Generalized travelling waves for reaction-diffusion equations

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ABSTRACT. In this paper, we introduce a generalization of travelling waves for evolution equations. We are especially interested in reaction-diffusion equations and systems in heterogeneous media (with general operators and general geometry). Our goal is threefold. First we give several definitions, for transition waves, fronts, pulses, global mean speed of propagation, etc. Next, we discuss the meaning of these definitions in various contexts. Then, we report on several results of [**4**] (of which this is a companion paper) about these notions. We further establish here several new properties. For this definition to be meaningful we need to show two things. First, that the definition covers and unifies all classical cases (and does not introduce spurious objects). Second, that it allows one to understand propagation fronts in completely new situations. In particular we report here on a result about travelling fronts passing an obstacle.

1. Classical notions of travelling fronts

1.1. Planar fronts. Travelling fronts form a specially important class of time-global solutions of reaction-diffusion equations. They arise and play an important role in various fields such as biology, population dynamics, ecology, physics, combustion... In many situations, they describe the transition between two different states.

Let us start with recalling the notion of classical travelling fronts in the homogeneous case, for the equation

$$(1.1) \qquad u_t = \Delta u + f(u) \text{ in } \mathbb{R}^N.$$

For basic properties of the linear heat equations, which allow one to derive existence and uniqueness of the Cauchy problem associated with (1.1), we refer to the classical text of H. Brezis [**14**].

In the case of (1.1), a planar travelling front connecting the uniform steady states 0 and 1 (assuming $f(0) = f(1) = 0$) is a solution which propagates in a given unit direction e with a speed c, and which can then be written as $u(t, x) = \phi(x \cdot e - ct)$ with $\phi(-\infty) = 1$ and $\phi(+\infty) = 0$. Two properties characterize such fronts: their

level sets are parallel hyperplanes which are orthogonal to the direction e, and the solution is invariant in the moving frame with speed c in the direction e. The profile ϕ of a planar front $\phi(x \cdot e - ct)$ satisfies the ordinary differential equation $\phi'' + c\phi' + f(\phi) = 0$ in $\mathbb{R}$. Existence and possible uniqueness of such fronts, formulæ for the speed(s) of propagation are well-known [1, 2, 16, 23] and depend upon the profile of the function f on $[0, 1]$.

1.2. Curved travelling fronts. Before introducing our general definition, let us recall the known extensions in non homogeneous cases. The first such extension is still one with classical travelling fronts but which are not planar anymore. Assume that the domain is a straight infinite cylinder of the type $\Omega = \mathbb{R} \times \omega$, where ω is a bounded smooth domain of $\mathbb{R}^{N-1}$. Denote $x = (x_1, y)$, with $y \in \omega$, the variables in Ω and consider the reaction-diffusion-advection equation

$$(1.2) \qquad u_t - \Delta u + \alpha(y)\frac{\partial u}{\partial x_1} = f(y, u)$$

with, say, Neumann boundary conditions on $\partial\Omega$. The functions α and f are given and may depend on the cross variables y. Assume that $f(y, 0) = f(y, 1) = 0$ for all $y \in \overline{\omega}$. In this context, a travelling front connecting 0 and 1 and propagating with speed c in the direction $e_1 = (1, 0, \ldots, 0)$ is a solution of the type $u(t, x_1, y) = \phi(x_1 - ct, y)$ such that $\phi(-\infty, y) = 1$ and $\phi(+\infty, y) = 0$ uniformly in $y \in \overline{\omega}$. These fronts are still invariant (in the moving frame with speed c in the direction e_1) and have a constant speed, but the profile ϕ is in general not planar anymore. It is a function of both variables $s = x_1 - ct \in \mathbb{R}$ and $y \in \overline{\omega}$, and it satisfies the elliptic partial differential equation

$$-\Delta\phi + (\alpha(y) - c)\frac{\partial\phi}{\partial s} = f(y, \phi) \text{ in } \Omega$$

with Neumann boundary conditions on $\partial\Omega$. Most of the known results which had been obtained on planar fronts for the homogeneous equation (1.1) have been extended, with PDE methods, to the case (1.2), see [2, 9, 10, 11, 27]. The case when ω is periodic in the variables y can also be treated similarly, see [3].

1.3. Curved fronts for (1.1) in $\mathbb{R}^N$. Non-planar fronts which arise in heterogeneous problems of the type (1.2) were recently shown to also exist even in the homogeneous case. Consider for instance the homogeneous equation (1.1) in $\mathbb{R}^N$ and call $r = (x_1^2 + \cdots + x_{N-1}^2)^{1/2}$. Assume that $f(0) = f(1) = 0$. For the main three classical classes of reaction terms f (combustion, bistable, monostable) and for any given angle $\alpha \in (0, \pi/2)$ equation (1.1) admits "conical-shaped" non-planar fronts of the type

$$u(t, x) = \phi(r, x_N - ct),$$

such that $\phi(r, s) \to 1$ (resp. 0) uniformly as $s - \psi(r) \to -\infty$ (resp. $+\infty$), where ψ satisfies: $\psi(r)/r \to \cot\alpha$ as $r \to +\infty$ (see [13, 15, 17, 19, 20, 25]). The profiles are still invariant in a moving frame with constant speed, but the level sets are not hyperplanes anymore. Conical-shaped fronts are also known to exist for systems of reaction-diffusion equations and for aperture angles α close to $\pi/2$ under some stability assumptions (see [21]). In the case when f is concave and positive on $(0, 1)$, then, many more non-planar travelling fronts also exist, which are not conical-shaped, see [20].

1.4. Pulsating travelling fronts, periodic media. Another important example of travelling fronts is for heterogeneous equations of the type

$$(1.3) \qquad u_t = \nabla \cdot (A(x)\nabla u) + q(x) \cdot \nabla u + f(x, u) \text{ in } \mathbb{R}^N,$$

where the uniformly elliptic matrix field A, the vector field q and the function f are smooth and periodic in $\mathbb{R}^N$. That is, there are $L_1, \dots, L_N > 0$ such that

$$(1.4) \qquad A(x + k) = A(x), \; q(x + k) = q(x), \; f(x + k, \cdot) = f(x, \cdot)$$

for all $x \in \mathbb{R}^N$ and $k = (k_1, \dots, k_N) \in L_1\mathbb{Z} \times \cdots \times L_N\mathbb{Z}$. Unlike all aforementioned cases, these equations in general are not invariant by translation in any direction. Assume that, say, $f(x, 1) = f(x, 0) = 0$ for all $x \in \mathbb{R}^N$. Given a unit vector $e \in \mathbb{S}^{N-1}$, a *pulsating travelling front* connecting 0 and 1, and propagating with speed $c \neq 0$ in the direction e is a solution $u(t, x)$ of (2.1) such that

$$(1.5) \qquad u\left(t + \frac{k \cdot e}{c}, x\right) = u(t, x - k)$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ and $k \in L_1\mathbb{Z} \times \cdots \times L_N\mathbb{Z}$, and $u(t, x) \to 1$ (resp. 0) as $x \cdot e \to -\infty$ (resp. $x \cdot e \to +\infty$) uniformly in t and in the variables which are orthogonal to e (see [**3, 30, 31**]). These fronts can be written as

$$u(t, x) = \phi(x \cdot e - ct, x)$$

where the function $(s, x) \mapsto \phi(s, x)$ is periodic in x in the sense of (1.4), and $\phi(-\infty, x) = 1$, $\phi(+\infty, x) = 0$ uniformly in x. The function ϕ satisfies a degenerate elliptic equation in the variables s and x. In the moving frame with speed c in the direction e, the profile of the front is not invariant anymore, but it is in general quasi-periodic in time. Observe that at each time t, each level set of u is trapped between two parallel hyperplanes which are orthogonal to e, but in general it is not planar. Existence results and formulæ for the speeds of propagation are given in [**3, 6, 7, 31**]. The case where the domain Ω satisfies

$$(1.6) \qquad \forall \, k \in L_1\mathbb{Z} \times \cdots L_N\mathbb{Z}, \; \Omega + k = \Omega,$$

(namely Ω has the same periodicity $(L_1, \dots, L_N)$ in the variables $(x_1, \dots, x_N)$ as the coefficients) has also been investigated, see [**3**]. For reaction-diffusion equations with time-dependent coefficients, pulsating fronts (which are defined in a similar way) are also known to exist (see [**18, 26**]). Moreover, the limiting states $p^{\pm}(t, x)$ may also depend on x or on t for space or time-periodic equations (see [**8, 22, 28**] for some examples).

1.5. Almost periodic case. Our last case deals with the almost-periodic framework. Consider the case where all coefficients of (1.3) are almost periodic. To make notations simpler, assume that (2.1) reduces to

$$(1.7) \qquad u_t = u_{xx} + b(x)f(u) \text{ in } \Omega = \mathbb{R}.$$

Assume that the closure $\mathcal{H}$, with respect to the uniform norm on $\mathbb{R}$, of the set of all translations $\sigma_y b$ (with $\sigma_y b(x) = b(x + y)$) of the coefficient b is compact. Assume moreover that $f(1) = f(0) = 0$. In this case, a new definition was introduced by H. Matano. Namely, a travelling wave (as defined in [**24**]) is a solution u for which

there exists a continuous map $w : \mathcal{H} \times \mathbb{R} \to \mathbb{R}$ and a function $\xi : \mathbb{R} \to \mathbb{R}$ such that $\xi(t) \to \pm\infty$ as $t \to \pm\infty$,

$$(1.8) \quad \begin{cases} u(t, x + \xi(t)) = w(\sigma_{\xi(t)}b, x), \\ w(z, s) \to 1 \text{ (resp. 0) as } s \to -\infty \text{ (resp. } s \to +\infty\text{) uniformly in } z \in \mathcal{H}. \end{cases}$$

In Matano's description, such a solution is a solution $u(t, x)$ such that the "profile", that is the function $x \mapsto u(t, \cdot)$ is a continuous function of the "landscape" (i.e. a shift of $b(x)$). This definition can also be given for more general equations (2.1) with spatially almost periodic (and time independent) coefficients in $\mathbb{R}^N$. The case of equations with coefficients which are almost periodic in time and independent of x has been dealt with in [**29**].

In all these examples of fronts which have been listed so far, the solutions always converge to 0 or 1 uniformly far away from their level sets. This simple but fundamental observation is what leads us to introduce a new fully general definition in the following section.

The paper is organized as follows. In the next section, we give the new general definitions of transition waves, fronts, pulses, invasions, global mean speeds of propagation, etc, and we explain why these definitions unify all abovementioned examples. In Sections 3 and 4, we report on some general results of [**4**] of which this article is a companion paper. Furthermore, we derive additional properties which are not in [**4**]. We especially show that under some assumptions the generalized fronts can reduce to the usual notions. In the remaining sections, we show that our new definitions can also take into account more general situations which are not covered by the classical notions, and we give some explicit examples of fronts which are not travelling fronts in the usual sense. We will see in particular that this definition accounts for fronts passing an obstacle.

2. Generalized fronts for heterogeneous media

In this section, we give a general single definition of transition waves which unifies all the classical examples of travelling fronts. We will see that some properties are intrinsiquely associated to the waves, and we also introduce additional specific notions.

2.1. The main definitions. The notion of travelling fronts or waves can be extended for very general heterogeneous reaction-diffusion-advection equations, or systems of equations, of the type

$$(2.1) \quad \begin{cases} u_t = \nabla_x \cdot (A(t, x)\nabla_x u) + q(t, x) \cdot \nabla_x u + f(t, x, u) \text{ in } \Omega, \\ B(t, x)[u] = 0 \text{ on } \partial\Omega. \end{cases}$$

Throughout the paper, Ω is a connected open subset of $\mathbb{R}^N$ which is locally uniformly smooth. Denote $\nu(x)$ its outward unit normal at a point $x \in \partial\Omega$. The unknown function u, defined in $\mathbb{R} \times \overline{\Omega}$, is in general a vector field $u = (u_1, \cdots, u_m) \in \mathbb{R}^m$. The boundary conditions $B(t, x)[u] = 0$ on $\partial\Omega$ may for instance be of the Dirichlet, Neumann or Robin types, or may be nonlinear as well. The diffusion matrix field $(t, x) \mapsto A(t, x) = (a_{ij}(t, x))_{1 \le i, j \le N}$ is assumed to be of class $C^{1,\beta}(\mathbb{R} \times \overline{\Omega})$ (with $\beta > 0$) and there exist $0 < \alpha_1 \le \alpha_2$ such that

$$\alpha_1|\xi|^2 \le a_{ij}(t, x)\xi_i\xi_j \le \alpha_2|\xi|^2 \text{ for all } (t, x) \in \mathbb{R} \times \overline{\Omega} \text{ and } \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N,$$

under the usual summation convention of repeated indices. We denote by $\cdot$ and $|\ |$ the scalar product and Euclidean norm in $\mathbb{R}^k$. The vector field $(t,x) \mapsto q(t,x)$ ranges in $\mathbb{R}^N$ and is of class $C^{0,\beta}(\mathbb{R} \times \overline{\Omega})$. Lastly, the map $(t,x,s) \mapsto f(t,x,s)$ is assumed to be of class $C^{0,\beta}$ in (t,x) locally in $s \in \mathbb{R}$, and locally Lipschitz-continuous in s, uniformly in $(t,x) \in \mathbb{R} \times \overline{\Omega}$.

Let d_Ω be the geodesic distance in $\overline{\Omega}$. For any two subsets A and B of $\overline{\Omega}$, denote

$$d_\Omega(A,B) = \inf\ \{d_\Omega(x,y);\ (x,y) \in A \times B\}.$$

For $x \in \overline{\Omega}$ and $r > 0$, we set

$$B(x,r) = \{y \in \overline{\Omega},\ d_\Omega(x,y) \le r\}\ \text{ and }\ S(x,r) = \{y \in \overline{\Omega},\ d_\Omega(x,y) = r\}.$$

We assume that we are given two classical solutions $p^{\pm}$ of (2.1), which are defined for all $(t,x) \in \mathbb{R} \times \overline{\Omega}$, as well as two families $(\Omega_t^-)_{t \in \mathbb{R}}$ and $(\Omega_t^+)_{t \in \mathbb{R}}$ of open disjoint nonempty subsets of Ω such that, for all $t \in \mathbb{R}$,

(2.2)
$$\begin{cases} \partial\Omega_t^- \cap \Omega = \partial\Omega_t^+ \cap \Omega =: \Gamma_t,\ \ \Omega_t^- \cup \Gamma_t \cup \Omega_t^+ = \Omega \\ \sup\ \{d_\Omega(x,\Gamma_t);\ t \in \mathbb{R},\ x \in \Omega_t^{\pm}\} = +\infty. \end{cases}$$

The first two properties mean somehow that Γ_t splits Ω into two parts, namely Ω_t^- and Ω_t^+. The last property especially implies that, for any given $t \in \mathbb{R}$, there is no $r > 0$ and $x \in \overline{\Omega}$ such that Ω_t^- or Ω_t^+ are included in $B(x,r)$.

DEFINITION 2.1. (Transition wave) For problem (2.1), a (generalized) transition wave between p^- and p^+ is a time-global classical solution u such that $u \not\equiv p^{\pm}$ and there exist some sets $\Omega_t^{\pm}$ as above with

$$u(t,x) - p^{\pm}(t,x) \to 0 \ \text{ uniformly in } t \in \mathbb{R} \text{ and } x \in \overline{\Omega_t^{\pm}} \text{ as } d_\Omega(x,\Gamma_t) \to +\infty.$$

Notice that in the above definition, a central role is played by the uniformity of the limits $u(t,x) - p^{\pm}(t,x) \to 0$.

Our second main definition and natural notion is that of global mean speed.

DEFINITION 2.2. (Global mean speed of propagation) We say that the transition wave u has global mean speed c (≥ 0) if

$$\frac{d_\Omega(\Gamma_t,\Gamma_s)}{|t-s|} \to c \ \text{ as } |t-s| \to +\infty.$$

We say that the transition wave u is almost-stationary if it has global mean speed $c = 0$, quasi-stationary if

$$\sup\ \{d_\Omega(\Gamma_t,\Gamma_s);\ (t,s) \in \mathbb{R}^2\} < +\infty,$$

and stationary if it does not depend on t.

2.2. Intrinsic properties. In the above general definitions, the sets $\Omega_t^{\pm}$ are not uniquely determined. Nevertheless, in the scalar case, under some assumptions on $p^{\pm}$ and $\Omega_t^{\pm}$, the sets Γ_t somewhat reflect the location of the level sets of u:

PROPOSITION 2.3. *Assume that $m = 1$, that $p^- < p^+$ are constant solutions of (2.1) and let u be a time-global classical solution of (2.1) such that $\{u(t,x),\ (t,x) \in \mathbb{R} \times \overline{\Omega}\} = (p^-,p^+)$ and $B(t,x)[u] = \mu(t,x) \cdot \nabla_x u(t,x) = 0$ on $\mathbb{R} \times \partial\Omega$, for some unit vector field $\mu \in C^{0,\beta}(\mathbb{R} \times \partial\Omega)$ (with $\beta > 0$) such that*

$$\inf\ \{\mu(t,x) \cdot \nu(x);\ (t,x) \in \mathbb{R} \times \partial\Omega\} > 0.$$

1. *Assume that u is a transition wave between p^- and p^+, that there is $\tau > 0$ such that*

(2.3) $$\sup\,\{d_\Omega(x,\Gamma_{t-\tau});\ t\in\mathbb{R},\ x\in\Gamma_t\} < +\infty,$$

and that

(2.4) $$\sup\,\{d_\Omega(y,\Gamma_t);\ y\in\overline{\Omega_t^\mp}\ \cap\ S(x,r)\} \underset{r\to+\infty}{\longrightarrow} +\infty \ \text{unif. in } t\in\mathbb{R},\ x\in\Gamma_t.$$

Then, for all $\lambda \in (p^-, p^+)$,

(2.5) $$\sup\,\{d_\Omega(x,\Gamma_t);\ u(t,x) = \lambda\} < +\infty,$$

and, for all $C \geq 0$,

(2.6) $$p^- < \inf\,\{u(t,x);\ d_\Omega(x,\Gamma_t) \leq C\} \leq \sup\,\{u(t,x);\ d_\Omega(x,\Gamma_t) \leq C\} < p^+.$$

2. *Conversely, if (2.5) and (2.6) hold for some choices of sets $(\Omega_t^\pm, \Gamma_t)_{t\in\mathbb{R}}$ satisfying (2.2) and if there is $d_0 > 0$ such that the sets*

$$\{(t,x) \in \mathbb{R}\times\overline{\Omega},\ x\in\overline{\Omega_t^\pm},\ d_\Omega(x,\Gamma_t) \geq d\}$$

are connected for all $d \geq d_0$, then u is a transition wave between p^- and p^+, or p^+ and p^-.

Roughly speaking the assumption (2.3) means that Γ_t and $\Gamma_{t-\tau}$ are not too far from each other. For instance, if all Γ_t are parallel hyperplanes in $\Omega = \mathbb{R}^N$, then the assumption means that the distance between Γ_t and $\Gamma_{t-\tau}$ is bounded independently of t, for some $\tau > 0$. The property (2.4) means that the sets $\Omega_t^\pm$ are in some sense wide enough, uniformly with respect to t.

Proposition 2.3 means that, under some assumptions, the boundedness of the distance between the sets Γ_t and the level sets of a transition wave is thus an intrinsic notion. It turns out that the global mean speed, if any, is also intrinsic.

PROPOSITION 2.4. *Let $p^\pm$ be two limiting states solving (2.1) and satisfying*

$$\inf\,\{|p^-(t,x) - p^+(t,x)|;\ (t,x) \in \mathbb{R}\times\overline{\Omega}\} > 0.$$

Let u be a transition wave between p^- and p^+ with a choice of sets $\Omega_t^\pm$ satisfying (2.2) and (2.4). If u has global mean speed c, then, for any other choice of sets $\widetilde{\Omega}_t^\pm$ satisfying (2.2) and (2.4), u has a global mean speed and this global mean speed is equal to c.

2.3. Further specifications. More specific notions of fronts, pulses, invasions (or travelling waves), almost planar waves can now be defined. These notions are related to some properties of the limiting states $p^\pm$ or of the sets $\Omega_t^\pm$, and are listed in the following definitions. Here u denotes a transition wave between p^- and p^+ in the sense of Definition 2.1.

DEFINITION 2.5. (Fronts and spatially extended pulses) Let $p^\pm = (p_1^\pm, \cdots, p_m^\pm)$. We say that the transition wave u is a front if either $p_i^-(t,x) < p_i^+(t,x)$ for all $(t,x) \in \mathbb{R}\times\overline{\Omega}$ and $1 \leq i \leq m$, or $p_i^-(t,x) > p_i^+(t,x)$ for all $(t,x) \in \mathbb{R}\times\overline{\Omega}$ and $1 \leq i \leq m$. The transition wave u is a spatially extended pulse if $p^-(t,x) = p^+(t,x)$ for all $(t,x) \in \mathbb{R}\times\overline{\Omega}$.

DEFINITION 2.6. (Invasions, or travelling waves) We say that p^+ invades p^- (resp. p^- invades p^+) if $\Omega_t^+ \supset \Omega_s^+$ (resp. $\Omega_t^- \supset \Omega_s^-$) for all $t \geq s$ and $d_\Omega(\Gamma_t, \Gamma_s) \to +\infty$ as $|t-s| \to +\infty$. Therefore, $u(t,x) - p^{\pm}(t,x) \to 0$ as $t \to \pm\infty$ (resp. $t \to \mp\infty$) locally uniformly in $\overline{\Omega}$ with respect to the distance d_Ω.

REMARK 2.7. A transition wave can thus be viewed as a spatial transition between the two limiting states p^- and p^+. The particular case of an invasion can also be viewed as a temporal connection between p^- and p^+.

DEFINITION 2.8. (Almost planar waves in the direction e) We say that the transition wave u is almost planar in the direction $e \in \mathbb{S}^{N-1}$ if, for all $t \in \mathbb{R}$, $\Omega_t^{\pm}$ can be chosen so that

$$\Gamma_t = \{x \in \Omega, \ x \cdot e = \xi_t\}$$

for some $\xi_t \in \mathbb{R}$.

DEFINITION 2.9. (Thin waves) In dimension $N = 1$, we say that the transition wave u is thin if Γ_t can be reduced to a singleton for each t. In dimensions $N \geq 2$, we say that the transition wave u is thin if there is an integer $k \geq 1$ such that, for each $t \in \mathbb{R}$, there are k open sets $\Omega_{i,t} \subset \mathbb{R}^{N-1}$, k continuous maps $\psi_{i,t} : \Omega_{i,t} \to \mathbb{R}$ and k rotations $R_{i,t}$ of $\mathbb{R}^N$ (for $1 \leq i \leq k$), such that

$$\Gamma_t \subset \bigcup_{1 \leq i \leq k} R_{i,t}\left(\{x_N = \psi_{i,t}(x_1, \ldots, x_{N-1}), \ (x_1, \ldots, x_{N-1}) \in \Omega_{i,t}\}\right).$$

Notice in particular that any almost planar wave is thin.

2.4. The classical examples. Let us now come back to the usual notions which were listed in Section 1. We shall see that they are all covered by the general definitions of transition waves and that they may correspond to some of the specific cases mentioned above.

For instance, for the homogeneous equation (1.1) in $\mathbb{R}^N$, if $f(0) = f(1) = 0$, the solutions $u(t,x) = \phi(x \cdot e - ct)$, with $\phi(-\infty) = 1$ and $\phi(+\infty) = 0$ are (almost) planar fronts connecting 1 and 0, with (global mean) speed $|c|$. The uniform stationary state $p^- = 1$ (resp. $p^+ = 0$) invades the uniform stationary state $p^+ = 0$ (resp. $p^- = 1$) if $c > 0$ (resp. $c < 0$). The sets $\Omega_t^{\pm}$ can for instance be defined as

$$(2.7) \qquad \Omega_t^{\pm} = \{x \in \mathbb{R}^N, \ \pm(x \cdot e - ct) > 0\}$$

For equation (1.2) in an infinite cylinder $\Omega = \mathbb{R} \times \omega$, the solutions $u(t, x_1, y) = \phi(x_1 - ct, y)$ such that $\phi(-\infty, y) = 1$ and $\phi(+\infty, y) = 0$ uniformly in $y \in \overline{\omega}$ are almost planar fronts connecting 1 and 0, and the sets $\Omega_t^{\pm}$ can be chosen as $\Omega_t^{\pm} = \{(x_1, y) \in \mathbb{R} \times \overline{\omega}, \ \pm(x_1 - ct) > 0\}$.

The curved fronts $u(t,x) = \phi(r, x_N - ct)$ exhibited in Section 1.3 for equation (1.1) can also be covered by Definition 2.1 with $p^- = 1$, $p^+ = 0$ and, say,

$$\Omega_t^{\pm} = \{x \in \mathbb{R}^N, \ \pm(x_N - ct - \psi(r)) > 0\}.$$

They are not almost planar as soon as $\psi(r)/r \not\to 0$ as $r = \sqrt{x_1^2 + \cdots + x_{N-1}^2} \to +\infty$.

The pulsating fronts which were mentioned in Section 1.4 also fall within the general definition of travelling fronts with $(p^-, p^+) = (1, 0)$ and, say, $\Omega_t^{\pm}$ given by (2.7) if $\Omega = \mathbb{R}^N$. But, in a general periodic domain satisfying (1.6), the global

mean speed (as defined in Definition 2.2) of a pulsating front solving (1.5) is equal to $\gamma|c|$, where $\gamma = \gamma(e) \geq 1$ is such that

$$(2.8) \quad \frac{d_\Omega(x,y)}{|x-y|} \to \gamma(e) \text{ as } |x-y| \to +\infty, \ (x,y) \in \overline{\Omega} \times \overline{\Omega} \text{ and } x-y \text{ is parallel to } e.$$

The constant $\gamma(e)$ is by definition larger than or equal to 1. It measures the asymptotic ratio of the geodesic and Euclidean distances along the direction e. If the domain Ω is invariant in the direction e, that is $\Omega = \Omega + se$ for all $s \in \mathbb{R}$, then $\gamma(e) = 1$.

Lastly, the almost-periodic case described in Section 1.5 is also a particular case of the general definitions. For instance, in the one-dimensional case (1.7) with $f(0) = f(1) = 0$, the solutions $u(t,x)$ satisfying (1.8) are transition waves with $(p^-, p^+) = (1,0)$, $\Omega_t^- = (-\infty, \xi(t))$ and $\Omega_t^+ = (\xi(t), +\infty)$.

To sum up, we have just seen that the general definitions given in this section generalize all the usual notions. Furthermore, what is also very important is that the new notions are both strong and wide. Indeed, first, we show in the following two sections that there is no abusive generalization since, under some assumptions, the transition waves can be reduced to the usual notions in some particular cases. Second, we will see that the transition waves can take into account other cases which cannot be covered by the classical definitions.

3. Applications of the definitions to the classical cases

In this section, we see how the general definitions can reduce to the usual notions in some particular cases. As an example of such results, we start in Section 3.1 with the proof of a one-dimensional symmetry property for almost planar bistable-type fronts in $\mathbb{R}^N$. A more general result is given in [**4**]. But we include the proof here because it is simple and explains clearly why this result is true. In Sections 3.2 to 3.4 we report on some results of [**4**] for generalized bistable-type fronts. Finally, we prove in Section 3.5 a new classification result for generalized monostable-type fronts which are trapped between two planar fronts.

3.1. Almost planar bistable transition waves. We consider classical time-global bounded real-valued solutions of

$$(3.1) \qquad\qquad u_t = \Delta u + f(u), \ x \in \mathbb{R}^N.$$

We assume here that the function $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz-continuous and

$$(3.2) \quad \begin{cases} f(0) = f(1) = 0, \\ \exists \, \delta > 0, \ f \text{ is non-increasing in } (-\infty, \delta] \text{ and in } [1-\delta, +\infty). \end{cases}$$

An example of such a function is the cubic nonlinearity $f(s) = s - s^3$ which arises in scalar Ginzburg-Landau equations (see [**12**]).

THEOREM 3.1. *Let u be a bounded almost planar transition wave solving* (3.1), *between $p^- = 0$ to $p^+ = 1$, and assume that there exist $e \in \mathbb{S}^{N-1}$, $c \geq 0$, $M \geq 0$ and a map $\mathbb{R} \ni t \mapsto \xi_t$ such that*

$$\begin{cases} \forall \, t \in \mathbb{R}, \ \Omega_t^\pm = \{x \in \mathbb{R}^N, \ \pm(x \cdot e - \xi_t) < 0\}, \\ \forall \, (t,s) \in \mathbb{R}^2, \ -M \leq |\xi_t - \xi_s| - c|t-s| \leq M. \end{cases}$$

Then there exist $\varepsilon \in \{-1, 1\}$ and a decreasing function $\phi : \mathbb{R} \to (0, 1)$ such that

$$\forall (t, x) \in \mathbb{R} \times \mathbb{R}^N, \ u(t, x) = \phi(x \cdot e - c\varepsilon t).$$

Roughly speaking, this result means that any almost planar transition wave is actually planar and invariant in the moving frame which propagates with speed c in the direction εe (actually, if $c = 0$, then u is stationary).

PROOF. Up to rotation of the frame, one can assume that $e = (1, 0, \ldots, 0)$. Denote $x' = (x_2, \ldots, x_N)$ and $x = (x_1, x')$. The assumption made on ξ_t provides the existence of $\varepsilon \in \{-1, 1\}$ such that the map

$$t \mapsto \zeta_t := \xi_t - c\varepsilon t$$

is bounded. Call

$$v(t, x) = u(t, x + c\varepsilon t e) = u(t, x_1 + c\varepsilon t, x')$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$. Our goal is to prove that v depends on x_1 only and that it is decreasing in x_1. The function v is a generalized transition wave between $p^- = 0$ and $p^+ = 1$, for the equation

$$(3.3) \qquad\qquad v_t = \Delta v + c\varepsilon e \cdot \nabla v + f(v),$$

and $\widetilde{\Omega}_t^{\pm} = \{x \in \mathbb{R}^N, \ \pm(x_1 - \zeta_t) < 0\}$. Since $t \mapsto \zeta_t$ is bounded, it follows that

(3.4) $v(t, x) \to 1$ (resp. 0) as $x_1 \to -\infty$ (resp. $+\infty$) unif. in $(t, x') \in \mathbb{R} \times \mathbb{R}^{N-1}$.

Thus, there exists $A > 0$ such that

$$(3.5) \qquad \begin{cases} v(t, x) \geq 1 - \delta & \text{for all } x_1 \leq -A \text{ and } (t, x') \in \mathbb{R} \times \mathbb{R}^{N-1}, \\ v(t, x) \leq \delta & \text{for all } x_1 \geq A \text{ and } (t, x') \in \mathbb{R} \times \mathbb{R}^{N-1}. \end{cases}$$

Notice that one can assume without loss of generality that $\delta \in (0, 1/2]$.

Choose now any $T \in \mathbb{R}$ and $\rho \in \mathbb{R}^{N-1}$. For all $s \in \mathbb{R}$ and $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, denote

$$w^s(t, x) = v(t + T, x_1 + s, x' + \rho),$$

and call $E = \{(t, x) \in \mathbb{R} \times \mathbb{R}^N, \ x_1 < -A\}$. Fix any $\sigma \geq 2A$. Since v and w are globally bounded (because u is), one has $v + \varepsilon \geq w^{\sigma}$ in $\overline{E}$ for all ε large enough. Define

$$\varepsilon^* = \inf \{\varepsilon > 0, \ v + \varepsilon \geq w^{\sigma} \text{ in } \overline{E}\}.$$

The real number ε^* is nonnegative and $v + \varepsilon^* \geq w^{\sigma}$ in $\overline{E}$. Assume $\varepsilon^* > 0$. Since $v, w^{\sigma} \to 1$ as $x_1 \to -\infty$ uniformly in (t, x') and since the Lipschitz-continuous functions v and w^{σ} satisfy $v \geq 1 - \delta \geq \delta \geq w^{\sigma}$ on $\partial E = \{x_1 = -A\}$ (because $\sigma \geq 2A$), there exist $x_{1,\infty} \in (-\infty, -A)$ and a sequence $(t_n, x_{1,n}, x'_n)_{n \in \mathbb{N}}$ in $\overline{E}$ such that

$$v(t_n, x_{1,n}, x'_n) + \varepsilon^* - w^{\sigma}(t_n, x_{1,n}, x'_n) \to 0 \text{ and } x_{1,n} \to x_{1,\infty} \text{ as } n \to +\infty.$$

From standard parabolic estimates, the functions $v_n(t, x_1, x') = v(t + t_n, x_1, x' + x'_n)$ converge locally uniformly, up to extraction of a subsequence, to a solution v_{∞} of (3.3) such that

$z(t, x_1, x') := v_{\infty}(t, x_1, x') + \varepsilon^* - v_{\infty}(t + T, x_1 + \sigma, x' + \rho) \geq 0$ for all $(t, x_1, x') \in \overline{E}$

and $z(0, x_{1,\infty}, 0) = 0$. Since $v_\infty \geq 1 - \delta$ in $\overline{E}$ and f is non-increasing in $[1 - \delta, +\infty)$, it follows that

$$
\begin{aligned}
z_t - \Delta z - c\varepsilon e \cdot \nabla z &= f(v_\infty(t, x)) - f(v_\infty(t + T, x_1 + \sigma, x' + \rho)) \\
&\geq f(v_\infty(t, x) + \varepsilon^*) - f(v_\infty(t + T, x_1 + \sigma, x' + \rho)) \\
&\geq -Bz \ \text{ in } \overline{E},
\end{aligned}
$$

for some constant B (remember that f is Locally Lipschitz-continuous and that v_∞ is bounded. The strong parabolic maximum principle implies that $z(t, x) = 0$ for all $t \leq 0$, $x_1 \leq -A$ and $x' \in \mathbb{R}^{N-1}$. But $z \geq \varepsilon^* > 0$ on ∂E, which leads to a contradiction.

Thus, $\varepsilon^* = 0$, whence $v \geq w^\sigma$ in $\overline{E}$. On the other hand, since $w^\sigma \leq \delta$ for $x_1 \geq -A$, $w^\sigma \leq v$ for $x_1 = -A$ and since f is non-increasing in $(-\infty, \delta]$, one can prove similarly that $w^\sigma \leq v$ in the set $\{x_1 \geq -A\}$.

To sum up, $w^\sigma \leq v$ in $\mathbb{R} \times \mathbb{R}^N$ for all $\sigma \geq 2A$. Call now

$$
\sigma^* = \inf \{\sigma \in \mathbb{R}, \ w^{\sigma'} \leq v \text{ in } \mathbb{R} \times \mathbb{R}^N \text{ for all } \sigma' \geq \sigma\}.
$$

One has $\sigma^* \leq 2A$ and $\sigma^* > -\infty$ because $v(t, -\infty, x') = 1 > 0 = v(t, +\infty, x')$. Moreover, $w^{\sigma^*} \leq v$ in $\mathbb{R} \times \mathbb{R}^N$. Assume $\sigma^* > 0$, and call $S = \{(t, x) \in \mathbb{R} \times \mathbb{R}^N, \ -A \leq x_1 \leq A\}$. If $\inf_S(v - w^{\sigma^*}) > 0$, then there exists $\eta_0 \in (0, \sigma^*)$ such that $v \geq w^{\sigma^* - \eta}$ in S for all $\eta \in [0, \eta_0]$. As above, one can then prove that $v \geq w^{\sigma^* - \eta}$ in $\overline{\overline{E}}$ for all $\eta \in [0, \eta_0]$. Similarly, $w^{\sigma^* - \eta} \leq \delta$ in $\{x_1 \geq A\}$ for all $\eta \in [0, \eta_0]$ (because of (3.5) and $\sigma^* - \eta_0 \geq 0$), whence $w^{\sigma^* - \eta} \leq v$ in $\{x_1 \geq A\}$. Therefore, $w^{\sigma^* - \eta} \leq v$ in $\mathbb{R} \times \mathbb{R}^N$ for all $\eta \in [0, \eta_0]$. This contradicts the minimality of σ^*. It follows then that

$$
\inf_S (v - w^{\sigma^*}) = 0.
$$

As a consequence, there exist $x_{1,\infty} \in [-A, A]$ and a sequence $(t_n, x_{1,n}, x'_n)_{n \in \mathbb{N}}$ such that

$$
x_{1,n} \to x_{1,\infty} \text{ and } v(t_n, x_{1,n}, x'_n) - w^{\sigma^*}(t_n, x_{1,n}, x'_n) \to 0 \text{ as } n \to +\infty.
$$

Call $v_n(t, x_1, x') = v(t + t_n, x_1, x' + x'_n)$. Up to extraction of a subsequence, the functions v_n converge locally uniformly to a solution v_∞ of (3.3) such that

$$
z(t, x) = v_\infty(t, x) - v_\infty(t + T, x_1 + \sigma^*, x' + \rho) \geq 0 \text{ in } \mathbb{R} \times \mathbb{R}^N
$$

and $z(0, x_{1,\infty}, 0) = 0$. From the strong parabolic maximum principle, it resorts as above that $z(t, x) = 0$ for all $t \leq 0$, and then $z \equiv 0$ in $\mathbb{R} \times \mathbb{R}^N$ by uniqueness of the solutions of the Cauchy problem for (3.3). Thus, $v_\infty(0, 0, 0) = v_\infty(kT, k\sigma^*, k\rho)$ for all $k \in \mathbb{Z}$. But $v_\infty(kT, k\sigma^*, k\rho) \to 1$ (resp. $\to 0$) as $k \to -\infty$ (resp. $k \to +\infty$) since $\sigma^* > 0$ and v_∞ still satisfies (3.4). One has then reached a contradiction.

Thus, $\sigma^* \leq 0$, whence

$$
v(t, x) \geq w^0(t, x) = v(t + T, x_1, x' + \rho) \text{ in } \mathbb{R} \times \mathbb{R}^N.
$$

Since $T \in \mathbb{R}$ and $\rho \in \mathbb{R}^{N-1}$ were arbitrary, one concludes that v depends on x_1 only, namely $v(t, x) = \phi(x_1)$. Furthermore, the arguments above actually imply that $\phi(x_1) \geq \phi(x + \sigma)$ for all $\sigma \geq 0$, and the strong (elliptic) maximum principle yields $\phi(x_1) > \phi(x_1 + \sigma)$ for all $x_1 \in \mathbb{R}$ and $\sigma > 0$. In particular, $\phi(x_1) \in (0, 1)$ for all $x_1 \in \mathbb{R}$. That completes the proof of Theorem 3.1.

3.2. Periodic framework. Our second result is concerned with periodic media. We assume here that Ω is a smooth periodic domain satisfying (1.6). Let u be a generalized transition wave between p^- and p^+ for equation (2.1), with boundary condition

$$(3.6) \qquad \mu(x) \cdot \nabla_x u(t,x) = \mu(x) \cdot \nabla_x p^\pm(x) = 0 \ \text{ on } \mathbb{R} \times \partial\Omega,$$

where μ is a uniformly locally $C^{0,\beta}(\partial\Omega)$ (with $\beta > 0$) unit vector field such that

$$(3.7) \qquad \inf \{\mu(x) \cdot \nu(x); \ x \in \partial\Omega\} > 0.$$

Assume that u and $p^\pm$ are globally bounded in $\mathbb{R} \times \overline{\Omega}$, that A, q, f, μ, $p^\pm$ do not depend on t, are periodic in x in the sense of (1.4), and there is $\delta > 0$ such that

$$(3.8) \qquad s \mapsto f(x,s) \text{ is nonincreasing in } (-\infty, p^-(x) + \delta] \text{ and } [p^+(x) - \delta, +\infty)$$

for all $x \in \overline{\Omega}$.

THEOREM 3.2. **[4]** *If u is an invasion of p^- by p^+ with*

$$(3.9) \qquad \kappa := \inf \{p^+(x) - p^-(x); \ x \in \overline{\Omega}\} > 0,$$

and if there exist $e \in \mathbb{S}^{N-1}$, $c \geq 0$ and a map $\mathbb{R} \ni t \mapsto \xi_t$ such that

$$(3.10) \qquad \sup \{ \mid d_\Omega(\Gamma_t, \Gamma_s) - c|t - s| \mid ; \ (t,s) \in \mathbb{R}^2\} < +\infty,$$

where

$$(3.11) \qquad \Gamma_t = \{x \in \Omega, \ x \cdot e - \xi_t = 0\} \text{ and } \Omega_t^\pm = \{x \in \Omega, \ \pm(x \cdot e - \xi_t) < 0\},$$

then u is a pulsating front. That is

$$u\left(t + \frac{\gamma \, k \cdot e}{c}, x\right) = u(t, x - k) \text{ for all } (t,x) \in \mathbb{R} \times \overline{\Omega} \text{ and } k \in L_1\mathbb{Z} \times \cdots \times L_N\mathbb{Z},$$

where $\gamma = \gamma(e) \geq 1$ is given in (2.8). Furthermore, u is unique up to shifts in t.

Remember that $\gamma(e)$ measures the asymptotic ratio of the geodesic distance and the Euclidean distance in the direction e, and $\gamma(e)$ is then automatically larger than or equal to 1. The speed $c/\gamma(e)$ is the "Euclidean" speed in the direction e, as if there were no obstacles, whereas c is the intrinsic geodesic speed which takes into account the geometry of the domain. Notice that $\gamma(e) = 1$ if $\Omega = \mathbb{R}^N$, or if Ω is invariant in the direction e.

Theorem 3.2 says that, under the above assumptions, our general definitions do not introduce new objects in the periodic framework : almost planar travelling fronts reduce to pulsating travelling fronts in the sense of Section 1.4.

3.3. Invariance in a moving frame. In Section 1.2, we mentioned several explicit examples of usual travelling fronts which are invariant in their direction of propagation. We gave in the previous subsections some conditions under which almost planar fronts are truly planar or pulsating in homogeneous or periodic frameworks. We here give a general characterization of fronts which are invariant in their moving frame, without assuming any periodicity in the medium.

We assume here that Ω is invariant in a direction $e \in \mathbb{S}^{N-1}$, that u is a generalized transition wave between p^- and p^+ for equation (2.1), that u and $p^\pm$ are globally bounded, that A, q, μ and $p^\pm$ depend only on the variables x' which are orthogonal to e, that $f = f(x', u)$ and that (3.8) and (3.9) hold.

PROPOSITION 3.3. [**4**] *If there exist $e \in \mathbb{S}^{N-1}$, $c \geq 0$ and a map $\mathbb{R} \ni t \mapsto \xi_t$ satisfying (3.10) and (3.11), then there exists $\varepsilon \in \{-1, 1\}$ such that*

$$u(t, x) = \phi(x \cdot e - c\varepsilon t, x')$$

for some function ϕ. Moreover, ϕ is decreasing in its first variable. If one further assumes that $c = 0$, then the conclusion holds good even if f and $p^{\pm}$ also depend on $x \cdot e$, provided that they are nonincreasing in $x \cdot e$. In particular, if u is quasi-stationary in the sense of Definition 2.2, then u is stationary.

As a consequence of Theorem 3.2 and Proposition 3.3, it follows that, if $\Omega = \mathbb{R}^N$ and if A, q, f, $p^{\pm}$ are independent of t and x, then u is a truly planar travelling front, that is :

$$u(t, x) = \phi(x \cdot e - ct),$$

where $\phi : \mathbb{R} \to (p^-, p^+)$ is decreasing and $\phi(\mp\infty) = p^{\pm}$. This result corresponds an immediate generalization of Theorem 3.1.

3.4. Invariance in the cross-directions. In the previous subsections, we gave some conditions under which the fronts reduce to planar, pulsating or usual travelling fronts. The fronts were assumed to have a global mean speed. The following result is concerned with the case of almost planar fronts which may not have any global mean speed and which may not be invasion fronts. It gives some conditions under which almost planar fronts actually reduce to one-dimensional fronts.

We assume here that $\Omega = \mathbb{R}^N$, that u is a generalized transition wave between p^- and p^+ for equation (2.1), that u and $p^{\pm}$ are globally bounded, that A and q depend only on t, that the limiting states $p^{\pm}$ depend only on t and $x \cdot e$ and are nonincreasing in $x \cdot e$, that $f = f(t, x \cdot e, u)$ is nonincreasing in $x \cdot e$, and that $\inf (p^+ - p^-) > 0$. Assume also that there is $\delta > 0$ such that, for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$,

$$s \mapsto f(t, x \cdot e, s) \text{ is nonincreasing in } (-\infty, p^-(t, x \cdot e) + \delta] \text{ and } [p^+(t, x \cdot e) - \delta, +\infty).$$

THEOREM 3.4. [**4**] *If u is almost planar in the direction $e \in \mathbb{S}^{N-1}$ with some sets Γ_t and $\Omega_t^{\pm}$ satisfying (3.11) and such that*

$$\forall \, \sigma \in \mathbb{R}, \quad \sup \{|\xi_{t+\sigma} - \xi_t|; \ t \in \mathbb{R}\} < +\infty,$$

then u is planar, that is u only depends on t and $x \cdot e$:

$$u(t, x) = \phi(t, x \cdot e)$$

for some function $\phi : \mathbb{R}^2 \to \mathbb{R}$. Furthermore,

$$(3.12) \qquad \forall \, (t, x) \in \mathbb{R} \times \mathbb{R}^N, \quad p^-(t, x \cdot e) < u(t, x) < p^+(t, x \cdot e)$$

and u is decreasing in $x \cdot e$.

Notice that the assumption that $\sup \{|\xi_{t+\sigma} - \xi_t|; \ t \in \mathbb{R}\} < +\infty$ for every $\sigma \in \mathbb{R}$ is clearly stronger than the property (2.3). But the map $t \mapsto \xi_t$ is not needed to be monotone and u may not be an invasion front.

Actually, if the inequalities (3.12) are assumed to hold a priori and if f is assumed to be nonincreasing in s for s in $[p^-(t, x \cdot e), p^-(t, x \cdot e) + \delta]$ and $[p^+(t, x \cdot e) - \delta, p^+(t, x \cdot e)]$ only, instead of $(-\infty, p^-(t, x \cdot e) + \delta]$ and $[p^+(t, x \cdot e) - \delta, +\infty)$, then the strict monotonicity of u in the variable $x \cdot e$ holds good.

As a consequence of Proposition 3.3 (with $c = 0$), the following property holds : in Theorem 3.4, if one further assumes that the function $t \mapsto \xi_t$ is bounded and

that A, q, f and $p^{\pm}$ do not depend on t, then u depends on $x \cdot e$ only, that is u is a stationary one-dimensional front. Roughly speaking, this means that any quasi-stationary front is truly stationary. This last result also corresponds to a generalization of Theorem 3.1 with $c = 0$.

3.5. Monostable transition waves which are trapped between two fronts. Subsections 3.1 to 3.4 were concerned with "bistable-type" transition waves, in the sense that the reaction term f was assumed to be nonincreasing in some neighbourhoods of the limiting states $p^{\pm}(t, x)$. Here, we give a clasification result for monostable fronts which are trapped between two given planar fronts. Namely, we assume that the function $f : [0, 1] \to \mathbb{R}$ is of class C^1 and that

$$(3.13) \qquad f(0) = f(1) = 0, \ f > 0 \text{ on } (0, 1), \ f'(0) > 0, \ f'(1) < 0.$$

It is known that the equation

$$(3.14) \qquad u_t = \Delta u + f(u), \ x \in \mathbb{R}^N$$

admits planar travelling fronts of the type $u(t, x) = \varphi_c(x \cdot e - ct)$, such that $\varphi_c : \mathbb{R} \to (0, 1)$ with $\varphi_c(-\infty) = 1$ and $\varphi_c(+\infty) = 0$, for all $e \in \mathbb{S}^{N-1}$ and for all $c \geq c^*$, where the minimal speed c^* is positive and does not depend on e (it is known that $c^* \geq 2\sqrt{f'(0)}$). Therefore, for a prescribed direction e, we cannot expect any uniqueness up to shifts. However, uniqueness (up to shifts) still holds for the transition waves which are trapped between two shifts of the same planar front.

THEOREM 3.5. *Assume that f satisfies (3.13). Let u be a bounded almost planar transition wave solving (3.14), between $p^- = 0$ and $p^+ = 1$, and satisfying*

$$(3.15) \qquad \forall \, (t, x) \in \mathbb{R} \times \mathbb{R}^N, \ \varphi_c(x \cdot e - ct) \leq u(t, x) \leq \varphi_c(x \cdot e - ct - a),$$

for some $c \geq c^*$, $e \in \mathbb{S}^{N-1}$ and $a \geq 0$, where $\varphi_c : \mathbb{R} \to (0, 1)$ solves $\varphi_c'' + \varphi_c' + f(\varphi_c) = 0$ in $\mathbb{R}$ with $\varphi_c(-\infty) = 1$ and $\varphi_c(+\infty) = 0$. Then there exists $b \in [0, a]$ such that

$$\forall \, (t, x) \in \mathbb{R} \times \mathbb{R}^N, \ u(t, x) = \varphi_c(x \cdot e - ct - b).$$

PROOF. As in the proof of Theorem 3.1, one can assume that $e = (1, 0, \ldots, 0)$. Call

$$v(t, x) = u(t, x + cte) = u(t, x_1 + ct, x')$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$. The function v solves

$$(3.16) \qquad v_t = \Delta v + ce \cdot \nabla v + f(v)$$

and

$$(3.17) \qquad \varphi_c(x_1) \leq v(t, x) \leq \varphi_c(x_1 - a) \text{ for all } (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

Our goal is to prove that v is a shift of the planar front $\varphi_c(x_1)$.

First, remember that φ_c is decreasing in $\mathbb{R}$. It is also well-known that

$$(3.18) \qquad \begin{cases} \varphi_c(s) \sim \alpha e^{-\lambda_c s} & \text{as } s \to -\infty \quad \text{if } c > c^*, \\ \varphi_c(s) \sim (\alpha s + \beta) e^{-\lambda_c s} & \text{as } s \to -\infty \quad \text{if } c = c^*, \end{cases}$$

where $\alpha > 0$ and $\lambda_c = (c - \sqrt{c^2 - 4f'(0)})/2$ if $c > c^*$. If $c = c^*$, then $\alpha > 0$, or $\alpha = 0$ and $\beta > 0$; furthermore, $\lambda_{c^*} = (c^* + \sqrt{(c^*)^2 - 4f'(0)})/2$.

Choose $\delta \in (0, 1)$ such that f is decreasing in $[1 - \delta, 1]$, and let $A > 0$ such that $\varphi_c(x_1) \geq 1 - \delta$ for all $x_1 \leq -A$. Call

$$F = \{(t, x) \in \mathbb{R} \times \mathbb{R}^N, \ x_1 > -A\}.$$

Fix any $T \in \mathbb{R}^*$ and $\rho \in \mathbb{R}^{N-1}$. For all $\sigma \in \mathbb{R}$ and $(t, x) \in \mathbb{R} \times \mathbb{R}^N$, denote $w^\sigma(t, x) = v(t + T, x_1 + \sigma, x' + \rho)$. Since φ_c is decreasing, (3.17) yields $w^\sigma \le v$ in $\mathbb{R} \times \mathbb{R}^N$ for all $\sigma \ge a$. Call now

$$\sigma^* = \inf \{\sigma \in \mathbb{R}, \ w^{\sigma'} \le v \text{ in } \mathbb{R} \times \mathbb{R}^N \text{ for all } \sigma' \ge \sigma\}.$$

The real number σ^* is well-defined because $\varphi_c(-\infty) = 1 > 0 = \varphi_c(+\infty)$, and one has

$$w^{\sigma^*} \le v \text{ in } \mathbb{R} \times \mathbb{R}^N.$$

Assume that $\sigma^* > 0$. Notice first that there exists no $\eta_0 > 0$ such that $w^{\sigma^* - \eta} \le v$ in $\overline{F}$ for all $\eta \in [0, \eta_0]$: otherwise, if such a η_0 exists, there would also hold that $w^{\sigma^* - \eta} \le v$ in $\{x_1 \le -A\}$ (as in the proof of Theorem 3.1, using that $v \ge \varphi_c(x_1) \ge 1 - \delta$ for all $x_1 \le -A$), whence $w^{\sigma^* - \eta} \le v$ in $\mathbb{R} \times \mathbb{R}^N$ for all $\eta \in [0, \eta_0]$. This contradicts the minimality of σ^*. Therefore, there exist two sequences $(\sigma_n)_{n \in \mathbb{N}}$ in $(\sigma^* - 1, \sigma^*)$ and $(t_n, x_n)_{n \in \mathbb{N}} = (t_n, x_{1,n}, x_n')_{n \in \mathbb{N}}$ in $\overline{F}$ such that

$$\sigma_n \to \sigma^* \text{ as } n \to +\infty \text{ and } w^{\sigma_n}(t_n, x_n) \ge v(t_n, x_n) \text{ for all } n \in \mathbb{N}.$$

Since $x_{1,n} \ge -A$ for all $n \in \mathbb{N}$, two cases may occur, up to extraction of a subsequence : either $x_{1,n} \to +\infty$, or $x_{1,n} \to x_{1,\infty} \in [-A, +\infty)$ as $n \to +\infty$. Let us first deal with the case when $x_{1,n} \to +\infty$ as $n \to +\infty$. From standard parabolic estimates and Harnack inequality, there are two positive constants C_1 and C_2 such that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
0 &\le v(t_n, x_{1,n}, x_n') - v(t_n + T, x_{1,n} + \sigma^*, x_n' + \rho) \\
&\le w^{\sigma_n}(t_n, x_{1,n}, x_n') - w^{\sigma^*}(t_n, x_{1,n}, x_n') \\
&\le C_1 \times (\sigma^* - \sigma_n) \times \max_{t_n - 1 \le t \le t_n, \ |x - x_n| \le 2} w^{\sigma^*}(t, x) \\
&\le C_1 C_2 \times (\sigma^* - \sigma_n) \times w^{\sigma^*}(t_n + 1, x_{1,n}, x_n') \\
&\le C_1 C_2 \times (\sigma^* - \sigma_n) \times \varphi_c(x_{1,n} - a + \sigma^*).
\end{aligned}$$

Let us first assume here that $T > 0$. Then there also exists a constant $C_3 > 0$ such that $(v - w^{\sigma^*})(t - T, x_1 - \sigma^*, x' - \rho) \le C_3 \times (v - w^{\sigma^*})(t, x_1, x')$ for all $(t, x_1, x') \in \mathbb{R} \times \mathbb{R}^N$. Thus,

$$\begin{aligned}
&v(t_n - kT, x_{1,n} - k\sigma^*, x_n' - k\rho) - v(t_n - (k-1)T, x_{1,n} - (k-1)\sigma^*, x_n' - (k-1)\rho) \\
&\qquad \le C_1 C_2 C_3^k \times (\sigma^* - \sigma_n) \times \varphi_c(x_{1,n} - a + \sigma^*)
\end{aligned}$$

for all $k \in \mathbb{N}$ and $n \in \mathbb{N}$, whence

$$\begin{aligned}
&v(t_n - kT, x_{1,n} - k\sigma^*, x_n' - k\rho) - v(t_n + T, x_{1,n} + \sigma^*, x_n' + \rho) \\
&\qquad \le C_1 C_2 (1 + C_3 + \cdots + C_3^k)(\sigma^* - \sigma_n) \varphi_c(x_{1,n} - a + \sigma^*).
\end{aligned}$$

From (3.17), it follows that

$$(3.19) \quad \varphi_c(x_{1,n} - k\sigma^*) \le \left[1 + C_1 C_2 (1 + C_3 + \cdots + C_3^k)(\sigma^* - \sigma_n)\right] \varphi_c(x_{1,n} - a + \sigma^*)$$

for all k and n in $\mathbb{N}$. Fix now $k \in \mathbb{N}$ such that $-k\sigma^* < -a + \sigma^*$ (this is possible since $\sigma^* > 0$). Because of (3.18), there is $\varepsilon > 0$ such that $\varphi_c(s - k\sigma^*) \ge (1 + \varepsilon)\varphi_c(s - a + \sigma^*)$ for all s large enough. Since $\varphi_c > 0$ and $x_{1,n} \to +\infty$, $\sigma_n \to \sigma^*$ as $n \to +\infty$, the inequalities (3.19) are impossible for n large enough. In the case $T < 0$, similarly, there exist two positive constants C_2' and C_3' such that

$$\begin{aligned}
&\varphi_c(x_{1,n}) - \varphi_c(x_{1,n} - a + k\sigma^*) \\
&\qquad \le v(t_n, x_{1,n}, x_n') - v(t_n + kT, x_{1,n} + k\sigma^*, x' + k\rho) \\
&\qquad \le C_1 C_2' (1 + C_3' + \cdots + (C_3')^{k-1})(\sigma^* - \sigma_n) \varphi_c(x_{1,n})
\end{aligned}$$

for all $n \in \mathbb{N}$ and $k \in \mathbb{N}$, $k \geq 1$. Choose k such that $-a + k\sigma^* > 0$ and, from (3.18), $\varepsilon > 0$ such that $(1 + \varepsilon)\varphi_c(s - a + k\sigma^*) \leq \varphi_c(s)$ for s large enough. Once again, one gets a contradiction as $n \to +\infty$ in the above inequalities.

Therefore, the sequence $(x_{1,n})_{n \in \mathbb{N}}$ has to be bounded. Up to extraction of a subsequence, one can assume that $x_{1,n} \to x_{1,\infty} \in [-A, +\infty)$ as $n \to +\infty$, and that the functions $v_n(t, x) = v(t + t_n, x_1, x' + x'_n)$ converge locally uniformly in $\mathbb{R} \times \mathbb{R}^N$ to a solution v_∞ of (3.3) such that

$$z(t, x) = v_\infty(t, x) - v_\infty(t + T, x_1 + \sigma^*, x' + \rho) \geq 0 \text{ in } \mathbb{R} \times \mathbb{R}^N,$$

with equality at $(0, x_{1,\infty}, 0)$. The strong maximum principle and the uniqueness of the Cauchy problem for (3.3) imply that $z \equiv 0$ in $\mathbb{R} \times \mathbb{R}^N$, that is $v_\infty(t, x) = v_\infty(t + T, x_1 + \sigma^*, x' + \rho)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$. But v_∞ still satisfies (3.17). Since $\sigma^* > 0$ and $\varphi_c(-\infty) = 1 > 0 = \varphi_c(+\infty)$, one has reached a contradiction.

As a conclusion, one has proved that $\sigma^* \leq 0$. Thus,

$$v(t, x) \geq w^\sigma(t, x) = v(t + T, x_1 + \sigma, x' + \rho) \text{ for all } \sigma \geq 0 \text{ and } (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$

Since $T \neq 0$ and $\rho \in \mathbb{R}^{N-1}$ were arbitrary, it follows that v can be written as a nonincreasing function $\phi(x_1)$ which depends on x_1 only. Because of (3.17), $0 = \phi(+\infty) < \phi(s) < \phi(-\infty) = 1$ for all $s \in \mathbb{R}$ and ϕ is decreasing from the strong maximum principle. Furthermore, the function ϕ satisfies $\phi'' + c\phi' + f(\phi) = 0$ in $\mathbb{R}$. Thus, by uniqueness, $\phi = \varphi(\cdot - b)$ for some $b \in \mathbb{R}$. Lastly, $0 \leq b \leq a$ because of (3.17) and φ_c is decreasing. This completes the proof of Theorem 3.5.

REMARK 3.6. 1. It is immediate to see that the conclusion of Theorem 3.5 still holds if, instead of $f > 0$ on $(0, 1)$ and $f'(0) > 0$ in (3.13), it is only assumed that all solutions $\phi : \mathbb{R} \to (0, 1)$ of $\phi'' + c\phi' + f(\phi) = 0$ in $\mathbb{R}$ with $\phi(-\infty) = 1$, $\phi(+\infty) = 0$ are equal to φ_c up to shifts, and that φ_c is decreasing and $\liminf_{s \to +\infty} \varphi_c(s - \tau)/\varphi_c(s) > 1$ for some $\tau > 0$.

2. The assumptions (3.15) imply that

(3.20) $0 < u < 1$ and $u(t, x) \to 1$ (resp. 0) unif. as $x \cdot e - ct \to -\infty$ (resp. $+\infty$),

that is u is an almost planar invasion front. Actually, if f is concave in $[0, 1]$ and if $c > c^*$ ($c^* = 2\sqrt{f'(0)}$ in this case), then the assumptions (3.20) are sufficient to ensure that u is of the type $u(t, x) = \varphi_c(x \cdot e - ct - b)$ for some $b \in \mathbb{R}$ (see [20]). However, this last result is open for general monostable nonlinearities f.

4. Further qualitative properties

We now proceed to further general qualitative properties of the generalized transition waves. Throughout this section, $m = 1$ and u denotes transition wave between p^- and p^+, for equation (2.1). We assume that u and $p^\pm$ are globally bounded in $\mathbb{R} \times \overline{\Omega}$ and that properties (2.3), (2.4), (3.6) and (3.7) are satisfied.

First, the following general monotonicity property holds.

THEOREM 4.1. [4] *Assume that A and q do not depend on t, that f and $p^\pm$ are nondecreasing in t and that there is $\delta > 0$ such that, for all $(t, x) \in \mathbb{R} \times \overline{\Omega}$,*

$$s \mapsto f(t, x, s) \text{ is nonincreasing in } (-\infty, p^-(t, x) + \delta] \text{ and } [p^+(t, x) - \delta, +\infty).$$

If u is an invasion of p^- by p^+ with $\inf_{\mathbb{R} \times \overline{\Omega}} (p^+ - p^-) > 0$, then

(4.1) $$\forall (t, x) \in \mathbb{R} \times \overline{\Omega}, \quad p^-(t, x) < u(t, x) < p^+(t, x).$$

and u is increasing in time t.

Notice that if (4.1) holds a priori and if f is assumed to be nonincreasing in s for s in $[p^-(t,x), p^-(t,x)+\delta]$ and $[p^+(t,x)-\delta, p^+(t,x)]$ only, instead of $(-\infty, p^-(t,x)+\delta]$ and $[p^+(t,x) - \delta, +\infty)$, then the strict monotonicity of u in t holds good.

Actually, Theorem 4.1 plays a crucial role in the uniqueness results of Sections 3.2 to 3.4. It says that the "bistable-type" invasion fronts are monotone in time. In the case of almost planar fronts, one can be more precise, that is one can compare any two fronts up to shifts in time.

THEOREM 4.2. [**4**] *Under the same conditions as in Theorem 4.1, assume furthermore that f and $p^\pm$ are independent of t, and that there exist $e \in \mathbb{S}^{N-1}$, $c \geq 0$ and a map $\mathbb{R} \ni t \mapsto \xi_t$ such that (3.10) and (3.11) are satisfied. Let $\widetilde{u}$ be another globally bounded invasion front of p^- by p^+ for equation (2.1) with the boundary condition (3.6), associated with*

$$\widetilde{\Gamma}_t = \{x \in \Omega, \ x \cdot e - \widetilde{\xi}_t = 0\} \ and \ \widetilde{\Omega}_t^\pm = \{x \in \Omega, \ \pm(x \cdot e - \widetilde{\xi}_t) < 0\}$$

and having global mean speed $\widetilde{c} \geq 0$ such that

$$\sup \ \{ \ | \ d_\Omega(\widetilde{\Gamma}_t, \widetilde{\Gamma}_s) - \widetilde{c}|t-s| \ |; \ (t,s) \in \mathbb{R}^2\} < +\infty.$$

Then $c = \widetilde{c}$ and there is (the smallest) $T \in \mathbb{R}$ such that

$$\widetilde{u}(t+T, x) \geq u(t,x) \ for \ all \ (t,x) \in \mathbb{R} \times \overline{\Omega}.$$

Furthermore, there exists a sequence $(t_n, x_n)_{n \in \mathbb{N}}$ in $\mathbb{R} \times \overline{\Omega}$ such that

$$(d_\Omega(x_n, \Gamma_{t_n}))_{n \in \mathbb{N}} \ is \ bounded \ and \ \widetilde{u}(t_n + T, x_n) - u(t_n, x_n) \to 0 \ as \ n \to +\infty.$$

Lastly, either $\widetilde{u}(t+T, x) > u(t,x)$ for all $(t,x) \in \mathbb{R} \times \overline{\Omega}$ or $\widetilde{u}(t+T, x) = u(t,x)$ for all $(t,x) \in \mathbb{R} \times \overline{\Omega}$.

This result is related to a uniqueness property. In many instances, uniqueness holds up to shifts but it is an open question to know under which general condition uniqueness holds. The proofs of Theorems 4.1 and 4.2 are rather lenghty and technical. We refer to [**4**] for more details.

5. Planar fronts which have no global mean speed

In Section 2, new general notions of transition waves were given. The examples in this section and in the following ones show how the new definitions include wave solutions which are not covered by the usual notions.

In this section, travelling fronts which have no global mean speed are constructed. This situation may happen even for very simple reaction-diffusion models. Namely, we consider here the homogeneous one-dimensional reaction-diffusion equation

$$(5.1) \qquad\qquad u_t = u_{xx} + f(u), \ x \in \mathbb{R},$$

where the function $f : [0,1] \to \mathbb{R}$ is assumed to be of class C^2 and

$$(5.2) \qquad f(0) = f(1) = 0, \ 0 < f(s) \leq f'(0)s \ \text{for all} \ s \in (0,1) \ \text{and} \ f \ \text{is concave.}$$

We recall that (5.1) admits *usual* travelling fronts solutions $\varphi_c(x - ct)$, for each speed $c \geq 2\sqrt{f'(0)}$, such that $0 < \varphi_c < 1$ in $\mathbb{R}$ and $\varphi_c(-\infty) = 1$, $\varphi_c(+\infty) = 0$. Furthermore, each function φ_c is decreasing and unique up to shifts.

PROPOSITION 5.1. *Under assumption* (5.2), *equation* (5.1) *admits invasion fronts connecting* 1 *and* 0, *which have no global mean speed. More precisely, given* $2\sqrt{f'(0)} \le c_- < c_+$, *equation* (5.1) *admits generalized transition waves u such that* $(p^-, p^+) = (1, 0)$, $\Omega_t^- = (-\infty, x_t)$, $\Omega_t^+ = (x_t, +\infty)$, $u(t, \cdot)$ *is decreasing in* $\mathbb{R}$ *for each* $t \in \mathbb{R}$ *and* $x_t/t \to c_\pm$ *as* $t \to \pm\infty$, *where, for each* $t \in \mathbb{R}$, x_t *is the unique real number such that, say,* $u(t, x_t) = 1/2$.

PROOF. For each $n \in \mathbb{N}$, call u_n the solution of the Cauchy problem

$$(5.3) \qquad \begin{cases} (u_n)_t &= (u_n)_{xx} + f(u_n), \ t > -n, \ x \in \mathbb{R}, \\ u_n(-n, x) &= u_{n,0}(x) := \max\left(\varphi_{c_-}(x + c_- n), \varphi_{c_+}(x + c_+ n)\right). \end{cases}$$

From the maximum principle, it follows that

$$\max\left(\varphi_{c_-}(x - c_- t), \varphi_{c_+}(x - c_+ t)\right) \le u_n(t, x) < 1$$

for all $n \in \mathbb{N}$, $t \ge -n$ and $x \in \mathbb{R}$. These estimates imply that $u_n(-m, \cdot) \ge u_m(-m, \cdot)$ in $\mathbb{R}$ as soon as $n \ge m$. Thus, the maximum principle implies that each sequence $(u_n(t, x))_{n \ge |t|}$ is nondecreasing. On the other hand, since $u_{n,0}$ is a subsolution of the associated elliptic equation, one gets that $u_n(t, x)$ is nondecreasing in $t \ (\ge -n)$ for each n and x.

Furthermore, since $u_{n,0}$ is decreasing in x, so is each function $u_n(t, \cdot)$ for $t \ge -n$. Lastly, since $f(a + b) \le f(a) + f(b)$ for all $(a, b) \in [0, 1] \times [0, 1]$ with $a + b \le 1$ by (5.2), the function

$$\min\left(\varphi_{c_-}(x - c_- t) + \varphi_{c_+}(x - c_+ t), 1\right)$$

is a super-solution of (5.3), whence

$$u_n(t, x) \le \min\left(\varphi_{c_-}(x - c_- t) + \varphi_{c_+}(x - c_+ t), 1\right)$$

for each $n \in \mathbb{N}$, $t \ge -n$ and $x \in \mathbb{R}$.

From standard parabolic estimates, one concludes that the functions u_n concerge locally uniformly in (t, x) to a classical solution u of (5.1) such that

$$(5.4) \qquad \begin{aligned} 0 < \max &\left(\varphi_{c_-}(x - c_- t), \varphi_{c_+}(x - c_+ t)\right) \\ &\le u(t, x) \le \min\left(\varphi_{c_-}(x - c_- t) + \varphi_{c_+}(x - c_+ t), 1\right) \end{aligned}$$

for all $(t, x) \in \mathbb{R}^2$. Thus, $u(t, -\infty) = 1$ and $u(t, +\infty) = 0$ for each $t \in \mathbb{R}$ and each function $u(t, \cdot)$ is decreasing in $\mathbb{R}$ from the strong maximum principle applied to u_x. By continuity, there exists then a unique real number $x_t \in \mathbb{R}$ such that $u(t, x_t) = 1/2$, for each $t \in \mathbb{R}$. One also gets immediately by passing to the limit that u is nondecreasing in t, and actually u is increasing in t from the strong maximum principle applied to u_t.

Let us now check that u satisfies all the conclusions of Proposition 5.1. Notice first that the map $t \mapsto x_t$ is continuous and increasing. Moreover, since $c_- < c_+$, the estimates (5.4) imply immediately that

$$(5.5) \qquad \begin{cases} u(t, x) - \varphi_{c_-}(x - c_- t) &\to \quad 0 \text{ as } t \to -\infty, \text{ uniformly in } x \in \mathbb{R}, \\ u(t, x) - \varphi_{c_+}(x - c_+ t) &\to \quad 0 \text{ as } t \to +\infty, \text{ uniformly in } x \in \mathbb{R}. \end{cases}$$

Therefore, $x_t/t \to c_\pm$ as $t \to \pm\infty$. One can even say that $\limsup_{t \to \pm\infty} |x_t - c_\pm t| < +\infty$. It also follows that

$$u(t, x) \to 1 \text{ uniformly as } x - x_t \to -\infty$$

and

$$u(t, x) \to 0 \text{ uniformly as } x - x_t \to +\infty.$$

As a conclusion, the function u is a generalized transition wave for (5.1), and it is an invasion front of 0 by 1 which has no global mean speed. This completes the proof of Proposition 5.1.

REMARK 5.2. The idea of the proof of Proposition 5.1 is inspired from [20]. Actually, given $2\sqrt{f'(0)} \leq c_- < c_+$, there exists an infinite-dimensional manifold of solutions satisfying (5.5) and the conclusions of Proposition 5.1.

With the same scheme as in the proof of Proposition 5.1, we can also prove the following

PROPOSITION 5.3. *Let $c_- \geq 2\sqrt{f'(0)}$ be given, and let $\zeta(t) : \mathbb{R} \to (0,1)$ be a solution of $\dot{\zeta}(t) = f(\zeta(t))$ for all $t \in \mathbb{R}$. Under assumption (5.2), equation (5.1) admits invasion fronts for which $(p^-, p^+) = (1, \zeta(t))$, $\Omega_t^- = (-\infty, x_t)$, $\Omega_t^+ = (x_t, +\infty)$, $u(t, \cdot)$ is decreasing in $\mathbb{R}$ for each $t \in \mathbb{R}$ and $x_t/t \to c_-$ as $t \to -\infty$ and $x_t/t \to +\infty$ as $t \to +\infty$.*

PROOF. Notice first that the function ζ is increasing, $\zeta(-\infty) = 0$, $\zeta(+\infty) = 1$ and there exists $h \in \mathbb{R}$ such that $\zeta(t) \sim e^{f'(0)(t+h)}$ as $t \to -\infty$. For each $n \in \mathbb{N}$, call u_n the solution of the Cauchy problem

$$\begin{cases} (u_n)_t & = & (u_n)_{xx} + f(u_n), \ t > -n, \ x \in \mathbb{R}, \\ u_n(-n, x) & = & u_{n,0}(x) := \max\left(\varphi_{c_-}(x + c_- n), \zeta(-n)\right). \end{cases}$$

As in Proposition 5.1, the maximum principle and standard parabolic estimates imply that the functions u_n converge locally uniformly in $(t, x) \in \mathbb{R}^2$ to a classical solution u of (5.1) such that

$$0 < \max\left(\varphi_{c_-}(x - c_- t), \zeta(t)\right) \leq u(t, x) \leq \min\left(\varphi_{c_-}(x - c_- t) + \zeta(t), 1\right)$$

for all $(t, x) \in \mathbb{R}^2$, and u is decreasing in x and increasing in t. The above estimates imply that

$$\begin{cases} u(t, x) - \varphi_{c_-}(x - c_- t) & \to & 0 \text{ as } t \to -\infty, \text{ uniformly in } x \in \mathbb{R}, \\ u(t, x) & \to & 1 \text{ as } t \to +\infty, \text{ uniformly in } x \in \mathbb{R}. \end{cases}$$

Furthermore, $u(t, -\infty) = 1$ and $u(t, +\infty) = \zeta(t)$ for each $t \in \mathbb{R}$.

Let $t_0 \in \mathbb{R}$ be such that $\zeta(t) \leq 1/4$ for all $t \leq t_0$. Therefore, for each $t \leq t_0$, there exists a unique $x_t \in \mathbb{R}$ such that $u(t, x_t) = 1/2$, and the map $(-\infty, t_0] \ni t \mapsto x_t$ is continuous, increasing, and $x_t/t \to c_-$ as $t \to -\infty$. Extend the map $t \mapsto x_t$ in $\mathbb{R}$ such that it is continuous and increasing in $\mathbb{R}$, and $x_t/t \to +\infty$ as $t \to +\infty$. The above observations yield $u(t, x) \to 1$ uniformly as $x - x_t \to -\infty$, and $u(t, x) - \zeta(t) \to 0$ uniformly as $x - x_t \to +\infty$. Define $\Omega_t^- = (-\infty, x_t)$ and $\Omega_t^+ = (x_t, +\infty)$ for each $t \in \mathbb{R}$. The function u is an invasion front which satisfies all conclusions of Proposition 5.3.

The conclusion of this section is that, even for the simple homogeneous one-dimensional equation (5.1), there are generalized transition waves which are not covered by the usual definitions. For instance, because of (5.5), the solutions which are constructed in Proposition 5.1 are not invariant or periodic in time in any moving frame. Neither can these solutions be written in the form (1.8). Roughly speaking, we could say that the instantaneous speed of these solutions is increasing in time.

6. Multidimensional invasion fronts whose directions of propagation change in time

In this section, we construct special solutions of the equation

$$(6.1) \qquad u_t = \Delta u + f(u), \ x \in \mathbb{R}^N,$$

in dimension $N \geq 2$, under the assumption (5.2). In Section 1.2, we recalled the existence of travelling fronts for various types of nonlinearities f. The level sets of the fronts were not planar in general, but the profile of each solution was invariant in time in a certain frame moving with constant speed in the direction of propagation. The aim of this section is to show that, roughly speaking, there are generalized travelling fronts for which the direction of propagation is not constant in time. For simplicity, we fix $N = 2$, but the same construction would hold in any dimension $N \geq 2$.

PROPOSITION 6.1. *Let $N = 2$ and $2\sqrt{f'(0)} \leq c_1^- < c_1^+$, $2\sqrt{f'(0)} \leq c_2^- < c_2^+$ be such that $c_1^+/c_1^- \neq c_2^+/c_2^-$. Choose any unit vectors ν_1 and ν_2 in $\mathbb{S}^1$ such that $\nu_1 \neq \pm\nu_2$. Call $c^\pm > 2\sqrt{f'(0)}$ and $\nu^\pm \in \mathbb{S}^1$ the speeds and unit vectors such that $c^\pm\nu^\pm \cdot \nu_i = c_i^\pm$ for $i = 1, 2$. Then $\nu^- \neq \pm\nu^+$ and equation (6.1) admits invasion fronts u connecting 1 and 0 (0 is invaded by 1) such that*

$$u(t, x + c^\pm\nu^\pm t) \to U^\pm(x) \text{ as } t \to \pm\infty, \text{ uniformly in } x \in \mathbb{R}^2,$$

where $0 < U^\pm < 1$ solve $\Delta U^\pm + c^\pm\nu^\pm \cdot \nabla U^\pm + f(U^\pm) = 0$ in $\mathbb{R}^2$ and $U^\pm$ are not equal up to shifts and rotations.

PROOF. Notice first that the existence and uniqueness of $c^\pm$ and $\nu^\pm$ are immediate since $\nu_1 \neq \pm\nu_2$. Furthermore, $\nu^- \cdot \nu^+ > 0$ and $\nu^- \neq \nu^+$ since $c_1^+/c_1^- \neq c_2^+/c_2^-$. Call

$$u_i^\pm(t, x) = \varphi_{c_i^\pm}(x \cdot \nu_i - c_i^\pm t)$$

for $i = 1, 2$, where $\varphi_{c_i^\pm} : \mathbb{R} \to (0, 1)$ denotes any solution of $\varphi''_{c_i^\pm} + c_i^\pm\varphi'_{c_i^\pm} + f(\varphi_{c_i^\pm}) = 0$ in $\mathbb{R}$ with $\varphi_{c_i^\pm}(-\infty) = 1$ and $\varphi_{c_i^\pm}(+\infty) = 0$. Each function $u_i^\pm$ is a solution of (6.1).

For each $\sigma \in \mathbb{R}$, call $u_\sigma^\pm$ and u_σ the solutions of the Cauchy problem associated to (6.1) for $t > -\sigma$ with initial data at time $-\sigma$ defined by

$$\begin{cases} u_\sigma^-(-\sigma, x) = u_{\sigma,0}^-(x) = \max\left(u_1^-(-\sigma, x), u_2^-(-\sigma, x)\right) \\ u_\sigma^+(-\sigma, x) = u_{\sigma,0}^+(x) = \max\left(u_1^+(-\sigma, x), u_2^+(-\sigma, x)\right) \\ u_\sigma(-\sigma, x) = u_{\sigma,0}(x) = \max\left(u_1^-(-\sigma, x), u_2^-(-\sigma, x), u_1^+(-\sigma, x), u_2^+(-\sigma, x)\right). \end{cases}$$

As in the proof of Proposition 5.1, the functions $u_\sigma^\pm$ and u_σ are nondecreasing in σ and they converge, as $\sigma \to +\infty$, locally uniformly in $(t, x) \in \mathbb{R} \times \mathbb{R}^2$ to three solutions $u^\pm$ and u of (6.1) such that

$$(6.2) \quad \begin{cases} \max\left(u_1^-(t, x), u_2^-(t, x)\right) \leq u^-(t, x) \leq \min\left(u_1^-(t, x) + u_2^-(t, x), 1\right) \\ \max\left(u_1^+(t, x), u_2^+(t, x)\right) \leq u^+(t, x) \leq \min\left(u_1^+(t, x) + u_2^+(t, x), 1\right) \\ \max\left(u^-(t, x), u^+(t, x)\right) \leq u(t, x) \leq \min\left(u^-(t, x) + u^+(t, x), 1\right) \end{cases}$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^2$. One also has $0 < u^\pm(t, x) < 1$ and $0 < u(t, x) < 1$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^2$ (the strict upper inequalities follow from the strong maximum principle).

On the other hand, for every $t_0 \in \mathbb{R}$ and $\sigma \in \mathbb{R}$, the function $(t,x) \mapsto v(t,x) = u_\sigma^-(t + t_0, x + c^-\nu^- t_0)$ is a solution of (6.1) with initial datum at time $-(t_0 + \sigma)$ given by

$$
\begin{aligned}
v(-(t_0 + \sigma), x) &= u_\sigma^-(-\sigma, x + c^-\nu^- t_0) \\
&= \max\left(\varphi_{c_1^-}(x \cdot \nu_1 + c^-\nu^- \cdot \nu_1 t_0 + c_1^- \sigma), \varphi_{c_2^-}(x \cdot \nu_2 + c^-\nu^- \cdot \nu_2 t_0 + c_2^- \sigma)\right) \\
&= \max\left(\varphi_{c_1^-}(x \cdot \nu_1 + c_1^-(t_0 + \sigma)), \varphi_{c_2^-}(x \cdot \nu_2 + c_2^-(t_0 + \sigma))\right) \\
&= u_{t_0+\sigma}^-(-(t_0 + \sigma), x)
\end{aligned}
$$

due to the definitions of c^- and ν^-. Thus, $u_\sigma^-(t + t_0, x + c^-\nu^- t_0) = u_{t_0+\sigma}^-(t,x)$ for all $t \geq -(t_0 + \sigma)$ and $x \in \mathbb{R}^2$. For a fixed $t_0 \in \mathbb{R}$, the passage to the limit as $\sigma \to +\infty$ yields

$$u^-(t + t_0, x + c^-\nu^- t_0) = u^-(t,x) \text{ for all } (t,x) \in \mathbb{R} \times \mathbb{R}^2.$$

The same property holds similarly after changing the minus sign into a plus sign. In other words, there exist two functions $U^\pm : \mathbb{R}^2 \to (0,1)$ such that

$$u^\pm(t,x) = U^\pm(x - c^\pm\nu^\pm t) \text{ for all } (t,x) \in \mathbb{R} \times \mathbb{R}^2.$$

The functions $U^\pm$ are classical solutions of

$$\Delta U^\pm + c^\pm\nu^\pm \cdot \nabla U^\pm + f(U^\pm) = 0 \text{ in } \mathbb{R}^2.$$

Furthermore, the inequalities (6.2) imply that

$$0 < \max\left(\varphi_{c_1^\pm}(x \cdot \nu_1), \varphi_{c_2^\pm}(x \cdot \nu_2)\right) \leq U^\pm(x) \leq \min\left(\varphi_{c_1^\pm}(x \cdot \nu_1) + \varphi_{c_2^\pm}(x \cdot \nu_2), 1\right)$$

for all $x \in \mathbb{R}^2$. Call τ_1 and τ_2 the unique unit vectors such that $\tau_1 \perp \nu_1$, $\tau_2 \perp \nu_2$, $\tau_1 \cdot \nu_2 > 0$ and $\tau_2 \cdot \nu_1 > 0$. The above estimates for $U^\pm$ imply that

$$U^\pm(x + r\tau_1) \to \varphi_{c_1^\pm}(x \cdot \nu_1) \text{ and } U^\pm(x + r\tau_2) \to \varphi_{c_2^\pm}(x \cdot \nu_2) \text{ as } r \to +\infty,$$

locally uniformly in $x \in \mathbb{R}^2$. Since $c_1^- < c_1^+$ and $c_2^- < c_2^+$, the limiting profiles of $U^\pm$ is the directions τ_1 and τ_2 are different, and the functions $U^\pm$ are not equal up to shifts and rotations.

For each $t \in (-\infty, 0]$, call

$$
\begin{cases}
\Omega_t^- = \left\{x \in \mathbb{R}^2,\ x \cdot \nu_1 < c_1^- t\right\} \cup \left\{x \in \mathbb{R}^2,\ x \cdot \nu_2 < c_2^- t\right\}, \\
\Omega_t^+ = \left\{x \in \mathbb{R}^2,\ x \cdot \nu_1 > c_1^- t\right\} \cap \left\{x \in \mathbb{R}^2,\ x \cdot \nu_2 > c_2^- t\right\},
\end{cases}
$$

and, for each $t \in [0, +\infty)$, call

$$
\begin{cases}
\Omega_t^- = \left\{x \in \mathbb{R}^2,\ x \cdot \nu_1 < c_1^+ t\right\} \cup \left\{x \in \mathbb{R}^2,\ x \cdot \nu_2 < c_2^+ t\right\}, \\
\Omega_t^+ = \left\{x \in \mathbb{R}^2,\ x \cdot \nu_1 > c_1^+ t\right\} \cap \left\{x \in \mathbb{R}^2,\ x \cdot \nu_2 > c_2^+ t\right\}.
\end{cases}
$$

With these choices of $\Omega_t^\pm$, it follows immediately from (6.2) that u is a generalized transition wave between 1 and 0, and that 1 invades 0.

Lastly, it is straightforward to check from the estimates (6.2) that $u(t,x) - u^-(t,x) \to 0$ as $t \to -\infty$, and $u(t,x) - u^+(t,x) \to 0$ as $t \to +\infty$, uniformly in $x \in \mathbb{R}^2$. In other words,

$$u(t, x + c^\pm\nu^\pm t) \to U^\pm(x) \text{ as } t \to \pm\infty, \text{ uniformly in } x \in \mathbb{R}^2.$$

This means that the solution u converges uniformly in $\mathbb{R}^2$ to two different profiles $U^\pm$ as $t \to \pm\infty$ in two different moving frames, which propagate with two different speeds $c^\pm$ into two different directions $\nu^\pm$. This completes the proof of Proposition 6.1.

7. Exterior domains and other examples

In this section, we describe another application of the general notions of transition waves. We deal here with propagation around an obstacle. More precisely, we assume that the domain Ω is a connected smooth open subset of $\mathbb{R}^N$ such that

$$\Omega = \mathbb{R}^N \backslash K,$$

where the obstacle K is non-empty and compact. We consider the following questions : given a planar front which is travelling in the direction of the obstable, can it propagate around the obstacle and, if the answer is positive, is its shape perturbed behind the obstacle, and is its profile shifted ?

We give answers to these questions for the reaction-diffusion problem

$$(7.1) \qquad \begin{cases} u_t - \Delta u & = & f(u) & \text{in } \Omega, \\ \nu \cdot \nabla u & = & 0 & \text{on } \partial\Omega, \end{cases}$$

where $\nu = \nu(x)$ denotes the outward unit normal to Ω at a point $x \in \partial\Omega$. We assume that the nonlinearity f is of the bistable type on $[0,1]$, that is f is of class $C^1([0,1])$, $f(0) = f(1) = f(\theta) = 0$, $f'(0) < 0$, $f'(1) < 0$, $f < 0$ on $(0,\theta)$, $f > 0$ on $(\theta, 1)$, where $\theta \in (0,1)$ is given. We also assume that $\int_0^1 f > 0$. It is well-known that equation (7.1) admits a unique planar front profile when $\Omega = \mathbb{R}^N$: there exist a unique speed c and a unique (up to shifts) function $\phi : \mathbb{R} \to (0,1)$ such that $\phi(-\infty) = 1$, $\phi(+\infty) = 0$ and $\phi(x \cdot e - ct)$ solves (7.1) when $\Omega = \mathbb{R}^N$, for any direction $e \in \mathbb{S}^{N-1}$. Furthermore, $c > 0$.

When $\Omega \neq \mathbb{R}^N$, these planar fronts $\phi(x \cdot e - ct)$ do not solve (7.1) anymore, because of the Neumann boundary conditions on $\partial\Omega$. This framework cannot be covered by the usual definitions of travelling waves. However, using our general definitions of transition waves, we can prove that propagation around the obstacle is still possible, under an additional geometrical condition on K.

THEOREM 7.1. [5] *Assume that K is strictly star-shaped. Given any direction $e \in \mathbb{S}^{N-1}$, there exists a solution $u(t,x)$ of (7.1) defined for all $(t,x) \in \mathbb{R} \times \overline{\Omega}$, and such that*

$$u(t,x) - \phi(x \cdot e - ct) \to 0 \ \text{as } t \to \pm\infty, \ \text{uniformly in } x \in \overline{\Omega}$$

and

$$u(t,x) - \phi(x \cdot e - ct) \to 0 \ \text{as } |x| \to +\infty, \ \text{uniformly in } t \in \mathbb{R}.$$

The proof of this theorem can be divided into three main steps, which correspond to the behavior of the front at very negative times, when it reaches the obstacle, and lastly when it recovers its shape at large times after passing the obstacle. Let us give a few words about each main step :

- firstly, the existence of a non trivial time-global solution $u(t,x)$ being close to the front $\phi(x \cdot e - ct)$ when $t \to -\infty$ is obtained as a limit as $n \to +\infty$ of a sequence of Cauchy problems starting at times $-n$;
- secondly, it is proved that $u(t,x) \to 1$ locally in x as $t \to +\infty$. Here, we use the fact that the obstacle K is strictly star-shaped, that is there exists $x_0 \in K$ such that $[x_0, x] \in K$ and $\nu(x) \cdot (x_0 - x) > 0$ for all $x \in \partial K = \partial\Omega$. We prove a useful result of independant interest, which says that any solution $0 \leq U(x) \leq 1$ of the stationary problem associated to (7.1) such that $U(x) \to 1$ as $|x| \to +\infty$, has to be identically equal to 1 if K is strictly star-shaped;

- lastly, we prove that the local deformations of the level sets of the front which are induced by the obstacle become negligible at large times. We use sub- and super-solutions with strong or weak diffusion in the directions which are orthogonal to e.

In equation (7.1), the obstacle K can then be viewed as a local perturbation of the uniform homogeneous medium. Other problems which are similar in nature can also be investigated. For instance, in equation (2.1), some coefficients may be locally perturbed. This is the case for instance in (1.7) when $b(x) - b_\infty$ has a compact support, or $b(x) \to b_\infty$ as $|x| \to +\infty$, for some constant b_∞, with $b(x) \not\equiv b_\infty$. This situation is not almost periodic. These problems are the purpose of current research.

Lastly, we mention that the generalized transition waves are the good tools to describe propagation in more complex geometrical situations, like spirals, curved cylinders with two different unbounded axes...

8. Further extensions

In the previous sections, the waves were defined as spatial transitions between *two* limiting states p^- and p^+. More generally speaking, waves with multiple transitions can be defined as follows:

DEFINITION 8.1. (Multiple transition waves) Let $k \geq 1$ be a given integer and let $p^1, \ldots, p^k$ be k time-global solutions of (2.1). A generalized transition wave between $p^1, \ldots, p^k$ is a time-global classical solution u of (2.1) such that $u \not\equiv p^j$ for all $1 \leq j \leq k$, and there exist k families $(\Omega_t^j)_{t \in \mathbb{R}}$ $(1 \leq j \leq k)$ of open pairwise disjoint nonempty subsets of Ω and a family $(\Gamma_t)_{t \in \mathbb{R}}$ of nonempty subsets of Ω, such that

$$\begin{cases} \forall\, t \in \mathbb{R}, \quad \bigcup_{1 \leq j \leq k} (\partial \Omega_t^j \cap \Omega) = \Gamma_t, \quad \Gamma_t \cup \bigcup_{1 \leq j \leq k} \Omega_t^j = \Omega, \\ \forall\, 1 \leq j \leq k, \quad \sup\{d_\Omega(x, \Gamma_t);\ t \in \mathbb{R},\ x \in \Omega_t^j\} = +\infty \end{cases}$$

and

$$u(t,x) - p^j(t,x) \to 0 \quad \text{uniformly in } t \in \mathbb{R} \text{ and } x \in \overline{\Omega_t^j} \text{ as } d_\Omega(x, \Gamma_t) \to +\infty$$

for all $1 \leq j \leq p$.

For instance, triple or more general multiple transition waves with constant limiting states p^j are known to exist in some reaction-diffusion problems. The above definition also covers the case of multiple wave trains.

Notice that the spatially extended pulses, as defined in Definition 2.5 with $p^- \equiv p^+$, correspond to the special case $k = 1$, $p^1 = p^\pm$ and $\Omega_t^1 = \Omega_t^- \cup \Omega_t^+$ in the above definition. We say that they are extended since, for each time t, the set Γ_t is unbounded in general. The usual notion of localized pulses can now be specified as a particular case of Definition 8.1:

DEFINITION 8.2. (Localized pulses) In the particular case where $k = 1$ and Γ_t is a singleton in Definition 8.1, then we say that u is a localized pulse.

We conclude this section with two important remarks.

REMARK 8.3. In all definitions of this paper, the time interval $\mathbb{R}$ can be replaced by any interval $I \subset \mathbb{R}$. The particular case $I = [0, T)$ with $0 < T \leq +\infty$ is of great

importance to describe the formation of waves and fronts for the solutions of Cauchy problems.

REMARK 8.4. All the general definitions of this paper can be still given for other types of evolution equations which are not of the parabolic type.

9. Open problems

There are natural questions that arise from this new notion of waves in several contexts. In each of these, the types of questions are: existence and uniqueness of fronts, range of front velocities –is there an interval of global mean speeds in KPP-type equations and a unique speed for bistable problems?–, stability of the fronts, etc. We mention here a non-exhaustive list of specific or more general open problems:

- For an equation like (1.7) where b is equal to the sum of a constant b_∞ and a compactly supported function, are there bistable or KPP generalized invasion fronts ? If the answer is positive, what is the possible phase shift which is induced by the local perturbation in the medium ? More general equations with locally perturbed coefficients can also be considered.
- The same questions can be asked when, say in dimension 1, the function b has two different limits when $x \to \pm\infty$? Obviously, all these questions can be extended to general parabolic operators, higher dimensions and more general geometries.
- The study of fronts for equations with time and space-dependent coefficients has just started and many fundamental questions about existence and dynamical properties of transition waves in this context remain open.
- Time-dependent domains can also be considered. The general definitions can indeed be easily extended to this framework.
- In Section 7, we reported on the existence of bistable almost-planar fronts passing an obstacle. Can the geometrical condition on the obstacle be removed ?

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