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EXISTENCE OF NONPLANAR SOLUTIONS OF A SIMPLE MODEL OF PREMIXED BUNSEN FLAMES*

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Abstract. This work deals with the existence of solutions of a reaction-diffusion equation in the plane $\mathbb{R}^2$. The problem, whose unknowns are the real c and the function u, is the following:

$$(P)\qquad \begin{cases} \Delta u - c\dfrac{\partial u}{\partial y} + f(u) = 0 & \text{in } \mathbb{R}^2, \\ \forall \vec{k} \in \mathcal{C}(-\vec{e}_2, \alpha), & u(\lambda \vec{k}) \underset{\lambda\to+\infty}{\longrightarrow} 0, \\ \forall \vec{k} \in \mathcal{C}(\vec{e}_2, \pi-\alpha), & u(\lambda \vec{k}) \underset{\lambda\to+\infty}{\longrightarrow} 1, \end{cases}$$

where $0 < \alpha \leq \pi/2$ is given, $\vec{e}_2 = (-1, 0)$, and, for any angle ϕ and any unit vector $\vec{e}$, $\mathcal{C}(\vec{e}, \phi)$ denotes the open half-cone with angle ϕ around the vector $\vec{e}$. The given function f is of the "ignition temperature" type. In this paper, we show the existence of a solution (c, u) of (P) and we give an explicit formula that relates the speed c and the angle α.

Key words. reaction-diffusion equations, sliding method, maximum principle, travelling waves, Bunsen flames

AMS subject classifications. 35B40, 35B50, 35J60, 35J65, 35Q35

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1. Introduction. Bunsen flames are usually made of two flames: a diffusion flame and a premixed flame (see Figure 1 and the papers by Buckmaster and Ludford [11], Joulin [23], Liñan [27], and Sivashinsky [31], [32]). In this paper, we focus on the study of the premixed Bunsen flame. Roughly speaking, the hot products of the chemical reactions are located above the flame and the fresh gaseous mixture (fuel and oxidant) is located below (see Figure 1). For the sake of simplicity, we can assume that a global chemical reaction takes place in the gaseous mixture:

$$\mathcal{R}: \qquad \text{Fuel} \ + O_2 \quad \rightarrow \quad \text{Products.}$$

The isotherms (level sets of the temperature) of the premixed Bunsen flame are conical in shape and, far away from the axis of symmetry, the flame is almost planar. The underlying subsonic mass flow goes upward from the fresh zone to the burnt gases with a uniform vertical velocity c.

In this paper, we deal with the stationary states of premixed flames that are invariant by translation in one of the directions orthogonal to the flow. Consequently, the mathematical problem only involves two variables (x, y) (see Figure 1). This situation occurs with Bunsen burners that have a thin rectangular cross section.

Under some additional physical conditions that correspond to the classical thermodiffusive model (see Berestycki and Larrouturou [4], Buckmaster and Ludford [11], Matkowsky and Sivashinsky [29]), the temperature $u(x, y)$, normalized in such a way

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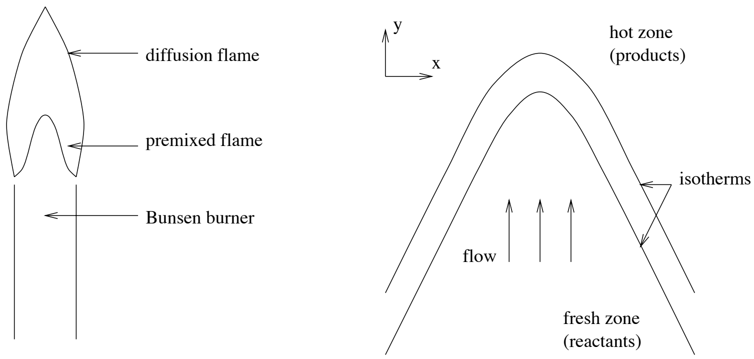

FIG. 1. *Bunsen flames (left) and the premixed flame (right).*

that $u \simeq 0$ in the fresh zone and $u \simeq 1$ in the hot zone far from the reaction sheet, solves the following reaction-diffusion equation in $\mathbb{R}^2 = \{(x,y),\ x \in \mathbb{R},\ y \in \mathbb{R}\}$:

$$\Delta u - c\frac{\partial u}{\partial y} + f(u) = 0 \quad \text{in } \mathbb{R}^2, \tag{1.1}$$

with the following limiting conditions at infinity:

$$\forall \vec{k} \in \mathcal{C}(-\vec{e}_2, \alpha), \qquad u(\lambda\vec{k}) \underset{\lambda \to +\infty}{\longrightarrow} 0, \tag{1.2}$$

$$\forall \vec{k} \in \mathcal{C}(\vec{e}_2, \pi - \alpha), \qquad u(\lambda\vec{k}) \underset{\lambda \to +\infty}{\longrightarrow} 1, \tag{1.3}$$

where α is a given angle such that $0 < \alpha \le \pi/2$. The vector $\vec{e}_2 = (0,1)$ is the unit vector in the direction $[Oy)$ and, for any unit vector $\vec{e}$ and any angle $\phi \in (0,\pi)$, $\mathcal{C}(\vec{e},\phi)$ denotes the open half-cone with aperture ϕ in the direction $\vec{e}$: $\mathcal{C}(\vec{e},\phi) = \{\vec{k} \in \mathbb{R}^2,\ \vec{k}\cdot\vec{e} > \|\vec{k}\|\ \|\vec{e}\| \cos\phi\}$. We also set $\mathcal{C}(z,\vec{e},\phi) = z + \mathcal{C}(\vec{e},\phi)$ for any point $z = (x,y) \in \mathbb{R}^2$.

The unknowns of this problem (1.1)–(1.3) are both the real c, which is like a nonlinear eigenvalue, and the function u, $0 < u < 1$, of class C^2 in $\mathbb{R}^2$. We shed light here on the fact that looking for the speed c, the angle α being known, is equivalent to looking for the angle α, the speed c being known, as is the case in experiments (see the comments after Theorem 1.2 below).

The function $1-u$ also represents the relative concentration of the reactant. In (1.1), the terms Δu, $c\frac{\partial u}{\partial y}$, and $f(u)$ are, respectively, the diffusion, transport, and source terms. The source term $f(u)$, which may take into account the Arrhénius law and the mass action law, is given and Lipschitz continuous in $[0,1]$. Furthermore, one assumes that it is of the "ignition temperature" type:

$$\exists \theta \in (0,1) \text{ such that } f \equiv 0 \text{ on } [0,\theta] \cup \{1\}, \quad f > 0 \text{ on } (\theta,1) \text{ and } f'(1) < 0. \tag{1.4}$$

For mathematical convenience, we extend f by 0 outside the interval $[0,1]$. The temperature θ is an ignition temperature, below which no chemical reaction happens.

In the one-dimensional case, the problem is reduced to

$$\begin{cases} u'' - c_0 u' + f(u) = 0, \\ u(-\infty) = 0,\ u(+\infty) = 1. \end{cases} \tag{1.5}$$

There have been many works devoted to the solutions of (1.5). We refer to the pioneering articles of Kolmogorov, Petrovsky, and Piskunov [26] for biological models, Zeldovich and Frank-Kamenetskii [37] for planar flames, as well as other papers by Aronson and Weinberger [2], Fife [14], Fife and McLeod [15], and Kanel' [24]. The main result is the following: if the function f fulfils (1.4), then there exist a unique real c_0 and a unique function $U(\xi)$ (up to translation with respect to ξ) which are solutions of (1.5). The real c_0 is positive and the function U is increasing in ξ. We may suppose that $U(0) = \theta$.

In more recent papers, multidimensional curved flames in infinite cylinders $\Sigma = \mathbb{R}\times\omega = \{(x_1, y), x_1 \in \mathbb{R}, y \in \omega\}$, with smooth cross sections ω, have been investigated. In this case, the temperature $u(x, y)$ solves the equations

$$(1.6)\qquad \begin{cases} \Delta u - (c+\alpha(y))\dfrac{\partial u}{\partial x_1} + f(u) &= 0 \quad \text{in } \Sigma,\\ u(-\infty,\cdot) = 0,\ u(+\infty,\cdot) &= 1,\\ \dfrac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Sigma, \end{cases}$$

where ν is the outward unit normal to $\partial\omega$ and $\alpha(y)$ is the x_1-component of the given underlying flow (see Berestycki and Larrouturou [5]; Berestycki, Larrouturou, and Lions [6]; Berestycki and Nirenberg [9]; Vega [33]; Volpert and Volpert [34]; and Xin [36] under periodic conditions). If $\alpha(y) = \alpha_0$ does not depend on y, it is known that (1.6) has a unique solution and that it is planar; namely, it depends only on the longitudinal variable x_1. If the function $y \mapsto \alpha(y)$ is not constant, the solution of (1.6) still exists and is unique, but it is not planar anymore (such solutions correspond to curved flames). Nonplanar flames may also be observed in infinite cylinders under different physical conditions: Glangetas and Roquejoffre [18] and Margolis and Sivashinsky [28] proved that if the single partial differential equation in (1.6) was replaced with a system of two reaction-diffusion equations, then a bifurcation toward nonplanar flames might occur.

Let us now come back to the question of the existence of solutions (c, u) of the problem (1.1)–(1.3). If $\alpha = \pi/2$, the couple (c_0, U) is obviously a solution. The question of the existence of solutions if $\alpha < \pi/2$ has so far remained open. In this paper, we show the existence of a speed c and of a nonplanar—if $\alpha < \pi/2$—function u defined in $\mathbb{R}^2$, which are solutions of (1.1)–(1.3). As a consequence, nonplanar flames exist for the model (1.1)–(1.3) although this model involves only one reaction-diffusion equation (and not two such equations) and although the underlying flow is uniform.

In this paper, we prove two main theorems. The first one states the existence of a solution (c, u) of (1.1)–(1.3) for any angle $0 < \alpha \le \pi/2$. The second one deals with the question of the speed c's uniqueness.

Theorem 1.1. *Let f fulfill* (1.4) *("ignition temperature" profile). For any $\alpha \in (0, \pi/2]$, there exists a solution (c, u) of* (1.1)–(1.3)*, namely,*

$$\begin{cases} & \Delta u - c\dfrac{\partial u}{\partial y} + f(u) = 0 \quad \text{in } \mathbb{R}^2,\\ \forall \vec{k} \in \mathcal{C}(-\vec{e}_2, \alpha), & u(\lambda\vec{k}) \underset{\lambda\to+\infty}{\longrightarrow} 0,\\ \forall \vec{k} \in \mathcal{C}(\vec{e}_2, \pi-\alpha), & u(\lambda\vec{k}) \underset{\lambda\to+\infty}{\longrightarrow} 1, \end{cases}$$

such that

$$(1.7)\qquad c = \frac{c_0}{\sin\alpha}.$$

Furthermore, $0 < u < 1$*,* u *is symmetric with respect to the variable* x*, and* u *is decreasing in any direction* $\vec{k} \in \mathcal{C}(-\vec{e}_2, \alpha)$*. The following limiting conditions, which are stronger than* (1.2)–(1.3)*, also hold:*

$$u(\lambda \vec{k}') \to 0 \quad \text{as } \lambda \to +\infty \text{ and } \vec{k}' \to \vec{k} \in \mathcal{C}(-\vec{e}_2, \alpha), \tag{1.8}$$

$$u(\lambda \vec{k}') \to 1 \quad \text{as } \lambda \to +\infty \text{ and } \vec{k}' \to \vec{k} \in \mathcal{C}(\vec{e}_2, \pi - \alpha). \tag{1.9}$$

Finally, for each $\lambda \in (0,1)$*, the level set* $\{(x,y),\ u(x,y) = \lambda\}$ *is a curve* $\{y = \varphi_\lambda(x), x \in \mathbb{R}\}$ *and it has two asymptotic directions that are directed by the vectors* $(\pm \sin\alpha, -\cos\alpha)$*. If* $x_n \to -\infty$*, then the functions* $u_n(x,y) = u(x + x_n, y + \varphi_\lambda(x_n))$ *converge locally to the planar function* $U(y \sin\alpha - x\cos\alpha + U^{-1}(\lambda))$*.*

THEOREM 1.2. *Let* f *fulfill* (1.4) *and* α *be an angle in* $(0, \pi/2]$*. If* (c, u) *is a solution of* (1.1) *and* (1.8)–(1.9)*, then*

$$c = \frac{c_0}{\sin\alpha}.$$

We can see that the speed $c = c_0/\sin\alpha$ of the nonplanar flame (for $\alpha < \pi/2$) is greater than the speed c_0 of the planar flame. Furthermore, the angle α is all the smaller as the speed c is larger. That is physically meaningful since the curvature of the flame increases with the speed of the fuel flow. It is worth noticing that the formula (1.7) has been known for a long time and had been formally derived from the planar behavior of the flame, far away from its center, along the directions $(\pm\sin\alpha, -\cos\alpha)$. This formula had been used in experiments to find the planar speed c_0: indeed, the vertical speed c of the gases at the exit of the Bunsen burner being known, one can measure the angle α and the one-dimensional speed c_0 is then given by the formula $c_0 = c\sin\alpha$ (see [31], Williams [35]).

Hence, the results of Theorems 1.1 and 1.2 are not surprising. Nevertheless, they are the first rigorous analysis of the conical premixed Bunsen flames.

REMARK 1.3. *From Theorem* 1.1*, there is a continuum of solutions* $(c_0/\sin\alpha, u)$ *solving* (1.1) *and satisfying the simple asymptotic limits* $u(x, -\infty) = 0$ *and* $u(x, +\infty) = 1$ *for all* $x \in \mathbb{R}$*. This is in contrast with problem* (1.6) *mentioned above. However, if the limits* $u(x, -\infty) = 0$ *and* $u(x, +\infty) = 1$ *are uniform with respect to* $x \in \mathbb{R}$*, then* (c_0, U) *will be the unique solution of* (1.1) *up to translation in the variables* (x, y) *for* U *(see Hamel and Monneau* [21]*).*

Open questions.

(1) For each fixed angle $\alpha \in]0, \pi/2]$, do all the solutions u of (1.1)–(1.3) have the same profile? What kind of a priori monotonicity or symmetry properties do they fulfill? Are they stable for the evolution problem $\partial_t u = \Delta u - c\partial_y u + f(u)$? Answers to some of those questions are given in [21].

(2) Is there any solution (c, u) to (1.1)–(1.3) if $\alpha > \pi/2$? The answer is no and is given in [21].

(3) Is there any solution (c, u) to the free boundary problem equivalent to (1.1)–(1.3) and obtained in the limit of "high activation energies"? The answer is yes (see Hamel and Monneau [22]).

(4) Are there three-dimensional flames and, if so, are they necessarily invariant by rotation?

Structure of the paper. Section 2 is devoted to solving problems that are similar to (1.1)–(1.3) but are set in finite rectangles $[-a, a] \times [-a\cot\gamma_a, a\cot\gamma_a]$ where

γ_a is an angle close to α. For those problems, some a priori estimates about the speeds c_a and the functions u_a are established. A technical lemma, which is proved in the Appendix (section 5), is devoted to determining the behavior of the functions u_a near the corners of the rectangles. In section 3, we pass to the limit $a \to \infty$ in the whole plane and we determine the shape of the level sets of the limit function u by resorting to arguments of the "sliding method" type. In section 4, we prove Theorem 1.2.

REMARK 1.4. *The proof of Theorem* 1.1*, which is detailed in the next sections, actually allows us to get an independent result about the following problem set in an infinite strip* $\Sigma = \{(x,y) \in (-L,L) \times \mathbb{R}\}$ *with oblique Neumann boundary conditions:*

$$(1.10) \qquad \begin{cases} \Delta u - c\partial_y u + f(u) = 0 \text{ in } \Sigma, \\ \forall y \in \mathbb{R}, \quad \partial_\tau u(-L,y) = \partial_{\tilde\tau} u(L,y) = 0, \\ u(\cdot,-\infty) = 0, \ u(\cdot,+\infty) = 1, \end{cases}$$

where $\tau = (-\sin\alpha, -\cos\alpha)$ *and* $\tilde\tau = (\sin\alpha, -\cos\alpha)$*. Namely, with the same method as for Theorem* 1.1*, it follows that there exists a solution* (c,u) *to* (1.10) *such that the function* u *is nondecreasing in each direction* $\rho \in \overline{\mathcal{C}(\vec{e}_2,\alpha)}$*.*

2. Solving equivalent problems in finite rectangles. Let us set any real $a > 1/\alpha^2$ and $\gamma_a = \alpha - 1/\sqrt{a}$. The angle γ_a is such that $0 < \gamma_a < \alpha$, $\gamma_a \to \alpha$ and $a(\cot\gamma_a - \cot\alpha) \to +\infty$ as $a \to +\infty$. Let Σ_a be the bounded and open rectangle $\Sigma_a = (-a,a) \times (-a\cot\gamma_a, a\cot\gamma_a)$. Call $\tau = (-\sin\alpha, -\cos\alpha)$ and $\tilde\tau = (\sin\alpha, -\cos\alpha)$ (see Figure 2). When there is no confusion, γ_a is often replaced with γ.

In this section, we focus on the questions of the existence and the uniqueness as well as on a priori estimates of the solutions (c_a, u_a) to the following problem:

$$(2.1) \qquad \begin{cases} \Delta u_a - c_a\partial_y u_a + f(u_a) = 0 \text{ in } \Sigma_a, \\ \forall x \in [-a,a], \quad u_a(x,-a\cot\gamma_a) = 0, \ u_a(x,a\cot\gamma_a) = 1, \\ \forall y \in (-a\cot\gamma_a, a\cot\gamma_a), \quad \dfrac{\partial u_a}{\partial\tau}(-a,y) = \dfrac{\partial u_a}{\partial\tilde\tau}(a,y) = 0 \end{cases}$$

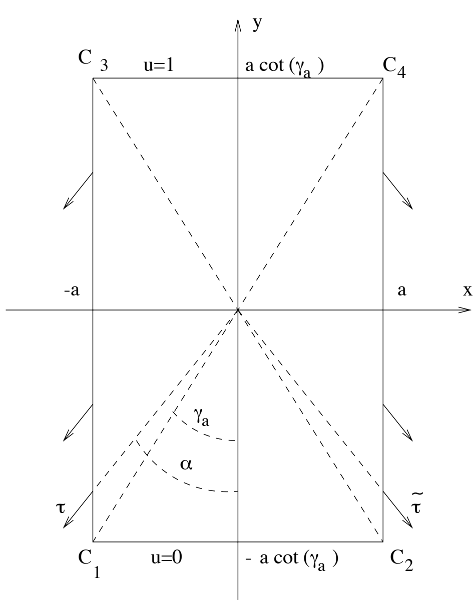

FIG. 2. *The rectangle* Σ_a.

under the following normalization condition:

$$\max_{\substack{y=-\cot\alpha\ |x|\\ -a\le x\le a}} u_a(x,y)=\theta. \tag{2.2}$$

2.1. Existence of solutions of (2.1)–(2.2) and a priori bounds for the speeds c_a.

2.1.1. On the solutions u_c of (2.1). Let c be any fixed real. Let us call $(C_i)_{1\le i\le 4}$ the four corners of Σ_a: $C_1=(-a,-a\cot\gamma)$, $C_2=(a,-a\cot\gamma)$, $C_3=(-a,a\cot\gamma)$, $C_4=(a,a\cot\gamma)$ (see Figure 2) and set $\tilde\Sigma_a=\overline{\Sigma_a}\setminus\cup_{i=1}^4\{C_i\}$.

Now consider the following Dirichlet–Neumann problem:

$$\begin{cases}\Delta u-c\partial_y u+f(u)=0 & \text{in } \Sigma_a,\\ \forall x\in[-a,a],\quad u(x,-a\cot\gamma)=0,\ u(x,a\cot\gamma)=1, & \\ \forall y\in(-a\cot\gamma,a\cot\gamma),\quad \partial_\tau u(-a,y)=\partial_{\tilde\tau}u(a,y)=0. & \end{cases} \tag{2.3}$$

This problem is the same as (2.1), but the speed c is given in (2.3) and only the function u is unknown. The following three lemmas are similar to some of the results in a paper by Berestycki and Nirenberg [7]. The proofs, which will be used several times in the sequel, are written for the sake of completeness.

Lemma 2.1. *For each speed $c\in\mathbb{R}$, we have that problem* (2.3) *has a solution u_c in $\cap_{p>1}W^{2,p}_{loc}(\tilde\Sigma_a)\cap C(\overline{\Sigma_a})$, where $C(\overline{\Sigma_a})$ is the space of all continuous functions in $\overline{\Sigma_a}$.*

Proof. Let $(\Sigma_{a,\varepsilon})_{\varepsilon>0}$ be a sequence of bounded and smooth domains such that, for each $\varepsilon>0$,

$$\Sigma_a\setminus\bigcup_{i=1}^4 B(C_i,\varepsilon)\ \subset\ \Sigma_{a,\varepsilon}\ \subset\ \Sigma_a,$$

where $B(C_i,\varepsilon)$ denotes the open ball centered on the point C_i with radius ε. Let $\varepsilon>0$ be small enough. Consider a smooth vector field $\rho_\varepsilon(x,y)$ defined on $\partial\Sigma_{a,\varepsilon}$ such that $\rho_\varepsilon\cdot\nu_\varepsilon\ge 0$ on $\partial\Sigma_{a,\varepsilon}$ (where ν_ε is the outward unit normal to $\partial\Sigma_{a,\varepsilon}$) $\rho_\varepsilon=\tau$ on $\{-a\}\times(-a\cot\gamma+\varepsilon,a\cot\gamma-\varepsilon)$, $\rho_\varepsilon=\tilde\tau$ on $\{a\}\times(-a\cot\gamma+\varepsilon,a\cot\gamma-\varepsilon)$, and $\rho_\varepsilon=\vec 0$ on $(-a+\varepsilon,a-\varepsilon)\times\{\pm a\cot\gamma\}$. Let $\sigma_{0,\varepsilon}(x,y)$ be a smooth nonnegative function defined on $\partial\Sigma_{a,\varepsilon}$ such that $\sigma_{0,\varepsilon}=1$ on $\partial\Sigma_{a,\varepsilon}\cap\{y\le -a\cot\gamma+\varepsilon\}$ and $\sigma_{0,\varepsilon}=0$ on $\partial\Sigma_{a,\varepsilon}\cap\{y\ge -a\cot\gamma+2\varepsilon\}$. Last, let $\sigma_{1,\varepsilon}$ be a smooth nonnegative function defined on $\partial\Sigma_{a,\varepsilon}$ such that $\sigma_{1,\varepsilon}=1$ on $\partial\Sigma_{a,\varepsilon}\cap\{y\ge a\cot\gamma-\varepsilon\}$ and $\sigma_{1,\varepsilon}=0$ on $\partial\Sigma_{a,\varepsilon}\cap\{y\le a\cot\gamma-2\varepsilon\}$. For each $\varepsilon>0$ small enough, the problem

$$\begin{cases}\Delta u_\varepsilon-c\partial_y u_\varepsilon+f(u_\varepsilon)=0 & \text{in } \Sigma_{a,\varepsilon},\\ \rho_\varepsilon\cdot\nabla u+\sigma_{0,\varepsilon}u+\sigma_{1,\varepsilon}(u-1)=0 & \text{on } \partial\Sigma_{a,\varepsilon}\end{cases}$$

has a solution u_ε such that $0\le u_\varepsilon\le 1$ since 0 and 1, respectively, are sub- and supersolutions (see Berestycki and Nirenberg [7]).

From the standard elliptic estimates up to the boundary (Agmon, Douglis, and Nirenberg [1]; Gilbarg and Trudinger [17]), up to extraction of some subsequence, the functions u_ε approach a function $u_c\in\bigcap_{p>1}W^{2,p}_{loc}(\tilde\Sigma_a)\cap C_{loc}(\tilde\Sigma_a)$ as $\varepsilon\to 0$. The function u_c is a solution of

$$\begin{cases}\Delta u_c-c\partial_y u_c+f(u_c)=0 & \text{in } \Sigma_a,\\ \forall x\in(-a,a),\quad u_c(x,-a\cot\gamma)=0,\ u_c(x,a\cot\gamma)=1, & \\ \forall y\in(-a\cot\gamma,a\cot\gamma),\quad \partial_\tau u_c(-a,y)=\partial_{\tilde\tau}u_c(a,y)=0. & \end{cases} \tag{2.4}$$

Furthermore, we claim that, for each $i \in \{1, \dots, 4\}$, there exists a function $\overline{v}_i$ defined in a neighborhood V_i of the corner C_i such that $\overline{v}_i(C_i) = 0$ and, for all $\varepsilon > 0$ small enough,

$$\begin{array}{llll} \text{if } i = 1 \text{ or } 2, & u_\varepsilon(x,y) & \le \overline{v}_i(x,y) & \\ \text{if } i = 3 \text{ or } 4, & 1 - u_\varepsilon(x,y) & \le \overline{v}_i(x,y) & \end{array} \text{ in } V_i \cap \overline{\Sigma_{a,\varepsilon}}. \tag{2.5}$$

The proof of this fact is temporarily postponed and will be given in Remark 5.2 in section 5.

As a consequence, the function u_c can be extended by continuity at the four corners C_i of Σ_a. In other words, $u_c \in \cap_{p>1} W^{2,p}_{loc}(\tilde{\Sigma}_a) \cap C(\overline{\Sigma_a})$. From the strong maximum principle and the Hopf lemma, it also follows that $0 < u_c < 1$ in $[-a,a] \times (-a\cot\gamma, a\cot\gamma)$. □

LEMMA 2.2. *The function u_c is increasing in y and it is the unique solution of (2.3) in $\cap_{p>1} W^{2,p}_{loc}(\tilde{\Sigma}_a) \cap C(\overline{\Sigma_a})$. Furthermore, if f is of class C^1, then $\partial_y u_c > 0$ in $\tilde{\Sigma}_a$.*

Proof. It is based on the sliding method (see [7]). Let u be any solution of (2.3) in $\cap_{p>1} W^{2,p}_{loc}(\tilde{\Sigma}_a) \cap C(\overline{\Sigma_a})$. For any $\lambda \in (0, 2a\cot\gamma)$, let v^λ be the function defined by $v^\lambda(x,y) = u(x, y-\lambda) - u(x,y)$ in the set

$$\Sigma_a^\lambda = (-a,a) \times (-a\cot\gamma + \lambda, a\cot\gamma). \tag{2.6}$$

Since u is uniformly continuous on the compact set $\overline{\Sigma_a}$ and since $u(\cdot, -a\cot\gamma) = 0$, $u(\cdot, a\cot\gamma) = 1$, there exists $\varepsilon > 0$ small enough such that v^λ is negative in $\overline{\Sigma_a^\lambda}$ for all λ in $[2a\cot\gamma - \varepsilon, 2a\cot\gamma)$.

Let us now decrease λ. Suppose that there exists $\lambda^* > 0$ such that $v^\lambda < 0$ in $\overline{\Sigma_a^\lambda}$ for all $\lambda \in (\lambda^*, 2a\cot\gamma)$ and $v^{\lambda^*} \le 0$ in $\overline{\Sigma_a^{\lambda^*}}$ with equality somewhere at a point $(\overline{x}, \overline{y}) \in \overline{\Sigma_a^{\lambda^*}}$. Since $0 < u < 1$ in $[-a,a] \times (-a\cot\gamma, a\cot\gamma)$, the function v^{λ^*} is negative at the "bottom" $[-a,a] \times \{-a\cot\gamma + \lambda^*\}$ of the boundary of $\Sigma_a^{\lambda^*}$. Similarly, the function v^{λ^*} is negative at the "top" $[-a,a] \times \{a\cot\gamma\}$ of the boundary of $\Sigma_a^{\lambda^*}$. We also have $\partial_\tau v^{\lambda^*}(-a,y) = \partial_{\tilde{\tau}} v^{\lambda^*}(a,y) = 0$ for all $y \in (-a\cot\gamma + \lambda^*, a\cot\gamma)$. The nonpositive function v^{λ^*} satisfies the elliptic equation

$$\Delta v^{\lambda^*} - c\partial_y v^{\lambda^*} + c(x,y) v^{\lambda^*} = 0 \ \text{ in } \Sigma_a^{\lambda^*},$$

where the function $c(x,y)$ is bounded in $\Sigma_a^{\lambda^*}$ because of the Lipschitz continuity of f. Since $v^{\lambda^*}(\overline{x}, \overline{y}) = 0$ at a point $(\overline{x}, \overline{y}) \in \overline{\Sigma_a^{\lambda^*}}$, we then conclude from the strong maximum principle (if $-a < \overline{x} < a$) or from the Hopf lemma (if $\overline{x} = \pm a$) that $v^{\lambda^*} \equiv 0$ in $\overline{\Sigma_a^{\lambda^*}}$. That is ruled out by the boundary conditions on $[-a,a] \times \{-a\cot\gamma + \lambda^*, a\cot\gamma\}$.

Hence, there is no such $\lambda^* > 0$. We finally conclude that

$$\forall 0 < \lambda < 2a\cot\gamma, \quad u^\lambda(x,y) = u(x, y-\lambda) < u(x,y) \ \text{ in } \overline{\Sigma_a^\lambda}.$$

This yields that for any $x \in [-a,a]$, the function $y \mapsto u(x,y)$ is strictly increasing with respect to $y \in [-a\cot\gamma, a\cot\gamma]$.

If f is of class C^1, we can differentiate the equation satisfied by u. From the strong maximum principle and the Hopf lemma, it follows that $\partial_y u > 0$ in $\tilde{\Sigma}_a$.

The second part of Lemma 2.2, namely, the uniqueness of the solution u_c of (2.3) in $\cap_{p>1} W^{2,p}_{loc}(\tilde{\Sigma}_a) \cap C(\overline{\Sigma_a})$, could be proved in the same way. Indeed, if there were two solutions u_c and u'_c, we would find as above that $u_c(x, y-\lambda) < u'_c(x,y)$ in $\overline{\Sigma_a^\lambda}$ for all $\lambda \in (0, 2a\cot\gamma)$, whence $u_c \le u'_c$ in $\overline{\Sigma_a}$. Changing u_c and u'_c, we have $u'_c \le u_c$ and finally $u_c = u'_c$. ◻

COROLLARY 2.3. *For each c, the function u_c is symmetric with respect to x.*

Proof. Indeed, if u_c denotes the unique solution of (2.3), the function $\tilde{u}(x,y) = u_c(-x,y)$ is also a solution. By uniqueness, we have $\tilde{u} = u_c$. ◻

LEMMA 2.4. *The functions u_c are decreasing and continuous, with respect to c, in the spaces $W^{2,p}_{loc}(\tilde{\Sigma}_a) \cap C(\overline{\Sigma_a})$ in the following sense: if $c < c'$, then $u_c > u_{c'}$ in $[-a,a] \times (-a\cot\gamma, a\cot\gamma)$ and if $c \to c_0$, then $u_c \to u_{c_0}$ in $\cap_{p>1} W^{2,p}_{loc}(\tilde{\Sigma}_a) \cap C(\overline{\Sigma_a})$.*

Proof. Choose any c and c' such that $c < c'$. We have to prove that $u_c > u_{c'}$ in $[-a,a] \times (-a\cot\gamma, a\cot\gamma)$. For each $0 < \lambda < 2a\cot\gamma$, we define the function $v^\lambda(x,y) = u_{c'}(x, y-\lambda) - u_c(x,y)$ in Σ_a^λ (see definition (2.6)).

If λ is close enough to $2a\cot\gamma$, we have $v^\lambda < 0$ in $\overline{\Sigma_a^\lambda}$ thanks to the boundary conditions fulfilled by u_c and $u_{c'}$. Let us now suppose that there exists $\lambda^* > 0$ such that $v^\lambda < 0$ in $\overline{\Sigma_a^\lambda}$ for all $\lambda \in (\lambda^*, 2a\cot\gamma)$ and $v^{\lambda^*} \le 0$ with equality somewhere in $\overline{\Sigma_a^{\lambda^*}}$. The function v^{λ^*} satisfies

$$(2.7) \qquad \begin{cases} \Delta v^{\lambda^*} - c\partial_y v^{\lambda^*} + c(x,y)v^{\lambda^*} &= (c'-c)\partial_y u_{c'}(x, y-\lambda^*) \ \text{ in } \Sigma_a^{\lambda^*}, \\ \partial_\tau v^{\lambda^*}(-a,y) = \partial_{\tilde{\tau}} v^{\lambda^*} &= 0 \quad \forall y \in (-a\cot\gamma + \lambda^*, a\cot\gamma) \end{cases}$$

for a bounded function $c(x,y)$. On the one hand, since $c < c'$ and $\partial_y u_{c'} \ge 0$ (from the first part of Lemma 2.2), it follows from the strong maximum principle and the Hopf lemma that $v^{\lambda^*} \equiv 0$ in $\overline{\Sigma_a^{\lambda^*}}$. On the other hand, since $0 < u_c, u_{c'} < 1$ in $[-a,a] \times (-a\cot\gamma, a\cot\gamma)$, we have $v^{\lambda^*} < 0$ on $[-a,a] \times \{-a\cot\gamma + \lambda^*, a\cot\gamma\}$. That eventually leads to a contradiction.

Hence, for all $\lambda \in (0, 2a\cot\gamma)$, we have

$$v^\lambda = u_{c'}(x, y-\lambda) - u_c(x,y) < 0 \text{ in } \overline{\Sigma_a^\lambda}.$$

Then, $u_c \ge u_{c'}$ in $\overline{\Sigma_a}$. Since $v^0 = u_{c'} - u_c$ satisfies equation (2.7), the strong maximum principle and the Hopf lemma yield that $u_c > u_{c'}$ in $[-a,a] \times (-a\cot\gamma, a\cot\gamma)$.

Now, consider a sequence (c_n) such that $c_n \to c_0 \in \mathbb{R}$ as $n \to +\infty$. From the standard elliptic estimates up to the boundary, and up to extraction of some subsequence, the functions u_{c_n} approach a function $\tilde{u}_{c_0} \in \cap_{p>1} W^{2,p}_{loc}(\tilde{\Sigma}_a) \cap C_{loc}(\tilde{\Sigma}_a)$. The function $\tilde{u}_{c_0}$ is a solution of (2.4) with the speed c_0. Furthermore, for each $i \in \{1, \dots, 4\}$, there exists a function $\overline{v}_i$ defined in a neighborhood V_i of the corner C_i, such that $\overline{v}_i(C_i) = 0$ and, for n large enough,

$$(2.8) \qquad \begin{array}{lrl} \text{if } i = 1 \text{ or } 2, & u_{c_n}(x,y) & \le \overline{v}_i(x,y) \\ \text{if } i = 3 \text{ or } 4, & 1 - u_{c_n}(x,y) & \le \overline{v}_i(x,y) \end{array} \quad \text{in } V_i \cap \overline{\Sigma_a}$$

(see Remark 5.2). Hence, the function $\tilde{u}_{c_0}$ can be extended by continuity at the four corners C_i. As a consequence, $\tilde{u}_{c_0} = u_{c_0}$. Furthermore, since the functions u_{c_n} approach u_{c_0} in any compact subset of $\tilde{\Sigma}_a$, the above estimates around the four corners C_i also imply that u_{c_n} approach u_{c_0} uniformly in $\overline{\Sigma_a}$. Finally, since the limit function u_{c_0} is unique, it follows that the whole sequence (u_{c_n}) approaches u_{c_0} as $n \to +\infty$. ◻

2.1.2. Estimating the speeds. In this subsection, we aim at establishing some a priori estimates for the speeds c_a of the possible solutions (c_a, u_a) of (2.1)–(2.2).

We first need some preliminary results about the speeds of some one-dimensional traveling fronts. Remember that the function f has been extended by 0 outside $[0,1]$.

Let $f'_-(1) = \lim_{t\to 1,\ t<1} \frac{f(t)}{t-1}$. For each $0 < \eta < \min(1-\theta, |f'_-(1)|)$, let f_η be a C^1 function in $[0,1]$, fulfilling (1.4) with the ignition temperature $\theta + \eta$, such that $f'_\eta(1) = f'_-(1) + \eta$, $f - \eta \le f_\eta \le f$ in $[0,1]$, and $f_\eta < f$ in $(\theta, 1)$. As for f, we also extend f_η by 0 outside $[0,1]$. From the results in [2], [9], [15] and [24], there exists a unique real c_0^η and a unique function u_η solving

$$\begin{cases} u''_\eta - c_0^\eta u'_\eta + f_\eta(u_\eta) = 0 \qquad \text{in } \mathbb{R}, \\ u_\eta(-\infty) = -\eta,\ u_\eta(0) = \theta,\ u_\eta(+\infty) = 1. \end{cases}$$

Moreover, $u'_\eta > 0$ in $\mathbb{R}$. With the same arguments as in the paper by Berestycki and Nirenberg [9], it also follows that $c_0^\eta \overset{\le}{\to} c_0$ as $\eta \to 0$ (remember that c_0 is the unique speed for which (1.5) has a solution).

LEMMA 2.5. *Under the above notation, there exists a real $a_1(\eta) > 0$ such that if $a \ge a_1(\eta)$ and if $c < c_0^\eta/\sin\alpha$, then $\theta < \max_{\substack{y=-\cot\alpha\ |x| \\ |x|\le a}} u_c$.*

Proof. Assume that c is such that $c < c_0^\eta/\sin\alpha$. Let u_c be the solution of (2.3) and set $v(x,y) = u_\eta(\cos\alpha\ x + \sin\alpha\ y)$ in Σ_a. We want to prove that if a is large enough, then this function v is a subsolution of problem (2.3).

We have

$$\begin{aligned} \Delta v - c\partial_y v + f(v) \quad &= u''_\eta - c\sin\alpha\ u'_\eta + f(u_\eta) \\ &= (c_0^\eta - c\sin\alpha) u'_\eta(\cos\alpha\ x + \sin\alpha\ y) + f(u_\eta) - f_\eta(u_\eta) \\ &> 0 \ \text{ in } \Sigma_a \end{aligned}$$

since $c < c_0^\eta/\sin\alpha$, $u'_\eta > 0$, and $f \ge f_\eta$. Furthermore, for all $y \in (-a\cot\gamma_a, a\cot\gamma_a)$, we can see that

$$\partial_\tau v(-a,y) = -2\sin\alpha\cos\alpha\ u'_\eta(-a\cos\alpha + \sin\alpha\ y) \le 0$$

and that $\partial_{\tilde\tau} v(a,y) = 0$. At the "top"of the boundary of Σ_a, we have $v(x, a\cot\gamma_a) < 1$ for all $x \in [-a,a]$. At the "bottom" of the boundary of Σ_a, the function v is equal to

$$v(x, -a\cot\gamma_a) = u_\eta(\cos\alpha\ x - a\cot\gamma_a\ \sin\alpha).$$

Since $|x| \le a$, it follows that

$$\cos\alpha\ x - a\cot\gamma_a \sin\alpha \le (\cos\alpha - \cot\gamma_a \sin\alpha)\ a \to -\infty \text{ as } a \to +\infty$$

since $\gamma_a = \alpha - 1/\sqrt{a}$ for $a > 1/\alpha^2$. On the other hand, the function u_η is increasing and $u_\eta(\xi) \to -\eta$ as $\xi \to -\infty$. Consequently, there exists a real $a_1(\eta)$ such that

$$(a \ge a_1(\eta)) \implies (\forall x \in [-a,a],\ v(x, -a\cot\gamma) < 0).$$

Hence, if $c < c_0^\eta/\sin\alpha$ and if $a \ge a_1(\eta)$, the function v is a subsolution of problem (2.3). Remember now that the function u_c is a solution of (2.3). As in the proof of the monotonicity result in Lemma 2.2, we can compare the functions v and u_c by using a sliding method. We would find that $v < u_c$ in Σ_a. This yields that $v(0,0) = \theta < u_c(0,0)$, whence $\theta < \max_{\substack{y=-\cot\alpha\ |x| \\ |x|\le a}} u_c$. That completes the proof of Lemma 2.5. □

The next lemma states that if the speed c is large enough, then the solution u_c of (2.3) will be below θ on the set $\{y = -\cot\alpha\ |x|,\ |x| \le a\}$. Before doing that, we need a few auxiliary notation. For any $\varepsilon \in (0,\theta)$, let f^ε be a C^1 function in

$[0, 1+\varepsilon]$ such that $f^\varepsilon \equiv 0$ in $(-\infty, \theta-\varepsilon] \cup [1+\varepsilon, +\infty)$, $f^\varepsilon > 0$ in $(\theta-\varepsilon, 1+\varepsilon)$, $(f^\varepsilon)'_-(1+\varepsilon) := \lim_{t\to 1+\varepsilon,\ t<1+\varepsilon} \frac{f^\varepsilon(t)}{t-1-\varepsilon}$ exists and is negative. In other words, f^ε fulfills the assertion (1.4) on the interval $[0, 1+\varepsilon]$ with the ignition temperature $\theta-\varepsilon$. Moreover, one assumes that $f \le f^\varepsilon \le f+\varepsilon$ in $\mathbb{R}$ and $f < f^\varepsilon$ in $[\theta, 1]$. From the results in [2], [9], [15] and [24], there exists a unique real $\tilde{c}_0^\varepsilon$ and a unique function u^ε defined in $\mathbb{R}$ such that

$$\begin{cases} (u^\varepsilon)'' - \tilde{c}_0^\varepsilon (u^\varepsilon)' + f^\varepsilon(u^\varepsilon) &= 0 \ \text{ in } \mathbb{R}, \\ u^\varepsilon(-\infty) = 0,\ u^\varepsilon(0) = \theta,\ u^\varepsilon(+\infty) &= 1+\varepsilon. \end{cases}$$

Moreover, one has $(u^\varepsilon)' > 0$ in $\mathbb{R}$ and $\tilde{c}_0^\varepsilon \overset{\geq}{\to} c_0$ as $\varepsilon \to 0$ (see [9]).

LEMMA 2.6. *There exists a real $a_2(\varepsilon)$ such that if $a \ge a_2(\varepsilon)$ and if $c > \tilde{c}_0^\varepsilon/\sin^2\alpha$, then $\theta > \max_{\substack{y=-\cot\alpha\ |x| \\ |x|\le a}} u_c$.*

Proof. Let c be a real such that $c > \tilde{c}_0^\varepsilon/\sin^2\alpha$. Let us set

$$\beta = \frac{3\cot\alpha}{2(c - \tilde{c}_0^\varepsilon/\sin^2\alpha)}$$

and choose $a > \beta$. Let us call φ the function defined in $\mathbb{R}$ by

$$\begin{cases} \varphi(x) = \dfrac{\cot\alpha}{8\beta^3}x^4 - \dfrac{3\cot\alpha}{4\beta}x^2 & \text{if } |x| \le \beta, \\ \varphi(x) = -|x|\cot\alpha + \dfrac{3}{8}\beta\cot\alpha & \text{if } \beta \le |x| \le a. \end{cases}$$

It is easy to see that the function φ is concave, is of class C^2 in $\mathbb{R}$, and that $|\varphi'(x)| \le \cot\alpha$, $|\varphi''(x)| \le c - \tilde{c}_0^\varepsilon/\sin^2\alpha$.

Let us now define the function $v(x,y) = u^\varepsilon(y - \varphi(x))$ in Σ_a and check that this function v is a supersolution of (2.3) for a large enough. We have

$$\partial_y v = (u^\varepsilon)'(y - \varphi(x))$$

and $$\Delta v = (1 + \varphi'(x)^2)(u^\varepsilon)''(y - \varphi(x)) - \varphi''(x)(u^\varepsilon)'(y - \varphi(x)).$$

Hence,

$$\begin{aligned} \Delta v - c\partial_y v + f(v) &= (1+\varphi'(x)^2)(u^\varepsilon)''(y-\varphi(x)) \\ &\quad -(c+\varphi''(x))(u^\varepsilon)'(y-\varphi(x)) + f(u^\varepsilon(y-\varphi(x))) \\ &= [\tilde{c}_0^\varepsilon(1+\varphi'(x)^2) - c - \varphi''(x)]\ (u^\varepsilon)'(y-\varphi(x)) \\ &\quad -\varphi'(x)^2 f^\varepsilon(u^\varepsilon(y-\varphi(x))) \\ &\quad +f(u^\varepsilon(y-\varphi(x))) - f^\varepsilon(u^\varepsilon(y-\varphi(x))). \end{aligned}$$

On the one hand, we know that $(u^\varepsilon)' > 0$ and that $0 \le f \le f^\varepsilon$. On the other hand, in view of the definition of φ, we infer that

$$\forall x \in \mathbb{R}, \quad \tilde{c}_0^\varepsilon(1+\varphi'(x)^2) - c - \varphi''(x) \le 0.$$

It follows that

$$\Delta v - c\partial_y v + f(v) \le 0 \ \text{ in } \Sigma_a.$$

Furthermore, one has, for all $y \in (-a\cot\gamma_a, a\cot\gamma_a)$,

$$\begin{aligned} \partial_\tau v(-a, y) &= (\sin\alpha\ \varphi'(-a) - \cos\alpha)\ (u^\varepsilon)'(y - \varphi(-a)) \\ &= 0 \end{aligned}$$

since $\varphi'(-a) = \cot\alpha$. Similarly, $\partial_{\tilde{\tau}} v(a,y) = 0$ for all $y \in (-a\cot\gamma_a, a\cot\gamma_a)$.

At the "bottom" of the boundary of Σ_a, one has $v(x, -a\cot\gamma_a) \geq 0$ for all $x \in [-a,a]$. At the "top" of the boundary of Σ_a, $v(x, a\cot\gamma_a) = u^\varepsilon(a\cot\gamma_a - \varphi(x))$ for all $x \in [-a,a]$ and

$$\forall x \in [-a,a], \quad |\varphi(x)| \leq a\cot\alpha - \frac{3}{8}\beta\cot\alpha \leq a\cot\alpha.$$

Since $(\cot\gamma_a - \cot\alpha)a \to +\infty$ as $a \to +\infty$ and since $u^\varepsilon(+\infty) = 1+\varepsilon$, it then follows that there exists a real $a_2(\varepsilon) > \beta$ such that if $a \geq a_2(\varepsilon)$ then $v(x, a\cot\gamma_a) > 1$ for all $x \in [-a,a]$.

Let us now choose $a \geq a_2(\varepsilon)$. The function v is a supersolution of problem (2.3). With the same arguments as in Lemma 2.2, we finally conclude that $v > u_c$ in $[-a,a] \times (-a\cot\gamma_a, a\cot\gamma_a)$. In particular, $u_c < v$ in $\{y = -|x|\cot\alpha,\ |x| \leq a\}$ since $0 < \gamma_a < \alpha$. As a consequence,

$$\max_{\substack{y=-\cot\alpha\ |x| \\ |x|\leq a}} u_c < \max_{\substack{y=-\cot\alpha\ |x| \\ |x|\leq a}} v = \max_{|x|\leq a} u^\varepsilon(-\cot\alpha\ |x| - \varphi(x)) = u_\varepsilon(0) = \theta. \qquad \square$$

We complete this section with the following proposition.

Proposition 2.7. *If ε and $\eta > 0$ are small enough, then there is a real $a_0(\eta,\varepsilon) \geq A_0$ such that, for any $a \geq a_0(\eta,\varepsilon)$, problem* (2.1)–(2.2) *has a unique solution (c_a, u_a). Furthermore, one has*

$$c_0^\eta / \sin\alpha \leq c_a \leq \tilde{c}_0^\varepsilon / \sin^2\alpha.$$

Proof. Proposition 2.7 is an immediate consequence of Lemmas 2.4, 2.5, and 2.6. Indeed, let us choose $\varepsilon > 0$ and $\eta > 0$ small enough and take $a_0(\eta,\varepsilon) = \max\,(a_1(\eta), a_2(\varepsilon))$: for $a \geq a_0(\eta,\varepsilon)$, if $c < c_0^\eta/\sin\alpha$, then $\max_{\substack{y=-\cot\alpha\ |x| \\ |x|\leq a}} u_c > \theta$ from Lemma 2.5 and if $c > \tilde{c}_0^\varepsilon/\sin^2\alpha$, then $\max_{\substack{y=-\cot\alpha\ |x| \\ |x|\leq a}} u_c < \theta$ from Lemma 2.6. From Lemma 2.4, the functions u_c are continuously increasing with respect to c. Hence, problem (2.1)–(2.2) has a unique solution (c_a, u_a) and $c_0^\eta/\sin\alpha \leq c_a \leq \tilde{c}_0^\varepsilon/\sin^2\alpha$. $\square$

2.2. Monotonicity properties of the solutions u_a. From Proposition 2.7, we assume from now on that a is large enough ($a \geq a(\eta_0, \varepsilon_0)$, where $\eta_0 > 0$, $\varepsilon_0 > 0$ are small enough) such that (2.1)–(2.2) has a unique solution (c_a, u_a). When there is no ambiguity, we call this solution (c,u). Set $\Sigma_a^- = (-a,0) \times (-a\cot\gamma_a, a\cot\gamma_a)$ and $\Sigma_a^+ = (0,a) \times (-a\cot\gamma_a, a\cot\gamma_a)$. Remember that C_i ($i = 1,\ldots,4$) are the four corners of the rectangle Σ_a.

Proposition 2.8. *For a large enough, the unique solution (c_a, u_a) of* (2.1)–(2.2) *is such that*

(i) *for any $\rho = (\cos\beta, \sin\beta)$ with $\pi/2 - \alpha \leq \beta \leq \pi$, one has $\partial_\rho u \geq 0$ in $\overline{\Sigma_a^-} \setminus \{C_1, C_3\}$;*

(ii) *for any $\rho = (\cos\beta, \sin\beta)$ with $0 \leq \beta \leq \pi/2 + \alpha$, one has $\partial_\rho u \geq 0$ in $\overline{\Sigma_a^+} \setminus \{C_2, C_4\}$.*

From this proposition we immediately get the following corollary.

Corollary 2.9. (i) *The function u is nonincreasing with respect to x in $\overline{\Sigma_a^-}$ and nondecreasing with respect to x in $\overline{\Sigma_a^+}$.*

(ii) *For any nonzero vector $\rho \in \overline{\mathcal{C}(\vec{e}_2, \alpha)}$, one has*

$$\partial_\rho u \geq 0 \quad in \quad \tilde{\Sigma}_a = \overline{\Sigma_a} \backslash \{C_1, C_2, C_3, C_4\}.$$

Proof of Proposition 2.8. By symmetry with respect to x and by continuity, it is sufficient to prove that $\partial_\rho u \geq 0$ in Σ_a^- for any vector $\rho = (\cos\beta, \sin\beta)$ such that $\pi/2 - \alpha < \beta < \pi$. Let ρ be such a vector.

Let us temporarily consider the case where the function f is of class C^1 in $[0,1]$. Let $z = (x,y)$ be the generic notation for the points of $\overline{\Sigma_a}$. For $\varepsilon > 0$ small enough, we are going to compare the functions $u(z)$ and $u(z+\varepsilon\rho)$ in the rectangular domain $R_\varepsilon = \overline{\Sigma_a^- \cap (\Sigma_a^- - \varepsilon\rho)}$ (see Figure 3).

Let us first show that

$$u(z) < u(z+\varepsilon\rho) \quad \text{on } \partial R_\varepsilon \tag{2.9}$$

for ε small enough. Indeed, consider first the "top" and "bottom" boundaries of R_ε. Set $\vec{e}_1 = (1,0)$. If $\rho \cdot \vec{e}_1 > 0$ (as drawn in Figure 3), then those parts of ∂R_ε are $[-a, -\varepsilon\rho\cdot\vec{e}_1] \times \{-a\cot\gamma\}$ and $[-a, -\varepsilon\rho\cdot\vec{e}_1] \times \{a\cot\gamma - \varepsilon\rho\cdot\vec{e}_2\}$. Since $\rho\cdot\vec{e}_2 > 0$, inequality (2.9) is satisfied there because $u = 0$ (resp., $u = 1$) on $[-a,a] \times \{-a\cot\gamma\}$ (resp., $[-a,a] \times \{a\cot\gamma\}$) and because $0 < u < 1$ in $[-a,a] \times (-a\cot\gamma, a\cot\gamma)$. The other case $\rho\cdot\vec{e}_1 \geq 0$ can be treated similarly.

On the other hand, on $\{0\} \times [-a\cot\gamma, a\cot\gamma]$, we have $\partial_y u > 0$ from Lemma 2.2 (remember that f is assumed here to be of class C^1) and $\partial_x u = 0$ since u is symmetric with respect to x (from Corollary 2.3). Hence, $\partial_\rho u > 0$ on the compact set $\{0\} \times [-a\cot\gamma, a\cot\gamma]$. Since the function $\partial_\rho u$ is uniformly continuous in a neighborhood of $\{0\} \times [-a\cot\gamma, a\cot\gamma]$, it follows from the finite increment theorem that there exists

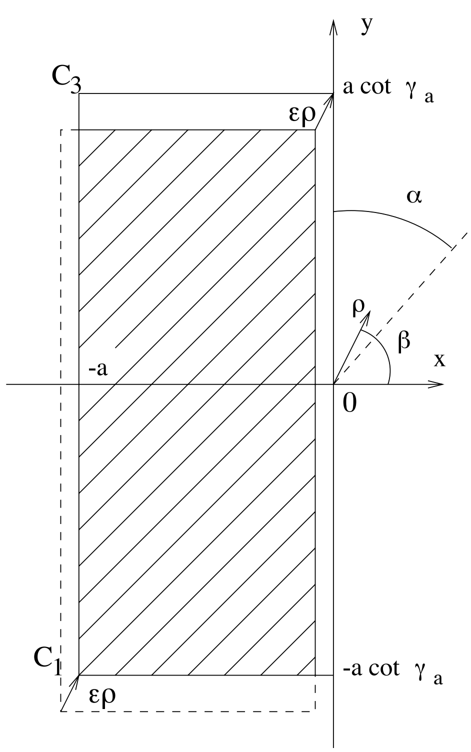

FIG. 3. *The rectangle R_ε.*

a real $\tilde{\varepsilon} > 0$ such that, if $0 < \varepsilon < \tilde{\varepsilon}$, then (2.9) is true on the right-hand side boundary of R_ε, namely, $\{-\varepsilon\rho \cdot \vec{e}_1\} \times [-a \cot \gamma, a \cot \gamma - \varepsilon\rho \cdot \vec{e}_2]$ if $\rho \cdot \vec{e}_1 \geq 0$ (as in Figure 3) or $\{0\} \times [-a \cot \gamma, a \cot \gamma - \varepsilon\rho \cdot \vec{e}_2]$ if $\rho \cdot \vec{e}_1 \leq 0$.

We now have to deal with the behavior of the function u on the left-hand boundary of R_ε and especially around the corners C_1 and C_3. We shall use the following lemma (notice that in this lemma the function f does not need to be of class C^1 in $[0, 1]$).

LEMMA 2.10. *For each $i = 1$ or 3, there exist a neighborhood V_i of C_i and a real $\varepsilon_i > 0$ such that*

$$\left(0 < \varepsilon < \varepsilon_i \text{ and } z,\ z + \varepsilon\rho\ \in V_i \cap \overline{\Sigma_a}\right) \implies (u(z) < u(z + \varepsilon\rho)).$$

This technical lemma is proved in section 5.

End of the proof of Proposition 2.8. For any point $z = (-a, y_0)$ on the left-hand boundary $\{-a\} \times (-a \cot \gamma, a \cot \gamma)$ of Σ_a, we have $\partial_\tau u = 0$ and $\partial_y u > 0$ from Lemma 2.2. Since $\tau = (-\sin\alpha, -\cos\alpha)$ and $\rho = (\cos\beta, \sin\beta)$ with $\pi/2 - \alpha < \beta < \pi$, it follows that $\partial_\rho u > 0$. Since u is of class C^1 near the point z, there exists a neighborhood V_z of z such that $\partial_\rho u(x, y) > 0$ for any $(x, y) \in V_z \cap \overline{\Sigma_a}$. Hence, from the finite increment theorem, there exists a real $\varepsilon_z > 0$ such that if $0 < \varepsilon < \varepsilon_z$ and if the point $z + \varepsilon\rho$ is in $\overline{V_z} \cap \overline{\Sigma_a}$, then

$$u(z) < u(z + \varepsilon\rho).$$

Without any restriction, the neighborhoods V_1 and V_3 of C_1 and C_3, which are given in Lemma 2.10, can be replaced with two open balls $B(C_i, \delta_i)$ centered on the points C_i and with radii δ_i ($i = 1$ or 3). Since $\{-a\} \times [-a \cot \gamma + \delta_1, a \cot \gamma - \delta_3]$ is a compact set, there exists a real $\overline{\varepsilon} > 0$ such that, if $0 < \varepsilon < \overline{\varepsilon}$, if $z = (x, y)$ where $y \in [-a \cot \gamma + \delta_1, a \cot \gamma - \delta_3]$, and $x = -a$ in the case $\rho \cdot \vec{e}_1 \geq 0$ (resp., $x = -a - \varepsilon\rho \cdot \vec{e}_1$ in the case $\rho \cdot \vec{e}_1 < 0$), then $z,\ z + \varepsilon\rho \in R_\varepsilon$ and

$$u(z) < u(z + \varepsilon\rho).$$

From Lemma 2.10, we conclude that, if $0 < \varepsilon < \min(\varepsilon_1, \varepsilon_3, \overline{\varepsilon})$, then (2.9) is true on the left-hand boundary of R_ε, namely, on $\{-a - \varepsilon\rho \cdot \vec{e}_1\} \times [-a \cot \gamma, a \cot \gamma - \varepsilon\rho \cdot \vec{e}_2]$ or $\{-a\} \times [-a \cot \gamma, a \cot \gamma - \varepsilon\rho \cdot \vec{e}_2]$ according to the sign of $\rho \cdot \vec{e}_1$.

Finally, we set $\varepsilon_0 = \min(\tilde{\varepsilon}, \varepsilon_1, \varepsilon_3, \overline{\varepsilon})$ (remember that $\tilde{\varepsilon}$ has been defined just before Lemma 2.10). For any $\varepsilon \in (0, \varepsilon_0)$ and for any $z \in \partial R_\varepsilon$, the points z and $z + \varepsilon\rho$ are in $\overline{\Sigma_a}$ and we have $u(z) < u(z + \varepsilon\rho)$. Next, as in the proof of Lemma 2.2, that is to say by using a sliding method along the direction $\vec{e}_2$ and the fact that u is increasing with respect to y, we find that

$$u(z) < u(z + \varepsilon\rho) \text{ in } R_\varepsilon.$$

This completes the proof of Proposition 2.8 in the case where the function f is of class C^1 in $[0, 1]$.

If f is not of class C^1 in $[0, 1]$, we can however approximate it by a sequence of functions f_n of class C^1 which are such that $\|f_n'\|_{L^\infty([0,1])} \leq C$, $\|f - f_n\|_{L^\infty([0,1])} \to 0$ as $n \to +\infty$ and which satisfy (1.4) with ignition temperature $\theta_n \to \theta$ as $n \to +\infty$. Under the notation of Lemmas 2.5 and 2.6, there exist two positive reals ε_1 and η_1 such that, for n large enough, we have $f_{\eta_1} \leq f_n \leq f^{\varepsilon_1}$, whence $f_{\eta_1} \leq f \leq f^{\varepsilon_1}$ by taking the limit $n \to +\infty$. Thus, as in the proof of Proposition 2.7, for n large enough and for $a \geq \max(a_1(\eta_1), a_2(\varepsilon_1))$, we get that there exists a unique solution (c_n, u_n) of

(2.1)–(2.2) with the source term f_n as well as a unique solution (c_a, u_a) of (2.1)–(2.2) with the source term f. Furthermore, one has $c_0^{\eta_1}/\sin\alpha \leq c_n \leq \tilde{c}_0^{\varepsilon_1}/\sin^2\alpha$.

Choose any $a \geq \max(a_1(\eta_1), a_2(\varepsilon_1))$. First of all, up to extraction of some subsequence, we can assume that $c_n \to \tilde{c} \in \mathbb{R}$. From the standard elliptic estimates up to the boundary, we can extract a subsequence $u_{n'}$ which approaches a solution u of (2.4) with the speed $\tilde{c}$ in the spaces $W^{2,p}_{loc}(\tilde{\Sigma}_a) \cap C_{loc}(\tilde{\Sigma}_a)$. Furthermore, for each $i \in \{1, \ldots, 4\}$, there exists a function $\overline{v}_i$ defined in a neighborhood V_i of the corner C_i such that $\overline{v}_i(C_i) = 0$ and, for all n' large enough,

$$\begin{array}{lrl} \text{if } i = 1 \text{ or } 2, & u_{n'}(x,y) & \leq \overline{v}_i(x,y) \\ \text{if } i = 3 \text{ or } 4, & 1 - u_{n'}(x,y) & \leq \overline{v}_i(x,y) \end{array} \quad \text{in } V_i \cap \overline{\Sigma_a} \tag{2.10}$$

(see Remark 5.2). As a consequence, the function $\tilde{u}$ can be extended by continuity at the four corners C_i. Hence, $\tilde{u}$ is the unique solution of (2.3) with the speed $\tilde{c}$. On the other hand, by passage to the limit $n' \to \infty$, the statements of Proposition 2.8 hold good for the function $\tilde{u}$. In particular, it follows that $\tilde{u}$ fulfills (2.2). Finally, from Lemma 2.4, we conclude that $(\tilde{c}, \tilde{u}) = (c_a, u_a)$. This completes the proof of Proposition 2.8. □

3. Passage to the limit in the whole plane. In the previous section, we proved the existence and the uniqueness of a solution (c_a, u_a) to problem (2.1)–(2.2) for a large enough. Moreover, we found several a priori bounds for the speeds c_a as well as a priori monotonicity properties for the functions u_a. We are now going to pass to the limit $a \to \infty$.

PROPOSITION 3.1. *There exists a sequence $a_n \to \infty$, a real c, and a function u such that $c_{a_n} \to c$ in $\mathbb{R}$ and $u_{a_n} \to u$ in $W^{2,p}_{loc}(\mathbb{R}^2)$ for all $p > 1$. Furthermore, the real c is such that*

$$\frac{c_0}{\sin\alpha} \leq c \leq \frac{c_0}{\sin^2\alpha}$$

and the function u satisfies

$$\Delta u - c\partial_y u + f(u) = 0 \textit{ in } \mathbb{R}^2, \tag{3.1}$$

$$0 < u < 1 \textit{ in } \mathbb{R}^2,$$

$$\forall (x,y) \in \mathbb{R}^2, \quad u(x,y) = u(-x,y),$$

$$\max_{\substack{y \leq -\cot\alpha \ |x| \\ x \in \mathbb{R}}} u = u(0,0) = \theta, \tag{3.2}$$

$$\begin{cases} \forall \rho = (\cos\beta, \sin\beta) \textit{ such that } \pi/2 - \alpha \leq \beta \leq \pi, & \partial_\rho u(x,y) \geq 0 \textit{ if } x \leq 0, \\ \forall \rho = (\cos\beta, \sin\beta) \textit{ such that } 0 \leq \beta \leq \pi/2 + \alpha, & \partial_\rho u(x,y) \geq 0 \textit{ if } x \geq 0. \end{cases} \tag{3.3}$$

COROLLARY 3.2. *For all $\rho = (\cos\beta, \sin\beta)$ with $\pi/2 - \alpha \leq \beta \leq \pi/2 + \alpha$, one has*

$$\partial_\rho u \geq 0 \quad \textit{in } \mathbb{R}^2.$$

Proof of Proposition 3.1. Under the notation of Proposition 2.7, choose $\varepsilon = \eta = 1/n$ where the integer n is large enough and set $a_n = a_0(1/n, 1/n)$. For n large enough, problem (2.1)–(2.2) has a unique solution (c_n, u_n) in Σ_{a_n} and one has $c_0^{1/n}/\sin\alpha \leq c_n \leq \tilde{c}_0^{1/n}/\sin^2\alpha$.

From the results of [9], we have $c_0^{1/n}$ and $\tilde{c}_0^{1/n} \to c_0$ as $n \to \infty$. Hence there exists a subsequence, that is still called (c_n), such that $c_n \to c \in [c_0/\sin\alpha, c_0/\sin^2\alpha]$. For any compact set K of $\mathbb{R}^2$, from the standard elliptic estimates, the sequence (u_{a_n}) is bounded in $W^{2,p}(K)$ (for a_n large enough such that $\overline{\Sigma_{a_n}} \subset \overset{o}{K}$). Hence, from the diagonal extraction process, there exists a subsequence that is still called (u_{a_n}) and a function u such that $u_{a_n} \to u$ in $W^{2,p}_{loc}(\mathbb{R}^2)$ for all $p > 1$. The function u satisfies (3.1). From the Sobolev injections and since f is Lipschitz continuous, the function u is in $C^{2,\mu}_{loc}(\mathbb{R}^2)$ for all $0 \le \mu < 1$.

Since $u(0,0) = \lim\ u_n(0,0) = \theta$ and since $0 \le u \le 1$, the strong maximum principle implies that $0 < u < 1$ in $\mathbb{R}^2$. The symmetry of u with respect to x derives from the symmetry of u_n. The assertions (3.3) come from Proposition 2.8. Together with (2.2), they yield the normalization condition (3.2). □

3.1. Exponential decay properties. For any $z = (x, y) \in \mathbb{R}^2$, let us define

$$T_z = (-|x|, |x|) \times (-\infty, y) \ \cup\ \mathcal{C}((x, y), -\vec{e}_2, \alpha) \ \cup\ \mathcal{C}((-x, y), -\vec{e}_2, \alpha).$$

Proposition 3.3. *Let x_0 be in $\mathbb{R}$.*

(i) *There exists a real $y_0 \in [-|x_0| \cot\alpha, 0]$ such that $u(x_0, y_0) = \theta$.*

(ii) *Set $z_0 = (x_0, y_0)$. The following exponential decay holds in $\overline{T_{z_0}}$:*

$$\forall z = (x,y) \in \overline{T_{z_0}},\quad u(z) \le 2\theta e^{-c\sin\alpha\cos\alpha\ |x_0|}\cosh(c\sin\alpha\cos\alpha\ x) e^{c\sin^2\alpha\ (y-y_0)} + \theta e^{c(y-y_0)}. \tag{3.4}$$

(iii) *A similar estimate is true in $\overline{\mathcal{C}(z_0, -\vec{e}_2, \alpha)}$. Namely, for all $\pi/2 - \alpha \le \varphi \le \pi/2 + \alpha$ and $\rho = (\cos\varphi, -\sin\varphi)$, we have*

$$\forall \lambda \ge 0,\quad u(z_0 + \lambda\rho) \le 2\theta\cosh(c\lambda\sin\alpha\cos\alpha\cos\varphi)\ e^{-c\lambda\sin^2\alpha\sin\varphi}. \tag{3.5}$$

Remark 3.4. *By taking $z_0 = (0, 0)$ and $\vec{k} \in \mathcal{C}(-\vec{e}_2, \alpha)$ in* (3.5)*, it follows that the function u fulfills* (1.2) *and* (1.8)*.*

Corollary 3.5. *The function u is increasing in y.*

Proof. From Corollary 3.2, we know that $u(x, y)$ is nondecreasing in y. Suppose that $u(x_0, y_0) = u(x_0, y_0')$ where $x_0 \in \mathbb{R}$ and $y_0 < y_0'$. It follows that u is equal to a constant u_0 in $\overline{\mathcal{C}((x_0, y_0), \vec{e}_2, \alpha)} \cap \overline{\mathcal{C}((x_0, y_0'), -\vec{e}_2, \alpha)}$. This constant u_0 is then a zero of the function f. Since $0 < u < 1$ in $\mathbb{R}^2$ and $f > 0$ on $(\theta, 1)$, we get $u_0 \in (0, \theta]$. The monotonicity properties imply that $u \le u_0$ in the cone $\mathcal{C} = \mathcal{C}((x_0, y_0'), -\vec{e}_2, \alpha)$ and that the function u satisfies

$$\Delta u - c\partial_y u = 0 \ \text{ in } \mathcal{C}.$$

In $\mathcal{C}$, the function u reaches its maximum u_0 at an interior point, for instance, $(x_0, (y_0 + y_0')/2)$. From the strong maximum principle, u is then equal to u_0 in $\overline{\mathcal{C}}$. This is impossible because $u(x_0, y) \to 0$ as $y \to -\infty$ from inequality (3.5). □

Proof of Proposition 3.3. From the symmetry of u with respect to x, we may suppose that $x_0 \ge 0$. Let now $a > x_0$. By Proposition 2.8, we have $u_a(x_0, 0) \ge \theta$ and $u_a(x_0, -x_0\cot\alpha) \le \theta$. Since u_a is continuous, there exists a real y_a in $[-x_0\cot\alpha, 0]$ such that $u_a(x_0, y_a) = \theta$. Since the y_a are bounded and since the functions u_a approach u in $C^1_{loc}(\mathbb{R}^2)$ (for a certain sequence $a \to +\infty$), then there exists a real y_0 in $[-x_0\cot\alpha, 0]$ such that $y_a \to y_0$ (for a sequence $a \to \infty$) and $u(x_0, y_0) = \theta$. This yields the assertion (i) of Proposition 3.3.

Let $z_0 = (x_0, y_0)$. Let us now consider the open trapezium D_a whose vertices are the four points $C_1 = (-a, -a\cot\gamma_a)$, $S_1 = (-x_0, y_a)$, $S_2 = (x_0, y_a)$, and $C_2 =$

$(a, -a\cot\gamma_a)$. The angles between $-\vec{e_2}$ and each side $[S_1, C_1]$ and $[S_2, C_2]$ are equal and, since $y_a \geq -x_0\cot\alpha \geq -x_0\cot\gamma_a$, they are not larger than γ_a and, a fortiori, they are less than α. Hence, from Proposition 2.8 we have

$$u_a \leq \theta \ \text{ in } \overline{D_a}$$

and

$$\Delta u_a - c_a\partial_y u_a = 0 \ \text{ in } D_a.$$

We are now going to compare u_a with the sum of three exponential functions in D_a. Choose any point $z_1 = (x_1, y_1)$ in the open set T_{z_0}. Since $y_a \to y_0$ and $\gamma_a \to \alpha$, there exists a positive real a_0 such that $z_1 \in D_a$ for all $a \geq a_0$. Let c' be a real in $(0, c\sin\alpha)$ – notice that this is possible since $\sin\alpha > 0$ and $c\sin\alpha \geq c_0 > 0$. Let us set $r_a = 1/\sqrt{(a\cot\gamma_a + y_a)^2 + (-a + x_0)^2}$ and define

$$w_a(x,y) = f_1(x,y) + f_2(x,y) + f_3(x,y),$$

where

$$\begin{cases} f_1(x,y) &= \theta e^{-c' r_a((a\cot\gamma_a + y_a)(x+x_0) + (x_0 - a)(y - y_a))}, \\ f_2(x,y) &= \theta e^{-c' r_a(-(a\cot\gamma_a + y_a)(x-x_0) + (x_0 - a)(y - y_a))}, \\ f_3(x,y) &= \theta e^{c'/\sin\alpha\ (y - y_a)}. \end{cases}$$

In particular, we have $w_a \geq \theta \geq u_a$ on ∂D_a. Moreover, a straightforward calculation gives

$$\Delta w_a - c_a\partial_y w_a = c'(c' - c_a r_a(a - x_0))(f_1 + f_2) + \frac{c'}{\sin^2\alpha}\ (c' - c_a\sin\alpha) f_3.$$

Since $c' > 0$ and since $c_a \to c > c'/\sin\alpha$, $r_a(a - x_0) \to \sin\alpha$ as $a \to \infty$, it follows that

$$\Delta w_a - c_a\partial_y w_a < 0 \ \text{ in } D_a$$

for a large enough. From the maximum principle, we deduce that $u_a < w_a$ in D_a. By passing to the limit $a \to \infty$, we obtain

$$\begin{aligned} u(x_1, y_1) &\leq \theta e^{-c'[\cos\alpha(x_1 + x_0) - \sin\alpha(y_1 - y_0)]} \\ &\quad + \theta e^{-c'[-\cos\alpha(x_1 - x_0) - \sin\alpha(y_1 - y_0)]} + \theta e^{c'/\sin\alpha\ (y_1 - y_0)}. \end{aligned}$$

Since this is true for any $c' < c\sin\alpha$, we can pass to the limit $c' \to c\sin\alpha$ and we get

$$u(x_1, y_1) \leq 2\theta\cosh(c\sin\alpha\cos\alpha\ x_1)\ e^{c\sin^2\alpha\ (y_1 - y_0) - c\sin\alpha\cos\alpha\ x_0} + \theta e^{c(y_1 - y_0)}.$$

This can be extended by continuity in $\overline{T_{z_0}}$. This gives assertion (ii) of Proposition 3.3.

In the same way, we could prove that for any $x_0 \geq 0$,

$$u(x,y) \leq 2\theta\cosh(c\sin\alpha\cos\alpha\ (x - x_0))e^{c\sin^2\alpha(y - y_0)} \ \text{ in } \overline{\mathcal{C}(z_0, -\vec{e_2}, \alpha)}$$

by comparing the function u_a with the sum of two suitable exponential functions in the triangles whose vertices are $S_1 = (-a + 2x_0, -a\cot\gamma_a)$, $S_2 = (x_0, y_0)$, and $S_3 = (a, -a\cot\gamma_a)$. This corresponds to assertion (iii) of Proposition 3.3. The case $x_0 \leq 0$ can be treated by symmetry. □

3.2. Estimating the speed c: Proof of formula (1.7). Consider now a sequence $x_n \to -\infty$ and, for any x_n, let y_n be the unique real such that $u(x_n, y_n) = \theta$. One has $x_n \cot\alpha \le y_n \le 0$. Move the origin at the point (x_n, y_n) and consider the functions

$$v_n(x,y) = u(x + x_n, y + y_n) \ \text{ in } \mathbb{R}^2.$$

From the standard elliptic estimates and the Sobolev injections, the functions v_n are bounded in $W^{2,p}_{loc}(\mathbb{R}^2)$ for all $1 < p < \infty$ and approach, up to extraction of some subsequence, a function $v \in \bigcap_{p>1} W^{2,p}_{loc}(\mathbb{R}^2)$, such that

$$\begin{cases} \Delta v - c\partial_y v + f(v) = 0 & \text{in } \mathbb{R}^2, \\ v(0,0) = \theta. \end{cases} \tag{3.6}$$

The function v has the following monotonicity properties.

LEMMA 3.6. *For any $\rho = (\cos\varphi, -\sin\varphi)$ such that $0 \le \varphi \le \pi/2 + \alpha$, one has the following:*

(i) *the function v is nonincreasing in the direction ρ;*

(ii) *it also holds that*

$$\forall \lambda \ge 0, \quad v(\lambda\rho) \le \theta e^{-c\lambda \sin\alpha \cos(\alpha - \varphi)} + \theta e^{-c\lambda \sin\varphi}. \tag{3.7}$$

Proof. Let ρ be as in the lemma above. Let $z = (x, y)$ be any point in $\mathbb{R}^2$ and let $\lambda > 0$. Consider both points z and $z + \lambda\rho$. Since $x_n \to -\infty$, we have $x + x_n \le 0$ and $x + x_n + \lambda\cos\varphi \le 0$ for n large enough. From (3.3), we have, for n large enough,

$$v_n(z) = u(x + x_n, y + y_n) \ge u(x + x_n + \lambda\cos\varphi, y + y_n - \lambda\sin\varphi) = v_n(z + \lambda\rho).$$

By taking the limit $n \to \infty$, it follows that $v(z) \ge v(z + \lambda\rho)$. This gives the assertion (i).

Consider the set

$$T_n = (-|x_n|, |x_n|) \times (-\infty, y_n) \ \cup \ \mathcal{C}((x_n, y_n), -\vec{e}_2, \alpha) \ \cup \ \mathcal{C}((-x_n, y_n), -\vec{e}_2, \alpha).$$

Under the notation of section 3.1, we have $T_n = T_{z_n = (x_n, y_n)}$. Since $x_n \to -\infty$, the points $(x_n, y_n) + \lambda\rho$ are in $\overline{T_n}$ for n large enough. Hence, inequality (3.4) implies that

$$\begin{aligned} v_n(\lambda\rho) \ &\le 2\theta e^{-c|x_n| \sin\alpha\cos\alpha} \cosh(c \sin\alpha\cos\alpha\ (x_n + \lambda\cos\varphi)) e^{-c\lambda \sin^2\alpha \sin\varphi} \\ &\quad + \theta e^{-c\lambda\sin\varphi}. \end{aligned}$$

Since $x_n \to -\infty$, we obtain at the limit $n \to \infty$

$$v(\lambda\rho) \le \theta e^{-c\lambda\sin\alpha\cos\alpha\cos\varphi} e^{-c\lambda\sin^2\alpha\sin\varphi} + \theta e^{-c\lambda\sin\varphi}.$$

This completes the proof of Lemma 3.6. □

PROPOSITION 3.7. *The speed c is equal to $c_0/\sin\alpha$.*

Proof. From (1.7), we already know that $c_0/\sin\alpha \le c \le c_0/\sin^2\alpha$. Let us suppose that $c > c_0/\sin\alpha$.

First step: Construction of a supersolution. As in the proof of Lemma 2.6, we use the same functions $f^\varepsilon \ge f$ such that $f^\varepsilon \equiv 0$ on $[0, \theta - \varepsilon] \cup \{1 + \varepsilon\}$, $f^\varepsilon > 0$ on

$(\theta - \varepsilon, 1 + \varepsilon)$, and $f^\varepsilon \to f$ as $\varepsilon \to 0$ uniformly in $[0, 1]$. For each $\varepsilon > 0$, there exists a unique solution $(\overline{c}_0^\varepsilon, U^\varepsilon)$ of

$$\text{(3.8)} \qquad \begin{cases} (U^\varepsilon)'' - \overline{c}_0^\varepsilon (U^\varepsilon)' + f^\varepsilon(U^\varepsilon) &= 0 \quad \text{in } \mathbb{R}, \\ U^\varepsilon(-\infty) = \varepsilon,\ U^\varepsilon(0) = \theta,\ U^\varepsilon(+\infty) &= 1 + \varepsilon. \end{cases}$$

From the results in [9], we have $\overline{c}_0^\varepsilon \to c_0$ as $\varepsilon \to 0$. Now choose $\varepsilon > 0$ such that

$$c > \overline{c}_0^\varepsilon / \sin\alpha$$

and denote by U the function U^ε.

Let us consider the new variables

$$X = y\cos\alpha + x\sin\alpha \quad \text{and} \quad Y = y\sin\alpha - x\cos\alpha.$$

The variables (X, Y) are obtained from (x, y) by a rotation of angle $\pi/2 - \alpha$ around the origin.

We are looking for a supersolution of (3.6) of the type

$$w(x, y) = U(Y - \phi(X)).$$

For such a function w, we have

$$\text{(3.9)} \quad \Delta w - c\partial_y w + f(w) = A(X)U'(Y - \phi(X)) + f(U) - f_\varepsilon(U) - \phi'^2 f_\varepsilon(U),$$

where

$$A(X) = \overline{c}_0^\varepsilon(1 + \phi'^2) - \phi'' - c(\sin\alpha - \cos\alpha\ \phi').$$

Since $f_\varepsilon \geq f \geq 0$ and $U' > 0$, in order to make the right-hand side of (3.9) nonpositive, it is sufficient to choose a function ϕ in such a way that $A(X) \leq 0$. Let ϕ be defined by

$$\phi(X) = -\frac{1}{c\sin\alpha}\ln(e^{-c\sin\alpha\tan\beta\ X} + e^{c\sin\alpha\cot(\alpha-\beta)X}),$$

where $\beta > 0$ shall be chosen later. Set $\delta = \cot(\alpha - \beta) + \tan\beta$. It is easy to check that

$$A(X) = \frac{1}{(1 + e^{c\sin\alpha\delta X})^2}\left[B(\beta)e^{2c\sin\alpha\delta X} + C(\beta)e^{c\sin\alpha\delta X} + D(\beta)\right],$$

where

$$\begin{cases} B(\beta) &= \overline{c}_0^\varepsilon - c\sin\alpha - c\cos\alpha\cot(\alpha - \beta) + \overline{c}_0^\varepsilon\cot^2(\alpha - \beta), \\ C(\beta) &= 2(\overline{c}_0^\varepsilon - c\sin\alpha) - c\cos\alpha\cot(\alpha - \beta) \\ & \quad + c\cos\alpha\tan\beta - 2\overline{c}_0^\varepsilon\tan\beta\cot(\alpha - \beta) + c\sin\alpha\ \delta^2, \\ D(\beta) &= \overline{c}_0^\varepsilon - c\sin\alpha + c\cos\alpha\tan\beta + \overline{c}_0^\varepsilon\tan^2\beta. \end{cases}$$

As $\beta \to 0$, we have $B(\beta) \to \overline{c}_0^\varepsilon/\sin^2\alpha - c/\sin\alpha < 0$, $C(\beta) \to 2(\overline{c}_0^\varepsilon - c\sin\alpha) < 0$, and $D(\beta) \to \overline{c}_0^\varepsilon - c\sin\alpha < 0$. Hence, we can choose $\beta \in (0, \alpha)$ small enough such that $B(\beta)$, $C(\beta)$, $D(\beta) < 0$.

Let β be chosen as above. The function $w(x, y)$ is then a supersolution of (3.6) in the sense that

$$\text{(3.10)} \qquad \Delta w - c\partial_y w + f(w) < 0 \ \text{ in } \mathbb{R}^2.$$

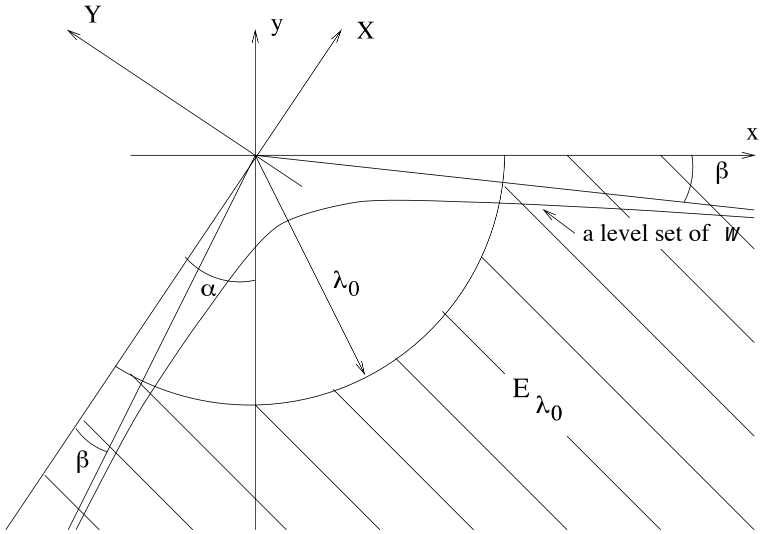

FIG. 4. *The set* E_{λ_0}.

Second step: Initialization of a sliding method. For any λ_0, we set

$$E_{\lambda_0} = \{z = (\lambda \cos \varphi, -\lambda \sin \varphi) \in \mathbb{R}^2,\ 0 \leq \varphi \leq \pi/2 + \alpha,\ \lambda \geq \lambda_0\} \tag{3.11}$$

(see Figure 4).

LEMMA 3.8. *There exists* $\lambda_0 > 0$ *such that*

$$w > v \quad in\ E_{\lambda_0}.$$

Proof. Assume that the previous conclusion is not true. There exist then two sequences $0 \leq \lambda_n \to +\infty$ and $z_n = (x_n, y_n) = (\lambda_n \cos \varphi_n,\ -\lambda_n \sin \varphi_n) \in E_{\lambda_n}$ such that $w(z_n) \leq v(z_n)$.

Set $X_n = y_n \cos \alpha + x_n \sin \alpha = \lambda_n \sin(\alpha - \varphi_n)$ and $Y_n = y_n \sin \alpha - x_n \cos \alpha = -\lambda_n \cos(\alpha - \varphi_n)$. From (3.6) and Lemma 3.6 (i), it follows that $v \leq \theta$ in E_{λ_0} and a fortiori in E_{λ_n} for n large enough. Hence, $w(z_n) = U(Y_n - \phi(X_n)) \leq \theta$. Since U is increasing and $U(0) = \theta$, we get that $Y_n - \phi(X_n) \leq 0$. On the other hand, from equation (3.8) satisfied by U, we have

$$\forall \xi \leq 0, \quad U(\xi) = \varepsilon + (\theta - \varepsilon) e^{\overline{c}_0^{\varepsilon} \xi}.$$

Hence,

$$w(z_n) = U(Y_n - \phi(X_n)) = \varepsilon + (\theta - \varepsilon) e^{\overline{c}_0^{\varepsilon}(Y_n - \phi(X_n))} \leq v(z_n). \tag{3.12}$$

Since $\varphi_n \in [0, \pi/2 + \alpha]$, up to extraction of some subsequence, the following two cases occur.

(i) $\varphi_n \to \varphi \in]0, \pi/2 + \alpha[$. In this case, inequality (3.7) implies that $v(z_n) \to 0$ as $n \to +\infty$, whereas the left-hand side of (3.12) is greater than the positive constant ε. Case (i) is then impossible.

(ii) $\varphi_n \to 0$ or $\pi/2 + \alpha$. Since $\beta > 0$ and since each level set of the function $Y - \phi(X)$ has two asymptotes directed by the vectors $\rho_1 = (\cos \beta, -\sin \beta)$ and $\rho_2 =$

$(\cos(\pi/2+\alpha-\beta), -\sin(\pi/2+\alpha-\beta))$, the distance between the points z_n and the half-lines $\mathbb{R}_+\rho_1$, $\mathbb{R}_+\rho_2$ necessarily approaches $+\infty$. This finally yields that $Y_n - \phi(X_n) \to +\infty$, whence $w(z_n) \to 1+\varepsilon$ as $n \to \infty$. This is ruled out by the inequality $w(z_n) \le v(z_n) < 1$.

This completes the proof of Lemma 3.8. □

Third step: The sliding method. We are now going to slide w in the Y-direction and compare it with the function v. For all $\tau \in \mathbb{R}$, we set

$$w_\tau(x,y) = U(\tau + Y - \phi(X)).$$

From Lemma 3.8, there exists a real λ_0 such that $w > v$ in E_{λ_0}, whence $w_\tau > v$ in E_{λ_0} for any $\tau \ge 0$ (remember that U is increasing).

The level set $\{Y - \phi(X) = 1+\varepsilon/2\}$ of w has two asymptotes directed by the vectors $(\cos\beta, -\sin\beta)$ and $(\cos(\pi/2+\alpha-\beta), -\sin(\pi/2+\alpha-\beta))$. Owing to the definition of E_{λ_0} and since $0<\beta$, there exists a real $\tau > 0$ such that the shifted level set $\{Y+\tau-\phi(X) = 1+\varepsilon/2\}$ in the direction Y is included in E_{λ_0}.

We now claim that

$$w_\tau > v \text{ in } \mathbb{R}^2.$$

Indeed, we already know that this is true in E_{λ_0}. But in $\mathbb{R}^2\backslash E_{\lambda_0}$, we have $w_\tau(x,y) = U(\tau+Y-\phi(X)) \ge 1+\varepsilon/2$ from the definition of τ. Hence,

$$w_\tau(x,y) \ge 1+\varepsilon/2 > v(x,y) \ \text{ in } \mathbb{R}^2\backslash E_{\lambda_0}.$$

Let us now slide w in the Y-direction. In other words, let us decrease τ and call

$$\tau^* = \inf\,\{\tau\in\mathbb{R},\ w_\tau > v \text{ in } \mathbb{R}^2\}.$$

This real is finite because $w_\tau(0,0) \to U(-\infty) = \varepsilon < \theta$ as $\tau \to -\infty$ and $v(0,0) = \theta$. Since U is increasing, we have $w_\tau > v$ for all $\tau > \tau^*$. By continuity, we find that

$$w_{\tau^*} \ge v \text{ in } \mathbb{R}^2.$$

Since the function w_{τ^*} satisfies (3.10), the nonnegative function $z = w_{\tau^*} - v$ is such that

$$\Delta z - c\partial_y z + c(x,y)z \le 0 \ \text{ in } \mathbb{R}^2$$

for some bounded function $c(x,y)$. From the strong maximum principle, one of the following two situations occurs:

(i) $w_{\tau^*} \equiv v$ in $\mathbb{R}^2$,

(ii) $w_{\tau^*} > v$ in $\mathbb{R}^2$.

Case (i) cannot occur since $w_{\tau^*} \to 1+\varepsilon$ as $Y \to +\infty$, whereas $v<1$ in $\mathbb{R}^2$. If case (ii) occurs, let us consider an increasing sequence $\tau_n \to \tau^*$. For each n, owing to the definition of τ^*, there exists a point $(x_n,y_n)\in\mathbb{R}^2$ such that $w_{\tau_n}(x_n,y_n) \le v(x_n,y_n)$. The points (x_n,y_n) cannot be bounded; otherwise there would exist a point $(\overline{x},\overline{y})\in\mathbb{R}^2$ such that $w_{\tau^*}(\overline{x},\overline{y}) \le v(\overline{x},\overline{y})$. The latter is impossible because of assumption (ii). Now, as in Lemma 3.8, there exists a real $\tilde\lambda_0$ such that $w_{\tau_0} > v$ in $E_{\tilde\lambda_0}$. Since the sequence (τ_n) is increasing, we have $w_{\tau_n} > v$ in $E_{\tilde\lambda_0}$. This implies that $(x_n,y_n)\notin E_{\tilde\lambda_0}$. On the other hand, since $0<\beta$ and since any level set of the function $Y-\phi(X)$ has two asymptotes directed by the vectors $\rho_1 = (\cos\beta,-\sin\beta)$ and $\rho_2 = (\cos(\pi/2+\alpha-\beta), -\sin(\pi/2+\alpha-\beta))$, it follows that $w_{\tau_n}(x_n,y_n) \to 1+\varepsilon$ as $n\to\infty$. This is impossible since $w_{\tau_n}(x_n,y_n) \le v(x_n,y_n) < 1$.

Finally, the assertion $c > c_0/\sin\alpha$ was impossible. Hence, $c = c_0/\sin\alpha$. This completes the proof of Proposition 3.7. □

3.3. Convergence of the function u to a planar wave far away from the axis of symmetry. The case $\alpha = \pi/2$ is treated separately. Indeed, in this case, from the uniqueness result in Lemma 2.2, the functions u_a only depend on y and they solve $u_a'' - c_a u_a' + f(u_a) = 0$, $u_a(-a\cot\gamma_a) = 0$, $u_a(0) = \theta$, and $u_a(a\cot\gamma_a) = 1$. From the construction given in [9], those functions u_a approach the solution $U(y)$ of (1.5) as $a \to +\infty$. This immediately yields the asymptotic limit (1.3) as well as the last assertion of Theorem 1.1.

In the case where $\alpha < \pi/2$, as in section 3.2, we again consider the function v, obtained as the limit of the functions $v_n(x,y) = u(x+x_n, y+y_n)$, where $x_n \to -\infty$ and $u(x_n,y_n) = \theta$. We know that the function v is nonincreasing in each direction $\rho = (\cos\varphi, -\sin\varphi)$ such that $0 \le \varphi \le \pi/2 + \alpha$. Furthermore, v has an exponential decay in the set $\{\lambda(\cos\varphi, -\sin\varphi),\ \lambda \ge 0, 0 \le \varphi \le \pi/2+\alpha\}$ of the type (3.7).

Our goal is to prove that v is actually equal to the planar wave $U(Y) = U(y\sin\alpha - x\cos\alpha)$. We divide the proof into four main steps.

First step: Construction of a supersolution. We still use the variables $X = y\cos\alpha + x\sin\alpha$ and $Y = y\sin\alpha - x\cos\alpha$. In the previous section, we considered a supersolution of (3.6) of the type $w(x,y) = U^{\varepsilon}(Y - \phi(X))$, which had two asymptotes directed by the two vectors $\rho_1 = (\cos\beta, -\sin\beta)$ and $\rho_2 = (\cos(\pi/2+\alpha-\beta), -\sin(\pi/2+\alpha-\beta))$ ($\beta > 0$ was a small angle).

Now, consider the function w defined by

$$w(x,y) = U(Y - \phi(X)),$$

where U is the unique solution of (1.5) such that $U(0) = \theta$ and where

$$\phi(X) = -\frac{1}{c_0}\ln(1 + e^{c_0 \cot\alpha\ X}).$$

Since $c = c_0/\sin\alpha$, we have

$$\Delta w - c\partial_y w + f(w) = -\phi'(X)^2 f(U(Y - \phi(X))) \le\not\equiv 0 \ \text{ in } \mathbb{R}^2. \tag{3.13}$$

Second step: Initialization of a sliding method. Let $h(X)$ be the function defined as follows:

$$h(X) = \begin{cases} 0 & \text{if } X \le 0, \\ -X\cot\alpha & \text{if } X \ge 0. \end{cases}$$

Set $E_0 = \{\lambda(\cos\varphi, -\sin\varphi),\ \lambda \ge 0, 0 \le \varphi \le \pi/2+\alpha\} = \{Y \le h(X)\}$ (this definition is the same as (3.11)). We claim that

$$w \ge v \ \text{ in } E_0. \tag{3.14}$$

Indeed, let $(x,y) = (\lambda\cos\varphi, -\lambda\sin\varphi) \in E_0$ with $\lambda \ge 0$ and $0 \le \varphi \le \pi/2 + \alpha$. We have $X = \lambda\sin(\alpha - \varphi)$, $Y = -\lambda\cos(\alpha-\varphi)$, and

$$w(x,y) = U(-\lambda\cos(\alpha-\varphi) - \phi(\lambda\sin(\alpha-\varphi))).$$

From Lemma 3.6 (i) and since $v(0,0) = \theta$, one has $v \le \theta$ in E_0. Hence, inequality (3.14) is immediately satisfied if $w \ge \theta$. Consider now the case where $w(x,y) \le \theta$. Since $U(\xi) = \theta e^{c_0\xi}$ for $\xi \le 0$, it follows that

$$\begin{aligned} w(x,y) &= U(-\lambda\cos(\alpha-\varphi) - \phi(\lambda\sin(\alpha-\varphi))) \\ &= \theta e^{c_0(-\lambda\cos(\alpha-\varphi) + \frac{1}{c_0}\ln(1+e^{c_0\lambda\cot\alpha\sin(\alpha-\varphi)}))} \\ &= \theta(e^{-c\lambda\sin\alpha\cos(\alpha-\varphi)} + e^{-c\lambda\sin\varphi}) \\ &\ge v(x,y) \quad \text{ by (3.7).} \end{aligned}$$

For any $\tau \in \mathbb{R}$, we set $w_\tau(x,y) = U(\tau + Y - \phi(X))$. Since U is increasing, we have

$$\forall \tau \geq 0, \quad w_\tau \geq v \text{ in } E_0. \tag{3.15}$$

On the half-line $\{Y = 0, X \leq 0\}$ of ∂E_0, we have $Y - \phi(X) = -\phi(X) \geq 0$. On the other half-line $\{Y = -\cot\alpha\ X, X \geq 0\}$ of ∂E_0, we have $Y - \phi(X) = -\cot\alpha\ X + 1/c_0 \ln(1 + e^{c_0 \cot\alpha\ X}) \geq 0$. Thus $w_\tau \geq U(\tau)$ on ∂E_0.

Since $f'_-(1) = \lim_{t\to 1,\ t<1} \frac{f(t)-f(1)}{t-1} < 0$ and $f \equiv 0$ on $[1,\infty[$, there exists a real $\varepsilon \in (0, 1-\theta)$ such that

$$(\ t \leq s \in [1-\varepsilon, 1]\ ) \implies \left(\ f(s) - f(t) \leq \frac{f'_-(1)}{2}\ (s-t) \leq 0\ \right). \tag{3.16}$$

Since U is increasing and approaches 1 at $+\infty$, there exists a real $\tau_1 \geq 0$ such that

$$\forall \tau \geq \tau_1, \quad w_\tau \geq 1 - \varepsilon \text{ on } \partial E_0. \tag{3.17}$$

Since the function w increases with respect to Y, we finally conclude from the definition of E_0 that

$$\forall \tau \geq \tau_1, \quad w_\tau \geq 1 - \varepsilon \text{ in } \mathbb{R}^2 \backslash E_0.$$

LEMMA 3.9. *For all $\tau \geq \tau_1$, $w_\tau \geq v$ in $\mathbb{R}^2$.*

Proof. Choose any $\tau \geq \tau_1$. By (3.15) and since $\tau_1 \geq 0$, we already know that $w_\tau \geq v$ in E_0.

Let $\tilde{\Omega}_+$ be the open set $\tilde{\Omega}_+ = \mathbb{R}^2 \backslash E_0 \cap \{w_\tau < v\}$. In order to prove Lemma 3.9, the only thing we still need to prove is that $\tilde{\Omega}_+$ is empty. Set $z = w_\tau - v$. From (3.6) and (3.13) we have

$$\Delta z - c\partial_y z \leq f(v) - f(w_\tau) \text{ in } \mathbb{R}^2.$$

In $\tilde{\Omega}_+$, the function v satisfies $1 \geq v > w_\tau \geq 1 - \varepsilon$ from (3.17). From the choice of ε (see (3.16)), we finally get

$$\Delta z - c\partial_y z + f'_-(1)/2\ z \leq 0 \text{ in } \tilde{\Omega}_+. \tag{3.18}$$

If $\tilde{\Omega}_+$ is not empty, define $-\delta = \inf_{\tilde{\Omega}_+} z$ (we have $-\varepsilon \leq -\delta < 0$) and consider a sequence $(x_n, y_n) \in \tilde{\Omega}_+$ such that $z(x_n, y_n) \to -\delta$ as $n \to \infty$. From the standard elliptic estimates, ∇z is bounded in $\mathbb{R}^2$. There exists then a real $r > 0$ such that the open ball $B((x_n, y_n), r)$ lies in $\tilde{\Omega}_+$ for n large enough. The functions $z_n(x,y) = z(x + x_n, y + y_n)$ approach, up to extraction of some subsequence, a function $\tilde{z}$ defined at least in $B((0,0), r)$. This function $\tilde{z}$ reaches its minimum $-\delta < 0$ at the point $(0,0)$ and it satisfies (3.18) in $B((0,0), r)$. This is clearly impossible since $f'_-(1) < 0$. Hence, $\tilde{\Omega}_+ = \emptyset$ and $w_\tau \geq v$ in $\mathbb{R}^2$ for all $\tau \geq \tau_1$. □

Third step: Sliding method. We now decrease τ and we are going to prove the following lemma.

LEMMA 3.10. *There exist two reals τ^*, $\overline{Y}$ and a sequence of points (x_n, y_n) such that the coordinates (X_n, Y_n) satisfy $X_n \to -\infty$, $Y_n \to \overline{Y}$, and*

$$v_n(x,y) = v(x + x_n, y + y_n) \ \to\ U(\tau^* + \overline{Y} + Y) \text{ as } n \to \infty$$

in the spaces $W^{2,p}_{loc}(\mathbb{R}^2)$ *for all* $p > 1$.

Proof. Call

$$\mathcal{E} = \{\tau,\ w_\tau \geq v \text{ in } \mathbb{R}^2\}.$$

The set $\mathcal{E}$ is not empty from Lemma 3.9. Let us define

$$\tau^* = \inf\ \mathcal{E}.$$

The real τ^* is finite since $w_\tau(x,y) \to 0$ as $\tau \to -\infty$ for any $(x,y) \in \mathbb{R}^2$. By continuity with respect to τ, we have

$$w_{\tau^*} \geq v.$$

Since the function w_{τ^*} is a strict supersolution of (3.1) in the sense that it satisfies (3.13), the strong maximum principle yields that $w_{\tau^*} > v$ in $\mathbb{R}^2$.

Remember that ε satisfies (3.16). Owing to the definition of w, there exists a real $A \geq 0$ such that

$$w_{\tau^*} \geq 1 - \varepsilon/2 \ \text{ on } \{Y = h(X) + A\}. \tag{3.19}$$

Let us set $\Omega_+ = \{Y \geq h(X) + A\}$ and $\Omega_- = E_0 = \{Y \leq h(X)\}$. By (3.6) and Lemma 3.6, we have already seen that $v \leq \theta$ in Ω_-. Last, let $\mathcal{B} = \{h(X) < Y < h(X) + A\} = \mathbb{R}^2 \backslash (\Omega_+ \cup \Omega_-)$ (see Figure 5).

Comparison of $w_{\tau^*-\delta}$ *and* v *on* $\partial\Omega_+$. Since the function w is Lipschitz continuous and fulfills (3.19), we have $w_{\tau^*-\delta} \geq 1 - \varepsilon$ on $\partial\Omega_+ = \{Y = h(X) + A\}$ if $\delta \in (0, \delta_0)$ for δ_0 small enough. Two cases may occur:

(i) There exists $\delta_1 \in (0, \delta_0)$ such that $w_{\tau^*-\delta_1} > v$ on $\partial\Omega_+$.

(ii) For n large enough, there exists a point $(x_n, y_n) \in \partial\Omega_+$ such that

$$w_{\tau^*-1/n}(x_n, y_n) \leq v(x_n, y_n). \tag{3.20}$$

Study of case (i). In this case, we argue as in the proof of Lemma 3.9 and conclude that $w_{\tau^*-\delta_1} \geq v$ in Ω_+. As a consequence, for all $\delta \in [0, \delta_1]$, one has $w_{\tau^*-\delta} \geq v$ in Ω_+.

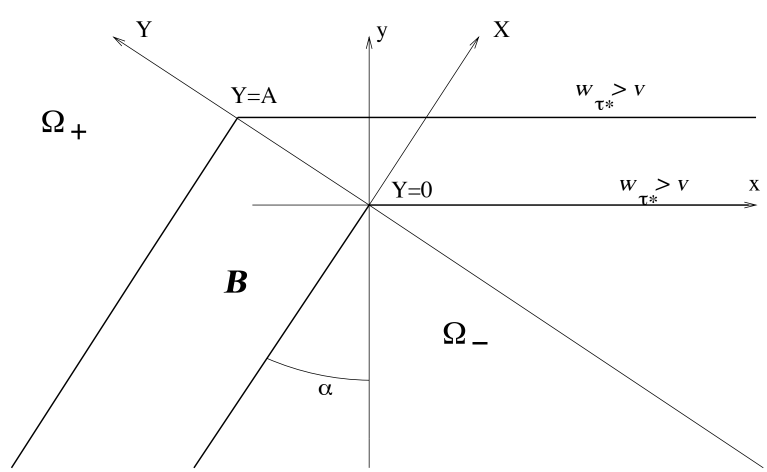

FIG. 5. *The sets* Ω_+, Ω_-, *and* $\mathcal{B}$.

Study of case (ii). In this case, the points (x_n, y_n) cannot be bounded; otherwise there exists a point $(\overline{x}, \overline{y}) \in \partial\Omega_+$ such that $w_{\tau^*}(\overline{x}, \overline{y}) = v(\overline{x}, \overline{y})$. But we have already seen that $w_{\tau^*} > v$ in $\mathbb{R}^2$. Hence one of the following situations occurs:

(ii)(a) There exists a subsequence of (x_n, y_n) such that $X_n \to -\infty$, and $Y_n = A$. We set

$$\begin{cases} w_n(x,y) &= w_{\tau^*}(x + x_n, y + y_n) \quad \text{in } \mathbb{R}^2, \\ v_n(x,y) &= v(x + x_n, y + y_n) \quad \text{in } \mathbb{R}^2. \end{cases}$$

Up to extraction of some subsequence, the functions v_n approach a solution v_∞ of (1.1) and the functions w_n approach the function $w_\infty = U(\tau^* + A + Y)$ in the spaces $W^{2,p}_{loc}(\mathbb{R}^2)$. At the limit $n \to +\infty$, we get

$$w_\infty \geq v_\infty \text{ in } \mathbb{R}^2. \tag{3.21}$$

Since the function w_τ has bounded derivatives, we conclude from (3.20) and (3.21) that $w_\infty(0,0) = v_\infty(0,0)$. Now, both functions v_∞ and w_∞ solve (1.1). From the strong maximum principle, we conclude that

$$v_\infty \equiv w_\infty = U(\tau^* + A + Y).$$

That gives the conclusion of Lemma 3.10.

(ii)(b) There exists a subsequence of (x_n, y_n) such that $x_n \to +\infty$, $y_n = A \sin\alpha$. We again normalize the functions w_{τ^*} and v as in case (ii)(a). Under the same notation as in case (ii)(a), we have $w_\infty = U((1/\sin\alpha)\ (y + A\sin\alpha) + \tau^*) \geq v_\infty$ and $w_\infty(0,0) = v_\infty(0,0)$. On the other hand, the function w_∞ is a solution of

$$\Delta w_\infty - c\partial_y w_\infty + f(w_\infty) = (1 - 1/\sin^2\alpha)\ f(U((1/\sin\alpha)\ (y + A\sin\alpha) + \tau^*)).$$

Since $\alpha < \pi/2$, the function w_∞ is then a strict supersolution of (1.1), whereas v_∞ is a solution. This is ruled out by the strong maximum principle.

As a conclusion of this part, only the cases (i) or (ii)(a) may occur and case (ii)(a) leads to the conclusion of Lemma 3.10.

Comparison of $w_{\tau^*-\delta}$ and v on $\partial\Omega_-$. As above, only two cases may occur:

(i′) There exists $\delta_2 \in (0, \delta_0)$ such that $w_{\tau^*-\delta_2} > v$ on $\partial\Omega_-$.

(ii′) For n large enough, there exists $(x_n, y_n) \in \partial\Omega_-$ such that

$$w_{\tau^*-1/n}(x_n, y_n) \leq v(x_n, y_n).$$

If case (i′) occurs, then, for any $0 \leq \delta \leq \delta_2$, we have $w_{\tau^*-\delta} > v$ on $\partial\Omega_-$. Since $f \equiv 0$ on $[0, \theta]$ and $v \leq \theta$ in Ω_-, with the same method as in the proof of Lemma 3.9, we would actually find that $w_{\tau^*-\delta} \geq v$ in Ω_- for all $0 \leq \delta \leq \delta_2$.

If case (ii′) occurs, we can argue word by word as in case (ii) above. That leads to the conclusion of Lemma 3.10.

Completion of the proof of Lemma 3.10. To complete the proof, the only thing left to consider is the case where both (i) and (i′) occur. Set $\delta_3 = \min(\delta_1, \delta_2)$. Thus

$$\forall \delta \in [0, \delta_3], \quad w_{\tau^*-\delta} \geq v \ \text{ in } \Omega_+ \cup \Omega_-. \tag{3.22}$$

From the definition of τ^*, for any $n \geq 1$, there exists a point (x_n, y_n) such that

$$w_{\tau^*-1/n}(x_n, y_n) < v(x_n, y_n).$$

By (3.22), the points (x_n, y_n) are in $\mathcal{B}$ for n large enough. Consequently, up to extraction of a subsequence, one of the following situations occurs:

(i,i′)(a) $X_n \to -\infty$, $Y_n \to \overline{Y} \in [0, A]$.

(i,i′)(b) $x_n \to +\infty$, $y_n \to \overline{y} \in [0, A\sin\alpha]$. The latter can be treated in the same way as the case (ii)(b) above: it is ruled out by the strong maximum principle.

Hence, only case (i,i′)(a) may occur and, as in the case (ii)(a), we get the conclusion of Lemma 3.10. □

Fourth step: Proving the planar behavior of u far away from the axis of symmetry. We are going to use here the (X, Y) coordinates. Fix a point $(X, Y) \in \mathbb{R}^2$. With the notation of Lemma 3.10, we have $X \geq X_n$ for n large enough. Since v is nondecreasing in the direction X, it follows that

$$v(X, Y) \geq v(X_n, Y) = v_n(0, Y - Y_n)$$

for n large enough. Since $Y_n \to \overline{Y}$ and since v has bounded derivatives, we conclude from Lemma 3.10 that

$$v(X_n, Y) \to U(\tau^* + Y) \text{ as } n \to \infty,$$

whence

$$v(X, Y) \geq U(\tau^* + Y).$$

On the other hand, from the definition of τ^*, we have

$$v(X, Y) \leq U(\tau^* + Y - \phi(X)).$$

By summarizing the previous results, it follows that

$$U(\tau^* + Y) \leq v(X, Y) \leq U(\tau^* + Y - \phi(X)) \text{ in } \mathbb{R}^2. \tag{3.23}$$

Now, for any $X_0 \geq 0$, consider the function

$$w^{X_0}(x, y) = U(Y - \phi(X - X_0)).$$

We could compare the functions w^{X_0} and v by arguing in the same way as above. First, the function w^{X_0} satisfies (3.13). Second, instead of (3.14), it is easy to check that

$$\forall \tau \geq X_0 \cot\alpha, \quad w_\tau^{X_0} := U(\tau + Y - \phi(X - X_0)) \geq v \text{ in } E_0.$$

Furthermore, we have $Y - \phi(X - X_0) \geq -X_0 \cot\alpha$ on ∂E_0. Hence, there exists a real $\tau_1' \geq 0$ that we can choose greater than $X_0 \cot\alpha$ such that

$$\forall \tau \geq \tau_1', \quad w_\tau^{X_0} \geq 1 - \varepsilon \text{ on } \partial E_0$$

with the same ε as in (3.16). As in Lemma 3.9, it follows that

$$\forall \tau \geq \tau_1', \quad w_\tau^{X_0} \geq v \text{ in } \mathbb{R}^2.$$

Lemma 3.10 can be applied to the function w^{X_0}. As for (3.23), we get the existence of a real $\tilde{\tau}^*$ such that

$$U(\tilde{\tau}^* + Y) \leq v(X, Y) \leq U(\tilde{\tau}^* + Y - \phi(X - X_0)) \text{ in } \mathbb{R}^2. \tag{3.24}$$

By taking the limit $X \to -\infty$ in (3.23) and (3.24) and by using the monotonicity of U, we conclude that $\tilde{\tau^*} = \tau^*$.

As a consequence, for all $X_0 \geq 0$, we have

$$U(\tau^* + Y) \leq v(X,Y) \leq U(\tau^* + Y - \phi(X - X_0)) \text{ in } \mathbb{R}^2.$$

We pass to the limit $X_0 \to +\infty$ and obtain

$$U(\tau^* + Y) \leq v(X,Y) \leq U(\tau^* + Y) \text{ in } \mathbb{R}^2.$$

Since $v(0,0) = U(0) = \theta$, it follows that $\tau^* = 0$. In other words, the function v is actually nothing but the planar function $U(Y)$. Last, the function v, which is the limit of a subsequence of the functions $v_n(x,y) = u(x + x_n, y + y_n)$, does not depend on the sequence $x_n \to -\infty$. We conclude that the whole sequence (u_n) approaches the function $U(Y)$.

So far, we have proved that, for any $x \in \mathbb{R}$, there existed a unique real $y = \varphi_\theta(x)$ such that $u(x,y) = \theta$. Furthermore, for any sequence $x_n \to -\infty$, the functions $u_n(x,y) = u(x + x_n, y + \varphi_\theta(x_n))$ approach the planar function $U(Y) = U(y \sin\alpha - x\cos\alpha)$.

Let $\lambda \in (0,1)$. We shall now prove that the level set $\{(x,y),\ u(x,y) = \lambda\}$ is a curve $\{y = \varphi_\lambda(x),\ x \in \mathbb{R}\}$.

First of all, the function u is increasing with respect to y. For each $x \in \mathbb{R}$, set $\psi(x) = \lim_{y\to+\infty} u(x,y)$. In the set $\Omega = \mathbb{R} \times (0,1)$, let us define the functions

$$\tilde{u}_n(x,y) = u(x, y+n) \text{ in } \Omega.$$

They still satisfy (3.1). From the standard elliptic estimates, those functions $\tilde{u}_n$ approach, up to extraction of some subsequence, a function u_∞ that is a solution of

$$\Delta u_\infty - c\partial_y u_\infty + f(u_\infty) = 0 \text{ in } \Omega.$$

But this function $v_\infty(x,y)$ is actually identically equal to the function $\psi(x)$. Hence, ψ fulfills

$$\psi'' + f(\psi) = 0 \text{ in } \mathbb{R}.$$

On the other hand, for any $y \in \mathbb{R}$, the function $x \mapsto u(x,y)$ is symmetric, nonincreasing in x for $x \leq 0$, and nondecreasing for $x \geq 0$. The same property holds well for the limit function ψ. Thus, 0 is a minimum point of ψ; whence $\psi''(0) \geq 0$. Furthermore, $\psi''(0) = -f(\psi(0)) \leq 0$. Hence, $\psi''(0) = f(\psi(0)) = 0$. In other words, $\psi(0)$ is a zero of the function f. Since $\psi(0) > u(0,0) = \theta$ and since f is positive on $(\theta, 1)$, we conclude that $\psi(0) = 1$ and finally that $\psi \equiv 1$.

Hence, for any $x \in \mathbb{R}$, $u(x,y) \to 1$ as $y \to +\infty$. Furthermore, $u(x,y) \to 0$ as $y \to -\infty$ from (3.5) applied in $z_0 = (0,0)$. Since u is continuous and increasing in y, we conclude that there exists a unique $y = \varphi_\lambda(x)$ such that $u(x, \varphi_\lambda(x)) = \lambda$.

Let (x_n) be a sequence such that $x_n \to -\infty$ as $n \to \infty$ and let K be the compact set

$$K = \{(X,Y) \in \mathbb{R}^2,\ |X| \leq 2\cot\alpha\ |U^{-1}(\lambda)|,\ |Y| \leq 2|U^{-1}(\lambda)|\}.$$

We know that the functions $u_n(x,y) = u(x + x_n, y + \varphi_\theta(x_n))$ approach the function $U(Y) = U(y\sin\alpha - x\cos\alpha)$ uniformly in K. For any $\varepsilon > 0$, there exists an integer n_0 such that if $n \geq n_0$, then

$$u_n(0, (1/\sin\alpha)\ U^{-1}(\lambda) - \varepsilon) < \lambda \text{ and } u_n(0, (1/\sin\alpha)\ U^{-1}(\lambda) + \varepsilon) > \lambda.$$

Hence, for $n \geq n_0$, one has

$$\varphi_\theta(x_n) + (1/\sin\alpha)\ U^{-1}(\lambda) - \varepsilon \leq \varphi_\lambda(x_n) \leq \varphi_\theta(x_n) + (1/\sin\alpha)\ U^{-1}(\lambda) + \varepsilon.$$

It then follows that

$$\varphi_\lambda(x_n) - \varphi_\theta(x_n) \to (1/\sin\alpha)\ U^{-1}(\lambda) \text{ as } n \to \infty.$$

Since this limit does not depend on the sequence $x_n \to -\infty$, we conclude that, for any $\lambda, \lambda' \in (0,1)$,

$$\varphi_\lambda(x) - \varphi_{\lambda'}(x) \to (1/\sin\alpha)\ (U^{-1}(\lambda) - U^{-1}(\lambda')) \text{ as } x \to -\infty.$$

The same limit also holds as $x \to +\infty$ by symmetry.

In particular, that implies that the functions $\tilde{u}_n(x,y) = u(x + x_n, y + \varphi_\lambda(x_n))$ approach the function $U(Y + U^{-1}(\lambda))$ in $W^{2,p}_{loc}(\mathbb{R}^2)$.

3.4. Asymptotic directions for the level sets of u. Let $\vec{k}$ be a vector in the open cone $\mathcal{C}(\vec{e_2}, \pi - \alpha)$. We are going to prove that the function u fulfills the limiting condition (1.3), namely, that $u(\lambda\vec{k}) \to 1$ as $\lambda \to +\infty$. By symmetry with respect to x and since $u(0,y) \to 1$ as $y \to +\infty$, it is enough to treat the case of a vector $\vec{k}$ such that $\vec{k} \cdot \vec{e}_1 < 0$. We can write $\vec{k} = (-\sin\beta, -\cos\beta)$ with $\alpha < \beta < \pi$ (β is the angle between $\vec{k}$ and $-\vec{e}_2$ if one goes clockwise).

Let $0 < \varepsilon < 1$. We shall show that, for λ large enough, we have

$$u(\lambda\vec{k}) \geq 1 - \varepsilon.$$

Consider the compact $K = [-1,1] \times [-2\cot\alpha, 2\cot\alpha]$ and the functions

$$u_n(x,y) = u(x - n, y + \varphi_{1-\varepsilon/2}(-n)).$$

From the previous sections, these functions u_n converge uniformly in K to the function $U(y\sin\alpha - x\cos\alpha + U^{-1}(1-\varepsilon/2))$.

Let S be the segment between the points $(0,0)$ and $(-1, -\cot\alpha)$. The functions u_n converge uniformly to $1 - \varepsilon/2$ on S. Since u is increasing in y, we deduce that there exists n_0 large enough such that

$$\text{(3.25)}\ \forall n \geq n_0, \quad \forall x \in [-n-1, -n], \quad \varphi_{1-\varepsilon}(x) \leq \varphi_{1-\varepsilon/2}(-n) + \cot\alpha\ (x+n).$$

Similarly, since $\alpha < \beta < \pi$ and since U is increasing, the sequence $(u_n(-1, -\cot((\alpha + \beta)/2)))$ approaches $1 - \eta$, as $n \to \infty$, with $0 < \eta < \varepsilon/2$. Hence, there exists $n_0' \geq n_0$ such that

$$\forall n \geq n_0', \quad \varphi_{1-\varepsilon/2}(-n-1) \leq \varphi_{1-\varepsilon/2}(-n) - \cot((\alpha+\beta)/2).$$

With an immediate induction, we get that

$$\text{(3.26)} \qquad \forall n \geq n_0', \quad \varphi_{1-\varepsilon/2}(-n) \leq \varphi_{1-\varepsilon/2}(-n_0') - \cot((\alpha+\beta)/2)(n - n_0').$$

Putting together (3.25) and (3.26), we have, for all $n \geq n_0'$ and for all $x \in [-n-1, -n]$,

$$\varphi_{1-\varepsilon}(x) \leq \varphi_{1-\varepsilon/2}(-n_0') + \cot\alpha\ (x+n) - \cot((\alpha+\beta)/2)\ (n - n_0').$$

Since $\cot\alpha \geq \cot((\alpha+\beta)/2)$ and since $x+n \leq 0$ in the previous inequality, we get

$$\forall x \leq -n_0', \quad \varphi_{1-\varepsilon}(x) \leq \varphi_{1-\varepsilon/2}(-n_0') + \cot((\alpha+\beta)/2)\ (x+n_0').$$

By putting $x = -\lambda\sin\beta$ in the last inequality, and since $\beta > \alpha$, we conclude that, for λ large enough,

$$\varphi_{1-\varepsilon}(-\lambda\sin\beta) \leq -\lambda\cos\beta.$$

Remember that $\vec{k} = (-\sin\beta, -\cos\beta)$ and that u is increasing with respect to y. It follows that $u(\lambda\vec{k}) \geq 1-\varepsilon$ for λ large enough. That implies the required formula (1.3).

Since (1.3) is true for any $\vec{k} \in \mathcal{C}(\vec{e}_2, \pi-\alpha)$ and since u is increasing with respect to y, the stronger limit (1.9) also holds.

Furthermore, for any $\rho \in \mathcal{C}(-\vec{e}_2, \alpha)$, we already know that u is nonincreasing in the direction ρ. Hence, for any $\tau > 0$, the function $z = u((x,y)+\tau\rho) - u(x,y)$ is nonpositive and it satisfies a linear elliptic equation of the type $\Delta z - c\partial_z + c(x,y)z = 0$ in $\mathbb{R}^2$ where $c(x,y)$ is a bounded function. Since $u(\lambda\rho) \to 0$ (resp., 1) as $\lambda \to +\infty$ (resp., $\lambda \to -\infty$), the function z cannot be identically 0. The strong maximum principle implies then that $z > 0$ in $\mathbb{R}^2$. In other words, the function u is decreasing in the direction ρ.

Last, the limiting conditions (1.2) and (1.3) imply that each level set $\{y = \varphi_\lambda(x),\ x \in \mathbb{R}\} = \{u = \lambda\}$ of the function u has two asymptotic directions that are directed by the vectors $(\pm\sin\alpha, -\cos\alpha)$.

4. Uniqueness of the speed *c*. In sections 2 and 3, we have proved the existence of a solution (c,u) of (1.1)–(1.3), (1.8)–(1.9) with the speed $c = c_0/\sin\alpha$ for any angle $\alpha \in (0,\pi/2]$.

Choose an angle $\alpha \in (0,\pi/2]$ and let (c,u) be a solution of (1.1)–(1.3), (1.8)–(1.9). First of all, since f is extended by 0 outside $[0,1]$, the strong maximum principle implies that $0 < u < 1$ in $\mathbb{R}^2$. We shall now prove the equality $c = c_0/\sin\alpha$. We divide the proof into three main steps.

(1) Let us consider the case where $0 < \alpha < \pi/2$ and let us suppose that $c < c_0/\sin\alpha$. For $\varepsilon > 0$ small enough, let f_ε be the function defined in $[-\varepsilon, 1-\varepsilon]$ by

$$f_\varepsilon(s) = \begin{cases} f(s) & \text{on } [-\varepsilon, 1-2\varepsilon], \\ \min\ (f(s), (1-\varepsilon-s)/\varepsilon\ f(1-2\varepsilon)) & \text{on } [1-2\varepsilon, 1-\varepsilon]. \end{cases}$$

Furthermore, we extend the functions f_ε by 0 outside $[-\varepsilon, 1-\varepsilon]$. For $\varepsilon > 0$ small enough, f_ε is Lipschitz continuous in $[-\varepsilon, 1-\varepsilon]$, $(f_\varepsilon)'_-(1-\varepsilon) := \lim_{t\to 1-\varepsilon,\ t<1-\varepsilon} \frac{f_\varepsilon(t)}{t-1+\varepsilon}$ exists and is negative, and f_ε fulfills (1.4) on $[-\varepsilon, 1-\varepsilon]$ with the ignition temperature θ. Moreover, we have $f_\varepsilon \leq f$ and the functions f_ε approach f uniformly in $[0,1]$ as $\varepsilon \to 0$. From the results in [2], [9], [15], [24], there exists a unique couple $(c_\varepsilon, u_\varepsilon)$ satisfying

$$(4.1) \qquad \begin{cases} u_\varepsilon'' - c_\varepsilon u_\varepsilon' + f_\varepsilon(u_\varepsilon) &= 0 \quad \text{in } \mathbb{R}, \\ u_\varepsilon(-\infty) = -\varepsilon,\ u_\varepsilon(0) = \theta,\ u_\varepsilon(+\infty) &= 1-\varepsilon. \end{cases}$$

Furthermore, we have $c_\varepsilon \leq c_0$ and $c_\varepsilon \to c_0$ as $\varepsilon \to 0$ [9].

Since $c < c_0/\sin\alpha$ and $0 < \alpha < \pi/2$, there exist a real $\varepsilon > 0$ small enough and an angle α' such that $0 < \alpha < \alpha' < \pi/2$ and $c < c_\varepsilon/\sin\alpha' < c_0/\sin\alpha$. Set

$$v(x,y) = u_\varepsilon(y\sin\alpha' - x\cos\alpha').$$

Let us first check that v is a subsolution of (1.1). Indeed,

$$\begin{aligned} \Delta v - c\partial_y v + f(v) &= u_\varepsilon'' - c\ \sin\alpha'\ u_\varepsilon' + f(u_\varepsilon) \\ &= (c_\varepsilon - c\sin\alpha')u_\varepsilon' + f(u_\varepsilon) - f_\varepsilon(u_\varepsilon) > 0 \ \text{ in } \mathbb{R}^2 \end{aligned} \tag{4.2}$$

since $c_\varepsilon > c\ \sin\alpha'$, $u_\varepsilon' > 0$, and $f \geq f_\varepsilon$.

We now claim that there exists $\tau \geq 0$ such that

$$v(x, y - \tau) < u(x, y) \ \text{ in } \mathbb{R}^2. \tag{4.3}$$

If not, then for any $n \in \mathbb{N}$, there exists a point $(x_n, y_n) \in \mathbb{R}^2$ such that

$$v(x_n, y_n - n) = u_\varepsilon(\sin\alpha'\ (y_n - n) - \cos\alpha'\ x_n)\ \geq\ u(x_n, y_n). \tag{4.4}$$

The points (x_n, y_n) are not bounded; otherwise the left-hand side of (4.4) approaches $-\varepsilon$, whereas the right-hand side is nonnegative. Write $(x_n, y_n) = \lambda_n(\sin\varphi_n, -\cos\varphi_n)$ with $-\pi < \varphi_n \leq \pi$: φ_n is the angle between (x_n, y_n) and the vector $-\vec{e}_2$ if one goes counterclockwise. We have $\lambda_n \to +\infty$. We can assume, up to extraction, that the sequence (φ_n) approaches $\varphi \in [-\pi, \pi]$ as $n \to +\infty$.

If $-\alpha' < \varphi < \pi - \alpha'$, then

$$v(x_n, y_n - n) = u_\varepsilon(-\lambda_n \sin(\alpha' + \varphi_n) - n\sin\alpha') \to -\varepsilon \text{ as } n \to \infty.$$

This is ruled out by (4.4) since $u > 0$.

In the other case, one has $-\pi \leq \varphi \leq -\alpha'$ or $\pi - \alpha' \leq \varphi \leq \pi$. In particular, $\varphi \in [-\pi, -\alpha) \cup (\alpha, \pi]$. The limiting condition (1.9) implies that $u(x_n, y_n) \to 1$ as $n \to \infty$. This contradicts (4.4) because $u_\varepsilon \leq 1 - \varepsilon$.

As a consequence, (4.3) is true. Next, decrease τ and define

$$\tau^* = \inf\{\tau \in \mathbb{R},\ v(x, y - \tau) < u(x, y) \text{ in } \mathbb{R}^2\}.$$

This real τ^* is finite because there are some points (x, y) where $u(x, y) < 1 - \varepsilon$ and $v(x, y - \tau) \to 1 - \varepsilon$ as $\tau \to -\infty$. For each $n \in \mathbb{N}^*$, there exists a point (x^n, y^n) such that

$$v(x^n, y^n - \tau^* + 1/n) = u_\varepsilon(\sin\alpha'\ (y^n - \tau^* + 1/n) - \cos\alpha'\ x^n) \geq u(x^n, y^n).$$

With the same arguments as above, we claim that the points (x^n, y^n) are bounded. Hence there exists a point $(\overline{x}, \overline{y}) \in \mathbb{R}^2$ such that $v(\overline{x}, \overline{y} - \tau^*) \geq u(\overline{x}, \overline{y})$. Moreover, owing to the definition of τ^*, we have $v(x, y - \tau^*) \leq u(x, y)$ in $\mathbb{R}^2$. The function $z(x, y) = v(x, y - \tau^*) - u(x, y)$ is nonpositive and reaches 0 somewhere in $\mathbb{R}^2$. Furthermore, from (1.1) and (4.2), it satisfies $\Delta z - c\partial_y z + f(v(x, y - \tau^*)) - f(u) \geq 0$ in $\mathbb{R}^2$. This implies that

$$\Delta z - c\partial_y z + c(x, y)z \geq 0$$

for a bounded function $c(x, y)$. The strong maximum principle yields that $z \equiv 0$ in $\mathbb{R}^2$; i.e., $v(x, y - \tau^*) = u_\varepsilon(\sin\alpha'\ (y - \tau^*) - \cos\alpha'\ x) \equiv u(x, y)$ in $\mathbb{R}^2$. This is impossible because $u_\varepsilon \leq 1 - \varepsilon$ and $\sup_{\mathbb{R}^2} u = 1$.

Eventually, that shows that if $0 < \alpha < \pi/2$, then $c \geq c_0/\ \sin\alpha$.

(2) In this part, we deal with the case $\alpha = \pi/2$, which has not been treated in part 1. Indeed, the sliding method used in part 1 no longer works for the limiting case $\alpha = \pi/2$.

Suppose that $c < c_0$. With the same notation as in part 1, there exists a real $\varepsilon > 0$, small enough and fixed, such that $c < c_\varepsilon$, where $(c_\varepsilon, u_\varepsilon)$ is the solution of (4.1). For some reals η, $\kappa > 0$ that will be chosen later, consider the function $v(x, y) = u_\varepsilon(y - \varphi(x))$, where $\varphi(x) = \sqrt{\eta^2 x^2 + \kappa^2}$.

Let us check that this function v is a subsolution of (1.1) if $\eta > 0$ and $\kappa > 0$ are suitably chosen. We have

$$\begin{aligned}\Delta v - c\partial_y v + f(v) &= (1 + \varphi'(x)^2)u_\varepsilon'' - \varphi''(x)u_\varepsilon' - cu_\varepsilon' + f(u_\varepsilon)\\ &= \varphi'(x)^2 u_\varepsilon'' + (c_\varepsilon - c - \varphi''(x))u_\varepsilon' + f(u_\varepsilon) - f_\varepsilon(u_\varepsilon).\end{aligned}$$

On the one hand, we have $f \geq f_\varepsilon$. On the other hand, since u_ε fulfills (4.1), it is well known that u_ε admits the following asymptotic behavior as $x_1 \to \pm\infty$: $u_\varepsilon(x_1) = -\varepsilon + (\theta + \varepsilon)e^{c_\varepsilon x_1}$ if $x_1 \leq 0$ and $u_\varepsilon(x_1) = 1 - \varepsilon - \alpha e^{\lambda' x_1} + o(e^{\lambda x_1})$, $u_\varepsilon'(x_1) = -\alpha\lambda e^{\lambda' x_1} + o(e^{\lambda x_1})$ as $x_1 \to +\infty$, where $\lambda = \frac{c_\varepsilon - \sqrt{c_\varepsilon^2 - 4(f_\varepsilon)'_-(1-\varepsilon)}}{2} < 0$. Furthermore, we have $u_\varepsilon'' = c_\varepsilon u_\varepsilon' - f_\varepsilon(u_\varepsilon)$ and $u_\varepsilon' > 0$ in $\mathbb{R}$. Finally, there exists a constant $C > 0$ such that $|u_\varepsilon''| \leq C u_\varepsilon'$ in $\mathbb{R}$. Remember now that $c_\varepsilon > c$. In order to have $\Delta v - c\partial_y v + f(v) \geq 0$ in $\mathbb{R}^2$, it is then sufficient to choose the function ϕ such that $|\varphi'^2|$ and $|\varphi''|$ are small enough. We have $|\varphi'^2| \leq \eta^2$ and $|\varphi''| \leq \eta^2/\kappa$. Hence, we can choose $\eta > 0$ and $\kappa > 0$ such that

$$\Delta v - c\partial_y v + f(v) \geq 0 \quad \text{in } \mathbb{R}^2.$$

To sum up, the function v is a subsolution of (1.1) and each of its level sets has two asymptotes directed by the vectors $(\pm 1, \arctan \eta)$.

We can now argue as in part 1: formula (4.3) is still true if τ is large enough. As in part 1, we can decrease τ, we can define τ^*, and we get a contradiction thanks to the maximum principle.

This eventually proves that if $\alpha = \pi/2$, then $c \geq c_0$.

(3) Choose now any angle $\alpha \in (0, \pi/2]$. We still have to prove that $c \leq c_0/\sin\alpha$. Suppose on the contrary that $c > c_0/\sin\alpha$. Let us consider some functions f^ε on $[\varepsilon, 1+\varepsilon]$ such that $f^\varepsilon = f$ on $[\varepsilon, 1-\varepsilon]$, $f^\varepsilon > 0$ on $(\theta, 1+\varepsilon)$, $f^\varepsilon(1+\varepsilon) = 0$, $(f^\varepsilon)'(1+\varepsilon)$ exists and is negative, $f^\varepsilon \geq f$ and $\|f^\varepsilon - f\|_\infty \to 0$ as $\varepsilon \to 0$. In particular, the function f^ε is of the ignition temperature type on the interval $[\varepsilon, 1+\varepsilon]$. For each $\varepsilon > 0$ small enough, there exists a unique couple $(c^\varepsilon, u^\varepsilon)$ fulfilling

$$\begin{cases} u^{\varepsilon\prime\prime} - c^\varepsilon u^{\varepsilon\prime} + f^\varepsilon(u^\varepsilon) = 0 & \text{in } \mathbb{R},\\ u^\varepsilon(-\infty) = \varepsilon,\ u^\varepsilon(0) = \theta,\ u^\varepsilon(+\infty) = 1 + \varepsilon. \end{cases}$$

Furthermore, $c^\varepsilon > c_0$ and $c^\varepsilon \to c_0$ as $\varepsilon \to 0$ (see [9]).

Choose α' and $\varepsilon > 0$ such that $0 < \alpha' < \alpha \leq \pi/2$ and $c > c^\varepsilon/\sin\alpha' > c_0/\sin\alpha$. From Theorem 1.1 applied to the function f^ε, there exists a solution $v(x, y)$ of

$$\begin{cases} \Delta v - c^\varepsilon/\sin\alpha'\ \partial_y v + f^\varepsilon(v) = 0 \quad \text{in } \mathbb{R}^2,\\ v(\lambda\vec{k}') \to \varepsilon \quad \text{as } \lambda \to +\infty \text{ and } \vec{k}' \to \vec{k} \in \mathcal{C}(-\vec{e}_2, \alpha'),\\ v(\lambda\vec{k}') \to 1 + \varepsilon \quad \text{as } \lambda \to +\infty \text{ and } \vec{k}' \to \vec{k} \in \mathcal{C}(\vec{e}_2, \pi - \alpha'). \end{cases}$$

Moreover, $\partial_y v \geq 0$. The function v is a supersolution of (1.1) in the sense that

$$\Delta v - c\partial_y v + f(v) = (c^\varepsilon/\sin\alpha' - c)\partial_y v + f(v) - f^\varepsilon(v) \leq 0 \quad \text{in } \mathbb{R}^2$$

since $c > c^\varepsilon/\sin\alpha'$, $\partial_y v \geq 0$, and $f \leq f^\varepsilon$.

We now claim that there exists $\tau \geq 0$ such that

$$v(x, y+\tau) > u(x, y) \text{ in } \mathbb{R}^2.$$

Otherwise, for each $n \in \mathbb{N}$, there exists a point $(x^n, y^n) \in \mathbb{R}^2$ such that $v(x^n, y^n + n) \leq u(x^n, y^n)$. As in part 1, by dealing successively with the cases where the sequence (x_n, y_n) is bounded or unbounded, we would get a contradiction.

Now, let us set

$$\tau^* = \inf \{\tau \in \mathbb{R},\ v(x, y+\tau) > u(x, y) \text{ in } \mathbb{R}^2\}.$$

As above, τ^* is finite and $v(x, y+\tau^*) \geq u(x, y)$ in $\mathbb{R}^2$ with equality somewhere. This is ruled out by the strong maximum principle.

Finally, it is always true that $c \leq c_0/\ \sin\alpha$. Together with parts 1 and 2, this inequality completes the proof of Theorem 1.2.

5. Appendix: Proof of Lemma 2.10. In this section, we actually deal with a more general situation than in Lemma 2.10. Let u be a bounded and positive function defined in the set

$$V = \{(x, y) \in \mathbb{R}^2, x > 0, y > 0, \sqrt{x^2 + y^2} < \delta\}$$

for a certain $\delta > 0$. We assume that the function u belongs to $W^{2,p}_{loc}(\overline{V}\backslash\{(0,0)\})$ for all $1 < p < \infty$ and that it is continuous in $\overline{V}$. We also suppose that that function v satisfies the following equations:

$$\left\{ \begin{aligned} \Delta u - c\partial_y u + f(u) = 0 &\quad \text{in } V, \\ u(x, 0) = 0 &\quad \text{for } 0 \leq x \leq \delta, \\ \partial_\tau u(0, y) = 0 &\quad \text{for } 0 < y \leq \delta, \end{aligned} \right. \tag{5.1}$$

where $\tau = (-\sin\alpha, -\cos\alpha)$. The given function f is Lipschitz continuous. Furthermore, $f(0) = 0$ and $f'_+(0) = \lim_{t\to 0,\ t>0} \frac{f(t)-f(0)}{t}$ exists.

Set $O = (0, 0)$. Choose any vector $\rho = (\cos\beta, \sin\beta)$ with $\pi/2 - \alpha < \beta < \pi$. We are going to determine the asymptotic behavior of u and ∇u in the neighborhood of the corner O. That behavior will imply the existence of a neighborhood $\tilde{V}$ of O and of a real $\varepsilon_1 > 0$ such that if $0 < \varepsilon \leq \varepsilon_1$ and if $z,\ z + \varepsilon\rho \in \tilde{V} \cap \overline{V}$, then $u(z) < u(z + \varepsilon\rho)$.

Before doing that, we briefly mention some papers and results that have been devoted to similar problems in the literature. In many works (see, e.g., Bernardi and Maday [10], Grisvard [19], Maz'ja and Plamenevskii [30]), the *linear* elliptic problem

$$\begin{gathered} Lu = f \text{ in } G, \\ Bu = g \text{ on } \partial G\backslash\{K\} \end{gathered} \tag{5.2}$$

has been investigated under the assumption that G is a subdomain of the plane $\mathbb{R}^2$ and that the boundary ∂G of G is Lipschitz continuous everywhere and smooth except at a corner K, say, $K = O$. Assume that L is an elliptic operator and B is a smooth linear function depending on the traces of u or ∇u on $\partial G\backslash\{K\}$. The function u belongs to some Sobolev spaces with weights but u, or its derivatives, may be singular at the point K. The general result is the following: in a neighborhood of the point $K = O$, the function u can be written as

$$u(r, \theta) = \sum_{k\geq 1} c_k r^{\alpha_k} \sum_{h=0}^{k} (-\ln r)^h \varphi_{k,h}(\theta), \tag{5.3}$$

where (r,θ) is the usual polar coordinate and where the complex numbers α_k have nondecreasing real parts. Thanks to the change of variables $r = e^t$ (see Kondrat'ev [25]), equation (5.2) becomes

$$\tilde{L}u = \tilde{f}$$

in a set containing an infinite strip of the type $(-\infty, \alpha] \times (0, \beta)$. The terms r^{α_k} become $e^{\alpha_k t}$ and the numbers α_k are given in terms of the eigenvalues of an operator L_0 depending on θ and on the principal part of L at the corner K.

In particular, for the Dirichlet problem

$$\begin{aligned} \Delta u = f \quad & \text{in } G = \{r > 0,\ 0 < \theta < \omega\}, \\ u = 0 \quad & \text{on } \partial G \backslash \{K\}, \end{aligned}$$

where $f \in W^{m,p}(G)$, it is known that, in a neighborhood of K, the function u is equal to

$$u(r,\theta) = \sum_{\pi/\omega \le k\pi/\omega < m+2-2/p} c_k r^{k\pi/\omega} \left\{ \begin{array}{l} \sin(k\pi\theta/\omega) \\ \text{or } (\ln r)\sin(k\pi\theta/\omega) + \theta\cos(k\pi\theta/\omega) \end{array} \right. + u_R,$$

where $u_R \in W^{m+2,p}(G)$ (see Geymonat and Grisvard [16], Grisvard [19], [20], or Dauge [13] for a three-dimensional situation).

Let us now come back to the elliptic problem (5.1) that is set in the domain V with the corner O. The boundary conditions on ∂V are of the Dirichlet and oblique-Neumann type. But, unlike the problems mentioned above, we have to deal with a *semilinear* problem. Then, we cannot a priori hope for an infinite asymptotic development of the type (5.3) for u. Nevertheless, we only need to know what u and its derivatives are equivalent to in the neighborhood of O.

In [9], [8], Berestycki and Nirenberg have emphasized the semilinear problem

$$\begin{aligned} Lu + f(x_1, u) = 0,\ u > 0 \quad & \text{in } \Sigma_- = \{(x_1, y),\ x_1 < 0,\ y \in \omega\}, \\ \partial_\nu u = 0 \quad & \text{on } (-\infty, 0) \times \partial\omega, \end{aligned}$$

where ω is a smooth domain with unit outward normal ν. If $u \to 0$ as $x_1 \to -\infty$ and if $|f(x_1,u)| = O(u^{1+\delta})$ as $u \to 0$ for a certain $\delta > 0$, then the nonlinear term $f(x_1,u)$ only makes small perturbations with respect to Δu. The asymptotic behavior of u as $x_1 \to -\infty$ is given in [8], [9].

If we come back to (5.1) and if we make the change of variables $r = e^t$, we can see that u fulfills

$$\Delta u - c\sin\theta \varepsilon^t \partial_t u - c\cos\theta\ e^t \partial_\theta u + e^{2t} f(u) = 0 \text{ in } (-\infty, \ln\delta) \times (0, \pi/2)$$

with Dirichlet and oblique-Neumann boundary conditions:

$$\begin{aligned} u = 0 \quad & \text{on } \{\theta = 0\}, \\ -\cos\alpha\ \partial_t u + \sin\alpha\ \partial_\theta u = 0 \quad & \text{on } \{\theta = \pi/2\}. \end{aligned}$$

To conclude this discussion, the semilinear problem (5.1) with mixed boundary conditions does not seem to have been treated so far in the literature. Hence, for the sake of completeness, we give a detailed proof of Lemma 5.1.

LEMMA 5.1. *Let $\gamma = (2/\pi)\ \alpha$. There exists a real $\lambda > 0$ such that*

$$\left\{ \begin{array}{ll} u - \lambda r^\gamma \sin(\gamma\theta) & = o(r^\gamma) \\ \nabla u - \lambda \nabla(r^\gamma \sin(\gamma\theta)) & = o(r^{\gamma-1}) \end{array} \right. \quad as\ r \overset{>}{\to} 0.$$

Proof of Lemma 2.10. Consider the behavior of u near the corner C_1 of Σ_a and call (r, θ) the polar coordinates with respect to the point C_1. From Lemma 5.1, one has

$$\nabla u \cdot \rho - \lambda \nabla(r^\gamma \sin(\gamma\theta)) \cdot \rho = o(r^{\gamma-1}) \quad \text{as } r \to 0. \tag{5.4}$$

Remember that $\rho = (\cos\beta, \sin\beta)$ with $\pi/2 - \alpha < \beta < \pi$. Thus,

$$\nabla(r^\gamma \sin(\gamma\theta)) \cdot \rho = \gamma r^{\gamma-1} \sin((\gamma-1)\theta + \beta).$$

For any point $z = (r, \theta) \in V$, we have

$$0 < \alpha - \pi/2 + \beta \leq (\gamma - 1)\theta + \beta \leq \beta < \pi.$$

As a consequence, there exists a real $\eta > 0$ such that

$$r^{-(\gamma-1)}\ \nabla(r^\gamma \sin(\gamma\theta)) \cdot \rho \geq \eta > 0.$$

From (5.4), it follows then that $\partial_\rho u > 0$ in a neighborhood V_1 of C_1. As far as the behavior of the function u near the corner C_1 of Σ_a is concerned, Lemma 2.10 is then a consequence of the finite increment theorem.

The other corner C_3 can be treated similarly. Indeed, after setting the origin in C_3 and making the change of variables $y \to -y$, $\tilde{u}(x, y) = u(x, -y)$, we find that

$$\left\{ \begin{array}{ll} (1 - \tilde{u}) - \lambda r^\gamma \sin(\gamma\theta) & = o(r^\gamma) \\ -\nabla\tilde{u} - \lambda\nabla(r^\gamma \sin(\gamma\theta)) & = o(r^{\gamma-1}) \end{array} \right. \quad \text{as } r \overset{>}{\to} 0,$$

where $\gamma = (2/\pi)\ (\pi - \alpha)$ and where λ is a positive real. The same calculations as above yield that, for any $\rho = (\cos\beta, \sin\beta)$ with $\pi/2 - \alpha < \beta < \pi$, the function u is such that $\partial_\rho u > 0$ in a neighborhood V_3 of C_3. Notice that, unlike the situation around the point C_1, the function $\partial_\rho u$ is bounded near C_3 since $\gamma \geq 1$. □

Proof of Lemma 5.1. Remember first that $V = \{0 < r < \delta,\ 0 < \theta < \pi/2\}$. We choose to work with the (r, θ) coordinates. Notice that everything works similarly with the coordinates (t, θ), where $r = e^t$. The following proof, similar to the one in [8], is divided into six main steps for the sake of clarity.

Step 1. Set $\gamma = (2/\pi)\ \alpha$; notice that $\gamma \in (0, 1]$. Let v be the function

$$v(r, \theta) = r^\gamma \sin(\gamma\theta) \quad \text{for } (r, \theta) \in (0, \delta] \times [0, \pi/2]$$

and $v(O) = 0$. It is easy to check that

$$\left\{ \begin{array}{rl} \Delta v = 0 & \text{in } V, \\ \partial_\tau v(0, y) = 0 & \text{if } 0 < y < \delta, \end{array} \right.$$

where $\tau = (-\sin\alpha, -\cos\alpha)$. Moreover, $v(x, 0) = 0$ for all $0 \leq x \leq \delta$ and $v(x, y) > 0$ if $y > 0$.

Step 2. We now want to construct two sub- and supersolutions $\underline{v}$ and $\overline{v}$ such that

$$\left\{ \begin{array}{rl} \Delta\underline{v} - c\partial_y\underline{v} + f(\underline{v}) \geq 0 & \text{in } V_0, \\ \underline{v}(x, 0) \leq 0 & \text{if } 0 \leq x < \delta_0, \\ \partial_\tau\underline{v}(0, y) < 0 & \text{if } 0 < y < \delta_0, \end{array} \right. \tag{5.5}$$

$$(5.6) \qquad \begin{cases} \Delta\overline{v} - c\partial_y\overline{v} + f(\overline{v}) \leq 0 & \text{in } V_0, \\ \overline{v}(x,0) \geq 0 & \text{if } 0 \leq x < \delta_0, \\ \partial_\tau\overline{v}(0,y) > 0 & \text{if } 0 < y < \delta_0, \end{cases}$$

in a small enough neighborhood V_0 of O of the type $V_0 = V \cap B(0,\delta_0)$, where the real $\delta_0 \in (0,\delta]$ will be chosen later.

Consider the functions

$$\begin{cases} \underline{g}(\theta) & = 1 - \cos(\underline{\beta}\theta) + \underline{A}\sin(\underline{\beta}\theta), \\ \overline{g}(\theta) & = -1 + \cos(\overline{\beta}\theta) + \overline{A}\sin(\overline{\beta}\theta), \end{cases}$$

and

$$\begin{cases} \underline{v} & = r^\gamma \sin(\gamma\theta) + r^{\underline{\beta}}\underline{g}(\theta), \\ \overline{v} & = r^\gamma \sin(\gamma\theta) + r^{\overline{\beta}}\overline{g}(\theta), \end{cases}$$

where $\underline{\beta}$ and $\overline{\beta}$ are two fixed reals, different from 1 and such that $\gamma < \underline{\beta}, \overline{\beta} < \gamma + 1$. The reals $\underline{A}$ and $\overline{A}$ will be chosen later. A straightforward computation gives

$$\begin{aligned} L\underline{v} := \; & \Delta\underline{v} - c\partial_y\underline{v} + f(\underline{v}) \\ = \; & \underline{\beta}^2 r^{\underline{\beta}-2} - c\gamma r^{\gamma-1}\cos((\gamma-1)\theta) \\ & -c\underline{\beta} r^{\underline{\beta}-1}[\sin\theta + \sin((\underline{\beta}-1)\theta) + \underline{A}\cos((\underline{\beta}-1)\theta)] + f(\underline{v}). \end{aligned}$$

Since $\underline{\beta} < \gamma + 1$ and $|f(t)| \leq M|t|$ for all t (with $M = \|f\|_{Lip} = \sup_{x,y\in[0,1],\ x\neq y} \frac{|f(x)-f(y)|}{|x-y|}$), it follows that there exists a real $\delta_1 \in (0,\delta]$ that depends only on α, $\underline{\beta}$, M, and $\underline{A}$ such that $L(\kappa\underline{v}) > 0$ in $V \cap B(O,\delta_1)$ for any $\kappa > 0$. On the other hand,

$$\forall 0 < y < \delta, \quad \partial_\tau\underline{v}(0,y) = \underline{\beta} r^{\underline{\beta}-1}[2\sin(\alpha - \underline{\beta}\pi/4)\sin(\underline{\beta}\pi/4) + \underline{A}\sin(\alpha - \underline{\beta}\pi/2)].$$

Since $(2/\pi)\,\alpha < \underline{\beta} < (2/\pi)\,\alpha + 1$, we can then choose a real $\underline{A}$ large enough, depending on α and $\underline{\beta}$, such that $\partial_\tau\underline{v}(0,y) < 0$ for all $0 < y < \delta_1$. Furthermore, we have $\underline{v}(x,y) = 0$ if $y = 0$ and $0 \leq x < \delta_1$. We then conclude that $\underline{v}$ satisfies (5.5) in $V \cap B(O,\delta_1)$.

Similarly, we can prove that there exists a real $\delta_2 \in (0,\delta]$ such that $\overline{v}$ satisfies (5.6) in $V \cap B(O,\delta_2)$. Eventually, by defining $\delta_0 = \min(\delta_1,\delta_2)$, it follows that $\underline{v}$ (resp., $\overline{v}$) satisfies (5.5) (resp., (5.6)) in $V_0 = V \cap B(0,\delta_0)$.

Step 3. Even if it means decreasing $\delta_0 > 0$, we can assume that $\underline{v}$ and $\overline{v}$ are positive in $\overline{V_0} \cap \{y > 0\}$. Indeed, this is possible because $\gamma < \underline{\beta}, \overline{\beta}$, because $\sin(\gamma\theta) > 0$ for $0 < \theta < \pi/2$ and because both functions $\underline{g}(\theta)/\sin(\gamma\theta)$ and $\overline{g}(\theta)/\sin(\gamma\theta)$ are bounded in the interval $\{0 \leq \theta \leq \pi/2\}$. On the other hand, we define a function

$$\varphi(x,y) = 2e^{\cos\alpha\ +\ \sin\alpha} - e^{1/\delta_0(\cos\alpha\ x\ -\ \sin\alpha\ y\ +\ \sin\alpha\ \delta_0)} \text{ in } V_0.$$

We observe that the function φ is positive in $\overline{V_0}$ and $\partial_\tau\varphi(0,y) = 0$ for all $0 < y < \delta_0$. Furthermore, we have

$$\Delta\varphi - c\partial_y\varphi + \|f\|_{Lip}\varphi \leq -1/\delta_0^2 + 1/\delta_0\ |c|\sin\alpha\ e^{\cos\alpha+\sin\alpha} + 2\|f\|_{Lip}e^{\cos\alpha+\sin\alpha}.$$

Even if it means decreasing again $\delta_0 > 0$, we may also assume that

$$\Delta\varphi - c\partial_y\varphi + \|f\|_{Lip}\varphi < 0 \ \text{ in} V_0.$$

Since u is positive in V_0 and satisfies (5.1), the maximum principle and the Hopf lemma yield that $u(x,y) > 0$ as soon as $y > 0$ and that $\partial_y u(x,0) > 0$ for all $x > 0$. Similarly, $\partial_y \overline{v}(x,0) > 0$ for all $x > 0$. Finally, there exist two reals ν, $\mu > 0$ such that

$$\forall (x,y) \in V \cap \{x^2 + y^2 = \delta_0^2\}, \quad \mu \underline{v}(x,y) < u(x,y) < \nu \overline{v}(x,y). \tag{5.7}$$

Let us now show that this last inequality (5.7) is actually true in the whole set V_0. Remember that u solves (5.1) and that $\mu \underline{v}$ satisfies inequality (5.5). Hence, the function $w = u - \mu \underline{v}$ satisfies

$$\tilde{L} w := \Delta w - c \partial_y w + c(x,y) w \leq 0 \ \text{ in } V_0,$$

where $c(x,y)$ is a bounded function in V_0 such that $\|c\|_\infty \leq \|f\|_{Lip}$. Set $g = w/\varphi$. One has

$$Mg := \Delta g + 2\frac{\nabla \varphi}{\varphi} \cdot \nabla g - c\partial_y g \leq -\frac{g}{\varphi}(\Delta\varphi - c\partial_y \varphi + c(x,y)\varphi) = -\frac{g}{\varphi}\tilde{L}\varphi.$$

In view of the properties fulfilled by φ, it follows that

$$\tilde{L}\varphi \leq \Delta\varphi - c\partial_y\varphi + \|f\|_{Lip}\varphi < 0 \ \text{ in } V_0.$$

If the set $\Omega_- = \{(x,y) \in V_0, \ g(x,y) < 0\}$ is not empty, we get that $Mg < 0$ in Ω_-. Since g is continuous in $\overline{V_0}$ (the function φ is positive and continuous in the compact set $\overline{V_0}$), let z_0 be a point in $\overline{\Omega_-}$ where g reaches its minimal value. If $z_0 \in V_0$, then $\nabla g(z_0) = 0$ and $\Delta g(z_0) \geq 0$. That is impossible because $Mg(z_0) < 0$. Now, since $w \geq 0$ on $\partial V_0 \cap (\{y = 0\} \cup \{x^2 + y^2 = \delta_0^2\})$, it follows that $z_0 = (0, y_0)$ with $0 < y_0 < \delta_0$. Furthermore, since $\partial_\tau \underline{v}(0,y_0) < 0$, we have $\partial_\tau w(z_0) = \partial_\tau u(z_0) - \mu \partial_\tau \underline{v}(z_0) > 0$ and

$$0 < \partial_\tau w(z_0) = g(z_0)\partial_\tau\varphi(z_0) + \varphi(z_0)\partial_\tau g(z_0).$$

The function φ is such that $\partial_\tau \varphi(z_0) = 0$ and $\varphi(z_0) > 0$. Hence, $\partial_\tau g(z_0) > 0$. The latter is ruled out by the Hopf lemma.

Finally, we have $\Omega_- = \emptyset$, whence $w \geq 0$; i.e., $\mu \underline{v} \leq u$ in V_0 and even $\mu \underline{v} < u$ in V_0 from the strong maximum principle. Similarly, we infer that $u < \nu \overline{v}$ in V_0.

So far, we have shown that

$$\mu \underline{v} < u < \nu \overline{v} \ \text{ in } V_0 = \{x > 0, \ y > 0, \ r < \delta_0\}.$$

Step 4. Let us now replace the variables (x,y) with $(\varepsilon x, \varepsilon y)$. Set $W_\varepsilon = \{(x,y) \in \mathbb{R}^2, \ (\varepsilon x, \varepsilon y) \in V_0\}$ and $u_\varepsilon(x,y) = \varepsilon^{-\gamma} u(\varepsilon x, \varepsilon y)$ for $(x,y) \in W_\varepsilon$. From the definitions of $\overline{v}$ and $\underline{v}$, we have

$$\mu \ (v + \varepsilon^{\underline{\beta}-\gamma} r^{\underline{\beta}} \underline{g}(\theta)) < u_\varepsilon(x,y) < \nu \ (v + \varepsilon^{\overline{\beta}-\gamma} r^{\overline{\beta}} \overline{g}(\theta)) \ \text{ in } W_\varepsilon, \tag{5.8}$$

where $r = \sqrt{x^2 + y^2}$. Let Π be the positive quadrant

$$\Pi = \{x > 0, y > 0\}.$$

Since $\gamma < \underline{\beta}, \overline{\beta}$, the left and the right sides of the inequality (5.8) uniformly approach μv and νv in any compact set $K \subset \overline{\Pi}$ as $\varepsilon \to 0$.

Furthermore, we have

$$\begin{cases} \Delta u_\varepsilon - \varepsilon c \partial_y u_\varepsilon & = -\varepsilon^{2-\gamma} f(u(\varepsilon x, \varepsilon y)) & \text{in } W_\varepsilon, \\ u_\varepsilon(x,0) & = 0 & \text{for all } 0 \leq x < \delta_0/\varepsilon, \\ \partial_\tau u_\varepsilon(0,y) & = 0 & \text{for all } 0 < y < \delta_0/\varepsilon. \end{cases}$$

Since $\gamma < 2$ and $f(u)$ is bounded in $\overline{V_0}$, the right side of the equation fulfilled by u_ε approaches 0 uniformly in any compact set $K \subset \overline{\Pi}$. The functions u_ε are defined in such a compact set K for ε small enough and they are also uniformly bounded in K from (5.8). Moreover, from the standard elliptic estimates up to the boundary, the functions (u_ε) are then bounded in $W^{2,p}(K)$ for any compact set $K \subset \overline{\Pi}\backslash\{O\}$ and for any $1 < p < \infty$. By a diagonal extraction process, it follows that there exists a continuous function u_0 defined in $\overline{\Pi}\backslash\{O\}$ such that, up to extraction of some subsequence, $u_\varepsilon \to u_0$ in $C^{1,\delta}_{loc}(\overline{\Pi}\backslash\{O\})$ for any $\delta \in (0,1)$. The function u_0 fulfills

$$\begin{cases} \Delta u_0 = 0 & \text{in } \Pi, \\ u_0(x,0) = 0 & \text{for all } x > 0, \\ \partial_\tau u_0(0,y) = 0 & \text{for all } y > 0. \end{cases} \tag{5.9}$$

Moreover, $\mu v \le u_0 \le \nu v$ in $\overline{\Pi}\backslash\{O\}$. In particular, the latter implies that the function u_0 can be extended by continuity at the point $O = (0,0)$ by setting $u_0(0,0) = 0$. Hence,

$$\mu v \le u_0 \le \nu v \ \text{ in } \overline{\Pi}.$$

From (5.8), for any $\eta > 0$, there exists $\delta' > 0$ such that $|u_\varepsilon| \le \eta$ in $\{(x,y) \in \overline{\Pi},\ \sqrt{x^2+y^2} \le \delta'\}$. It follows that, up to extraction of some subsequence, the functions u_ε also approach u_0 uniformly in any compact set $K \subset \overline{\Pi}$.

Step 5. We now aim at proving that $u_0 = \lambda v$ for a certain λ such that $\mu \le \lambda \le \nu$. Define $\overline{\mu}$ and $\overline{\nu}$ by $\overline{\mu} = \sup\ \{\mu,\ \mu v \le u_0 \text{ in } \overline{\Pi}\}$ and $\overline{\nu} = \inf\ \{\nu,\ u_0 \le \nu v \text{ in } \overline{\Pi}\}$. We have $\overline{\mu} v \le u_0 \le \overline{\nu} v$ in $\overline{\Pi}$ and $\overline{\mu} \le \overline{\nu} \in \mathbb{R}$.

Let us now suppose that $\overline{\mu} < \overline{\nu}$. The strong maximum principle then yields that $\overline{\mu} v < u_0 < \overline{\nu} v$ in Π. For every $R > 0$, let us call $C(R) = \{(x,y) \in \overline{\Pi},\ x^2+y^2 = R^2\}$ and $B(R) = \{(x,y) \in \overline{\Pi},\ x^2+y^2 \le R^2\}$. Choose any $R > 0$. On $C(R)$, we have $v > 0$ and $\overline{\mu} \le u_0/v \le \overline{\nu}$. There exists then a subset $\Gamma \subset C(R)$ such that $|\Gamma|/|C(R)| \ge 1/2$ ($|\Gamma|$ is the length of Γ) and one of the following assertions occurs:

$$\text{(i)} \quad \frac{\overline{\mu}+\overline{\nu}}{2} \le \frac{u_0}{v} \text{ on } \Gamma, \quad \text{i.e., } u_0 - \overline{\mu} v \ge \frac{\overline{\nu}-\overline{\mu}}{2} v,$$

$$\text{(ii)} \quad \frac{u_0}{v} \le \frac{\overline{\mu}+\overline{\nu}}{2} \text{ on } \Gamma, \quad \text{i.e., } \overline{\nu} v - u_0 \ge \frac{\overline{\nu}-\overline{\mu}}{2} v.$$

Suppose that case (i) occurs. Since $u_0 - \overline{\mu} v > 0$ in Π, since both u_0 and v fulfill (5.9), and since (5.9) is invariant by stretching the variables, a straightforward application of the Harnack inequality up to the boundary leads to the existence of a real $\varepsilon > 0$, which does not depend on R, such that

$$u_0 - \overline{\mu} v \ge \varepsilon v \ \text{ on } C(R/2)$$

(see also Berestycki, Caffarelli, and Nirenberg [3] and Caffarelli [12] for related problems). Hence, as in Step 3, we get

$$u_0 - \overline{\mu} v \ge \varepsilon v \ \text{ in } B(R/2).$$

Since (i) or (ii) occurs for each $R > 0$, we may suppose, say, that there is a sequence $R_n \to +\infty$ such that (i) occurs for each R_n. As a consequence, $u_0 - \overline{\mu} v \ge \varepsilon v$ in $B(R_n/2)$, whence

$$u_0 - \overline{\mu} v \ge \varepsilon v \ \text{ in } \overline{\Pi}.$$

That is ruled out by the definition of $\overline{\mu}$.

We conclude that $\overline{\mu} = \overline{\nu} =: \lambda$, that is to say that $u_0 \equiv \lambda v$ in $\overline{\Pi}$.

Step 6. Conclusion: we have to prove that

$$u - \lambda r^\gamma \sin(\gamma\theta) = o(r^\gamma) \quad \text{as } r \overset{>}{\to} 0, \tag{5.10}$$

$$\nabla u - \lambda \nabla(r^\gamma \sin(\gamma\theta)) = o(r^{\gamma-1}) \quad \text{as } r \overset{>}{\to} 0. \tag{5.11}$$

Let K be the compact defined by $K = \{(x,y) \in \overline{\Pi}, 1 \le \sqrt{x^2+y^2} \le 2\}$ and let η be any positive number. We know that $u_\varepsilon \to \lambda v$ as $\varepsilon \to 0$, uniformly in K. Hence, there exists a real $\varepsilon_0 \in (0,1)$ such that: $\forall 0 < \varepsilon \le \varepsilon_0$, $\forall (x,y) \in K$, $|u_\varepsilon - \lambda v| \le \eta$. Owing to the definitions of the function u_ε and v, we get

$$\forall (x,y) \in K,\ \forall \varepsilon \le \varepsilon_0, \quad |u(\varepsilon x, \varepsilon y) - \lambda(\varepsilon r)^\gamma \sin(\gamma\theta)| \le \eta\varepsilon^\gamma \le \eta(\varepsilon r)^\gamma.$$

In other words, for each $(x,y) \in \overline{\Pi}$ such that $0 < r = \sqrt{x^2+y^2} \le 2\varepsilon_0$, we have $|u(x,y) - \lambda r^\gamma \sin(\gamma\theta)| \le \eta r^\gamma$. Since $\eta > 0$ was arbitrary, we have thus shown the formula (5.10).

Assertion (5.11) can be proved with the same arguments as above. That completes the proof of Lemma 5.1. □

Remark 5.2. Let $\overline{v}$ be defined as in Step 2 by

$$\overline{v} = r^\gamma \sin(\gamma\theta) + r^{\overline{\beta}}\overline{g}(\theta),$$

where $\overline{g}(\theta) = -1 + \cos(\overline{\beta}\theta) + \overline{A}\sin(\overline{\beta}\theta)$ and where (r,θ) are the polar coordinates with respect to the corner $C_1 = (-a, -a\cot\gamma)$ of Σ_a. We choose $\overline{A}$ such that (5.6) holds in $V_0 = \{x > 0,\ y > 0,\ 0 < r < \delta_0\}$ for some δ_0 small enough. In particular, for $\varepsilon \in (0, \delta_0)$, we have $\partial_\tau \overline{v} = \nabla\overline{v} \cdot \tau > 0$ at the point $(-a, -a\cot\gamma + \varepsilon)$. Hence, under the notation of Lemma 2.1, one can require that the vector field ρ_ε fulfill $\rho_\varepsilon = \tau$ on $\{-a\} \times (-a\cot\gamma + \varepsilon, -a\cot\gamma + \delta_0)$ and $\rho_\varepsilon \cdot \nabla\overline{v} \ge 0$ on $\partial\Sigma_{a,\varepsilon} \cap B(C_1, \delta_0)$. For instance, choose a function $\eta(x,y)$ defined on $\partial\Sigma_{a,\varepsilon} \cap B(C_1,\delta_0)$ such that $0 \le \eta \le 1$, $\eta = 1$ on $\{-a\} \times (-a\cot\gamma + \varepsilon, -a\cot\gamma + \delta_0)$, $\eta = 0$ on $\partial\Sigma_{a,\varepsilon} \cap \{x > -a + \varepsilon^2\}$ (for $\varepsilon > 0$ small enough). Next, take $\rho_\varepsilon(x,y) = \eta(x,y)\tau$ on $\partial\Sigma_{a,\varepsilon} \cap B(C_1,\delta_0)$. Finally, the function $\overline{v}$ fulfills

$$\rho_\varepsilon \cdot \nabla\overline{v} + \sigma_{0,\varepsilon}\overline{v} \ge 0 \quad \text{on } \partial\Sigma_{a,\varepsilon} \cap B(C_1,\delta_0),$$

whereas the function u_ε fulfills

$$\rho_\varepsilon \cdot \nabla u_\varepsilon + \sigma_{0,\varepsilon} u_\varepsilon = 0 \quad \text{on } \partial\Sigma_{a,\varepsilon} \cap B(C_1,\delta_0)$$

(remember that $\sigma_{1,\varepsilon} = 0$ on $\partial\Sigma_{a,\varepsilon} \cap B(C_1,\delta_0)$ for $\varepsilon > 0$ and $\delta_0 > 0$ small enough).

Furthermore, since $\partial_y u_\varepsilon(-a+\delta_0, -a\cot\gamma) \to \partial_y u_c(-a+\delta_0, -a\cot\gamma) < +\infty$ as $\varepsilon \to 0$ and $u_\varepsilon \le 1$ in $\overline{\Sigma_{a,\varepsilon}}$, there exists then a constant $\nu > 0$ such that, as in Step 3,

$$\forall (x,y) \in \overline{\Sigma_{a,\varepsilon}} \cap \{r = \delta_0\}, \quad u_\varepsilon(x,y) \le \nu\overline{v}(x,y)$$

for all $\varepsilon > 0$ small enough. Next, we choose the same function φ as in Step 3. In particular, in view of the choice of ρ_ε, we have $\rho_\varepsilon \cdot \nabla\varphi = 0$ and $\rho_\varepsilon \cdot \nu_\varepsilon \ge 0$ on $\partial\Sigma_{a,\varepsilon} \cap B(C_1,\delta_0)$ for $\varepsilon > 0$ small enough (ν_ε is the outward unit normal to $\partial\Sigma_{a,\varepsilon}$). As in Step 3, it follows then that if the function $g = \frac{w}{\varphi} := \frac{\nu\overline{v} - u_\varepsilon}{\varphi}$ reaches a negative minimal value at a point z_0 in $\overline{\Sigma_{a,\varepsilon}} \cap \overline{B(C_1, \delta_0)}$, then $z_0 = (x_0, y_0)$ lies necessarily on $\partial\Sigma_{a,\varepsilon} \cap B(C_1, \delta_0)$. At the point z_0, one has $\rho_\varepsilon \cdot \nabla w + \sigma_{0,\varepsilon} w \geq 0$, whence

$$g(z_0)\ \rho_\varepsilon(z_0) \cdot \nabla\varphi(z_0) + \varphi(z_0)\ \rho_\varepsilon(z_0) \cdot \nabla g(z_0) + \sigma_{0,\varepsilon}(z_0) g(z_0) \varphi(z_0) \geq 0. \tag{5.12}$$

The first term of (5.12) is equal to 0 because $\rho_\varepsilon \cdot \nabla\varphi = 0$. The second and third terms are nonpositive because $\varphi > 0$, $\rho_\varepsilon \cdot \nabla g \leq 0$ (from the Hopf lemma), $g(z_0) < 0$, and $\sigma_{0,\varepsilon} \geq 0$. Furthermore, if $y_0 \geq -a \cot\gamma + \varepsilon$, then $\rho_\varepsilon(z_0) = \tau$ whence $\rho_\varepsilon(z_0) \cdot \nabla g(z_0) < 0$, and if $y_0 \leq -a \cot\gamma + \varepsilon$, then $\sigma_{0,\varepsilon}(z_0) = 1$. Hence, all the three terms of (5.12) are nonpositive and at least one is negative. This is impossible.

We conclude that

$$u_\varepsilon(x, y) \leq \nu \overline{v}(x, y) \quad \text{in } \overline{\Sigma_{a,\varepsilon}} \cap \overline{B(C_1, \delta_0)}$$

for all $\varepsilon > 0$ small enough. This gives the required estimate (2.5) around the point C_1. The other corners C_2, C_3, C_4 can be treated similarly.

The proofs of the estimates (2.8) and (2.10) resort to the same arguments. As far as (2.8) is concerned, the function $\overline{v}$ can be chosen as in Step 2 such that (5.6) is true for each c_n because the reals c_n are bounded. As far as (2.10) is concerned, the function $\overline{v}$ can be chosen as in Step 2 such that (5.6) is true for each f_n because the norms $\|f_n\|_{Lip}$ are bounded.

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REFERENCES

[1] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*, Comm. Pure Appl. Math., 12 (1959), pp. 623–727; 17 (1964), pp. 35–92.

[2] D. G. Aronson and H. F. Weinberger, *Multidimensional nonlinear diffusions arising in population genetics*, Adv. Math., 30 (1978), pp. 33–76.

[3] H. Berestycki, L. Caffarelli, and L. Nirenberg, *Uniform estimates for regularisation of free boundary problems*, in C. Sadosky & M. Decker, eds., Anal. and Part. Diff. Eq., 1990, pp. 567–617.

[4] H. Berestycki and B. Larrouturou, *Quelques aspects mathématiques de la propagation des flammes prémélangées*, in Nonlinear P.D.E. and their Applications, Collège de France seminar, 10, Brézis & Lions, eds., Pitman Longman, Harbow, UK, 1990.

[5] H. Berestycki and B. Larrouturou, *A semilinear elliptic equation in a strip arising in a two-dimensional flame propagation model*, J. Reine Angew. Math., 396 (1989), pp. 14–40.

[6] H. Berestycki, B. Larrouturou, and P. L. Lions, *Multidimensional traveling-wave solutions of a flame propagation model*, Arch. Rational Mech. Anal., 111 (1990), pp. 33–49.

[7] H. Berestycki and L. Nirenberg, *On the method of moving planes and the sliding method*, Bol. Soc. Brasil. Mat., 22 (1991), pp. 1–37.

[8] H. Berestycki and L. Nirenberg, *Asymptotic behaviour via Harnack inequality*, in Nonlinear Analysis, A Tribute in Honour of Giovanni Prodi., Scuola Normale Superiore, Pisa, Quaderni, Univ. di Pisa, Pisa, 1991, pp. 135–144.

[9] H. Berestycki and L. Nirenberg, *Travelling fronts in cylinders*, Ann. Inst. H. Poincaré, Anal. Non Linéaire, 9 (1992), pp. 497–572.

[10] C. Bernardi and Y. Maday, *Properties of some weighted Sobolev spaces and application to spectral approximates*, SIAM J. Numer. Anal., 26 (1988), pp. 769–829.

[11] J. D. Buckmaster and G. S. S. Ludford, *Lectures on Mathematical Combustion*, CBMS-NSF Conf. Ser. Appl. Math. 43, SIAM, Philadelphia, PA, 1983.

[12] L. Caffarelli, *A Harnack inequality approach to the regularity of free boundaries, Part* II*: Flat free boundaries are Lipschitz*, Comm. Pure Appl. Math., XLII (1989), pp. 55–78.

[13] M. Dauge, *Elliptic boundary value problems on corners domains*, Lecture Notes in Math., 1341 Springer, Paris, 1988.

[14] P. C. Fife, *Mathematical aspects of reacting and diffusing systems*, Lecture Notes in Biomath. 28, Springer, New York, 1979.

[15] P. C. Fife and J. B. McLeod, *The approach of solutions of non-linear diffusion equations to traveling front solutions*, Arch. Rational Mech. Anal., 65 (1977), pp. 335–361.

[16] G. Geymonat and P. Grisvard, *Eigenfunctions expansions for non self-adjoint operators and separations of variables*, in Singularities and Constructive Methods fot their Treatment, P. Grisvard, W.L. Wendland, J.R. Whiteman, eds., Lecture Notes in Math., 1121, Springer, New York, 1985.

[17] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 1983.

[18] L. Glangetas and J. M. Roquejoffre, *Bifurcations of travelling waves in the thermo-diffusive model for flame propagation*, Arch. Rational Mech. Anal., 134 (1996), pp. 341–402.

[19] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston, MA, 1985.

[20] P. Grisvard, *Singularities in boundary value problems*, Res. Notes Appl. Math., Springer-Verlag, Berlin, 1992.

[21] F. Hamel and R. Monneau, *Solutions of semilinear elliptic equations in $\mathbb{R}^N$ with conical-shaped level sets*, Preprint Labo. Ana. Num. Paris VI, R98029 (1998), submitted.

[22] F. Hamel and R. Monneau, *Existence and uniqueness of solutions of a conical shaped free boundary problem in $\mathbb{R}^2$*, manuscript, 1999.

[23] G. Joulin, *Dynamique des fronts de flammes*, in Modélisation de la combustion, Images des Mathématiques, CNRS, 1996 (in French).

[24] Ya. I. Kanel', *Certain problems of burning-theory equations*, Sov. Math. Dokl., 2 (1961), pp. 48–51.

[25] V. A. Kondrat'ev, *Boundary problems for elliptic equations in domains with conical or angular points*, Trans. Moscow Math. Soc., 16 (1967), pp. 227–313.

[26] A. N. Kolmogorov, I. G. Petrovsky, and N. S. Piskunov, *Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, Bulletin Université d'Etat à Moscou (Bjul. Moskowskogo Gos. Univ.), Série internationale, section A 1 (1937), pp. 1–26 (in French).

[27] A. Liñan, *The structure of diffusion flames*, in Fluid Dynamical Aspects of Combustion Theory, Pitman Res. Notes Math. Ser. 223, Longman Sci. Tech., Harlow, 1991, pp. 11–29. S.B. MARGOLIS, Bifurcation phenomena in burner-stabilized premixed flames, Comb.

[28] S. B. Margolis and G. I. Sivashinsky, *Flame propagation in vertical channels: Bifurcation to bimodal cellular flames*, SIAM J. Appl. Math., 44 (1984), pp. 344–368.

[29] B. J. Matkowsky and G. I. Sivashinsky, *An asymptotic derivation of two models in flame theory associated with the constant density approximation*, SIAM J. Appl. Math., 37 (1979), pp. 686–699.

[30] V. G. Maz'ja and B. A. Plamenevskii, *On the coefficients in the asymptotics of solutions of elliptic boundary-value problems near conical points*, Soviet Math. Dokl., 15 (1974), pp. 1574–1575.

[31] G. I. Sivashinsky, *The structure of Bunsen flames*, J. Chem. Phys., 62 (1975), pp. 638–643.

[32] G. I. Sivashinsky, *The diffusion stratification effect in Bunsen flames*, J. Heat Transfer, Transactions of ASME, 11 (1974), pp. 530–535.

[33] J. M. Vega, *Multidimensional travelling fronts in a model from combustion theory and related problems*, Differential Integral Equations, 6 (1993), pp. 131–155.

[34] A. I. Volpert and V. A. Volpert, *Application of the Leray-Schauder degree to investigation of traveling wave solutions of parabolic systems*, in Elliptic and Parabolic Problems, Pont-à-Mousson 1994, Pitman Res. Notes in Math. Series 325, 1994, pp. 224–239.

[35] F. Williams, *Combustion Theory*, Addison-Wesley, Reading, MA, 1983.

[36] X. Xin, *Existence and uniqueness of travelling waves in a reaction-diffusion equation with combustion nonlinearity*, Idiana Univ. Math. J., 40 (1991), pp. 985–1008.

[37] J. B. Zeldovich and D. A. Frank-Kamenetskii, *A theory of thermal propagation of flame*, Acta Phys. URSS, 9 (1938), pp. 341–350 (in Russian). Dynamics of curved fronts, R. Pelcé Ed., Perspectives in Physics Series, Academic Press, New York, 1988, pp. 131–140 (in English).