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NON-EXISTENCE OF TRAVELLING FRONT SOLUTIONS OF SOME BISTABLE REACTION-DIFFUSION EQUATIONS

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Abstract. This work deals with travelling fronts solutions of some reaction-diffusion equations in an infinite cylinder in dimension ≥ 2. The problem is set in $\Sigma = \{(x_1, y) \in \mathbb{R} \times \omega\}$ where $\omega \subset \mathbb{R}^{N-1}$ is a bounded and smooth domain with outward normal ν. The equations, with unknowns $c \in \mathbb{R}$ and $u \in C^2(\overline{\Sigma})$, are

$$(P) \qquad \begin{cases} \Delta u - (c + \alpha(y))\, \partial_1 u + f(u) = 0 & \text{in } \Sigma = \mathbb{R} \times \omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Sigma = \mathbb{R} \times \partial\omega \\ u(-\infty, \cdot) = 0 \text{ and } u(+\infty, \cdot) = 1 \end{cases}$$

The function $\alpha \in C^0(\overline{\omega})$ is given. The nonlinearity f is assumed to be of the "bistable type": it changes sign once in $(0, 1)$. Berestycki and Nirenberg [8] proved that if ω is convex then the problem has a solution. Here, by using the invariance by translation and the sliding method, we construct an example of a non-convex domain ω and of a function α for which we prove that (P) has no solutions. This is in sharp contrast with other types of nonlinearities for which solutions exist whatever ω may be.

1. Introduction. This work is primarily concerned with a non existence result for the following semilinear elliptic problem set in an infinite cylinder $\Sigma = \mathbb{R} \times \omega = \{x = (x_1, y) \in \mathbb{R}^N; x_1 \in \mathbb{R}, y \in \omega\}$ in dimension $N \geq 2$:

$$\begin{cases} \Delta u - (c + \alpha(y))\partial_1 u + f(u) = 0 & \text{in } \Sigma \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Sigma \\ u(-\infty, \cdot) = 0 < u < u(+\infty, \cdot) = 1. \end{cases} \tag{1.1}$$

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We assume that $\omega \subset \mathbb{R}^{N-1}$ is a smooth bounded domain with outward unit normal ν. The continuous function $\alpha \in C^0(\overline{\omega})$ and the Lipschitz-continuous nonlinearity f are given. The unknowns are the parameter $c \in \mathbb{R}$ and the function u defined in $\overline{\Sigma}$. We denote by $\partial_1 u$ the derivative $\frac{\partial u}{\partial x_1}$. Throughout this paper, the limits as $x_1 \to \pm\infty$ are understood to be uniform with respect to $y \in \overline{\omega}$.

The purpose of this work is to construct, for a function f of the so-called "bistable" type, an example of a domain ω which is nonconvex and of a function α for which problem (1.1) has no solution, whereas it always has a solution for other types of nonlinearities f. The results presented in this paper have been announced in [3].

Let us first recall the origin of the problem and some of the known results. In a few words, this kind of problem, set either in infinite cylinders, in the real axis $\mathbb{R}$ or in the whole space $\mathbb{R}^N$, arises in several physical situations. In particular, these reaction-diffusion equations arise in combustion or in biological models according to the type of nonlinearity f (*see e.g.* Aronson, Weinberger [2], Fife [11], Fisher [13], Hadeler, Rothe [15], Kanel' [19], Kolmogorov, Petrovsky, Piskunov [21], Stokes [30], Zeldovic, Frank-Kamenetskii [37]). For instance, in the thermo-diffusive model for premixed equidiffusional flames (*see* Berestycki, Larrouturou [4]), the function u represents a normalized temperature in a mixture including a reactant and a product. A flame propagates with the speed c in this mixture and equation (1.1) is the equation for the temperature in the frame moving with the speed c to the left. The nonlinear source term $f(u)$ may take into account the mass action law and the Arrhenius's law. In biological models, u is the concentration of a species. Generally speaking, the parameter c is the speed of a front. The Neumann condition on $\partial\Sigma$ means that there is no flow across the walls of the cylinder.

The known results for problem (1.1) highly depend on the profile of the nonlinearity f. In this article, we mainly investigate the case of a source term f of the "bistable" type. Namely, we assume that there exists $\theta \in (0,1)$ such that:

$$f < 0 \text{ on } (0,\theta),\ \ f > 0 \text{ on } (\theta,1) \text{ and } f(0) = f(\theta) = f(1). \tag{1.2}$$

Moreover, we assume that the function f satisfies the following hypotheses:

$$f \in C^{1,\delta}([0,1]) \text{ for some } 0 < \delta < 1 \tag{1.3}$$

$$f'(0) < 0 \text{ and } f'(1) < 0. \tag{1.4}$$

Such a function f is called bistable because it has two zeros, 0 and 1, which are stable for the dynamical system $\dot{X} = f(X)$. It also has another zero, θ, which is unstable.

In dimension 1, problem (1.1) is reduced to

$$u'' - cu' + f(u) = 0, \quad u(-\infty) = 0,\ u(+\infty) = 1. \tag{1.5}$$

For a function f satisfying (1.2) and (1.4), Aronson, Weinberger [2], Fife, McLeod [12] and Kanel' [19] showed the existence and the uniqueness of a solution (c, u) of (1.5). They also proved the stability of this wave u for the evolution problem $v_t = v'' - cv' + f(v)$ under a large class of initial conditions. With the same kind of nonlinearities, similar results have also been obtained for systems of differential equations (*see* for instance Volpert, Volpert, Volpert [34]).

In higher dimension, *i.e.,* in infinite cylinders Σ with convex sections ω, most of the results related to equation (1.5) were generalized for equation (1.1) by Berestycki, Nirenberg [8] and Roquejoffre [29] (*see* also Papanicolaou, Xin [26] and Xin [35] for similar problems in periodic media). We especially mention the following theorem of Berestycki and Nirenberg:

Theorem 1.1. ([8] Th 1.3, Th 1.1') *Let f satisfy* (1.2)-(1.4). *If the domain ω is convex, then there exists a solution* (c, u) *of problem* (1.1). *Furthermore, whatever the domain ω is, if there exists a solution* (c, u) *of* (1.1), *then $\partial_1 u > 0$ in $\overline{\Sigma}$ and* (c, u) *is unique* (*up to translations in the x_1-direction for* u).

Papanicolaou, Xin [26] and Xin [35] got the same result for problems in periodic media. On the other hand, Hamel [17] proved the existence of an interval (c_-, c^+) of speeds which are solutions of (1.1) if the nonlinearity f depends on x_1 and is nondecreasing in x_1.

Statement of our main result. The purpose of this paper is to understand the role of the assumption of convexity of ω. *Throughout the paper, except in section 4, we suppose that f satisfies* (1.2)-(1.4).

For problem (1.1), in the case of a general (nonconvex) domain ω, Berestycki and Nirenberg proved in [8] the existence of a solution (c, u) of

$$\begin{cases} \Delta u - (c + \alpha(y))\partial_1 u + f(u) = 0 & \text{in } \Sigma \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Sigma \end{cases} \tag{1.6}$$

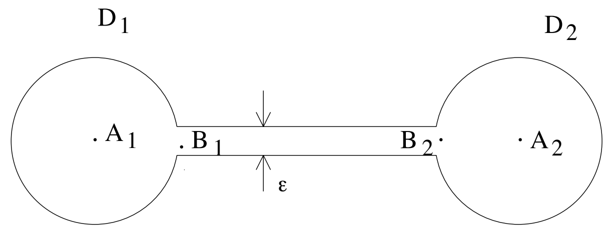

Figure 1: Domain ω_ε

such that

$$u(-\infty,\cdot)=0 \text{ and } u(+\infty,y)=\psi(y) \quad \text{for } y\in\overline{\omega}, \tag{1.7}$$

where ψ is a function satisfying the following problem:

$$\left\{\begin{aligned} \Delta\psi+f(\psi)=0 &\quad \text{in } \omega \\ \partial_\nu\psi=0 &\quad \text{on } \partial\omega. \end{aligned}\right. \tag{1.8}$$

Besides, either $\psi\equiv 1$ or $0<\psi<1$ in $\overline{\omega}$. Notice that ψ is otherwise not specified. It was not known in general if one thus obtained a solution of problem (1.1), that is, one which also satisfied $u(+\infty,\cdot)=\psi=1$.

Consequently, the existence of a solution for problem (1.1) in a general domain ω was an open question. The purpose of the present work is to give a negative answer to this question. Indeed, we construct examples of non-convex domains ω for which there is no solution of problem (1.1). In other words, in such cylinders $\Sigma=\mathbb{R}\times\omega$, every solution of the equations (1.6)-(1.7) will be such that $0<u(+\infty,\cdot)=\psi<1$ in $\overline{\omega}$.

Let us now describe more precisely the construction. We will use a family of smooth domains ω_ε, and cylinders $\Sigma_\varepsilon=\mathbb{R}\times\omega_\varepsilon$, where the ω_ε's have a familiar "dumbbell" shape (*see* Figure 1). We define them as follows: let us choose two points A_1 and A_2 in $\mathbb{R}^{N-1}$ such that $d(A_1,A_2)>2$. For $i=1,2$, we denote by D_i the open ball $D(A_i,1)$ in $\mathbb{R}^{N-1}$ with center A_i and radius 1. The open connected sets ω_ε are such that for $\varepsilon>0$ small enough ($\varepsilon<\varepsilon_0$)

$$D_1\cup D_2\ \subset\ \omega_\varepsilon\ \subset\ D_1\cup D_2\cup\{x\in\mathbb{R}^{N-1};\ d(x,(A_1A_2))<\varepsilon\},$$

where $d(x,(A_1A_2))$ is the distance from a point $x\in\mathbb{R}^{N-1}$ to the segment $[A_1,A_2]$. We call B_i ($i=1,2$) the points at the intersection of ∂D_i with $[A_1,A_2]$. We suppose moreover that the family $(\omega_\varepsilon)_{\varepsilon>0}$ satisfies:

$\omega_\varepsilon \subset \omega_{\varepsilon'}$ if $0 < \varepsilon < \varepsilon' < \varepsilon_0$ and $\bigcap_{0<\varepsilon<\varepsilon_0} \omega_\varepsilon = D_1 \cup D_2 \cup [B_1, B_2]$. As a consequence, the domains ω_ε are not convex for $0 < \varepsilon < 1$.

Lastly, we consider a family of functions $\alpha_\varepsilon \in C^0(\overline{\omega_\varepsilon})$ such that

$$0 \leq \alpha_\varepsilon(y) \leq 1 \text{ in } \overline{\omega_\varepsilon},\ \alpha_\varepsilon = 1 \text{ in } \overline{D_1} \text{ and } \alpha_\varepsilon = 0 \text{ in } \overline{D_2}.$$

Our main result is the following

Theorem 1.2. *For $\varepsilon > 0$ small enough, there does not exist any solution (c, u) of problem (1.1) in the cylinder $\Sigma_\varepsilon = \mathbb{R} \times \omega_\varepsilon$ for the choice $\alpha = \alpha_\varepsilon$.*

Comments. Theorem 1.2 implies in particular the existence of a nonconstant solution ψ of (1.8) for any nonlinearity f of the bistable type and in any domain ω_ε for $\varepsilon > 0$ small enough. Notice that this last result had been independently proved by Matano [22] and Matano, Mimura [23] for elliptic equations or systems in similar domains.

Theorem 1.2 stands in sharp contrast with other types of nonlinearities f that are often considered in the literature. We are going to emphasize two special types of functions f. The first case consists in assuming that there exists a real $\theta \in (0,1)$ such that $f = 0$ on $[0,\theta] \cup \{1\}$, $f > 0$ on $(\theta, 1)$ and $f'(1) < 0$ (θ is said to be an ignition temperature). This case arises in combustion models. The one-dimensional model was investigated by Kanel' [19] and by Berestycki, Nicolaenko, Scheurer [7]. As far as the multidimensional problem (1.1) is concerned, Berestycki, Larrouturou, Lions, Nirenberg and Vega [5], [6], [8], [32] proved the existence and the uniqueness of a solution (c, u) of (1.1), whatever the domain ω may be. For the second type of functions f, we assume that $f > 0$ on $(0,1)$, $f(0) = f(1) = 0$ and $f'(0) > 0$, $f'(1) < 0$ (so-called KPP or ZFK cases). Then, whatever the domain ω may be, Berestycki and Nirenberg proved that there exists a half-line $[c^*, +\infty[$ of speeds which are solutions of (1.1) and that the profiles u are unique for any $c \geq c^*$ (this result generalized to the higher dimensions well-known one-dimensional results: Aronson, Weinberger [2], Freidlin [14], Hadeler, Rothe [15], Kolmogorov, Petrovsky, Piskunov [21], Stokes [30], Uchiyama [31]).

In the case of a bistable nonlinearity f, if $\alpha(y) = \alpha$ is constant (or also if $\Sigma = \mathbb{R}$), then there exists a solution (c, u) of (1.1) and the function u depends only on x_1. In this case, the geometry of the section ω does not play a role. Hence, the nonexistence result given in theorem 1.2 is due to an higher dimensional effect in the sense that both the dependence of α on y and the geometry of ω play a role.

Lastly, let us mention several papers that also shed light, in other frameworks, on the role played by the velocity field in non-propagation phenomena. First, for a bistable nonlinearity, in the one-dimensional case, Xin proved the non-existence of travelling waves if the coefficients of the diffusion and convection terms are periodic in the direction of propagation and vary enough from their mean values (*see* [36]). Second, in the limit of the large activation energies for slowly varying flames, Berestycki and Sivashinsky proved that the flames quench if the flow field is periodic and has oscillations with large amplitude (*see* [9]). Similarly, Pauwelussen [27] and next Ikeda, Mimura [18] proved some wave-blocking phenomena for one-dimensional equations or systems with highly varying diffusion coefficients.

Open questions. The open sets ω_ε and the functions α_ε can be chosen so that the ω_ε's depend continuously on $\varepsilon > 0$ and are convex for $\varepsilon > 0$ large enough (and non-convex for $\varepsilon > 0$ small enough). Hence, problem (1.1) has no solution for small $\varepsilon > 0$ whereas it has a solution for large $\varepsilon > 0$. The question of the bifurcation between the existence and the non-existence of solutions (c, u) of (1.1) is still open. On the other hand, if the domain ω is convex, the solutions u of (1.1) are stable under a large class of perturbations (Roquejoffre [29]). If $\varepsilon > 0$ is small enough, the question of the stability of the solution u of (1.6)-(1.8), with $\psi < 1$, is also open.

In conclusion, we shall say that, although the classification of all the stable and unstable solutions of (1.8) in domains of the type ω_ε (for small $\varepsilon > 0$) is now known (*see* Hale, Vegas [16], Mimura, Ei, Fang [24], Vegas [33]), the knowledge of all the properties of the travelling waves solutions of (1.6)-(1.8) in infinite cylinders with sections ω_ε has not been reached yet. Our paper at least shows that some of the facts about the travelling waves in cylinders with convex sections ω or with constant velocity fields α no longer hold for the travelling waves in the cylinders $\mathbb{R} \times \omega_\varepsilon$ with the velocity fields α_ε.

Outline of the proof of theorem 1.2. Let us describe the main arguments to derive this nonexistence result. We argue by contradiction. Suppose that for some sequence $\varepsilon = \varepsilon_n \to 0$ (we omit the subscript n), there exists a solution $(c_\varepsilon, u_\varepsilon)$ of problem (1.1) in the cylinder $\Sigma_\varepsilon = \mathbb{R} \times \omega_\varepsilon$ for the choice $\alpha = \alpha_\varepsilon$. These couples $(c_\varepsilon, u_\varepsilon)$ are unique, up to translation in the x_1-direction for u_ε. The proof is based on several steps:

1) by comparing the functions u_ε's with special solutions depending only on x_1, we show that the speeds c_ε's are bounded,

2) by passing to the limit $c_{\varepsilon'} \to c$ for some subsequence $\varepsilon' \to 0$ and by normalizing suitably the functions $u_{\varepsilon'}$'s in $\overline{D_1}$, we get the existence of a solution (c, u) of

$$\begin{cases} \Delta u - (c+1)\, \partial_1 u + f(u) = 0 & \text{in } \mathbb{R} \times D_1 \\ \partial_\nu u = 0 & \text{on } \mathbb{R} \times \partial D_1 \\ u(-\infty, \cdot) = 0 \text{ and } u(+\infty, \cdot) = 1 & \text{uniformly in } \overline{D_1}. \end{cases}$$

Since this problem does not depend on the transversal coordinate $y \in D_1$, theorem 1.1 implies that u depends only on x_1 and is solution of

$$\ddot{u} - (c+1)\dot{u} + f(u) = 0 \text{ in } \mathbb{R}, \quad u(-\infty) = 0,\ u(+\infty) = 1.$$

3) by normalizing next the functions $u_{\varepsilon'}$'s in $\overline{D_2}$, we get the existence of a solution (c, u') of

$$\ddot{u}' - (c+1)\dot{u}' + f(u') = 0 \text{ in } \mathbb{R}, \quad u'(-\infty) = 0,\ u'(+\infty) = 1.$$

4) we then get the existence of two couples $(c+1, u)$ and (c, u') which are solutions of the same problem of the type (1.1) in the "cylinder" $\Sigma = \mathbb{R}$. Since this problem admits a unique couple solution by theorem 1.1, this leads to a contradiction.

Structure of the paper. The paper is organized as follows. In section 2, we recall some preliminary results. Section 3 is concerned with the proof of theorem 1.2. The goal of section 4 is to explain why different types of nonlinearities f, for which there exist solutions $(c_\varepsilon, u_\varepsilon)$ with the same choice of ω_ε and α_ε, give rise to radically different limiting behaviours by passing to the limit $\varepsilon \to 0$.

2. Some preliminary results.

2.1. Asymptotic behaviour as $x_1 \to \pm\infty$. In this paper, one essential tool is to know the asymptotic behaviour, as $x_1 \to \pm\infty$, of the possible solutions u_ε of problem (1.1) in the cylinders Σ_ε.

In this subsection, we recall some results of [1], [8], [28] which are used later in the proofs. These results deal mainly with the asymptotic behaviour as $x_1 \to -\infty$ of *positive* solutions u of

$$\begin{cases} \Delta u - \beta(y)\partial_1 u + f(y, u) = 0 & \text{in } \Sigma^- = (-\infty, 0) \times \omega \\ \partial_\nu u(x_1, y) = 0 & \forall\ x_1 < 0,\ y \in \partial\omega \end{cases} \tag{2.1}$$

such that $u(x_1, y) \to 0$ as $x_1 \to -\infty$ uniformly in y. Here the function $f(y, s)$ is assumed to be of class $C^{1,\delta}$ with respect to s in a neighbourhood of $s = 0$, and $f(y, 0) = 0$ for all $y \in \overline{\omega}$. The function $\beta : \overline{\omega} \to \mathbb{R}$ is continuous. The study of the asymptotic behaviour as $x_1 \to +\infty$ systematically boils down to the previous study by changing the variables $x_1 \to -x_1$.

Consider the linearized problem of (2.1) around the function 0:

$$\begin{cases} \Delta w - \beta(y)\partial_1 w - a(y)w = 0 & \text{in } \Sigma^- \\ \partial_\nu w = 0 & \forall\ x_1 < 0,\ y \in \partial\omega \end{cases} \tag{2.2}$$

with $a(y) = -f_s(y, 0)$. In various cases which are developed below, this problem has "exponential" solutions of the form $w(x_1, y) = e^{\lambda x_1}\phi(y)$ for a real $\lambda > 0$ and a function $\phi > 0$ on $\overline{\omega}$. The real λ and the function ϕ are said to be a principal eigenvalue and a principal eigenfunction. They are solutions of

$$\begin{cases} -\Delta\phi + a(y)\phi & = (\lambda^2 - \lambda\beta(y))\phi & \text{in } \omega \\ \partial_\nu\phi & = 0 & \text{on } \partial\omega. \end{cases} \tag{2.3}$$

Generally speaking, if $a(y)$ is a bounded function on $\overline{\omega}$, we call μ_1 the first eigenvalue of the problem

$$\begin{cases} (-\Delta + a(y))\sigma & = \mu_1\sigma & \text{in } \omega \\ \partial_\nu\sigma & = 0 & \text{on } \partial\omega. \end{cases} \tag{2.4}$$

The solutions of the elliptic equation (2.1) can be expressed in terms of the special exponential solutions of the linearized problem (2.2):

Lemma 2.1. ([8] Th 2.1 and 4.4) *Let u be a positive solution of* (2.1) *with $u(-\infty, \cdot) = 0$ and call μ_1 the first eigenvalue of problem* (2.4) *with $a(y) = -f_s(y, 0)$.*

1) *If $\mu_1 \neq 0$, then*

(*i*) $$u(x_1, y) = \alpha e^{\lambda x_1}\phi(y) + o(e^{\lambda x_1}) \text{ as } x_1 \to -\infty \quad \text{or}$$

(*ii*) $$u(x_1, y) = \alpha e^{\lambda x_1}(-x_1\phi(y) + \phi_0(y)) + o(e^{\lambda x_1}) \text{ as } x_1 \to -\infty.$$

In (*i*) *and* (*ii*), *α is a positive constant, $\lambda > 0$ and ϕ are respectively principal eigenvalue and eigenfunction of* (2.3). *Furthermore the case* (*ii*) *may only occur if $\mu_1 < 0$ and if the principal positive eigenvalue λ solution of* (2.3) *is unique.*

2) *If* $\mu_1 > 0$, *then* (2.3) *admits exactly one positive and one negative principal eigenvalue. For each one, there exists a unique positive eigenfunction* ϕ *solving* (2.3) *up to multiplication by a positive constant. Furthermore, if* $\beta \leq \overline{\beta}$, $\beta \not\equiv \overline{\beta}$, *then the respective principal positive eigenvalues* λ *and* $\overline{\lambda}$ *in (2.3) are such that* $0 < \lambda < \overline{\lambda}$.

3) *If* $\mu_1 < 0$, *then* (2.3) *admits* 0, 1 *or* 2 *principal eigenvalues. If two exist, they have the same sign.*

Now return to problem (1.6). We assume that f is of class $C^{1,\delta}$ in the neighborhood of 0 with $f(0) = 0$ and $f'(0) < 0$. We have the

Lemma 2.2. ([8] Lemma 4.1) *Let* u *and* u' *be positive solutions of* (1.6) *in* Σ^- *with the same* c. *Assume that* $u \geq u'$ *and that* (*i*) *is true for both* u *and* u' *with the same values of* α, λ *and* ϕ. *Then* $u \equiv u'$ *in* Σ^-.

2.2. Properties of the solutions ψ of (1.8). In this subsection, we assume that f is of the bistable type, *i.e.,* f satisfies (1.2)-(1.4). As mentioned in the introduction, for a general domain ω, Berestycki and Nirenberg proved the existence of a solution (c, u) of (1.6) such that $u(-\infty, \cdot) = 0$, $u(+\infty, y) = \psi(y)$ where ψ is some solution of (1.8). If the section ω is convex, the function ψ is identically equal to 1 ([8]). This last result is based on the following propositions proved by Berestycki, Nirenberg [8], Casten, Holland [10] and Matano [22]. They will be used in Section 3.2 of this paper.

Proposition 2.3. ([10], [22]) *Let* ψ *be a non-constant solution of* (1.8). *If* ω *is convex and* f *is* C^1, *then* ψ *is unstable in the sense that the principal eigenvalue* $\mu_1(\psi)$ *of the operator* $-\Delta_y - f'(\psi(y))$ *with Neumann boundary conditions on* $\partial\omega$, *is negative.*

We mention here that this last result was generalized by Kishimoto and Weinberger for reaction-diffusion systems [20].

Proposition 2.4. ([8]) *Under the assumptions of Proposition* 2.3, *let* $\psi_- \leq \psi_+$ *be two nonconstant solutions of* (1.8). *If there does not exist any zero of* f *between* ψ_- *and* ψ_+, *then* $\psi_- \equiv \psi_+$ *in* $\overline{\omega}$.

3. Proof of Theorem 1.2. We argue by contradiction. Consider the domains ω_ε and the functions α_ε defined in the introduction and suppose that, for some sequence $\varepsilon \to 0$, there exist solutions $(c_\varepsilon, u_\varepsilon)$ of (1.1) in the cylinders $\Sigma_\varepsilon = \mathbb{R} \times \omega_\varepsilon$ for the choice $\alpha = \alpha_\varepsilon$. We first establish a priori bounds for the speeds c_ε and next pass to the limit in both infinite cylinders $\mathbb{R} \times D_1$ and $\mathbb{R} \times D_2$.

3.1. Step 1: bounds for the speeds c_ε's. In this subsection, we fix $\varepsilon > 0$ and we call as above $(c_\varepsilon, u_\varepsilon)$ a solution of problem (1.1). Remember that such a couple $(c_\varepsilon, u_\varepsilon)$ is unique, up to translation in the x_1-direction for u_ε, and that $\partial_1 u_\varepsilon > 0$ in $\overline{\Sigma_\varepsilon}$. We may then assume that

$$\max_{y \in \overline{\omega_\varepsilon}} u_\varepsilon(0, y) = \theta.$$

From Lemma 2.1, since $f'(0) < 0$ and since f is $C^{1,\delta}$ near 0, there exist a real $\lambda_\varepsilon > 0$ and a function $\phi_\varepsilon > 0$ on $\overline{\omega_\varepsilon}$ such that

$$u_\varepsilon(x_1, y) = e^{\lambda_\varepsilon x_1} \phi_\varepsilon(y) + o(e^{\lambda^\varepsilon x_1}) \text{ as } x_1 \to -\infty,$$

where the function $w = e^{\lambda_\varepsilon x_1} \phi_\varepsilon(y)$ is a solution of $\Delta w - (c_\varepsilon + \alpha_\varepsilon(y))\partial_1 w + f'(0)w = 0$ in $\Sigma_\varepsilon = \mathbb{R} \times \omega_\varepsilon$ with Neumann boundary conditions on $\partial\Sigma_\varepsilon$. Similarly, the behaviour of u_ε as $x_1 \to +\infty$ is given by:

$$u_\varepsilon(x_1, y) = 1 - e^{\mu_\varepsilon x_1} \psi_\varepsilon(y) + o(e^{\mu_\varepsilon x_1}) \text{ as } x_1 \to +\infty$$

where μ_ε is negative and $\psi_\varepsilon > 0$ is a positive function defined on $\overline{\omega_\varepsilon}$. The function $w' = e^{\mu_\varepsilon x_1} \psi_\varepsilon(y)$ is a solution of $\Delta w' - (c_\varepsilon + \alpha_\varepsilon(y))\, \partial_1 w' + f'(1) w' = 0$ in Σ_ε with Neumann boundary conditions on $\partial\Sigma_\varepsilon$.

From theorem 1.1 applied to dimension 1, there exist a unique $k \in \mathbb{R}$ and a unique function v defined in $\mathbb{R}$ such that

$$v'' - kv' + f(v) = 0, \quad v(-\infty) = 0, \; v(+\infty) = 1, \quad v(0) = \theta.$$

Moreover $v' > 0$ in $\mathbb{R}$. Let us then set $v(x_1, y) = v(x_1)$. As above, we can write the behavior of the function v

$$\begin{cases} v(x_1, y) = v(x_1) & = Ce^{\lambda x_1} + o(e^{\lambda x_1}) \qquad \text{as } x_1 \to -\infty \\ v(x_1, y) = v(x_1) & = 1 - C' e^{\mu x_1} + o(e^{\mu x_1}) \quad \text{as } x_1 \to +\infty \end{cases}$$

where the constants $C, C' > 0$ and the eigenvalues $\lambda > 0$, $\mu < 0$ are solutions of $\lambda^2 - \lambda k + f'(0) = 0$ and $\mu^2 - \mu k + f'(1) = 0$.

Lemma 3.1. *With the same notations as above, we have $k - 1 < c_\varepsilon < k$.*

Proof. Let us argue by contradiction. *Suppose that $c_\varepsilon \leq k - 1$.* For all y in $\overline{\omega_\varepsilon}$, it follows that $c_\varepsilon + \alpha_\varepsilon(y) \leq c_\varepsilon + 1 \leq k$ and the inequality is strict somewhere in $\overline{\omega_\varepsilon}$ (at least in $\overline{D_2}$). From Lemma 2.1, it follows that $0 < \lambda_\varepsilon < \lambda$, $\mu_\varepsilon < \mu < 0$. Therefore, there exists $R > 0$ such that

$$u_\varepsilon(x_1, y) > v(x_1) \text{ if } |x_1| \geq R, \; \forall\, y \in \overline{\omega_\varepsilon}.$$

On the other hand, there exist reals $0 < \alpha \leq \beta < 1$ such that $\alpha \leq v(x_1) \leq \beta$ if $|x_1| \leq R$. Since $u_\varepsilon(x_1, y) \to 1$ as $x_1 \to +\infty$ uniformly in $y \in \overline{\omega_\varepsilon}$, there exists a real $\tau > 0$ such that $u_\varepsilon(x_1 + \tau, y) > \beta \geq v(x_1)$ for any $|x_1| \leq R$, $y \in \overline{\omega_\varepsilon}$. We also have that $u_\varepsilon(x_1+\tau, y) > u_\varepsilon(x_1, y)$ because $\partial_1 u_\varepsilon > 0$. Hence,

$$u_\varepsilon(x_1 + \tau, y) > v(x_1, y) \quad \forall (x_1, y) \in \overline{\Sigma_\varepsilon}.$$

We can then shift u_ε to the right until a position $\overline{\tau}$ such that $u_\varepsilon(x_1+\overline{\tau}, y) \geq v(x_1)$ in $\overline{\Sigma_\varepsilon}$ with equality somewhere. This necessarily happens because u_ε and v have exponential behaviours as $x_1 \to \pm\infty$ with different powers. The function $w(x_1, y) := v(x_1) - u_\varepsilon(x_1 + \bar{\tau}, y) \leq 0$ satisfies

$$\begin{cases} \Delta w - (c_\varepsilon + \alpha_\varepsilon(y))\partial_1 w + c(x_1, y) w = (k - c_\varepsilon - \alpha_\varepsilon(y)) v' \geq 0 & \text{in } \Sigma_\varepsilon \\ \partial_\nu w = 0 & \text{on } \partial\Sigma_\varepsilon \\ w(-\infty, y) = 0 \text{ and } w(+\infty, y) = 0 & \end{cases}$$

for some function $c \in L^\infty$ (because f is Lipschitz-continuous). Moreover, $w \leq 0$ in $\overline{\Sigma_\varepsilon}$ and $w(\overline{x_1}, \overline{y}) = 0$ for some $(\overline{x_1}, \overline{y})$ in $\overline{\Sigma_\varepsilon}$. The strong maximum principle and the Hopf lemma yield then that $w \equiv 0$ in $\overline{\Sigma_\varepsilon}$, *i.e.*, $u_\varepsilon(x_1+\overline{\tau}, y) = v(x_1)$ in $\overline{\Sigma_\varepsilon}$. This is ruled out by the behaviours of u_ε and v as $x_1 \to \pm\infty$.

Finally, we conclude that $c_\varepsilon > k - 1$ and similarly we could prove that $c_\varepsilon < k$. This completes the proof of Lemma 3.1.

3.2. Step 2: passage to the limit in $\mathbb{R} \times D_1$. From Lemma 3.1, there exists a sub-sequence of (c_ε), which we still name (c_ε), such that $c_\varepsilon \to c \in [k-1, k]$ as $\varepsilon \to 0$. *In what follows, we only consider sub-sequences of this sequence.*

Since $\partial_1 u_\varepsilon > 0$ and $u_\varepsilon(-\infty, \cdot) = 0$, $u_\varepsilon(+\infty, \cdot) = 1$, there exists a unique τ_ε such that $\max_{y \in \overline{D_1}} u_\varepsilon(\tau_\varepsilon, y) = \theta$ (notice that $\tau_\varepsilon \geq 0$ because $\max\limits_{y \in \overline{\omega_\varepsilon}} u_\varepsilon(0, y) = \theta$). Let us set $v_\varepsilon(x_1, y) = u_\varepsilon(x_1 + \tau_\varepsilon, y)$. Because of the invariance of problem (1.1) by translation in the x_1-direction, the function v_ε also satisfies (1.1). Since $\overline{D_1} \subset \overline{\omega_\varepsilon}$, we can now consider the restriction w_ε of v_ε to $\mathbb{R} \times \overline{D_1}$. We have that $|w_\varepsilon| \leq 1$, $|f(w_\varepsilon)| \leq M$ and $|c_\varepsilon + \alpha_\varepsilon(y)| \leq |k| + 1$ for all ε. From the standard a priori elliptic estimates up to the boundary and the Sobolev injections, the functions w_ε's are bounded in $W^{2,p}_{loc}(\mathbb{R} \times D_1)$ for all $1 < p < \infty$ and there exist a sub-sequence $\varepsilon' \to 0$, *which we rename* ε, and a function u such that $w_\varepsilon \overset{weak}{\longrightarrow} u$ in $W^{2,p}_{loc}(\mathbb{R} \times D_1)$ (for all $1 < p < \infty$). Remember now that $\alpha_\varepsilon = 1$ on $\overline{D_1}$. The function u satisfies

$$\Delta u - (c+1)\partial_1 u + f(u) = 0 \text{ in } \mathbb{R} \times D_1 \tag{3.1}$$
$$\partial_\nu u = 0 \text{ on } \mathbb{R} \times \partial D_1 \tag{3.2}$$
$$\max_{y \in \overline{D_1}} u(0,y) = \theta \tag{3.3}$$
$$\partial_1 u \geq 0 \text{ in } \mathbb{R} \times \overline{D_1}. \tag{3.4}$$

The Neumann boundary condition (3.2) is immediately fulfilled at any point on $\mathbb{R} \times (\partial D_1 \backslash \{B_1\})$ (because any such point is on $\mathbb{R} \times \partial\omega_\varepsilon$ for ε small enough). Besides, since the functions w_ε's and $f(w_\varepsilon)$'s are uniformly bounded in $\mathbb{R} \times \overline{D_1}$, it follows from the standard elliptic estimates up to the boundary that the singularities at the points on $\mathbb{R} \times \{B_1\}$ are removable (*see* Omrani [25]). Hence, the function u is of class C^1 in $\mathbb{R} \times \overline{D_1}$ and $\partial_\nu u = 0$ on $\mathbb{R} \times \{B_1\}$.

In order to prove that $(c+1, u)$ is solution of (1.1) in $\mathbb{R} \times D_1$, we only have to show the following lemma:

Lemma 3.2. *The limits of u as $x_1 \to \pm\infty$ are:*

$$u(-\infty, \cdot) = 0 \text{ and } u(+\infty, \cdot) = 1 \text{ uniformly in } y \in \overline{D_1}$$

Proof. From the standard elliptic estimates and since u is increasing in x_1, there exist two continuous functions $\psi_\pm$ defined on $\overline{D_1}$ such that

$$\left\{ \begin{array}{rcl} u(x_1,y) & \underset{x_1 \to \pm\infty}{\longrightarrow} & \psi_\pm(y) \\ \nabla_y u(x_1,y) & \underset{x_1 \to \pm\infty}{\longrightarrow} & \nabla_y \psi_\pm(y) \\ \partial_1 u(x_1,y) & \underset{x_1 \to \pm\infty}{\longrightarrow} & 0 \end{array} \right. \qquad \text{uniformly in } y \in \overline{D_1}$$

and

$$\left\{ \begin{array}{rl} \Delta\psi_\pm + f(\psi_\pm) = 0 & \text{in } D_1 \\ \partial_\nu \psi_\pm = 0 & \text{on } \partial D_1 \end{array} \right. \tag{3.5}$$

By (3.3) and (3.4), it follows that $\psi_- \leq \theta$ since $u \leq \theta$ in $\mathbb{R}_- \times \overline{D_1}$. On the other hand, $f < 0$ in $(0, \theta)$ and $f(0) = f(\theta) = 0$. By integration of the equation (3.5) satisfied by ψ_-, we conclude that ψ_- is a constant, namely 0 or θ.

Proof of $\psi_- \equiv 0$. Suppose on the contrary that $\psi_- = \theta$. As $\partial_1 u \geq 0$ and $\max_{y \in \overline{D_1}} u(0,y) = \theta$, the strong maximum principle and the Hopf lemma yield that $u \equiv \theta$ in $\mathbb{R}\times\overline{D_1}$. The sequence (w_ε) converges then to the constant θ uniformly on each compact subset of $\mathbb{R}\times\overline{D_1}$.

On the other hand, there exists $\tau'_\varepsilon > 0$ such that $\max\limits_{y\in\overline{D_1}} w_\varepsilon(-\tau'_\varepsilon, y) = \theta/2$. Let us set $z_\varepsilon(x_1,y) := w_\varepsilon(x_1-\tau'_\varepsilon, y)$. After the extraction of a sub-sequence ε', that we rename ε, there exists a function z such that $z_\varepsilon \overset{weak}{\longrightarrow} z$ in $W^{2,p}_{loc}(\mathbb{R}\times D_1)$ (for all $1<p<\infty$). The function z satisfies

$$\left\{\begin{aligned} \Delta z-(c+1)\;\partial_1 z+f(z) &= 0 \text{ in } \mathbb{R}\times D_1\\ \partial_\nu z &= 0 \text{ on } \mathbb{R}\times\partial D_1\\ \max_{y\in\overline{D_1}} z(0,y) &= \theta/2\\ \partial_1 z &\geq 0 \text{ in } \mathbb{R}\times\overline{D_1}. \end{aligned}\right. \tag{3.6}$$

By using the same arguments as above, we get that $z(-\infty,\cdot)=0$ and that $z(x_1,y)\to\psi'_+(y)$ as $x_1\to+\infty$ uniformly in $y\in\overline{D_1}$ where ψ'_+ is a solution of (3.5). As $\tau'_\varepsilon\geq 0$ and $\partial_1 w_\varepsilon\geq 0$, we have that $z_\varepsilon\leq w_\varepsilon$ everywhere in $\overline{\Sigma_\varepsilon}$ whence $z\leq\theta$ and $\psi'_+\leq\theta$. Since $f<0$ on $(0,\theta)$, by integration over D_1 of the equation (3.5) satisfied by ψ'_+, we conclude this time that $\psi'_+\equiv\theta$. If we sum up, we have that

$$z(-\infty,\cdot)=0 \text{ and } z(+\infty,\cdot)=\theta. \tag{3.7}$$

Similarly, we can translate the functions w_ε (defined on $\mathbb{R}\times\overline{D_1}$) to the left and introduce $\tau^\varepsilon>0$ such that $\max_{y\in\overline{D_1}} w_\varepsilon(\tau^\varepsilon,y)=(1+\theta)/2$. Let us set $\tilde{z}^\varepsilon(x_1,y) := w_\varepsilon(x_1+\tau^\varepsilon,y)$. As above, for some sub-sequence that we still rename $\tilde{z}^\varepsilon$, there exists a function $\tilde{z}$ such that $\tilde{z}^\varepsilon\overset{weak}{\longrightarrow}\tilde{z}$ in $W^{2,p}_{loc}(\mathbb{R}\times D_1)$ ($\forall\ 1<p<\infty$). The function $\tilde{z}$ satisfies

$$\left\{\begin{aligned} \Delta\tilde{z}-(c+1)\;\partial_1\tilde{z}+f(\tilde{z}) &= 0 \text{ in } \mathbb{R}\times D_1\\ \partial_\nu\tilde{z} &= 0 \text{ on } \mathbb{R}\times\partial D_1\\ \max_{y\in\overline{D_1}} \tilde{z}(0,y) &= (1+\theta)/2. \end{aligned}\right. \tag{3.8}$$

For all (x_1,y) in $\mathbb{R}\times\overline{D_1}$, we have that $z^\varepsilon(x_1,y)=w_\varepsilon(x_1+\tau^\varepsilon,y)\geq w_\varepsilon(x_1,y)$ because $\tau^\varepsilon>0$ and $\partial_1 w_\varepsilon\geq 0$. By passing to the limit $\varepsilon\to 0$, it then follows that $\tilde{z}(x_1,y)\geq\theta$. Since $f>0$ on $(\theta,1)$, we conclude as above that

$$\tilde{z}(-\infty,\cdot)=\theta \text{ and } \tilde{z}(+\infty,\cdot)=1. \tag{3.9}$$

Next, with the same notations as in [8], we define two functions $\overline{f}$ and $\underline{f}$ on $[0,1]$ by

$$\overline{f} = \begin{cases} 0 \text{ in } [0,\theta] \\ f \text{ in } [\theta,1] \end{cases}, \quad \underline{f} = \begin{cases} f \text{ in } [0,\theta] \\ 0 \text{ in } [\theta,1] \end{cases}.$$

Let α be such that $0 < \alpha < \min\,(\theta, 1-\theta)$. Since the function $\overline{f}$ is of ignition temperature type on $[\theta-\alpha, 1]$, there exists then a couple (c_α, u_α) fulfilling $\theta - \alpha < u_\alpha < 1$, $\partial_1 u_\alpha > 0$ in $\mathbb{R}\times\overline{D_1}$ and which is a solution of

$$\begin{cases} \Delta u_\alpha - (c_\alpha + 1)\,\partial_1 u_\alpha + \overline{f}(u_\alpha) = 0 & \text{in } \mathbb{R}\times D_1 \\ \partial_\nu u_\alpha = 0 & \text{on } \mathbb{R}\times\partial D_1 \\ u_\alpha(-\infty,\cdot) = \theta-\alpha \text{ and } u_\alpha(+\infty,\cdot) = 1. \end{cases}$$

Notice that this couple is unique, up to translation in x_1-direction for u_α and that u_α actually depends only on x_1 by theorem 1.1. Similarly, there exists a unique couple (c^α, u^α) fulfilling $0 < u^\alpha < \theta + \alpha$, $\partial_1 u^\alpha > 0$ in $\mathbb{R}\times\overline{D_1}$ and which is a solution of

$$\begin{cases} \Delta u^\alpha - (c^\alpha + 1)\partial_1 u^\alpha + \underline{f}(u^\alpha) = 0 & \text{in } \mathbb{R}\times D_1 \\ \partial_\nu u^\alpha = 0 & \text{on } \mathbb{R}\times\partial D_1 \\ u^\alpha(-\infty,\cdot) = 0 \text{ and } u^\alpha(+\infty,\cdot) = \theta+\alpha. \end{cases}$$

We now claim that

$$c^\alpha < c_\alpha \tag{3.10}$$

Indeed, if we suppose that $c^\alpha \geq c_\alpha$, then, by using a sliding method and the maximum principle as in section 3.1, we would reach a contradiction.

Similarly, since $z(-\infty,\cdot) = u^\alpha(-\infty,\cdot) = 0$, $z(+\infty,\cdot) = \theta < \theta+\alpha = u^\alpha(+\infty,\cdot)$ and since $f = \underline{f}$ on $[0,\theta]$, it follows that $c < c^\alpha$. By comparing $\tilde{z}$ and u_α, we would also get that $c_\alpha < c$. Finally, this implies that $c_\alpha < c^\alpha$. This is in contradiction with (3.10).

This assumption $\psi_- \equiv \theta$ was impossible. Hence, one concludes that $\psi_- = u(-\infty,\cdot) = 0$.

Proof of $\psi_+ \equiv 1$. The function u is a solution of

$$\begin{cases} \Delta u - (c+1)\partial_1 u + f(u) = 0 & \text{in } \mathbb{R}\times D_1 \\ \partial_\nu u = 0 & \text{on } \mathbb{R}\times\partial D_1 \\ u(-\infty,\cdot) = 0,\ u(+\infty,y) = \psi_+(y) & \\ \partial_1 u \geq 0 & \text{in } \mathbb{R}\times\overline{D_1} \\ \max_{\overline{D_1}} u(0,\cdot) = \theta & \end{cases}$$

and the function ψ_+ is a solution of (3.5). From the maximum principle and the Hopf lemma, we infer that either $\psi_+ \equiv 1$ or $\psi_+ < 1$. If $\psi_+ \equiv 1$, Lemma 3.2 is proved.

Suppose that $\psi_+ < 1$. The strong maximum principle implies that $\theta < \max\ \psi_+$ (otherwise, $u \equiv \theta$ in $\mathbb{R} \times \overline{D_1}$) and next that $\min\ \psi_+ < \theta$ (otherwise, $\psi_+ \equiv 1$ by integration of (3.5) over D_1). Let d be such that $\theta < \max\ \psi_+ < d < 1$. There exists a unique shift $\tilde{\tau}_\varepsilon > 0$ such that $\min_{\overline{D_1}} w_\varepsilon(\tilde{\tau}_\varepsilon, \cdot) = d$. Up to extraction of some subsequence, the functions $\tilde{w_\varepsilon}(x_1, y) = w_\varepsilon(x_1 + \tilde{\tau}_\varepsilon, y)$ defined in $\mathbb{R} \times \overline{D_1}$ converge to a function $\tilde{u}$ which is a solution of

$$\begin{cases} \Delta\tilde{u} - (c+1)\partial_1\tilde{u} + f(\tilde{u}) = 0 & \text{in } \mathbb{R} \times D_1 \\ \partial_\nu \tilde{u} = 0 & \text{on } \mathbb{R} \times \partial D_1 \\ \tilde{u}(-\infty, y) = \tilde{\psi}_-(y),\ \tilde{u}(+\infty, \cdot) = 1 & \\ \partial_1 \tilde{u} \geq 0 & \text{in } \mathbb{R} \times \overline{D_1} \\ \min\limits_{\overline{D_1}} \tilde{u}(0, \cdot) = d & \end{cases}$$

where $\tilde{\psi}_-$ is a solution of (3.5). Since $u \leq u(+\infty, y) = \psi_+(y) < d$ in $\mathbb{R} \times D_1$, we deduce that $\tilde{\tau}_\varepsilon \to +\infty$ as $\varepsilon \to 0$. For any $(x_1, y) \in \mathbb{R} \times D_1$ and any $A > 0$, we have that $\tilde{u_\varepsilon}(x_1, y) = u_\varepsilon(x_1 + \tilde{\tau}_\varepsilon, y) \geq u_\varepsilon(x_1 + A, y)$ for ε small enough. By successively passing to the limits $\varepsilon \to 0$ and $A \to +\infty$, we find that $\tilde{u}(x_1, y) \geq \psi_+(y)$ in $\mathbb{R} \times \overline{D_1}$. Hence, $\psi_+ \leq \tilde{\psi}_-$ in $\overline{D_1}$. In particular, the inequality $\theta < \max\ \psi_+$ implies that $\tilde{\psi}_-$ cannot be identically equal to the constant θ. Furthermore, $\tilde{\psi}_-$ cannot be identically equal to the constant 1 since $\min\ \tilde{\psi}_- \leq d < 1$. Eventually, $\tilde{\psi}_-$ is not constant, as well as ψ_+, and there cannot be any zero of f between ψ_+ and $\tilde{\psi}_-$. Proposition 2.4 yields then that $\psi_+ \equiv \tilde{\psi}_-$.

By analyzing the asymptotic behaviour of the function u as $x_1 \to +\infty$ and of the function $\tilde{u}$ as $x_1 \to -\infty$, we are going to reach a contradiction. First of all, since the function $\psi_+ \equiv \tilde{\psi}_-$ is a non-constant solution of (3.5), proposition (2.3) states that the first eigenvalue $\mu_1(\psi_+)$ of the linearized problem of (3.5) is negative (we use here the convexity of D_1).

Let us now emphasize the behaviour of u as $x_1 \to +\infty$. The function $w(x_1, y) = \psi_+(y) - u(-x_1, y)$ is positive, goes to 0 as $x_1 \to -\infty$ and satisfies the equation

$$\Delta w + (c+1)\partial_1 w + g(y, w) = 0 \text{ in } \Sigma$$

where $g(y, w) = f(\psi_+(y)) - f(\psi_+(y) - w)$. We have $g(y, 0) = 0$, $g_w(y, 0) = f'(\psi_+(y))$ and the first eigenvalue of $-\Delta_y - g_w(y, 0)$ with Neumann boundary conditions, namely $\mu_1(\psi_+)$, is negative. From Lemma 2.1, there exist a positive principal eigenvalue $\lambda > 0$ and an eigenfunction $\phi(y) > 0$ in $\overline{D_1}$, which are solutions of the eigenvalue problem

$$\begin{cases} -\Delta_y \phi - f'(\psi_+)\phi &= (\lambda^2 + \lambda(c+1))\phi \quad \text{in } D_1 \\ \partial_\nu \phi &= 0 \quad \text{on } \partial D_1. \end{cases} \tag{3.11}$$

The behaviour of u as $x_1 \to +\infty$ is given by $u(x_1, y) = \psi_+(y) - \alpha e^{-\lambda x_1}\phi(y) + o(e^{-\lambda x_1})$ as $x_1 \to +\infty$ or by $u(x_1, y) = \psi_+(y) - \alpha e^{-\lambda x_1}(x_1\phi(y) + \phi_0(y)) + o(e^{-\lambda x_1})$ as $x_1 \to +\infty$ where α and λ are positive and ϕ is a positive function in $\overline{D_1}$ satisfying (3.11).

Let us now study the behaviour of $\tilde{u}$ as $x_1 \to -\infty$. By applying again Lemma 2.1, it follows that $\tilde{u}(x_1, y) = \psi_+(y) + \alpha' e^{\lambda' x_1}\phi'(y) + o(e^{\lambda' x_1})$ as $x_1 \to -\infty$ or $\tilde{u}(x_1, y) = \psi_+(y) + \alpha' e^{\lambda' x_1}(-x_1\phi'(y) + \phi_0'(y)) + o(e^{\lambda' x_1})$ as $x_1 \to -\infty$, where α', λ' are positive and ϕ' is a positive function defined in $\overline{D_1}$ and solving

$$\begin{cases} -\Delta_y \phi' - f'(\psi_+)\phi' &= ({\lambda'}^2 - \lambda'(c+1))\phi \quad \text{in } D_1 \\ \partial_\nu \phi' &= 0 \quad \text{on } \partial D_1. \end{cases}$$

Therefore, the same eigenvalue problem (3.11) admits one positive principal eigenvalue, λ, and one negative principal eigenvalue, $-\lambda'$. Since $\mu_1(\psi_+) < 0$, Lemma 2.1 asserts that the principal eigenvalues of (3.11) necessarily have the same sign. We have then reached a contradiction. This proves that $\psi_+ \equiv 1$ and completes the proof of Lemma 3.2.

We can summarize the previous results as follows: there exists a solution u of

$$\begin{cases} \Delta u - (c+1)\ \partial_1 u + f(u) = 0 \quad \text{in } \mathbb{R} \times D_1 \\ \partial_\nu u = 0 \quad \text{on } \mathbb{R} \times \partial D_1 \\ u(-\infty, \cdot) = 0 \text{ and } u(+\infty, \cdot) = 1. \end{cases}$$

Since this problem does not depend on the transversal coordinate y, theorem 1.1 yields that u does not depend on y and that $(c+1, u) = (c^*, \varphi)$, up to a translation in the x_1-direction for u, where φ is the unique profile solving

$$\varphi'' - c^*\varphi' + f(\varphi) = 0 \ \text{ in } \mathbb{R}, \quad \varphi(-\infty) = 0,\ \varphi(+\infty) = 1. \tag{3.12}$$

3.3. Step 3: passage to the limit in $\mathbb{R} \times D_2$ and completion of the proof of theorem 1.2. Consider the same sub-sequence $c_\varepsilon \to c$ and normalize this time the functions u_ε's in $\overline{D_2}$. More precisely, there exists

$\tau'_\varepsilon \geq 0$ such that $\max_{y \in \overline{D_2}} u_\varepsilon(\tau'_\varepsilon, y) = \theta$. Let $v'_\varepsilon(x_1, y) := u_\varepsilon(x_1 + \tau'_\varepsilon, y)$ and w'_ε be the restriction of v'_ε in $\mathbb{R} \times \overline{D_2}$.

As in the previous subsection, there exists a sub-sequence, which we rename $\varepsilon \to 0$ and a function u' defined in $\mathbb{R} \times \overline{D_2}$ such that $w'_\varepsilon \xrightarrow{weak} u'$ in $W^{2,p}_{loc}(\mathbb{R} \times D_2)$ ($\forall\ 1 < p < \infty$). The function u' is a solution of

$$\begin{cases} \Delta u' - c\ \partial_1 u' + f(u') = 0 & \text{in } \mathbb{R} \times D_2 \\ \partial_\nu u' = 0 & \text{on } \mathbb{R} \times \partial D_2 \\ u'(-\infty, \cdot) = 0 \text{ and } u'(+\infty, \cdot) = 1 \end{cases}$$

(indeed, $\alpha_\varepsilon = 0$ in $\overline{D_2}$). Thus, up to a translation in the x_1-direction for u', it is the case that $(c, u') = (c^*, \varphi)$. Finally, we get that $c^* = c = c + 1$. This is impossible.

Hence, the existence of the solutions $(c_\varepsilon, u_\varepsilon)$ of problem (1.1) for the choice $\omega = \omega_\varepsilon$ and $\alpha = \alpha_\varepsilon$ cannot be valid if $\varepsilon > 0$ is small enough. This completes the proof of Theorem 1.2.

4. Two other types of nonlinearities f.

4.1. Ignition temperature case. In section 3, we showed that, for a bistable nonlinearity f and for the choice $\omega = \omega_\varepsilon$ and $\alpha = \alpha_\varepsilon$, two passages to the limit in two cylinders lead to two solutions of the same problem, but with two different speeds. It is natural to investigate what the same process can lead to if the source term $f(u)$ is of "ignition temperature type". Namely, let us assume that there exists $\theta \in (0, 1)$ such that

$$f \equiv 0 \text{ on } [0, \theta],\ \ f > 0 \text{ on } (\theta, 1) \text{ and } f(1) = 0$$

(*see* [19] for the derivation of this model). The function f is assumed to be lipschitz-continuous, $C^{1,\delta}$ in the neighbourhood of 1 (for some $0 < \delta < 1$) and such that $f'(1) < 0$. We again consider the same cylinders Σ_ε with the sections ω_ε and the same functions α_ε as defined in the introduction.

From theorems 1.1 and 1.1' in [8], there exist a unique real c_ε and a unique profile $0 < u_\varepsilon < 1$ solution of

$$\begin{cases} \Delta u_\varepsilon - (c_\varepsilon + \alpha_\varepsilon(y))\partial_1 u_\varepsilon + f(u_\varepsilon) = 0 & \text{in } \mathbb{R} \times \omega_\varepsilon \\ \partial_\nu u_\varepsilon = 0 & \text{on } \mathbb{R} \times \partial\omega_\varepsilon \\ u_\varepsilon(-\infty, \cdot) = 0,\ u_\varepsilon(+\infty, \cdot) = 1. \end{cases}$$

In the one-dimensional case, there exists a unique pair (k, v) solving

$$\begin{cases} v'' - kv' + f(v) = 0 & \text{in } \mathbb{R} \\ v(-\infty) = 0 \text{ and } v(+\infty) = 1. \end{cases}$$

As in section 3.1, we find that $k-1 < c_\varepsilon < k$. We can then extract a subsequence $(c_{\varepsilon'})$, *that we rename* (c_ε), such that $c_\varepsilon \to c \in [k-1, k]$. For any $\gamma \in [0, \theta)$, let $(K(\gamma), v_\gamma)$ be the unique pair solving

$$\begin{cases} v_\gamma'' - K(\gamma) v_\gamma' + f(v_\gamma) = 0 \text{ in } \mathbb{R} \\ v_\gamma(-\infty) = \gamma < v_\gamma < v_\gamma(+\infty) = 1. \end{cases} \tag{4.1}$$

We also have that $v_\gamma' > 0$ and we can see that $K(0) = k$.

On the other hand, from Theorem 1.4 of [8] (see also [2], [15], [21], [30], [31] for the similar problem in dimension 1), there exists a "minimal speed" c^* which satisfies the following property: there exist solutions (c', v) of

$$\begin{cases} \Delta v - c'\partial_1 v + f(v) = 0 & \text{in } \mathbb{R} \times D_2 \\ \partial_\nu v = 0 & \text{on } \mathbb{R} \times \partial D_2 \\ v(-\infty, \cdot) = \theta \text{ and } v(+\infty, \cdot) = 1 \end{cases} \tag{4.2}$$

if and only if $c' \geq c^*$.

By the device of a sliding method as in Section 3.1, it immediately follows that $K(\gamma) \leq c^*$ for all $\gamma \in [0, \theta[$. Further properties of the function $\gamma \mapsto K(\gamma)$, defined on $[0, \theta[$, are given in the following lemma:

Lemma 4.1. (*i*) *the function* $\gamma \mapsto K(\gamma)$ *is continuous.* (*ii*) *This function is strictly increasing.* (*iii*) $\lim_{\gamma \nearrow \theta} K(\gamma) = c^*$.

Proof. Assertion (i) was proved in [8] (Proposition 7.1) and follows from a uniform exponential decay of the functions v_γ's near $-\infty$ if γ belongs to a small enough neighborhood of a given point $\gamma_0 \in [0, \theta[$.

The fact that the function K is nondecreasing can be proved by *reductio ad absurdum* as in Section 3.1 (by using a sliding method and the exponential decay of the solutions v_γ's). The *strict* growth of the function K follows from Lemma 2.2 given in Section 2.

In order to prove (iii), remember first that the v_γ's are solutions of (4.1). Set $v_\gamma(x_1, y) := v_\gamma(x_1)$ in $\mathbb{R} \times \overline{D_2}$ and shift these functions v_γ's if necessary so that $\max_{\overline{D_2}} v_\gamma(0, \cdot) = (\theta + 1)/2$. Let us now pass to the limit $\gamma \nearrow \theta$. Since the function K is increasing and $K(\gamma) \leq c^*$ for all $\gamma < \theta$, there exists

$$\lim_{\gamma \nearrow \theta} K(\gamma) := c' \leq c^*.$$

Moreover, from the classical *a priori* elliptic estimates and the Sobolev injections, there exists a function v, limit of a sub-sequence (v_γ), such that

$\partial_1 v \geq 0$ and

$$\begin{cases} \Delta v - c' \ \partial_1 v + f(v) = 0 & \text{in } \mathbb{R} \times D_2 \\ \partial_\nu v = 0 & \text{on } \mathbb{R} \times \partial D_2 \\ \theta \leq v \leq 1 & \\ \max_{\overline{D_2}} v(0, \cdot) = (\theta + 1)/2. & \end{cases}$$

As usual, $v(-\infty, \cdot)$ and $v(+\infty, \cdot)$ are constants and zeros of f. Since $f > 0$ in $(\theta, 1)$ and from the normalization condition on $\{0\} \times \overline{D_2}$, it follows that $v(-\infty, \cdot) = \theta$ and $v(+\infty, \cdot) = 1$. Hence, owing to the definition of c^*, we deduce that $c' \geq c^*$ and finally we conclude that $c' = c^*$. This completes the proof of Lemma 4.1.

Let us now proceed as in section 3 and study the limit of the pairs $(c_\varepsilon, u_\varepsilon)$ *in* $\mathbb{R} \times \overline{D_2}$ *as* $\varepsilon \to 0$. Remember first that $c_\varepsilon \to c \in [k-1, k]$ as $\varepsilon \to 0$.

For any $h \in (0,1)$, let us call u_h^ε the shifted function of u_ε such that $\max_{\overline{D_2}} u_h^\varepsilon(0, \cdot) = h$. Next, pass to the limit $\varepsilon \to 0$. As in Section 3.2, the restrictions of the functions u_h^ε to $\mathbb{R} \times \overline{D_2}$ converge to a solution u_h of

$$\begin{cases} \Delta u_h - c \ \partial_1 u_h + f(u_h) = 0 & \text{in } \mathbb{R} \times D_2 \\ \partial_\nu u_h = 0 & \text{on } \mathbb{R} \times \partial D_2 \\ \max_{\overline{D_2}} u_h(0, \cdot) = h & \end{cases}$$

such that $\partial_1 u_h \geq 0$. Besides, as $f \geq 0$ on $[0, 1]$, we find that $u_h(-\infty, \cdot) = \gamma_-$ and $u_h(+\infty, \cdot) = \gamma_+$ where the $\gamma_\pm$ are two zeros of f such that $\gamma_- \leq h \leq \gamma_+$.

If $h > \theta$, u_h is then a connection between a constant $\gamma_- \leq \theta$ and 1. Two cases can occur: 1) $0 \leq \gamma_- < \theta$, whence $c = K(\gamma_-)$ and then $u_h(x_1, y) = v_{\gamma_-}(x_1)$ up to translation, *or* 2) $\gamma_- = \theta$ and then $c \geq c^*$. From Lemma 4.1 and since $K(0) = k$, we necessarily have that $c \geq k$ in both cases 1) and 2). But we have shown that $c \in [k-1, k]$. Thus $c = k$ and that the whole sequence (c_ε) has only one accumulation point, namely k, and then converges to k as $\varepsilon \to 0$.

Furthermore, if $\gamma_+ \leq \theta$, then u_h is a solution of

$$\begin{cases} \Delta u_h - c \ \partial_1 u_h = 0 & \text{in } \mathbb{R} \times D_2 \\ \partial_\nu u_h = 0 & \text{on } \mathbb{R} \times \partial D_2. \end{cases}$$

By taking the derivative of this equation with respect to x_1, we get from the strong maximum principle and the Hopf lemma that $\partial_1 u_h \equiv 0$. Hence

$\Delta_y u_h = 0$ in $\mathbb{R}\times D_2$, $\partial_\nu u = 0$ on $\mathbb{R}\times\partial D_2$ and we conclude that $u_h \equiv cte = h$ in $\mathbb{R}\times\overline{D_2}$.

If $h > \theta$, we have seen that u_h is a connection between a constant less than or equal to θ and 1, with the speed k. From Lemma 4.1, part (ii), and since $K(0) = k$, we necessarily have that $u_h(-\infty,\cdot) = 0$. Moreover, since the functions u_h are nondecreasing with h (because the functions u_ε are increasing in x_1), it follows that, for any $h \in]0,\theta]$, the function u_h is not constant and thus goes to 1 as $x_1 \to +\infty$. In other words, for any $h \in (0,1)$, the function u_h is equal to a translation of the function v_0.

We can now do the same passages to the limit in $\mathbb{R}\times\overline{D_1}$ and call v_h the limit functions. These functions v_h's are solutions of

$$\begin{cases} \Delta v_h - (k+1)\ \partial_1 v_h + f(v_h) = 0 & \text{in } \mathbb{R}\times D_1 \\ \partial_\nu v_h = 0 & \text{on } \mathbb{R}\times\partial D_1 \\ \max\limits_{\overline{D_1}} v_h(0,\cdot) = h & \end{cases}$$

such that $v_h(\pm\infty,\cdot) = \gamma'_\pm$ (both constants). Let E be the set $E = \{\gamma \in [0,\theta[$ such that $K(\gamma) = k+1\}$. From Lemma 4.1, E is either empty or reduced to a single point.

If $E = \{\gamma_0\}$ (where $\gamma_0 \in [0,\theta[$), then each function v_h which is not constant must be equal to some translation of the function v_{γ_0}. Since the functions v_h's are nondecreasing with respect to h, it follows that, for any $h > \gamma_0$, v_h is not constant and goes to 1 as $x_1 \to +\infty$, whence v_h is a translation of the function v_{γ_0}. Besides, for any $h \le \gamma_0$, $v_h \equiv h$.

If $E = \emptyset$ (this happens by keeping $\alpha_\varepsilon = 0$ in $\overline{D_2}$ and $\alpha_\varepsilon = \alpha$ large enough in $\overline{D_1}$ so that $K + \alpha \ge c^*$), then it follows that $v_h \equiv h$ for any $h \le \theta$ and that v_h is a connection between θ and 1 with speed $k+1 \ge c^*$ for any $h > \theta$. As a conclusion, the passages to the limit in $\mathbb{R}\times\overline{D_1}$ and in $\mathbb{R}\times\overline{D_2}$ do not lead to any contradiction if the function f is of the ignition temperature type.

4.2. The “KPP” case: $f > 0$ on $(0,1)$. Assume that f is lipschitz-continuous, $f(0) = f(1) = 0$, $f'(1) < 0$ and that $f > 0$ in $(0,1)$. The question of the existence of travelling waves for this kind of nonlinearity has been widely treated in the literature in the one-dimensional case since the pioneering paper of Kolmogorov, Petrovsky, Piskunov [21]. In the multidimensional case, Berestycki and Nirenberg proved in [8] that for any $\varepsilon > 0$, there exists a minimal speed c_ε^* and there are solutions $0 < u^\varepsilon < 1$ of

$$\begin{cases} \Delta u^\varepsilon - (c + \alpha_\varepsilon(y))\, \partial_1 u^\varepsilon + f(u^\varepsilon) = 0 & \text{in } \mathbb{R} \times \omega_\varepsilon \\ \partial_\nu u^\varepsilon = 0 & \text{on } \mathbb{R} \times \partial\omega_\varepsilon \\ u^\varepsilon(-\infty, \cdot) = 0,\ u^\varepsilon(+\infty, \cdot) = 1 & \end{cases} \tag{4.3}$$

if and only if $c \geq c_\varepsilon^*$. Let us now try to get *a priori* estimates for the speeds c_ε^*. First of all, let us call c^* the minimal speed solution of the same problem as (4.3), but set in $\mathbb{R} \times D_2$ with $\alpha(y) \equiv 0$.

Let $\theta > 0$ be small enough. As in [8], let χ_θ be a smooth and nondecreasing function defined in $[0, 1]$, equal to 0 in $[0, \theta]$ and equal to 1 in $[2\theta, 1]$. Set $f_\theta = f\chi_\theta$. Thus $f_\theta = 0$ in $[0, \theta]$ and f_θ is of the ignition temperature type on $[0, 1]$. Call c_θ^ε the unique speed for which there exists a solution u_θ^ε of the above problem (4.3) with the nonlinearity f_θ instead of f. It has been shown in [8] that $c_\theta^\varepsilon \nearrow c_\varepsilon^*$ as $\theta \searrow 0$. Similarly, we have $c_\theta \nearrow c^*$ as $\theta \searrow 0$. Lastly, from the arguments used in section 3.1, it follows that $c_\theta - 1 < c_\theta^\varepsilon < c_\theta$ and the passage to the limit $\theta \to 0$ gives that $c^* - 1 \leq c_\varepsilon^* \leq c^*$. Let us argue now as in Section 3.2. Let c_ε^* be a subsequence such that $c_\varepsilon^* \to c \in [c^* - 1, c^*]$ as $\varepsilon \to 0$. For any $\varepsilon > 0$, there exists a function $0 < u^\varepsilon < 1$, increasing in x_1, and which is a solution of

$$\begin{cases} \Delta u^\varepsilon - (c_\varepsilon^* + \alpha_\varepsilon(y))\, \partial_1 u^\varepsilon + f(u^\varepsilon) = 0 & \text{in } \mathbb{R} \times \omega_\varepsilon \\ \partial_\nu u^\varepsilon = 0 & \text{on } \mathbb{R} \times \partial\omega_\varepsilon \\ u^\varepsilon(-\infty, \cdot) = 0 \text{ and } u^\varepsilon(+\infty, \cdot) = 1 & \end{cases}$$

and we may suppose that $\max_{\overline{D_2}} u^\varepsilon(0, \cdot) = 1/2$. If we pass to the limit $\varepsilon \to 0$ in $\mathbb{R} \times \overline{D_2}$, we get a solution $0 \leq u \leq 1$ of

$$\begin{cases} \Delta u - c\, \partial_1 u + f(u) = 0 & \text{in } \mathbb{R} \times D_2 \\ \partial_\nu u = 0 & \text{on } \mathbb{R} \times \partial D_2 \\ \max\limits_{\overline{D_2}} u(0, \cdot) = 1/2 & \end{cases}$$

Furthermore, $\partial_1 u \geq 0$. As $f \geq 0$ in $[0, 1]$, it is the case that $u(\pm\infty, \cdot) = \gamma_\pm$, where $\gamma_\pm$ are constants and zeros of f. Since $\gamma_- \leq 1/2 \leq \gamma_+$, we have $\gamma_- = 0$ and $\gamma_+ = 1$. It then follows that $c \geq c^*$. From the inequalities $c^* - 1 \leq c \leq c^*$, we conclude that $c = c^*$.

If we resort to the same limiting process as in section 3, namely if we consider a sequence of solutions $(c_\varepsilon, u_\varepsilon)$ of (4.3) such that c_ε $(\geq c_\varepsilon^*) \to c$ $(\geq c^*)$ and if we fix the value of $\max u_\varepsilon(0, \cdot)$ in $\overline{D_2}$ or $\overline{D_1}$, we would get two solutions (c, u) or $(c+1, v)$ of two equivalent problems set in the same cylinders $\mathbb{R} \times D_2$ and $\mathbb{R} \times D_1$. But this does not lead to any contradiction because the set of the admissible speeds is the half-line $[c^*, +\infty[$.

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