Strong asymptotics for multiple Laguerre polynomials *

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Abstract

We consider multiple Laguerre polynomials $l_{\vec{n}}$ of degree $2n$ orthogonal on $(0, \infty)$ with respect to the weights $x^\alpha e^{-\beta_1 x}$ and $x^\alpha e^{-\beta_2 x}$, where $-1 < \alpha$, $0 < \beta_1 < \beta_2$, and we study their behavior in the large n limit. The analysis differs among three different cases which correspond to the ratio β_2/β_1 being larger, smaller, or equal to some specific critical value κ. In this paper, the first two cases are investigated and strong uniform asymptotics for the scaled polynomials $l_{\vec{n}}(nz)$ are obtained in the entire complex plane by using the Deift–Zhou steepest descent method for a 3×3–matrix Riemann–Hilbert problem.

Asymptotique forte pour les polynômes de Laguerre multiples

Résumé

Nous considérons les polynômes de Laguerre multiples $l_{\vec{n}}$ de degré $2n$ orthogonaux sur $(0, \infty)$ par rapport aux poids $x^\alpha e^{-\beta_1 x}$ et $x^\alpha e^{-\beta_2 x}$, avec $-1 < \alpha$, $0 < \beta_1 < \beta_2$, et nous décrivons leur comportement asymptotique pour n grand. Le problème se décompose en trois cas, suivant que le rapport β_2/β_1 est plus grand, plus petit, ou égal à une certaine valeur critique κ. Dans cet article, les deux premiers cas sont étudiés et l'asymptotique forte des polynômes $l_{\vec{n}}(nz)$ est obtenue localement uniformément dans toute région du plan complexe. On utilise pour cela la méthode de descente de Deift et Zhou pour un problème de Riemann–Hilbert matriciel de dimensions 3×3.

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1 Introduction

Multiple orthogonal polynomials $q_{\vec{n}}(z)$ of type II of vector index $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ are polynomials of one variable of degree less than or equal to $|\vec{n}| = n_1 + \cdots + n_r$, which satisfy the following orthogonality conditions with respect to r weights w_j supported on contours γ_j,

$$\int_{\gamma_j} q_{\vec{n}}(z) z^k w_j(z) dz = 0, \qquad \text{for } k = 0, \ldots, n_j - 1, \quad j = 1, \ldots, r.$$

These polynomials naturally appear in such areas as number theory, special functions, rational approximation, spectral and scattering theory for higher order difference operators, cf. [3, 6, 7, 27], and are closely related to simultaneous Padé approximation with common denominator, also known as Hermite–Padé approximation of type II, see [21, Chapter 4] and [15, 23, 24].

Multiple Laguerre polynomials $l_{\vec{n}}(x)$ correspond to the case of weights

$$w_j(x) = x^\alpha e^{-\beta_j x}, \quad -1 < \alpha, \quad 0 < \beta_1 < \cdots < \beta_r,$$

on the semi–infinite real axis $(0, \infty)$. Hence, they satisfy the relations

$$\int_0^\infty l_{\vec{n}}(x) x^{k+\alpha} e^{-\beta_j x} dx = 0, \qquad \text{for } k = 0, \ldots, n_j - 1, \quad j = 1, \ldots, r. \tag{1.1}$$

These multiple Laguerre polynomials $l_{\vec{n}}(x)$ first appeared in the book by E. M. Nikishin and V. N. Sorokin [21]. Since the system of functions $e^{-\beta_1 x}, \ldots, e^{-\beta_r x}$ is an algebraic Chebyshev system on $(0, \infty)$, each vector index $\vec{n}$ is normal, cf. [21, Theorem 4.3], meaning that the degree of $l_{\vec{n}}$ is exactly equal to $|\vec{n}|$. In particular, this entails that the polynomial $l_{\vec{n}}$ is unique, up to normalization. All its zeros are simple and lie in $(0, \infty)$. Similar to classical ones, multiple Laguerre polynomials can be represented by a Rodrigues formula. For the monic polynomial $l^*_{\vec{n}}$, we have

$$l^*_{\vec{n}}(x) = x^{-\alpha} \prod_{j=1}^{r} \left(1 - \frac{1}{\beta_j} \frac{d}{dx}\right)^{n_j} x^{|\vec{n}|+\alpha}$$

where the product of differential operators can be taken in any order, see [21, p.160].

Algebraic properties of multiple orthogonal polynomials for classical weights w satisfying Pearson's differential equation

$$(\phi(z) w(z))' + \varphi(z) w(z) = 0,$$

with polynomial coefficients ϕ and φ were recently obtained in [6, 28]. In particular, explicit recurrence relations and linear differential equation for the case of multiple Laguerre polynomials $l_{\vec{n}}(x)$ were found, see [6, Corollary 1].

In this paper, we consider the scaled monic polynomials of degree $2n$,

$$L_n(z) = n^{-2n} l^*_{\vec{n}}(nz), \tag{1.2}$$

for the case of 2 weights ($r = 2$) and diagonal indices (n, n), satisfying

$$\int_0^\infty L_n(x) x^{k+\alpha} e^{-n\beta_1 x} dx = 0, \qquad \int_0^\infty L_n(x) x^{k+\alpha} e^{-n\beta_2 x} dx = 0,$$

for $k = 0, \ldots, n-1$. Throughout we assume that $0 < \beta_1 < \beta_2$.

The aim of this work is to obtain strong asymptotic estimates, as n becomes large, for the polynomials L_n in every regions of the complex plane. For this, we use the Deift–Zhou steepest descent method for Riemann–Hilbert boundary value problems, see [10, 12, 13, 17]. Among others, this method is currently applied in the theory of integrable systems [16], and for universality questions in random matrix theory [11]. Its use in the study of multiple orthogonal polynomials first appeared in [29].

As in [23, 19, 8] an appropriate Riemann surface plays an important role in the analysis. It is found in the following way. From [6, Corollary 1], we get, after some calculations, the differential equation satisfied by the polynomials $L_n(z)$, namely,

$$\begin{aligned} z^2L_n^{'''}(z) - z[nz(\beta_1+\beta_2) - 2(\alpha+1)]L_n^{''}(z) \\ + [\beta_1\beta_2n^2z^2 + (\beta_1+\beta_2)(\alpha+1-n)nz + \alpha(\alpha+1)]L_n^{'}(z) \\ - n^2[2\beta_1\beta_2nz + \alpha(\beta_1+\beta_2)]L_n(z) = 0. \end{aligned} \tag{1.3}$$

On the other hand, we expect the asymptotics for $L_n(z)$ outside of its zeros to be of the form

$$L_n(z) = F(z)\Phi^n(z)(1+\mathcal{O}(1/n)), \quad \text{as } n\to\infty.$$

Plugging the previous estimate in (1.3) and taking care only of the largest terms with respect to n, that is those terms of order n^3, we end up with the following algebraic equation for $\psi(z) = \Phi'/\Phi(z)$,

$$z\psi^3(z) - (\beta_1+\beta_2)z\psi^2(z) + (\beta_1\beta_2z + \beta_1+\beta_2)\psi(z) - 2\beta_1\beta_2 = 0.$$

Hence, we consider the Riemann surface $\mathcal{R}$ associated with the algebraic equation of degree 3,

$$zw^3 - (\beta_1+\beta_2)zw^2 + (\beta_1\beta_2z + \beta_1+\beta_2)w - 2\beta_1\beta_2 = 0, \tag{1.4}$$

which can equivalently be rewritten as

$$z = \frac{2\beta_1\beta_2 - (\beta_1+\beta_2)w}{w(w-\beta_1)(w-\beta_2)} \tag{1.5}$$

or

$$z = \frac{2}{w} - \frac{1}{w-\beta_1} - \frac{1}{w-\beta_2}.$$

The Riemann surface has three sheets $\mathcal{R}_0$, $\mathcal{R}_1$, and $\mathcal{R}_2$, that are identified with copies of the complex plane. We denote the restrictions of ψ to $\mathcal{R}_0$, $\mathcal{R}_1$ and $\mathcal{R}_2$ by ψ_0, ψ_1 and ψ_2 respectively. So $\psi_0(z)$, $\psi_1(z)$ and $\psi_2(z)$ are the three solutions of (1.4), and, in particular, we have

$$\psi_0(z) + \psi_1(z) + \psi_2(z) = \beta_1+\beta_2. \tag{1.6}$$

In view of (1.5), we can compute the asymptotic behavior of these three maps near infinity. We choose the three sheets of $\mathcal{R}$ in such a way that, as $z\to\infty$,

$$\psi_0(z) = \frac{2}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \tag{1.7}$$

$$\psi_1(z) = \beta_1 - \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \tag{1.8}$$

$$\psi_2(z) = \beta_2 - \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right). \tag{1.9}$$

The Riemann surface has four branch points $z_i = z(w_i)$, $i = 1, \ldots, 4$, which are the roots of the discriminant of equation (1.4). They correspond by the map ψ to the points w_i, $i = 1, \ldots, 4$, for which $z'(w_i) = 0$. According to the value of the ratio β_2/β_1 with respect to the critical value κ,

$$\kappa = \frac{7 + 3\sqrt{3}}{2} + \sqrt{\frac{36 + 21\sqrt{3}}{2}} = 12.1136... \tag{1.10}$$

the following cases arise:

- Case 1: for $\kappa < \beta_2/\beta_1$. The Riemann surface $\mathcal{R}$ has four simple real branch points $0 < b < c < d$.
- Case 2: $1 < \beta_2/\beta_1 < \kappa$. The Riemann surface $\mathcal{R}$ has four simple branch points, two of them are real, 0 and $d > 0$. Two of them are complex conjugates. We still denote them by b and c with $c = \overline{b}$ and we assume that $\operatorname{Im} c > 0$.
- Case 3: $\beta_2/\beta_1 = \kappa$. This is the critical case where the Riemann surface $\mathcal{R}$ has three real branch points, one of them being degenerate. The two branch points, 0 and $d = 5.9151\ldots$, are simple. The third one, $c = 0.29006\ldots$, is of order 2.

The analysis differs among these three cases. In this paper, we will consider Case 1 and Case 2. For the study of Case 2, we use as a special ingredient the so–called global opening of lenses that was first devised in [5].

Case 3, the critical case, will not be considered here. In this connection, we note that the paper [9] studies the critical case for multiple Hermite polynomials. It uses, among others, special functions known as Pearcey integrals which are third order analogs of Airy functions.

The following references are also connected to our work. In [8, 5], an analysis of multiple Hermite polynomials is performed and applied to establish the universal limiting behavior of local eigenvalue correlations for Gaussian random matrices with external source. In [2], multiple orthogonal polynomials with respect to Nikishin systems are studied and strong asymptotics are established. In [4], methods closely related to ours are used to obtain deep results on the asymptotics of orthogonal polynomials with varying complex weights. Finally, in the recent paper [30], the Deift–Zhou method is applied to obtain Plancherel–Rotach asymptotics of Laguerre–type orthogonal polynomials with applications to random matrix theory.

2 Statement of results

We first need to describe the Riemann surface in some more details.

2.1 The Riemann surface

We start with Case 1. The Riemann surface $\mathcal{R}$ has two cuts, that we choose as follows. One of them connects the two branch points 0 and b. We take for this cut the interval $[0, b]$. The other cut is the interval $[c, d]$ which connects the two other branch points c and d. The sheets $\mathcal{R}_0$ and $\mathcal{R}_1$ are glued together along the cut $[c, d]$. The sheets $\mathcal{R}_0$ and $\mathcal{R}_2$ are glued together along the segment $[0, b]$. We set

$$\Delta_2 = [0, b], \quad \Delta_1 = [c, d], \quad \Delta = \Delta_2 \cup \Delta_1.$$

We assume that Δ_1 and Δ_2 are oriented in the positive direction. The sheet structure is shown in Figure 1. The functions ψ_0, ψ_1, ψ_2 are defined on the sheets $\mathcal{R}_0$, $\mathcal{R}_1$, and $\mathcal{R}_2$

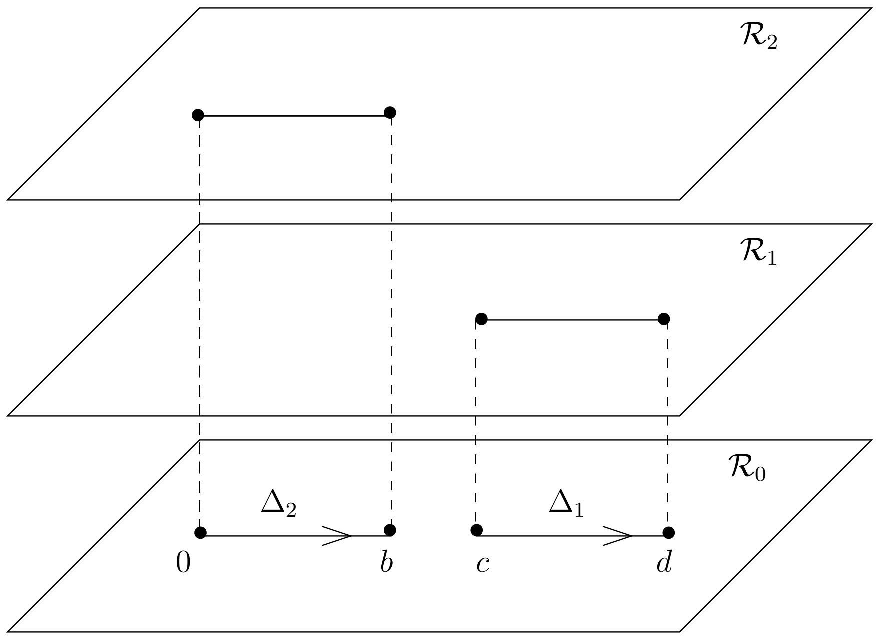

Figure 1: The Riemann surface $\mathcal{R}$ for Case 1: $\kappa < \beta_2/\beta_1$

respectively, and we have the jump relations:

$$\begin{aligned} \psi_{0\pm}(z) &= \psi_{1\mp}(z), \quad z \in \Delta_1, \\ \psi_{0\pm}(z) &= \psi_{2\mp}(z), \quad z \in \Delta_2. \end{aligned} \tag{2.1}$$

Together the three functions ψ_j, $j = 1, 2, 3$, constitute a conformal map from $\mathcal{R}$ onto the Riemann sphere. Near the origin, one may check that,

$$\psi_0(z) = -\frac{i\sqrt{\beta_1+\beta_2}}{\sqrt{z}} + \frac{1}{2}\frac{\beta_1^2+\beta_2^2}{\beta_1+\beta_2} + \mathcal{O}\left(\sqrt{z}\right), \tag{2.2}$$

$$\psi_1(z) = \frac{2\beta_1\beta_2}{\beta_1+\beta_2} + \mathcal{O}\left(z\right), \tag{2.3}$$

$$\psi_2(z) = \frac{i\sqrt{\beta_1+\beta_2}}{\sqrt{z}} + \frac{1}{2}\frac{\beta_1^2+\beta_2^2}{\beta_1+\beta_2} + \mathcal{O}\left(\sqrt{z}\right), \tag{2.4}$$

as $z \to 0$. The square root in (2.2) and (2.4) is defined with a branch cut on the positive real semi-axis $\mathbb{R}_+$ and it sends the negative real semi-axis $\mathbb{R}_-$ on $i\mathbb{R}_+$. The branch point 0 is mapped by ψ to the point at infinity in the w-plane. We respectively denote by w_b, w_c, and w_d the images of the branch points b, c, and d. The images of the different sheets of $\mathcal{R}$ in the w-plane are shown in Figures 2 and for the particular values $\beta_1 = 1$ and $\beta_2 = 40$ of the parameters.

As the ratio β_2/β_1 decreases, the branch points b and c in the z-plane come closer one to each others on the real line. When this ratio equals the value κ given in (1.10), the two branch points coalesce. This is Case 3.

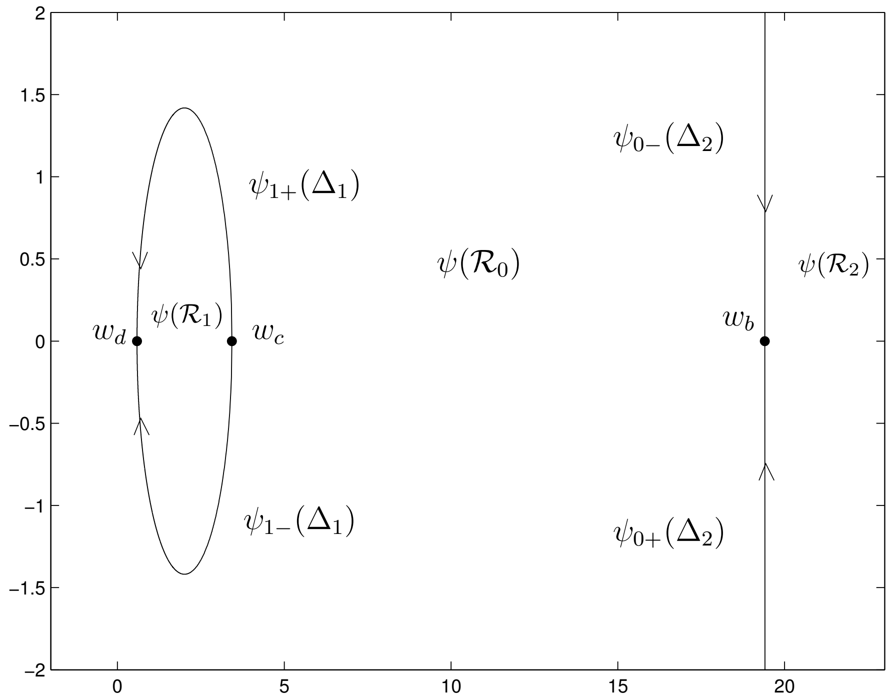

Figure 2: ψ-image of the Riemann surface $\mathcal{R}$ (Case 1: $\beta_1 = 1$ and $\beta_2 = 40$)

Then, for smaller values of β_2/β_1, which correspond to Case 2, one recovers again two non–degenerate branch points, which are this time conjugate in the complex plane. We denote them by b and $c = \overline{b}$ with $\operatorname{Im} c > 0$. Now, the Riemann surface has a different sheet structure, which is as follows. The sheets $\mathcal{R}_1$ and $\mathcal{R}_2$ are glued together along an oriented smooth curve Γ which goes from b to c. We assume that this cut intersects the real axis at some point $a \in (0, d)$. Then, the sheets $\mathcal{R}_0$ and $\mathcal{R}_2$ are glued together along the segment $(0, a)$ while the sheets $\mathcal{R}_0$ and $\mathcal{R}_1$ are glued together along the segment (a, d). We set

$$\Delta_2 = [0, a], \quad \Delta_1 = [a, d], \quad \Delta = \Delta_2 \cup \Delta_1.$$

The following jump relations hold true,

$$\begin{aligned} \psi_{0\pm}(z) &= \psi_{1\mp}(z), \quad z \in \Delta_1, \\ \psi_{0\pm}(z) &= \psi_{2\mp}(z), \quad z \in \Delta_2, \\ \psi_{1\pm}(z) &= \psi_{2\mp}(z), \quad z \in \Gamma. \end{aligned} \tag{2.5}$$

The sheet structure is shown in Figure 3. As in Case 1, the branch point 0 is mapped by ψ to the point at infinity in the w-plane. We still denote by w_b, w_c, and w_d the images of the branch points b, c, and d. The expansions (1.7)–(1.9) at infinity and (2.2)–(2.4) at the origin of the ψ–functions remain unchanged in Case 2.

Figure 4 depicts the ψ-image of the Riemann surface for the values $\beta_1 = 1$ and $\beta_2 = 8$ of the parameters. Note that for the moment, there is much freedom in the choice of the cut Γ. Later on, we shall put slight restrictions on this choice, see Section 6.1.

Independently from the case under consideration, we note that when z is real, the equation (1.4) has real coefficients so that among $\psi_0(z)$, $\psi_1(z)$ and $\psi_2(z)$ there can only be a real root

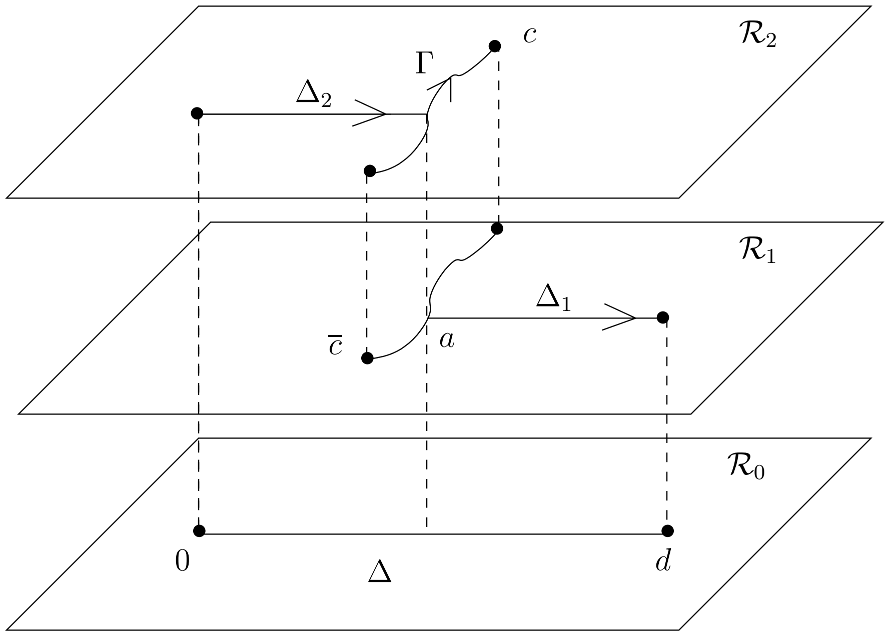

Figure 3: The Riemann surface $\mathcal{R}$ for Case 2: $1 < \beta_2/\beta_1 < \kappa$

and a pair of complex conjugate roots or three real roots. On the cuts Δ_2 and Δ_1, the following actually holds,

$$\psi_{j\pm}(x) = \overline{\psi_{0\pm}(x)}, \quad x \in \Delta_j, \quad j = 1, 2. \tag{2.6}$$

For later use, note also that in view of (1.7)–(1.9) and from the residue theorem as applied to the exterior of positively oriented closed contours around the cuts $\Delta_2 \cup \Delta_1$, Δ_1 and Δ_2 in Case 1 and around the cuts Δ, $\Delta_1 \cup \Gamma$ and $\Delta_2 \cup \Gamma$ in Case 2, we have

$$\oint \psi_0(s)ds = 4i\pi, \quad \oint \psi_1(s)ds = -2i\pi, \quad \oint \psi_2(s)ds = -2i\pi. \tag{2.7}$$

2.2 Measures and λ–functions

Let us define two measures μ_1 and μ_2 on Δ_1 and Δ_2 respectively by

$$d\mu_1(s) = \frac{1}{2\pi i}(\psi_1 - \psi_0)_+(s)ds, \qquad s \in \Delta_1, \tag{2.8}$$

$$d\mu_2(s) = \frac{1}{2\pi i}(\psi_2 - \psi_0)_+(s)ds, \qquad s \in \Delta_2, \tag{2.9}$$

Remark 2.1. *Since w_d is a non degenerate critical point of $z(w)$, we have that, as z tends to d,*

$$\begin{aligned} \psi_1(z) &= w_d + \alpha_d(z-d)^{1/2} + \mathcal{O}(z-d), \\ \psi_0(z) &= w_d - \alpha_d(z-d)^{1/2} + \mathcal{O}(z-d), \end{aligned} \tag{2.10}$$

where α_d is some non-zero constant and similarly near b and c in Case 1. Near the origin, we also have in view of (2.2) and (2.4) that

$$(\psi_2 - \psi_0)(z) = \alpha_0 z^{-1/2} + \mathcal{O}(\sqrt{z}), \tag{2.11}$$

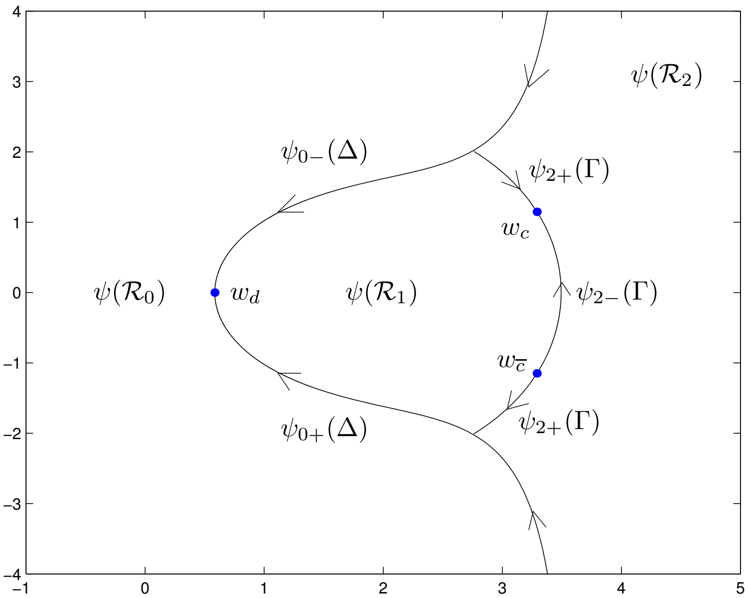

Figure 4: ψ-image of the Riemann surface $\mathcal{R}$ (Case 2: $\beta_1 = 1$, $\beta_2 = 8$). For this drawing, the vertical segment from $b = \overline{c}$ to c has been chosen as the cut Γ on $\mathcal{R}$

where $\alpha_0 = 2i\sqrt{\beta_1 + \beta_2}$. Hence, the measure μ_1 has a density with respect to arclength of Δ_1 which vanishes like a square root at the point d (and in Case 1, at the point c and μ_2 at the point b as well). On the contrary, the measure μ_2 has a density which grows to infinity like the inverse of a square root at the origin. These behaviors are characteristic from the so–called soft and hard edges of Δ_1 and Δ_2.

Proposition 2.2. *In Case 1, the measures μ_1 and μ_2 are probability measures on Δ_1 and Δ_2 respectively. In Case 2, the measure $\mu_1 + \mu_2$ is a positive measure on Δ of mass 2.*

For every polynomial p of exact degree n, we denote by ν_p the normalized zero counting measure,

$$\nu_p = \frac{1}{n} \sum_{p(z)=0} \delta_z,$$

where each zero is counted according to its multiplicity.

The asymptotic distribution of the zeros of the scaled Laguerre polynomials L_n is given by the following theorem.

Theorem 2.3. *We have*

$$\nu_{L_n} \xrightarrow{*} \frac{1}{2}(\mu_1 + \mu_2),$$

as $n \to \infty$, where the convergence is in the sense of weak* convergence of measures.

Finally, we introduce the analytic functions

$$\lambda_0(z) = \int_d^z \psi_0(s) ds, \tag{2.12}$$

$$\lambda_1(z) = \int_d^z \psi_1(s)ds, \tag{2.13}$$

$$\lambda_2(z) = \int_{0_+}^z \psi_2(s)ds + \lambda_{0_-}(0). \tag{2.14}$$

In Case 1, the functions λ_0 and λ_1 are taken with a cut on $(-\infty, d]$ and the function λ_2 with a cut on $(-\infty, b]$ while in Case 2, the function λ_0 is taken with a cut on $(-\infty, d]$, the function λ_1 with a cut on $(-\infty, d] \cup \Gamma$ and the function λ_2 with a cut on $(-\infty, a] \cup \Gamma$. The integration in the definition of λ_2 starts from 0, on the upper side of its cut.

Note that from (1.7)–(1.9), we get the following expansions at infinity:

$$\begin{aligned} \lambda_0(z) = & \quad 2\log z + \ell_0 + \mathcal{O}(1/z), \\ \lambda_1(z) = & \quad \beta_1 z - \log z + \ell_1 + \mathcal{O}(1/z), \\ \lambda_2(z) = & \quad \beta_2 z - \log z + \ell_2 + \mathcal{O}(1/z), \end{aligned} \tag{2.15}$$

where ℓ_j, $j = 1, 2, 3$, are some constants. Also, we have

$$\lambda_{j\pm}(x) = \overline{\lambda_{0\pm}(x)}, \quad x \in \Delta_j, \quad j = 1, 2. \tag{2.16}$$

Indeed, when $j = 1$, (2.16) follows from (2.6) and the definitions of λ_0 and λ_1. When $j = 2$, this follows from (2.6) and the definitions of λ_0 and λ_2, observing that

$$\lambda_0(z) = \int_{0_+}^z \psi_0(s)ds + \lambda_{0_+}(0).$$

Since

$$(\lambda_2 - \lambda_0)(z) = \int_{0+}^z (\psi_2 - \psi_0)(s)ds - \int_\gamma \psi_0(s)ds,$$

where γ is a positively oriented closed contour around Δ, we get, in view of the first equality in (2.7), that

$$(\lambda_2 - \lambda_0)(z) = \int_{0+}^z (\psi_2 - \psi_0)(s)ds - 4i\pi. \tag{2.17}$$

2.3 Strong asymptotics away from the zeros

Unless otherwise specified, we define z^α in the complex plane with a branch cut along the negative real axis. Thus $z^\alpha = |z|^\alpha e^{i\alpha \arg z}$ with $\arg z \in (-\pi, \pi)$. We will also need the polynomial

$$D(w) = 2(\beta_1 + \beta_2)w^3 - (\beta_1^2 + 8\beta_1\beta_2 + \beta_2^2)w^2 + 4(\beta_1 + \beta_2)\beta_1\beta_2 w - 2\beta_1^2\beta_2^2, \tag{2.18}$$

which is obtained by differentiating the right-hand side of (1.5) with respect to w and by taking the numerator. The roots of D are the three points w_b, w_c, and w_d.

The square root of $D(w)$, which branches at these three points, is defined with a cut on $\psi_{0+}(\Delta_2) \cup \psi_{0+}(\Delta_1)$ in Case 1, and with a cut on $\psi_{0+}(\Delta) \cup \psi_{1+}(\Gamma)$ in Case 2. In both cases, we assume the square root is positive for large positive w.

We are now ready to state the strong asymptotics results for the scaled Laguerre polynomials L_n, away from their zeros.

Theorem 2.4. *Uniformly for z in compact subsets of $\mathbb{C} \setminus \Delta$, we have as $n \to \infty$,*

$$L_n(z) = -2^{\alpha+\frac{1}{2}} i \frac{(\psi_0(z) - \beta_1)(\psi_0(z) - \beta_2)}{(z\psi_0(z))^\alpha \sqrt{D(\psi_0(z))}} e^{n(\lambda_0(z)-\ell_0)} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right). \tag{2.19}$$

Remark 2.5. *Note that, in Case 2, the previous expansion is also valid at the two complex conjugate branch points b and c.*

2.4 Strong asymptotics near the segments Δ_2 and Δ_1

We now give the asymptotics of L_n near the curves Δ_2 and Δ_1. As we know from Theorem 2.3 these are the segments where the zeros of L_n accumulate.

Theorem 2.6. *Uniformly for z on the $\pm$-side of Δ_j, $j = 1, 2$, away from the endpoints, we have as $n \to \infty$,*

$$\begin{aligned} 2^{-\alpha-\frac{1}{2}} i L_n(z) = &\left[\frac{(\psi_0(z) - \beta_1)(\psi_0(z) - \beta_2)}{(z\psi_0(z))^\alpha \sqrt{D(\psi_0(z))}} + \mathcal{O}\left(\frac{1}{n}\right)\right] e^{n(\lambda_0(z)-\ell_0)} \\ &\pm \left[\frac{(\psi_j(z) - \beta_1)(\psi_j(z) - \beta_2)}{(z\psi_j(z))^\alpha \sqrt{D(\psi_j(z))}} + \mathcal{O}\left(\frac{1}{n}\right)\right] e^{n(\lambda_j(z)-\ell_0)}, \quad j = 1, 2. \end{aligned} \tag{2.20}$$

Remark 2.7. *Note that, in Case 2, a given z can as well be on the left or on the right of the cut Γ, depending on the choice we make for Γ. However, in view of the jump relations $\psi_{1\pm} = \psi_{2\mp}$ across Γ, the two corresponding expressions given by (2.20) for $j = 1, 2$ coincide. Still in Case 2, note that the intersection a of Δ with Γ is not a special point, since by deformation of Γ, we may always suppose that z in (2.20) is away from a neighborhood of a.*

Remark 2.8. *As in the expansion for multiple Hermite polynomials given in* [8, Section 10 (2)], *we may rewrite (2.20) when $z = x \in \Delta_j$, $j = 1, 2$, away from the endpoints, as*

$$L_n(x) = 2^{\alpha+\frac{1}{2}} x^{-\alpha} \left\{A(x) \sin\left[n \operatorname{Im} \lambda_{0+}(x) + \varphi(x)\right] + \mathcal{O}\left(\frac{1}{n}\right)\right\} e^{n(\operatorname{Re} \lambda_{0+}(x)-\ell_0)}, \tag{2.21}$$

where

$$A(x) = 2\left|\frac{(\psi_{0+}(x) - \beta_1)(\psi_{0+}(x) - \beta_2)}{\psi_{0+}^\alpha(x)\sqrt{D(\psi_{0+}(x))}}\right|,$$

and

$$\varphi(x) = \arg \frac{(\psi_{0+}(x) - \beta_1)(\psi_{0+}(x) - \beta_2)}{\psi_{0+}^\alpha(x)\sqrt{D(\psi_{0+}(x))}}.$$

Equation (2.21) is a simple consequence of (2.20) using the relations (2.6), (2.16), and

$$\sqrt{D(\psi_{j\pm}(x))} = \mp\overline{\sqrt{D(\psi_{0\pm}(x))}}, \quad x \in \Delta_j, \quad j = 1, 2.$$

Note that (2.21) is independent from the index j.

Moreover, in view of the definitions (2.8)–(2.9) of the measures μ_1 and μ_2, along with (2.6), (2.17), and the equality

$$(\lambda_1 - \lambda_0)(z) = \int_d^z (\psi_1 - \psi_0)(s)ds,$$

the expression (2.21) also rewrites on Δ_j, away from the branch points, as

$$L_n(x) = -2^{\alpha+\frac{1}{2}} x^{-\alpha} \left\{ A(x) \sin \left[n\pi \int_{x_j}^{x} d\mu_j - \varphi(x) \right] + \mathcal{O}\left(\frac{1}{n}\right) \right\} e^{n(\operatorname{Re}\lambda_{0+}(x)-\ell_0)}, \quad j = 1, 2,$$

where $x_j = d$ for $j = 1$ and $x_j = 0$ for $j = 2$. This last formula shows the oscillating behavior of L_n on Δ_j, $j = 1, 2$.

2.5 Strong asymptotics near the real branch points

We begin with the asymptotics of L_n near the branch point d. It involves Airy functions. Recall that the Airy function $\operatorname{Ai}(z)$ is the unique solution of $y''(z) = zy(z)$ with asymptotics as $z \to \infty$ given by

$$\begin{aligned} \operatorname{Ai}(z) &= \frac{1}{2\sqrt{\pi}} z^{-1/4} e^{-\frac{2}{3}z^{3/2}} \left(1 + \mathcal{O}\left(\frac{1}{z^{3/2}}\right)\right), \\ \operatorname{Ai}'(z) &= \frac{-1}{2\sqrt{\pi}} z^{1/4} e^{-\frac{2}{3}z^{3/2}} \left(1 + \mathcal{O}\left(\frac{1}{z^{3/2}}\right)\right), \end{aligned}$$

which holds for $|\arg z| < \pi$, where $z^{1/4}$ and $z^{3/2}$ are defined with principal branch (i.e., with a cut along the negative real axis).

From (2.10) and the definitions of λ_0 and λ_1, we get that

$$(\lambda_1 - \lambda_0)(z) = (z-d)^{3/2} h_d(z), \quad z \in D_d \setminus \Delta_1,$$

where D_d is a small neighborhood of d and h_d is analytic and without zeros in D_d. The function $(z-d)^{3/2}$ has a branch cut along Δ_1.

The function $f_d(z)$ is defined such that

$$(\lambda_1 - \lambda_0)(z) = \frac{4}{3} \left[f_d(z) \right]^{3/2}. \tag{2.22}$$

It is a biholomorphic map in D_d. The relation (2.22) determines f_d only up to one of the multiplicative constants 1, $e^{2i\pi/3}$, $e^{4i\pi/3}$. Note also that, since $\lambda_1 - \lambda_0$ has a jump on Δ_1, the 3/2-power in the previous equality has a branch on $f_d(\Delta_1)$. We choose the multiplicative constant so that f_d is real-valued on the real axis near d, and maps Δ_1 onto a part of the negative real axis.

Then we have the following result.

Theorem 2.9. *Uniformly for z in a small neighborhood D_d of d, we have as $n \to \infty$,*

$$\begin{aligned} L_n(z) = -2^{\alpha+\frac{1}{2}} \sqrt{\pi} \Big[& n^{1/6} B_d(z) \operatorname{Ai}(n^{2/3} f_d(z))(1 + \mathcal{O}(n^{-1})) \\ & + n^{-1/6} C_d(z) \operatorname{Ai}'(n^{2/3} f_d(z))(1 + \mathcal{O}(n^{-1})) \Big] e^{\frac{n}{2}((\lambda_0+\lambda_1)(z) - 2\ell_0)}, \end{aligned} \tag{2.23}$$

where B_d and C_d are two analytic functions in D_d, which have explicit expressions

$$B_d(z) = f_d(z)^{1/4} \left(\frac{(\psi_1(z) - \beta_1)(\psi_1(z) - \beta_2)}{z^\alpha \psi_1(z)^\alpha \sqrt{D(\psi_1(z))}} + i \frac{(\psi_0(z) - \beta_1)(\psi_0(z) - \beta_2)}{(z\psi_0(z))^\alpha \sqrt{D(\psi_0(z))}} \right),$$

and

$$C_d(z) = f_d(z)^{-1/4}\left(\frac{(\psi_1(z)-\beta_1)(\psi_1(z)-\beta_2)}{z^\alpha\psi_1(z)^\alpha\sqrt{D(\psi_1(z))}} - i\frac{(\psi_0(z)-\beta_1)(\psi_0(z)-\beta_2)}{(z\psi_0(z))^\alpha\sqrt{D(\psi_0(z))}}\right).$$

The branch of the fourth root of $f_d^{1/4}$ in B_d and C_d is taken with a cut along Δ_1.

In Case 1, similar results can be given for the behavior of the polynomials L_n near the branch points b and c.

Finally, we give the behavior of L_n near the branch point 0. This time it involves Bessel functions for which a standart reference is [1, Chapter 9]. In the sequel, J_α denotes the usual Bessel function of order α with a branch cut on the negative real axis.

We need to introduce the functions $\widetilde{\lambda}_0$ and $\widetilde{\lambda}_2$ which are defined as λ_0 and λ_2 in (2.12) and (2.14), except for the choice of the branches. More precisely,

$$\widetilde{\lambda}_0(z) = \int_{d_+}^z \psi_0(s)ds, \quad \widetilde{\lambda}_2(z) = \int_0^z \psi_2(s)ds + \widetilde{\lambda}_0(0), \tag{2.24}$$

where we choose for $\widetilde{\lambda}_0$ the branch on $[0,\infty)$ and for $\widetilde{\lambda}_2$ the branch on $[0,\infty)$ in Case 1 and the branch on $[0,\infty)\cup\Gamma$ in Case 2. Hence, locally, $\widetilde{\lambda}_2-\widetilde{\lambda}_0$ is analytic in a punctured disk around 0 with a branch on Δ_2. Using (2.7), one easily checks that

$$\begin{aligned}\widetilde{\lambda}_0(z) &= \lambda_0(z), & \widetilde{\lambda}_2(z) &= \lambda_2(z)+4i\pi, \quad \text{for } \mathrm{Im}\,(z)>0,\\ \widetilde{\lambda}_0(z) &= \lambda_0(z)+4i\pi, & \widetilde{\lambda}_2(z) &= \lambda_2(z)+2i\pi, \quad \text{for } \mathrm{Im}\,(z)<0.\end{aligned} \tag{2.25}$$

From (2.11) and (2.24), we get that

$$(\widetilde{\lambda}_2-\widetilde{\lambda}_0)(z) = z^{1/2}h_0(z), \quad z\in D_0\setminus\Delta_2,$$

where D_0 is a small neighborhood of 0 and h_0 is analytic and without zeros in D_0. The function $z^{1/2}$ has a branch cut along Δ_2. Then the function $f_0(z)$ is defined by the relation

$$(\widetilde{\lambda}_2-\widetilde{\lambda}_0)(z) = -4\left[f_0(z)\right]^{1/2}. \tag{2.26}$$

It is a biholomorphic map in D_0 which maps Δ_2 on the negative real axis. Since $\widetilde{\lambda}_2-\widetilde{\lambda}_0$ has a jump on Δ_2, the 1/2–power in the previous equality has a branch on $f_0(\Delta_2)$, that is on the negative real axis.

We can now assert the theorem.

Theorem 2.10. *Uniformly for z in a small neighborhood D_0 of 0, we have as $n\to\infty$,*

$$L_n(z) = -2^{\alpha+\frac{1}{2}}i\sqrt{\pi n}f_0(z)^{1/4}e^{\mp i\alpha\pi/2}\Big[B_0(z)J_\alpha(2n(-f_0(z))^{1/2})(1+\mathcal{O}(n^{-1}))$$

$$\pm C_0(z)J'_\alpha(2n(-f_0(z))^{1/2})(1+\mathcal{O}(n^{-1}))\Big]\,e^{\frac{n}{2}((\widetilde{\lambda}_0+\widetilde{\lambda}_2)(z)-2\ell_0)}, \tag{2.27}$$

where

$$B_0(z) = \frac{(\psi_0(z)-\beta_1)(\psi_0(z)-\beta_2)}{(z\psi_0(z))^\alpha\sqrt{D(\psi_0(z))}} + i\frac{(\psi_2(z)-\beta_1)(\psi_2(z)-\beta_2)}{(-z)^\alpha\psi_2(z)^\alpha\sqrt{D(\psi_2(z))}},$$

and

$$C_0(z) = i\frac{(\psi_0(z)-\beta_1)(\psi_0(z)-\beta_2)}{(z\psi_0(z))^\alpha\sqrt{D(\psi_0(z))}} + \frac{(\psi_2(z)-\beta_1)(\psi_2(z)-\beta_2)}{(-z)^\alpha\psi_2(z)^\alpha\sqrt{D(\psi_2(z))}}.$$

In (2.27), the factor $\mathrm{e}^{-i\alpha\pi/2}$ *and the* $+$ *sign holds for* z *in the upper half–plane while the factor* $\mathrm{e}^{i\alpha\pi/2}$ *and the* $-$ *sign holds for* z *in the lower half–plane. The functions* $f_0^{1/4}$ *and* $(-f_0)^{1/2}$ *are respectively taken with branch cuts along the real positive and real negative semi–axis.*

Remark 2.11. *In Case 2, when the ratio* β_2/β_1 *tends to 1, one recovers for* L_n *the usual (scaled) Laguerre polynomials. Though the passage to the limit is not justified, one can check that the asymptotics in (2.19), (2.21), (2.23), and (2.27), then respectively agree (possibly with slightly weaker error terms) with the classical Perron, Fejer, Plancherel–Rotach and Hilb formulas, see Theorems 8.22.3, 8.22.1, 8.22.8 (c), and 8.22.4 in* [26]. *Note that the definition of the Airy function, denoted by* $A(t)$*, in this last reference is different from ours, namely the relation between* $A(t)$ *and our Airy function* $\mathrm{Ai}(t)$ *is* $A(t) = 3^{-1/3}\pi\,\mathrm{Ai}(-3^{-1/3}t)$.

3 Properties of the measures and functions associated with the Riemann surface

In this section, we will prove Proposition 2.2 on the measures μ_1 and μ_2, then give some properties of the λ–functions and finally display an expression of λ_0 in terms of the complex logarithmic potentials associated to μ_1 and μ_2.

Lemma 3.1. *In Case 1, we have*

$$\frac{1}{2\pi i}\int_{\Delta_1}(\psi_1-\psi_0)_+(s)\,ds = 1, \qquad \frac{1}{2\pi i}\int_{\Delta_2}(\psi_2-\psi_0)_+(s)\,ds = 1, \tag{3.1}$$

and in Case 2, we have

$$\frac{1}{2\pi i}\int_{\Delta_1}(\psi_1-\psi_0)_+(s)\,ds + \frac{1}{2\pi i}\int_{\Delta_2}(\psi_2-\psi_0)_+(s)\,ds = 2. \tag{3.2}$$

Proof. We start wit Case 1. Let γ be a closed contour on the sheet $\mathcal{R}_2$ going around Δ_2 once in the positive direction. Then the residue theorem for the exterior of γ gives

$$\frac{1}{2\pi i}\int_{\gamma}\psi_2(s)\,ds = -1,$$

because ψ_2 is analytic outside γ and we have (1.9). If we shrink γ to Δ_2, then the integral becomes

$$\frac{1}{2\pi i}\int_{\Delta_2}(\psi_{2-}-\psi_{2+})(s)\,ds = -1.$$

Taking into account that $(\psi_2)_- = (\psi_0)_+$, we obtain (3.1). The reasoning is similar for the proof of the second equality in (3.1), where we use a closed contour going around Δ_1 on $\mathcal{R}_1$ and the behavior (1.8) of ψ_1 at infinity.

In Case 2, we consider a closed contour γ_1 around Δ_1 and Γ on $\mathcal{R}_1$ and a closed contour γ_2 around Δ_2 and Γ on $\mathcal{R}_2$, both in the positive direction. Then, we have

$$\frac{1}{2\pi i}\int_{\gamma_1}\psi_1(s)\,ds = -1, \qquad \frac{1}{2\pi i}\int_{\gamma_2}\psi_2(s)\,ds = -1.$$

Shrinking γ_1 to $\Delta_1 \cup \Gamma$ and γ_2 to $\Delta_2 \cup \Gamma$, we get

$$\frac{1}{2\pi i} \int_{\Delta_1} (\psi_{1-} - \psi_{1+})(s)\, ds + \frac{1}{2\pi i} \int_{\Gamma} (\psi_{1+} - \psi_{1-})(s)\, ds = -1,$$

and

$$\frac{1}{2\pi i} \int_{\Delta_2} (\psi_{2-} - \psi_{2+})(s)\, ds + \frac{1}{2\pi i} \int_{\Gamma} (\psi_{2+} - \psi_{2-})(s)\, ds = -1.$$

Adding both equations and using the relations $\psi_{2-} = \psi_{0+}$ on Δ_2, $\psi_{1-} = \psi_{0+}$ on Δ_1, and $\psi_{2\pm} = \psi_{1\mp}$ on Γ, we obtain (3.2). $\square$

Proof of Proposition 2.2. We consider Case 1. The segment Δ_2 is such that for $z \in \Delta_2$, the integral $\frac{1}{2\pi i} \int_0^z (\psi_2 - \psi_0)_+(s)\, ds$ is real. For $z = 0$, it has the value 0, and for $z = b$ it has the value 1 by (3.1). The derivative of

$$t \mapsto \frac{1}{2\pi i} \int_0^t (\psi_2 - \psi_0)_+(s)\, ds \tag{3.3}$$

is equal to $\frac{1}{2\pi i} (\psi_2(t) - \psi_0(t))$ and this is different from 0 for $t \in (0, b)$. Thus (3.3) is strictly increasing from 0 for $t = 0$ to 1 for $t = b$. This immediately implies that μ_2 defined by (2.9) is a positive measure of mass 1, hence a probability measure on Δ_2. Similarly μ_1 defined in (2.8) is a probability measure on Δ_1.

In Case 2, the proof that $\mu_1 + \mu_2$ is a positive measure of mass 2 on Δ is also similar. $\square$

We now give some properties of the λ–functions introduced in (2.12)–(2.14). From (2.1), (2.5), and (2.7), we check that the following jump relations hold true,

$$\lambda_{0+} = \lambda_{0-} + 4i\pi, \quad \lambda_{1+} = \lambda_{1-} - 2i\pi, \quad \lambda_{2+} = \lambda_{2-} - 2i\pi \quad \text{on }]-\infty, 0). \tag{3.4}$$

Moreover, in Case 1, we have

$$\begin{aligned} &\lambda_{0+} = \lambda_{2-} + 2i\pi, \quad \lambda_{0-} = \lambda_{2+}, \quad \lambda_{1+} = \lambda_{1-} - 2i\pi \quad \text{on } \Delta_2 = [0, b], \\ &\lambda_{0+} = \lambda_{0-} + 2i\pi, \quad \lambda_{1+} = \lambda_{1-} - 2i\pi \quad \text{on } [b, c], \\ &\lambda_{1\pm} = \lambda_{0\mp} \quad \text{on } \Delta_1 = [c, d]. \end{aligned} \tag{3.5}$$

while in Case 2, we have

$$\begin{aligned} &\lambda_{0+} = \lambda_{2-} + 2i\pi, \quad \lambda_{0-} = \lambda_{2+}, \quad \lambda_{1+} = \lambda_{1-} - 2i\pi, \quad \text{on } \Delta_2 = [0, a], \\ &\lambda_{1\pm} = \lambda_{0\mp}, \quad \text{on } \Delta_1 = [a, d], \\ &\lambda_{1\pm} = \lambda_{2\mp}, \quad \text{on } \Gamma_+ = \{z \in \Gamma, \operatorname{Im} z > 0\}, \\ &\lambda_{1\pm} = \lambda_{2\mp} + 4i\pi, \quad \text{on } \Gamma_- = \{z \in \Gamma, \operatorname{Im} z < 0\}. \end{aligned} \tag{3.6}$$

In the next proposition, we show that the function λ_0 can be expressed in terms of the complex logarithmic potentials $\mathcal{V}_1$ and $\mathcal{V}_2$ respectively associated to the measures μ_1 and μ_2,

$$\mathcal{V}_1(z) = -\int_{\Delta_1} \log(z - s) d\mu_1(s), \quad \mathcal{V}_2(z) = -\int_{\Delta_2} \log(z - s) d\mu_2(s).$$

This will be useful in the proof of Theorem 2.3, see Section 7. Note that the potentials $\mathcal{V}_1$ and $\mathcal{V}_2$ are multi–valued functions, depending on the specific choice of the branches of the logarithmic functions. In Case 1, since μ_1 and μ_2 have total masses 1, $\mathcal{V}_1$ and $\mathcal{V}_2$ are defined modulo $2i\pi$ in $\mathbb{C} \setminus \Delta_2$ and $\mathbb{C} \setminus \Delta_1$ respectively. In Case 2, $\mathcal{V}_1 + \mathcal{V}_2$ is defined modulo $4i\pi$ in $\mathbb{C} \setminus \Delta$.

Proposition 3.2. *The following relation modulo $2i\pi$ holds true*

$$\lambda_0(z) + \mathcal{V}_1(z) + \mathcal{V}_2(z) = \ell_0, \quad z \in \mathbb{C} \setminus \Delta. \tag{3.7}$$

Proof. We compute the derivatives of $\mathcal{V}_1$ and $\mathcal{V}_2$. We consider Case 1 first. The derivative of $\mathcal{V}_1$ is

$$\mathcal{V}_1'(z) = \frac{1}{2\pi i} \int_{\Delta_1} \frac{1}{z-s} (\psi_0 - \psi_1)_+(s)\, ds.$$

If γ is a closed contour going around Δ_1 on $\mathcal{R}_1$ in the positive direction but with z outside γ, then, since $\psi_{0+} = \psi_{1-}$,

$$\mathcal{V}_1'(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{\psi_1(s)}{z-s}\, ds.$$

The integral over γ can be calculated with the residue theorem for the exterior of γ, for which there are residues at z and ∞. This proves that $\mathcal{V}_1'(z) = \psi_1(z) - \beta_1$, in $\mathbb{C} \setminus \Delta_1$. The proof that $\mathcal{V}_2'(z) = \psi_2(z) - \beta_2$ in $\mathbb{C} \setminus \Delta_2$ is similar.

Next, in Case 2, we have since $\psi_{2\pm} = \psi_{1\mp}$ on Γ,

$$\begin{aligned}
\mathcal{V}_1'(z) + \mathcal{V}_2'(z) &= \frac{1}{2i\pi} \int_{\Delta_2} \frac{1}{z-s} (\psi_0 - \psi_2)_+(s)\, ds + \frac{1}{2i\pi} \int_{\Gamma} \frac{1}{z-s} (\psi_{2+} - \psi_{2-})(s)\, ds \\
&\quad + \frac{1}{2i\pi} \int_{\Delta_1} \frac{1}{z-s} (\psi_0 - \psi_1)_+(s)\, ds + \frac{1}{2i\pi} \int_{\Gamma} \frac{1}{z-s} (\psi_{1+} - \psi_{1-})(s)\, ds \\
&= \frac{1}{2i\pi} \int_{\gamma_2} \frac{\psi_2(s)}{z-s} ds + \frac{1}{2i\pi} \int_{\gamma_1} \frac{\psi_1(s)}{z-s} ds \\
&= \psi_2(z) - \beta_2 + \psi_1(z) - \beta_1.
\end{aligned} \tag{3.8}$$

In the second equality, γ_2 denotes a positively oriented contour around Δ_2 and Γ on $\mathcal{R}_2$ and γ_1 a positively oriented contour around Δ_1 and Γ on $\mathcal{R}_1$. Equation (3.8) holds true in $\mathbb{C} \setminus (\Delta \cup \Gamma)$. However, since all the functions involved are analytic on $\Gamma \setminus \{a\}$, it holds true in $\mathbb{C} \setminus \Delta$.

In both cases, as the derivative of λ_0 is ψ_0, we get in view of (1.6) that the derivative of $\lambda_0 + \mathcal{V}_1 + \mathcal{V}_2$ vanishes in $\mathbb{C} \setminus \Delta$. Hence, $\lambda_0 + \mathcal{V}_1 + \mathcal{V}_2$ is a constant C, whose value can be obtained from the behavior at infinity. Since $\mathcal{V}_j(z) = -\log z + \mathcal{O}(1/z)$, $j = 1, 2$, and in view of the first expansion in (2.15), we get $C = \ell_0$ modulo $2i\pi$. ☐

4 The Riemann–Hilbert problem and its solution

Let us first introduce the so–called functions of the second kind

$$r_{\vec{n},j}(z) = \frac{n^{-\alpha}}{2\pi i} \int_0^{\infty} \frac{l_{\vec{n}}(x) x^{\alpha} e^{-\beta_j x} dx}{x - z}, \qquad j = 1, 2, \tag{4.1}$$

which are defined up to the normalization of $l_{\vec{n}}(x)$. Expanding the kernel $1/(x-z)$ in the representation (4.1) in powers of z and using the orthogonality relations (1.1) for $l_{\vec{n}}$, we immediately get

$$r_{\vec{n},j}(z) = \mathcal{O}\left(\frac{1}{z^{n_j+1}}\right), \quad j = 1, 2, \tag{4.2}$$

as $z \to \infty$.

Now, we consider the following Riemann–Hilbert problem,

1. Y is analytic in $\mathbb{C} \setminus \mathbb{R}_+$.

2. Y possesses continuous boundary values for $z \in \mathbb{R}_+$ denoted by Y_+ and Y_-, where Y_+ and Y_- denote the limiting values of $Y(z')$ as z' approaches z from the left and the right, according to the orientation on $\mathbb{R}_+$, and

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^\alpha e^{-\beta_1 n z} & z^\alpha e^{-\beta_2 n z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad z \in \mathbb{R}_+, \tag{4.3}$$

3. $Y(z)$ has the following behavior at infinity:

$$Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^{2n} & 0 & 0 \\ 0 & z^{-n} & 0 \\ 0 & 0 & z^{-n} \end{pmatrix}, \quad z \to \infty, \quad z \in \mathbb{C} \setminus \mathbb{R}_+. \tag{4.4}$$

4. $Y(z)$ has the following behavior near the origin, as $z \to 0$, $z \in \mathbb{C} \setminus \mathbb{R}_+$,

$$Y(z) = \mathcal{O}\begin{pmatrix} 1 & h(z) & h(z) \\ 1 & h(z) & h(z) \\ 1 & h(z) & h(z) \end{pmatrix}, \text{ with } h(z) = \begin{cases} |z|^\alpha, & \text{if} \quad -1 < \alpha < 0, \\ \log|z|, & \text{if} \quad \alpha = 0, \\ 1, & \text{if} \quad 0 < \alpha, \end{cases} \tag{4.5}$$

   where z^α is defined with a branch on $\mathbb{R}_+$. The $\mathcal{O}$ condition in (4.5) is to be taken entrywise.

Remark 4.1. *Item 4, that describes the behavior of Y near the origin, is needed to ensure uniqueness of a solution to the Riemann–Hilbert problem (see the proof of the next theorem).*

Theorem 4.2. *The solution of the above Riemann–Hilbert problem for Y is unique and is given by*

$$Y(z) = \begin{pmatrix} L_n(z) & R_{n,1}(z) & R_{n,2}(z) \\ l_{(n-1,n)}(nz) & r_{(n-1,n),1}(nz) & r_{(n-1,n),2}(nz) \\ l_{(n,n-1)}(nz) & r_{(n,n-1),1}(nz) & r_{(n,n-1),2}(nz) \end{pmatrix}, \tag{4.6}$$

with $L_n(z)$ the scaled monic Laguerre polynomial (1.2), and

$$R_{n,1}(z) = r_{(n,n),1}(nz), \quad R_{n,2}(z) = r_{(n,n),2}(nz),$$

where $r_{(n,n),1}$ and $r_{(n,n),2}$ equal the expression in (4.1), $j = 1, 2$, corresponding to the choice of the Laguerre polynomial $l_{(n,n)} = n^{-2n} l^*_{(n,n)}$. Furthermore, the normalization of $l_{(n-1,n)}$ and $l_{(n,n-1)}$ are chosen in such a way that

$$z^n r_{(n-1,n),1}(nz) \to 1, \quad z^n r_{(n,n-1),2}(nz) \to 1,$$

as $z \to \infty$.

Proof. It is clear that the given Y is analytic outside of $\mathbb{R}_+$. The jump condition on $\mathbb{R}_+$ for the first row of Y is trivial, in accordance with the fact that its entries are analytic on $\mathbb{R}_+$. For the first entry in the second row, it reads

$$(Y_{1,2})_+(z) = z^\alpha e^{-n\beta_1 z} (Y_{1,1})_-(z) + (Y_{1,2})_-(z),$$

which is indeed so, as follows from the Sokhotskii–Plemelj formula applied to the integral representation of $R_{n,1}(z)$ which is easily derived from that of $r_{(n,n),1}(z)$, see (4.1). The jump conditions on $\mathbb{R}_+$ for the second and third entries in the second row follow in a similar way, as well as the entries in the third row when β_1 is replaced with β_2.

The fact that the matrix (4.6) has a behavior at infinity given by (4.4) follows from the respective degrees $2n$, $2n-1$, $2n-1$ of the polynomials $L_n(z)$, $l_{n-1,n}(z)$, and $l_{n,n-1}(z)$, the chosen normalization of $L_n(z)$, $r_{(n-1,n),1}(nz)$, $r_{(n,n-1),2}(nz)$, and the estimates (4.2).

From the behavior of a Cauchy type integral at the endpoint of the contour, see [14, §8.1], we know that, for $j = 1, 2$,

$$r_{\vec{n},j}(z) = \begin{cases} \mathcal{O}(\log|z|), & \text{if} \quad \alpha = 0, \\ \mathcal{O}(1), & \text{if} \quad 0 < \alpha, \end{cases}$$

as $z \to 0$, $z \in \mathbb{C} \setminus \mathbb{R}_+$. Hence, when $0 \le \alpha$, the matrix in (4.6) indeed satisfies the condition (4.5). When $-1 < \alpha < 0$, we also know, see [14, §8.4], that, for $j = 1, 2$,

$$r_{\vec{n},j}(z) = -\frac{e^{-\alpha i\pi} l_{\vec{n}}(0)}{2i\pi \sin \alpha\pi} z^\alpha + \mathcal{O}(1),$$

as $z \to 0$, $z \in \mathbb{C} \setminus \mathbb{R}_+$, where z^α is defined with a branch cut on $\mathbb{R}_+$. Consequently, the matrix in (4.6) also satisfies the condition (4.5) when $-1 < \alpha < 0$. This finishes the proof that the matrix Y in (4.6) is a solution to the Riemann–Hilbert problem (4.3)–(4.5).

We now show that a solution Y is unique. For that, let us introduce the matrix

$$\widetilde{Y}(z) = Y(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{(\beta_1-\beta_2)nz} \\ 0 & 0 & 1 \end{pmatrix},$$

where Y satisfies the conditions (4.3)–(4.5). The jump condition (4.3) translates into the following condition for $\widetilde{Y}$,

$$\widetilde{Y}_+(z) = \widetilde{Y}_-(z) \begin{pmatrix} 1 & z^\alpha e^{-\beta_1 nz} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad z \in \mathbb{R}_+. \tag{4.7}$$

We see from (4.7) that the last column of $\widetilde{Y}$ has no jump on $\mathbb{R}_+$, hence its entries are analytic in $\mathbb{C} \setminus \{0\}$. Since they behave like $\mathcal{O}(h(z))$ as $z \to 0$, and h has at most a singularity of order $\alpha > -1$, we deduce that the singularity is removable and the entries in the last column of $\widetilde{Y}$ are entire functions. Hence, condition (4.5) for $\widetilde{Y}$ near the origin can actually be replaced with

$$\widetilde{Y}(z) = \mathcal{O}\begin{pmatrix} 1 & h(z) & 1 \\ 1 & h(z) & 1 \\ 1 & h(z) & 1 \end{pmatrix}, \tag{4.8}$$

which implies that $\det Y(z) = \det \widetilde{Y}(z) = \mathcal{O}(h(z))$ as $z \to 0$, $z \in \mathbb{C} \setminus \mathbb{R}_+$. On the other hand, it follows from the jump condition (4.3) that $\det Y(z)$ is analytic in $\mathbb{C} \setminus \{0\}$. Consequently, the singularity at 0 is removable and $\det Y(z)$ is an entire function in $\mathbb{C}$. From (4.4), $\det Y(z) \to 1$ as $z \to \infty$ so that, by Liouville's theorem, $\det Y(z)$ (as well as $\det X(z)$) is constant and equal to 1. In particular, $Y^{-1}(z)$ is well defined and analytic in $\mathbb{C} \setminus \mathbb{R}_+$.

Let Y_1 be a second solution of the RHP for Y, and let

$$\widetilde{Y}_1(z) = Y_1(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{(\beta_1-\beta_2)nz} \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, $J(z) = Y_1(z)Y^{-1}(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}_+$ and, from (4.3), has no jump across $\mathbb{R}_+$. Hence, it is analytic in $\mathbb{C} \setminus \{0\}$. Moreover, $J(z) = \widetilde{Y}_1(z)\widetilde{Y}^{-1}(z)$ and since $\det \widetilde{Y}(z) = 1$, the inverse of $\widetilde{Y}$ equals its adjugate, which shows, together with (4.8), that, as $z \to 0$, $z \in \mathbb{C}\setminus\mathbb{R}_+$,

$$\widetilde{Y}^{-1}(z) = \mathcal{O}\begin{pmatrix} h(z) & h(z) & h(z) \\ 1 & 1 & 1 \\ h(z) & h(z) & h(z) \end{pmatrix}.$$

Hence, for each entry J_{ij} of J, we have

$$J_{ij}(z) = \mathcal{O}(h(z)) \text{ with } h(z) = \begin{cases} |z|^\alpha, & \text{if} \quad -1 < \alpha < 0, \\ \log|z|, & \text{if} \quad \alpha = 0, \\ 1, & \text{if} \quad 0 < \alpha. \end{cases}$$

Thus, J has a removable singularity at the origin so that it is analytic in $\mathbb{C}$. Since $J(z) \to I$ as $z \to \infty$, it follows, again from Liouville's theorem, that $J = I$ and $Y_1 = Y$. □

5 The Riemann–Hilbert analysis for Case 1

Before starting the analysis we need to compare the real parts of the λ–functions introduced in (2.12)–(2.14). From (2.16), we know that $\operatorname{Re}\lambda_{2\pm} = \operatorname{Re}\lambda_{0\pm}$ on the segment Δ_2 and $\operatorname{Re}\lambda_{1\pm} = \operatorname{Re}\lambda_{0\pm}$ on the segment Δ_1. The next lemma extends these relations by stating inequalities between the real parts of the functions λ_j, $j = 1, 2, 3$, on $\mathbb{R}_+$ and on neighborhoods in the complex plane of the segments Δ_2 and Δ_1. These inequalities will be used in the sequel.

Lemma 5.1. (a) *The following inequalities hold true,*

$$\operatorname{Re}\lambda_0 \ (= \operatorname{Re}\lambda_2) < \operatorname{Re}\lambda_1 \quad on\ [0, b], \tag{5.1}$$

$$\operatorname{Re}\lambda_0 < \operatorname{Re}\lambda_1 \ on\ [b, c), \quad \operatorname{Re}\lambda_0 < \operatorname{Re}\lambda_2 \quad on\ (b, c], \tag{5.2}$$

$$\operatorname{Re}\lambda_0 \ (= \operatorname{Re}\lambda_1) < \operatorname{Re}\lambda_2 \quad on\ [c, d], \tag{5.3}$$

$$\operatorname{Re}\lambda_0 < \operatorname{Re}\lambda_1, \quad \operatorname{Re}\lambda_0 < \operatorname{Re}\lambda_2 \quad on\ (d, +\infty). \tag{5.4}$$

(b) *The open interval $(0, b)$ has a neighborhood U_2 in the complex plane such that* $\operatorname{Re}\lambda_2 < \operatorname{Re}\lambda_0$ *for $z \in U_2 \setminus (0, b)$.*
(c) *The open interval (c, d) has a neighborhood U_1 in the complex plane such that* $\operatorname{Re}\lambda_1 < \operatorname{Re}\lambda_0$ *for $z \in U_1 \setminus (c, d)$.*

Remark 5.2. *Note that* $\operatorname{Re}\lambda_0$ *is continuous across the cut $(-\infty, d]$. Indeed, on $[c, d]$, it follows from the fact that* $\operatorname{Re}\lambda_{0\pm} = \operatorname{Re}\lambda_{1\pm}$ *and the jump relation on the fourth line of (3.5). On $]-\infty, 0)$ and on $[b, c]$, it follows from the jump relations for λ_0 on the first and third lines of (3.5). On $[0, b]$, it follows from the fact that* $\operatorname{Re}\lambda_{0\pm} = \operatorname{Re}\lambda_{2\pm}$ *and the jump relations for λ_0 on the second line of (3.5). Similarly,* $\operatorname{Re}\lambda_1$ *is continuous across the cut $(-\infty, d]$ and* $\operatorname{Re}\lambda_2$ *is continuous across the cut $(-\infty, b]$. This explains that in part (a) of Lemma 5.1 it is not necessary to specify $\pm$ subscripts in the λ_j, $j = 1, 2, 3$, functions.*

Proof. For $z \in [0, b]$, we already know that $\operatorname{Re}\lambda_0 = \operatorname{Re}\lambda_2$. Moreover, ψ_1 is real and $\psi_1 < \operatorname{Re}\psi_0$. Hence, from the definitions of λ_0 and λ_1, we get $\operatorname{Re}\lambda_0 < \operatorname{Re}\lambda_1$ on $[0, b]$.

For $z \in (b, c)$, the three roots are real and $\psi_1 < \psi_0 < \psi_2$. Hence, in view of (2.12)–(2.14) and (2.17), we obtain $\operatorname{Re}\lambda_0 < \operatorname{Re}\lambda_1$ and $\operatorname{Re}\lambda_0 < \operatorname{Re}\lambda_2$ on (b, c).

For $z \in [c, d]$, ψ_0 and ψ_1 have the same real parts, less than or equal to w_c while ψ_2 is real and larger than $w_b > w_c$. Moreover,

$$\operatorname{Re}(\lambda_2 - \lambda_0)(z) = \operatorname{Re}(\lambda_2 - \lambda_0)(c) + \int_c^z \operatorname{Re}(\psi_2 - \psi_0)(s)ds,$$

and we know from what precedes that $\operatorname{Re}(\lambda_2 - \lambda_0)(c) > 0$. Consequently, $\operatorname{Re}\lambda_0 = \operatorname{Re}\lambda_1 < \operatorname{Re}\lambda_2$ on $[c, d]$.

For $z \in (d, +\infty)$, the three roots are real and $\psi_0 < \psi_1 < \psi_2$. Hence $\operatorname{Re}\lambda_0 < \operatorname{Re}\lambda_1$. Also

$$\operatorname{Re}(\lambda_2 - \lambda_0)(z) = \operatorname{Re}(\lambda_2 - \lambda_0)(d) + \int_d^z \operatorname{Re}(\psi_2 - \psi_0)(s)ds,$$

and we know from what precedes that $\operatorname{Re}(\lambda_2 - \lambda_0)(d) \geq 0$. Consequently, $\operatorname{Re}\lambda_0 < \operatorname{Re}\lambda_2$ on $(d, +\infty)$, which finishes the proof of part (a) of the lemma.

On the $+$ side of Δ_2, $(\lambda_2 - \lambda_0)_+$ is purely imaginary. Its derivative $(\psi_2 - \psi_0)_+(z)$ is purely imaginary as well, with positive imaginary part. Hence by the Cauchy-Riemann equations the real part of $(\lambda_2 - \lambda_0)(z)$ decreases as z moves into the upper half-plane, so that $\operatorname{Re}\lambda_2 < \operatorname{Re}\lambda_0$ for z near Δ_2 in the upper half-plane. Similarly, $\operatorname{Re}\lambda_2 < \operatorname{Re}\lambda_0$ for z near Δ_2 in the lower half-plane. The proof that $\operatorname{Re}\lambda_1 < \operatorname{Re}\lambda_0$ in $U_1 \setminus (c, d)$ where U_1 is a neighborhood of (c, d) in the complex plane is similar. ☐

For completeness, we show in Figure 5 the curves $\operatorname{Re}(\lambda_j - \lambda_k) = 0$, $j \neq k$, in the complex plane for the particular values $\beta_1 = 1$ and $\beta_2 = 40$ of the parameters. Figure 5 has been produced with Matlab and is the analog in Case 1 to Figures 10, 11 and 12 in Case 2, see Section 6.1. Note that the middle part of the segment Δ_1 in Figure 5 has been cut out in order to show all the mentioned curves on the same picture.

5.1 First transformation

The goal of the first transformation is to obtain a Riemann–Hilbert problem with a solution U which is normalized at infinity, and whose jump matrix is suitable for further analysis. We define

$$U(z) = L^n Y(z) \begin{pmatrix} e^{-n\lambda_0(z)} & 0 & 0 \\ 0 & e^{-n(\lambda_1(z) - \beta_1 z)} & 0 \\ 0 & 0 & e^{-n(\lambda_2(z) - \beta_2 z)} \end{pmatrix}, \tag{5.5}$$

where L is the constant diagonal matrix

$$L = \begin{pmatrix} e^{\ell_0} & 0 & 0 \\ 0 & e^{\ell_1} & 0 \\ 0 & 0 & e^{\ell_2} \end{pmatrix}. \tag{5.6}$$

The matrix U is analytic in $\mathbb{C} \setminus \mathbb{R}_+$ since $e^{\lambda_0(z)}$, $e^{\lambda_1(z)}$ and $e^{\lambda_2(z)}$ are analytic, single–valued, and non–zero outside of $\Delta_2 \cup \Delta_1$, Δ_1 and Δ_2 respectively, see (3.5).

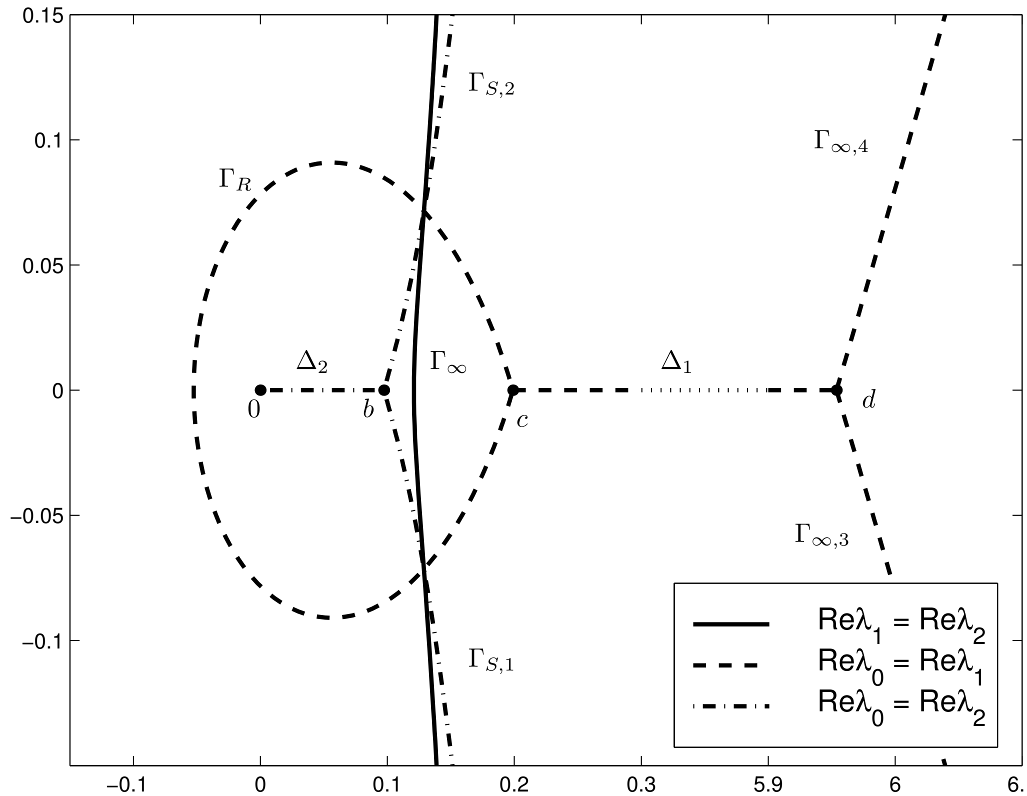

Figure 5: Case 1: curves $\mathrm{Re}\,(\lambda_j - \lambda_k) = 0$, $j \neq k$, for the values $\beta_1 = 1$ and $\beta_2 = 40$ of the parameters

From (2.15), we deduce that

$$U(z) = I + \mathcal{O}\left(\frac{1}{z}\right), \quad z \to \infty,$$

and U is normalized at infinity.

The jump relation for U on $\mathbb{R}_+$ is given by

$$U_+(z) = U_-(z) \begin{pmatrix} e^{n(\lambda_{0-}(z)-\lambda_{0+}(z))} & z^\alpha e^{n(\lambda_{0-}(z)-\lambda_{1+}(z))} & z^\alpha e^{n(\lambda_{0-}(z)-\lambda_{2+}(z))} \\ 0 & e^{n(\lambda_{1-}(z)-\lambda_{1+}(z))} & 0 \\ 0 & 0 & e^{n(\lambda_{2-}(z)-\lambda_{2+}(z))} \end{pmatrix}.$$

Making use of the jump relations (3.5), one checks easily that $U(z)$ is the solution of the following Riemann–Hilbert problem:

1. $U(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}_+$.

2. $U(z)$ possesses continuous boundary values for $z \in \mathbb{R}_+$ denoted by U_+ and U_-, where U_+ and U_- denote the limiting values of $U(z')$ as z' approaches z from the left and the

right, according to the orientation on $\mathbb{R}_+$, and

$$U_+(z) = U_-(z) \begin{pmatrix} e^{n(\lambda_2-\lambda_0)_+(z)} & z^\alpha e^{n(\lambda_2-\lambda_1)_+(z)} & z^\alpha \\ 0 & 1 & 0 \\ 0 & 0 & e^{n(\lambda_2-\lambda_0)_-(z)} \end{pmatrix}, \quad z \in \Delta_2 = [0,b], \tag{5.7}$$

$$U_+(z) = U_-(z) \begin{pmatrix} 1 & z^\alpha e^{n(\lambda_0-\lambda_1)_+(z)} & z^\alpha e^{n(\lambda_{0-}-\lambda_2)(z)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in \mathbb{R}_+ \setminus (\Delta_2 \cup \Delta_1), \tag{5.8}$$

$$U_+(z) = U_-(z) \begin{pmatrix} e^{n(\lambda_1-\lambda_0)_+(z)} & z^\alpha & z^\alpha e^{n(\lambda_{1+}-\lambda_2)(z)} \\ 0 & e^{n(\lambda_1-\lambda_0)_-(z)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in \Delta_1 = [c,d]. \tag{5.9}$$

3. $U(z)$ has the following behavior near infinity:

$$U(z) = I + \mathcal{O}(1/z), \qquad z \to \infty, \quad z \in \mathbb{C} \setminus \mathbb{R}_+. \tag{5.10}$$

4. $U(z)$ has the same behavior as $Y(z)$ at the origin, see (4.5).

Remark 5.3. *The behavior of U at the origin given in item 4 is obtained from the definitions of the λ–functions and the behaviors (2.2)–(2.4) of the three roots ψ_0, ψ_1 and ψ_2 near the origin. Because of the prescribed behavior at the origin, it is not completely obvious that the RHPs for Y and U are equivalent. Actually they are, and this may be checked as in [18, Lemma 4.1]. In particular, the equivalence shows that the solution of the RHP for U is unique.*

The set $\mathbb{R}_+ \setminus (\Delta_2 \cup \Delta_1)$ lies in a region where $\mathrm{Re}\,(\lambda_1 - \lambda_0)$ and $\mathrm{Re}\,(\lambda_2 - \lambda_0)$ are positive, see (5.2) and (5.4). Hence, the jump matrix in (5.8) is the identity matrix I plus a matrix whose entries tend exponentially fast to zero as $n \to \infty$. Moreover, $(\lambda_2 - \lambda_0)_+ = -(\lambda_2 - \lambda_0)_-$ is purely imaginary on Δ_2 and $(\lambda_1 - \lambda_0)_+ = -(\lambda_1 - \lambda_0)_-$ is purely imaginary on Δ_1, so that the diagonal elements of the jump matrices in (5.7) and (5.9) are oscillatory.

5.2 Second transformation

The second transformation is based on the following factorizations of the jump matrices respectively on Δ_2 and Δ_1,

$$\begin{pmatrix} e^{n(\lambda_2-\lambda_0)_+(z)} & z^\alpha e^{n(\lambda_2-\lambda_1)_+(z)} & z^\alpha \\ 0 & 1 & 0 \\ 0 & 0 & e^{n(\lambda_2-\lambda_0)_-(z)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ z^{-\alpha} e^{n(\lambda_2-\lambda_0)_-(z)} & -e^{n(\lambda_2-\lambda_1)_-(z)} & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} 0 & 0 & z^\alpha \\ 0 & 1 & 0 \\ -z^{-\alpha} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ z^{-\alpha} e^{n(\lambda_2-\lambda_0)_+(z)} & e^{n(\lambda_2-\lambda_1)_+(z)} & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} e^{n(\lambda_1-\lambda_0)_+(z)} & z^\alpha & z^\alpha e^{n(\lambda_1-\lambda_2)_+(z)} \\ 0 & e^{n(\lambda_1-\lambda_0)_-(z)} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ z^{-\alpha} e^{n(\lambda_1-\lambda_0)_-(z)} & 1 & -e^{n(\lambda_1-\lambda_2)_-(z)} \\ 0 & 0 & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} 0 & z^\alpha & 0 \\ -z^{-\alpha} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ z^{-\alpha} e^{n(\lambda_1-\lambda_0)_+(z)} & 1 & e^{n(\lambda_1-\lambda_2)_+(z)} \\ 0 & 0 & 1 \end{pmatrix}.$$

We open up a lens around Δ_2. It consists of two contours $\Delta_2^- \cup \Delta_2^+$ connecting 0 and b such that Δ_2^- is on the minus side of Δ_2 and Δ_2^+ is on the plus side of Δ_2. We assume that both Δ_2^- and Δ_2^+ lie in the region where $\mathrm{Re}\,(\lambda_2 - \lambda_0) < 0$, see Lemma 5.1, part (b), and in the region where $\mathrm{Re}\,(\lambda_2-\lambda_1) < 0$, which is possible by (5.1) and the continuity of the λ–functions. Similarly, we open up a lens around Δ_1. It consists of two contours $\Delta_1^- \cup \Delta_1^+$ connecting c and d such that Δ_1^- is on the minus side of Δ_1 and Δ_1^+ is on the plus side of Δ. We also assume that both Δ_1^- and Δ_1^+ lie in the region where $\mathrm{Re}\,(\lambda_1 - \lambda_0) < 0$, see Lemma 5.1, part (c), and in the region where $\mathrm{Re}\,(\lambda_1 - \lambda_2) < 0$, which is possible by (5.3) and the continuity of the λ–functions. All these contours are shown in Figure 6. They determine 5 regions in the plane. We define the second transformation $U \mapsto T$ as follows. For z in the domain bounded

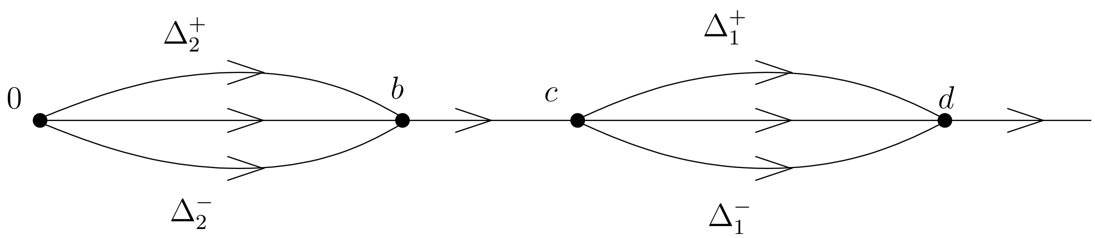

Figure 6: Deformation of contours around Δ_2 and Δ_1

by $\Delta_2^\pm$ and Δ_2,

$$T(z) = U(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mp z^{-\alpha} e^{n(\lambda_2-\lambda_0)(z)} & -e^{n(\lambda_2-\lambda_1)(z)} & 1 \end{pmatrix}, \tag{5.11}$$

for z in the domain bounded by $\Delta_1^\pm$ and Δ_1,

$$T(z) = U(z) \begin{pmatrix} 1 & 0 & 0 \\ \mp z^{-\alpha} e^{n(\lambda_1-\lambda_0)(z)} & 1 & -e^{n(\lambda_1-\lambda_2)(z)} \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.12}$$

and finally we let $T = U$ in the remaining region.

Then, straightforward calculations show that $T(z)$ is a solution of the following Riemann–Hilbert problem.

1. T is analytic in each of the 5 regions.

2. T has a jump $T_+(z) = T_-(z) j_T(z)$ on each of the 8 contours, given by

$$j_T(z) = \begin{pmatrix} 0 & z^\alpha & 0 \\ -z^{-\alpha} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in \Delta_1,$$

$$j_T(z) = \begin{pmatrix} 1 & z^\alpha e^{n(\lambda_{0-}-\lambda_{1+})(z)} & z^\alpha e^{n(\lambda_{0-}-\lambda_2)(z)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in \mathbb{R}_+ \setminus (\Delta_2 \cup \Delta_1),$$

$$j_T(z) = \begin{pmatrix} 0 & 0 & z^\alpha \\ 0 & 1 & 0 \\ -z^{-\alpha} & 0 & 0 \end{pmatrix}, \quad z \in \Delta_2,$$

$$j_T(z) = \begin{pmatrix} 1 & 0 & 0 \\ z^{-\alpha} e^{n(\lambda_1-\lambda_0)(z)} & 1 & \pm e^{n(\lambda_1-\lambda_2)(z)} \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in \Delta_1^\pm,$$

$$j_T(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ z^{-\alpha} e^{n(\lambda_2-\lambda_0)(z)} & \pm e^{n(\lambda_2-\lambda_1)(z)} & 1 \end{pmatrix}, \quad z \in \Delta_2^\pm,$$

3. $T(z)$ has the following behavior near infinity:

$$T(z) = I + \mathcal{O}\left(\frac{1}{z}\right), \quad \text{as } z \to \infty.$$

4. For $-1 < \alpha < 0$, $T(z)$ behaves near the origin like:

$$T(z) = \mathcal{O}\begin{pmatrix} 1 & |z|^\alpha & |z|^\alpha \\ 1 & |z|^\alpha & |z|^\alpha \\ 1 & |z|^\alpha & |z|^\alpha \end{pmatrix}, \quad \text{as } z \to 0. \tag{5.13}$$

For $\alpha = 0$, $T(z)$ behaves near the origin like:

$$T(z) = \mathcal{O}\begin{pmatrix} \log|z| & \log|z| & \log|z| \\ \log|z| & \log|z| & \log|z| \\ \log|z| & \log|z| & \log|z| \end{pmatrix}, \quad \text{as } z \to 0. \tag{5.14}$$

For $0 < \alpha$, $T(z)$ behaves near the origin like:

$$T(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, & \text{as } z \to 0 \text{ outside the lens}, \\ \mathcal{O}\begin{pmatrix} |z|^{-\alpha} & 1 & 1 \\ |z|^{-\alpha} & 1 & 1 \\ |z|^{-\alpha} & 1 & 1 \end{pmatrix}, & \text{as } z \to 0 \text{ inside the lens}. \end{cases} \tag{5.15}$$

Again, because of the prescribed behavior at the origin, it is not completely obvious that the RHPs for U and T are equivalent. Reasoning as in [18, Lemma 4.1] still shows that they are. In particular, the solution of the RHP for T is unique.

5.3 Model Riemann–Hilbert problem

In view of Lemma 5.1 the jump matrices for $T(z)$ tend to the identity matrix exponentially fast as $n \to \infty$, except for $z \in \Delta_2 \cup \Delta_1$. Thus we expect that the main contribution to the asymptotics of $T(z)$ is described by a solution N of the following model Riemann–Hilbert problem on the contours $\Delta_2 \cup \Delta_1$.

1. N is analytic in $\mathbb{C} \setminus (\Delta_2 \cup \Delta_1)$.

2. N has jumps on Δ_2 and Δ_1 given by

$$N_+(z) = N_-(z) \begin{pmatrix} 0 & z^\alpha & 0 \\ -z^{-\alpha} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in \Delta_1, \tag{5.16}$$

$$N_+(z) = N_-(z) \begin{pmatrix} 0 & 0 & z^\alpha \\ 0 & 1 & 0 \\ -z^{-\alpha} & 0 & 0 \end{pmatrix}, \quad z \in \Delta_2. \tag{5.17}$$

3. $N(z)$ behaves near infinity like

$$N(z) = I + \mathcal{O}\left(\frac{1}{z}\right), \quad z \to \infty.$$

For a solution to the above problem, we shall need the polynomial $D(w)$ and its square root that were defined at the beginning of Section 2.3, see (2.18).

Proposition 5.4. *A solution of the Riemann–Hilbert problem for N is given by*

$$N(z) = \begin{pmatrix} F_1(\psi_0(z)) & F_1(\psi_1(z)) & F_1(\psi_2(z)) \\ F_2(\psi_0(z)) & F_2(\psi_1(z)) & F_2(\psi_2(z)) \\ F_3(\psi_0(z)) & F_3(\psi_1(z)) & F_3(\psi_2(z)) \end{pmatrix} \tag{5.18}$$

where

$$\begin{aligned} F_1(w) &= -2^{\alpha+1/2} i (w-\beta_1)(w-\beta_2) G(w) D(w)^{-\frac{1}{2}}, && (5.19) \\ F_2(w) &= \beta_1^\alpha w (w-\beta_2) G(w) D(w)^{-\frac{1}{2}}, && (5.20) \\ F_3(w) &= \beta_2^\alpha w (w-\beta_1) G(w) D(w)^{-\frac{1}{2}}, && (5.21) \end{aligned}$$

and

$$G(w) = \begin{cases} \left(\frac{(w-\beta_1)(w-\beta_2)}{2\beta_1\beta_2 - (\beta_1+\beta_2)w}\right)^\alpha = (zw)^{-\alpha}, & w \in \psi(\mathcal{R}_0) \\ \\ w^{-\alpha}, & w \in \psi(\mathcal{R}_1) \cup \psi(\mathcal{R}_2). \end{cases} \tag{5.22}$$

Proof. We solve this problem by the same method as in [19], reducing the matrix Riemann–Hilbert problem to three scalar ones, by way of the bijective mapping ψ.

Let us consider the first row (N_{11}, N_{12}, N_{13}) of N. From (5.16) we get the following jumps on Δ_1

$$\begin{cases} (N_{11})_+(z) = -z^{-\alpha}(N_{12})_-(z), \\ (N_{12})_+(z) = z^\alpha (N_{11})_-(z), \\ (N_{13})_+(z) = (N_{13})_-(z), \end{cases} \quad z \in \Delta_1, \tag{5.23}$$

and from (5.17) the following jumps on Δ_2

$$\begin{cases} (N_{11})_+(z) = -z^{-\alpha}(N_{13})_-(z), \\ (N_{12})_+(z) = (N_{12})_-(z), \\ (N_{13})_+(z) = z^\alpha (N_{11})_-(z), \end{cases} \quad z \in \Delta_2. \tag{5.24}$$

Clearly N_{13} is analytic on Δ_1 and N_{12} is analytic on Δ_2. Hence, we may see N_{11} as a function on the sheet $\mathcal{R}_0$ of the Riemann surface $\mathcal{R}$, N_{12} as a function on $\mathcal{R}_1$ and N_{13} as a function on $\mathcal{R}_2$. Then we transform the problem from $\mathcal{R}$ with the variable z, to the complex w-plane, via the mapping $\psi : \mathcal{R} \to \mathbb{C}$. The variables z and w are connected by (1.5).

Now we transplant the functions N_{11}, N_{12}, and N_{13} from the Riemann surface to the w-plane, by defining F_1 as follows.

$$F_1(w) = \begin{cases} N_{11}\left(\frac{2\beta_1\beta_2-(\beta_1+\beta_2)w}{w(w-\beta_1)(w-\beta_2)}\right), & w \in \psi(\mathcal{R}_0), \\ N_{12}\left(\frac{2\beta_1\beta_2-(\beta_1+\beta_2)w}{w(w-\beta_1)(w-\beta_2)}\right), & w \in \psi(\mathcal{R}_1), \\ N_{13}\left(\frac{2\beta_1\beta_2-(\beta_1+\beta_2)w}{w(w-\beta_1)(w-\beta_2)}\right), & w \in \psi(\mathcal{R}_2). \end{cases} \tag{5.25}$$

Then F_1 is analytic in $\mathbb{C} \setminus (\psi_{0\pm}(\Delta_1) \cup \psi_{0\pm}(\Delta_2))$. The jumps that F_1 should satisfy can be determined from (5.23)–(5.24) and are given by

$$\begin{cases} F_{1+}(w) = -z^{-\alpha} F_{1-}(w), & w \in \psi_{0+}(\Delta_1), \\ F_{1+}(w) = z^{-\alpha} F_{1-}(w), & w \in \psi_{0-}(\Delta_1), \\ F_{1+}(w) = -z^{-\alpha} F_{1-}(w), & w \in \psi_{0+}(\Delta_2), \\ F_{1+}(w) = z^{-\alpha} F_{1-}(w), & w \in \psi_{0-}(\Delta_2), \end{cases} \tag{5.26}$$

where $z = z(w) = \frac{2\beta_1\beta_2-(\beta_1+\beta_2)w}{w(w-\beta_1)(w-\beta_2)}$.

The asymptotic condition on N implies that $N_{11}(z) \to 1$, $N_{12}(z) \to 0$, $N_{13}(z) \to 0$ as $z \to \infty$. For F_1, this means that

$$F_1(0) = 1, \qquad F_1(\beta_1) = 0, \qquad F_1(\beta_2) = 0. \tag{5.27}$$

We now seek F_1 in the form

$$F_1(w) = -2^{\alpha+1/2}(w-\beta_1)(w-\beta_2)G(w)D(w)^{-\frac{1}{2}}. \tag{5.28}$$

Then G should be analytic in $\mathbb{C} \setminus (\psi_{0\pm}(\Delta_1) \cup \psi_{0\pm}(\Delta_2))$ with jumps

$$G_+(w) = z^{-\alpha} G_-(w), \quad w \in \psi_{0\pm}(\Delta_1) \cup \psi_{0\pm}(\Delta_2), \tag{5.29}$$

with $z = z(w)$. The normalization for G is

$$G(0) = -\sqrt{D(0)}/(2^{\alpha+1/2}\beta_1\beta_2). \tag{5.30}$$

It is straightforward to check that G given by (5.22) indeed satisfies (5.29) and (5.30). Then by (5.28) it follows that F_1 has the correct jumps (5.26) and normalization (5.27). Then from (5.25) we recover N_{11}, N_{12}, and N_{13} in terms of F_1 by

$$N_{11}(z) = F_1(\psi_0(z)), \quad N_{12}(z) = F_1(\psi_1(z)), \quad N_{13}(z) = F_1(\psi_2(z)).$$

Then the jumps (5.23) and (5.24) are satisfied, and in addition the normalization at infinity is correct. So we have found the first row of N.

The proof for the second and third rows is similar. The only difference is that we have a different normalization at infinity, which leads to the construction of functions F_2 and F_3 that satisfy the same jumps (5.26) as F_1, but are normalized by

$$F_2(0) = 0, \qquad F_2(\beta_1) = 1, \qquad F_2(\beta_2) = 0,$$

and

$$F_3(0) = 0, \qquad F_3(\beta_1) = 0, \qquad F_3(\beta_2) = 1.$$

Similar calculations then lead to the formulas (5.20) and (5.21) with the same function G. $\square$

Near the branch points, the matrix TN^{-1} is not bounded which means that N is not a good approximation to T. Hence we need a local analysis around these points.

5.4 Parametrices near the branch points b, c, and d (soft edges)

The parametrices near b, c, and d can be constructed in a similar way. We consider the branch point d in detail. Let D_d be a small disk centered at d and of fixed radius $r > 0$. We look for a local parametrix $P^{(d)}$ defined within D_d, such that

1. $P^{(d)}$ is analytic in $D_d \setminus (\mathbb{R}_+ \cup \Delta_1^{\pm})$,
2. $P^{(d)}$ has the jumps
$$P_+^{(d)}(z) = P_-^{(d)}(z) j_T(z), \qquad z \in (\mathbb{R}_+ \cup \Delta_1^{\pm}) \cap D_d,$$
3. On the boundary ∂D_d of the disk D_d, we have that $P^{(d)}$ matches N in the sense that
$$P^{(d)}(z) = \left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) N(z) \tag{5.31}$$
uniformly for $z \in \partial D_d$.

Recall that the function f_d was defined at the beginning of Section 2.5, see (2.22). Since $f_d'(d) \neq 0$, the function f_d is a conformal map from D_d onto a convex neighborhood of 0 (we may have to shrink D_d, if necessary). Recall also that f_d is real-valued on the real axis near d, and maps Δ_1 onto a part of the negative real axis.

We have some freedom in the choice of Δ_1^- and Δ_1^+. We take Δ_1^- and Δ_1^+ so that they are mapped by f_d onto the rays in the complex plane of constant arguments $-2\pi/3$ and $2\pi/3$ respectively. Then $\Delta_1^{\pm}$ and $\mathbb{R}$ divide the disk D_d into four regions whose images by the conformal map f_d are contained in the four regions I, II, III, and IV shown in Figure 7.

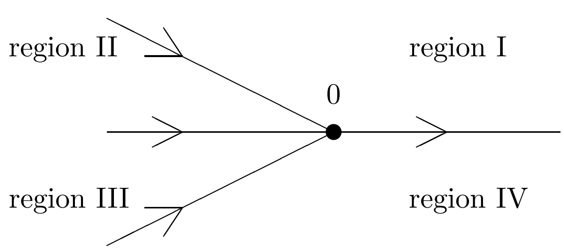

Figure 7: The regions I, II, III, and IV (case of soft edges)

Now we write

$$\widetilde{P}^{(d)} = \begin{cases} P^{(d)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{n(\lambda_1-\lambda_2)} \\ 0 & 0 & 1 \end{pmatrix} & \text{in regions } f_d^{-1}(I) \text{ and } f_d^{-1}(IV) \\ P^{(d)} & \text{in regions } f_d^{-1}(II) \text{ and } f_d^{-1}(III). \end{cases} \tag{5.32}$$

Then the jumps for $\widetilde{P}^{(d)}$ are $\widetilde{P}^{(d)}_+ = \widetilde{P}^{(d)}_- j_{\widetilde{P}^{(d)}}$ with

$$j_{\widetilde{P}^{(d)}} = \begin{pmatrix} 0 & z^{\alpha} & 0 \\ -z^{-\alpha} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } [d-r, d) \tag{5.33}$$

$$j_{\widetilde{P}^{(d)}} = \begin{pmatrix} 1 & z^{\alpha} e^{n(\lambda_0-\lambda_1)(z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } (d, d+r] \tag{5.34}$$

and

$$j_{\widetilde{P}^{(d)}} = \begin{pmatrix} 1 & 0 & 0 \\ z^{-\alpha} e^{n(\lambda_1-\lambda_0)(z)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } \Delta_1^{\pm} \cap D_d. \tag{5.35}$$

From (5.3) in Lemma 5.1, $\operatorname{Re}(\lambda_1 - \lambda_2)(d) < 0$. Thus, by continuity, this inequality is still true in $\overline{D}_d$ for sufficiently small $r > 0$. Hence, the matching condition for $\widetilde{P}^{(d)}$ remains the same as for $P^{(d)}$, namely

$$\widetilde{P}^{(d)}(z) = \left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) N(z) \tag{5.36}$$

uniformly for $z \in \partial D_d$.

Only the entries in the 2×2 upper block of the jump matrices are non-trivial, hence the Riemann–Hilbert problem is a 2×2 problem. Moreover, the jumps have a standard form and a local parametrix can be built out of Airy functions, cf. [12, p. 1523-1525].

We define Φ by

$$\Phi(s) = \begin{pmatrix} \operatorname{Ai}(s) & -\omega_3^2 \operatorname{Ai}(\omega_3^2 s) & 0 \\ \operatorname{Ai}'(s) & -\omega_3 \operatorname{Ai}'(\omega_3^2 s) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad s \in I, \tag{5.37}$$

$$\Phi(s) = \begin{pmatrix} -\omega_3 \operatorname{Ai}(\omega_3 s) & -\omega_3^2 \operatorname{Ai}(\omega_3^2 s) & 0 \\ -\omega_3^2 \operatorname{Ai}'(\omega_3 s) & -\omega_3 \operatorname{Ai}'(\omega_3^2 s) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad s \in II, \tag{5.38}$$

$$\Phi(s) = \begin{pmatrix} -\omega_3^2 \operatorname{Ai}(\omega_3^2 s) & \omega_3 \operatorname{Ai}(\omega_3 s) & 0 \\ -\omega_3 \operatorname{Ai}'(\omega_3^2 s) & \omega_3^2 \operatorname{Ai}'(\omega_3 s) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad s \in III, \tag{5.39}$$

$$\Phi(s) = \begin{pmatrix} \operatorname{Ai}(s) & \omega_3 \operatorname{Ai}(\omega_3 s) & 0 \\ \operatorname{Ai}'(s) & \omega_3^2 \operatorname{Ai}'(\omega_3 s) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad s \in IV, \tag{5.40}$$

where $\omega_3 = e^{2\pi i/3}$ is a primitive third root of unity. Then we choose $\widetilde{P}^{(d)}$ in the form

$$\widetilde{P}^{(d)}(z) = E_n^{(d)}(z)\Phi(n^{2/3} f_d(z)) \begin{pmatrix} z^{-\alpha/2} e^{n(\lambda_1-\lambda_0)(z)/2} & 0 & 0 \\ 0 & z^{\alpha/2} e^{-n(\lambda_1-\lambda_0)(z)/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.41}$$

With the above definitions of Φ and f_d it may then be shown that for any analytic prefactor $E_n^{(d)}$ the matrix $\widetilde{P}^{(d)}$ defined by (5.41) satisfies the jump conditions (5.33)–(5.35). The extra factor $E_n^{(d)}$ has to be chosen in such a way that $\widetilde{P}^{(d)}$ satisfies the matching condition on ∂D_d as well. It is given by

$$E_n^{(d)}(z) = \sqrt{\pi} N(z) \begin{pmatrix} z^{\alpha/2} & 0 & 0 \\ 0 & z^{-\alpha/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & -1 & 0 \\ -i & -i & 0 \\ 0 & 0 & \pi^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} n^{1/6} f_d(z)^{\frac{1}{4}} & 0 & 0 \\ 0 & n^{-1/6} f_d(z)^{-\frac{1}{4}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.42}$$

where the fourth root in $f_d(z)^{\frac{1}{4}}$ is defined with a cut along Δ_1. With this choice of $E_n^{(d)}$ one may then check that the matching condition (5.36) holds uniformly on ∂D_d. One may also check that the jumps of $N(z)$ and $f_d(z)^{\frac{1}{4}}$ on Δ_1 annihilate so that $E_n^{(d)}$ has no jump, hence is analytic, across Δ_1. From (5.42) and the fact that the entries of N have at most fourth root singularities at d, we see that the entries of $E_n^{(d)}$ have at most a square root singularity at d. Since $E_n^{(d)}$ is analytic in $D_d \setminus \{d\}$, the singularity at d is removable, and this proves that $E_n^{(d)}$ is analytic in the full D_d. This completes the construction of the parametrix $P^{(d)}$ in the neighborhood D_d of d.

As said before, we can construct the parametrices $P^{(b)}$ and $P^{(c)}$ near the branch points b and c in a similar way.

5.5 Parametrix near the branch point 0 (hard edge)

We consider a small disk D_0 centered at the branch point 0 and of fixed radius $r > 0$. Similarly to the previous section, we look for a local parametrix $P^{(0)}$ defined within the disk D_0, such that

1. $P^{(0)}$ is analytic in $D_0 \setminus (\Delta_2 \cup \Delta_2^{\pm})$,
2. $P^{(0)}$ has the jumps

$$P_+^{(0)}(z) = P_-^{(0)}(z) j_T(z), \qquad z \in (\Delta_2 \cup \Delta_2^{\pm}) \cap D_0,$$

3. On the boundary ∂D_0 of the disk D_0, we have that $P^{(0)}$ matches N in the sense that

$$P^{(0)}(z) = \left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) N(z) \tag{5.43}$$

uniformly for $z \in \partial D_0$.

4. $P^{(0)}(z)$ behaves near the origin like $T(z)$, see (5.13), (5.14), and (5.15).

Recall that the function f_0 was defined in Section 2.5, see (2.26). Since $f_0'(0) \neq 0$, we can choose r so small that the function f_0 is a conformal map from D_0 onto a convex neighborhood of 0. As $\mathrm{Re}\,(\widetilde{\lambda}_2 - \widetilde{\lambda}_0) = 0$ on Δ_2, f_0 maps Δ_2 onto a part of the negative real axis. We now *define* $\Delta_2^- \cap D_0$ and $\Delta_2^+ \cap D_0$ in D_0 as the preimages by f_0 of the rays γ_2^- and γ_2^+ in the complex plane of constant arguments $2\pi/3$ and $-2\pi/3$ respectively. Then $\Delta_2^{\pm}$ and Δ_2 divide the disk D_0 into three regions whose images by the conformal map f_0 are the three regions I, II, and III shown in Figure 8.

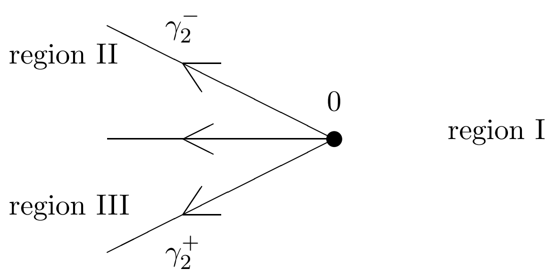

Figure 8: The regions I, II, and III (case of hard edge at 0)

Next we write

$$\widetilde{P}^{(0)} = \begin{cases} P^{(0)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -e^{n(\lambda_2-\lambda_1)} \end{pmatrix} & \text{in region } f_0^{-1}(I) \\ P^{(0)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \text{in regions } f_0^{-1}(II) \text{ and } f_0^{-1}(III). \end{cases} \tag{5.44}$$

Then $\widetilde{P}^{(0)}$ should satisfy the following RHP.

1. $\widetilde{P}^{(0)}$ is analytic in $D_0 \setminus (\Delta_2 \cup \Delta_2^{\pm})$,

2. $\widetilde{P}^{(0)}$ has the jumps $\widetilde{P}^{(0)}_+ = \widetilde{P}^{(0)}_- j_{\widetilde{P}^{(0)}}$ with

$$j_{\widetilde{P}^{(0)}}(z) = \begin{pmatrix} 0 & z^{\alpha} & 0 \\ -z^{-\alpha} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } [0, r) \tag{5.45}$$

and

$$j_{\widetilde{P}^{(0)}}(z) = \begin{pmatrix} 1 & 0 & 0 \\ z^{-\alpha} e^{n(\lambda_2-\lambda_0)(z)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } \Delta_2^{\pm} \cap D_0. \tag{5.46}$$

3. On the boundary ∂D_0 of the disk D_0, $\widetilde{P}^{(0)}$ satisfies the matching condition

$$\widetilde{P}^{(0)}(z) = \left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) \widetilde{N}(z) \tag{5.47}$$

uniformly for $z \in \partial D_0$, where

$$\widetilde{N}(z) = N(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5.48}$$

4. The behavior of $\widetilde{P}^{(0)}(z)$ near the origin is the same as the one of $P^{(0)}(z)$, i.e. as in (5.13)–(5.15).

Note that the matching condition for $\widetilde{P}^{(0)}$ in item 3 follows easily from that of $P^{(0)}$. Indeed, from (5.1) in Lemma 5.1, we know that $\mathrm{Re}\,(\lambda_2 - \lambda_1)(0) < 0$, and by continuity this inequality still holds in $\overline{D}_0$ for sufficiently small $r > 0$. Note also that item 4 is a direct consequence of the behavior of $P^{(0)}$ near the origin and (5.44).

Let $W(z) = (-z)^{\alpha/2}$ be defined and analytic for $z \in \mathbb{C} \setminus \mathbb{R}^+$ and positive for $z < 0$. Recall that the function z^α was defined with a branch cut along the negative real axis. Thus, we have

$$W(z) = \begin{cases} e^{-i\alpha\pi/2} z^{\alpha/2}, & \text{for } \mathrm{Im}\,(z) > 0, \\ e^{i\alpha\pi/2} z^{\alpha/2}, & \text{for } \mathrm{Im}\,(z) < 0. \end{cases} \tag{5.49}$$

In particular,

$$W_+(z) W_-(z) = z^\alpha, \quad \text{for } z \in \Delta_2. \tag{5.50}$$

We seek $\widetilde{P}^{(0)}$ in the form

$$\widetilde{P}^{(0)}(z) = E_n^{(0)}(z) \widehat{P}^{(0)}(z) \begin{pmatrix} W^{-1}(z) e^{\frac{n}{2}(\widetilde{\lambda}_2 - \widetilde{\lambda}_0)(z)} & 0 & 0 \\ 0 & W(z) e^{-\frac{n}{2}(\widetilde{\lambda}_2 - \widetilde{\lambda}_0)(z)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.51}$$

where $E_n^{(0)}(z)$ is some matrix analytic in a neighborhood of D_0 and $\widetilde{\lambda}_0$ and $\widetilde{\lambda}_2$ have been defined in (2.24). Then one checks that the RHP for $\widehat{P}^{(0)}$ is as follows:

1. $\widehat{P}^{(0)}$ is analytic in $D_0 \setminus (\Delta_2 \cup \Delta_2^\pm)$.

2. $\widehat{P}^{(0)}$ has jumps on the parts of Δ_2 and $\Delta_2^\pm$ lying in D_0 given by

$$\widehat{P}_+^{(0)}(z) = \widehat{P}_-^{(0)}(z) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in [0, r),$$

$$\widehat{P}_+^{(0)}(z) = \widehat{P}_-^{(0)}(z) \begin{pmatrix} 1 & 0 & 0 \\ e^{\mp i\alpha\pi} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in \Delta_2^\pm \cap D_0.$$

3. For $-1 < \alpha < 0$, $\widehat{P}^{(0)}(z)$ behaves near the origin like:

$$\widehat{P}^{(0)}(z) = \mathcal{O} \begin{pmatrix} |z|^{\alpha/2} & |z|^{\alpha/2} & |z|^\alpha \\ |z|^{\alpha/2} & |z|^{\alpha/2} & |z|^\alpha \\ |z|^{\alpha/2} & |z|^{\alpha/2} & |z|^\alpha \end{pmatrix}, \quad \text{as } z \to 0. \tag{5.52}$$

For $\alpha = 0$, $\widehat{P}^{(0)}(z)$ behaves near the origin like:

$$\widehat{P}^{(0)}(z) = \mathcal{O}\begin{pmatrix} \log|z| & \log|z| & \log|z| \\ \log|z| & \log|z| & \log|z| \\ \log|z| & \log|z| & \log|z| \end{pmatrix}, \quad \text{as } z \to 0. \tag{5.53}$$

For $0 < \alpha$, $\widehat{P}^{(0)}(z)$ behaves near the origin like:

$$\widehat{P}^{(0)}(z) = \begin{cases} \mathcal{O}\begin{pmatrix} |z|^{\alpha/2} & |z|^{-\alpha/2} & 1 \\ |z|^{\alpha/2} & |z|^{-\alpha/2} & 1 \\ |z|^{\alpha/2} & |z|^{-\alpha/2} & 1 \end{pmatrix}, & \text{as } z \to 0 \text{ outside the lens,} \\ \mathcal{O}\begin{pmatrix} |z|^{-\alpha/2} & |z|^{-\alpha/2} & 1 \\ |z|^{-\alpha/2} & |z|^{-\alpha/2} & 1 \\ |z|^{-\alpha/2} & |z|^{-\alpha/2} & 1 \end{pmatrix}, & \text{as } z \to 0 \text{ inside the lens.} \end{cases} \tag{5.54}$$

The computation of the jumps in item 2 uses the jumps (5.45)–(5.46) of $\widetilde{P}^{(0)}$, the jump relations between λ_0 and λ_2 in the second line of (3.5) and relation (5.50). The behavior of $\widehat{P}^{(0)}$ near the origin follows from that of $P^{(0)}$ together with the behavior near 0 of the modulus of $W(z)$ which is like $|z|^{\alpha/2}$ and the fact that the modulus of $(\lambda_2 - \lambda_0)(z)$ remains bounded below and above near $z = 0$.

Since only the entries in the 2×2 upper block of the jump matrices are non-trivial, the Riemann–Hilbert problem again simplifies to a 2×2 problem. In the case of a hard edge, it is known that a local parametrix can be built out of modified Bessel functions, cf. [18, Section 6] and [20]. We denote by I_α and K_α the modified Bessel functions of order α, and by $H_\alpha^{(1)}$ and $H_\alpha^{(2)}$ the Hankel functions of order α, see [1, Chapter 9]. These functions are defined and analytic in the complex plane with a branch cut along the negative real axis. Following [18, Section 6], we define Ψ by

$$\Psi(s) = \begin{pmatrix} I_\alpha(2s^{1/2}) & -\frac{i}{\pi}K_\alpha(2s^{1/2}) & 0 \\ -2\pi i s^{1/2} I_\alpha'(2s^{1/2}) & -2s^{1/2}K_\alpha'(2s^{1/2}) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s \in I, \tag{5.55}$$

$$\Psi(s) = \begin{pmatrix} \frac{1}{2}H_\alpha^{(1)}(2(-s)^{1/2}) & -\frac{1}{2}H_\alpha^{(2)}(2(-s)^{1/2}) & 0 \\ -\pi s^{1/2}(H_\alpha^{(1)})'(2(-s)^{1/2}) & \pi s^{1/2}(H_\alpha^{(2)})'(2(-s)^{1/2}) & 0 \\ 0 & 0 & 1 \end{pmatrix} e^{\frac{1}{2}\alpha\pi i\sigma_3}, \quad s \in II, \tag{5.56}$$

$$\Psi(s) = \begin{pmatrix} \frac{1}{2}H_\alpha^{(2)}(2(-s)^{1/2}) & \frac{1}{2}H_\alpha^{(1)}(2(-s)^{1/2}) & 0 \\ \pi s^{1/2}(H_\alpha^{(2)})'(2(-s)^{1/2}) & \pi s^{1/2}(H_\alpha^{(1)})'(2(-s)^{1/2}) & 0 \\ 0 & 0 & 1 \end{pmatrix} e^{-\frac{1}{2}\alpha\pi i\sigma_3}, \quad s \in III, \tag{5.57}$$

where $s^{1/2}$ has a branch cut on the real negative axis and $\sigma_3 = \mathrm{diag}(1, -1, 0)$. As in the proof of [18, Theorem 6.3] where the same RHP (up to the directions on the contours which are reversed) has been considered, we can show that $\widehat{P}^{(0)}(z) = \Psi(n^2 f_0(z))$ is a solution of the RHP for $\widehat{P}^{(0)}$. Then, $\widetilde{P}^{(0)}$ as given in (5.51), with $E_n^{(0)}$ an analytic matrix valued function, satisfies the items 1, 2, and 4 of the RHP for $\widetilde{P}^{(0)}$. We define $E_n^{(0)}$ by

$$E_n^{(0)}(z) = \frac{1}{\sqrt{2}}\widetilde{N}(z)\begin{pmatrix} W(z) & 0 & 0 \\ 0 & W^{-1}(z) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & i & 0 \\ i & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} (2\pi n)^{1/2} f_0(z)^{\frac{1}{4}} & 0 & 0 \\ 0 & (2\pi n)^{-1/2} f_0(z)^{-\frac{1}{4}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (5.58)$$

where $\widetilde{N}$ is given in (5.48) and the fourth root in $f_0(z)^{\frac{1}{4}}$ is defined with a cut along Δ_2. Then, it is possible to check that $E_n^{(0)}$ is analytic in D_0 and that the matching condition (5.47) for $\widetilde{P}^{(0)}$ is satisfied. This completes the construction of the local parametrix $P^{(0)}$ near the origin.

5.6 Final transformation

We now introduce the final matrix. Let $D = D_0 \cup D_b \cup D_c \cup D_d$. We set

$$S(z) = T(z)N(z)^{-1}, \qquad z \in \mathbb{C} \setminus D, \quad (5.59)$$

$$S(z) = T(z)P^{(j)}(z)^{-1}, \qquad z \in D_j, \quad (5.60)$$

where j stands for one of the symbols 0, b, c, or d. Note that the matrices N and $P^{(j)}$, $j \in \{0, b, c, d\}$ are invertible since their determinant is equal to 1. For N, it follows from Liouville's theorem together with the fact that the determinant of a solution to the Riemann–Hilbert problem defined at the beginning of Section 5.3 is an entire function which tends to 1 as $z \to \infty$. For $P^{(d)}$ it can be seen directly from its definition. Since $P^{(b)}$ and $P^{(c)}$ are constructed in the same way than $P^{(d)}$, they have inverse as well. For the fact that $\det P^{(0)} = 1$ we refer to [18, Remark 7.1].

Inside D_j, $j \in \{0, b, c, d\}$, the matrices T and $P^{(j)}$ have the same jumps, hence S has no jumps inside D. Outside of D the matrices T and N have the same jumps on Δ_2 and Δ_1. Hence S has no jump on Δ_2 and Δ_1. Consequently, S solves a Riemann–Hilbert problem on the reduced system of curves Σ_S shown in Figure 9.

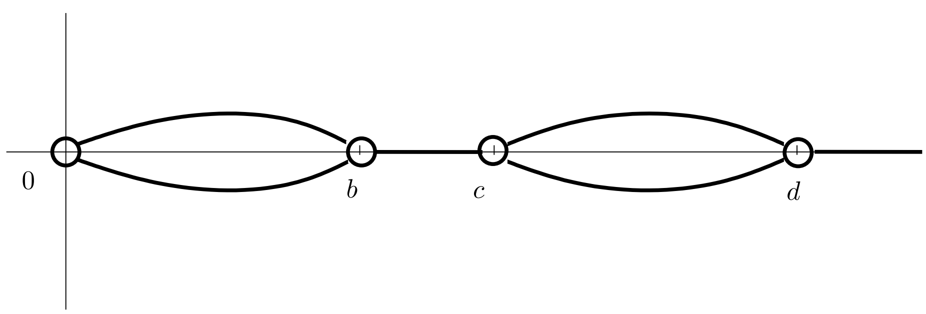

Figure 9: Contours of the RHP for S (bold lines)

The matrix S is analytic outside of the above system of contours. Let us prove that the possible isolated singularity of S at the origin is removable. If $\alpha > 0$, it follows from (5.60), (5.15), $\det P^{(0)} = 1$, and the fact that $P^{(0)}$ behaves near the origin like T, that all the entries of $S(z)$ behave like $\mathcal{O}(1)$ when $z \to 0$ from outside the lens and like $\mathcal{O}(|z|^{-\alpha})$ when $z \to 0$ from inside the lens. The entries of S cannot have a pole at 0 as they remain bounded when they approach this point from outside the lens. Moreover, since for any integer $m > \alpha$, $z^m S(z)$ is bounded near 0, the entries of S cannot have an essential singularity as well, and the singularity at 0 is removable.

If $\alpha = 0$, it similarly follows from (5.60) and (5.14) that all the entries of $S(z)$ behave like $\mathcal{O}((\log|z|)^3)$ as $z \to 0$, so that the singularity at 0 is again removable in this case.

If $\alpha < 0$, (5.60) and (5.13) imply that all the entries of $S(z)$ behave like $\mathcal{O}(|z|^{2\alpha})$ as $z \to 0$ which shows that the singularity is removable in case $\alpha > -1/2$ only. To handle all values $-1 < \alpha < 0$, we introduce the matrix $\widetilde{T}$ which is defined similarly to T, see (5.11)– (5.12), except that

$$\widetilde{T}(z) = T(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -e^{n(\lambda_2-\lambda_1)(z)} & 1 \end{pmatrix},$$

for z outside the lenses around Δ_2 and Δ_1. Then, it is straightforward to check that the entries in the second column of $\widetilde{T}$ have no jump around the origin. Since, in view of (5.13), these entries behave like $\mathcal{O}(|z|^{\alpha})$, $-1 < \alpha < 0$, the singularity at 0 is removable and

$$\widetilde{T}(z) = \mathcal{O} \begin{pmatrix} 1 & 1 & |z|^{\alpha} \\ 1 & 1 & |z|^{\alpha} \\ 1 & 1 & |z|^{\alpha} \end{pmatrix}, \qquad \text{as } z \to 0.$$

From the definition (5.44) of $\widetilde{P}^{(0)}$, we deduce that in D_0,

$$S = T(P^{(0)})^{-1} = \widetilde{T} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} (\widetilde{P}^{(0)})^{-1}. \tag{5.61}$$

On the other hand, it follows from the definition (5.55)–(5.57) of $\Psi(s)$ that an improved version of formula (5.52) holds true, namely

$$\widehat{P}^{(0)}(z) = \mathcal{O} \begin{pmatrix} |z|^{\alpha/2} & |z|^{\alpha/2} & 1 \\ |z|^{\alpha/2} & |z|^{\alpha/2} & 1 \\ |z|^{\alpha/2} & |z|^{\alpha/2} & 1 \end{pmatrix}, \quad \text{as } z \to 0.$$

Together with (5.49), (5.51), and the fact that $E_n^{(0)}$ is analytic in a neighborhood of D_0, this leads to

$$\widetilde{P}^{(0)}(z) = \mathcal{O} \begin{pmatrix} 1 & |z|^{\alpha} & 1 \\ 1 & |z|^{\alpha} & 1 \\ 1 & |z|^{\alpha} & 1 \end{pmatrix}, \quad \text{as } z \to 0,$$

which shows, along with (5.61) and $\det \widetilde{P}^{(0)} = 1$, that all entries of $S(z)$ behave like $\mathcal{O}(|z|^{\alpha})$ as $z \to 0$. Hence, it also follows in case $-1 < \alpha < 0$ that the singularity at 0 is removable.

Finally, we notice that the matrix S is normalized at infinity: $S(z) = I + \mathcal{O}(1/z)$, as z tends to infinity.

Theorem 5.5. *The matrix $S(z)$ has the behavior*

$$S(z) = I + \mathcal{O}\left(\frac{1}{n}\right), \qquad n \to \infty, \tag{5.62}$$

uniformly on $\mathbb{C} \setminus \Sigma_S$, hence, by deformation of the contours, also on $\mathbb{C}$.

Proof. The jumps on all of the contours are uniformly of the form $I + \mathcal{O}(e^{-cn})$ with some fixed $c > 0$, except for the jumps on the four circles ∂D_j, $j \in \{0, b, c, d\}$, where we have

$$S_+(z) = S_-(z) P^{(j)}(z) N(z)^{-1}, \qquad z \in \partial D_j.$$

Because of the matching conditions (5.31) and (5.43), $S(z)$ solves a Riemann–Hilbert problem, normalized at ∞ with jumps close to the identity matrix up to $\mathcal{O}(1/n)$, uniformly on the contours Σ_S. We can then use arguments as those leading to Theorem 7.171 in [10] to obtain (5.62), see also Theorem 3.1 in [17]. $\square$

6 The Riemann–Hilbert analysis for Case 2

We again consider the RH problem for the 3×3 matrix Y that was defined and studied in Section 4. Recall that its solution is given by (4.6) in terms of the Laguerre polynomials and the functions of the second kind.

6.1 First transformation

We perform the first transformation $Y \to U$ as in (5.5). Then we get for U the same RH problem as the one obtained at the end of Section 5.1 except that U is now only analytic in $\mathbb{C} \setminus (\mathbb{R}_+ \cup \Gamma)$ and that, in addition to the three jumps (5.7)–(5.9), there is a jump on Γ. More precisely, the jumps for U now become

$$U_+(z) = U_-(z) \begin{pmatrix} e^{n(\lambda_2-\lambda_0)_+(z)} & z^\alpha e^{n(\lambda_2-\lambda_1)_+(z)} & z^\alpha \\ 0 & 1 & 0 \\ 0 & 0 & e^{n(\lambda_2-\lambda_0)_-(z)} \end{pmatrix}, \quad z \in \Delta_2 = [0, a], \tag{6.1}$$

$$U_+(z) = U_-(z) \begin{pmatrix} 1 & z^\alpha e^{n(\lambda_0-\lambda_1)(z)} & z^\alpha e^{n(\lambda_0-\lambda_2)(z)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in \mathbb{R}_+ \setminus \Delta, \tag{6.2}$$

$$U_+(z) = U_-(z) \begin{pmatrix} e^{n(\lambda_1-\lambda_0)_+(z)} & z^\alpha & z^\alpha e^{n(\lambda_{1+}-\lambda_2)(z)} \\ 0 & e^{n(\lambda_1-\lambda_0)_-(z)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in \Delta_1 = [a, d], \tag{6.3}$$

$$U_+(z) = U_-(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-n(\lambda_{1+}-\lambda_{1-})(z)} & 0 \\ 0 & 0 & e^{-n(\lambda_{2+}-\lambda_{2-})(z)} \end{pmatrix}, \quad z \in \Gamma. \tag{6.4}$$

To proceed, we need to know the signs of $\mathrm{Re}\,(\lambda_j - \lambda_k)$, $j \neq k$, in the complex domain. To this end we study the curves where $\mathrm{Re}\,(\lambda_j - \lambda_k) = 0$, $j \neq k$. From the definitions of the functions λ_j, $j = 1, 2, 3$, and the fact that the roots $\psi_1(z)$ (resp. $\psi_2(z)$) and $\psi_0(z)$ are conjugate on Δ_1 (resp. Δ_2), one deduces that Δ_1 and Δ_2 respectively belongs to the curves where $\mathrm{Re}\,(\lambda_1 - \lambda_0) = 0$ and $\mathrm{Re}\,(\lambda_2 - \lambda_0) = 0$. One may also observe that the complex branch points b and c belong to the curves where $\mathrm{Re}\,(\lambda_2-\lambda_1)=0$. Indeed, we have

$$(\lambda_2-\lambda_1)(z)=\int_0^z(\psi_2-\psi_0)(s)ds+\int_d^z(\psi_0-\psi_1)(s)ds-4i\pi \tag{6.5}$$

$$=\int_{a_+}^z(\psi_2-\psi_0)(s)ds+\int_{a_-}^z(\psi_0-\psi_1)(s)ds+\int_\gamma\psi_0(s)ds-4i\pi \tag{6.6}$$

$$=\int_{a_+}^z(\psi_2-\psi_0)(s)ds+\int_{a_-}^z(\psi_0-\psi_1)(s)ds. \tag{6.7}$$

In (6.6), we have used the jump relations $\psi_{2+}=\psi_{0-}$ across Δ_2 and $\psi_{1+}=\psi_{0-}$ across Δ_1. Still in (6.6), in the first (resp. second) integral, the path of integration starts from a on the upper side of Δ and on the left (resp. right) side of the cut Γ. In the third integral, the integration is on a closed and positively oriented contour γ around Δ. In (6.7), we have used the first equality in (2.7). From (6.7), the continuity of ψ_0 across Γ and the jump relation $\psi_{2+}=\psi_{1-}$ on Γ, we get

$$(\lambda_2-\lambda_1)(c)=0, \tag{6.8}$$

and in particular $\mathrm{Re}\,(\lambda_2-\lambda_1)(c)=0$. In the same manner, we can show that $(\lambda_2-\lambda_1)(b)=0$.

The geometry of the curves where $\mathrm{Re}\,(\lambda_j-\lambda_k)=0$, $j\neq k$, differs according to the value of the ratio β_2/β_1. From plotting these curves with Matlab, one sees that three different cases may happen, namely:

- Case 2.1: $1<\beta_2/\beta_1\leq\kappa_1=1.2649\ldots$
- Case 2.2: $\kappa_1<\beta_2/\beta_1\leq\kappa_2=3+2\sqrt{2}=5.8284\ldots$
- Case 2.3: $\kappa_2<\beta_2/\beta_1<\kappa=12.1136\ldots$

The values of κ_1 and κ_2 are respectively obtained by solving for β_2/β_1 in the equations $\mathrm{Re}\,\lambda_2(d)=0$ and $\mathrm{Re}\,(\lambda_1(0)-\lambda_{0-}(0))=0$. It seems difficult to get a simple closed form for κ_1.

Figures 10, 11, 12 show the curves $\mathrm{Re}\,(\lambda_j-\lambda_k)=0$, $j\neq k$, for each of these three cases.

Let us briefly describe the change in the geometry among these different cases. In Case 2.1, the curves denoted by Γ_R and Γ_R^* in Figure 10 respectively stay on the left and on the right of the segment Δ. For $\beta_2/\beta_1=\kappa_1$, the curves Γ_R^* and Γ_S go through the right endpoint d of Δ_1. For larger values of β_2/β_1, Γ_R^* intersects Δ_1. This can be seen in Figure 11 where Γ_R^* is now the union of two analytic curves $\Gamma_{R,1}^*$ and $\Gamma_{R,2}^*$ intersecting Δ_1 at the point denoted by x_1. Note also that the curve Γ_S now consists of two pieces $\Gamma_{S,1}$ and $\Gamma_{S,2}$ that are the continuations across Δ_1 of $\Gamma_{R,1}^*$ and $\Gamma_{R,2}^*$ respectively. Then, when β_2/β_1 becomes equal to κ_2 the curve Γ_R goes through the left endpoint 0 of Δ_2. For larger values of β_2/β_1, Γ_R intersects Δ_2 at a point denoted by x_2 and a new loop around 0 appears as can be seen in Figure 12. As in Figure 5, the middle part of the segment Δ_1 has been cut out in order to show all the curves on the same picture. When β_2/β_1 equals κ, which is the critical case, the points c, $b=\overline{c}$, x_1 and a coalesce. The curves $\Gamma_{R,1}^*$ and $\Gamma_{R,2}^*$ disappear and the curves $\Gamma_{\infty,1}$ and $\Gamma_{\infty,2}$ join to give the curve Γ_∞ from Figure 5 (which corresponds to Case 1).

The assumption on the curve Γ is that it cuts once the segment Δ (at the point a) and that it stays in the domain bounded by Γ_R and Γ_R^* in Case 2.1, by Γ_R, $\Gamma_{R,1}^*$, and $\Gamma_{R,2}^*$ in Case 2.2, and by $\Gamma_{R,1}^*$, $\Gamma_{R,2}^*$ and the two pieces of Γ_R joining the points c with x_2, x_2 with $\overline{c}$ in Case 2.3.

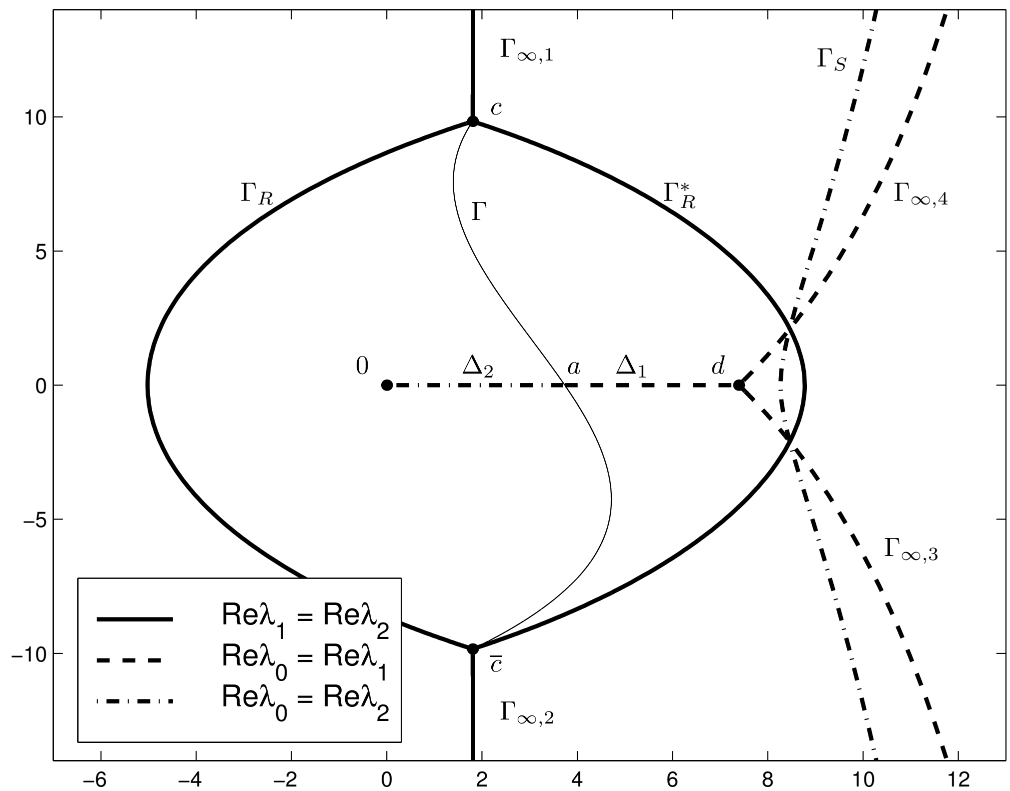

Figure 10: Case 2.1: curves $\mathrm{Re}\,(\lambda_j - \lambda_k) = 0$, $j \neq k$, and a cut Γ (particular values for this drawing: $\beta_1 = 1$ and $\beta_2 = 1.2$)

From the curves depicted in Figures 10, 11, and 12, one can find the signs of $\mathrm{Re}\,(\lambda_j - \lambda_k)$, $j \neq k$, in any domain of $\mathbb{C}$. Indeed, with (2.15) we know that $\mathrm{Re}\,\lambda_0 < \mathrm{Re}\,\lambda_1 < \mathrm{Re}\,\lambda_2$ in the infinite region on the right of d. Then, each time a curve $\mathrm{Re}\,(\lambda_j - \lambda_k)$ is crossed the ordering of $\mathrm{Re}\,\lambda_j$ and $\mathrm{Re}\,\lambda_k$ is just interchanged. Note also that when crossing a cut of a function λ_j, $j = 1, 2, 3$, the corresponding jump relation has to be taken into account.

The non-constant diagonal entries in the jump matrices in (6.1) and (6.3) are of modulus 1 and rapidly oscillating for large n.

From the signs of $\mathrm{Re}\,(\lambda_j - \lambda_k)$, $j \neq k$, one sees that the $(1, 2)$ entry in the jump matrix in (6.1) is exponentially increasing except in Case 2.3 when $z \in (0, x_2)$ where it is exponentially decreasing. Similarly, the $(1, 3)$ entry in the jump matrix in (6.3) is exponentially increasing except in Cases 2.2 and 2.3 when $z \in (x_1, d)$ where it is exponentially decreasing. The off-diagonal entries in the jump matrix in (6.2) are exponentially decreasing except for the (1,3) entry which is exponentially increasing in Case 2.1 when z belongs to the real segment joining d with the point of intersection of Γ_S with $\mathbb{R}$. Finally, the (2,2) entry in the jump matrice (6.4) is exponentially increasing while the $(3, 3)$ entry is exponentially decreasing.

Because of the exponentially increasing entries, we need to modify U. This is the aim of the next section where the global opening mentioned in the introduction will be performed. Fortunately, it will be possible to work out Cases 2.1, 2.2, and 2.3 simultaneously.

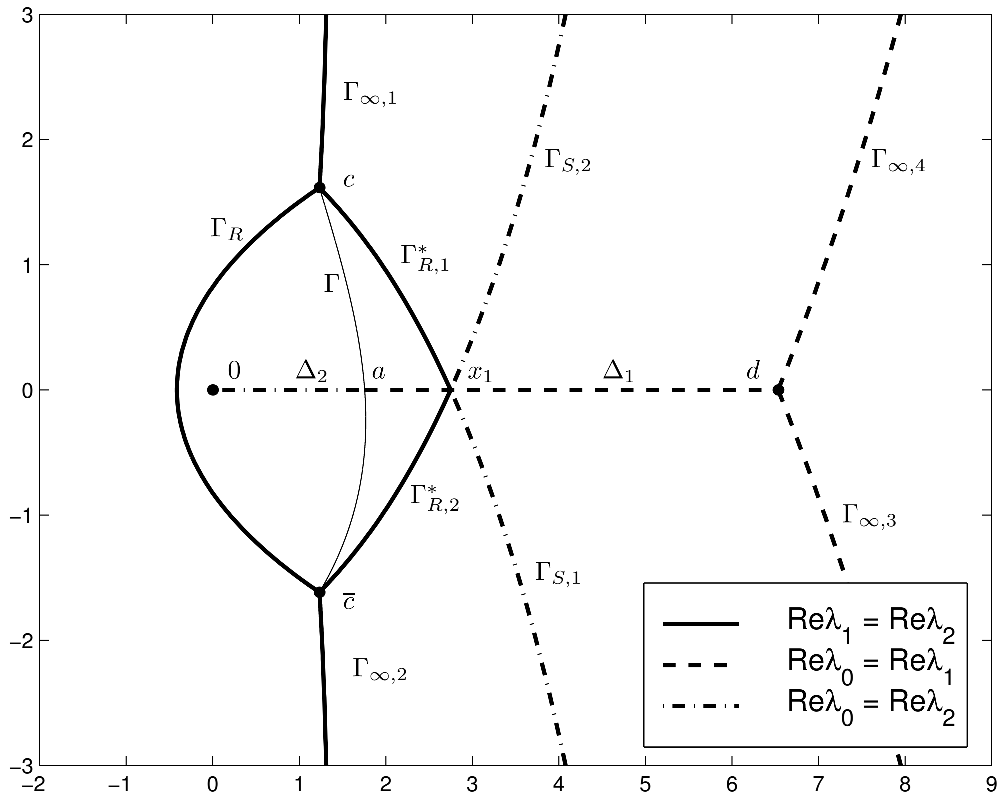

Figure 11: Case 2.2: curves $\mathrm{Re}\,(\lambda_j - \lambda_k) = 0$, $j \neq k$, and a cut Γ (particular values for this drawing: $\beta_1 = 1$ and $\beta_2 = 2$)

6.2 Second transformation

For this transformation, we consider a closed curve $\Sigma = \Sigma_1 \cup \Sigma_2$, oriented clockwise, going through b and c such that Σ_1 lies to the right of the vertical line through b and c and Σ_2 lies to its left. The curve Σ_1 is chosen so that it belongs to the region where $\mathrm{Re}\,\lambda_1 < \mathrm{Re}\,\lambda_2$ while Σ_2 is chosen so that it belongs to the region where $\mathrm{Re}\,\lambda_2 < \mathrm{Re}\,\lambda_1$. We assume also that the contour Σ encloses the segment Δ as well as the curves Γ_R and Γ^*_R. To illustrate, we have depicted in Figure 13 the contour Σ together with the curves $\mathrm{Re}\,(\lambda_j - \lambda_k) = 0$, $j \neq k$, for the Case 2.1. Now, we set

$$R(z) = U(z) \quad \text{for } z \text{ outside of } \Sigma, \tag{6.9}$$

$$R(z) = U(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -e^{n(\lambda_2-\lambda_1)(z)} & 1 \end{pmatrix} \quad \text{for } z \text{ in the region bounded by } \Sigma_2 \text{ and } \Gamma, \tag{6.10}$$

$$R(z) = U(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{n(\lambda_1-\lambda_2)(z)} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for } z \text{ in the region bounded by } \Sigma_1 \text{ and } \Gamma. \tag{6.11}$$

Then, R satisfies the following RH problem.

1. R is analytic in $\mathbb{C} \setminus (\mathbb{R}_+ \cup \Gamma \cup \Sigma)$.

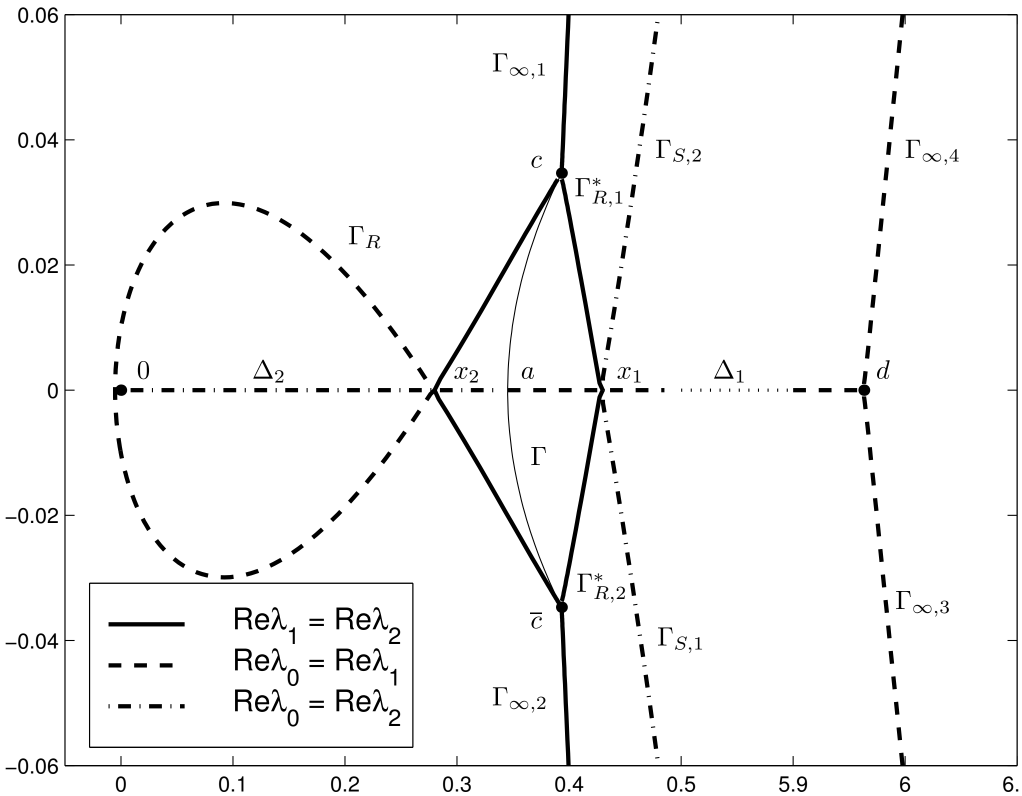

Figure 12: Case 2.3: curves $\mathrm{Re}\,(\lambda_j - \lambda_k) = 0$, $j \neq k$, and a cut Γ (particular values for this drawing: $\beta_1 = 1$ and $\beta_2 = 8$)

2. R has a jump $R_+(z) = R_-(z) j_R(z)$ given by

$$j_R(z) = \begin{pmatrix} e^{n(\lambda_2-\lambda_0)_+(z)} & 0 & z^\alpha \\ 0 & 1 & 0 \\ 0 & 0 & e^{n(\lambda_2-\lambda_0)_-(z)} \end{pmatrix}, \quad z \in \Delta_2,$$

$$j_R(z) = \begin{pmatrix} e^{n(\lambda_1-\lambda_0)_+(z)} & z^\alpha & 0 \\ 0 & e^{n(\lambda_1-\lambda_0)_-(z)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in \Delta_1,$$

$$j_R(z) = \begin{pmatrix} 1 & z^\alpha e^{n(\lambda_0-\lambda_1)(z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in [d, x_1^*),$$

$$j_R(z) = \begin{pmatrix} 1 & z^\alpha e^{n(\lambda_0-\lambda_1)(z)} & z^\alpha e^{n(\lambda_0-\lambda_2)(z)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in [x_1^*, +\infty),$$

$$j_R(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & e^{n(\lambda_1-\lambda_2)_+(z)} \end{pmatrix}, \quad z \in \Gamma,$$

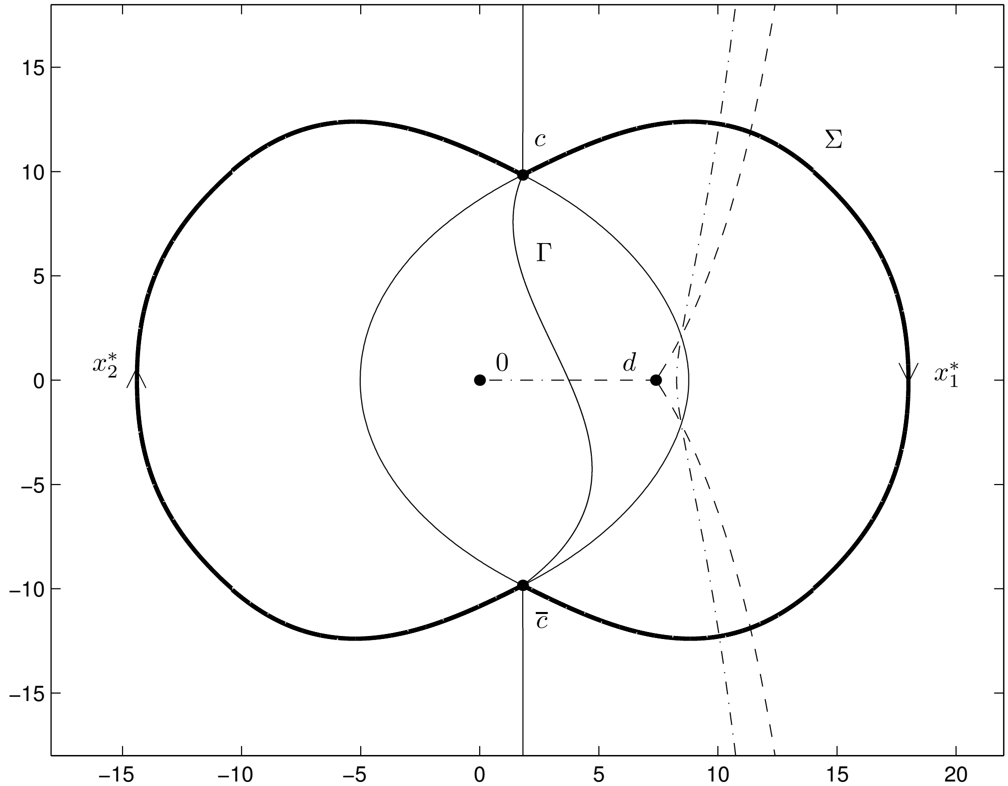

Figure 13: Contour Σ together with the curves $\mathrm{Re}\,(\lambda_j - \lambda_k) = 0$, $j \neq k$, and a cut Γ for the Case 2.1

$$j_R(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & e^{n(\lambda_2 - \lambda_1)(z)} & 1 \end{pmatrix}, \quad z \in \Sigma_2,$$

$$j_R(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{n(\lambda_1 - \lambda_2)(z)} \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in \Sigma_1,$$

3. $R(z)$ has the following behavior near infinity:

$$R(z) = I + \mathcal{O}\left(\frac{1}{z}\right), \quad \text{as } z \to \infty.$$

4. $R(z)$ has the same behavior as $Y(z)$ at the origin, see (4.5).

Note that the jumps for R are more convenient than those of U in the sense that no entries in the jumps for R on Δ_2 and Δ_1 may exponentially increase with n as was the case for the jumps of U. Also, all others jumps of R tend exponentially fast to the identity matrix except the one on Γ that tends to the constant matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{6.12}$$

6.3 Third transformation

The aim of this transformation is to change the oscillating entries on the diagonals of the jump matrices for R on Δ_2 and Δ_1 into constant entries (with respect to n). Hence, similarly to the transformation performed in Section 5.2, we open a lens around $\Delta = \Delta_2 \cup \Delta_1$, see

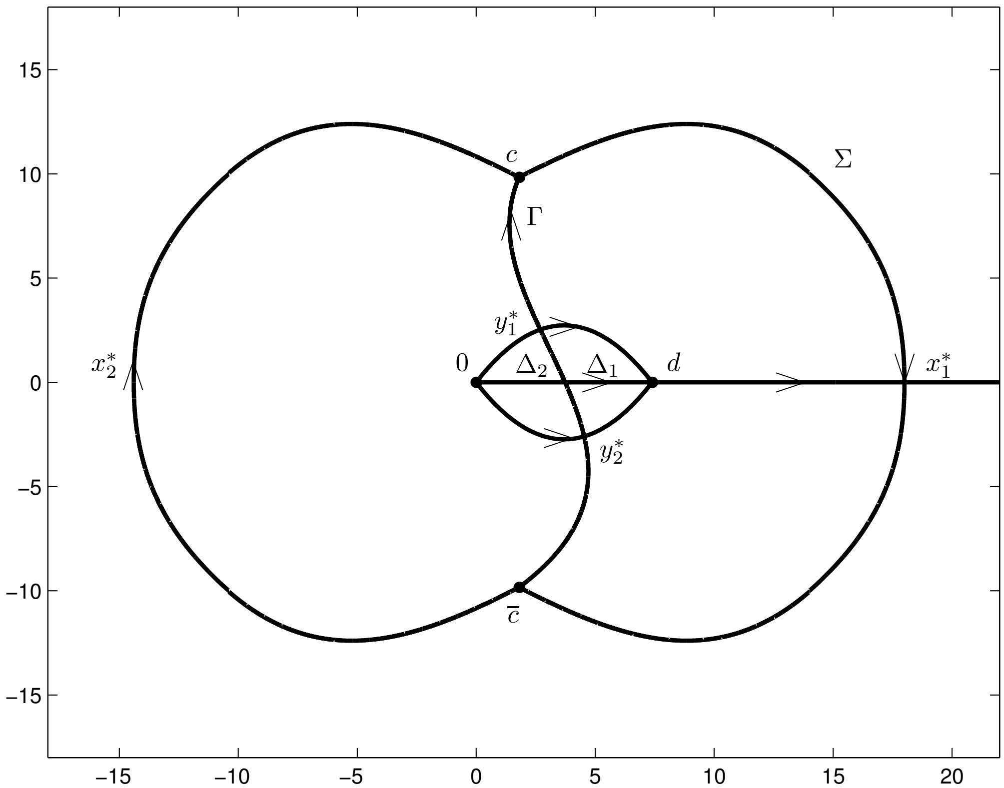

Figure 14: Opening of lens around Δ. The matrix T has jumps on the semi-axis $\mathbb{R}_+$, the cut Γ, the contour Σ, and the upper and lower parts of the lens

Figure 14, and we put

$$T(z) = R(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -z^{-\alpha} e^{n(\lambda_2 - \lambda_0)(z)} & 0 & 1 \end{pmatrix}, \quad \text{in the upper part of the lens to the left of } \Gamma, \tag{6.13}$$

$$T(z) = R(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ z^{-\alpha} e^{n(\lambda_2 - \lambda_0)(z)} & 0 & 1 \end{pmatrix}, \quad \text{in the lower part of the lens to the left of } \Gamma, \tag{6.14}$$

$$T(z) = R(z) \begin{pmatrix} 1 & 0 & 0 \\ -z^{-\alpha} e^{n(\lambda_1 - \lambda_0)(z)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{in the upper part of the lens to the right of } \Gamma, \tag{6.15}$$

$$T(z) = R(z) \begin{pmatrix} 1 & 0 & 0 \\ z^{-\alpha} e^{n(\lambda_1 - \lambda_0)(z)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{in the lower part of the lens to the right of } \Gamma. \tag{6.16}$$

We let $T = R$ in the four remaining regions, see Figure 14. The resulting Riemann–Hilbert problem for T is as follows.

1. T is analytic in each of the 8 regions.

2. T has a jump $T_+(z) = T_-(z) j_T(z)$ on 12 different contours, given by

$$j_T(z) = \begin{pmatrix} 0 & 0 & z^{\alpha} \\ 0 & 1 & 0 \\ -z^{-\alpha} & 0 & 0 \end{pmatrix}, \quad z \in \Delta_2, \quad j_T(z) = \begin{pmatrix} 0 & z^{\alpha} & 0 \\ -z^{-\alpha} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in \Delta_1, \tag{6.17}$$

$$j_T(z) = \begin{pmatrix} 1 & z^{\alpha} e^{n(\lambda_0 - \lambda_1)(z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in [d, x_1^*), \tag{6.18}$$

$$j_T(z) = \begin{pmatrix} 1 & z^{\alpha} e^{n(\lambda_0 - \lambda_1)(z)} & z^{\alpha} e^{n(\lambda_0 - \lambda_2)(z)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in [x_1^*, +\infty), \tag{6.19}$$

$$j_T(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & e^{n(\lambda_1 - \lambda_2)_+(z)} \end{pmatrix}, \quad \text{on the subarcs } (b, y_2^*) \text{ and } (y_1^*, c) \text{ of } \Gamma, \tag{6.20}$$

$$j_T(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -z^{-\alpha} e^{n(\lambda_{1+} - \lambda_0)(z)} & -1 & e^{n(\lambda_2 - \lambda_1)_-(z)} \end{pmatrix}, \quad \text{on the subarc } (a, y_1^*) \text{ of } \Gamma, \tag{6.21}$$

$$j_T(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ z^{-\alpha} e^{n(\lambda_{2-} - \lambda_0)(z)} & -1 & e^{n(\lambda_2 - \lambda_1)_-(z)} \end{pmatrix}, \quad \text{on the subarc } (y_2^*, a) \text{ of } \Gamma, \tag{6.22}$$

$$j_T(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & e^{n(\lambda_2 - \lambda_1)(z)} & 1 \end{pmatrix}, \quad z \in \Sigma_2, \quad j_T(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{n(\lambda_1 - \lambda_2)(z)} \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in \Sigma_1, \tag{6.23}$$

$$j_T(z) = \begin{pmatrix} 1 & 0 & 0 \\ z^{-\alpha} e^{n(\lambda_1 - \lambda_0)(z)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{on the lips of the lens to the right of } \Gamma, \tag{6.24}$$

$$j_T(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ z^{-\alpha} e^{n(\lambda_2 - \lambda_0)(z)} & 0 & 1 \end{pmatrix}, \quad \text{on the lips of the lens to the left of } \Gamma. \tag{6.25}$$

3. $T(z)$ has the following behavior near infinity:

$$T(z) = I + \mathcal{O}\left(\frac{1}{z}\right), \quad \text{as } z \to \infty.$$

4. $T(z)$ behaves near the origin like in (5.13)–(5.15).

Now, the jumps for T are nice since they all tend to the identity matrix, except for the jumps on Δ_2 and Δ_1 which are independent of n and for the jump on Γ which tends to the constant matrix (6.12) as n becomes large. In the limit we get a RH problem for a matrix N which we now explicitly solve.

6.4 Model Riemann–Hilbert problem

We expect that the main contribution for the asymptotics of the matrix T is given by a solution N of the following RH problem.

1. N is analytic in $\mathbb{C} \setminus (\Delta_2 \cup \Delta_1 \cup \Gamma)$.
2. N has jumps on Δ_2, Δ_1 and Γ given by

$$N_+(z) = N_-(z) \begin{pmatrix} 0 & z^{\alpha} & 0 \\ -z^{-\alpha} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in \Delta_1,$$

$$N_+(z) = N_-(z) \begin{pmatrix} 0 & 0 & z^{\alpha} \\ 0 & 1 & 0 \\ -z^{-\alpha} & 0 & 0 \end{pmatrix}, \quad z \in \Delta_2,$$

$$N_+(z) = N_-(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad z \in \Gamma.$$

3. $N(z)$ behaves near infinity like

$$N(z) = I + \mathcal{O}\left(\frac{1}{z}\right), \quad z \to \infty.$$

We can solve this problem by using the same technique as in Section 5.3. There are two differences with the RH problem considered there. The first one is the change in the branch points and in the definition of the cuts for the Riemann surface $\mathcal{R}$ and the second one is the extra jump on the cut Γ. However, it happens that the solution of the present RH problem for N remains the same as the solution of the RH problem in Section 5.3.

Proposition 6.1. *A solution to the RH problem for N is given by the formulas (5.18)–(5.22) from Proposition 5.4. The square root of $D(w)$ is now defined in these formulas with a cut on $\psi_{0+}(\Delta) \cup \psi_{1+}(\Gamma)$.*

The proof is similar to the proof of Proposition 5.4.

6.5 Parametrices near the branch points

Near the branch points, the matrix SN^{-1} is not bounded and a local analysis is needed. Near 0 and d we recover the same situations as the ones in Case 1. So, for the construction of the local parametrices at these two points, we just refer to Sections 5.4 and 5.5.

We now consider a small disk D_c around c. There, we want that the local parametrix $P^{(c)}$ satisfies

1. $P^{(c)}$ is analytic in $D_c \setminus (\Gamma \cup \Sigma)$,

2. $P^{(c)}$ has the jumps

$$P_+^{(c)}(z) = P_-^{(c)}(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & e^{n(\lambda_1-\lambda_2)_+(z)} \end{pmatrix}, \quad z \in \Gamma \cap D_c, \tag{6.26}$$

$$P_+^{(c)}(z) = P_-^{(c)}(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & e^{n(\lambda_2-\lambda_1)(z)} & 1 \end{pmatrix}, \quad z \in \Sigma_2 \cap D_c, \tag{6.27}$$

$$P_+^{(c)}(z) = P_-^{(c)}(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{n(\lambda_1-\lambda_2)(z)} \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in \Sigma_1 \cap D_c. \tag{6.28}$$

3. On the boundary ∂D_c of the disk D_c, we have that $P^{(c)}$ matches N in the sense that

$$P^{(c)}(z) = \left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) N(z) \tag{6.29}$$

uniformly for $z \in \partial D_c$.

Since w_c is a non degenerate critical point of $z(w)$, we have that as z tends to c,

$$\psi_2(z) = w_c + \alpha_c(z-c)^{1/2} + \mathcal{O}(z-c),$$

$$\psi_1(z) = w_c - \alpha_c(z-c)^{1/2} + \mathcal{O}(z-c),$$

where α_c is some non-zero constant. Hence, recalling the definitions of λ_1 and λ_2 along with (6.8), we get that

$$(\lambda_2 - \lambda_1)(z) = (z-c)^{3/2} h_c(z), \quad z \in D_c \setminus \Gamma,$$

with h_c analytic and without zeros in D_c. The function $(z-c)^{3/2}$ is defined with a branch cut along Γ.

The function $f_c(z)$ is defined, up to one of the multiplicative constants 1, $e^{2i\pi/3}$, $e^{4i\pi/3}$, by

$$(\lambda_2 - \lambda_1)(z) = \frac{4}{3}\left[f_c(z)\right]^{3/2}.$$

Recall that $\mathrm{Re}\,(\lambda_2 - \lambda_1) = 0$ on Γ_R. Hence we may (and do) choose the multiplicative constant such that f_c maps Γ_R to the real negative axis. This uniquely determines f_c. It is a conformal map from D_c onto a neighborhood of 0. Now, we still have some freedom in the choice of Γ and Σ. We choose them such that f_c respectively maps Γ, Σ_1, and Σ_2 on the rays of constant arguments $-2\pi/3$, 0, and $2\pi/3$ (this means that Γ, Σ_1 and Σ_2 are locally analytic continuations of $\Gamma_{\infty,1}$, Γ_R and Γ_R^*).

Then Γ and Σ divide the disk D_c into three regions whose images by the conformal map f_c are the three regions I, II, and III shown in Figure 15.

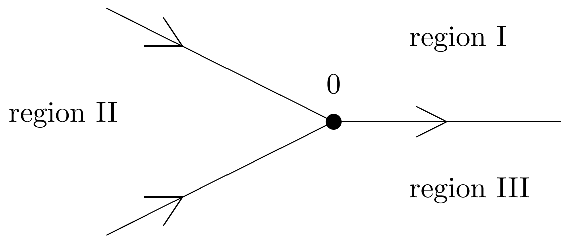

Figure 15: The regions I, II, and III (case of the complex branch point c)

As in Section 5.4, we use Airy functions to solve the RH problem for $P^{(c)}$. Following [5, Section 7], we first define Φ by

$$\Phi(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Ai}(z) & -\omega_3^2 \,\mathrm{Ai}(\omega_3^2 z) \\ 0 & \mathrm{Ai}'(z) & -\omega_3 \,\mathrm{Ai}'(\omega_3^2 z) \end{pmatrix}, \qquad z \in I,$$

$$\Phi(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\omega_3 \,\mathrm{Ai}(\omega_3 z) & -\omega_3^2 \,\mathrm{Ai}(\omega_3^2 z) \\ 0 & -\omega_3^2 \,\mathrm{Ai}'(\omega_3 z) & -\omega_3 \,\mathrm{Ai}'(\omega_3^2 z) \end{pmatrix}, \qquad z \in II,$$

$$\Phi(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Ai}(z) & \omega_3 \,\mathrm{Ai}(\omega_3 z) \\ 0 & \mathrm{Ai}'(z) & \omega_3^2 \,\mathrm{Ai}'(\omega_3 z) \end{pmatrix}, \qquad z \in III.$$

Then we choose $P^{(c)}$ in the form

$$P^{(c)}(z) = E_n^{(c)}(z)\Phi(n^{2/3} f_c(z)) \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{n}{2}(\lambda_2-\lambda_1)(z)} & 0 \\ 0 & 0 & e^{\frac{n}{2}(\lambda_1-\lambda_2)(z)} \end{pmatrix}. \tag{6.30}$$

With the above definitions of Φ and f_c one checks that for any analytic prefactor $E_n^{(c)}$ the matrix $P^{(c)}$ defined by (6.30) satisfies the jump conditions (6.26)–(6.28). The extra factor $E_n^{(c)}$ has to be chosen in such a way that $P^{(c)}$ satisfies the matching condition on ∂D_c as well. It is given by

$$E_n^{(c)}(z) = \sqrt{\pi} N(z) \begin{pmatrix} \pi^{-\frac{1}{2}} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -i & -i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & n^{1/6} f_c(z)^{\frac{1}{4}} & 0 \\ 0 & 0 & n^{-1/6} f_c(z)^{-\frac{1}{4}} \end{pmatrix}, \tag{6.31}$$

where the fourth root in $(f_c(z))^{\frac{1}{4}}$ is defined with a cut along Γ. With this choice of $E_n^{(c)}$ one may then check that the matching condition (6.29) holds uniformly on ∂D_c. One may also check that the jumps of N and $(f_c(z))^{\frac{1}{4}}$ on Γ interact in such a way that $E_n^{(c)}$ has no jump, hence is analytic, across Γ. The fact that $E_n^{(c)}$ is analytic in the full of D_c is proved by the same argument as at the end of Section 5.4. This completes the construction of the parametrix $P^{(c)}$ in the neighborhood D_c of c.

In a similar way, we can construct the parametrix $P^{(b)}$ in a neighborhood D_b of b.

6.6 Final transformation

As in Section 5.6 we define a matrix S by

$$S(z) = \begin{cases} T(z)N(z)^{-1} & \text{away from the branch points,} \\ T(z)P^{(j)}(z)^{-1} & \text{near the branch point } j, \end{cases} \tag{6.32}$$

where j stands for one of the symbols $0, b, c, d$.

Comparing the jumps of T, N, and $P^{(j)}$ on the different contours, we see that S solves a RH problem on the reduced system of curves shown in Figure 16. Moreover, in view of the

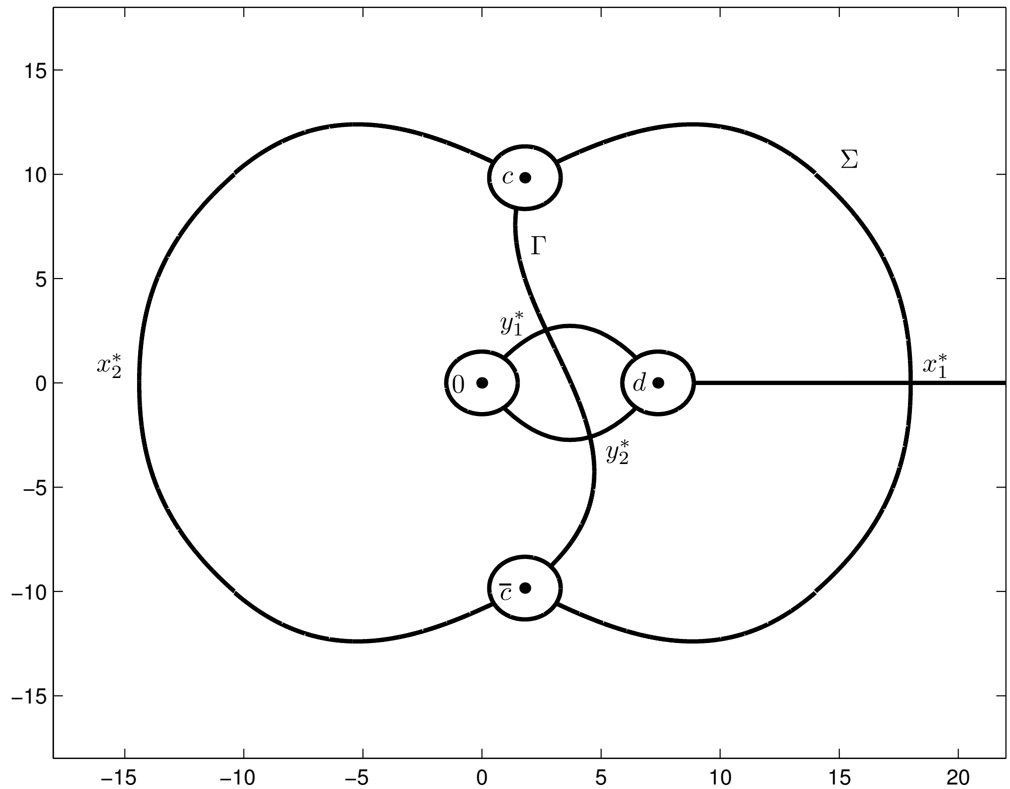

Figure 16: Contours of the RHP for S

jumps (6.18)–(6.25) for T and the matching conditions for $P^{(j)}$, the jumps for S are uniformly close to the identity matrix up to $\mathcal{O}(1/n)$. Hence, as in the proof of Theorem 5.5, we get that

$$S(z) = I + \mathcal{O}\left(\frac{1}{n}\right), \quad n \to \infty, \tag{6.33}$$

uniformly for $z \in \mathbb{C}$.

7 Proofs of the asymptotic formulas

In this last section, we show the asserted asymptotics for the scaled Laguerre polynomials L_n. We consider Case 1 and Case 2 simultaneously. We use the sequence of transformations

$Y \to U$ $(\to R) \to T \to S$ and the asymptotics (5.62) and (6.33) for S.

Proof of Theorem 2.4 Let $z \in \mathbb{C} \setminus \Delta$. Recalling that we used the same transformation $Y \to U$ in Case 1 and Case 2, we obtain from (5.5) and (5.6) that

$$L_n(z) = Y_{11}(z) = e^{n(\lambda_0(z)-\ell_0)} U_{11}(z).$$

In Case 2, we need to condider the extra transformation $U \to R$. However, from (6.9)–(6.11), we see that, independently from the location of z with respect to the curves Σ and Γ, we have $U_{11}(z) = R_{11}(z)$.

Then, from the definition of T, and since it is always possible to assume that z does not belong to the lenses around Δ_1 and Δ_2 in Case 1 and to the lens around Δ in Case 2, we get $U_{11}(z) = T_{11}(z)$. Finally, $T = SN$, except in Case 2 when $z \in D_b \cup D_c$. Then, $T = SP$. In view of the definition (6.30) and (6.31) of P, we get in either case

$$T_{11}(z) = S_{11}(z) + N_{11}(z) + S_{12}(z)N_{21}(z) + S_{13}(z) + N_{31}(z),$$

as the first columns of N and P agree. Since $S = I + \mathcal{O}(1/n)$ and since N_{11} does not vanish in a neighborhood of z, we get

$$L_n(z) = e^{n(\lambda_0(z)-\ell_0)} N_{11}(z) \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right),$$

locally uniformly in $\mathbb{C} \setminus \Delta$. Using the formula for N_{11} in Propositions 5.4 and 6.1, we get (2.19). □

Proof of Theorem 2.3 The limit for the counting measure ν_{L_n} follows from the strong convergence result given in Theorem 2.4 and from the expression (3.7) for λ_0 in terms of complex logarithmic potentials. The proof uses the unicity theorem for logarithmic potentials, see [22, Theorem II.2.1], and is similar to the proof of [25, Theorem 2.1]. □

Proof of Theorem 2.6 We consider $z_0 \in \Delta_2$, away from the endpoints, and prove that (2.20) holds uniformly in a disk D_{z_0} centered at z_0. We may choose this disk so that it is disjoint from D_0, D_b and is contained in the lens around Δ_1 (resp. D_0, Γ, and is contained in the lens around Δ) in Case 1 (resp. Case 2), see Figures 9 and 16. Let $z \in D_{z_0}$. Then, as in the proof of Theorem 2.4, we have $L_n(z) = e^{n(\lambda_0(z)-\ell_0)} U_{11}(z)$ in Case 1 and $L_n(z) = e^{n(\lambda_0(z)-\ell_0)} R_{11}(z)$ in Case 2.

In view of the relations (5.11) between T and U in Case 1 and the relations (6.13)–(6.14) between T and R in Case 2, we get

$$L_n(z) = e^{n(\lambda_0(z)-\ell_0)} (T_{11}(z) \pm z^{-\alpha} e^{n(\lambda_2(z)-\lambda_0(z))} T_{13}(z)),$$

where the $+$ sign holds in the upper part of the lens and the $-$ sign holds in the lower part. Since $T = SN$ with $S = I + \mathcal{O}(\frac{1}{n})$, and since the entries of N do not vanish in D_{z_0}, we obtain

$$L_n(z) = e^{n(\lambda_0(z)-\ell_0)} \left[N_{11}(z) + \mathcal{O}\left(\frac{1}{n}\right)\right] \pm z^{-\alpha} e^{n(\lambda_2(z)-\ell_0)} \left[N_{13}(z) + \mathcal{O}\left(\frac{1}{n}\right)\right],$$

where we also have used the fact that $N_{11}(z)$ and $N_{13}(z)$ are bounded in D_{z_0}. Using the formulas for N_{11} and N_{13} in Propositions 5.4 and 6.1, we get (2.20).

The proof when $z_0 \in \Delta_1$ is similar. $\square$

Proof of Theorem 2.9 We will use the parametrix $P^{(d)}$ constructed in Section 5.4. Assume z belongs to D_d and $f_d(z)$ lies in regions I or IV. From (5.60), (6.32), and (5.32), we get

$$T = SP^{(d)} = S\widetilde{P}^{(d)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{n(\lambda_1-\lambda_2)(z)} \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that, since $f_d(z)$ lies in regions I or IV, $T = U$ in Case 1 and $T = R$ in Case 2. Then, by (5.41), (5.42), and (6.9)–(6.11), we deduce that

$$\begin{aligned} U(z)\widetilde{I}(z) = \sqrt{\pi} S(z) N(z) &\begin{pmatrix} z^{\alpha/2} n^{1/6} f_d(z)^{1/4} & -z^{\alpha/2} n^{-1/6} f_d(z)^{-1/4} & 0 \\ -iz^{-\alpha/2} n^{1/6} f_d(z)^{1/4} & -iz^{-\alpha/2} n^{-1/6} f_d(z)^{-1/4} & 0 \\ 0 & 0 & \pi^{-1/2} \end{pmatrix} \\ &\times \Phi(n^{2/3} f_d(z)) \operatorname{diag}(z^{-\alpha/2} e^{\frac{n}{2}(\lambda_1-\lambda_0)(z)}, z^{\alpha/2} e^{-\frac{n}{2}(\lambda_1-\lambda_0)(z)}, 1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{n(\lambda_1-\lambda_2)(z)} \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned} \tag{7.1}$$

where

$$\widetilde{I}(z) = \begin{cases} I & \text{in Case 1,} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{n(\lambda_1-\lambda_2)(z)} \\ 0 & 0 & 1 \end{pmatrix} & \text{in Case 2.} \end{cases}$$

Restricting equation (7.1) to the $(1,1)$ entry, making use of (5.5), (5.62) or (6.33), along with the expressions (5.37) or (5.40) for $\Phi(s)$ and the expression for N given in Propositions 5.4 leads to (2.23). Assuming that $f_d(z)$ lies in regions II or III and making use of the corresponding expressions (5.38) or (5.39) for $\Phi(s)$ would lead to (2.23) as well. The fact that the functions B_d and C_d in (2.23) are analytic across Δ_1 can be verified from the jump relations

$$\psi_{0\pm}(x) = \psi_{1\mp}(x), \quad \sqrt{D(\psi_{0\pm}(x))} = \mp\sqrt{D(\psi_{1\mp}(x))}, \quad (f_d^{1/4})_+(x) = i(f_d^{1/4})_-(x), \quad x \in \Delta_1.$$

One may also check that B_d and C_d are analytic at d from the fact that both $f_d(z)^{1/4}$ and $\sqrt{D(\psi_j(z))}$, $j = 0, 1$, have a fourth root zero at d. $\square$

Proof of Theorem 2.10 It is similar to the proof of Theorem 2.9 where we now use the parametrix $P^{(0)}$ constructed in Section 5.5. Assume z belongs to D_0 and $f_0(z)$ lies in region III, so that z actually belongs to the upper part of the lens on the right of 0. From (5.60), (6.32), and (5.44), we get

$$T = SP^{(0)} = S\widetilde{P}^{(0)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

By (5.51) and (6.9)–(6.11), we deduce that

$$\begin{aligned} U(z)\widehat{I}(z)V(z) &= S(z) E_n^{(0)}(z) \\ &\times \Psi(n^2 f_0(z)) \begin{pmatrix} W(z)^{-1} e^{\frac{n}{2}(\widetilde{\lambda}_2-\widetilde{\lambda}_0)(z)} & 0 & 0 \\ 0 & W(z) e^{-\frac{n}{2}(\widetilde{\lambda}_2-\widetilde{\lambda}_0)(z)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \end{aligned} \tag{7.2}$$

where

$$\widehat{I}(z) = \begin{cases} I & \text{in Case 1,} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -e^{n(\lambda_2-\lambda_1)(z)} & 1 \end{pmatrix} & \text{in Case 2,} \end{cases}$$

and V is the matrix used in the transformation from U to T in Case 1, see (5.11), and from R to T in Case 2, see (6.13).

A few computations show that (7.2) implies

$$U(z)\begin{pmatrix}1\\0\\0\end{pmatrix} = z^{-\alpha/2}e^{\frac{n}{2}(\widetilde{\lambda}_2-\widetilde{\lambda}_0)(z)}S(z)E_n^{(0)}(z)\Psi(n^2 f_0(z))\begin{pmatrix}e^{i\alpha\frac{\pi}{2}}\\ e^{-i\alpha\frac{\pi}{2}}\\ 0\end{pmatrix}, \tag{7.3}$$

where in deriving (7.3) we have used (2.25), (5.49) and the fact that $\mathrm{Im}\,(z) > 0$. From the expression (5.57) for $\Psi(s)$ in region III and [1, formulas 9.1.3 and 9.1.4], we have

$$\Psi(n^2 f_0(z))\begin{pmatrix}e^{i\alpha\frac{\pi}{2}}\\ e^{-i\alpha\frac{\pi}{2}}\\ 0\end{pmatrix} = \begin{pmatrix} J_\alpha(2n(-f_0(z))^{1/2}) \\ 2\pi n f_0(z)^{1/2}J'_\alpha(2n(-f_0(z))^{1/2}) \\ 0\end{pmatrix}.$$

We plug this in equation (7.3) and restrict it to the $(1,1)$ entry. Then, making use of (5.62) in Case 1, (6.33) in Case 2 along with (5.5), (5.58), (5.48) and the expression for N given in Proposition 5.4 leads to (2.27) when $z \in f_0^{-1}(III)$. Assuming that $f_0(z)$ lies in regions I or II and making use of the expressions (5.55) or (5.56) for $\Psi(s)$ would lead to (2.27) when z belongs to the corresponding regions in D_0. □

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