \qquad z \in \overline{\mathbb{C}} \setminus \gamma^{-1}, \tag{6.14}$$

where j_n is some Blaschke product of finite degree. Because the support of ν is infinite by hypothesis, we saw in Section 5.2 that $F \notin H_n^\infty$; hence $s_n(A_F) \neq 0$, and since in addition $\overline{j_n(\bar z)}$ has all its zeros in $\mathbb{D}$ (conjugate to those of j_n) we deduce from (6.14) that $\overline{w_n(\bar z)}$ extends holomorphically to $\overline{\mathbb{C}} \setminus \gamma^{-1}$. On replacing everywhere b by T, the computation that leads from equation (142) to equation (145) in [23] (compare (129)-(130) in that reference) takes us from (6.14) to

$$s_n(A_f) P_{H^2}\left(T(e^{-i\theta})\,\overline{j_n(e^{-i\theta}) w_n(e^{-i\theta})}\right) = \int_\gamma \frac{T(t) b_{k_n}^2(t) w_n(t)}{1 - e^{i\theta} t}\, d\nu(t). \tag{6.15}$$

Comparing (6.15) and (6.13) and canceling out $s_n(A_F)$, we deduce that

$$P_{H^2}\big(T(e^{-i\theta})\overline{j_n(e^{-i\theta}) w_n(e^{-i\theta})}\big),$$

which is in H^2 by construction, extends to a holomorphic function in $\overline{\mathbb{C}} \setminus \gamma^{-1}$ which is locally uniformly bounded there, *independently of* n. Besides, since it is the projection on $\overline{H}^{2,0}$ of a function whose $L^2(\mathbb{T})$-norm is bounded independently of n (recall that $|j_n| = 1$ on $\mathbb{T}$ and that w_n has unit $L^2(\mathbb{T})$-norm), we have that

$$P_{\overline{H}^{2,0}}\big(T(e^{-i\theta})\overline{j_n(e^{-i\theta}) w_n(e^{-i\theta})}\big)$$

extends to a holomorphic function in $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ which is also locally uniformly bounded independently of n by the Cauchy formula. Adding up, we see that $T(e^{-i\theta})\overline{j_n(e^{-i\theta}) w_n(e^{-i\theta})}$ extends holomorphically in $\overline{\mathbb{C}} \setminus \{\gamma^{-1} \cup \overline{\mathbb{D}}\}$ to a function which is locally uniformly bounded independently of n. By analytic continuation this function is nothing but $T(1/z)\overline{j_n(\bar z) w_n(\bar z)}$, and since $\overline{j_n(\bar z)}$ has modulus greater than 1 if $|z| > 1$ because it is a Blaschke product, we get that $T(1/z)\overline{w_n(\bar z)}$ is locally uniformly bounded on $\overline{\mathbb{C}} \setminus \{\gamma^{-1} \cup \overline{\mathbb{D}}\}$, independently of n. Since it is in fact analytic in $\overline{\mathbb{C}} \setminus \gamma^{-1}$, it must be locally uniformly bounded there, independently of n, by the maximum principle. Therefore, on any Jordan subdomain of $\overline{\mathbb{C}} \setminus \gamma^{-1}$ whose boundary contains no zero of $T(1/z)$, we conclude that $\overline{w_n(\bar{z})}$ (which is holomorphic as we already know) is bounded independently of n by the maximum principle. This proves the normality of $\{\overline{w_n(\bar{z})}\}$ thus also of $\{w_n\}$ on $\overline{\mathbb{C}} \setminus \gamma^{-1}$, and since $\|w_n\|_{L^2(\mathbb{T})} = 1$ no limit function of the family $\{w_n\}$ can be the zero function. But as w_n has no zero on $\mathbb{D}$, it follows from a classical theorem of Hurwitz that a limit function of $\{w_n\}$ is either the zero function or does not vanish on $\mathbb{D}$, and since we ruled out the first possibility the latter necessarily holds. ■

Propositions 6.1-6.3 and Corollary 6.2 show that when F is as in (5.3) and is not already meromorphic, the poles of a best (or local best) approximant g_n to F from $\mathcal{E}_n^2(D)$ or $\mathcal{E}_n^\infty(D)$ must cluster to γ when the latter is a geodesic arc, at least when ν has an argument of bounded variation. To make this more precise, let us introduce the *counting measure of the poles* of g_n, denoted by μ_{g_n}, which is the discrete probability measure having equal mass at each poles, counting multiplicities. We also recall that a sequence of compactly supported measures μ_n converges weak* to a measure μ if $\int f d\mu_n \to \int f d\mu$ for every continuous function f with compact support in $\mathbb{C}$. For a geometric interpretation, observe that μ_{g_n} converges weak* to μ if, and only if the proportion of the poles of g_n contained in an open set $U \subset \mathbb{C}$ converges to $\mu(U)$ as $n \to \infty$ (remember all the poles lie in D so that no mass can go to infinity).

To state a weak* convergence result for μ_{g_n}, we need to introduce the hyperbolic equilibrium distribution of a compact subset $K \subset D$. We merely state the basic definitions, and refer the reader to [65, Ch. II] for a detailed treatment. Given any probability measure μ with support in K, the Green potential of μ with respect to the domain D is the superharmonic function

$$p_D(\mu, z) = \int g_D(z, \zeta) d\mu(\zeta) \geq 0, \quad z \in D,$$

and the Green energy of μ is

$$I_D(\mu) = \int p_D(\mu, z) d\mu(z)$$

which is a non-negative number or $+\infty$. If K is so thin that no μ has finite Green energy on K, then we say that it is *polar*; polar sets look very bad: for instance they are totally disconnected. If on the contrary there is a probability measure on K of finite Green energy, then there also exists a unique probability measure $\omega_{D,K}$ of minimal Green energy called the *hyperbolic equilibrium distribution of* K. For instance if $[a, b] \subset (-1, 1)$ is a real segment, then

$$d\omega_{\mathbb{D},[a,b]} = \frac{C\, dt}{\sqrt{(1 - bt)(b - t)(t - a)(1 - at)}},$$

where C is some normalizing constant. For the approach to inverse Dirichlet-Neumann problems considered in the next section, it is an important feature of the equilibrium distribution that it charges more the corner points and endpoints of K. In this respect, the previous example where the density is infinite at the endpoints of $[a, b]$ is rather typical. The conformal invariance of the Green function immediately results in the conformal invariance of the Green equilibrium measure, that is if Ψ maps D conformally onto D' and takes K to K', then $\omega_{D,K} = \omega_{D',K'} \circ \Psi$.

The quantity $C(\Gamma, K) = 1/I_D(\omega_{D,K})$ is called the *capacity of the condenser* (Γ, K), and by convention this capacity is zero when K is polar. We say that K is *regular* if $p_D(\omega_{D,K}, z)$ is continuous on D; all nice compact sets are regular, in particular all whose boundary has no connected component that reduces to a point.

In the statement of the theorem below, we use $B(z_0, r)$ to denote the open ball centered at z_0 of radius r, and we let $|\cdot|$ indicate the linear measure induced by arclength on γ.

Theorem 6.4 *Let D satisfy hypotheses* **H1**-**H2** *and γ be a hyperbolic geodesic arc in D. Assume that ν is a complex measure on γ whose support S is regular and whose polar decomposition (6.5) satisfies $V(\varphi, S) < \infty$ and*

$$\mu\left(\overline{B(x,\delta)} \cup S\right) \geq c \left|\overline{B(x,\delta)} \cap S\right|^L \qquad \text{for all } x \in S \text{ and for all } \delta \in (0,1), \tag{6.16}$$

where c, L are positive constants. Suppose that F is given by (5.3) where $q = 2$ (resp. $q = \infty$). If for each $n \in \mathbb{N}$ we let g_n be a best or local best approximant to F from $\mathcal{E}_n^2$ (resp. a best approximant to F from $\mathcal{E}_n^\infty$), then the sequence μ_{g_n} of counting measures of the poles of g_n converges weak to the Green equilibrium distribution $\omega_{D,\gamma}$ as n tends to ∞.*

Proof: By Propositions 5.4, 5.7, and the conformal invariance of Green equilibrium distributions, it is enough to prove the result when $D = \mathbb{D}$. Then, it becomes a consequence of Theorem 5.3 and [17, Thm. 5.1] when $q = 2$, and of Theorem 5.6, Proposition 6.3 and [17, cor. 6.2] when $q = \infty$. ∎

7 Crack detection

In this section we return to the inverse Dirichlet-Neumann problem posed in Section 2. When the crack γ is a geodesic arc, the preceding results may be used to localize its endpoints. Indeed, if we assume that $\Phi \in L^p(\Gamma)$ for some p such that $1 < p < 2$, the solution u to (2.1) can be obtained from Theorem 4.1 on setting $\phi^+ = \phi^- = 0$, and by Theorem 4.4 we have that $u = \mathrm{Re} f$ where f is of the form (5.3) for all q. Since $\partial u/\partial z \in \mathcal{E}^p(D \setminus \gamma)$ by Theorem 4.1, the nontangential convergence of $\nabla u . n_\gamma^\pm$ to zero a.e. on γ means that $\mathrm{Re}\{n_\gamma \partial u^\pm/\partial z\} = 0$ a.e. on γ in the sense of nontangential limits, where we have kept the notation n_γ to indicate the complex number whose real and imaginary parts are the coordinates of the unit normal to γ. Now, if we write $f = u + iv$, the Cauchy Riemann equations (that remain valid a.e. on γ when taking nontangential limits from each side) imply that $\partial u^\pm/\partial z = i\partial v^\pm/\partial z$, and since $\pm t_\gamma = i n_\gamma^\pm$ we get $\nabla v^\pm . t_\gamma = \mathrm{Re}\{t_\gamma \partial v^\pm/\partial z\} = 0$ a.e. on γ. Thus $v^\pm$ is constant on γ (remember $f^\pm$ is absolutely continuous). By the reflection principle, it follows that $f^\pm$ locally extends holomorphical across $\overset{\circ}{\gamma}$; hence the density σ in (4.45) is analytic except for branched singularities at γ_0, γ_1, and the assumptions of Theorem 6.4 are satisfied (see the detailed argument in the proof of Theorem 7.1(ii) below) unless $\sigma = 0$, that is, unless γ is a level line of the solution to the Neumann problem in the sane domain D. In the latter case, the flux which is used cannot identify the crack and should be modified. In the former, one can look for clusters of the poles of best meromorphic approximants to f in order to locate γ, as indicated in Section 3. Note that the constant functions v^+ and v^- must agree, because from Theorem 4.4 we know that $\sigma = f^+ - f^-$ vanishes at γ_0 and γ_1. Hence σ is in fact real on γ where it is equal to $u^+ - u^-$.

Of course the assumption "γ is a hyperbolic geodesic arc", which is of the same type as the assumption "γ is line segment" made in the reciprocity gap method as described in Section 2, is overly strong. Now, if γ is no longer a geodesic arc, it is natural to ask whether it can be deformed into such an arc while keeping the endpoints γ_0 and γ_1 fixed, without changing the value of (4.45). By Cauchy's theorem this will be possible if σ extends holomorphically to a sufficiently large domain, and then one could use what precedes to recover at least γ_0 and γ_1. Still we note that this involves strong assumptions on γ: it must be an analytic arc, being a level line of the imaginary part of the holomorphic function σ. Conversely, as soon as γ is analytic, $\sigma = f^+ - f^-$ extends locally holomorphically across $\mathring{\gamma}$ because both f^+ and f^- do by reflection. The whole point when γ is analytic is therefore to give conditions under which the domain of analyticity of σ contains the geodesic arc linking γ_0 and γ_1 in D. The next section explores that issue.

7.1 Close-to-geodesic analytic cracks

Let us first consider as an example the case where γ is a line segment in D. Denote by Π_+ and Π_- the positive and negative half-planes cut-out by the line supporting this segment. Since f is analytic in $\Pi_\pm \cap D$ and extends continuously to $\gamma^\pm$ where it has a constant imaginary part as we have seen, it can be analytically continued from each side across the interior of γ by the Schwarz reflection principle. Hence, $\sigma = f^+ - f^-$ has an analytic continuation all the way to $\mathcal{G}$ if, and only if the domain of analyticity of f contains the reflection of $\mathcal{G}$ across γ. Note that this reflection may well lie partly outside of D, and therefore additional requirements both on how γ sits in D and on the regularity of f may be in order. The next result formalizes this idea.

Theorem 7.1 *Let D satisfy assumptions* **H1**-**H2** *and $\Phi \in L^p(\Gamma)$ with $1 < p < 2$. Let further $P : \mathcal{O} \to \mathcal{U}$ be a conformal map between simply connected domains in $\mathbb{C}$, where $\mathcal{O}$ is bounded and contains $[0,1]$. Suppose that $\gamma := P([0,1])$ is included in D, with $\gamma_0 = P(0)$ and $\gamma_1 = P(1)$, and denote by $\mathcal{G}$ be the hyperbolic geodesic arc between γ_0 and γ_1 in D. Assume that $\mathcal{G} \subset \mathcal{U}$ and that the reflection of $P^{-1}(\mathcal{G})$ across the real axis is included in $\mathcal{O}$. Finally, let us make the hypothesis that the function f obtained from Theorem 4.1 with $\phi^+ = \phi^- = 0$ extends holomorphically to $\mathcal{U} \setminus \gamma$ (this is automatic if $\mathcal{U} \subset D$) and that $\sigma = f^+ - f^-$ is not the zero function on γ. Then:*

(i) The function $\sigma = f^+ - f^-$, initially defined on γ, can be analytically continued over a two-sheeted Riemann surface lying above an open set $\mathcal{V}$ containing the interior of both γ and $\mathcal{G}$ but excluding their endpoints γ_0 and γ_1. At these two points, the function σ has limit zero. The contours γ and $\mathcal{G}$ are homotopic in $\mathcal{V}$ so that, in the complement of any simply connected domain $\mathcal{D}$ containing γ and $\mathcal{G}$, the singular part of f in (4.45) can be rewritten as

$$\frac{1}{2i\pi} \int_{\mathcal{G}} \frac{\sigma(\xi)}{\xi - z} d\xi, \quad z \notin \mathcal{G}. \tag{7.1}$$

(ii) The holomorphic continuation of σ to $\mathcal{G} \setminus \{\gamma_0, \gamma_1\}$ has an argument of bounded variation, and the measure μ having density $|\sigma|$ with respect to arclength on $\mathcal{G}$ has support $S = \mathcal{G}$ and meets (6.16).

Proof: The function $h = f \circ P$ is analytic in $\mathcal{O} \setminus [0,1]$ and the limits h^+ and h^- from above and below on $[0,1]$ are continuous, with constant imaginary part. Thus, by the reflection principle, h continues analytically across $(0,1)$ from above and below, according to the formula $h(z) = \overline{h(\overline{z})}$. Since applying twice this reflection rule gives back $h(z)$, the latter is naturally defined on a two-sheeted Riemann surface above $\mathcal{O} \cap \mathcal{O}^s \setminus \{0,1\}$, where $\mathcal{O}^s$ denotes the reflection of $\mathcal{O}$ across the real axis. Of necessity $\lambda = h^+ - h^-$ also has an analytic continuation to that Riemann surface, and applying P we get an analytic continuation of σ to some two-sheeted Riemann surface above $P(\mathcal{O} \cap \mathcal{O}^s) \setminus \{\gamma_0, \gamma_1\}$. Moreover, we know from Theorem 4.4 that σ vanishes continuously at γ_0 and γ_1. Now, there exists a continuous homotopy from $[0,1]$ to $P^{-1}(\mathcal{G})$ in $\mathcal{O} \cap \mathcal{O}^s$, since each connected component of the latter is simply connected (as is the case for the intersection of two bounded simply connected domains) and since $[0,1]$ and $P^{-1}(\mathcal{G})$ lie in the same component for they have the same endpoints. Therefore, on applying P, we see there is a continuous homotopy from γ to $\mathcal{G}$

in $P(\mathcal{O} \cap \mathcal{O}^s)$. By Cauchy's theorem, this implies that (7.1) is indeed equal to the Cauchy integral in (4.45), and achieves the proof of assertion (i).
Next, since σ is a nonzero analytic function with branchpoints at γ_0 and γ_1 by (i), it has a Puiseux expansion at γ_j of the form

$$\sigma(z) = (z - \gamma_j)^{k_j/2} L_j\left((z - \gamma_j)^{1/2}\right), \qquad L_j(0) \neq 0, \qquad j = 0, 1, \tag{7.2}$$

where L_j is holomorphic in a neighborhood of 0. From (7.2) it is clear that $\arg \sigma(z)$ has a smooth limit as $z \to \gamma_j$ along $\mathcal{G}$ (remember that $\mathcal{G}$ is smooth), and that σ has only finitely many zeros on $\mathcal{G} \setminus \{\gamma_0, \gamma_1\}$, say, $z_1, \ldots, z_m$. Each z_j contributes either π or 2π to the variation of the argument, depending whether the order m_j of z_j is odd or even, and the function

$$\frac{\sigma(z)}{\prod_{j=1}^{m}(z - z_j)^{m_j}}$$

is clearly smooth and non vanishing on $\mathcal{G} \setminus \{\gamma_0, \gamma_1\}$ so it has a smooth argument there. Finally, from (7.2) and the analyticity of σ on $\mathcal{G} \setminus \{\gamma_0, \gamma_1\}$, it follows easily that the support S of μ, which is included in $\mathcal{G}$ by definition, is in fact equal to it and that (6.16) holds. This proves (ii). ■

In order to apply Theorem 7.1 to a given analytic crack γ, we need to know that:
(i) the mapping $P : [0, 1] \to \gamma$ which parameterizes γ can be holomorphically continued to a sufficiently large simply connected domain $\mathcal{O}$, containing the reflection of $P^{-1}(\mathcal{G})$ with respect to the real axis. In particular, this will be satisfied if γ is parametrized through an entire function, *e.g.* a polynomial.
(ii) the function f admits an analytic continuation to $P(\mathcal{O}) \setminus \gamma$. This is automatic if $P(\mathcal{O}) \subset D$, but may otherwise be delicate for it requires choosing the flux Φ carefully in connection with the singularities of Γ and making prior assumptions on the location of γ.

Specific applications of this principle in the case where $D = \mathbb{D}$ can be found in [54]. Here, we rather illustrate the fact that if γ is globally analytic in D (*i.e.* the image of a real segment under a conformal map which is *onto* D) and sufficiently close to a geodesic arc in some sense, then Theorem 7.1 can be applied.

Theorem 7.2 *Let D satisfy* **H1**-**H2** *and P conformally map a bounded domain Ω containing $[0, 1]$ onto D. Set $\gamma = P([0, 1])$ and denote by $\mathcal{G}$ the hyperbolic geodesic arc in D linking $\gamma_0 = P(0)$ and $\gamma_1 = P(1)$. For the assumptions of Theorem 7.1 to hold, it is sufficient that one of the following two conditions be satisfied.*

(i) The hyperbolic distance in D between any two consecutive intersection points of $\mathcal{G}$ and γ is less than c_D, where $c_D > 0$ depends only on D; a possible value for $c_{\mathbb{D}}$ is 0.2856.

(ii) The hyperbolic distance in D from any point of $\mathcal{G}$ to γ is less than some constant K, with $K > 0.17328$.

Remark: In particular (i) or (ii) is satisfied if the hyperbolic length of γ is less than c_D or K.

Proof: The statement being conformally invariant, it is enough to consider the case where $D = \mathbb{D}$. Let $h = P^{-1}$ which is conformal from $\mathbb{D}$ onto Ω. Since $P(\Omega) = \mathbb{D}$, point (ii) above the statement of the Theorem is obvious, so we need only show that the reflection $h(\mathcal{G})^s$ of $h(\mathcal{G})$ across the real axis is included in Ω if either condition (i) or (ii) holds. We first prove the sufficiency of (i), for which we may assume that $\gamma \cap \mathcal{G}$ is a finite set, otherwise $\gamma = \mathcal{G}$ by analyticity thus $h(\mathcal{G}) = [0, 1] = h(\mathcal{G})^s$.

For $0 \leq k \leq m$, denote by a_k, the intersection points of γ and $\mathcal{G}$ ordered along the oriented arc $\mathcal{G}$, with $a_0 = \gamma_0$ and $a_m = \gamma_1$. Letting $\mathcal{G}_k$ be the subarc of $\mathcal{G}$ linking a_k and a_{k+1} for $0 \leq k \leq m-1$, we will show that $h(\mathcal{G}_k)^s \subset \Omega$ if $\lambda(a_k, a_{k+1}) < 0.2856$ and this will prove (i) is sufficient. Fix $k \in \{0, \ldots m-1\}$ and let φ be an automorphism of $\mathbb{D}$, of the form (6.2), such that for some $c \in (0,1)$ we have:

$$\varphi(-c) = a_k, \quad \varphi(c) = a_{k+1}, \quad \text{and } \varphi([-c,c]) = \mathcal{G}_k.$$

Note from (6.3) that

$$\lambda(a_k, a_{k+1}) = \lambda(-c, c) = \operatorname{Arctanh} \frac{2c}{1-c^2}. \tag{7.3}$$

by the conformal invariance of the hyperbolic metric.

Now, a sufficient condition for the inclusion $h(\mathcal{G}_k)^s \subset \Omega$ is that the (Euclidean) length $l(h(\mathcal{G}_k))$ of $h(\mathcal{G}_k)$ is less than the length of any rectifiable path from $h(a_{k-1})$ to $h(a_k)$ that intersects the boundary $\partial\Omega$ of Ω. Setting $\mathcal{H} = h \circ \varphi$, this last condition is implied by the following one:

$$l(\mathcal{H}([-c,c])) < \operatorname{dist}(\mathcal{H}(-c), \partial\Omega) + \operatorname{dist}(\mathcal{H}(c), \partial\Omega), \tag{7.4}$$

where "dist" means Euclidean distance. By Koebe's distortion Theorem [60, Theorem 1.3] one has

$$\left|\mathcal{H}'(0)\right| \frac{1-|z|}{(1+|z|)^3} \leq \left|\mathcal{H}'(z)\right| \leq \left|\mathcal{H}'(0)\right| \frac{1+|z|}{(1-|z|)^3}, \qquad z \in \mathbb{D}, \tag{7.5}$$

and from [60, Cor. 1.4] we get

$$\frac{1}{4}\left(1-|z|^2\right)\left|\mathcal{H}'(z)\right| \leq \operatorname{dist}(\mathcal{H}(z), \partial\Omega) \leq \left(1-|z|^2\right)\left|\mathcal{H}'(z)\right|; \tag{7.6}$$

hence

$$\operatorname{dist}(\mathcal{H}(-c), \partial\Omega) + \operatorname{dist}(\mathcal{H}(c), \partial\Omega) \geq \frac{1}{4}(1-c^2)\left(\left|\mathcal{H}'(-c)\right| + \left|\mathcal{H}'(c)\right|\right). \tag{7.7}$$

Applying respectively (7.5) for $z = \pm c$ and $z = \tau \in (0, c)$, we get

$$\frac{1}{4}(1-c^2)\left(\left|\mathcal{H}'(-c)\right| + \left|\mathcal{H}'(c)\right|\right) \geq \frac{1}{2}\left|\mathcal{H}'(0)\right| \left(\frac{1-c}{1+c}\right)^2 \tag{7.8}$$

and

$$l(\mathcal{H}([-c,c])) = \int_{-c}^{c} \left|\mathcal{H}'(\tau)\right| d\tau \leq 2\int_0^c \left|\mathcal{H}'(0)\right| \frac{1+\tau}{(1-\tau)^3}\, d\tau = \left|\mathcal{H}'(0)\right| \frac{2c}{(1-c)^2}. \tag{7.9}$$

Thus, in order for(7.4) to hold, it is enough in view of (7.7)-(7.9) that

$$\frac{2c}{(1-c)^2} < \frac{1}{2}\left(\frac{1-c}{1+c}\right)^2$$

which is equivalent to

$$(1-c)^4 - 4c(1+c)^2 > 0. \tag{7.10}$$

The left-hand side of (7.10), when viewed as a polynomial in c, has a unique root $c^* \in (0,1)$. Thus (7.10) will hold provided that $c < c^*$, and by (7.3) this amounts to:

$$\lambda(a_k, a_{k+1}) < \operatorname{Arctanh} \frac{2c^*}{1-(c^*)^2}.$$

But it is easily checked that $c^* > 0.13697$ and subsequently that the above right-hand side is larger than 0.2856, as desired.

To prove (ii), we appeal to the inequality

$$|T(z) - T(0)| \leq |T'(0)| \, \frac{|z|}{(1-|z|)^2}, \qquad z \in \mathbb{D}, \tag{7.11}$$

which is valid for every conformal map T from $\mathbb{D}$ into $\mathbb{C}$ [60, Thm. 1.3]. Let $z_1 \in \mathcal{G}$ and $z_0 \in \gamma$. If φ_{z_0} denotes the Möbius transformation (6.2) with $\xi_0 = 1$, we get on applying (7.11) with $T = h \circ \varphi_{z_0}^{-1}$ and $z = \varphi_{z_0}(z_1)$ that

$$|h(z_1) - h(z_0)| \leq |h'(z_0)| \, (1 - |z_0|^2) \frac{|z|}{(1-|z|)^2},$$

hence from (7.6) with $\mathcal{H}$ replaced by h:

$$|h(z_1) - h(z_0)| \leq 4 \operatorname{dist}(h(z_0), \partial\Omega) \, \frac{|z|}{(1-|z|)^2}.$$

As $h(z_0)$ is real, the previous equation implies that $\overline{h(z_1)} \in \Omega$ as soon as $|z|/(1-|z|)^2 < 1/4$, that is, as soon as $|z| < x_0$ where x_0 is the unique root in $(0,1)$ of the equation $(1-x)^2 - 4x = 0$. But from (6.3) we know that $\lambda(z_1, z_0) = \operatorname{Arctanh}|z|$, so we conclude that $h(\mathcal{G})^s \subset \Omega$ provided that

$$\min_{z_0 \in \gamma} \lambda(z_1, z_0) < \operatorname{Arctanh} x_0 \stackrel{\Delta}{=} K, \qquad z_1 \in \mathcal{G}.$$

Numerical estimation shows that $x_0 > 0.17157$ and then that $K > 0.17328$, as desired. ∎

The hypothesis that the crack is globally analytically in D is of course quite strong. To obviate this a little, let us point out the following corollary.

Corollary 7.3 *Let the assumptions and notations be as in Theorem 7.2, except that P need not be* onto *D. Let $\mathcal{O} \subset D$ be the image of P, and assume that $\mathcal{G} \subset \mathcal{O}$. If the hyperbolic distance* in $\mathcal{O}$ *from any point of $\mathcal{G}$ to γ is less than K, then the assumptions of Theorem 7.1 do hold.*

Proof: In the proof of Theorem 7.2, we never used that $\mathcal{G}$ was a geodesic arc when showing the sufficiency of (ii). ∎

Theorems 6.4, 7.1, and 7.2 team up in the following and last result:

Theorem 7.4 *Suppose the assumptions of Theorem 7.1 are met; in particular this is the case when D satisfies* **H1**-**H2** *and either (i) or (ii) of Theorem 7.2 holds, while γ is not a level line of the solution to the Neumann problem on D with flux Φ. If $\mu_{2,n}$ (resp. $\mu_{\infty,n}$) is the counting measure of the poles of best or local best (resp. best) approximants from $\mathcal{E}_n^2$ (resp. from $\mathcal{E}_n^\infty$) to f in (4.45), then both $\mu_{2,n}$ and $\mu_{\infty,n}$ converge weak* as $n \to \infty$ to the hyperbolic equilibrium distribution $\omega_{D,\mathcal{G}}$, where $\mathcal{G}$ is the hyperbolic geodesic arc linking the endpoints of γ.*

Remark: It should be observed that the limit distribution of the poles of both type of approximants is *independent* from the particular flux Φ which has been prescribed on Γ.

7.2 More general cracks

The scope of the mechanism behind Theorem 7.1 broadens significantly if we consider *piecewise analytic* cracks, that is, if γ is a concatenation of finitely many arcs each of which is parametrized by a map P as in the theorem; this time there is no need to assume that γ is connected, *i.e. there may be several cracks*. In this case the hyperbolic geodesic arc between the endpoints is no longer the right object that attracts the poles, and it is to be replaced by a certain system of analytic arcs whose endpoints comprise the endpoints and the edges of γ. This system of arcs, that we call $\mathcal{S}$, is the solution to the extremal problem of minimizing $C(\Gamma, \mathcal{S})$ while keeping σ single-valued in $D \setminus \mathcal{S}$¶. Much like in Theorem 7.2, geometric conditions can be given for σ to extend analytically to $\mathcal{S}$ so that the singular part of f in (4.45) can be rewritten as

$$\frac{1}{2i\pi} \int_{\mathcal{S}} \frac{\sigma(\xi)}{\xi - z} d\xi, \quad z \notin \mathcal{S}, \tag{7.12}$$

and then one can establish an analog of Theorem 7.4 where $\mathcal{G}$ gets replaced by $\mathcal{S}$. Such generalization would take us too far afield; let us simply mention that a detailed study of the geometry of the so called *symmetric contour* $\mathcal{S}$ can be found in [69], that the weak* convergence of the counting measure of the poles of best meromorphic approximants to (7.12) toward the Green equilibrium distribution on $\mathcal{S}$ depends on unpublished work [24], and that no analog to the non-asymptotic relations (6.6), (6.11) is available at present. Some of the numerical experiments in the next section illustrate this more general situation. Here again, the equilibrium distribution charges the endpoints and the edges of γ, so that clusters of poles enable one in principle to locate them (see figure 6).

Let us also point out that the geometric conditions on γ set forth in Theorem 7.2, that enabled us to apply Theorem 7.1, can be weakened considerably if one chooses the flux Φ in a more specific manner. For instance, let us consider the case where γ is the injective image of a real segment by an entire function P (*e.g.* a polynomial) which needs not, however extend *injectively* onto D (as was the case in Theorem 7.2); by Corollary 4.3, we may assume that $D = \mathbb{D}$. Now, if $\mathcal{A}$ is a geodesic arc in $\mathbb{D}$ that does not meet γ and $\mathcal{A}$ cuts out $\mathbb{D}$ in two domains D_1, D_2 with, say, $\gamma \subset D_1$, we may choose $\Phi = 0$ on $\mathbb{T} \cap \partial D_1$ so that f will continue analytically across this arc, allowing for further deformation of γ within the domain of analyticity of σ denoted by $\mathcal{W}$. There are two sources of difficulty here: the first is that we cannot ascertain *a priori* that a given arc $\mathcal{A}$ does not meet γ, and only retrospectively may we check such an assumption. This is common in inverse problems. The second difficulty is that we may this time encounter critical values of P when trying to deform γ into $\mathcal{G}$ within $\mathcal{W}$. In this case the critical values of P become branch points for σ, and we end up again deforming γ into a system of arcs $\mathcal{S}$ of the type mentioned above.

Finally, the authors are willing to conjecture that endpoints and edges *always* attract a positive proportion of the poles, even if γ is not piecewise analytic, provided that it is piecewise smooth.

8 Numerical experiments

In this last section, we produce numerical experiments that illustrate the above theoretical results. In view of Corollary 4.3 we fix the domain D to be the unit disk, in which we explicitly embed various cracks γ, see below. The data, *i.e.* the functional pair (Φ, u) on $\mathbb{T}$, is obtained by choosing Φ analytically and then numerically computing u at equispaced points on $\mathbb{T}$, using finite elements

¶This, actually, is what the geodesic arc does when σ can be continued analytically over $D \setminus \{\gamma_0, \gamma_1\}$.

methods from the NAG and the MATLAB libraries. Within this approach, for numerical reasons, γ gets approximated by a thin surface element. Typically 1000 values of u were estimated, and they were subsequently splined to order three on $\mathbb{T}$ before computing the Fourier coefficients by means of quadrature formulas.

8.1 Numerical implementation of the AAK algorithm

In practice, the singular value decomposition of the Hankel operator A_f can be made constructive only when it has finite rank, that is when $f = H + R$ where $H \in H^\infty$ and R is rational (see *e.g.* [57, Thm. 3.11]). Thus a preliminary (non-optimal) rational approximation to f on $\mathbb{T}$ (usually of high degree) has to be performed, and the issue of continuity of best approximants with respect to f arises naturally. It is known [56] that the best approximant from H_n^∞ to $f \in C(\mathbb{T})$ is discontinuous at every $f \notin H_n^\infty$. Therefore, in that preliminary step, one needs to approximate f with respect to some stronger norm than the $L^\infty(\mathbb{T})$-norm. Such an approximation can generically be obtained in the Wiener norm, by simply truncating the Fourier series of F which is absolutely convergent as pointed out at the end of Section 4. Indeed, it is proved in [40] that the operator of best approximation from H_n^∞ (mapping f to g_n according to (5.22)) is continuous in Wiener norm provided that the $(n+1)$–th singular value of A_f is simple. Numerically the assumption of non-multiplicity cannot be verified, but it is generically true [19].
Of course this truncation technique is justified only if the Fourier coefficients can be computed accurately, whereas u is only estimated at a discrete set of equispaced points on $\mathbb{T}$ due to our use of finite elements to simulate the experiments. To avoid difficulties here, we chose smooth fluxes to the effect that u is likewise smooth, and then the error on each Fourier coefficients when using M interpolation points and cubic splines is of the order of $1/M^2$, while the Fourier coefficients themselves decay polynomially fast. This way the truncation error can be kept small.

To evaluate the degree of approximation, up to which the location of the poles remains meaningful, one can observe the magnitude of the singular values as well as their rate of decrease, namely the quotient s_n/s_{n+1} of two consecutive singular values. Indeed, it is known [37] that the rate of decrease for functions like (4.45) is geometric, so when the quotient s_n/s_{n+1} approaches 1 one may suspect that the numerical precision becomes insufficient and the results no more significant.

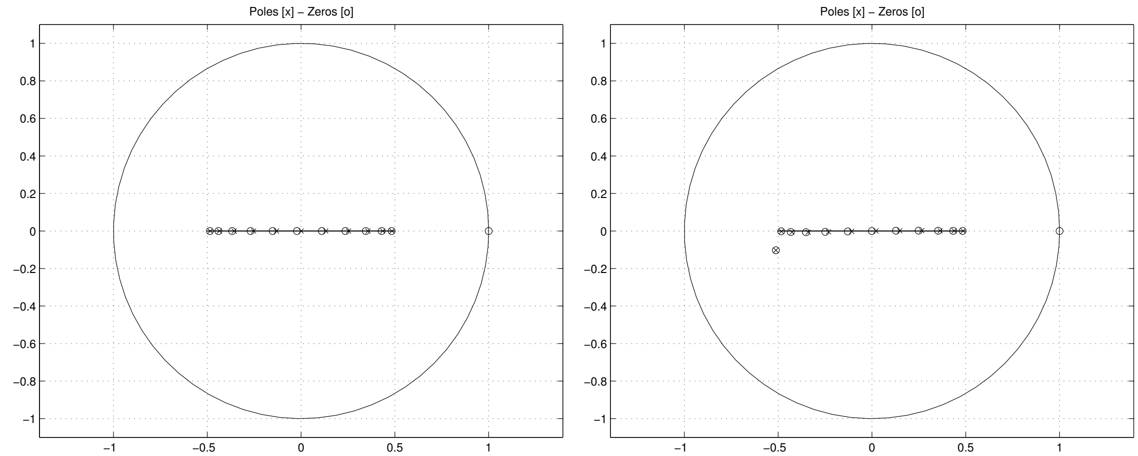
8.2 Numerical implementation of the H^2 rational approximation

Here, again, the function f is considered to be a trigonometric polynomial of large degree. As explained above, computing this representation entails various difficulties. However, there is a main difference with the previous case since, as follows from [13], H^2 rational approximation is continuous with respect to the L^2 norm. From a practical point of view, we use the *hyperion* software described in [38]. Note that the computation in quadruple precision of the Fourier series of the splines constructed from the data is available in the *hyperion* software. Here, we increase the degree of approximation until the criterion reaches the numerical precision of the computer.

8.3 Numerical experiments

1) Crack lying on a diameter with a positive jump of temperature
We choose the crack to be the line segment $(-1/2, 1/2)$, and the flux to be $\Phi(\theta) = \sin\theta$, $\theta \in [0, 2\pi]$. Because the jump of temperature across γ is positive, the poles should lie *on* γ in this case [18]. The values of u were collected at 1000 points on $\mathbb{T}$, and the Fourier series of the resulting f was truncated from degree -70 to 70. The decrease of the singular values is regular up to the degree 10, while

$s_{11} \simeq 10^{-14}$. Since the L^∞ norm of f is approximatively 1, we see that the ratio of s_{11} with the norm of f corresponds to the double precision used for the computations. In Figure 1, poles and zeros of the AAK approximants of degree 11 and 12 are plotted. In Figure 2, poles of the H^2 approximant



(a) Poles of the approximant of degree 11 (b) Poles of the approximant of degree 12

Figure 1: Behavior of the AAK poles for $\Phi(\theta) = \sin\theta$.

of degree 6 is represented. In this case, it is difficult to compute approximants of higher degree, because at degree 6 the criterion is very small already: $\|f - p/q\|^2_{L^2(\mathbb{T})} \,/\, \|f\|^2_{L^2(\mathbb{T})} = 4.561.10^{-15}$.

2) Crack lying on a diameter with a change of sign in the jump of temperature

We now choose the flux $\Phi(\theta) = \cos\theta + 2\cos 2\theta + 2\sin 2\theta$. Then, the temperature has one change of sign on the crack. Note that the crack γ as well as the terms $\cos\theta$ and $2\cos 2\theta$ in the definition of Φ are invariant under the symmetry with respect to the real axis. Hence, these terms induce limits u^+ and u^- for the temperature on γ that are equal and thus induce no jump of temperature. Consequently, it is equivalent to consider the flux $\Phi_s(\theta) = 2\sin 2\theta$. Since it is symmetric with respect to the origin, we deduce that $u^+(z) = u^-(-z)$, $z \in \gamma$, so that the jump of temperature

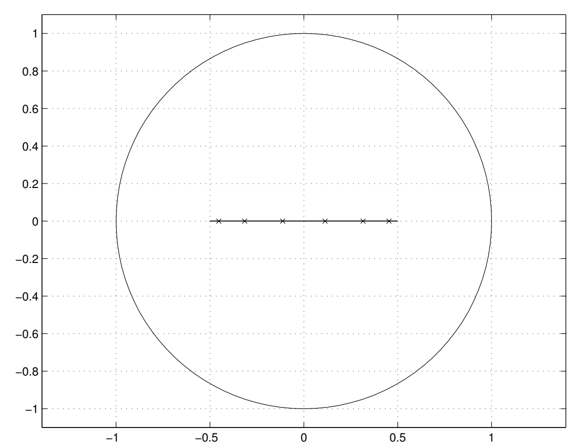
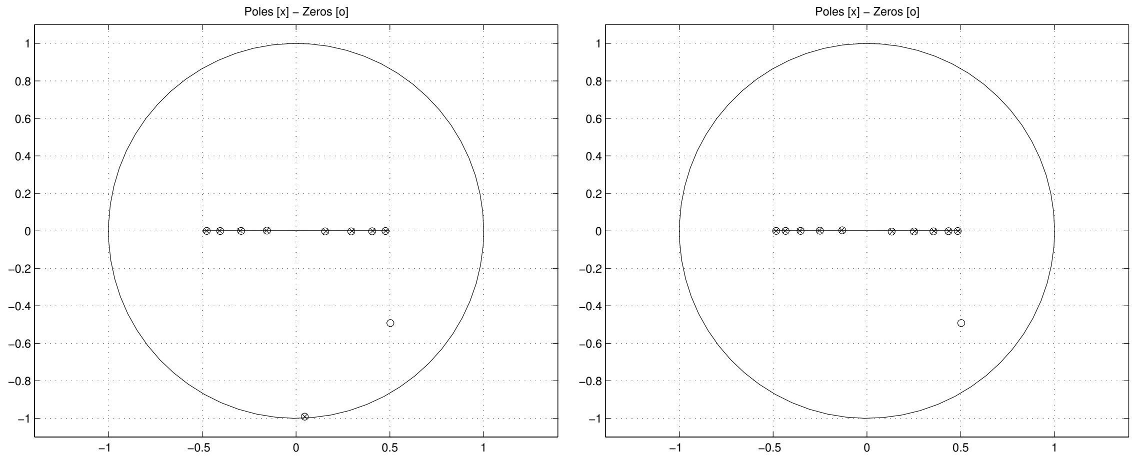


Figure 2: The crack and the poles of the H^2 approximant of degree 6 for $\Phi(\theta) = \sin\theta$



(a) Poles of the approximant of degree 9 (b) Poles of the approximant of degree 10

Figure 3: Behavior of the AAK poles for $\Phi(\theta) = \cos\theta + 2\cos 2\theta + 2\sin 2\theta$

σ is an odd function of z on γ, and the integral in (7.1) is an even function of z in D_γ, say $S(z)$. If $g_n(z)$ is the best AAK approximant to $S(z)$, so is $g_n(-z)$. By uniqueness, g_n is an even function, in particular g_n can only have an even number of poles. Hence, $g_{2p+1} = g_{2p}$, $p \geq 0$, and all singular values have multiplicity 2. Of course, numerically, the singular values do not repeat exactly from odd degree to even degree, but the poles in odd degrees do not bring any new information with respect to poles in even degrees. This can be verified on Figure 3, where one sees that the approximant of degree 9 has a pole near $\mathbb{T}$ that almost coincide with a zero, suggesting that they should cancel each other. In degree 10, such phenomenon does not occur. Finally, note that s_{10} has a magnitude of 10^{-14} while the L^∞ norm of f equals approximatively 3.36.

3) Crack not lying on a diameter

We assume that the flux is given by $\Phi(\theta) = \sin\theta$ and that the crack γ is a line segment joining the two endpoints $\gamma_0 = (-1/5, 1/5)$ and $\gamma_1 = (4/5, 1/5)$. Then, with a truncation of the Fourier series between the degrees -150 and 150, the first thirteen singular values decrease geometrically. The poles (and zeros) of the AAK approximant of degree 12 are shown in Figure 4. As foreseen by the theoretical results, they indeed approach quite well the geodesic joining γ_0 and γ_1. The magnitude of s_{12} is about 10^{-14}. The results for L^2 approximants are quite good as well, see the poles of the approximant of degree 6 in Figure 5. The corresponding criterion $\|f - p/q\|^2_{L^2(\mathbb{T})} / \|f\|^2_{L^2(\mathbb{T})}$ is already very small, equal to $3.51.10^{-13}$.

4) Piecewise rectilinear cracks

Finally we take up two examples of piecewise rectilinear cracks, using the same flux as in Figure 3. The results of [24] mentioned (but not proved) in Section 7.2 predict that the counting measure of the poles should converge to the Green equilibrium distribution of the compact subset of minimal Green capacity outside of which the function in (7.1) is holomorphic and single-valued. In the present case, this set coincides with the continuum of minimum Green capacity connecting the endpoints of γ. Figure 6 illustrates this fact.

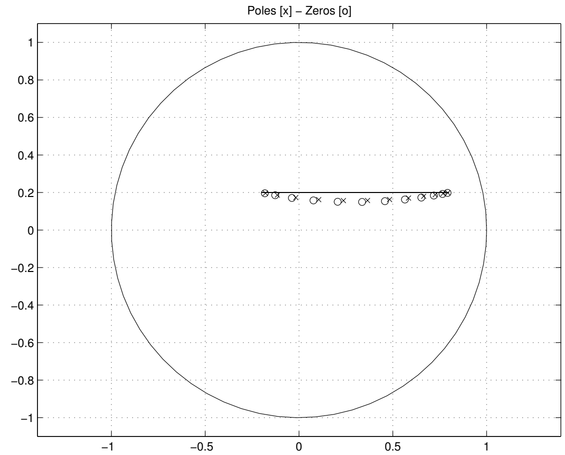


Figure 4: Poles of the AAK approximant of degree 12, with 150 Fourier coefficients

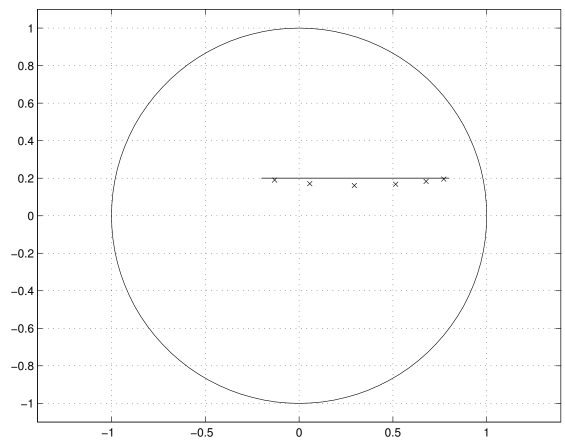
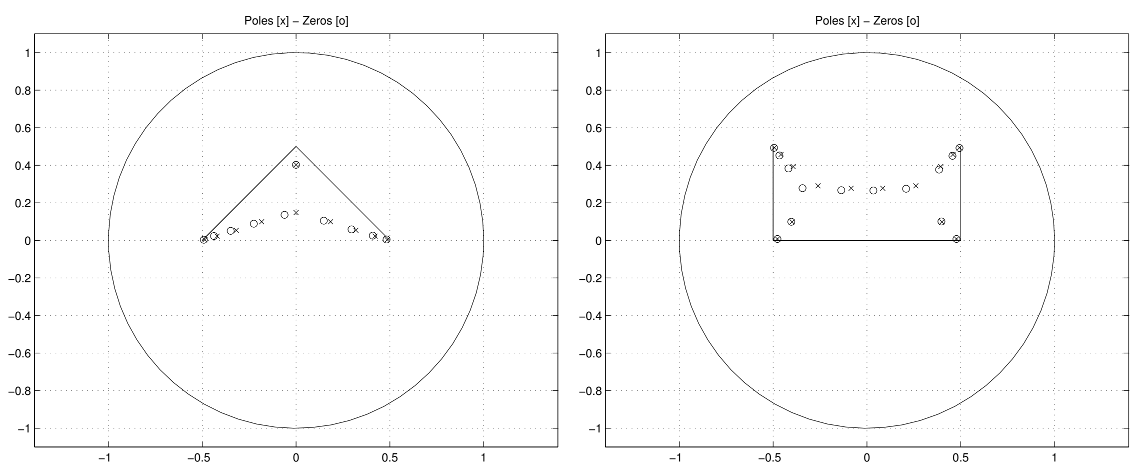


Figure 5: Poles of the L^2 approximant of degree 6



(a) 3 branch points (AAK with 10 poles)

(b) 4 branch points (AAK with 14 poles)

Figure 6: Numerical experiments with 3 and 4 branch points

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