Bistable travelling waves around an obstacle

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Abstract

We consider travelling waves for a nonlinear diffusion equation with a bistable or multistable nonlinearity. The goal is to study how a planar travelling front interacts with a compact obstacle that is placed in the middle of the space $\mathbb{R}^N$. As a first step we prove the existence and uniqueness of an entire solution that behaves like a planar wave front approaching from infinity and eventually reaching the obstacle. This causes disturbance on the shape of the front, but we show that the solution will gradually recover its planar wave profile and continue to propagate in the same direction, leaving the obstacle behind. Whether the recovery is uniform in space or not is shown to depend on the shape of the obstacle.

Contents

1 Introduction and main results

This work is concerned with semi-linear parabolic problems

$$\begin{cases} u_t = \Delta u + f(u) & \text{for } x \in \Omega = \mathbb{R}^N \setminus K \subset \mathbb{R}^N, \\ \nu \cdot \nabla u = 0 & \text{for } x \in \partial\Omega = \partial K \end{cases} \tag{1.1}$$

set in exterior domains. Here, K is a compact set which is the closure of an open set with smooth boundary and f is a *bistable* or a multistable type nonlinearity. Our goal is to study how a propagating planar wave front interacts with the obstacle K. In order to formulate this as a proper mathematical question, we have to start with showing that such an object really exists. This amounts to constructing *entire*, i.e. time global, solutions of problem (1.1) that are asymptotic to classical planar travelling front solutions for large negative time, when the front is far away from the obstacle. Once this is done, we then study how the front approaches the obstacle, goes past it and eventually recovers the shape of a planar travelling front for large positive time, far behind the obstacle.

The solutions we study here can be viewed as generalized travelling front solutions – or *transition waves* – for reaction-diffusion equations in exterior domains. Exterior domains represent a class of heterogeneous media for which such generalized travelling front solutions have been hitherto unknown. We also show that the actual long time behaviour – locally in space – is influenced by the geometry of the obstacle.

Travelling front solutions for homogeneous equations

$$u_t = \Delta u + f(u) \quad \text{in} \quad \mathbb{R}^N$$

have long been studied since the seminal article of Kolmogorov, Petrovsky and Piskunov [14]. More recently, front propagation in spatially varying environments has been receiving growing attention because of its relevance in various fields of science. Much work has been devoted to extending the classical notions of travelling fronts to those in *inhomogeneous* media where the coefficients of the equation or the underlying domain have a spatial dependence. For instance, in the case of problems of the type (1.1), as soon as Ω is not the whole space (or a straight cylinder) classical travelling fronts do not exist any more. A typical – and so far the best-understood – case is when Ω is periodic in the direction of propagation. There, one can naturally extend the notion of travelling front to that of pulsating (or periodic) travelling front. Definitions and results about this case can be found in [3], [5], [6], [7], [19], [20] and [21]. In particular, we refer to [7] for an exhaustive discussion of the nature such travelling fronts and their extensions.

Beyond the periodic case, one is required to consider further generalizations of the notion of travelling front. Such extensions have been introduced by Matano for the almost periodic as well as recurrent frameworks [17] and by Shen [18] for random one-dimensional media. A fully general notion of travelling front for non homogeneous settings has been recently introduced by Berestycki and Hamel in [4], [5] and [6]. These are called *Transition Waves*. In particular, it is shown that while covering the classical cases, this definition allows one to consider many new situations within a unified framework. One important case, in particular, that fits into this general framework (but not into the previous ones), is that of *local perturbations* of the homogeneous case. In this paper, we construct for the first time generalized transition waves for a problem precisely of this kind. More detailed description of this object will be given in the theorem below. Its characterization as the generalized transition wave as defined in [5] is indicated in the theorem at the end of the section.

Let us now formulate our problem more precisely. As mentioned before, we are considering an exterior domain $\Omega = \mathbb{R}^N \setminus K$, where K is a compact set with smooth boundary. On the boundary $\partial\Omega$ $(= \partial K)$, the Neumann boundary condition is imposed. We assume that f is of class $C^{1,1}([0,1])$ and is such that

$$f(0) = f(1) = 0, \qquad f'(0) < 0,\ f'(1) < 0. \tag{1.2}$$

Furthermore, we suppose that there exists a solution ϕ of

$$\begin{cases} \phi''(z) - c\phi'(z) + f(\phi(z)) = 0 \quad (z \in \mathbb{R}), \\ \quad \phi(-\infty) = 0, \quad \phi(+\infty) = 1, \\ \quad 0 < \phi(z) < 1 \quad (z \in \mathbb{R}), \end{cases} \tag{1.3}$$

with $c > 0$. It follows from (1.2) and (1.3) that ϕ is unique up to shifts, and that $\phi' > 0$ in $\mathbb{R}$. The existence of a solution (c, ϕ) of (1.3) with $c > 0$ implies that

$$\int_s^1 f(\tau)d\tau > 0 \quad \text{for all} \ \ 0 \le s < 1, \tag{1.4}$$

as is easily seen by multiplying the equation (1.3) by ϕ' and integrating it over $[z, +\infty)$. But the condition (1.4) is not sufficient to guarantee the existence of (c, ϕ) in general. However, if f is of the bistable type with positive mass, namely if there exists $\theta \in (0,1)$ such that (1.2) holds and

$$f < 0 \text{ on } (0,\theta), \quad f > 0 \text{ on } (\theta,1), \quad \int_0^1 f(\tau)d\tau > 0, \tag{1.5}$$

then the condition (1.3) with $c > 0$, as well as (1.4), is automatically fulfilled.

Our goal is to prove the existence of an entire solution that behaves like

$$u(x,t) \sim \phi(x_1 + ct) \qquad \text{as } t \to \pm\infty.$$

The results differ in some details depending on the shape of the obstacle K. For some of the results, we consider two types of geometrical shapes as described in the following definitions:

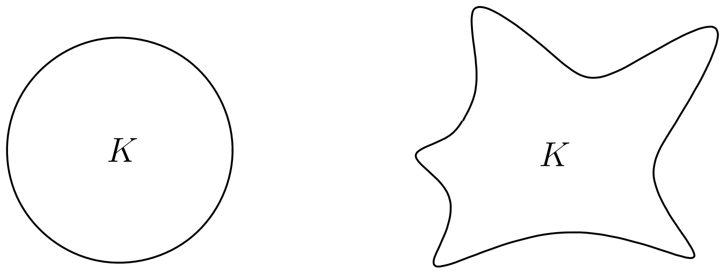

Figure 1: Two examples of star-shaped obstacles

Definition 1.1 By a *star-shaped* obstacle, we mean that either $K = \emptyset$, or there is $x \in \overset{\circ}{K}$ such that, for all $y \in \partial K$ and $t \in [0,1)$, the point $x + t(y-x)$ lies in $\overset{\circ}{K}$ and $\nu_K(y) \cdot (y-x) \geq 0$, where $\nu_K(y)$ denotes the outward unit normal to K at y.

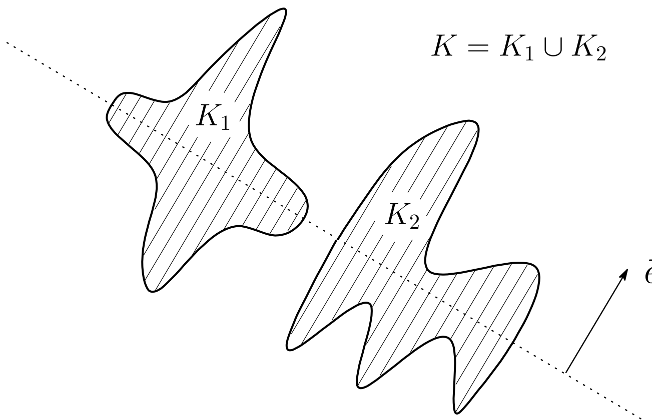

Figure 2: A directionally convex obstacle

Definition 1.2 K is called *directionally convex with respect to a hyperplane* P, if there exists a hyperplane $P = \{x \in \mathbb{R}^N,\ x \cdot e = a\}$, where e is a unit vector and a is some real number, such that

- for every line Σ parallel to e, the set $K \cap \Sigma$ is either a single line segment or empty,
- $K \cap P = \pi(K)$, where $\pi(K)$ is the orthogonal projection of K onto P.

Note that the above condition is slightly more stringent than the usual notion of "directional convexity" because of the second condition $K \cap P = \pi(K)$.

If the obstacle belongs to either one of these classes, we can construct generalized transition waves connecting 1 and 0, as stated in the following theorem:

Theorem 1 *Assume f satisfies (1.2) and that (1.3) holds with $c > 0$. Let the obstacle K be compact and be either star-shaped or directionally convex with respect to some heyperplane P. Then there exists an entire solution $u(t,x)$ of (1.1) in $\Omega = \mathbb{R}^N \setminus K$, such that $0 < u(t,x) < 1$ and $u_t(t,x) > 0$ for all $(t,x) \in \mathbb{R} \times \overline{\Omega}$ and*

$$u(t,x) - \phi(x_1 + ct) \to 0$$

as $t \to \pm\infty$ uniformly in $x \in \overline{\Omega}$, and as $|x| \to +\infty$ uniformly in $t \in \mathbb{R}$.

In this theorem, one has $u(t,x) \to 1$ (resp. 0) uniformly as $x_1 + ct \to +\infty$ (resp. as $x_1 + ct \to -\infty$) (see Theorem 3 below). The solution u is thus a generalized transition front between 1 and 0 with (global mean) speed c in the sense of the definitions given in [4], [5] and [6]. Furthermore, in the terminology of these articles, it is an almost-planar invasion front in the direction $-e_1$. In other words, the level sets of u for a given level value $\lambda \in (0,1)$ stay within a finite distance from the hyperplanes $\{x_1 + ct = 0\}$, uniformly in time (precise statements are given in Theorem 3 below). In particular, $u(t,x)$ converges to 1 as $t \to +\infty$ for each $x \in \Omega$. In the general case when the obstacle may not be star-shaped or directionally convex, the following theorem still establishes the existence of a generalized front, which connects 0 to a stationary solution $u_\infty(x) > 0$, which may now be less than 1.

Theorem 2 (general case) *Assume f satisfies (1.2) and that (1.3) holds with $c > 0$. Let the obstacle K be compact. Then there exists an entire solution $u(t,x)$ of (1.1) in $\Omega = \mathbb{R}^N \setminus K$, such that $0 < u(t,x) < 1$ and $u_t(t,x) > 0$ for all $(t,x) \in \mathbb{R} \times \overline{\Omega}$ and*

$$u(t,x) - \phi(x_1 + ct) \to 0 \quad \text{as} \quad t \to -\infty \quad \text{uniformly in} \quad x \in \overline{\Omega}, \tag{1.6}$$

and as $|x| \to +\infty$ uniformly in $t \in \mathbb{R}$. Furthermore, there exists a classical solution u_∞ of

$$\begin{cases} \Delta u_\infty + f(u_\infty) = 0 & \text{in } \Omega, \\ \nu \cdot \nabla u_\infty = 0 & \text{on } \partial\Omega, \\ 0 < u_\infty(x) \le 1 & \text{for all } x \in \overline{\Omega}, \\ \lim_{|x| \to +\infty} u_\infty(x) = 1 & \end{cases} \tag{1.7}$$

such that

$$u(t,x) - \phi(x_1 + ct) u_\infty(x) \to 0 \quad \text{as} \quad t \to +\infty \quad \text{uniformly in} \quad x \in \overline{\Omega}. \tag{1.8}$$

In particular, $u(t,x) \to u_\infty(x)$ as $t \to +\infty$ locally uniformly in $x \in \overline{\Omega}$.

Remark 1.3 As we will see in Theorem 2.1 below, the entire solution $0 < u(t,x) < 1$ satisfying the condition (1.6) is unique.

Note that if the obstacle K is either star-shaped or directionally convex, then $u_\infty = 1$ (see Section 6), hence Theorem 2 reduces to Theorem 1 in this case.

The following theorem, which easily follows from Theorem 2, states that the solution u is a generalized transition front between u_∞ and 0 in the sense of [5].

Theorem 3 *The solution $u(t,x)$ given in Theorems 2 is a generalized transition almost-planar invasion front between u_∞ and 0 with (global mean) speed c, in the sense that*

$$\sup_{(t,x)\in\mathbb{R}\times\overline{\Omega},\ x_1+ct\geq A} |u(t,x) - u_\infty(x)| \underset{A\to+\infty}{\longrightarrow} 0 \quad \textit{and} \quad \sup_{(t,x)\in\mathbb{R}\times\overline{\Omega},\ x_1+ct\leq -A} u(t,x) \underset{A\to+\infty}{\longrightarrow} 0. \tag{1.9}$$

2 Time before reaching the obstacle

In this section, as a preparation for Theorems 1 and 2, we prove the existence of an entire solution of (1.1) that is monotone increasing in t and converges to the planar wave solution $\phi(x_1 + ct)$ as $t \to -\infty$ uniformly in $x \in \mathbb{R}^N \setminus K$. We also show in the next section that it is uniquely determined from the condition as $t \to -\infty$ - a kind of initial value problem at $-\infty$.

For this result, we assume that f satisfies (1.2) and ϕ satisfies (1.3) with $c > 0$, but we do not need to assume the compactness of K. Instead, we simply assume that K is a closed set with uniformly smooth boundary satisfying

$$K \subset \{\, x \in \mathbb{R}^N \,|\, x_1 \leq 0 \,\}. \tag{2.1}$$

This entire solution satisfies $u(x,t) \approx \phi(x_1 + ct)$ until its wave front comes close to the obstacle K. Then a significant disturbance occurs when the front hits K. What happens afterwards depends on the nature of the obstacle K.

For example, if K is compact, as assumed in the present paper except in this section, then the incidental disturbance caused by the collision will die out eventually, as stated in Theorems 1 and 2. On the other hand, if K is a perforated wall stretching over the region $\{x \in \mathbb{R}^N \,|\, a < x_1 < b\}$, then the effect of collision may remain forever after the front penetrates through the wall. This latter case will be studied in our forthcoming paper [8].

In what follows, we denote by θ_0 the largest positive constant such that

$$f(\tau) \leq 0 \qquad \text{for} \;\; 0 \leq \tau \leq \theta_0. \tag{2.2}$$

By the assumption (1.2), we have $0 < \theta_0 < 1$. We normalize the function ϕ in (1.3) by

$$\phi(0) = \theta_0. \tag{2.3}$$

This condition and (1.3) determines ϕ uniquely.

The main result of this section is the following:

Theorem 2.1 *Let K satisfy* (2.1). *Then there exists an entire solution $u(t,x)$ of* (1.1) *in $\Omega = \mathbb{R}^N \setminus K$, such that $0 < \bar{u}(t,x) < 1$ and $\bar{u}_t(t,x) > 0$ for all $(t,x) \in \mathbb{R} \times \overline{\Omega}$ and that*

$$\bar{u}(t,x) - \phi(x_1 + ct) \to 0 \qquad \textit{as} \;\; t \to -\infty \qquad \textit{uniformly in} \;\; x \in \overline{\Omega}. \tag{2.4}$$

Furthermore, condition (2.4) *determines a unique entire solution of* (1.1).

Uniqueness will be proved in Section 3 below. The rest of the present section is devoted to the proof of the existence part in the above theorem. The general procedure is inspired from the construction of new entire solutions of the KPP equation by Hamel and Nadirashvili [12]. The method involves in a crucial way the construction of suitable super- and subsolutions. For the supersolution, we rely in part on a technique of Guo and Morita [11], in which they used a similar supersolution to study an entire solution having a pair of mutually annihilating fronts.

2.1 Super- and subsolutions before the encounter

The supersolution $w^+(x,t)$ we will use is defined by

$$w^+(x,t) = \begin{cases} \phi(x_1 + ct + \xi(t)) + \phi(-x_1 + ct + \xi(t)) & (x_1 \geq 0) \\ 2\,\phi(ct + \xi(t)) & (x_1 < 0), \end{cases} \tag{2.5}$$

while the subsolution $w^-(x,t)$ is given by

$$w^-(x,t) = \begin{cases} \phi(x_1 + ct - \xi(t)) - \phi(-x_1 + ct - \xi(t)) & (x_1 \geq 0) \\ 0 & (x_1 < 0). \end{cases} \tag{2.6}$$

Here $\xi(t)$ is a solution of the following ordinary differentail equation:

$$\dot{\xi} = M\, e^{\lambda\,(ct+\xi)} \qquad (t < -T), \qquad \xi(-\infty) = 0, \tag{2.7}$$

where M, T are positive constants to be specified later, and λ is the positive root of the equation

$$\lambda^2 - c\lambda + f'(0) = 0.$$

More precisely,

$$\lambda = \frac{1}{2}(c + \sqrt{c^2 - 4f'(0)}\,) \tag{2.8}$$

and

$$\xi(t) = \frac{1}{\lambda}\,\log \frac{1}{1 - c^{-1}M\, e^{\lambda\, c\, t}}\,. \tag{2.9}$$

In order for the above function to be defined, one must have $1 - c^{-1}M\, e^{\lambda\, c\, t} > 0$. We also impose that

$$ct + \xi(t) \leq 0 \qquad \text{for} \quad -\infty < t \leq T.$$

Thus we set

$$T := \frac{1}{\lambda\, c}\,\log \frac{c}{c + M}.$$

The main ingredient in the proof of the existence theorem lies in the property of w^+ and w^- being a super- and a sub-solution which we state next.

Lemma 2.2 *For $M > 0$ sufficiently large, w^+ and w^- are, respectively, a supersolution and a subsolution of* (1.1) *in the time range $-\infty < t \le T_1$ for some $T_1 \in (-\infty, T]$.*

The proof of this lemma will be given in Subsection 2.3.

2.2 Basic estimates

As is shown in [9, 11], it is easily seen that $\phi(z)$ satisfies

$$\begin{aligned} \alpha_0\, e^{\lambda z} \le \phi(z) \le \beta_0\, e^{\lambda z} \qquad & (z \le 0) \\ \alpha_1\, e^{-\mu z} \le 1 - \phi(z) \le \beta_1\, e^{-\mu z} \qquad & (z > 0), \end{aligned} \tag{2.10}$$

where $\alpha_0, \alpha_1, \beta_0, \beta_1$ are some positive constants, λ is as in (2.8) and μ is defined by

$$\mu = \frac{1}{2}(-c + \sqrt{c^2 - 4f'(1)}\,). \tag{2.11}$$

The derivative $\phi'(z)$ satisfies the following estimates for some constants $\gamma_0, \gamma_1, \delta_0, \delta_1 > 0$:

$$\begin{aligned} \gamma_0\, e^{\lambda z} \le \phi'(z) \le \delta_0\, e^{\lambda z} \qquad & (z \le 0), \\ \gamma_1\, e^{-\mu z} \le \phi'(z) \le \delta_1\, e^{-\mu z} \qquad & (z > 0) \end{aligned} \tag{2.12}$$

The following estimate will also be useful later:

$$|f(u+v) - f(u) - f(v)| \le Luv \qquad (0 \le u, v \le 1), \tag{2.13}$$

where L is some constant. (Here the $C^{1,1}$ character of f is used.)

2.3 Proof of Lemma 2.2

First, we observe that since $K \subset \{x_1 \le 0\}$ and both w^+ and w^- have been constructed, so that they are independent of x in the region $\{x_1 \le 0\}$, w^+ and w^- satisfy

$$\nu \cdot \nabla w^+ = \nu \cdot \nabla w^- = 0 \qquad \text{on} \;\; \partial\Omega.$$

Therefore, it is enough to show that $\mathcal{L}\, w^+ \ge 0$ and $\mathcal{L}\, w^- \le 0$ where $\mathcal{L}$ is the operator defined by

$$\mathcal{L}\, w := w_t - \Delta w - f(w).$$

This is carried out in the next computations.

2.3.1 Supersolution

A straightforward computation shows that

$$\mathcal{L}\, w^+ = \begin{cases} 2(c + \dot{\xi})\, \phi'(ct + \xi(t)) - f(2\phi(ct + \xi(t))) & (x_1 < 0), \\ \dot{\xi}(t)\big(\phi'(z_+) + \phi'(z_-)\big) + G(t, x_1) & (x_1 > 0), \end{cases}$$

where $z_+ := x_1 + ct + \xi(t)$, $z_- = -x_1 + ct + \xi(t)$ and

$$G(t, x_1) = f(\phi(z_+)) + f(\phi(z_-)) - f(\phi(z_+) + \phi(z_-)).$$

Using (2.7), this can be rewritten as

$$\mathcal{L}\, w^+ = \begin{cases} 2\big(c + M\, e^{\lambda\,(ct+\xi)}\big)\, \phi'(ct + \xi(t)) - f(2\phi(ct + \xi(t))) & (x_1 < 0), \\ M\, e^{\lambda\,(ct+\xi)} \big(\phi'(z_+) + \phi'(z_-)\big) + G(t, x_1) & (x_1 > 0), \end{cases}$$

Since w^+ is C^2 for $x_1 \neq 0$ and C^1 for $x_1 \in \mathbb{R}$, in order to show that w^+ is a supersolution, it suffices to check that $\mathcal{L}w^+ \geq 0$ both for $x_1 < 0$ and for $x_1 > 0$.

In the range $x_1 < 0$, we clearly have $\mathcal{L}\, w^+ > 0$ so long as $T_1 \in (-\infty, T]$ is chosen sufficiently negative so that

$$\phi(ct + \xi(t)) \leq \frac{\theta_0}{2} \qquad \text{for} \quad -\infty < t \leq T_1, \tag{2.14}$$

where θ_0 is as in (2.2). In the range $0 < x_1 \leq -(ct + \xi(t))$, the inequality (2.13) and the estimates (2.10), (2.12) yield

$$\begin{aligned} \mathcal{L}\, w^+ &\geq M e^{\lambda\,(ct+\xi)} \big(\phi'(z_+) + \phi'(z_-)\big) - L\phi(z_+)\phi(z_-) \\ &\geq M\gamma_0\, e^{\lambda\,(ct+\xi)} e^{\lambda\,(x_1+ct+\xi)} - L\beta_0^2\, e^{\lambda\,(x_1+ct+\xi)} e^{\lambda\,(-x_1+ct+\xi)} \\ &= e^{2\lambda(ct+\xi)} \big(M\gamma_0\, e^{\lambda x_1} - L\beta_0^2\big). \end{aligned}$$

Thus we have $\mathcal{L}\, w^+ > 0$ provided that

$$M\gamma_0 > L\beta_0^2. \tag{2.15}$$

It remains to prove $\mathcal{L}\, w^+ > 0$ in the range $x_1 > -(ct + \xi(t))$ ($\geq 0 \geq ct + \xi(t)$). Observe that

$$\begin{aligned} \mathcal{L}\, w^+ &\geq M e^{\lambda\,(ct+\xi)} \big(\phi'(z_+) + \phi'(z_-)\big) - L\phi(z_+)\phi(z_-) \\ &\geq M\gamma_1\, e^{\lambda\,(ct+\xi)} e^{-\mu\,(x_1+ct+\xi)} - L\beta_0\, e^{\lambda\,(-x_1+ct+\xi)} \\ &\geq e^{\lambda(ct+\xi)} \big(M\gamma_1\, e^{-\mu(x_1+ct+\xi)} - L\beta_0\, e^{-\lambda x_1}\big). \end{aligned} \tag{2.16}$$

Here we consider the case $\lambda \geq \mu$ and the case $\lambda < \mu$ separately. First, in the case $\lambda \geq \mu$, from the above inequalities one gets $\mathcal{L}\, w^+ > 0$ provided that

$$M\gamma_1 > L\beta_0. \tag{2.17}$$

In the case where $\lambda < \mu$, we have

$$m_0 := -f'(0) < -f'(1) =: m_1.$$

In this case we remark that

$$f(u) + f(v) - f(u + v) = (m_1 - m_0)\, v + O\big(v^2\big) + O\big(|v(1 - u)|\big) \tag{2.18}$$

for $u \approx 1$ and $v \approx 0$; hence $G(t,x_1) \geq 0$ if $z_+ \gg 1$ and $z_- \ll -1$. Consequently, there exists a constant $L_1 > 0$ such that (recall that $ct + \xi(t) \leq 0$)

$$\mathcal{L}\, w^+ \geq 0 \qquad \text{if} \;\; x_1 \in [-(ct+\xi(t)) + L_1,\, \infty).$$

Finally, in the range $x_1 \in [-(ct+\xi(t)),\, -(ct+\xi(t)) + L_1]$, we see from (2.16) that

$$\begin{aligned}\mathcal{L}\, w^+ \;&\geq e^{\lambda(ct+\xi)}\big(M\gamma_1\, e^{-\mu(x_1+ct+\xi)} - L\beta_0\, e^{-\lambda x_1}\big)\\ &\geq e^{\lambda(ct+\xi)}\big(M\gamma_1\, e^{-\mu L_1} - L\beta_0\, e^{-\lambda x_1}\big).\end{aligned}$$

Thus we have $\mathcal{L}\, w^+ > 0$ if

$$M\gamma_1\, e^{-\mu L_1} > L\beta_0. \tag{2.19}$$

Combining these, we see that w^+ is a supersolution of (1.1) provided that the constant $M > 0$ and $T_1 \in (-\infty, T]$ are chosen so that (2.15), (2.17), (2.19) and (2.14) hold.

2.3.2 Subsolution

Next we show that w^- is a subsolution. Most of the argument here goes in parallel with the previous argument for w^+, but some details are different.

A straightforward computation and (2.7) show

$$\mathcal{L}\, w^- = \begin{cases} 0 & (x_1 < 0),\\ -M\, e^{\lambda\,(ct+\xi)}\big(\phi'(y_+) - \phi'(y_-)\big) + H(t,x_1) & (x_1 > 0),\end{cases}$$

where $y_+ := x_1 + ct - \xi(t)$, $y_- = -x_1 + ct - \xi(t)$ and

$$H(t,x_1) = f(\phi(y_+)) - f(\phi(y_-)) - f(\phi(y_+) - \phi(y_-)).$$

Note that w^- is C^2 except at $x_1 = 0$ and that w^- has a positive derivative gap at $x_1 = 0$. Therefore, in order to show that w^- is a subsolution, it suffices to check that $\mathcal{L}w^- \leq 0$ both for $x_1 < 0$ and for $x_1 > 0$. Since the range $x_1 < 0$ is trivial, we consider the range $x_1 > 0$.

First, in the range $0 < x_1 \leq -(ct - \xi(t))$, we note that

$$\phi'(y_+) - \phi'(y_-) = \int_{y_-}^{y_+} \phi''(z)dz = \int_{y_-}^{y_+} \big(c\phi'(z) - f(\phi(z))\big)dz.$$

Since $y_- < y_+ < 0$ in this range, we have $\phi(z) < \theta_0/2$ for $z \in [y_-, y_+]$ by virtue of (2.2). This implies $f(\phi(z)) \leq 0$. Hence

$$\phi'(y_+) - \phi'(y_-) \geq c\big(\phi(y_+) - \phi(y_-)\big). \tag{2.20}$$

By (2.13), we have

$$|H(t,x_1)| \leq L\phi(y_-)\big(\phi(y_+) - \phi(y_-)\big).$$

Combining these, we obtain

$$\begin{aligned}\mathcal{L}\, w^- &\le -cM\, e^{\lambda\,(ct+\xi)}\big(\phi(y_+)-\phi(y_-)\big)+L\phi(y_-)\big(\phi(y_+)-\phi(y_-)\big)\\ &\le \big(-cMe^{\lambda(ct+\xi)}+L\phi(y_-)\big)\big(\phi(y_+)-\phi(y_-)\big)\\ &\le \big(-cMe^{\lambda(ct+\xi)}+L\beta_0 e^{\lambda(-x_1+ct-\xi(t))}\big)\big(\phi(y_+)-\phi(y_-)\big)\\ &= e^{\lambda ct}\big(-cMe^{\lambda\xi}+L\beta_0 e^{\lambda(-x_1-\xi(t))}\big)\big(\phi(y_+)-\phi(y_-)\big).\end{aligned}$$

Therefore $\mathcal{L}\, w^- < 0$ in this range provided that

$$cM > L\beta_0. \tag{2.21}$$

Next we prove $\mathcal{L}\, w^- < 0$ in the range $x_1 > -(ct-\xi(t))$ ($\ge 0 \ge ct-\xi(t)$). We consider the case $\lambda \ge \mu$ and the case $\lambda < \mu$ separately. First, in the case $\lambda \ge \mu$, we have

$$\begin{aligned}\mathcal{L}\, w^- &\le -Me^{\lambda\,(ct+\xi)}\big(\phi'(y_+)-\phi'(y_-)\big)+L\phi(y_-)\big(\phi(y_+)-\phi(y_-)\big)\\ &\le -M\, e^{\lambda\,(ct+\xi)}\big(\gamma_1 e^{-\mu\,(x_1+ct-\xi)}-\delta_0 e^{\lambda\,(-x_1+ct-\xi)}\big)+L\beta_0\, e^{\lambda\,(-x_1+ct-\xi)}\\ &= -Me^{\lambda\,(-x_1+ct+\xi)}\big(\gamma_1 e^{-\mu\,(ct-\xi)+(\lambda-\mu)x_1}-\delta_0 e^{\lambda\,(ct-\xi)}-M^{-1}L\beta_0 e^{-2\lambda\xi}\big)\\ &\le -Me^{\lambda\,(-x_1+ct+\xi)}\big(\gamma_1 e^{-\mu\,(ct-\xi)}-\delta_0 e^{\lambda\,(ct-\xi)}-M^{-1}L\beta_0\big).\end{aligned} \tag{2.22}$$

Thus we have $\mathcal{L}\, w^- < 0$ provided that $T_1 \in (-\infty, T]$ is chosen sufficiently negative so that

$$\gamma_1 e^{-\mu\,(ct-\xi)}-\delta_0 e^{\lambda\,(ct-\xi)}-M^{-1}L\beta_0 > 0. \tag{2.23}$$

In the case where $\lambda < \mu$, we again use the estimate (2.18). Then we see that

$$H(t,x_1) = -(m_1-m_0)\phi(y_-)+O\big(\phi^2(y_-)\big)+O\big(\phi(y_-)(1-\phi(y_+))\big) \le -\mu\phi(y_-)$$

for some constant $\mu > 0$, provided that $y_+ \gg 1$ and $y_- \ll -1$. Consequently, there exists a constant $L_2 > 0$ such that

$$H(t,x_1) \le -\mu\beta_0 e^{\lambda(-x_1+ct-\xi)} \qquad \text{if} \;\; x_1 \in [-(ct-\xi(t))+L_2,\, \infty).$$

Hence

$$\mathcal{L}w^- \le M\, e^{\lambda\,(ct+\xi)}\phi'(y_-)+H(t,x_1) \le e^{\lambda(-x_1+ct-\xi)}\big(M\delta_0 e^{\lambda\,(ct+\xi)}-\mu\beta_0\big).$$

It follows that $\mathcal{L}w^- < 0$ provided that $T_1 \in (-\infty, T]$ is chosen sufficiently negative so that

$$M\delta_0 e^{\lambda\,(ct+\xi)} < \mu\beta_0 \qquad \text{for} \quad -\infty < t \le T_1. \tag{2.24}$$

Finally, in the range $x_1 \in [-(ct-\xi(t)),\, -(ct-\xi(t))+L_2]$, we see from the third line in (2.22) that

$$\mathcal{L}\, w^- \;\le -Me^{\lambda\,(-x_1+ct+\xi)}\big(\gamma_1 e^{-\lambda\,(ct-\xi)-(\mu-\lambda)L_2}-\delta_0 e^{\lambda\,(ct-\xi)}-M^{-1}L\beta_0 e^{-2\lambda\xi}\big)$$

Thus we have $\mathcal{L}\, w^- < 0$ in this range provided that $T_1 \in (-\infty, T]$ is chosen sufficiently negative so that

$$\gamma_1 e^{-\lambda\,(ct-\xi)-(\mu-\lambda)L_2}-\delta_0 e^{\lambda\,(ct-\xi)}-M^{-1}L\beta_0 > 0 \qquad \text{for} \quad -\infty < t \le T_1. \tag{2.25}$$

Combining these, we see that w^- is a subsolution of (1.1) provided that the constant $M > 0$ and $T_1 \in (-\infty, T]$ are chosen so that (2.21), (2.23), (2.24) and (2.25) hold.

2.4 Construction of the entire solution

This can be done by constructing a sequence of solutions defined for $-n \le t < \infty$ and letting $n \to \infty$. Let $u_n(x,t)$ be the solution of (1.1) for $t \ge -n$ with initial data

$$u_n(-n,x) = w^-(-n,x).$$

Observe that $w^- \le w^+$. Then, since $w^-(-n,x) = u_n(-n,x) < w^+(-n,x)$, the comparison principle implies

$$w^-(t,x) \le u_n(t,x) \le w^+(t,x) \qquad \text{for } \ t \in [-n, T_1],\ x \in \Omega. \tag{2.26}$$

Setting $t = -(n-1)$ in the above inequality yields

$$u_n(-n+1,x) \ge w^-(-n+1,x) = u_{n-1}(-n+1,x).$$

Applying again the comparison principle, we obtain

$$u_n(t,x) \ge u_{n-1}(t,x) \qquad \text{for } \ t \in [-n+1, T_1],\ x \in \Omega. \tag{2.27}$$

Hence the sequence $u_n(t,x)$ is monotone increasing in n. Letting $n \to \infty$ and using parabolic estimates, we see that this sequence converges to an entire solution defined for $t \in \mathbb{R}$, $x \in \Omega$, which we denote by $\bar{u}(x,t)$. Letting $n \to \infty$ in (2.26) gives

$$w^-(t,x) \le \bar{u}(t,x) \le w^+(t,x) \qquad \text{for } \ t \in (-\infty, T_1],\ x \in \Omega. \tag{2.28}$$

Next we show that

$$\bar{u}_t > 0 \qquad \text{for } \ t \in \mathbb{R},\ x \in \Omega. \tag{2.29}$$

First, we note that $w^-(t,x)$ is monotone increasing in t for t sufficiently negative. In fact,

$$w_t^-(t,x) = (c - \dot{\xi}(t))\big(\phi'(y_+) - \phi'(y_-)\big).$$

Since $\dot{\xi}(t) \to 0$ as $t \to -\infty$, we have $c - \dot{\xi}(t) > 0$ for t sufficiently negative. The monotonicity of ϕ and the inequality $y_+ > y_-$ then imply $w_t^- > 0$. This and (2.26) shows $(u_n)_t(-n,x) > 0$ for all sufficiently large n. Applying the maximum principle to u_t then yields that

$$(u_n)_t(t,x) > 0 \qquad \text{for } \ t \in (-n, +\infty),\ x \in \Omega.$$

Letting $n \to \infty$, we get

$$u_t(t,x) \ge 0 \qquad \text{for } \ t \in \mathbb{R},\ x \in \Omega.$$

Since u_t is clearly not identially equal to 0, the strong maximum principle implies (2.29).

3 Uniqueness of the entire solution

In this section we prove the uniqueness of the entire solution under the condition (2.4). Let us first introduce some notation. Given $\eta \in (0, \frac{1}{2}]$, we define, for each $t \in \mathbb{R}$,

$$\Omega_\eta(t) := \{x \in \Omega \mid \eta \leq \bar{u}(t,x) \leq 1-\eta\}. \tag{3.1}$$

Roughly speaking, $\Omega_\eta(t)$ denotes the region where the 'front' of $\bar{u}$ is located at time t. By the condition (2.4), for any $\eta \in (0, \frac{1}{2}]$ we can find $T_\eta \in \mathbb{R}$ and $M_\eta \geq 0$ such that

$$\Omega_\eta(t) \subset \{x \in \Omega \mid |ct + x_1| \leq M_\eta\} \subset \{x \in \mathbb{R}^N \mid x_1 \geq 1\} \quad \text{for} \quad -\infty < t \leq T_\eta. \tag{3.2}$$

Lemma 3.1 *For any $\eta \in (0, \frac{1}{2}]$, there exists $\delta > 0$ such that*

$$\bar{u}_t(t,x) \geq \delta \quad \textit{for} \;\; t \in (-\infty, T_\eta], \;\; x \in \Omega_\eta(t). \tag{3.3}$$

Proof. Suppose that (3.3) does not hold. Then there exists a sequence $t_k \in (-\infty, T_\eta]$ and $x_k := (x_{k,1}, x_{k,2}, \cdots, x_{k,N}) \in \Omega_\eta(t)$ such that

$$\bar{u}_t(t_k, x_k) \to 0 \quad \text{as} \;\; k \to \infty.$$

Without loss of generality, we may assume that either t_k converges to some $t^* \in (-\infty, T_\eta]$ or $t_k \to -\infty$ as $k \to \infty$. In the former case, by (3.2), $x_{k,1}$ remains bounded, so we may assume that $x_{k,1} \to x_1^*$ as $k \to \infty$. In this case, we set

$$\bar{u}_k(t,x) := \bar{u}(t, x + x_k).$$

Then, by (2.1) and (3.2), each $\bar{u}_k$ is defined for all $(t,x) \in (-\infty, T_\eta] \times \{x_1 \geq -1\}$ and, by parabolic estimates, we can choose a subsequence — again denoted by $\{\bar{u}_k\}$ — such that

$$\bar{u}_k(t,x) \to \bar{u}^*(t,x) \quad \text{in} \;\; C^{1,2}_{loc}((-\infty, T_\eta] \times \{x_1 \geq -1\}).$$

The limit function u^* satisfies the equation (1.1) on $(-\infty, T_\eta] \times \{x_1 \geq -1\}$ and, by (2.29) and the above convergence, we have

$$\bar{u}_t^*(t^*, 0) = 0, \qquad \bar{u}_t^*(t,x) \geq 0 \quad \text{for} \;\; (t,x) \in (-\infty, T_\eta] \times \{x_1 \geq -1\}$$

Applying the strong maximum principle to $\bar{u}_t^*$, we obtain $\bar{u}_t^* \equiv 0$ for $t \leq t^*$, but this is impossible since (2.4) implies

$$\bar{u}^*(t,x) - \phi(x_1 + x_1^* + ct) \to 0 \quad \text{as} \;\; t \to -\infty \qquad \text{uniformly in} \;\; x \in \{x_1 \geq -1\}.$$

Next, in the case where $t_k \to -\infty$ as $k \to \infty$, we set

$$\bar{u}_k(t,x) := \bar{u}(t + t_k, x + x_k).$$

Then by a similar argument, we can find a convergent subsequence $\bar{u}_k \to u^*$, where u^* satisfies $u_t^*(0,0) = 0$, hence $u_t^*(t,x) \equiv 0$ for $t \leq 0$, but this is impossible because

$$u^*(t,x) = \phi(x_1 + ct + \alpha)$$

for some $\alpha \in [-M_\eta, M_\eta]$. This contradiction proves the lemma. $\square$

Now let us prove the uniqueness of the entire solution. Suppose there exists another entire solution v of (1.1) satisfying (2.4). Choose $\eta > 0$ sufficiently small so that

$$f'(s) \le -\beta \qquad \text{for} \ \ s \in [-2\eta, 2\eta] \cup [1-2\eta, 1+2\eta] \tag{3.4}$$

for some $\beta > 0$. Then for any $\varepsilon \in (0, \eta)$ we can find $t_\varepsilon \in \mathbb{R}$ such that

$$\|v(t,\cdot) - \bar{u}(t,\cdot)\|_{L^\infty(\Omega)} < \varepsilon \qquad \text{for} \ \ -\infty < t \le t_\varepsilon. \tag{3.5}$$

Modifying the idea of [9], for each $t_0 \in (-\infty, T_\eta - \sigma\varepsilon]$ we define

$$W^+(t,x) := \bar{u}(t_0+t+\sigma\varepsilon(1-e^{-\beta t}), x) + \varepsilon e^{-\beta t}, \quad W^-(t,x) := \bar{u}(t_0+t-\sigma\varepsilon(1-e^{-\beta t}), x) - \varepsilon e^{-\beta t},$$

where the constant $\sigma > 0$ is to be specified later. Then, by (3.5),

$$W^-(0,x) \le v(t_0, x) \le W^+(0,x) \qquad \text{for} \ \ x \in \Omega. \tag{3.6}$$

Next we show that W^+ and W^- are a super- and a subsolution in the time range $t \in [0, T_\eta - t_0 - \sigma\varepsilon]$.

$$\begin{aligned} \mathcal{L} W^+ &:= W_t^+ - \Delta W^+ - f(W^+) \\ &= \sigma\varepsilon\beta e^{-\beta t}\bar{u}_t - \varepsilon\beta e^{-\beta t} + f(\bar{u}) - f(\bar{u} + \varepsilon e^{-\beta t}) \\ &= \varepsilon e^{-\beta t}\big(\sigma\beta\bar{u}_t - \beta - f'(\bar{u} + \theta\varepsilon e^{-\beta t})\big), \end{aligned}$$

where $\theta = \theta(t,x)$ is some function satisfying $0 < \theta < 1$ and $\bar{u} = \bar{u}(t_0 + t + \sigma\varepsilon(1 - e^{-\beta t}), x)$, $\bar{u}_t = \bar{u}_t(t_0 + t + \sigma\varepsilon(1 - e^{-\beta t}), x)$. If $x \in \Omega_\eta(t + t_0 + \sigma\varepsilon(1 - e^{-\beta t}))$, then by (3.3),

$$\mathcal{L} W^+ \ge \varepsilon e^{-\beta t}\big(\sigma\beta\delta - \beta - \max_{0 \le s \le 1} f'(s)\big).$$

Therefore $\mathcal{L} W^+ > 0$ if σ is chosen sufficiently large, independently of $\varepsilon > 0$. On the other hand, if $x \notin \Omega_\eta(t + t_0 + \sigma\varepsilon(1 - e^{-\beta t}))$, then

$$\bar{u} + \theta\varepsilon e^{-\beta t} \in [0, 2\eta] \cup [1 - \eta, 1 + \eta].$$

Consequently, by (3.4), one sees that $f'(\bar{u} + \theta\varepsilon e^{-\beta t}) \le -\beta$; hence

$$\mathcal{L} W^+ \ge \varepsilon e^{-\beta t}(-\beta + \beta) = 0.$$

Combining these, we see that $\mathcal{L} W^+ \ge 0$ for all $t \in [0, T_\eta - t_0 - \sigma\varepsilon]$, $x \in \Omega$. Similarly, we have $\mathcal{L} W^- \le 0$ in this region. In view of this and (3.6), we see that

$$W^-(t,x) \le v(t_0 + t, x) \le W^+(t,x) \qquad \text{for} \ \ t \in [0, T_\eta - t_0 - \sigma\varepsilon], \ x \in \Omega.$$

Rewriting $t_0 + t$ by t, respectively, we can rewrite this inequality as

$$\bar{u}(t - \sigma\varepsilon(1 - e^{-\beta(t-t_0)}), x) - \varepsilon e^{-\beta(t-t_0)} \le v(t,x) \le \bar{u}(t + \sigma\varepsilon(1 - e^{-\beta(t-t_0)}), x) + \varepsilon e^{-\beta(t-t_0)}.$$

for $t \in [t_0, T_\eta - \sigma\varepsilon]$ and $t_0 \in (-\infty, T_\eta - \sigma\varepsilon]$. Letting $t_0 \to -\infty$, we obtain

$$\bar{u}(t - \sigma\varepsilon, x) \le v(t,x) \le \bar{u}(t + \sigma\varepsilon, x) \tag{3.7}$$

for all $t \in (-\infty, T_\eta - \sigma\varepsilon]$, $x \in \Omega$. Hence, by the comparison principle, the above inequalities hold for all $t \in \mathbb{R}$, $x \in \Omega$. Letting $\varepsilon \to 0$, we get $v \equiv \bar{u}$. This proves the uniqueness of the entire solution satisfying (2.4).

4 Intermediate time: behaviour near the horizon

In this section, we are concerned with the behaviour near the horizon of a time-global solution of (1.1) which behaves like a planar front for very negative time. What we mean by the horizon is the limit as $|x'| \to +\infty$ when x_1 stays bounded, where $x' = (x_2, \dots, x_N)$ denotes the variables which are orthogonal to the direction x_1.

Proposition 4.1 *Let f be a $C^1([0,1])$ function satisfying* (1.2) *and assume that there exists a solution (c, ϕ) of* (1.3). *Let Ω be a smooth domain of $\mathbb{R}^N$ (with $N \geq 2$) with outward unit normal ν, and assume that $K = \mathbb{R}^N \backslash \Omega$ is compact. Let $u = u(t,x)$ be a classical solution of*

$$\begin{cases} u_t &= \Delta u + f(u), & x \in \Omega, & t \in \mathbb{R}, \\ \nu \cdot \nabla u &= 0, & x \in \partial\Omega, & t \in \mathbb{R}, \\ 0 \leq & u(t,x) \leq 1, & x \in \Omega, & t \in \mathbb{R}, \end{cases} \tag{4.1}$$

such that

$$\sup_{x \in \overline{\Omega}} |u(t,x) - \phi(x_1 + ct)| \to 0 \ \text{as } t \to -\infty. \tag{4.2}$$

Then, for any sequence $(x'_n)_{n\in\mathbb{N}} \in \mathbb{R}^{N-1}$ such that $|x'_n| \to +\infty$ as $n \to +\infty$, there holds

$$u(t, x_1, x' + x'_n) \underset{n\to+\infty}{\to} \phi(x_1 + ct), \ \textit{locally uniformly in } (t,x) = (t, x_1, x') \in \mathbb{R} \times \mathbb{R}^N.$$

Proof. Under the assumptions of Proposition 4.1, let $(x'_n)_{n\in\mathbb{N}} \in \mathbb{R}^{N-1}$ be a sequence such that $|x'_n| \to +\infty$ as $n \to +\infty$, and call

$$u_n(t,x) = u(t, x_1, x' + x'_n)$$

for each $t \in \mathbb{R}$ and $x = (x_1, x') \in \Omega - (0, x'_n)$. Since $0 \leq u \leq 1$ and $\mathbb{R}^N \backslash \Omega$ is compact, it follows from standard parabolic estimates that, up to extraction of a subsequence, the functions u_n converge as $n \to +\infty$, locally uniformly in $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, to a solution $U(t,x)$ of

$$U_t = \Delta U + f(U), \ t \in \mathbb{R}, \ x \in \mathbb{R}^N$$

with $0 \leq U(t,x) \leq 1$ for all $(t,x) \in \mathbb{R}^N$. Furthermore, since $u(t,x) - \phi(x_1 + ct) \to 0$ as $t \to -\infty$ uniformly in $x \in \overline{\Omega}$, the function U satisfies

$$\sup_{x \in \mathbb{R}^N} |U(t,x) - \phi(x_1 + ct)| \to 0 \text{ as } t \to -\infty.$$

The remaining part of the proof is inspired from the seminal paper by Fife and McLeod [9]. Extend the function f by

$$f(s) = f'(0)s \text{ for } s \leq 0, \ \text{and } f(s) = f'(1)(s-1) \text{ for } s \geq 1. \tag{4.3}$$

Define

$$\omega = \min\left(\frac{|f'(0)|}{2}, \frac{|f'(1)|}{2}\right) > 0$$

and let $\rho > 0$ be such that $f'(s) \le -\omega$ for all $s \in (-\infty, 2\rho] \cup [1-2\rho, +\infty)$. Let $A > 0$ be chosen so that $\phi(\xi) \le \rho$ for all $\xi \le -A$, and $\phi(\xi) \ge 1-\rho$ for all $\xi \ge A$. It follows from the sliding method that the (continuous) function ϕ' is positive in $\mathbb{R}$, whence

$$\delta := \min_{\xi \in [-A,A]} \phi'(\xi) > 0.$$

Fix now any arbitrary ε such that $\varepsilon \in (0, \rho)$, and let $T \in \mathbb{R}$ be such that

$$|U(t,x) - \phi(x_1 + ct)| \le \varepsilon \text{ for all } t \le T \text{ and for all } x \in \mathbb{R}^N. \tag{4.4}$$

Let t_0 be any time such that $t_0 \le T$, and define

$$\underline{u}(t,x) = \phi(\xi(t,x)) - \varepsilon e^{-\omega(t-t_0)} \text{ for all } t \ge t_0 \text{ and } x \in \mathbb{R}^N,$$

where

$$\xi(t,x) = x_1 + ct - 2\varepsilon \|f'\|_\infty \delta^{-1} \omega^{-1} (1 - e^{-\omega(t-t_0)}).$$

It follows from (4.4) that

$$\underline{u}(t_0, x) \le U(t_0, x) \text{ for all } x \in \mathbb{R}^N.$$

Furthermore, for all $t \ge t_0$ and $x \in \mathbb{R}^N$,

$$\begin{aligned}
\underline{u}_t - \Delta \underline{u} - f(\underline{u}) &= c\phi'(\xi(t,x)) - 2\varepsilon \|f'\|_\infty \delta^{-1} e^{-\omega(t-t_0)} \phi'(\xi(t,x)) + \varepsilon \omega e^{-\omega(t-t_0)} \\
&\quad -\phi''(\xi(t,x)) - f(\underline{u}(t,x)) \\
&= f(\phi(\xi(t,x))) - f(\underline{u}(t,x)) - 2\varepsilon \|f'\|_\infty \delta^{-1} e^{-\omega(t-t_0)} \phi'(\xi(t,x)) \\
&\quad +\varepsilon \omega e^{-\omega(t-t_0)}.
\end{aligned}$$

If $\xi(t,x) \le -A$, then $\underline{u}(t,x) = \phi(\xi(t,x)) - \varepsilon e^{-\omega(t-t_0)} \le \phi(\xi(t,x)) \le \rho$, whence

$$f(\phi(\xi(t,x))) - f(\underline{u}(t,x)) \le -\omega \varepsilon e^{-\omega(t-t_0)}$$

and

$$\underline{u}_t - \Delta \underline{u} - f(\underline{u}) \le 0$$

since $\phi' > 0$. If $\xi(t,x) \ge A$, then $\phi(\xi(t,x)) \ge 1-\rho$ and $\underline{u}(t,x) \ge \phi(\xi(t,x)) - \varepsilon \ge 1-2\rho$, whence $\underline{u}_t - \Delta \underline{u} - f(\underline{u}) \le 0$ as in the above case. Lastly, if $\xi(t,x) \in [-A, A]$, then $\phi'(\xi(t,x)) \ge \delta$, whence

$$\underline{u}_t - \Delta \underline{u} - f(\underline{u}) \le \|f'\|_\infty \varepsilon e^{-\omega(t-t_0)} - 2\varepsilon \|f'\|_\infty e^{-\omega(t-t_0)} + \varepsilon \omega e^{-\omega(t-t_0)} \le 0$$

since $\omega \le \|f'\|_\infty$.

The maximum principle then yields

$$U(t,x) \ge \underline{u}(t,x) \text{ for all } t \ge t_0 \text{ and } x \in \mathbb{R}^N.$$

Fixing $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ and passing to the limit as $t_0 \to -\infty$ in the above inequality leads to

$$U(t,x) \ge \phi(x_1 + ct - 2\varepsilon \|f'\|_\infty \delta^{-1} \omega^{-1}),$$

and then $U(t,x) \ge \phi(x_1 + ct)$ since ε was arbitrary in $(0, \rho)$.

Similarly, by constructing suitable super-solutions, one can prove that $U(t,x) \le \phi(x_1+ct)$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. As a conclusion, $U(t,x) \equiv \phi(x_1 + ct)$ and, since the limit is uniquely determined, the whole sequence $(u_n(t,x))_{n \in \mathbb{N}}$ converges to $\phi(x_1 + ct)$ locally in (t,x) as $n \to +\infty$. The proof of Proposition 4.1 is complete. □

Remark 4.2 Notice that no assumption on the sign of the speed c of the planar front ϕ is made in Proposition 4.1.

5 Large time: convergence to 1 far away from the obstacle

In this section, we are concerned with the large time behaviour far away from the obstacle, for a time-global solution $u(t,x)$ of (1.1) which behaves like a planar front for very negative time. By using spherically symmetric sub-solutions which expand at about the speed c, we prove that, for any $\varepsilon>0$, the region where $u>1-\varepsilon$ can completely surround the obstacle.

Proposition 5.1 *Let f be a $C^1([0,1])$ function satisfying* (1.2) *and assume that there exists a solution (c,ϕ) of* (1.3) *with $c>0$. Let Ω be a smooth domain of $\mathbb{R}^N$ (with $N\ge 2$) with outward unit normal ν, and assume that $K=\mathbb{R}^N\backslash\Omega$ is compact. Let $u=u(t,x)$ be a classical solution of* (4.1) *satisfying* (4.2) *and $u_t(t,x)\ge 0$ for all $(t,x)\in\mathbb{R}\times\overline{\Omega}$. Then there exists a classical solution u_∞ of*

$$\left\{\begin{array}{rcll}\Delta u_\infty+f(u_\infty)&=&0&\text{in }\Omega,\\ \nu\cdot\nabla u_\infty&=&0&\text{on }\partial\Omega,\\ 0\ <\ u_\infty(x)&\le&1&\text{for all }x\in\overline{\Omega}\end{array}\right.\tag{5.1}$$

such that $u(t,x)\to u_\infty(x)$ as $t\to+\infty$, locally uniformly in $x\in\overline{\Omega}$, and

$$u_\infty(x)\to 1\ \text{ as }|x|\to+\infty.$$

The above result is based on an auxiliary lemma, the proof of which is postponed at the end of this section. In the sequel, for any $R>0$ and $x\in\mathbb{R}^N$, we define

$$B_R(x)=\{y\in\mathbb{R}^N,\ |y-x|<R\}$$

the open euclidean ball of centre x and radius R. Since $f(1)=0$ and $f'(1)<0$, there exists $\theta\in(0,1)$ such that

$$f>0\text{ on }(\theta,1).\tag{5.2}$$

Lemma 5.2 *Under the assumptions of Proposition* 5.1*, for any $\eta\in(\theta,1)$, there exist four positive real numbers $R_1=R_1(\eta)$, $R_2=R_2(\eta)$, $R_3=R_3(\eta)$ and $T=T(\eta)$ such that $R_3>R_2>R_1>0$,*

$$R_2-R_1>\frac{cT}{4}\tag{5.3}$$

and, if $B_{R_3}(x_0)\subset\Omega$ and $u(t_0,x)\ge\eta$ for all $x\in B_{R_1}(x_0)$, for some $x_0\in\Omega$ and $t_0\in\mathbb{R}$, then

$$u(t_0+T,x)\ge\eta\ \text{ for all }x\in B_{R_2}(x_0).$$

Proof of Proposition 5.1. Since $u_t \geq 0$ and $0 \leq u \leq 1$, one has that $u(t,x) \to u_\infty(x) \in [0,1]$ as $t \to +\infty$, for all $x \in \overline{\Omega}$. Because of (4.2) and the strong parabolic maximum principle, $0 < u(t,x) < 1$ for all $(t,x) \in \mathbb{R} \times \overline{\Omega}$, whence $u_\infty(x) > 0$ for all $x \in \overline{\Omega}$. Furthermore, standard parabolic estimates imply that the convergence is (at least) locally uniform, and that u_∞ is a classical solution of (5.1).

Let us now prove that $u_\infty(x) \to 1$ as $|x| \to +\infty$. First, let $R_0 > 0$ be such that $K \subset B_{R_0}(0)$. Fix any $\varepsilon \in (0, 1-\theta)$, where $\theta \in (0,1)$ is given in (5.2). From (4.2) and the limit $\phi(+\infty) = 1$, there exist $T_0 \in \mathbb{R}$ and $\xi_0 \in \mathbb{R}$ such that

$$H = \{x \in \mathbb{R}^N,\ x_1 \geq \xi_0\} \subset \Omega \text{ and } u(T_0, \cdot) \geq 1 - \varepsilon \text{ in } H. \tag{5.4}$$

Let $R_1 = R_1(1-\varepsilon)$, $R_2 = R_2(1-\varepsilon)$, $R_3 = R_3(1-\varepsilon)$ and $T = T(1-\varepsilon)$ be the four positive real numbers given in Lemma 5.2 with the parameter $\eta = 1 - \varepsilon \in (\theta, 1)$. Let now $x \in \mathbb{R}^N$ be any point such that $|x| > R_0 + R_3 - R_2$ (in particular, $x \in \Omega$). It is straigthforward to see that there exist then an integer $k \geq 1$ and k points $x^1, \ldots, x^k$ in $\mathbb{R}^N$ such that

$$\begin{cases} B_{R_1}(x^1) \subset H, & \\ B_{R_3}(x^i) \subset \Omega & \text{for } 1 \leq i \leq k, \\ B_{R_1}(x^{i+1}) \subset B_{R_2}(x^i) & \text{for } 1 \leq i \leq k-1, \\ x \in B_{R_2}(x^k). & \end{cases}$$

From (5.4) and Lemma 5.2, it follows that

$$u(T_0 + T, \cdot) \geq 1 - \varepsilon \text{ in } B_{R_2}(x^1).$$

Since $B_{R_1}(x^2) \subset B_{R_2}(x^1)$ and $B_{R_3}(x^2) \subset \Omega$, another application of Lemma 5.2 yields

$$u(T_0 + 2T, \cdot) \geq 1 - \varepsilon \text{ in } B_{R_2}(x^2).$$

By immediate induction, there holds $u(T_0 + kT, \cdot) \geq 1 - \varepsilon$ in $B_{R_2}(x^k)$, whence $u(T_0 + kT, x) \geq 1 - \varepsilon$ and $u_\infty(x) \geq 1 - \varepsilon$.

One has shown that

$$u_\infty(x) \geq 1 - \varepsilon \text{ for all } x \in \mathbb{R}^N \text{ such that } |x| > R_0 + R_3 - R_2.$$

Since $u_\infty \leq 1$ in $\overline{\Omega}$, the proof of Proposition 5.1 is complete. □

Proof of Lemma 5.2. Let $\eta \in (\theta, 1)$ be fixed. As usual, extend the function f by (4.3). For $0 < \delta < \min(1/2, 1-\theta)$, let f_δ be the function defined in $\mathbb{R}$ by

$$f_\delta(s) = \begin{cases} f(\delta) \times \dfrac{s+\delta}{2\delta} & \text{if } s \leq \delta \\ f(s) & \text{if } \delta < s < 1 - 2\delta, \\ f(1-2\delta) \times \dfrac{1-\delta-s}{\delta} & \text{if } s \geq 1 - 2\delta. \end{cases}$$

It is immediate to check that for $\delta > 0$ small enough, the function f_δ has two stable zeroes at $-\delta$ and $1-\delta$, namely $f_\delta(-\delta) = f_\delta(1-\delta) = 0$, $f'(0) < f'_\delta(-\delta) < 0$ and $f'_\delta(1-\delta) < f'(1) < 0$.

Furthermore, for $\delta > 0$ small enough, $f_\delta \le f$ on $\mathbb{R}$ and there exists a unique (up to shifts) solution (c_δ, ϕ_δ) of (1.3) with the nonlinearity f_δ and the conditions

$$-\delta = \phi_\delta(-\infty) < \phi_\delta < \phi_\delta(+\infty) = 1 - \delta.$$

It is also known that $c_\delta \to c\ (> 0)$ as $\delta \to 0^+$. One can then fix a positive δ such that

$$c_\delta > \frac{c}{2} > 0$$

and all above conditions are fulfilled. One can also assume without loss of generality that $\delta < 1 - \eta$.

It follows then from Aronson and Weinberger [1] that there exists $R_1 > 0$ such that the solution $\underline{u}$ of the Cauchy problem

$$\begin{cases} \underline{u}_t &= \Delta \underline{u} + f_\delta(\underline{u}),\ t > 0,\ x \in \mathbb{R}^N, \\ \underline{u}(0,x) &= \begin{cases} \eta & \text{if } x \in B_{R_1}(0), \\ -\delta & \text{if } x \in \mathbb{R}^N \backslash B_{R_1}(0) \end{cases} \end{cases}$$

satisfies $\underline{u}(t,x) \to 1 - \delta$ as $t \to +\infty$ locally uniformly in $x \in \mathbb{R}^N$, and moreover

$$\lim_{t \to +\infty} \min_{|x| \le c' t} \underline{u}(t,x) = 1 - \delta \text{ for all } c' \in [0, c_\delta).$$

Therefore, there exist $R_2 > R_1$ and $T > 0$ such that

$$\frac{R_2 - R_1}{T} > \frac{c_\delta}{2} > \frac{c}{4}$$

and

$$\underline{u}(T, \cdot) \ge \eta \text{ in } B_{R_2}(0). \tag{5.5}$$

Fix $R_2 > R_1 > 0$, $T > 0$ and $\underline{u}$ as above. Let now $\xi_0 \in \mathbb{R}$ and $R_3 > R_2$ be chosen so that $\phi_\delta(-R_1 + \xi_0) \ge \eta$ and $\phi_\delta(-R_3 + c_\delta T + \xi_0) \le 0$. Therefore, for all unit vector e and for all $x \in B_{R_1}(0)$, one has

$$\underline{u}(0,x) = \eta \le \phi_\delta(-R_1 + \xi_0) \le \phi_\delta(x \cdot e + \xi_0)$$

since ϕ is increasing. The comparison $\underline{u}(0,x) \le \phi_\delta(x \cdot e + \xi_0)$ is also true by construction for all $x \in \mathbb{R}^N \backslash B_{R_1}(0)$. The maximum principle implies that $\underline{u}(t,x) \le \phi_\delta(x \cdot e + c_\delta t + \xi_0)$ for all $t \ge 0$, $x \in \mathbb{R}^N$ and $|e| = 1$. In particular, since ϕ_δ is increasing, there holds

$$\forall\ t \in [0,T],\ \forall\ x \in \mathbb{R}^N \backslash B_{R_3}(0), \quad \underline{u}(t,x) \le \phi_\delta(-R_3 + c_\delta T + \xi_0) \le 0. \tag{5.6}$$

Under the assumptions of Lemma 5.2, u is such that $u(t_0, \cdot) \ge \eta$ in $B_{R_1}(x_0)$ and $B_{R_3}(x_0) \subset \Omega$ for some $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^N$. One shall compare $u(t,x)$ with

$$w(t,x) = \underline{u}(t - t_0, x - x_0)$$

for $t_0 \le t \le t_0 + T$ and $x \in B_{R_3}(x_0)$. Observe first that

$$u(t_0, x) \ge w(t_0, x) \text{ for all } x \in B_{R_3}(x_0).$$

This is indeed true in $B_{R_1}(x_0)$ since $u(t_0,\cdot) \geq \eta = w(t_0,\cdot)$ in $B_{R_1}(x_0)$, and also in $B_{R_3}(x_0)\backslash B_{R_1}(x_0)$ since $u(t_0,\cdot) \geq 0 \geq -\delta = w(t_0,\cdot)$ in $B_{R_3}(x_0)\backslash B_{R_1}(x_0)$. Furthermore,

$$\forall\ t_0 \leq t \leq t_0 + T,\ \forall\ x \in \partial B_{R_3}(x_0), \quad w(t,x) \leq 0 \leq u(t,x)$$

from (5.6). Lastly,

$$w_t = \Delta w + f_\delta(w) \leq \Delta w + f(w),\ t_0 < t \leq t_0 + T,\ x \in B_{R_3}(x_0)$$

from the choice of δ. One concludes from the parabolic maximum principle that

$$u(t,x) \geq w(t,x) \text{ for all } t_0 \leq t \leq t_0 + T \text{ and } x \in B_{R_3}(x_0).$$

In particular,

$$u(t_0 + T, x) \geq \underline{u}(T, x - x_0) \geq \eta \text{ for all } x \in B_{R_2}(x_0)$$

from (5.5). The proof of Lemma 5.2 is now complete. □

Remark 5.3 *The above arguments imply that, in the statement of Lemma* 5.2*, given any* $\varepsilon \in (0,c)$*, the real numbers* $R_i = R_i(\eta,\varepsilon)$ *and* $T = T(\eta,\varepsilon)$ *can be chosen so that* $R_2 - R_1 > (c-\varepsilon)T$*, instead of* (5.3).

6 The stationary problem

This section is concerned with the study of the stationary solutions $0 \leq u(x) \leq 1$ of (5.1) (or (6.2) below), which converge to 1 as $|x| \to +\infty$. First, we prove two Liouville type results, that is $u \equiv 1$, when the obstacle K is compact and either star-shaped or directionally convex. Then we construct explicit counter-examples to this property for some obstacles K which are neither star-shaped nor directionally convex.

6.1 Star-shaped obstacles

The following result, which is of independent interest, is slightly more general than what we really need, and it uses the fact that the obstacle K is star-shaped.

Theorem 6.1 *Let* f *be a Lipschitz-continuous function in* $[0,1]$ *such that* $f(0) = f(1) = 0$ *and* f *is nonincreasing in* $[1-\delta,1]$ *for some* $\delta > 0$*. Assume that* f *satisfies* (1.4)*, that is*

$$\forall\ 0 \leq s < 1, \quad \int_s^1 f(\tau)d\tau > 0. \tag{6.1}$$

Let Ω *be a smooth open connected subset of* $\mathbb{R}^N$ *(with* $N \geq 2$*) with outward unit normal* ν*, and assume that* $K = \mathbb{R}^N\backslash\Omega$ *is compact. Let* $0 \leq u \leq 1$ *be a classical solution of*

$$\left\{\begin{array}{rcll} -\Delta u & = & f(u) & \text{in } \Omega, \\ \nu\cdot\nabla u & = & 0 & \text{on } \partial\Omega, \\ u(x) & \to & 1 & \text{as } |x| \to +\infty. \end{array}\right. \tag{6.2}$$

If K *is star-shaped, then*

$$u \equiv 1 \text{ in } \overline{\Omega}. \tag{6.3}$$

Remark 6.2 Condition (6.1) is equivalent to the existence of a function $U \in C^2(\mathbb{R}_+)$ such that

$$U(0)=0, \quad U(+\infty)=1, \quad U''(\xi)+f(U(\xi))=0 \text{ and } U'(\xi)>0 \text{ for all } \xi\geq 0. \tag{6.4}$$

These two equivalent properties are satisfied under assumption (1.2) for f and if there exists a planar front ϕ solving (1.3) with $c>0$. They are also guaranteed in the important bistable case with positive mass, that is when f satisfies (1.2) and there is $\theta \in (0,1)$ such that (1.5) holds (see [9]).

Proof of Theorem 6.1. Let us first extend f by 0 in $[1,+\infty)$. The function f is then Lipschitz-continuous in $[0,+\infty)$. Up to a shift of the origin, one can assume without loss of generality that K –if not empty– is star-shaped with respect to 0. In what follows, B_r denotes the open Euclidean ball of center 0 and radius $r>0$.

Let us first observe that $u>0$ in $\overline{\Omega}$ from the strong maximum principle and Hopf lemma. Next, let $r_0>0$ be such that $K\subset B_{r_0}$ (i.e. $\mathbb{R}^N\backslash B_{r_0} \subset \Omega$) and

$$u(x)\geq 1-\delta \text{ for all } |x|\geq r_0.$$

We claim that

$$u(x)\geq U(|x|-r_0) \text{ for all } |x|\geq r_0, \tag{6.5}$$

where U solves (6.4). Let

$$\varepsilon^*=\inf\{\varepsilon>0;\ u^\varepsilon(x):=u(x)+\varepsilon\geq U(|x|-r_0) \text{ for all } |x|\geq r_0\}.$$

Since u and U are bounded, ε^* is a nonnegative real number, and one has

$$u^{\varepsilon^*}(x)=u(x)+\varepsilon^*\geq U(|x|-r_0) \text{ for all } |x|\geq r_0. \tag{6.6}$$

One shall prove that $\varepsilon^*=0$. Assume that $\varepsilon^*>0$. There exist then a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive numbers and a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ such that $\varepsilon_n\to\varepsilon^*$ as $n\to+\infty$ and

$$|x_n|\geq r_0, \quad u(x_n)+\varepsilon_n<U(|x_n|-r_0) \text{ for all } n\in\mathbb{N}.$$

Since both $u(x)$ and $U(|x|-r_0)$ converge to 1 as $|x|\to+\infty$, it follows that the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded, and one can then assume, up to extraction of some subsequence, that $x_n\to\overline{x}\in\overline{\Omega}$ as $n\to+\infty$, where $|\overline{x}|\geq r_0$. Thus, $u(\overline{x})+\varepsilon^*\leq U(|\overline{x}|-r_0)$ and then

$$u(\overline{x})+\varepsilon^*=U(|\overline{x}|-r_0)$$

because of (6.6). But

$$u^{\varepsilon^*}(x)-U(|x|-r_0)=u(x)+\varepsilon^*-U(|x|-r_0)>u(x)-U(|x|-r_0)\geq 0 \text{ on } \partial B_{r_0}$$

since $u\geq 0$ in $\overline{\Omega}$ and $U(0)=0$. Hence, $|\overline{x}|>r_0$. Observe now that

$$\Delta u^{\varepsilon^*}+f(u^{\varepsilon^*})=f(u+\varepsilon^*)-f(u)\leq 0 \text{ in } \mathbb{R}^N\backslash B_{r_0}\ (\subset\Omega)$$

because $u \geq 1-\delta$ in this set, and f is nonincreasing in $[1-\delta,+\infty)$. On the other hand, the function $\tilde{U}(x) = U(|x|-r_0)$ satisfies

$$\begin{aligned} \Delta\tilde{U}(x) + f(\tilde{U}(x)) &= U''(|x|-r_0) + \frac{N-1}{|x|}U'(|x|-r_0) + f(U(|x|-r_0)) \\ &= \frac{N-1}{|x|}U'(|x|-r_0) \ > \ 0 \text{ in } \mathbb{R}^N\backslash B_{r_0} \end{aligned}$$

because U is increasing in $\mathbb{R}_+$. Notice actually that the above calculations imply that the function $U(|\cdot|-r)$ is a strict sub-solution of (1.1) in $\{x,\ |x| > \max(r,0)\}$ for all $r \in \mathbb{R}$. The function $z := u^{\varepsilon^*} - \tilde{U}$ then satisfies

$$\Delta z + b(x)z \leq 0 \text{ in } \mathbb{R}^N\backslash B_{r_0}$$

for some globally bounded function b, because f is Lipschitz-continuous in $[0,+\infty)$. But z is nonnegative in $\mathbb{R}^N\backslash B_{r_0}$ and it vanishes at the interior point $\overline{x}$. The strong maximum principle implies that $z(x) = 0$, namely $u(x) + \varepsilon^* = U(|x|-r_0)$, for all $|x| \geq r_0$. This is impossible for $|x| = r_0$, as already underlined.

Thus $\varepsilon^* = 0$ and the claim (6.5) follows. Actually, since U is increasing in $\mathbb{R}_+$, one gets that

$$u(x) \geq U(|x|-r) \text{ for all } r \geq r_0 \text{ and } |x| \geq r.$$

Define now

$$r^* = \inf\ \{r \in \mathbb{R};\ u(x) \geq U(|x|-r) \text{ for all } x \in \overline{\Omega} \text{ and } |x| \geq r\}.$$

One has $r^* \leq r_0$ and our goal is to prove that $r^* = -\infty$ (which will then yield $u \equiv 1$ in $\overline{\Omega}$). Assume that $r^* > -\infty$. One has

$$u(x) \geq U(|x|-r^*) \text{ for all } |x| \geq r^* \text{ and } x \in \overline{\Omega}$$

by continuity. Two cases may occur.

Case 1: assume here that

$$\inf\ \{u(x) - U(|x|-r^*);\ x \in \overline{\Omega},\ r^* \leq |x| \leq r_0\} > 0.$$

Since both ∇u and U' are globally bounded (in $\overline{\Omega}$ and $\mathbb{R}_+$ respectively), there exists $r_* < r^*$ $(\leq r_0)$ such that

$$u(x) \geq U(|x|-r_*) \text{ for all } x \in \overline{\Omega} \text{ and } r_* \leq |x| \leq r_0.$$

Notice that this property holds whatever the sign of r^* is. Since $u(x) \geq 1-\delta$ for all $|x| \geq r_0$ and $u(x) \geq U(|x|-r_*)$ for all $|x| = r_0$, one concludes with the same arguments as above that

$$u(x) \geq U(|x|-r_*) \text{ for all } |x| \geq r_0.$$

Therefore, $u(x) \geq U(|x|-r_*)$ for all $x \in \overline{\Omega}$ and $|x| \geq r_*$. That contradicts the minimality of r^* and case 1 is then ruled out.

Case 2: one then has that

$$\inf \{u(x) - U(|x| - r^*);\ x \in \overline{\Omega},\ r^* \leq |x| \leq r_0\} = 0.$$

By continuity, there exists then $\overline{x} \in \overline{\Omega}$ such that

$$u(\overline{x}) = U(|\overline{x}| - r^*) \text{ and } r^* \leq |\overline{x}| \leq r_0.$$

Moreover,

$$|\overline{x}| > r^*$$

because of (6.4) and $u > 0$ in $\overline{\Omega}$.

Assume first that $\overline{x} \in \Omega$ and $|\overline{x}| > 0$. Observe that the set $\tilde{\Omega} = \Omega \cap \{x;\ |x| > r^*\}$ is connected because K –if not empty– is star-shaped with respect to 0. The strong maximum principle implies that

$$u(x) = U(|x| - r^*) \text{ for all } x \in \overline{\Omega} \text{ and } |x| \geq r^*,$$

because $u \geq U(|\cdot| - r^*)$ in this set and

$$\tilde{U}(x) = U(|x| - r^*)$$

is a subsolution of (1.1) in $\tilde{\Omega} \backslash \{0\}$. But since $\tilde{U}$ is actually a strict subsolution of (1.1) in this set, one has reached a contradiction.

Therefore, either $\overline{x} \in \partial\Omega$ or $\overline{x} = 0$. In the latter case, then $r^* < 0$ and $\Omega = \mathbb{R}^N$ (remember that K –if not empty– is assumed to be star-shaped with respect to the origin). Since $u \in C^1(\mathbb{R}^N)$ satisfies $u \geq \tilde{U} = U(|\cdot| - r^*)$ in $\mathbb{R}^N$ (in this case) with $U'(-r^*) > 0$, it follows that $u(0) > \tilde{U}(0)$, which is impossible.

As a consequence, $\overline{x} \in \partial\Omega$ (whence $K \neq \emptyset$ and $|\overline{x}| > 0$), and

$$u(x) > U(|x| - r^*) \text{ for all } x \in \Omega \text{ and } |x| \geq r^*.$$

Remember that $u(\overline{x}) = U(|\overline{x}| - r^*)$ and $|\overline{x}| > r^*$. The nonnegative function $u - \tilde{U}$ satisfies $\Delta(u - \tilde{U}) + b(x)(u - \tilde{U}) \leq 0$ in $\Omega \cap \{|x| > r^*\}$, for some globally bounded function b. Hopf lemma then implies that

$$\nu(\overline{x}) \cdot \nabla(u - \tilde{U})(\overline{x}) < 0.$$

Hence,

$$0 < \nu(\overline{x}) \cdot \nabla\tilde{U}(\overline{x}) = \left(\nu(\overline{x}) \cdot \frac{\overline{x}}{|\overline{x}|}\right) \times U'(|\overline{x}| - r^*).$$

But the last term of the inequality is nonpositive because $U' > 0$ in $\mathbb{R}_+$ and K is star-shaped (notice indeed that $\nu(\overline{x}) = -\nu_K(\overline{x})$, where $\nu_K(\overline{x})$ is the outward unit normal to K at the point $\overline{x}$). Case 2 is then ruled out too.

As a conclusion, $r^* = -\infty$, and then $u(x) \geq U(|x| - r)$ for all $r \in \mathbb{R}$ and $x \in \overline{\Omega}$ with $|x| \geq r$, because U is increasing in $\mathbb{R}_+$. For each $x \in \overline{\Omega}$, one then has $u(x) \geq U(|x| - r)$ for all $r \leq |x|$, whence $u(x) \geq 1$ by taking the limit as $r \to -\infty$. Since $u \leq 1$ in $\overline{\Omega}$, one then concludes that $u \equiv 1$ in $\overline{\Omega}$. That completes the proof of Theorem 6.1. □

Remark 6.3 In dimension $N = 1$ with $\Omega = \mathbb{R}$, the same arguments can be adapted straightforwardly, and the conclusion of Theorem 6.1 holds.

6.2 Directionally convex obstacles

The following result is the analogue of Theorem 6.1 when the obstacle K is directionally convex, in the sense of Definition 1.2.

Theorem 6.4 *If in Theorem* 6.1 *the obstacle* K *is assumed to be directionally convex instead of star-shaped, then the conclusion* (6.3) *still holds.*

Proof. First, there is $r_0 > 0$ such that $K \subset \{x \in \mathbb{R}^N,\ |x\cdot e - a| \le r_0\}$ and $u(x) \ge 1-\delta$ for all x such that $|x\cdot e - a| \ge r_0$. Notice that, here, for any $r \in \mathbb{R}$, the function $x \mapsto U(|x\cdot e-a|-r)$ is a stationary solution of (1.1) in the domain $\{x \in \mathbb{R}^N,\ |x \cdot e - a| > \max(r, 0)\}$. As in the proof of Theorem 6.1, one gets that

$$u(x) \ge U(|x \cdot e - a| - r_0) \text{ for all } x \in \mathbb{R}^N \text{ such that } |x \cdot e - a| \ge r_0,$$

where U solves (6.4). Actually, this property holds good if we replace r_0 by any larger real number.

Define now

$$r^* = \inf\ \{r \in \mathbb{R};\ u(x) \ge U(|x \cdot e - a| - r) \text{ for all } x \in \overline{\Omega} \text{ such that } |x \cdot e - a| \ge r\}.$$

One has $r^* \le r_0$ and our goal is to prove that $r^* = -\infty$, which will then imply that $u \equiv 1$ in $\overline{\Omega}$. Assume that $r^* > -\infty$. By continuity, one has

$$u(x) \ge U(|x \cdot e - a| - r^*) \text{ for all } x \in \overline{\Omega} \text{ such that } |x \cdot e - a| \ge r^*.$$

Two cases may occur.

Case 1: assume here that

$$\inf\ \{u(x) - U(|x \cdot e - a| - r^*);\ x \in \overline{\Omega},\ r^* \le |x \cdot e - a| \le r_0\} > 0.$$

With similar arguments as in the proof of Theorem 6.1, there exists then $r_* < r^*$ such that $u(x) \ge U(|x \cdot e - a| - r_*)$ for all $x \in \overline{\Omega}$ with $|x \cdot e - a| \ge r_*$. That contradicts the minimality of r^* and case 1 is then ruled out.

Case 2: one then has that

$$\inf\ \{u(x) - U(|x \cdot e - a| - r^*);\ x \in \overline{\Omega},\ r^* \le |x \cdot e - a| \le r_0\} = 0.$$

There exists then a sequence of points $(x_n)_{n\in\mathbb{N}} = (x_{1,n}, \cdots, x_{N,n})_{n\in\mathbb{N}}$ of $\overline{\Omega}$ such that

$$u(x_n) - U(|x_n \cdot e - a| - r^*) \to 0 \text{ as } n \to +\infty, \text{ and } r^* \le |x_n \cdot e - a| \le r_0 \text{ for all } n \in \mathbb{N}.$$

Up to extraction of a subsequence, two cases may occur: either $x_n \to \overline{x} \in \overline{\Omega}$ or $|\pi(x_n)| \to +\infty$ as $n \to +\infty$, where π denotes the orthogonal projection onto the hyperplane P.

Subcase 1: $x_n \to \overline{x} \in \overline{\Omega}$ as $n \to +\infty$. Therefore,

$$u(\overline{x}) = U(|\overline{x} \cdot e - a| - r^*) \text{ and } r^* \le |\overline{x} \cdot e - a| \le r_0.$$

Moreover,

$$|\overline{x} \cdot e - a| > r^*$$

because of (6.4) and $u > 0$ in $\overline{\Omega}$.

Assume first that $\overline{x} \in \Omega$ and $|\overline{x} \cdot e - a| > 0$ (that is $\overline{x} \notin P$). Observe that the sets $\Omega_{\pm} = \Omega \cap \{x;\ \pm(x \cdot e - a) > \max(r^*, 0)\}$ are connected because K is directionally convex with respect to the direction e. Define $\Omega_\varepsilon \in \{\Omega_+, \Omega_-\}$ the connected set containing $\overline{x}$. The strong maximum principle implies that

$$u(x) = U(|x \cdot e - a| - r^*) \text{ for all } x \in \overline{\Omega_\varepsilon} \text{ such that } |x \cdot e - a| \geq r^*,$$

because $u(x) \geq U(|x \cdot e - a| - r^*)$ in this set, with equality at the interior point $\overline{x}$, and

$$\tilde{U}(x) = U(|x \cdot e - a| - r^*)$$

is a stationary solution of (1.1) in Ω_ε. But since $u(x) \to 1$ as $|x| \to +\infty$ (thanks to (6.2)) and $\limsup U(|x \cdot e - a| - r^*) < 1$ as $|\pi(x)| \to +\infty$ and $|x \cdot e - a| - r^*$ is fixed, one gets a contradiction.

Therefore, either $\overline{x} \in \partial\Omega$ and $|\overline{x} \cdot e - a| > 0$, or $|\overline{x} \cdot e - a| = 0$. In the latter case, then $r^* < 0$ and since $u \in C^1(\overline{\Omega})$ satisfies $u(x) \geq \tilde{U}(x) = U(|x \cdot e - a| - r^*)$ for all $x \in \overline{\Omega}$ (in this case) with $U'(-r^*) > 0$ and $\overline{x} + \mathbb{R}e \subset \overline{\Omega}$ (from Definition 1.2), it follows that $u(\overline{x}) > \tilde{U}(\overline{x})$, which is impossible.

As a consequence, $\overline{x} \in \partial\Omega$ (whence $K \neq \emptyset$), $|\overline{x} \cdot e - a| > 0$, and

$$u(x) > U(|x \cdot e - a| - r^*) \text{ for all } x \in \Omega \text{ and } |x \cdot e - a| \geq r^*.$$

Remember that $u(\overline{x}) = U(|\overline{x} \cdot e - a| - r^*)$ and $|\overline{x} \cdot e - a| > r^*$. The nonnegative function $u - \tilde{U}$ satisfies $\Delta(u - \tilde{U}) + b(x)(u - \tilde{U}) = 0$ in $\Omega \cap \{|x \cdot e - a| > r^*\}$, for some globally bounded function b. Hopf lemma then implies that

$$\nu(\overline{x}) \cdot \nabla(u - \tilde{U})(\overline{x}) < 0.$$

Hence,

$$0 < \nu(\overline{x}) \cdot \nabla\tilde{U}(\overline{x}) = (\nu(\overline{x}) \cdot e) \times \frac{\overline{x} \cdot e - a}{|\overline{x} \cdot e - a|} \times U'(|\overline{x} \cdot e - a| - r^*).$$

But the last term of the inequality is nonpositive because $U' > 0$ in $\mathbb{R}_+$ and K is directionally convex in the direction e. Subcase 1 is then ruled out too.

Subcase 2: $|\pi(x_n)| \to +\infty$ as $n \to +\infty$. Then $u(x_n) \to 1$ because of (6.2), while $\limsup_{n \to +\infty} U(|x_n \cdot e - a| - r^*) < 1$ since $r^* \leq |x_n \cdot e - a| \leq r_0$. One has then reached a contradiction.

As a conclusion, $r^* = -\infty$, whence $u \equiv 1$ in $\overline{\Omega}$. The proof of Theorem 6.4 is complete.□

6.3 A counter-example to the Liouville result

Here, we construct a special domain which is neither star-shaped nor directionally convex and for which the previous Liouville property fails for the exterior problem (6.2):

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^N \setminus K = \Omega, \\ \dfrac{\partial u}{\partial \nu} := \nu \cdot \nabla u = 0 & \text{on } \partial K = \partial\Omega, \\ u(x) \to 1 & \text{as } |x| \to \infty. \end{cases} \tag{6.7}$$

We assume that f satisfies (1.2) and (6.1). Notice that, as in Theorems 6.1 and 6.4, we do not need to assume the existence of a solution (c, ϕ) of (1.3).

We construct examples of compact obstacles K for which there exist solutions u of (6.7) such that $0 < u < 1$ in $\overline{\Omega}$. For this we consider a family of smooth almost annular regions K_ε into which a small channel of width $\varepsilon > 0$ is pierced as in the Figure below. Then we prove the existence of a local minimizer of the associated energy functional in a suitable functional space, when the width ε of the channel is small enough. We refer to [16] and [2] for other properties of non-trivial solutions of elliptic equations in convex or strongly non-convex (dumbbell shaped) domains, and also to [13] for the construction of local minimizers of a family of energy functionals which approximate an appropriate perimeter functional in a fixed domain.

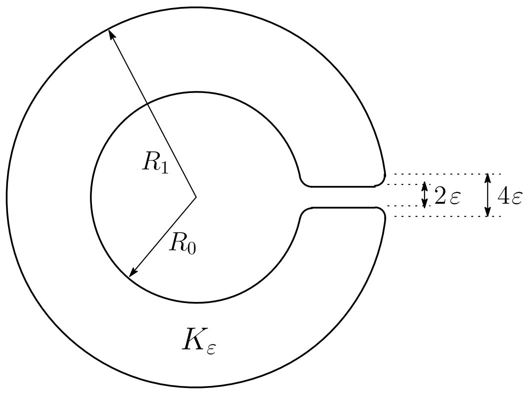

Figure 3: Obstacle K_ε

Theorem 6.5 *For small enough $\varepsilon > 0$, problem (6.7) with $K = K_\varepsilon$ admits a solution u with $0 < u < 1$.*

It is enough to show that for any R such that $B_R \supset K_\varepsilon$ (actually, R is otherwise arbitrary), for small enough ε, the following problem has a solution:

$$\begin{cases} -\Delta w = f(w) & \text{in } B_R \setminus K_\varepsilon =: \Omega_\varepsilon, \\ \nu \cdot \nabla w = 0 & \text{on } \partial K_\varepsilon, \\ w = 1 & \text{on } \partial B_R. \end{cases} \tag{6.8}$$

Indeed, then, w extended by 1 outside B_R is a supersolution. In the previous section, we showed that $U(|x| - R)$ is a subsolution where $U : \mathbb{R}^+ \to (0, 1)$ satisfies $U'' + f(U) = 0$ in $\xi > 0$, $U(0) = 0$, $U'(\xi) > 0$, $\forall\, \xi \geq 0$, $U(+\infty) = 1$ (remember that the existence of such U is equivalent to the condition (6.1)). Define $\psi(x) = U(|x| - R)$ if $|x| \geq R$ and $\psi(x) = 0$ if $|x| \leq R$. Then, ψ is a global subsolution. Since $w \equiv 1$ on $\mathbb{R}^N \setminus B_R$, we know that $\psi \leq w$. Therefore, there exists a solution u of (6.7) such that $\psi \leq u \leq w$. This implies that $u \not\equiv 1$ and $u \not\equiv 0$, whence, by the maximum principle, that $0 < u < 1$.

From now on we work with (6.8). Let $v = 1 - u$; it satisfies:

$$\left\{ \begin{array}{rcl} -\Delta v &=& -f(1-v) =: g(v) \ \text{ in } B_R \setminus K_\varepsilon, \\ \nu \cdot \nabla v &=& 0 \ \text{ on } \partial K_\varepsilon, \\ v &=& 0 \ \text{ on } \partial B_R. \end{array} \right. \tag{6.9}$$

and we look for a solution v such that $v \not\equiv 0$. Extend f by values: $f(s) > 0$ for $s \leq 0$ and $f(s) < 0$ for $s \geq 1$. More precisely, we take f to be C^1 with f linear on $s \geq 1$ and on $s \leq 0$, that is we extend f by (4.3).

By the maximum principle a solution is such that $0 \leq v \leq 1$ and, if $v \not\equiv 0$, then $0 < v < 1$. The function $g(v)$ still satisfies (1.2), but its mass has opposite sign: $\displaystyle\int_0^1 g(s)ds < 0$.

Thus, our goal is to prove that for some classes of domains, solutions $v \not\equiv 0$ of (6.9) can be found.

We now indicate the type of domains we construct. Choose (arbitrarily) two radii R_0, R_1 such that $0 < R_0 < R_1 < R$. Let $\mathcal{A}$ denote the annular region corresponding to these two radii: $\mathcal{A} = \{x, R_0 \leq |x| \leq R_1\}$. For small enough ε, we consider a smooth subdomain K_ε of $\mathcal{A}$ such that:

$$\left\{ \begin{array}{l} \mathcal{A} \cap \{x; x_1 \leq 0\} \subset K_\varepsilon, \\ \mathcal{A} \cap \{x; x_1 > 0, |x'| > 2\,\varepsilon\} \subset K_\varepsilon, \\ K_\varepsilon \subset [\mathcal{A} \cap \{x, x_1 > 0, |x'| > \varepsilon\}] \cup [\mathcal{A} \cap \{x, x_1 \leq 0\}]. \end{array} \right. \tag{6.10}$$

where $x' = (x_2, ..., x_N)$. Thus K_ε is the annulus $\mathcal{A}$ into which a channel of width between 2ε and 4ε has been pierced.

Let D_0 be the inner ball of radius R_0:

$$D_0 = B_{R_0} \subset \Omega_\varepsilon \quad \forall\, \varepsilon > 0 \text{ sufficiently small.}$$

For any domain $D \subset \Omega_\varepsilon$, let us consider the energy:

$$J_D(w) = \int_D \left\{ \frac{1}{2} |\nabla w|^2 - G(w) \right\} \, dx$$

defined for functions of $H^1(D)$, where

$$G(t) = \int_0^t g(s)ds.$$

We start with the following observation:

Proposition 6.6 *In $D_0, w_0 \equiv 1$ is a strict local minimum of J_{D_0} in the space $H^1(D_0)$. More precisely, there exist $\alpha > 0$ and $\delta > 0$ for which:*

$$J_{D_0}(w) \geq J_{D_0}(w_0) + \alpha ||w - w_0||^2_{H^1(D_0)}$$

for all $w \in H^1(D_0)$ such that $||w - w_0||_{H^1(D_0)} \leq \delta$.

Proof. Here, we use that G is C^2 (so f is assumed C^1 – but this argument could be made more general). Expand G to get:

$$G(s) = G(1) + \frac{(s-1)^2}{2} G''(1) + \eta(s-1)[s-1]^2$$

where $\eta(s)$ is continuous and $\eta(s-1) \to 0$ as $s \to 1$. From the construction of the extension of f, it follows that G has quadratic growth at infinity. This implies that η is bounded: $|\eta(t)| \leq C,\ \forall\ t \in \mathbb{R}$, for some constant $C > 0$. Then, recalling that $w_0 \equiv 1$, we get:

$$J_{D_0}(w_0) = -G(1)|D_0|$$

and

$$J_{D_0}(w) = \int_{D_0} \frac{1}{2}|\nabla w|^2 \ - \ G(1)|D_0| \ - \ \int_{D_0} \left\{ \frac{(w-w_0)^2}{2} G''(1) + \eta(w-w_0)(w-w_0)^2 \right\}.$$

Set $w - w_0 = z$ so that:

$$J_{D_0}(w) - J_{D_0}(w_0) = \int_{D_0} \left\{ \frac{1}{2}|\nabla z|^2 - \frac{G''(1)}{2} z^2 \right\} + \int_{D_0} \eta(z) z^2. \tag{6.11}$$

Now use that $G''(1) < 0$, that is, $f'(0) < 0$. Since η is bounded and $\eta(s) \to 0$ as $s \to 0$, standard integration arguments and Sobolev embeddings yield the following lemma:

Lemma 6.7

$$\left| \int_{D_0} \eta(z) z^2 \right| \leq \varepsilon(z) ||z||^2_{H^1(D_0)}$$

where $\varepsilon(z) \to 0$ *as* $z \to 0$ *in* $H^1(D_0)$.

Proof of Lemma 6.7. For any $\varepsilon > 0$, let $\delta(\varepsilon) > 0$ be such that $|\eta(t)| \leq \varepsilon$ if $|t| \leq \delta(\varepsilon)$. Therefore:

$$|\eta(t)| \leq \varepsilon + \frac{C}{(\delta(\varepsilon))^p}\ |t|^p, \quad \forall\ t \in \mathbb{R}.$$

Hence,

$$\int_{D_0} \eta(z) z^2 \leq \varepsilon \int_{D_0} z^2 + \frac{C}{(\delta(\varepsilon))^p} \int_{D_0} |z|^{2+p}.$$

By choosing $p > 0$ sufficiently small, Sobolev embedding theorem yields:

$$\int_{D_0} |z|^{2+p} \leq C ||z||^{2+p}_{H^1(D_0)}.$$

Therefore,

$$\left| \int_{D_0} \eta(z) z^2 \right| \leq \varepsilon ||z||^2_{H^1(D_0)} + \frac{C}{\delta(\varepsilon)^p}\ ||z||^{2+p}_{H^1(D_0)} \leq \left\{ \varepsilon + \frac{C}{\delta(\varepsilon)^p}\ ||z||^p_{H^1(D_0)} \right\} ||z||^2_{H^1(D_0)}.$$

Since this holds for all $\varepsilon > 0$, it implies the lemma. $\square$

Inequality (6.11) then yields:

$$J_{D_0}(w) - J_{D_0}(w_0) \geq \alpha||w - w_0||^2_{H^1(D_0)} \quad \text{if} \quad ||w - w_0||_{H^1(D_0)} \leq \delta. \tag{6.12}$$

The proof of Proposition 6.6 is complete. $\square$

Let us now look at the domain $\Omega_\varepsilon = B_R \setminus K_\varepsilon$ for $\varepsilon > 0$ sufficiently small. First, we extend w_0 as follows: we set

$$\begin{cases} w_0(x) = 1 \text{ if } x \in B_{R_1} \backslash K_\varepsilon \ \cap \ \left\{x; x_1 \leq \dfrac{2R_0 + R_1}{3}\right\}, \\ w_0(x) = \dfrac{3}{R_1 - R_0}\left(\dfrac{R_0 + 2R_1}{3} - x_1\right) \text{ if } x \in B_{R_1} \backslash K_\varepsilon \ \cap \ \left\{x; \dfrac{2R_0 + R_1}{3} \leq x_1 \leq \dfrac{R_0 + 2R_1}{3}\right\}, \\ w_0(x) = 0 \text{ if } x \in [B_R \backslash B_{R_1}] \cup \left[B_{R_1} \backslash K_\varepsilon \ \cap \ \left\{x; x_1 \geq \dfrac{R_0 + 2R_1}{3}\right\}\right]. \end{cases}$$

Proposition 6.8 *There exist $\beta > 0$ and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, and $w \in H^1(\Omega_\varepsilon) \cap \{w = 0 \text{ on } \partial B_R\} =: E_\varepsilon$ such that $||w - w_0||_{H^1(\Omega_\varepsilon)} = \delta$, then:*

$$J_{\Omega_\varepsilon}(w_0) < J_{\Omega_\varepsilon}(w) - \beta.$$

Proof. Define $C_\varepsilon := \Omega_\varepsilon \cap \{x;\ R_0 \leq |x| \leq R_1\}$, $D_\varepsilon := D_0 \cup C_\varepsilon$ and $F := \Omega_\varepsilon \setminus D_\varepsilon = B_R \setminus B_{R_1}$.

Consider now $w \in H^1(\Omega_\varepsilon)$ with $w = 0$ on ∂B_R and such that $||w - w_0||_{H1(\Omega_\varepsilon)} = \delta$. Since $||w - w_0||_{H^1(D_0)} \leq \delta$ (as $D_0 \subset \Omega_\varepsilon$), we know that:

$$J_{D_0}(w) \geq J_{D_0}(w_0) + \alpha||w - w_0||^2_{H^1(D_0)} \geq J_{D_0}(w_0).$$

From the conditions $f(1) = 0$, $f'(1) < 0$, and (6.1), it follows that there exists a constant $\kappa > 0$ such that:

$$G(s) \leq -\kappa s^2, \quad \forall\ s \geq 0.$$

Hence, for any domain D,

$$J_D(w) \geq \ \nu||w||^2_{H^1(D)}$$

where $\nu = \min\{1/2, \kappa\} > 0$.

Therefore, we derive the following lower bounds

$$J_{C_\varepsilon}(w) \geq \nu||w||^2_{H^1(C_\varepsilon)}, \quad J_F(w) \geq \nu||w||^2_{H^1(F)}. \tag{6.13}$$

Since $|\nabla w_0| \leq C$ (for some generic constant $C > 0$) and $w_0 \leq 1$, we have:

$$J_{C_\varepsilon}(w_0) \leq C|C_\varepsilon| \leq C\varepsilon^{N-1}.$$

Lastly, we know that $J_F(w_0) = 0$ as $w_0 \equiv 0$ outside $D_0 \cup C_\varepsilon$. Splitting the energy in Ω_ε into the pieces in D_0, C_ε and $\Omega_\varepsilon \setminus D_\varepsilon = F$ respectively, we get:

$$J_{\Omega_\varepsilon} = J_{D_0} + J_{C_\varepsilon} + J_F.$$

From the previous estimates, we thus infer that:

$$\begin{aligned} J_{\Omega_\varepsilon}(w) - J_{\Omega_\varepsilon}(w_0) \;\geq\; & \alpha||w-w_0||^2_{H^1(D_0)} \;-\; C\varepsilon^{N-1} + \\ & +\frac{\nu}{2}||w-w_0||^2_{H^1(C_\varepsilon)} - \; C\varepsilon^{N-1} + \nu||w-w_0||^2_{H^1(F)}. \end{aligned}$$

Hence,

$$J_{\Omega_\varepsilon}(w) - J_{\Omega_\varepsilon}(w_0) \geq \beta||w-w_0||^2_{H^1(\Omega_\varepsilon)} - C\varepsilon^{N-1}.$$

for some constants $\beta,\ C>0$. Indeed, notice that the square of the $H^1(\Omega_\varepsilon)$ norm breaks into three pieces as well, in $D_0,\ C_\varepsilon$ and F.

From this, the proposition follows. □

Conclusion: proof of Theorem 6.5. The functional J_{Ω_ε} admits a *local minimum* in the ball of radius δ about w_0 in E_ε. This yields a (stable) solution v of (6.9) for small enough $\varepsilon>0$. Furthermore, provided δ is chosen small enough, this solution does not coincide neither with 1 nor with 0, hence $0<v<1$ in Ω_ε. The proof of the theorem is thereby complete. □

7 Large time behaviour for star-shaped or directionally convex obstacles

In this section, we prove that, at large time, the generalized front converges to the planar front uniformly in $\overline{\Omega}$ when the obstacle K is star-shaped or directionally convex.

First, we prove a result of independent interest, about the uniform convergence of some solutions $u(t,x)$ of (1.1) to the planar front $\phi(x_1+ct)$ as $t\to+\infty$. The solutions are assumed to be close to the planar front for large $|x|$ at some time, and to converge to 1 locally in x as $t\to+\infty$.

Then, assuming that the obstacle is star-shaped or directionally convex, we complete the proof of Theorem 1.

7.1 Uniform convergence to the flat front when $\lim_{t\to+\infty} u(t,x)=1$

Theorem 7.1 *Let f be a $C^1([0,1])$ function satisfying* (1.2)*, and assume that there exists a solution (c,ϕ) of* (1.3) *with $c>0$. Let Ω be a smooth domain of $\mathbb{R}^N$ (with $N\geq 2$) with outward unit normal ν, and assume that $K=\mathbb{R}^N\backslash\Omega$ is compact. Let $t_0\in\mathbb{R}$ and $u=u(t,x)$ be a classical solution of*

$$\begin{cases} \qquad\quad u_t \;=\; \Delta u + f(u), & x\in\Omega, \quad t\in[t_0,+\infty), \\ \qquad \nu\cdot\nabla u \;=\; 0, & x\in\partial\Omega, \ t\in[t_0,+\infty), \\ 0 \;\leq\; u(t,x) \;\leq\; 1, & x\in\Omega, \quad t\in[t_0,+\infty). \end{cases} \tag{7.1}$$

Assume that, for any $\varepsilon>0$, there exist $t_\varepsilon\geq t_0$ and a compact set $C_\varepsilon\subset\overline{\Omega}$ such that

$$|u(t_\varepsilon,x)-\phi(x_1+ct_\varepsilon)|\leq\varepsilon \ \ \textit{for all } x\in\overline{\Omega}\backslash C_\varepsilon, \tag{7.2}$$

and

$$u(t,x) \geq 1-\varepsilon \ \ \textit{for all } t \geq t_\varepsilon \textit{ and } x \in \partial\Omega = \partial K. \tag{7.3}$$

Then

$$\sup_{x \in \overline{\Omega}} |u(t,x) - \phi(x_1 + ct)| \to 0 \ \textit{as } t \to +\infty.$$

Proof. The proof is divided into several steps. Some technical lemmas will be used in the proof. In order to keep the arguments clear for the reader's convenience, we postpone the proofs of these lemmas to an Appendix in Section 10 at the end of the paper.

The main idea of the proof of Theorem 7.1 is to construct suitable sub- and super-solutions which strongly diffuse, with slow decay rates, in the directions $x_2, \ldots, x_N$. Let us first define some auxiliary constants and functions.

Step 1: Some preliminary notations. First, up to a shift in x_1, one can assume that $K \subset \{x \in \mathbb{R}^N,\ x_1 \leq 0\}$ and that $\phi(0) = 1/2$. Set

$$L = \max_{x=(x_1,\ldots,x_N) \in \overline{K}} \left(\max_{1 \leq i \leq N} |x_i| \right) \tag{7.4}$$

if $K \neq \emptyset$, and set $L = 0$ if $K = \emptyset$ (that is, $\Omega = \mathbb{R}^N$).

Next, observe that the function ϕ is (at least) of class C^3 and ϕ' satisfies

$$(\phi')'' - c(\phi')' + f'(\phi)\phi' = 0$$

and $\phi' > 0$ in $\mathbb{R}$. Since the function $z \mapsto f'(\phi(z))$ is itself bounded, it follows then from standard interior estimates and Harnack inequality that the function ϕ''/ϕ' is bounded. Namely, there exists $C_1 > 0$ such that

$$|\phi''(z)| \leq C_1 \phi'(z) \ \text{ for all } z \in \mathbb{R}. \tag{7.5}$$

Even if it means increasing C_1, one can assume without loss of generality that

$$C_1 \geq \max\left(1, \frac{(N-1)L^2}{4c}, \frac{(N-1)L^2}{4}\right). \tag{7.6}$$

Furthermore, it is known that there exists $C_2 > 0$ such that

$$\left\{ \begin{array}{rcl} 1-\phi(z) & \underset{z \to +\infty}{\sim} & C_2 e^{-\lambda z} \\ \phi'(z) & \underset{z \to +\infty}{\sim} & C_2 \lambda e^{-\lambda z} \end{array} \right. \quad \text{with } \lambda = \frac{-c + \sqrt{c^2 - 4f'(1)}}{2} > 0. \tag{7.7}$$

Set

$$C_3 = 2\lambda\sqrt{C_2} > 0.$$

It follows immediately that there exists $0 < \kappa_0 \leq 1/2$ such that

$$[\ \phi(z) \geq 1 - \kappa \geq 1 - \kappa_0\ ] \implies [\ \phi'(z) \leq C_3 \sqrt{\kappa}\ e^{-\lambda z/2}\ ]. \tag{7.8}$$

Set

$$\omega = \min\left(\frac{|f'(0)|}{4}, \frac{|f'(1)|}{4}, \frac{\lambda c}{2}, 1\right) > 0. \tag{7.9}$$

Let us now extend f by (4.3). The function f is now of class $C^1(\mathbb{R})$. Let $\rho > 0$ be such that

$$\begin{cases} |f'(s) - f'(0)| & \le & \omega \text{ for all } s \le \rho, \\ |f'(s) - f'(1)| & \le & \omega \text{ for all } s \ge 1 - \rho, \end{cases} \tag{7.10}$$

and let $A > 0$ be such that

$$\begin{cases} \phi(z) & \ge & 1 - \dfrac{\rho}{2} & \text{for all } z \ge A, \\ \phi(z) & \le & \rho & \text{for all } z \le -A. \end{cases} \tag{7.11}$$

Since ϕ' is continuous and positive in $\mathbb{R}$, one has

$$\delta := \min_{z \in [-A,A]} \phi'(z) > 0. \tag{7.12}$$

In the case when $\Omega \neq \mathbb{R}^N$, let $\tilde{\zeta}$ be a function of class $C^2(\overline{\Omega})$, with compact support in $\overline{\Omega}$, and such that $\nu \cdot \nabla\tilde{\zeta} = 1$ on $\partial\Omega$. The functions $\Delta\tilde{\zeta}$ and $\tilde{\zeta}$ are continuous and compactly supported in $\overline{\Omega}$ and they are then bounded. There exists then a constant $C_4 > 0$ such that

$$\zeta = \tilde{\zeta} + C_4 \ge 1 \text{ in } \overline{\Omega}$$

and

$$\left\|\frac{\Delta\zeta}{\zeta}\right\|_{L^\infty(\Omega)} \le \omega. \tag{7.13}$$

In the case when $\Omega = \mathbb{R}^N$, we define $\zeta = 1$ in $\mathbb{R}^N$, and (7.13) obviously holds.

Lastly, set

$$\begin{cases} C_5 & = & C_3\, e^{\lambda(L+2)/2}\left(1 + \dfrac{(N-1)L}{4}\right) > 0, \\ C_6 & = & (2+\omega)(48+20\omega)^{-1}\omega\delta\|f'\|_\infty^{-1}\|\zeta\|_\infty^{-1}\|\phi'\|_\infty^{-1} > 0, \end{cases} \tag{7.14}$$

where the notation $\|F\|_\infty$ stands for the L^∞ norm of a function F on its domain of definition.

Step 2: Some other coefficients and functions depending on η. Our goal is to prove that $u(t,x)$ is uniformly bounded from below, as close as we want, by the planar front $\phi(x_1 + ct)$ for large time (we shall also show later the boundedness from above). To do so, we fix an arbitrary positive real number

$$\eta > 0$$

and we will prove that

$$\liminf_{t\to+\infty}\left[\inf_{x\in\overline{\Omega}}(u(t,x) - \phi(x_1+ct))\right] \ge -\eta.$$

In this step, we define some auxiliary coefficients and functions depending on η, which will be used in Step 3 when we construct a sub-solution.

Even if it means decreasing η, one can assume that

$$\eta \le \min\left(\|\phi'\|_\infty, \frac{\rho}{2C_6\|\zeta\|_\infty}, \frac{\kappa_0}{C_6\underline{\zeta}}, \frac{(2+\omega)^2 C_5^2 \underline{\zeta}}{4C_6}\right), \tag{7.15}$$

where $\underline{\zeta} = \min_{x\in\overline{\Omega}} \zeta(x) > 0$ and κ_0, ω, ρ, ζ, C_5 and C_6 are given in (7.8), (7.9), (7.10), (7.13) and (7.14). Then, denote

$$\mu = \min\left(\frac{\rho}{2\|\zeta\|_\infty}, C_6\eta\right) = C_6\eta > 0, \tag{7.16}$$

and

$$\varepsilon = \min\left(\kappa_0, \mu\underline{\zeta}, 4\mu^2(2+\omega)^{-2}C_5^{-2}\right) = 4\mu^2(2+\omega)^{-2}C_5^{-2} = 4C_6^2\eta^2(2+\omega)^{-2}C_5^{-2} > 0. \tag{7.17}$$

The explicit calculation of ε in the right-hand side of (7.17) is a consequence of (7.15) and (7.16).

Apply now the main assumption of Theorem 7.1 with the positive number $\varepsilon/2 > 0$. There exist then $t_{\varepsilon/2} \ge t_0$ and a compact set $C_{\varepsilon/2} \subset \overline{\Omega}$ such that

$$\begin{cases} |u(t_{\varepsilon/2}, x) - \phi(x_1 + ct_{\varepsilon/2})| \le \dfrac{\varepsilon}{2} & \text{for all } x \in \overline{\Omega}\backslash C_{\varepsilon/2}, \\ u(t,x) \ge 1 - \dfrac{\varepsilon}{2}\ (\ge 1-\varepsilon) & \text{for all } t \ge t_{\varepsilon/2} \text{ and } x \in \partial K. \end{cases} \tag{7.18}$$

Since both u and $\phi(x_1 + ct)$ are bounded and satisfy (1.1) in Ω, there exist a time $t_\varepsilon > t_{\varepsilon/2}$ and a compact set $C_\varepsilon \subset \overline{\Omega}$ such that (7.18) holds with ε instead of $\varepsilon/2$. Define

$$\tilde{u}(t,x) = u(t - 1 + t_\varepsilon, x) \text{ and } \tilde{\phi}(t,x) = \phi(x_1 + c(t - 1 + t_\varepsilon)).$$

Thus,

$$|\tilde{u}(1,x) - \tilde{\phi}(1,x)| \le \varepsilon \quad \text{for all } x \in \overline{\Omega}\backslash C_\varepsilon \tag{7.19}$$

and

$$\tilde{u}(t,x) \ge 1 - \varepsilon \quad \text{for all } t \ge 1 \text{ and } x \in \partial K. \tag{7.20}$$

By assumption, u is not identically equal to 0 or 1 in $[t_0, +\infty) \times \overline{\Omega}$. Thus, $0 < u(t,x) < 1$ for all $t > t_0$ and $x \in \overline{\Omega}$ from the strong maximum principle and Hopf lemma. In particular, $\min_{C_\varepsilon} \tilde{u}(1,\cdot) = \min_{C_\varepsilon} u(t_\varepsilon, \cdot) > 0$ because C_ε is compact and $t_\varepsilon > t_0$. Furthermore, since $\phi(-\infty) = 0$, there exists $\beta > 0$ such that

$$\phi\left(x_1 + ct_\varepsilon - \beta e^{-|x'|^2}\right) \le \tilde{u}(1,x) \quad \text{for all } x \in C_\varepsilon, \tag{7.21}$$

where $x' = (x_2, \ldots, x_N)$ and $|x'|^2 = x_2^2 + \cdots + x_N^2$. Since ϕ is increasing, β can also be chosen large enough (even if it means increasing β) so that

$$\begin{cases} \beta \ge \max\left(\dfrac{N-1}{4}, \dfrac{\gamma_0}{8C_1}, 1\right), \\ c(t_\varepsilon + 2 + 2\omega^{-1}) - \beta(3 + 2\omega^{-1})^{-1} e^{-\frac{(N-1)L^2}{3+2\omega^{-1}}} < 0, \end{cases} \tag{7.22}$$

where $\gamma_0 > 0$ is such that

$$\exp\left(-\frac{(N-1)L^2}{s}\right) \geq 1 - \frac{2(N-1)L^2}{s} \quad \text{for all } s \geq \gamma_0. \tag{7.23}$$

Define

$$\left\{\begin{array}{rcl} \alpha & = & \dfrac{N-1}{4C_1\beta} > 0, \\ \gamma & = & \max(\gamma_0, 8, 8C_1\beta, 2\beta(N-1)L^2c^{-1}, 2\beta(N-1)L^2) \;=\; 8C_1\beta \;\geq\; 1. \end{array}\right. \tag{7.24}$$

The calculation of γ follows from (7.6) and (7.22). From (7.6) and (7.22), one also has $0 < \alpha \leq 1$.

Let g be the function defined by

$$g(t) = c(t - 1 + t_\varepsilon) - \beta t^{-\alpha} e^{-\frac{(N-1)L^2}{\gamma t}} \quad \text{for } t \geq 1. \tag{7.25}$$

From the choice of γ, one obtains, for all $t \geq 1$,

$$\begin{array}{rcl} g'(t) & = & c + \beta t^{-\alpha-1} e^{-\frac{(N-1)L^2}{\gamma t}} (\alpha - (N-1)L^2\gamma^{-1}t^{-1}) \\ & \geq & c - \beta(N-1)L^2\gamma^{-1} \;\geq\; \dfrac{c}{2}. \end{array} \tag{7.26}$$

Since $\beta > 0$, $0 < \alpha \leq 1$, $\gamma \geq 1$ and $3 + 2\omega^{-1} \geq 1$, it follows that

$$\begin{array}{rcl} g(3+2\omega^{-1}) & = & c(t_\varepsilon + 2 + 2\omega^{-1}) - \beta(3+2\omega^{-1})^{-\alpha} e^{-\frac{(N-1)L^2}{\gamma(3+2\omega^{-1})}} \\ & \leq & c(t_\varepsilon + 2 + 2\omega^{-1}) - \beta(3+2\omega^{-1})^{-1} e^{-\frac{(N-1)L^2}{3+2\omega^{-1}}}. \end{array}$$

Therefore, $g(3+2\omega^{-1}) < 0$ because of the second inequality in (7.22). On the other hand, g is continuous in $[1, +\infty)$ and $g' \geq c/2 > 0$ because of (7.26). There exists then a unique

$$t_1 > 3 + 2\omega^{-1} \text{ such that } g(t_1) = 0. \tag{7.27}$$

Define

$$t_2 = t_1 - 2 - \frac{2\omega^{-1}C_5\varepsilon^{1/2}}{C_5\varepsilon^{1/2} + \mu e^{2\omega+2-\omega(t_1-1)}} \;\in\; (t_1 - 2 - 2\omega^{-1}, t_1 - 2) \tag{7.28}$$

and

$$\left\{\begin{array}{rcl} \sigma_1 & = & \omega^2(C_5\varepsilon^{1/2} + \mu e^{2\omega+2-\omega(t_1-1)}) > 0, \\ \sigma_2 & = & C_5\varepsilon^{1/2}\mu e^{2\omega+2-\omega(t_1-1)} > 0, \end{array}\right.$$

where ω, C_5, μ and ε where defined in (7.9), (7.14), (7.16) and (7.17).

Let now v be function defined for all $t \geq 1$ by

$$v(t) = \left\{\begin{array}{ll} \mu e^{-\omega(t-1)} & \text{for } t \in [1, t_1 - 2 - 2\omega^{-1}], \\ C_5\varepsilon^{1/2} + \dfrac{\sigma_1}{4}[(t-t_2)^2 - (t_1 - 2 - t_2)^2] & \text{for } t \in [t_1 - 2 - 2\omega^{-1}, t_1 - 2], \\ C_5\varepsilon^{1/2} + \dfrac{C_5\omega\varepsilon^{1/2}}{2}[1 - (t - t_1 + 1)^2] & \text{for } t \in [t_1 - 2, t_1], \\ C_5\varepsilon^{1/2} e^{-\omega(t-t_1)} & \text{for } t \geq t_1. \end{array}\right.$$

It is straightforward to check that the function v is of class C^1 on $[1,+\infty)$ and that it is decreasing on $[1,t_2]$, increasing on $[t_2,t_1-1]$ and decreasing on $[t_1-1,+\infty)$. Furthermore, $v>0$ on $[t_1,+\infty)$ and

$$\min_{t\in[1,t_1]} v(t)=v(t_2)=C_5\varepsilon^{1/2}-\frac{\sigma_1}{4}(t_1-2-t_2)^2=\omega^2\sigma_2\sigma_1^{-1}>0.$$

Hence, $v>0$ on $[1,+\infty)$. On the other hand, $v(1)=\mu$ and $v(t_1-1)=C_5\varepsilon^{1/2}(1+\omega/2)=\mu$ because of (7.17). Therefore,

$$\max_{t\geq 1} v(t)=\mu. \tag{7.29}$$

Lemma 7.2 *The function v satisfies*

$$-v'(t)\leq 2\omega v(t) \ \ \textit{for all } t\geq 1. \tag{7.30}$$

The proof of this lemma is postponed to the Appendix, Section 10. We continue with the proof of Theorem 7.1.

Let now V be the function defined in $[1,+\infty)$ by

$$V(t)=4\|\zeta\|_\infty\|f'\|_\infty\delta^{-1}\int_t^{+\infty} v(\tau)d\tau. \tag{7.31}$$

Then, V is decreasing in $[1,+\infty)$ and $V(+\infty)=0$. Let us now estimate $V(1)$. Owing to the definition of v, one has

$$\begin{array}{rcl} V(1) & = & 4\|f'\|_\infty\|\zeta\|_\infty\delta^{-1}\left[\displaystyle\int_1^{t_1-2-2\omega^{-1}} v(\tau)d\tau+\int_{t_1-2-2\omega^{-1}}^{t_1} v(\tau)d\tau+\int_{t_1}^{+\infty} v(\tau)d\tau\right] \\ & \leq & 4\|f'\|_\infty\|\zeta\|_\infty\delta^{-1}[\mu\omega^{-1}+(2+2\omega^{-1})\mu+C_5\varepsilon^{1/2}\omega^{-1}] \quad \text{(because of (7.29))} \\ & \leq & 4\|f'\|_\infty\|\zeta\|_\infty\delta^{-1}\omega^{-1}(5\mu+C_5\varepsilon^{1/2}) \quad \text{(because } 0<\omega\leq 1) \\ & = & 4\|f'\|_\infty\|\zeta\|_\infty\delta^{-1}\omega^{-1}(5C_6+2(2+\omega)^{-1}C_6)\eta. \end{array}$$

Thus,

$$V(1)\leq \|\phi'\|_\infty^{-1}\eta, \tag{7.32}$$

because of the definition of C_6 in (7.14).

Step 3: Construction of a sub-solution. Let us now define the function $\underline{u}$ by

$$\underline{u}(t,x)=\phi(\xi)-v(t)\zeta(x) \text{ for all } t\geq 1 \text{ and } x\in\overline{\Omega},$$

where

$$\xi=\xi(t,x)=x_1+c(t-1+t_\varepsilon)-\beta t^{-\alpha}e^{-\frac{|x'|^2}{\gamma t}}+V(t)-V(1),$$

and let us check that this function is a sub-solution of (1.1), for $t\geq 1$.

First, compare $\underline{u}(1,\cdot)$ and $\tilde{u}(1,\cdot)$ in $\overline{\Omega}$. If $x\in C_\varepsilon$, then

$$\underline{u}(1,x)\leq\phi\left(x_1+ct_\varepsilon-\beta e^{-\frac{|x'|^2}{\gamma}}\right)\leq\phi\left(x_1+ct_\varepsilon-\beta e^{-|x'|^2}\right)\leq\tilde{u}(1,x)=u(t_\varepsilon,x)$$

because $\phi' \geq 0$, $\beta > 0$, $\gamma \geq 1$ and because of (7.21). If $x \in \overline{\Omega}\backslash C_\varepsilon$, then

$$\underline{u}(1,x) \leq \phi(x_1 + ct_\varepsilon) - \mu\underline{\zeta} \leq \phi(x_1 + ct_\varepsilon) - \varepsilon \leq \tilde{u}(1,x) = u(t_\varepsilon, x)$$

because $v(1) = \mu > 0$, $\underline{\zeta} > 0$ and because of (7.17) and (7.19). As a consequence,

$$\forall\ x \in \overline{\Omega}, \quad \underline{u}(1,x) \leq \tilde{u}(1,x).$$

Set

$$\mathcal{L}\underline{u} = \underline{u}_t - \Delta\underline{u} - f(\underline{u}).$$

Lemma 7.3 *The function* $\underline{u}$ *satisfies*

$$\mathcal{L}\underline{u}(t,x) \leq 0 \ \ \textit{for all } t \geq 1 \textit{ and } x \in \overline{\Omega}.$$

Let us now check the boundary conditions on $\partial\Omega = \partial K$. Remember first that $\tilde{u}(t,x) \geq 1 - \varepsilon$ for all $t \geq 1$ and $x \in \partial K$ from (7.20). Therefore, if $\underline{u}(t,x) \leq 1 - \varepsilon$, with $t \geq 1$ and $x \in \partial K$, then $\underline{u}(t,x) \leq \tilde{u}(t,x)$. Furthermore, the following lemma holds.

Lemma 7.4 *If* (t,x) *is such that*

$$t \geq 1, \quad x \in \partial K \ \ \textit{and} \ \ \underline{u}(t,x) \geq 1 - \varepsilon, \tag{7.33}$$

then $\nu(x) \cdot \nabla\underline{u}(t,x) \leq 0$.

It then follows from the parabolic maximum principle that

$$\underline{u}(t,x) \leq \tilde{u}(t,x) \ \text{ for all } t \geq 1 \text{ and } x \in \overline{\Omega}.$$

Therefore, for all $t \geq t_\varepsilon$,

$$\begin{aligned}
&\inf_{x\in\overline{\Omega}} [u(t,x) - \phi(x_1 + ct)] \\
&\qquad = \inf_{x\in\overline{\Omega}} [\tilde{u}(t+1-t_\varepsilon, x) - \phi(x_1 + ct)] \\
&\qquad \geq \inf_{x\in\overline{\Omega}} [\phi(\xi(t+1-t_\varepsilon, x)) - v(t+1-t_\varepsilon)\zeta(x) - \phi(x_1 + ct)] \\
&\qquad \geq -[\beta(t+1-t_\varepsilon)^{-\alpha} + V(1) - V(t+1-t_\varepsilon)]\ \|\phi'\|_\infty - v(t+1-t_\varepsilon)\|\zeta\|_\infty
\end{aligned}$$

But the right-hand side converges, as $t \to +\infty$, to $-V(1)\|\phi'\|_\infty \geq -\eta$ by (7.32).
Since $\eta > 0$ was arbitrary, one concludes that

$$\liminf_{t\to+\infty} \left\{ \inf_{x\in\overline{\Omega}} [u(t,x) - \phi(x_1 + ct)] \right\} \geq 0.$$

Step 4: Construction of a super-solution. It is a bit simpler than for the sub-solution. We define L as in (7.4), and then C_1 such that (7.5) holds and

$$C_1 \geq \max\left(1, \frac{N-1}{2c}\right). \tag{7.34}$$

From (7.7), there exists a constant $C_2' > 0$ such that

$$\phi'(z) \le C_2' e^{-\lambda z} \text{ for all } z \in \mathbb{R}. \tag{7.35}$$

Let $\omega > 0$ and $\rho > 0$ be as in (7.9) and (7.10), and let $A' > 0$ be such that

$$\begin{cases} \phi(z) & \ge & 1-\rho & \text{for all } z \ge A', \\ \phi(z) & \le & \dfrac{\rho}{2} & \text{for all } z \le -A' \end{cases}$$

and define $\delta' > 0$ by

$$\delta' = \min_{z \in [-A', A']} \phi'(z).$$

Choose ζ as in Step 1 and let

$$C_6' = \frac{\omega \delta' \|f'\|_\infty^{-1} \|\zeta\|_\infty^{-1} \|\phi'\|_\infty^{-1}}{3} > 0.$$

Let $\eta > 0$ be any positive number such that

$$0 < \eta \le \frac{\rho}{2 C_6' \|\zeta\|_\infty}.$$

Let $\mu' > 0$ and $\varepsilon' > 0$ be defined by

$$\mu' = \min\left(\frac{\rho}{2\|\zeta\|_\infty}, C_6'\eta\right) = C_6'\eta > 0$$

and

$$\varepsilon' = \mu' \underline{\zeta} = C_6' \eta \underline{\zeta} > 0.$$

Apply the main assumption of Theorem 7.1 with the positive number $\varepsilon'/2 > 0$. As in Step 2, there exist then $t_{\varepsilon'} > t_0$ and a compact set $C_{\varepsilon'} \subset \overline{\Omega}$ such that

$$|u(t_{\varepsilon'}, x) - \phi(x_1 + ct_{\varepsilon'})| \le \varepsilon' \text{ for all } x \in \overline{\Omega} \backslash C_{\varepsilon'}.$$

Since $\phi(+\infty) = 1$ and $\max_{C_{\varepsilon'}} u(t_{\varepsilon'}, \cdot) < 1$ ($C_{\varepsilon'}$ is compact), there exists $\beta' > 0$ such that

$$\phi\left(x_1 + ct_{\varepsilon'} + \beta' e^{-|x'|^2}\right) \ge u(t_{\varepsilon'}, x) \text{ for all } x \in C_{\varepsilon'}.$$

Since ϕ is increasing, β' can also be chosen large enough (even if it means increasing β') so that

$$\beta' \ge \max\left(1, \frac{N-1}{4}\right) \ge \max\left(1, \frac{N-1}{4C_1}\right)$$

and

$$C_2' \left(1 + \frac{(N-1)L}{4}\right) e^{-\lambda(-L + ct_{\varepsilon'} + \beta' e^{-(N-1)L^2})} \le \mu'. \tag{7.36}$$

Now define the super-solution $\overline{u}$ by, for $t \ge 1$ and $x \in \overline{\Omega}$,

$$\overline{u}(t,x) = \phi(\tilde{\xi}(t,x)) + \tilde{v}(t)\zeta(x),$$

where

$$\left\{\begin{array}{rcl}\tilde\xi & = & \tilde\xi(t,x)=x_1+c(t-1+t_{\varepsilon'})+\beta' t^{-\alpha'}e^{-\frac{|x'|^2}{\gamma' t}}+\tilde V(1)-\tilde V(t),\\ \tilde v(t) & = & \mu' e^{-\omega(t-1)},\\ \tilde V(t) & = & 3\|f'\|_\infty\|\zeta\|_\infty\delta'^{-1}\omega^{-1}\mu' e^{-\omega(t-1)}=\|\phi'\|_\infty^{-1}\eta e^{-\omega(t-1)}\end{array}\right. \tag{7.37}$$

and

$$\alpha'=\frac{N-1}{4C_1\beta'}\in(0,1],\quad \gamma'=\max(8,8C_1\beta')=8C_1\beta'\geq 1.$$

This function $\overline{u}$ is of the same type as $\underline{u}$ in step 3, but with some opposite signs and simpler definitions for $\tilde v$ and $\tilde V$.

With these choices of parameters and functions, it is then straightforward to check that

$$\overline{u}(1,x)\geq u(t_{\varepsilon'},x)\ \text{ for all } x\in\overline{\Omega}.$$

Furthermore, with the same calculations as in Step 3 (see the proof of Lemma 7.3), one gets that

$$\mathcal{L}\overline{u}\geq f(\phi(\tilde\xi))-f(\phi(\tilde\xi)+\tilde v(t)\zeta(x))+\tilde v'(t)\zeta(x)-\tilde v(t)\Delta\zeta(x)-\tilde V'(t)\phi'(\xi),$$

and then $\mathcal{L}\overline{u}\geq 0$ in each of the cases $\tilde\xi\leq -A'$, $\tilde\xi\geq A'$ and $|\tilde\xi|\leq A'$.

Lemma 7.5 *For all $t\geq 1$ and $x\in\partial\Omega$, one has $\nu(x)\cdot\nabla\overline{u}(t,x)\geq 0$.*

The parabolic maximum principle yields

$$\overline{u}(t,x)\geq u(t-1+t_{\varepsilon'},x)\ \text{ for all } t\geq 1 \text{ and } x\in\overline{\Omega}.$$

Therefore, for all $t\geq t_{\varepsilon'}$,

$$\begin{array}{l}\sup\limits_{x\in\overline{\Omega}}\,[u(t,x)-\phi(x_1+ct)]\\ \qquad\leq \sup\limits_{x\in\overline{\Omega}}\,[\phi(\tilde\xi(t+1-t_{\varepsilon'},x))+\tilde v(t+1-t_{\varepsilon'})\zeta(x)-\phi(x_1+ct)]\\ \qquad\leq (\beta'(t+1-t_{\varepsilon'})^{-\alpha'}+\tilde V(1)-\tilde V(t+1-t_{\varepsilon'}))\ \|\phi'\|_\infty+\tilde v(t+1-t_{\varepsilon'})\|\zeta\|_\infty\end{array}$$

But the right-hand side converges, as $t\to+\infty$, to $\tilde V(1)\|\phi'\|_\infty=\eta$ by (7.37).

Since $\eta>0$ was arbitrary, one concludes that

$$\limsup_{t\to+\infty}\left\{\sup_{x\in\overline{\Omega}}\,[u(t,x)-\phi(x_1+ct)]\right\}\leq 0.$$

Step 5: Conclusion. It follows from Step 3 and Step 4 that

$$\sup_{x\in\overline{\Omega}}\,|u(t,x)-\phi(x_1+ct)|\to 0 \text{ as } t\to+\infty.$$

That completes the proof of Theorem 7.1. □

Remark 7.6 The above proof is valid when there is no obstacle, that is when $\Omega = \mathbb{R}^N$, and in that case, the condition (7.3) is not needed. When $\Omega = \mathbb{R}^N$, assumption (7.2) is satisfied in particular if $u(t_0, x) - \phi(x_1 + ct_0) \to 0$ as $|x| \to +\infty$. Moreover, even without obstacle, Theorem 7.1 can be viewed as a generalization of some earlier results (see for instance Levermore and Xin [15]). Paper [15] was concerned with the stability of the planar front $\phi(x_1 + ct)$ for the Cauchy problem (1.1) in $\mathbb{R}^N$ under the assumption that the initial condition is a localized and small perturbation of ϕ. This assumption is not satisfied in Theorem 7.1 (even in the case $\Omega = \mathbb{R}^N$) since, in any bounded region, the difference between u and ϕ may be large at initial time. As already emphasized, one of the basic ideas was to construct explicit sub- and super-solutions $\underline{u}$ and $\overline{u}$ such that the phase shifts in ϕ diffuse strongly in the variables x' and relax weakly in time. In such a way, the initial perturbations diffuse and become negligible at large time.

7.2 Proof of the main convergence result when K is star-shaped or directionally convex

By means of Theorem 7.1 and of the results of the previous sections, we are now able to complete the proof of the main Theorem 1 which corresponds to the propagation around a star-shaped or directionally convex obstacle.

Proof of Theorem 1. Assume that the function $f \in C^1([0,1])$ satisfies (1.2), that there is a solution (c, ϕ) of (1.3) with $c > 0$, and that the compact obstacle K is star-shaped or directionally convex. From Section 2, there exists a time-global solution u of (4.1) such that $u_t(t,x) > 0$ for all $(t,x) \in \mathbb{R} \times \overline{\Omega}$, and which satisfies:

$$u(t,x) - \phi(x_1 + ct) \to 0 \text{ as } t \to -\infty, \quad \text{uniformly in } x \in \overline{\Omega}, \tag{7.38}$$

where ϕ is the unique solution of (1.3) (with, say, the normalization condition $\phi(0) = 1/2$). The strong maximum principle also implies that $0 < u(t,x) < 1$ for all $(t,x) \in \mathbb{R} \times \overline{\Omega}$.

From Proposition 5.1,

$$u(t,x) \to u_\infty(x) \text{ as } t \to +\infty,$$

locally uniformly in $x \in \overline{\Omega}$, and $u_\infty(x) \to 1$ as $|x| \to +\infty$. The function u_∞ is a classical solution of (5.1). Since the obstacle K is assumed to be star-shaped or directionally convex, it follows then from Theorems 6.1, 6.4 and Remark 6.2 that

$$u_\infty \equiv 1 \text{ in } \overline{\Omega}.$$

We are going to check the assumptions of Theorem 7.1, with say, $t_0 = 0$. Let $\varepsilon > 0$ be an arbitrary positive real number. Since $u(t,x) \to u_\infty(x) = 1$ as $t \to +\infty$ locally uniformly in $x \in \overline{\Omega}$, there is a time $t_\varepsilon \geq 0$ such that (7.3) holds.

Let us now deal with property (7.2). From (7.38), there exists a time $T_1 \leq 0$ such that $|u(T_1, x) - \phi(x_1 + cT_1)| \leq \varepsilon/2$ for all $x \in \overline{\Omega}$. Since $\phi(+\infty) = 1$, there is then $\xi_+ \geq 0$ such that

$$H_+ = \{x \in \mathbb{R}^N,\ x_1 \geq \xi_+\} \subset \Omega$$

and $1-\varepsilon \le u(T_1,x) \le 1$ for all $x \in H_+$, whence

$$1-\varepsilon \le u(t_\varepsilon,x) \le 1 \text{ for all } x \in H_+$$

since $t_\varepsilon \ge 0 \ge T_1$ and $u_t > 0$. Similarly, since $c > 0$ and $\phi' > 0$, there holds:

$$1-\varepsilon/2 \le \phi(x_1+cT_1) \le \phi(x_1+ct_\varepsilon) \le 1 \quad \text{for all} \quad x \in H_+.$$

As a consequence,

$$|u(t_\varepsilon,x)-\phi(x_1+ct_\varepsilon)| \le \varepsilon \text{ for all } x \in H_+. \tag{7.39}$$

For $0 < \delta < \min(1/2, \varepsilon/2)$, call g_δ the function defined in $\mathbb{R}$ by

$$g_\delta(s) = \begin{cases} f(2\delta) \times \dfrac{s-\delta}{\delta} & \text{if } s \le 2\delta \\ f(s) & \text{if } 2\delta < s < 1-\delta, \\ f(1-\delta) \times \dfrac{1+\delta-s}{2\delta} & \text{if } s \ge 1-\delta. \end{cases}$$

It is immediate to check that for $\delta > 0$ small enough, the function g_δ has two stable zeroes at δ and $1+\delta$, namely $g_\delta(\delta) = g_\delta(1+\delta) = 0$, $g_\delta'(\delta) < f'(0) < 0$, $f'(1) < g_\delta'(1+\delta) < 0$. Furthermore, for $\delta > 0$ small enough, $g_\delta \ge f$ on $\mathbb{R}$ (f is assumed to be extended in $\mathbb{R}$ by (4.3), as usual) and there exists a unique (up to shifts) solution $(\gamma_\delta, w_\delta)$ of (1.3) with the nonlinearity g_δ and the conditions

$$\delta = w_\delta(-\infty) < w_\delta < w_\delta(+\infty) = 1+\delta.$$

It is also known that $\gamma_\delta > c$, whence $\gamma_\delta > 0$. Let $T_2 \le 0$ be such that $|u(T_2,x)-\phi(x_1+cT_2)| \le \delta/2$ for all $x \in \overline{\Omega}$. Since $\phi(-\infty) = 0$, there exists $\tilde{\xi}_- \le 0$ such that

$$\tilde{H}_- = \{x \in \mathbb{R}^N,\ x_1 \le \tilde{\xi}_-\} \subset \Omega$$

and

$$0 \le u(T_2,x) \le \delta \text{ for all } x \in \tilde{H}_-.$$

Since $w_\delta(+\infty) = 1+\delta$, there is $\xi_0 \in \mathbb{R}$ such that $w_\delta(\tilde{\xi}_- + \gamma_\delta T_2 + \xi_0) \ge 1$ for all $x_1 \ge \tilde{\xi}_-$. One shall compare $u(t,x)$ with $W(t,x) = w_\delta(x_1+\gamma_\delta t+\xi_0)$ in $\tilde{H}_-$ for $t \ge T_2$. First, $u(T_2,x) \le \delta \le W(T_2,x)$ for all $x \in \tilde{H}_-$. Observe also that, for all $t \ge T_2$ and $x \in \partial\tilde{H}_-$ (that is, $x_1 = \tilde{\xi}_-$),

$$W(t,x) = w_\delta(x_1+\gamma_\delta t+\xi_0) \ge w_\delta(\tilde{\xi}_- + \gamma_\delta T_2 + \xi_0) \ge 1 \ge u(t,x).$$

Lastly,

$$W_t = \Delta W + g_\delta(W) \ge \Delta W + f(W)$$

from the choice of δ. The parabolic maximum principle yields

$$0 \le u(t,x) \le w_\delta(x_1+\gamma_\delta t+\xi_0) \text{ for all } t \ge T_2 \text{ and } x \in \tilde{H}_-.$$

In particular, $0 \le u(t_\varepsilon, x) \le w_\delta(x_1 + \gamma_\delta t_\varepsilon + \xi_0)$ for all $x \in \tilde{H}_-$, whence

$$\limsup_{x_1 \to -\infty,\ x' \in \mathbb{R}^{N-1}} u(t_\varepsilon, x) \le \delta \le \frac{\varepsilon}{2},$$

since $w_\delta(-\infty) = \delta \le \varepsilon/2$. Since $\phi(-\infty) = 0$, there exists then $\xi_- \le 0$ such that

$$H_- = \{x \in \mathbb{R}^N,\ x_1 \le \xi_-\} \subset \Omega$$

and

$$|u(t_\varepsilon, x) - \phi(x_1 + ct_\varepsilon)| \le \varepsilon \text{ for all } x \in H_-. \tag{7.40}$$

Notice also that, since δ can be chosen arbitrarily small, the above arguments imply that, for all $\tau \ge 0$,

$$\sup_{x \in \overline{\Omega},\ |t| \le \tau,\ x_1 \le -A} u(t, x) \to 0 \text{ as } A \to +\infty, \tag{7.41}$$

and this property holds *even if the obstacle K is not star-shaped or directionally convex.* On the other hand, it follows from Proposition 4.1 that there exists $B \ge 0$ such that

$$E = \{x \in \mathbb{R}^N,\ \xi_- \le x_1 \le \xi_-,\ |x'| \ge B\} \subset \Omega$$

and

$$|u(t_\varepsilon, x) - \phi(x_1 + ct_\varepsilon)| \le \varepsilon \text{ for all } x \in E. \tag{7.42}$$

We now set

$$C = C_\varepsilon = \{x \in \mathbb{R}^N,\ \xi_- \le x_1 \le \xi_+,\ |x'| \le B\} \cap \overline{\Omega}.$$

The set C_ε is a compact subset of $\overline{\Omega}$ and

$$|u(t_\varepsilon, x) - \phi(x_1 + ct_\varepsilon)| \le \varepsilon \text{ for all } x \in \overline{\Omega} \backslash C_\varepsilon$$

from (7.39), (7.40) and (7.42). In other words, property (7.2) is fulfilled.

Theorem 7.1 then implies that

$$u(t, x) - \phi(x_1 + ct) \to 0 \text{ as } t \to +\infty$$

uniformly in $x \in \overline{\Omega}$.

Lastly, let us show that $u(t, x) - \phi(x_1 + ct) \to 0$ as $|x| \to +\infty$, uniformly in $t \in \mathbb{R}$. Let $\varepsilon > 0$ be an arbitrary positive real number. Since $u(t, x) - \phi(x_1 + ct) \to 0$ as $t \to \pm\infty$ uniformly in $x \in \overline{\Omega}$, there exists $\tau \ge 0$ such that $|u(t, x) - \phi(x_1 + ct)| \le \varepsilon$ for all $|t| \ge \tau$ and for all $x \in \overline{\Omega}$. On the other hand, the same arguments as above yield the existence of a compact set $D_\varepsilon \subset \overline{\Omega}$ such that

$$|u(t, x) - \phi(x_1 + ct)| \le \varepsilon \text{ for all } |t| \le \tau \text{ and for all } x \in \overline{\Omega} \backslash D_\varepsilon.$$

Indeed, this is exactly what we just proved when $t = t_\varepsilon$ was fixed, but the arguments immediately work locally in time.

As a conclusion,

$$u(t, x) - \phi(x_1 + ct) \to 0 \text{ as } |x| \to +\infty$$

uniformly in $t \in \mathbb{R}$. The proof of Theorem 1 is complete. □

8 Large time behaviour for a general compact obstacle

In this section, we deal with the general case, that is when the compact obstacle K is not assumed to be star-shaped or directionally convex anymore. The convergence of the solution $u(t,x)$ to the planar front $\phi(x_1+ct)$ as $t\to+\infty$ only takes place in half-spaces $\{x_1\le\zeta(t)\}$, provided that $\lim_{t\to+\infty}\zeta(t)=-\infty$.

Proof of Theorem 2. First, as in the beginning of the proof of Theorem 1, there exists a time-global solution $u(t,x)$ of (1.1) such that $0<u(t,x)<1$, $u_t(t,x)>0$ for all $(t,x)\in\mathbb{R}\times\overline{\Omega}$ and such that the limit (7.38) holds uniformly in $x\in\overline{\Omega}$. Furthermore, $u(t,x)\to u_\infty(x)$ as $t\to+\infty$ locally uniformly in $x\in\overline{\Omega}$, where $0<u_\infty\le 1$ is a solution of (5.1) such that $u_\infty(x)\to 1$ as $|x|\to+\infty$.

Let us now fix an arbitrary map $t\mapsto\zeta(t)$ such that $\lim_{t\to+\infty}\zeta(t)=-\infty$ and let us prove that

$$\sup_{x\in\overline{\Omega},\ x_1\le\zeta(t)}|u(t,x)-\phi(x_1+ct)|\to 0 \text{ as } t\to+\infty. \tag{8.1}$$

The proof has some similarities with the one of Theorem 7.1, but the fact that, in general, $u(t,x)\not\to 1$ as $t\to+\infty$ introduces new difficulties.

Step 1 : Some constants which depend on f and ϕ only. First fix a positive real number $\sigma>0$ such that

$$\max\left(2\|\phi'\|_\infty\sigma+\sigma^2,c\sigma\right)<\min\left(\frac{2|f'(0)|}{3},\frac{3|f'(1)|}{8}\right). \tag{8.2}$$

Extend f by (4.3), set

$$\omega=\frac{c\sigma}{4}\ (<\min\left(|f'(0)|,|f'(1)|\right)) \tag{8.3}$$

and choose $0<\rho<1/2$ so that (7.10) holds. Let $A>0$ be such that

$$\begin{cases}\phi(z) & \ge & 1-\dfrac{\rho}{2} & \text{for all } z\ge A,\\ \phi(z) & \le & \dfrac{\rho}{2} & \text{for all } z\le -A\end{cases} \tag{8.4}$$

and let $\delta>0$ be defined as in (7.12). Lastly, fix a constant $\kappa>0$ such that

$$\max_{[0,1]}|f|+3\|f'\|_\infty+2\|\phi'\|_\infty\sigma+\sigma^2+2\omega<\kappa\delta\omega\theta\ (<\kappa\delta\omega), \tag{8.5}$$

where $\theta\in(0,1)$ is chosen so that

$$f>0 \text{ on } (\theta,1).$$

All above constants are fixed throughout the proof. They will be used in the construction of lower and upper functions in Step 3 and 4.

Step 2 : Expansion of the set where $u(t,x)\ge 1-\varepsilon$ in the $-x_1$-direction. Let ε be any positive real number such that

$$0<\varepsilon<\min(1-\theta,\rho/8). \tag{8.6}$$

Let $R_\varepsilon > 0$ be such that

$$K \subset B_{R_\varepsilon}(0)$$

and

$$u_\infty(x) \geq 1 - \frac{\varepsilon}{2} \text{ for all } |x| \geq R_\varepsilon. \tag{8.7}$$

Let now $R_1 = R_1(1-\varepsilon)$, $R_2 = R_2(1-\varepsilon)$, $R_3 = R_3(1-\varepsilon)$ and $T = T(1-\varepsilon)$ be as in Lemma 5.2. Let $t_0 > 0$ be such that

$$\phi(-R_\varepsilon - R_1 - R_3 + ct_0) \geq 1 - \frac{\varepsilon}{2}. \tag{8.8}$$

From Proposition 4.1, there holds

$$u(t_0, x_1, x') \underset{|x'| \to +\infty}{\to} \phi(x_1 + ct_0) \text{ uniformly in } x_1 \in [-R_\varepsilon - R_1 - R_3, -R_\varepsilon].$$

Since ϕ is increasing, there exists then $R'_\varepsilon > 0$ such that

$$u(t_0, x_1, x') \geq 1 - \varepsilon \text{ for all } x_1 \in [-R_\varepsilon - R_1 - R_3, -R_\varepsilon] \text{ and } |x'| \geq R'_\varepsilon. \tag{8.9}$$

On the other hand, it follows from the definition of u_∞ and (8.7) that there exists t_ε such that

$$t_\varepsilon \geq t_0 \tag{8.10}$$

and

$$u(t_\varepsilon, x_1, x') \geq 1 - \varepsilon \text{ for all } x_1 \in [-R_\varepsilon - R_1 - R_3, -R_\varepsilon] \text{ and } |x'| \leq R'_\varepsilon.$$

Together with (8.9) and the time-monotonicity of u, one gets that

$$u(t, x_1, x') \geq 1 - \varepsilon \text{ for all } t \geq t_\varepsilon,\ x_1 \in [-R_\varepsilon - R_1 - R_3, -R_\varepsilon] \text{ and } x' \in \mathbb{R}^{N-1}. \tag{8.11}$$

Remember that the half-space $\{x \in \mathbb{R}^N,\ x_1 \leq -R_\varepsilon\}$ is included in $\overline{\Omega}$. For all $x' \in \mathbb{R}^{N-1}$, it follows from (8.11) and Lemma 5.2 that

$$u(t + T, y) \geq 1 - \varepsilon \text{ for all } t \geq t_\varepsilon \text{ and for all } y \in B_{R_2}(-R_\varepsilon - R_3, x').$$

Therefore,

$$\begin{aligned} u(t, x_1, x') \geq 1 - \varepsilon \quad & \text{for all } t \geq t_\varepsilon + T,\ x' \in \mathbb{R}^{N-1} \\ & \text{and } x_1 \in [-R_\varepsilon - R_3 - R_2, -R_\varepsilon] = [-R_\varepsilon - R_3 - R_1 - (R_2 - R_1), -R_\varepsilon]. \end{aligned}$$

By immediate induction, we get that, for all $k \in \mathbb{N}$,

$$u(t, x_1, x') \geq 1 - \varepsilon \text{ for all } t \geq t_\varepsilon + kT,\ x' \in \mathbb{R}^{N-1} \text{ and } x_1 \in [-R_\varepsilon - R_3 - R_1 - k(R_2 - R_1), -R_\varepsilon].$$

In particular,

$$\begin{aligned} u(t, x) \geq 1 - \varepsilon \quad & \text{for all } t \geq t_\varepsilon,\ x' \in \mathbb{R}^{N-1} \\ & \text{and } x_1 \in \left[-R_\varepsilon - R_3 - R_1 + R_2 - R_1 - \frac{R_2 - R_1}{T} \times (t - t_\varepsilon), -R_\varepsilon\right]. \end{aligned}$$

Let $R_4 = R_\varepsilon + R_3 - R_2 + 2R_1$ and observe that

$$-R_\varepsilon - R_3 - R_1 < -R_4 < -R_\varepsilon. \tag{8.12}$$

Since $R_2 - R_1 > cT/4$ from (5.3), it follows in particular that

$$u(t,x) \geq 1 - \varepsilon \text{ for all } t \geq t_\varepsilon,\ x' \in \mathbb{R}^{N-1} \text{ and } -R_4 - \frac{c(t-t_\varepsilon)}{4} \leq x_1 \leq -R_4. \tag{8.13}$$

Denote

$$H = \{x \in \mathbb{R}^N,\ x_1 \leq -R_4\} \subset \overline{\Omega}.$$

As in the proof of Theorem 1, there exists a compact set

$$C_\varepsilon \subset H$$

such that

$$|u(t_\varepsilon, x) - \phi(x_1 + ct_\varepsilon)| \leq \varepsilon \text{ for all } x \in H \backslash C_\varepsilon. \tag{8.14}$$

Since C_ε is compact, one has $0 < \min_{C_\varepsilon} u(t_\varepsilon, \cdot) \leq \max_{C_\varepsilon} u(t_\varepsilon, \cdot) < 1$ and there is a constant $\beta = \beta_\varepsilon > 0$ such that

$$\phi(x_1 + ct_\varepsilon - \beta e^{-|x'|^2}) \leq u(t_\varepsilon, x) \leq \phi(x_1 + ct_\varepsilon + \beta e^{-|x'|^2}) \text{ for all } x \in C_\varepsilon, \tag{8.15}$$

because $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. Let

$$\gamma = \max(8, 8\beta C_1) \text{ and } \alpha = 2(N-1)\gamma^{-1}, \tag{8.16}$$

where the constant C_1 is given by (7.5).

Step 3 : Construction of a lower function $\underline{u}$. For $t \geq 1$ and $x \in H$, define

$$\tilde{u}(t,x) = u(t - 1 + t_\varepsilon, x)$$

and

$$\underline{u}(t,x) = \phi(\xi) w(x_1) - 2\varepsilon e^{-\omega(t-1)},$$

where

$$\begin{cases} \xi = \xi(t,x) &= x_1 + c(t - 1 + t_\varepsilon) - \beta t^{-\alpha} e^{-\frac{|x'|^2}{\gamma t}} + \kappa\varepsilon e^{-\omega(t-1)} - \kappa\varepsilon, \\ w(x_1) &= 1 - \varepsilon e^{\sigma(x_1 + R_4)}, \end{cases}$$

and the constants ω, κ and σ were chosen in Step 1.

If $x \in H \backslash C_\varepsilon$, then

$$\underline{u}(1,x) \leq \phi(x_1 + ct_\varepsilon) - 2\varepsilon \leq u(t_\varepsilon, x) - \varepsilon = \tilde{u}(1,x) - \varepsilon$$

because of (8.14). On the other hand, since $\gamma \geq 1$ and $\phi' > 0$, it follows that if $x \in C_\varepsilon$, then

$$\underline{u}(1,x) \leq \phi(x_1 + ct_\varepsilon - \beta e^{-|x'|^2}) - 2\varepsilon \leq u(t_\varepsilon, x) - 2\varepsilon = \tilde{u}(1,x) - 2\varepsilon$$

from (8.15). As a consequence,

$$\underline{u}(1,x) \le \tilde{u}(1,x) - \varepsilon \text{ for all } x \in H. \tag{8.17}$$

Our goal here is to prove that $\underline{u}(t,x) \le \tilde{u}(t,x)$ for all $t \ge 1$ and $x \in H$. Actually, it is not clear that $\underline{u}$ is a sub-solution of (1.1) in $[1,+\infty) \times H$, but this property shall at least be true in the set where $\underline{u}(t,x) \ge \tilde{u}(t,x)$. This will be enough to get the desired result. Define

$$E = \{(t,x) \in [1,+\infty) \times H,\ \underline{u}(t,x) > \tilde{u}(t,x)\} \tag{8.18}$$

and assume that E is not empty. Then, set

$$\underline{t} = \inf\ \{t \ge 1,\ \exists\ x \in H,\ (t,x) \in E\}.$$

Then $\underline{t} \in [1,+\infty)$ and there exists a sequence $(t_n,x_n)_{n\in\mathbb{N}} = (t_n,x_{1,n},x'_n)_{n\in\mathbb{N}}$ in $[1,+\infty) \times H$ such that $t_n \to \underline{t}$ as $n \to +\infty$, and

$$\underline{u}(t_n,x_n) > \tilde{u}(t_n,x_n) \text{ for all } n \in \mathbb{N}. \tag{8.19}$$

We claim that the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded. If $x_{1,n} \to -\infty$ as $n \to +\infty$ (up to extraction of a subsequence), then

$$\tilde{u}(t_n,x_n) = u(t_n - 1 + t_\varepsilon, x_n) \to 0 \text{ as } n \to +\infty$$

from property (7.41). But $\limsup_{n\to+\infty} \underline{u}(t_n,x_n) \le -2\varepsilon e^{-\omega(\underline{t}-1)} < 0$ by definition of $\underline{u}$ and since $\phi(-\infty) = 0$. This contradicts (8.19) and therefore $x_{1,n} \to \underline{x}_1 \in (-\infty,-R_4]$ as $n \to +\infty$ (for a subsequence). Now, if $|x'_n| \to +\infty$ as $n \to +\infty$ (up to extraction of another subsequence), then

$$\tilde{u}(t_n,x_n) = u(t_n - 1 + t_\varepsilon, x_n) \to \phi(\underline{x}_1 + c(\underline{t} - 1 + t_\varepsilon)) \text{ as } n \to +\infty$$

from Proposition 4.1. But

$$\limsup_{n\to+\infty}\ \underline{u}(t_n,x_n) \le \phi(\underline{x}_1 + c(\underline{t} - 1 + t_\varepsilon)) - 2\varepsilon e^{-\omega(\underline{t}-1)} < \phi(\underline{x}_1 + c(\underline{t} - 1 + t_\varepsilon))$$

and one has reached a contradiction too. As a consequence, one can also assume that $x'_n \to \underline{x}' \in \mathbb{R}^{N-1}$ as $n \to +\infty$.

Set $\underline{x} = (\underline{x}_1, \underline{x}') \in H$. Passing to the limit as $n \to +\infty$ in (8.19) yields $\underline{u}(\underline{t},\underline{x} \ge \tilde{u}(\underline{t},\underline{x})$. In particular, $\underline{t} > 1$ from (8.17). Therefore, $\underline{u}(t,x) \le \tilde{u}(t,x)$ for all $1 \le t \le \underline{t}$ and $x \in H$, whence

$$\underline{u}(\underline{t},\underline{x}) = \tilde{u}(\underline{t},\underline{x})$$

and

$$\underline{u}_t(\underline{t},\underline{x}) \ge \tilde{u}_t(\underline{t},\underline{x}).$$

Furthermore, for all $x' \in \mathbb{R}^{N-1}$,

$$\underline{u}(\underline{t},-R_4,x') \le 1 - \varepsilon - 2\varepsilon e^{-\omega(\underline{t}-1)} < 1 - \varepsilon \le u(\underline{t} - 1 + t_\varepsilon, -R_4, x') = \tilde{u}(\underline{t},-R_4,x')$$

from (8.13). As a consequence, $\underline{x}_1 < -R_4$, whence

$$\Delta \underline{u}(\underline{t},\underline{x}) \le \Delta \tilde{u}(\underline{t},\underline{x}).$$

Finally,

$$\underline{u}_t(\underline{t},\underline{x}) - \Delta \underline{u}(\underline{t},\underline{x}) - f(\underline{u}(\underline{t},\underline{x})) \ge \tilde{u}_t(\underline{t},\underline{x}) - \Delta \tilde{u}(\underline{t},\underline{x}) - f(\tilde{u}(\underline{t},\underline{x})) = 0. \tag{8.20}$$

Let

$$\mathcal{L}\underline{u}(t,x) = \underline{u}_t(t,x) - \Delta \underline{u}(t,x) - f(\underline{u}(t,x))$$

for all $t \ge 1$ and $x \in H$. One has

$$\mathcal{L}\underline{u}(t,x) = L_1 + L_2 + L_3 + L_4,$$

where

$$\left\{\begin{array}{rcl} L_1 & = & c\phi'(\xi)w(x_1) - \phi''(\xi)w(x_1) + [2\phi'(\xi)\sigma + \phi(\xi)\sigma^2]\varepsilon e^{\sigma(x_1+R_4)} \\ & & -f\left(\phi(\xi)w(x_1) - 2\varepsilon e^{-\omega(t-1)}\right), \\ L_2 & = & 2\omega\varepsilon e^{-\omega(t-1)} - \omega\kappa\varepsilon\phi'(\xi)w(x_1)e^{-\omega(t-1)}, \\ L_3 & = & \beta(\alpha - 2(N-1)\gamma^{-1})t^{-\alpha-1}e^{-\frac{|x'|^2}{\gamma t}}\phi'(\xi)w(x_1), \\ L_4 & = & \beta\gamma^{-1}(4\gamma^{-1}-1)t^{-\alpha-2}|x'|^2 e^{-\frac{|x'|^2}{\gamma t}}\phi'(\xi)w(x_1) - 4\beta^2\gamma^{-2}t^{-2\alpha-2}|x'|^2 e^{-\frac{2|x'|^2}{\gamma t}}\phi''(\xi)w(x_1). \end{array}\right.$$

Notice first that $L_3 = 0$ because of (8.16). Moreover, $L_4 \le 0$ from (7.5), (8.16) and since $t \ge 1$, $\phi' > 0$ and $w > 0$. Therefore,

$$\begin{array}{rcl} \mathcal{L}\underline{u}(t,x) & \le & f(\phi(\xi))w(x_1) - f\left(\phi(\xi)w(x_1) - 2\varepsilon e^{-\omega(t-1)}\right) \\ & & +(2\|\phi'\|_\infty\sigma + \sigma^2)\varepsilon e^{\sigma(x_1+R_4)} + [2 - \kappa\phi'(\xi)w(x_1)]\omega\varepsilon e^{-\omega(t-1)} \end{array} \tag{8.21}$$

for all $t \ge 1$ and $x \in \overline{\Omega}$.

Lemma 8.1 *One actually has* $\mathcal{L}\underline{u}(\underline{t},\underline{x}) < 0$.

The proof is postponed to the Appendix in Section 11. From Lemma 8.1, one gets a contradiction with (8.20). As a conclusion, the set E defined in (8.18) is empty, whence

$$\underline{u}(t,x) \le \tilde{u}(t,x) = u(t-1+t_\varepsilon, x) \text{ for all } t \ge 1 \text{ and } x \in H.$$

Therefore, for all $t \ge t_\varepsilon$,

$$\begin{array}{l} \inf_{x\in H}[u(t,x) - \phi(x_1+ct)] \\ \qquad \ge \inf_{x\in H}[\underline{u}(t+1-t_\varepsilon,x) - \phi(x_1+ct)] \\ \qquad = \inf_{x\in H}\left[\phi\left(x_1 + ct - \beta(t+1-t_\varepsilon)^{-\alpha}e^{-\frac{|x'|^2}{\gamma(t+1-t_\varepsilon)}} + \kappa\varepsilon e^{-\omega(t-t_\varepsilon)} - \kappa\varepsilon\right)\right. \\ \qquad\qquad \left.\times\left(1 - \varepsilon e^{-\sigma(x_1+R_4)}\right) - 2\varepsilon e^{-\omega(t-t_\varepsilon)} - \phi(x_1+ct)\right] \\ \qquad \ge -\|\phi'\|_\infty\left(\beta(t+1-t_\varepsilon)^{-\alpha} + \kappa\varepsilon\right) - \varepsilon - 2\varepsilon e^{-\omega(t-t_\varepsilon)} \end{array}$$

since $0 \le \phi \le 1$. As a consequence,

$$\liminf_{t\to+\infty} \left\{ \inf_{x\in H} [u(t,x) - \phi(x_1+ct)] \right\} \ge -(\kappa\|\phi'\|_\infty + 1)\varepsilon.$$

Since this is true for all ε small enough, since the constant κ does not depend on ε, and since $\zeta(t) \to -\infty$ as $t \to +\infty$, one concludes that

$$\liminf_{t\to+\infty} \left\{ \inf_{x\in\overline{\Omega},\ x_1\le\zeta(t)} [u(t,x) - \phi(x_1+ct)] \right\} \ge 0. \tag{8.22}$$

Step 4 : Construction of an upper function $\overline{u}$. For $t \ge 1$ and $x \in H$, set

$$\overline{u}(t,x) = \phi(\tilde{\xi})\tilde{w}(x_1) + 2\varepsilon e^{-\omega(t-1)},$$

where

$$\begin{cases} \tilde{\xi} = \tilde{\xi}(t,x) &= x_1 + c(t-1+t_\varepsilon) + \beta t^{-\alpha} e^{-\frac{|x'|^2}{\gamma t}} + \kappa\varepsilon - \kappa\varepsilon e^{-\omega(t-1)}, \\ \tilde{w}(x_1) &= 1 + \varepsilon e^{\sigma(x_1+R_4)}, \end{cases}$$

and the constants ω, κ and σ (resp. α, β and γ) were chosen in Step 1 (resp. Step 2).

From (8.14), (8.15), and since $\gamma \ge 1$ and $\phi' > 0$, it follows as in Step 3 that

$$\overline{u}(1,x) \ge \tilde{u}(1,x) + \varepsilon \text{ for all } x \in H.$$

Observe also that, for all $t \ge 1$ and $x' \in \mathbb{R}^{N-1}$,

$$\overline{u}(t,-R_4,x') \ge \phi(-R_4 + c(t-1+t_\varepsilon)) \times (1+\varepsilon) \ge \left(1 - \frac{\varepsilon}{2}\right) \times (1+\varepsilon) > 1 \ge \tilde{u}(t,-R_4,x').$$

The inequality $\phi(-R_4 + c(t-1+t_\varepsilon)) \ge 1 - \varepsilon/2$ indeed follows from the positivity of ϕ' and from (8.8), (8.10), (8.12). Let

$$\tilde{E} = \{(t,x) \in [1,+\infty) \times H,\ \overline{u}(t,x) < \tilde{u}(t,x)\} \tag{8.23}$$

and assume that $\tilde{E}$ is not empty. As in Step 3, it follows from the above observations that there exists a point $(\overline{t},\overline{x}) = (\overline{t},\overline{x}_1,\overline{x}')$ such that $\overline{t} > 1$, $\overline{x}_1 < -R_4$,

$$\overline{u}(\overline{t},\overline{x}) = \tilde{u}(\overline{t},\overline{x})$$

and

$$\mathcal{L}\overline{u}(\overline{t},\overline{x}) \le 0. \tag{8.24}$$

On the other hand, the following lemma holds

Lemma 8.2 *One actually has* $\mathcal{L}\overline{u}(\overline{t},\overline{x}) > 0$.

Therefore, one has reached a contradiction. As a consequence, the set $\tilde{E}$ defined in (8.23) is empty, whence $\overline{u}(t,x) \geq \tilde{u}(t,x) = u(t-1+t_\varepsilon, x)$ for all $t \geq 1$ and $x \in H$. One concludes as in Step 3 that

$$\liminf_{t\to+\infty} \left\{ \inf_{x\in H} [\phi(x_1+ct) - u(t,x)] \right\} \geq -(\kappa\|\phi'\|_\infty + 1)\varepsilon$$

and then that

$$\liminf_{t\to+\infty} \left\{ \inf_{x\in\overline{\Omega},\ x_1\leq\zeta(t)} [\phi(x_1+ct) - u(t,x)] \right\} \geq 0.$$

Together with (8.22), we get the claim (8.1).

Let us finally prove that

$$u(t,x) - \phi(x_1+ct) \to 0 \text{ as } |x| \to +\infty, \text{ uniformly in } t \in \mathbb{R}. \tag{8.25}$$

Fix any arbitrary positive real number ε. From (7.38), there exists $t_1 < 0$ such that

$$|u(t,x) - \phi(x_1+ct)| \leq \frac{\varepsilon}{2} \text{ for all } t \leq t_1 \text{ and for all } x \in \overline{\Omega}. \tag{8.26}$$

Since $\phi' > 0$ and $\phi(+\infty) = 1$, there exists $r_1 > 0$ such that

$$H_1 = \{x_1 \geq r_1\} \subset \overline{\Omega}$$

and $\phi(r_1+ct_1) \geq 1-\varepsilon/2$, whence $1 \geq \phi(x_1+ct) \geq 1-\varepsilon/2$ for all $x_1 \geq r_1$ and $t \geq t_1$. As a consequence of (8.26), it follows that $(1 \geq)\ u(t_1,x) \geq 1-\varepsilon$ for all $x_1 \geq r_1$, whence $1 \geq u(t,x) \geq 1-\varepsilon$ for all $t \geq t_1$ and $x_1 \geq r_1$ since $u_t > 0$. Therefore,

$$|u(t,x) - \phi(x_1+ct)| \leq \varepsilon \text{ for all } t \in \mathbb{R} \text{ and } x_1 \geq r_1.$$

From the arguments of the proof of (8.1), there exist $t_2 > 0$ and $r_2 < 0$ such that

$$H_2 = \{x_1 \leq r_2\} \subset \overline{\Omega}$$

and $|u(t,x) - \phi(x_1+ct)| \leq \varepsilon$ for all $t \geq t_2$ and $x \in H_2$. From (7.41) and $\phi(-\infty) = 0$, there is $r_3 \leq r_2$ such that $0 \leq \phi(r_3+ct_2) \leq \varepsilon$ and $0 \leq u(t_2,x) \leq \varepsilon$ for all $x_1 \leq r_3$. Since $\phi' > 0$, $u_t > 0$ and $u > 0$, it follows that $|u(t,x) - \phi(x_1+ct)| \leq \varepsilon$ for all $t \leq t_2$ and $x_1 \leq r_3$. Therefore,

$$|u(t,x) - \phi(x_1+ct)| \leq \varepsilon \text{ for all } t \in \mathbb{R} \text{ and for all } x_1 \leq r_3.$$

On the other hand, there is $t_3 > 0$ such that $\phi(r_3+ct_3) \geq 1-\varepsilon/2$, whence

$$\phi(x_1+ct) \geq 1 - \frac{\varepsilon}{2} \text{ for all } t \geq t_3 \text{ and } x_1 \geq r_3. \tag{8.27}$$

From Proposition 4.1, there exists $r' > 0$ such that $K \subset B_{r'}(0)$ and

$$|u(t,x) - \phi(x_1+ct)| \leq \frac{\varepsilon}{2} \text{ for all } t_1 \leq t \leq t_3,\ r_3 \leq x_1 \leq r_1 \text{ and } |x'| \geq r'. \tag{8.28}$$

This holds also for all $t \leq t_1$ because of (8.26). Lastly, for all $t \geq t_3$ and $r_3 \leq x_1 \leq r_1$, one has that $1 \geq \phi(x_1 + ct) \geq 1 - \varepsilon/2$. Together with (8.28) and $u < 1$, it follows that $1 - \varepsilon \leq u(t_3, x) \leq 1$ for all $r_3 \leq x_1 \leq r_1$ and $|x'| \geq r'$. Finally, since $u_t > 0$, one gets that $|u(t,x) - \phi(x_1 + ct)| \leq \varepsilon$ for all $t \geq t_3$, $r_3 \leq x_1 \leq r_1$ and $|x'| \geq r'$.

One concludes that $|u(t,x) - \phi(x_1 + ct)| \leq \varepsilon$ for all $t \in \mathbb{R}$ and for all $x \in \overline{\Omega}$ such that either $x_1 \geq r_1$, or $x_1 \leq r_3$, or $r_3 \leq x_1 \leq r_1$ and $|x'| \geq r'$. Since $\varepsilon > 0$ was arbitrary, we have shown (8.25). Property (1.8) follows from (8.25) and Proposition 5.1. The proof of Theorem 2 is complete. □

9 The solutions $u(t, x)$ are generalized fronts

This section is devoted to the proof of Theorem 3. One shall prove that the limits (1.9) hold uniformly in (t, x) as $x_1 + ct \to \pm\infty$, where u_∞ solves (1.7).

If the obstacle is star-shaped or directionally convex, then $u_\infty = 1$ from Theorems 6.1 or 6.4. In these cases, one still has that $u(t,x) \to u_\infty = 1$ at $t \to +\infty$ locally in x, because $\phi(+\infty) = 1$ and $\sup_{x\in\overline{\Omega}} |u(t,x) - \phi(x_1 + ct)| \to 0$ as $t \to +\infty$. Actually, the limit $\lim_{t\to+\infty} u(t,x) = 1$ then holds uniformly in any family of half-spaces $\{x_1 \geq \xi(t)\}$ such that $\xi(t) + ct \to +\infty$ as $t \to +\infty$.

The proof of (1.9) is done below in the general case of a compact obstacle, which may or may not be star-shaped or directionally convex.

Assume that the first limit in (1.9) does not hold. Then, there exist $\varepsilon > 0$ and a sequence of points $(t_n, x_n)_{n\in\mathbb{N}} = (t_n, x_{1,n}, \ldots, x_{N,n})_{n\in\mathbb{N}}$ in $\mathbb{R}\times\overline{\Omega}$ such that $x_{1,n}+ct_n \to +\infty$ as $n \to +\infty$ and

$$\forall\, n \in \mathbb{N}, \quad u_\infty(x_n) - u(t_n, x_n) \geq \varepsilon. \tag{9.1}$$

We here used the fact $|u_\infty(x) - u(t,x)| = u_\infty(x) - u(t,x)$, since $u_t > 0$ in $\mathbb{R} \times \overline{\Omega}$. Up to extraction of a subsequence, two cases may occur: either $x_n \to x_\infty \in \overline{\Omega}$ or $|x_n| \to +\infty$ as $n \to +\infty$.

Case 1: $x_n \to x_\infty \in \overline{\Omega}$ as $n \to +\infty$. Then $t_n \to +\infty$, $u(t_n, x_n) \to u_\infty(x_\infty)$ and $u_\infty(x_n) \to u_\infty(x_\infty)$ as $n \to +\infty$, which is impossible due to (9.1).

Case 2: $|x_n| \to +\infty$ as $n \to +\infty$. Then

$$u_\infty(x_n) \to 1 \text{ as } n \to +\infty. \tag{9.2}$$

Up to extraction of another subsequence, three subcases may occur.

Subcase 2.1: $t_n \to -\infty$ as $n \to +\infty$. Then $u(t_n, x_n) - \phi(x_{1,n} + ct_n) \to 0$ from (1.6). But $x_{1,n} + ct_n \to +\infty$, whence $\phi(x_{1,n} + ct_n) \to 1$ and $u(t_n, x_n) \to 1$. Together with (9.1) and (9.2), one gets a contradiction.

Subcase 2.2: $t_n \to t_\infty \in \mathbb{R}$ as $n \to +\infty$. Thus, $x_{1,n} \to +\infty$. Let $T < 0$ be such that $\sup_{x\in\overline{\Omega}} |u(T,x) - \phi(x_1 + cT)| \leq \varepsilon/2$, and $T < t_n$ for all $n \in \mathbb{N}$. Since $u_t > 0$, one gets

$$\forall\, n \in \mathbb{N}, \quad 1 \geq u(t_n, x_n) \geq u(T, x_n) \geq \phi(x_{1,n} + cT) - \frac{\varepsilon}{2},$$

whence $\liminf_{n\to+\infty} u(t_n, x_n) \geq 1 - \varepsilon/2$. Therefore, $\limsup_{n\to+\infty} u_\infty(x_n) - u(t_n, x_n) \leq \varepsilon/2$, which contradicts (9.1).

Subcase 2.3: $t_n \to +\infty$ as $n \to +\infty$. If, up to extraction of another subsequence, $x_{1,n} \to -\infty$, then $u(t_n, x_n) - \phi(x_{1,n} + ct_n) \to 0$ from (8.1) and one reaches a contradiction as in Subcase 2.1. If $x_{1,n} \to x_{1,\infty} \in \mathbb{R}$, then, for any $T \in \mathbb{R}$, one has, for n large enough,

$$1 \geq u(t_n, x_n) \geq u(T, x_n).$$

But $u(T, x_n) \to \phi(x_{1,\infty} + cT)$ as $n \to +\infty$ because of Proposition 4.1, and $\phi(x_{1,\infty} + cT) \to 1$ as $T \to +\infty$. Hence $u(t_n, x_n) \to 1$ as $n \to +\infty$ and a contradiction is reached as in Subcase 2.1. Lastly, if $x_{1,n} \to +\infty$, then argue as in Subcase 2.2 to get a contradiction.

As a conclusion, the first limit in (1.9) has been proved.

Assume now that the second limit in (1.9) does not hold. Then, there exist $\varepsilon > 0$ and a sequence of points $(t_n, x_n)_{n\in\mathbb{N}} = (t_n, x_{1,n}, \ldots, x_{N,n})_{n\in\mathbb{N}}$ in $\mathbb{R} \times \overline{\Omega}$ such that $x_{1,n} + ct_n \to -\infty$ as $n \to +\infty$ and

$$\forall\, n \in \mathbb{N}, \quad u(t_n, x_n) \geq \varepsilon \tag{9.3}$$

(remember that $u > 0$). Up to extraction of a subsequence, three cases may occur.

Case 1: $t_n \to -\infty$ as $n \to +\infty$. Then $u(t_n, x_n) - \phi(x_{1,n} + ct_n) \to 0$ because of (1.6). But $x_{1,n} + ct_n \to -\infty$, whence $\phi(x_{1,n} + ct_n) \to 0$ and $u(t_n, x_n) \to 0$, which contradicts (9.3).

Case 2: $t_n \to t_\infty \in \mathbb{R}$ as $n \to +\infty$. Then $x_{1,n} \to -\infty$ and $u(t_n, x_n) \to 0$ because of property (7.41), which holds for general compact obstacles which may or may not be star-shaped or directionally convex. Case 2 is then ruled out too.

Case 3: $t_n \to +\infty$ as $n \to +\infty$. Then $x_{1,n} \to -\infty$ and $u(t_n, x_n) - \phi(x_{1,n} + ct_n) \to 0$ from (8.1) and one gets a contradiction as in Case 1.

Therefore, the second limit in (1.9) has been checked and the proof of Theorem 3 is complete. □

10 Appendix A: proofs of Lemmas 7.2–7.5

Proof of Lemma 7.2. On the interval $[t_2, t_1 - 1]$, the function v is increasing and (always) positive, whence $-v'(t) \leq 0 \leq 2\omega v(t)$ for all $t \in [t_2, t_1 - 1]$. By definition of v, one has $-v'(t) = \omega v(t) \leq 2\omega v(t)$ for all $t \in [1, t_1 - 2 - 2\omega^{-1}] \cup [t_1, +\infty)$. For all $t \in [t_1 - 1, t_1]$, one has

$$\frac{|v'(t)|}{v(t)} = \frac{\omega(t - t_1 + 1)}{1 + \dfrac{\omega}{2}\,(1 - (t - t_1 + 1)^2)} \leq \omega \leq 2\omega.$$

Hence, it only remains to prove that $-v'(t) = |v'(t)| \leq 2\omega v(t)$ for all $t \in I = [t_1 - 2 - 2\omega^{-1}, t_2]$, where t_2 was defined in (7.28). Define

$$h(t) = -\frac{v'(t)}{v(t)} = \frac{|v'(t)|}{v(t)} \ \text{ for } t \in I.$$

One has

$$h'(t) = -\frac{\sigma_1}{2} v(t)^{-2} i(t) \ \text{ for all } t \in I,$$

where

$$i(t) = C_5 \varepsilon^{1/2} - \frac{\sigma_1}{4}(t - t_2)^2 - \frac{\sigma_1}{4}(t_1 - 2 - t_2)^2.$$

The function i, which can be defined in $\mathbb{R}$, is such that $i(t_2) = v(t_2) = \omega^2\sigma_2\sigma_1^{-1} > 0$, whence

$$i(t) = 0 \text{ if and only if } t = t_\pm = t_2 \pm (i(t_2)4\sigma_1^{-1})^{1/2} = t_2 \pm 2\omega\sigma_2^{1/2}\sigma_1^{-1}.$$

If $t_- < t_1-2-2\omega^{-1}$, then $i(t) > 0$ and $h'(t) < 0$ on I, whence $h(t) \le h(t_1-2-2\omega^{-1}) = \omega \le 2\omega$ on I. Otherwise, $t_- \ge t_1 - 2 - 2\omega^{-1}$, that is

$$t_1 - 2 - 2\omega^{-1} \le t_- = t_1 - 2 - \frac{2\omega^{-1}C_5\varepsilon^{1/2} + 2\omega^{-1}\sqrt{C_5\varepsilon^{1/2}\mu e^{2\omega+2-\omega(t_1-1)}}}{C_5\varepsilon^{1/2} + \mu e^{2\omega+2-\omega(t_1-1)}},$$

whence $\mu e^{2\omega+2-\omega(t_1-1)} \ge C_5\varepsilon^{1/2}$ and

$$\sigma_2^{1/2} \ge C_5\varepsilon^{1/2}. \tag{10.1}$$

Furthermore, $h' \ge 0$ on $[t_1 - 2 - 2\omega^{-1}, t_-]$, $h' \le 0$ on $[t_-, t_2]$ and then

$$\begin{aligned}\max_I h = h(t_-) = \frac{\sigma_1\omega^{-1}\sigma_2^{-1/2}}{2} &\le \frac{\omega(C_5\varepsilon^{1/2} + \mu e^{2\omega+2-\omega(t_1-1)})}{2C_5\varepsilon^{1/2}} \\ &\le \frac{\omega}{2} + \frac{\omega\mu}{2C_5\varepsilon^{1/2}} = \frac{\omega}{2} + \frac{\omega(2+\omega)}{4} \le 2\omega\end{aligned}$$

because of (7.9), (7.16), (7.17), (7.27) and (10.1).

As a conclusion, inequality (7.30) always holds and the proof of Lemma 7.2 is complete.□

Proof of Lemma 7.3. One has

$$\mathcal{L}\underline{u} = L_1 + L_2 + L_3 + L_4,$$

where

$$\left\{\begin{array}{lcl} L_1 &=& c\phi'(\xi) - \phi''(\xi) - f(\phi(\xi) - v(t)\zeta(x)) \\ L_2 &=& -v'(t)\zeta(x) + v(t)\Delta\zeta(x) + V'(t)\phi'(\xi) \\ L_3 &=& \beta(\alpha - 2(N-1)\gamma^{-1})t^{-\alpha-1}e^{-\frac{|x'|^2}{\gamma t}}\phi'(\xi) \\ L_4 &=& \beta\gamma^{-1}(4\gamma^{-1} - 1)t^{-\alpha-2}|x'|^2e^{-\frac{|x'|^2}{\gamma t}}\phi'(\xi) - 4\beta^2\gamma^{-2}t^{-2\alpha-2}|x'|^2e^{-2\frac{|x'|^2}{\gamma t}}\phi''(\xi). \end{array}\right.$$

Notice first that $L_1 = f(\phi(\xi)) - f(\phi(\xi) - v(t)\zeta(x))$ by definition of ϕ. Next, $L_3 = 0$ because of (7.24). Furthermore, since $t \ge 1$ and $\phi' \ge 0$, it follows from (7.5), (7.6), (7.22) and (7.24) that

$$L_4 \le \beta\gamma^{-1}(4\gamma^{-1} - 1 + 4\beta\gamma^{-1}C_1)t^{-\alpha-2}|x'|^2e^{-\frac{|x'|^2}{\gamma t}}\phi'(\xi) \le 0.$$

Therefore,

$$\mathcal{L}\underline{u} \le f(\phi(\xi)) - f(\phi(\xi) - v(t)\zeta(x)) - v'(t)\zeta(x) + v(t)\Delta\zeta(x) + V'(t)\phi'(\xi) \tag{10.2}$$

for all $t \ge 1$ and $x \in \overline{\Omega}$.

Let $A > 0$ be given as in (7.11). One shall consider three cases : $\xi(t,x) \le -A$, $\xi(t,x) \ge A$ and $-A \le \xi(t,x) \le A$.

First case : $\xi = \xi(t,x) \leq -A$. Then

$$\phi(\xi) - v(t)\zeta(x) \leq \phi(\xi) \leq \rho$$

(remember that v and ζ are positive functions). Hence

$$[\ \xi(t,x) \leq -A\ ] \Longrightarrow [\ f(\phi(\xi)) - f(\phi(\xi) - v(t)\zeta(x)) \leq (f'(0)+\omega)v(t)\zeta(x)\ ]$$

because of (7.9) and (7.10). Furthermore, $\phi' \geq 0$, and $V' \leq 0$ owing to the definition of V in (7.31). Therefore,

$$\begin{array}{rcl}[\ \xi(t,x) \leq -A\ ] \Longrightarrow [\ \mathcal{L}\underline{u} & \leq & (f'(0)+\omega)v(t)\zeta(x) - v'(t)\zeta(x) + v(t)\Delta\zeta(x) \\ & \leq & (f'(0)+4\omega)v(t)\zeta(x) \leq 0\ ]\end{array}$$

because of (7.9), (7.13) and (7.30).

Second case : $\xi = \xi(t,x) \geq A$. Then

$$\phi(\xi) \geq \phi(\xi) - v(t)\zeta(x) \geq 1 - \frac{\rho}{2} - \mu\|\zeta\|_\infty \geq 1 - \rho$$

because of (7.11), (7.16) and (7.29). Using again (7.9), (7.10), (7.13) and (7.30), one concludes as above that $\mathcal{L}\underline{u} \leq 0$.

Third case : $-A \leq \xi = \xi(t,x) \leq A$. Then $\phi'(\xi) \geq \delta > 0$ by (7.12). Since $V'(t) \leq 0$, it then follows from (10.2) that

$$L\underline{u} \leq \|f'\|_\infty v(t)\zeta(x) - v'(t)\zeta(x) + v(t)\Delta\zeta(x) + \delta V'(t).$$

Using again (7.13) and (7.30), together with the fact that $\omega \leq \|f'\|_\infty$, from (7.9), one gets that

$$\mathcal{L}\underline{u} \leq (\|f'\|_\infty + 3\omega)v(t)\zeta(x) + \delta V'(t) \leq 4\|f'\|_\infty\|\zeta\|_\infty v(t) + \delta V'(t) = 0$$

from the definition of V in (7.31). □

Proof of Lemma 7.4. From (7.33) and (7.17), one has

$$\phi(\xi) = \underline{u}(t,x) + v(t)\zeta(x) \geq 1 - \varepsilon \geq 1 - \kappa_0.$$

Thus, (7.8) yields

$$\phi'(\xi) \leq C_3\varepsilon^{1/2}e^{-\lambda\xi/2}. \tag{10.3}$$

Furthermore, κ_0 was chosen so that $\kappa_0 \leq 1/2$, whence $\phi(\xi) \geq 1 - \kappa_0 \geq 1/2$. Since ϕ is increasing and $\phi(0) = 1/2$ (by normalization), one gets that $\xi \geq 0$. On the other hand, since $x_1 \leq 0$ on ∂K and since $V(t) \leq V(1)$, it follows from the definition of L in (7.4), that

$$\xi = \xi(t,x) \leq c(t-1+t_\varepsilon) - \beta t^{-\alpha}e^{-\frac{(N-1)L^2}{\gamma t}} = g(t),$$

where g was defined in (7.25). Thus, $g(t) \geq 0$. But $g' \geq c/2 > 0$ by (7.26), and $t_1 > 1$ was chosen in (7.27) so that $g(t_1) = 0$. Hence, $t \geq t_1$. Define the function j by

$$j(\tau) = -L + c(\tau - 1 + t_\varepsilon) - \beta\tau^{-\alpha} - V(1) \text{ for all } \tau \geq 1,$$

where $L \geq 0$ is such that $x_1 \geq -L$ on ∂K, by (7.4). The function j is such that

$$j'(\tau) = c + \alpha\beta\tau^{-\alpha-1} \geq c$$

and then

$$\xi \geq -L + c(t - 1 + t_\varepsilon) - \beta t^{-\alpha} - V(1) = j(t) \geq j(t_1) + c(t - t_1). \tag{10.4}$$

On the other hand,

$$\begin{array}{rcl} j(t_1) & = & j(t_1) - g(t_1) \\ & = & -L - V(1) + \beta t_1^{-\alpha}\left(e^{-\frac{(N-1)L^2}{\gamma t_1}} - 1\right) \\ & \geq & -L - V(1) + \beta t_1^{-\alpha}\left(e^{-\frac{(N-1)L^2}{\gamma}} - 1\right) \quad \text{(because } t_1 \geq 1\text{)} \\ & \geq & -L - V(1) - 2\beta t_1^{-\alpha}(N-1)L^2\gamma^{-1} \quad \text{(because of (7.23) and (7.24))} \\ & \geq & -L - V(1) - 1 \quad \text{(because of (7.24) and } t_1 \geq 1\text{)} \\ & \geq & -L - 2. \end{array}$$

The last line follows from (7.15) and (7.32). Putting together with (10.3) and (10.4), one obtains

$$\phi'(\xi) \leq C_3 e^{\lambda(L+2)/2} \varepsilon^{1/2} e^{-\lambda c(t-t_1)/2}. \tag{10.5}$$

On ∂K, the function ζ satisfies $\nu \cdot \nabla\zeta = 1$, where $\nu = (\nu_1, \ldots, \nu_N)$ denotes the outward unit normal on $\partial\Omega$. Hence, still assuming (7.33), one has

$$\begin{array}{rcl} \nu \cdot \nabla \underline{u} & = & \left(\nu_1 + \displaystyle\sum_{i=2}^{N} \nu_i \beta t^{-\alpha-1} 2x_i \gamma^{-1} e^{-\frac{|x'|^2}{\gamma t}}\right)\phi'(\xi) - v(t) \\ & \leq & \left(1 + \dfrac{(N-1)L}{4}\right)\phi'(\xi) - v(t) \quad \text{(because } \phi' \geq 0,\ t \geq 1 \text{ and } \gamma = 8C_1\beta \geq 8\beta\text{)} \\ & \leq & \left(1 + \dfrac{(N-1)L}{4}\right) C_3 e^{\lambda(L+2)/2}\varepsilon^{1/2}e^{-\lambda c(t-t_1)/2} - v(t) \quad \text{(because of (10.5))} \\ & = & C_5 \varepsilon^{1/2} e^{-\lambda c(t-t_1)/2} - C_5 \varepsilon^{1/2} e^{-\omega(t-t_1)} \quad \text{(because of (7.14) and } t \geq t_1\text{)} \\ & \leq & 0 \quad \text{(because of (7.9) and } t \geq t_1\text{).} \end{array}$$

That completes the proof of Lemma 7.4. □

Proof of Lemma 7.5. For $t \geq 1$ and $x \in \partial\Omega$, one can check that

$$\begin{array}{rcl} \nu \cdot \nabla \overline{u}(t,x) & \geq & -\left(1 + \dfrac{(N-1)L}{4}\right)\phi'(\tilde{\xi}) + \tilde{v}(t) \\ & \geq & -C_2'\left(1 + \dfrac{(N-1)L}{4}\right)e^{-\lambda\tilde{\xi}} + \tilde{v}(t) \qquad \text{(from (7.35)).} \end{array}$$

On the other hand, by definition of L in (7.4),

$$\forall\ t \geq 1,\ \forall\ x \in \partial\Omega, \quad \tilde{\xi}(t,x) \geq -L + c(t - 1 + t_{\varepsilon'}) + \beta' t^{-\alpha'} e^{-\frac{(N-1)L^2}{\gamma' t}} =: k(t)$$

and

$$k'(t) \geq c - \alpha'\beta' t^{-\alpha'-1} e^{-\frac{(N-1)L^2}{\gamma' t}} \geq c - \alpha'\beta' = c - \frac{N-1}{4C_1} \geq \frac{c}{2}$$

by (7.34). Therefore, for all $t \geq 1$ and $x \in \partial\Omega$, one has

$$\tilde{\xi}(t,x) \geq k(t) \geq k(1) + \frac{c}{2} \times (t-1) \geq -L + ct_{\varepsilon'} + \beta' e^{-(N-1)L^2} + \frac{c}{2} \times (t-1)$$

and

$$\begin{array}{rcl} \nu \cdot \nabla \overline{u}(t,x) & \geq & -C_2' \left(1 + \dfrac{(N-1)L}{4}\right) e^{-\lambda(-L+ct_{\varepsilon'}+\beta' e^{-(N-1)L^2}) - \lambda c(t-1)/2} + \mu' e^{-\omega(t-1)} \\ & \geq & 0 \end{array}$$

from (7.36) and because of the inequality $\omega \leq \lambda c/2$. □

11 Appendix B: proofs of Lemmas 8.1 and 8.2

Proof of Lemma 8.1. Set

$$\underline{\xi} = \xi(\underline{t}, \underline{x}).$$

One shall consider three cases : $\underline{\xi} \leq -A$, $-A \leq \underline{\xi} \leq A$ and $\underline{\xi} \geq A$.

First case : $\underline{\xi} \leq -A$. Thus, $\phi(\underline{\xi}) \leq \rho/2 \leq \rho \leq 1 - \varepsilon$, whence

$$u(\underline{t} - 1 + t_\varepsilon, \underline{x}) = \tilde{u}(\underline{t}, \underline{x}) = \underline{u}(\underline{t}, \underline{x}) \leq (1-\varepsilon)(1 - \varepsilon e^{\sigma(\underline{x}_1 + R_4)}) - 2\varepsilon e^{-\omega(\underline{t}-1)} < 1 - \varepsilon.$$

Therefore, $\underline{x}_1 < -R_4 - (c/4)(\underline{t} - 1)$ from (8.13), and

$$e^{\sigma(\underline{x}_1 + R_4)} \leq e^{-\frac{c\sigma}{4}(\underline{t}-1)} = e^{-\omega(\underline{t}-1)} \tag{11.1}$$

from (8.3). Furthermore, $\phi(\underline{\xi}) w(\underline{x}_1) - 2\varepsilon e^{-\omega(\underline{t}-1)} \leq \phi(\underline{\xi}) \leq \rho/2 \leq \rho$. It then follows from (7.10), (8.21), (11.1) and the positivity of ϕ' and w that

$$\begin{array}{rcl} \mathcal{L}\underline{u}(\underline{t}, \underline{x}) & \leq & f(\phi(\underline{\xi}))w(\underline{x}_1) - f(\phi(\underline{\xi})) + f(\phi(\underline{\xi})) - f\left(\phi(\underline{\xi})w(\underline{x}_1) - 2\varepsilon e^{-\omega(\underline{t}-1)}\right) \\ & & +(2\|\phi'\|_\infty \sigma + \sigma^2)\varepsilon e^{-\omega(\underline{t}-1)} + 2\omega\varepsilon e^{-\omega(\underline{t}-1)} \\ & \leq & [1 - w(\underline{x}_1)]\ [-f'(0) + \omega]\ \phi(\underline{\xi}) + [f'(0) + \omega] \left\{\phi(\underline{\xi})[1 - w(\underline{x}_1)] + 2\varepsilon e^{-\omega(\underline{t}-1)}\right\} \\ & & +(2\|\phi'\|_\infty \sigma + \sigma^2 + 2\omega)\varepsilon e^{-\omega(\underline{t}-1)} \\ & \leq & 2\omega\phi(\underline{\xi})\varepsilon e^{\sigma(\underline{x}_1 + R_4)} + [2f'(0) + 4\omega + 2\|\phi'\|_\infty \sigma + \sigma^2]\varepsilon e^{-\omega(\underline{t}-1)} \\ & \leq & [2f'(0) + 6\omega + 2\|\phi'\|_\infty \sigma + \sigma^2]\varepsilon e^{-\omega(\underline{t}-1)}. \end{array}$$

Therefore,

$$\mathcal{L}\underline{u}(\underline{t}, \underline{x}) \leq \left[2f'(0) + \frac{3c\sigma}{2} + 2\|\phi'\|_\infty \sigma + \sigma^2\right] \varepsilon e^{-\omega(\underline{t}-1)} < 0$$

from (8.2) and (8.3).

Second case : $-A \le \underline{\xi} \le A$. Then, as previously,

$$\phi(\underline{\xi}) \le 1 - \frac{\rho}{2} \le 1 - \varepsilon \quad \text{and} \quad e^{\sigma(\underline{x}_1 + R_4)} \le e^{-\omega(\underline{t}-1)}.$$

Furthermore, $\phi'(\underline{\xi}) \ge \delta$ from (7.12). It follows from (8.21) that

$$\begin{array}{rcl} \mathcal{L}\underline{u}(\underline{t},\underline{x}) & \le & f(\phi(\underline{\xi}))w(\underline{x}_1) - f(\phi(\underline{\xi})) + f(\phi(\underline{\xi})) - f\left(\phi(\underline{\xi})w(\underline{x}_1) - 2\varepsilon e^{-\omega(\underline{t}-1)}\right) \\ & & +(2\|\phi'\|_\infty \sigma + \sigma^2)\varepsilon e^{-\omega(\underline{t}-1)} + 2\omega\varepsilon e^{-\omega(\underline{t}-1)} - \kappa\delta\omega\, w(\underline{x}_1)\varepsilon e^{-\omega(\underline{t}-1)} \\ & & \\ & \le & \left(\max\limits_{[0,1]} |f|\right) [1 - w(\underline{x}_1)] + \|f'\|_\infty [1 - w(\underline{x}_1) + 2\varepsilon e^{-\omega(\underline{t}-1)}] \\ & & +[2\|\phi'\|_\infty \sigma + \sigma^2 + 2\omega - \kappa\delta\omega\, w(\underline{x}_1)]\varepsilon e^{-\omega(\underline{t}-1)}. \end{array}$$

Since $1 - w(\underline{x}_1) = \varepsilon e^{\sigma(\underline{x}_1 + R_4)} \le \varepsilon e^{-\omega(\underline{t}-1)}$ and $w(\underline{x}_1) \ge 1 - \varepsilon \ge \theta$ from (8.6), it follows that

$$\mathcal{L}\underline{u}(\underline{t},\underline{x}) \le \left(\max\limits_{[0,1]} |f| + 3\|f'\|_\infty + 2\|\phi'\|_\infty \sigma + \sigma^2 + 2\omega - \kappa\delta\omega\theta\right) \varepsilon e^{-\omega(\underline{t}-1)} < 0$$

from (8.5).

Third case : $\underline{\xi} \ge A$. Then $\phi(\underline{\xi}) \ge 1 - \rho/2$ and

$$\phi(\underline{\xi}) \ge \phi(\underline{\xi})w(\underline{x}_1) - \varepsilon e^{-\omega(\underline{t}-1)} \ge (1-\varepsilon)\phi(\underline{\xi}) - \varepsilon \ge \phi(\underline{\xi}) - 2\varepsilon \ge 1 - \frac{\rho}{2} - 2\varepsilon \ge 1 - \rho$$

from (8.6). In particular, $f(\phi(\underline{\xi})) \ge 0$ from (7.10) and (8.3), whence $f(\phi(\underline{\xi}))[w(\underline{x}_1) - 1] \le 0$. As a consequence, it follows from (7.10), (8.21) and the positivity of ϕ' and w that

$$\begin{array}{rcl} \mathcal{L}\underline{u}(\underline{t},\underline{x}) & \le & f(\phi(\underline{\xi}))w(\underline{x}_1) - f(\phi(\underline{\xi})) + f(\phi(\underline{\xi})) - f\left(\phi(\underline{\xi})w(\underline{x}_1) - 2\varepsilon e^{-\omega(\underline{t}-1)}\right) \\ & & +[2\|\phi'\|_\infty \sigma + \sigma^2]\varepsilon e^{\sigma(\underline{x}_1 + R_4)} + 2\omega\varepsilon e^{-\omega(\underline{t}-1)} \\ & & \\ & \le & [f'(1) + \omega]\left\{\phi(\underline{\xi})[1 - w(\underline{x}_1)] + 2\varepsilon e^{-\omega(\underline{t}-1)}\right\} \\ & & +[2\|\phi'\|_\infty \sigma + \sigma^2]\varepsilon e^{\sigma(\underline{x}_1 + R_4)} + 2\omega\varepsilon e^{-\omega(\underline{t}-1)} \\ & & \\ & \le & \{[f'(1) + \omega]\phi(\underline{\xi}) + 2\|\phi'\|_\infty \sigma + \sigma^2\}\varepsilon e^{\sigma(\underline{x}_1 + R_4)} + [2f'(1) + 4\omega]\varepsilon e^{-\omega(\underline{t}-1)}. \end{array}$$

Since $f'(1) + \omega \le 0$ from (8.3) and $\phi(\underline{\xi}) \ge 1 - \rho/2 \ge 3/4$, one gets that

$$\mathcal{L}\underline{u}(\underline{t},\underline{x}) \le \left[\frac{3f'(1)}{4} + \frac{3c\sigma}{16} + 2\|\phi'\|_\infty \sigma + \sigma^2\right] \varepsilon e^{\sigma(\underline{x}_1 + R_4)} + [2f'(1) + c\sigma]\varepsilon e^{-\omega(\underline{t}-1)} < 0$$

from (8.2). □

Proof of Lemma 8.2. The same calculations as in the proof of Lemma 8.1 yield

$$\begin{array}{rcl} \mathcal{L}\overline{u}(\overline{t},\overline{x}) & \ge & f\left(\phi(\overline{\xi})\right)\tilde{w}(\overline{x}_1) - f\left(\phi(\overline{\xi})\tilde{w}(\overline{x}_1) + 2\varepsilon e^{-\omega(\overline{t}-1)}\right) \\ & & -(2\|\phi'\|_\infty \sigma + \sigma^2)\varepsilon e^{\sigma(\overline{x}_1 + R_4)} + \left[-2 + \kappa\phi'(\overline{\xi})\tilde{w}(\overline{x}_1)\right]\omega\varepsilon e^{-\omega(\overline{t}-1)}, \end{array}$$

where

$$\overline{\xi} = \tilde{\xi}(\overline{t}, \overline{x}).$$

First case : $\overline{\xi} \leq -A$. Then $0 \leq \phi(\overline{\xi}) \leq \rho/2$ and

$$\phi(\overline{\xi}) \leq \phi(\overline{\xi})\left(1+\varepsilon e^{\sigma(\overline{x}_1+R_4)}\right) + 2\varepsilon e^{-\omega(\overline{t}-1)} = \overline{u}(\overline{t}, \overline{x}) \leq \frac{\rho}{2} + \frac{\rho\varepsilon}{2} + 2\varepsilon \leq \frac{\rho}{2} + 3\varepsilon \leq \rho < 1-\varepsilon$$

from (8.6). Consequently, $u(\overline{t}-1+t_\varepsilon, \overline{x}) = \tilde{u}(\overline{t}, \overline{x}) = \overline{u}(\overline{t}, \overline{x}) < 1-\varepsilon$ and thus $\overline{x}_1 < -R_4 - (c/4)(\overline{t}-1)$ from (8.13). Therefore, $e^{\sigma(\overline{x}_1+R_4)} < e^{-\omega(\overline{t}-1)}$. With the same calculations as in Step 3, one gets from (7.10) and the positivity of ϕ' that

$$\mathcal{L}\overline{u}(\overline{t}, \overline{x}) \geq \left[2|f'(0)| - 2\|\phi'\|_\infty\sigma - \sigma^2 - 6\omega\right]\varepsilon e^{-\omega(\overline{t}-1)} > 0$$

from (8.2) and (8.3).

Second case : $-A \leq \overline{\xi} \leq A$. Then, $\phi(\overline{\xi}) \leq 1-\rho/2$ and

$$u(\overline{t}-1+t_\varepsilon, \overline{x}) = \tilde{u}(\overline{t}, \overline{x}) = \overline{u}(\overline{t}, \overline{x}) \leq \left(1-\frac{\rho}{2}\right)(1+\varepsilon) + 2\varepsilon \leq 1 - \frac{\rho}{2} + 3\varepsilon < 1-\varepsilon$$

from (8.6). Therefore, $e^{\sigma(\overline{x}_1+R_4)} < e^{-\omega(\overline{t}-1)}$ and, as in Step 3, one gets that

$$\mathcal{L}\overline{u}(\overline{t}, \overline{x}) \geq \left(\kappa\delta\omega - \max_{[0,1]}|f| - 3\|f'\|_\infty - 2\|\phi'\|_\infty\sigma - \sigma^2 - 2\omega\right)\varepsilon e^{-\omega(\overline{t}-1)} > 0$$

from (8.5).

Third case : $\overline{\xi} \geq A$. One then has

$$\frac{3}{4} \leq 1 - \frac{\rho}{2} \leq \phi(\overline{\xi}) \leq \phi(\overline{\xi})\tilde{w}(\overline{x}_1) + 2\varepsilon e^{-\omega(\overline{t}-1)}$$

and it follows from (7.10) and (8.3) that $f(\phi(\overline{\xi})) \geq 0$, whence $f(\phi(\overline{\xi}))w(\overline{x}_1) \geq f(\phi(\overline{\xi}))$. Together with the positivity of ϕ', one gets that

$$\begin{aligned}\mathcal{L}\overline{u}(\overline{t}, \overline{x}) &\geq [-f'(1)-\omega]\left[\phi(\overline{\xi})\varepsilon e^{\sigma(\overline{x}_1+R_4)} + 2\varepsilon e^{-\omega(\overline{t}-1)}\right] \\ &\quad -[2\|\phi'\|_\infty\sigma + \sigma^2]\varepsilon e^{\sigma(\overline{x}_1+R_4)} - 2\omega\varepsilon e^{-\omega(\overline{t}-1)} \\ &\geq \left[\frac{3|f'(1)|}{4} - \frac{3\omega}{4} - 2\|\phi'\|_\infty\sigma - \sigma^2\right]\varepsilon e^{\sigma(\overline{x}_1+R_4)} + [2|f'(1)| - 4\omega]\varepsilon e^{-\omega(\overline{t}-1)} > 0\end{aligned}$$

from (8.2) and (8.3). □

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